A Character Analogue of Ramanujan's formula for Odd Zeta Values

M.Sc. Thesis

by

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A Character Analogue of Ramanujan's formula for Odd Zeta Values

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **A Character Analogue of Ramanujan's formula for Odd Zeta Values** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF MATHEMATICS**, **Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2021 to June 2022 under the supervision of **Dr. Bibekananda Maji**, Assistant Professor, Department of Mathematics, IIT Indore. The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Abstract

Around two decades ago, three Japanese mathematicians Kanemitsu, Tanigawa, and Yoshimoto investigated an infinite series of the following form:

$$\sum_{m=1}^{\infty} \frac{m^{N-2h}}{\exp(m^N x) - 1},$$

where $N \in \mathbb{N}$ and $h \in \mathbb{Z}$ with some restriction on h. Recently, Dixit and Maji pointed out that this series is already present in the lost notebook of Ramanujan with a more general form. Although, Ramanujan did not provide any transformation identity for it. Dixit and Maji found an elegant generalization of Ramanujan's celebrated identity for $\zeta(2m+1)$ while extending the results of Kanemitsu et al. Later, Kanemitsu et al. also studied another extended version of the aforementioned series, namely,

$$\sum_{r=1}^{q} \sum_{n=1}^{\infty} \frac{\chi(r) n^{N-2h} \exp\left(-\frac{r}{q} n^N x\right)}{1 - \exp(-n^N x)},$$

where χ denotes the Dirichlet character modulo q. They have studied this series for $N \in 2\mathbb{N}$ and with some restriction on the variable h. In this thesis, we investigate the same series for any $N \in \mathbb{N}$ and $h \in \mathbb{Z}$. Moreover, we obtain interesting formulas for the Dirichlet L-function at rational arguments.

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List of Symbols

Symbol	Description
$\mathbb{N}$	The set of positive integers
$2\mathbb{N}$	The set of all positive even integers
$\mathbb{Z}$	The ring of integers
$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The field of complex numbers
$\int_{(c_0)}$	$\int_{c_0-i\infty}^{c_0+i\infty}$
$(\mathbb{Z}/n\mathbb{Z})^*$	The set of all units of the ring $\mathbb{Z}/n\mathbb{Z}$
(a,b)	The greatest common divisor of a and b
$\lfloor x \rfloor$	$\max\{k \in \mathbb{Z} \mid k \leq x\}$
p	A rational prime number
P	The set of all primes
γ	Euler's constant
$\Re(s)$	Real part of a complex number s
$\Im(s)$	Imaginary part of a complex number s
$g_1(x) = O(g_2(x))$	$\lvert g_1(x)\rvert \leq M g_2(x)$ for x large enough, where M is a positive constant
$g_1(x) \ll g_2(x)$	$g_1(x) = O(g_2(x))$

Chapter 1

Introduction

It is well-known that the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^x}$ converges if and only if $x > 1$. Now the natural question is that where does it converge for a fixed x? Can we obtain some explicit formula for this series? Can we say something about the algebraic nature of the values of $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\sum_{n=1}^{\infty} \frac{1}{n^3}$, $\sum_{n=1}^{\infty} \frac{1}{n^4}, \cdots$? Are these values algebraic or transcendental? The Basel problem was, about finding the exact value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$, first introduced by Pietro Mengoli in 1644. This problem remained open for almost 90 years. It was Euler who first showed that the exact value of the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is $\frac{\pi^2}{6}$. Moreover, he found a remarkable generalization of this identity. Mainly, he showed that, for any $m \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \frac{(-1)^{m-1}(2\pi)^{2m}}{2(2m)!} B_{2m} = \pi^{2m} \times r, \tag{1.1}$$

where B_{2m} denotes the $2m$th Bernoulli number, and r is some rational number. Transcendentality of π together with the above formula, one can conclude that the series $\sum_{n=1}^{\infty} \frac{1}{n^{2m}}$ converges to a transcendental number. In 19th century, Bernhard Riemann studied the same series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ with complex variable s on his ground-breaking article [23]. Riemann denoted this series by $\zeta(s)$, and due to that, now it is popularly known as the Riemann zeta function. We can observe that Euler's formula (1.1) tells us about the transcendental nature of even zeta values. The next instinctive question arises about the arithmetic nature of odd zeta values. Is there any formula for $\zeta(2m+1)$ that is similar to the formula (1.1)? The nature of $\zeta(2m+1)$, $m \in \mathbb{N}$, remains mystery except $\zeta(3)$. In 1979, Roger Apéry [2] demonstrated that $\zeta(3)$ is irrational. But, we are still unaware of the algebraic nature of $\zeta(3)$. In 2001, Ball and Rioval [4] provided a breakthrough result. They showed that there are infinitely many $\zeta(2m+1)$ which are linearly independent over the rational. This indicates the existence of infinitely many irrational odd zeta values. Strikingly, Zudilin [26] gave an impressive result about the same period. He showed that "At least one among $\zeta(5), \zeta(7), \zeta(9)$ and $\zeta(11)$ is irrational". This is one of the finest results in this field.

Ramanujan in his 2nd Notebook [22, p. 173, Entry 21(i)] as well as in his Lost Notebook [21, p. 320, Formula (28)], noted down the following appealing identity for $\zeta(2m+1)$:

Let $\alpha, \beta \in \mathbb{R}^+$ with $\alpha\beta = \pi^2$. For every non-zero integer m,

$$\begin{aligned}
&\alpha^{-m}\left(\frac{1}{2}\zeta(2m+1) + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{(e^{2n\alpha}-1)}\right) \\
&= (-\beta)^{-m}\left(\frac{1}{2}\zeta(2m+1) + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{(e^{2n\beta}-1)}\right) \\
&- 2^{2m}\sum_{k=0}^{m+1}(-1)^k \frac{B_{2k}B_{2m+2-2k}}{(2k)!(2m+2-2k)!}\alpha^{m+1-k}\beta^k. \qquad (1.2)
\end{aligned}$$

This identity does not provide an explicit formula for $\zeta(2m+1)$, unlike Euler's identity (1.1), but it is considered as another outstanding discovery of Ramanujan that has grabbed the attention of several mathematicians. Malurkar [18], who had no knowledge of Ramanujan's Notebook, gave the first published proof in 1925. Berndt [5], in 1977, showed that the Euler's identity (1.1) and Ramanujan's identity (1.2) can be deduced from a single formula associated to an extended Eisenstien series. The Fourier series expansion of Eisenstien series has a close connection with the Ramanujan's identity (1.2). One can learn more about this connection in [6], [7].

Around two decades ago, three Japanese mathematicians Kanemitsu, Tanigawa, and Yoshimoto [14] investigated an infinite series of the following form:

$$\sum_{m=1}^{\infty} \frac{m^{N-2h}}{\exp(m^N x) - 1}, \tag{1.3}$$

where $N \in \mathbb{N}$ and h is an integer lying in the interval $[1, N/2]$. They were able to derive interesting identities for $\zeta(s)$ at rational arguments while studying the above infinite series. For example, they obtained Ramanujan's identity for $\zeta(1/2)$. Recently, Dixit and Maji [10] noticed that the series (1.3) is in fact present in the Lost Notebook of Ramanujan with a more general form. Although, Ramanujan did not provide any transformation identity for it. This motivated Dixit and Maji [10] to study (1.3) further. Quite unexpectedly, they found an astonishing generalization of Ramanujan's celebrated identity for $\zeta(2m+1)$ while extending the results of Kanemitsu et al. Mainly, Dixit and Maji [10, Theorem 1.2] obtained the following generalized identity which connects two distinct odd zeta values:

Suppose N is an odd natural number and $\alpha, \beta \in \mathbb{R}^+$ with $\alpha\beta^N = \pi^{N+1}$. Then for $m \in \mathbb{Z} - \{0\}$, we have

$$\begin{aligned}
&\alpha^{-\frac{2Nm}{N+1}} \left(\frac{1}{2}\zeta(2Nm+1) + \sum_{n=1}^{\infty} \frac{n^{-2Nm-1}}{\exp\left((2n)^N \alpha\right) - 1} \right) = \left(-\beta^{\frac{2N}{N+1}}\right)^{-m} \frac{2^{2m(N-1)}}{N} \\
&\times \left(\frac{1}{2}\zeta(2m+1) + (-1)^{\frac{N+3}{2}} \sum_{j=-\frac{(N-1)}{2}}^{\frac{N-1}{2}} (-1)^j \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{\exp\left((2n)^{\frac{1}{N}} \beta e^{\frac{i\pi j}{N}}\right) - 1} \right) \\
&+ (-1)^{m+\frac{N+3}{2}} 2^{2Nm} \sum_{j=0}^{\left\lfloor \frac{N+1}{2N} + m \right\rfloor} \frac{(-1)^j B_{2j} B_{N+1+2N(m-j)}}{(2j)!(N+1+2N(m-j))!} \alpha^{\frac{2j}{N+1}} \beta^{N+\frac{2N^2(m-j)}{N+1}}. \qquad (1.4)
\end{aligned}$$

Here we emphasize that the following integral representation of (1.3) was the starting point of the work of Dixit and Maji:

$$\sum_{m=1}^{\infty} \frac{m^{N-2h}}{\exp(m^N x) - 1} = \frac{1}{2\pi i} \int_{c_0 - i\infty}^{c_0 + i\infty} \Gamma(s)\zeta(s)\zeta(Ns - (N-2h)) x^{-s} \mathrm{d}s, \tag{1.5}$$

where c_0 is some large positive quantity. Motivated from this representation, recently, Gupta and Maji [12] also studied a similar integral to find an extension of Ramanujan's identity (1.2) in a different direction. In a subsequent paper, Dixit et al. [9] also studied the above integral (6.1) associated to the Hurwitz zeta function $\zeta(s,a)$, inspired from another work of Kanemitsu et al. [15]. Mainly, they investigated the following infinite series and its integral representation:

$$\sum_{m=1}^{\infty} \frac{m^{N-2h} \exp\left(-a\, m^N x\right)}{\exp(m^N x) - 1} = \frac{1}{2\pi i} \int_{c_0 - i\infty}^{c_0 + i\infty} \Gamma(s)\zeta(s,a)\zeta(Ns - (N-2h)) x^{-s} \mathrm{d}s. \tag{1.6}$$

While studying (1.6), Dixit et al. [9, Theorem 2.4] obtained a gigantic two-variable extension of Ramanujan's identity (1.2) by which they were able to connect many odd zeta values from their identity.

A few years later, the trio, Kanemitsu et al. further explored a character analogue of the series (1.3), namely, the following infinite series and its integral representation:

$$\sum_{r=1}^{q} \sum_{n=1}^{\infty} \frac{\chi(r) n^{N-2h} \exp\left(-\frac{r}{q} n^N x\right)}{1 - \exp(-n^N x)} = \frac{1}{2\pi i} \int_{(c_0)} \Gamma(s) L(s,\chi)\zeta(Ns - N + 2h) \left(\frac{x}{q}\right)^{-s} \mathrm{d}s, \tag{1.7}$$

where χ is a Dirichlet character modulo q, and for some large positive c_0.

For any $N \in 2\mathbb{N}$ and $h \in \mathbb{Z}$ with some restriction on h, they were able to obtain many interesting identities for the Dirichlet L-function $L(s,\chi)$ at different rational arguments from a transformation formula for the infinite series (1.7). For example, they gave formulae for $L(1/2,\chi)$ and $L(1/4,\chi)$ analogous to Ramanujan's famous identity for $\zeta(1/2)$.

In the current thesis, we further explore the same series (1.7) for any $N \in \mathbb{N}$ and $h \in \mathbb{Z}$ without any restriction on h. Interestingly, we obtain a new character analogue of Ramanujan's identity (1.2). We also found an identity for $L(1/3,\chi)$.

Chapter 2

Preliminaries

In this chapter, we will define some basic definitions as well as certain well-known results that are relevant to the thesis's major goal.

Preliminaries from Number Theory

Definition 1. (Dirichlet characters) Let N and q be positive integers. A *Dirichlet character modulo* q is a homomorphism $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ with the following properties:

(i) $\chi(m+q) = \chi(m)$ for all $m \in \mathbb{N}$,

(ii) $\chi(mn) = \chi(m)\chi(n)$, for all $m, n \in \mathbb{N}$.

The character $\chi_0 = \chi_{0,q}$ defined by

$$\chi_0(n) = \begin{cases} 0, & \text{if } (n,q) = 1, \\ 1, & \text{otherwise}, \end{cases} \tag{2.1}$$

is called the *principal character modulo* q. A character χ is called a *trivial character* if $\chi(m) = 1$ for all $m \in \mathbb{N}$.

The *conductor* c_χ of a Dirichlet character χ is the smallest positive integer c such that χ can be generated from $(\mathbb{Z}/c\mathbb{Z})^*$.

$$\begin{tikzcd}
(\mathbb{Z}/q\mathbb{Z})^* \arrow[rr, "\chi"] \arrow[dr, "\text{modulo } c"'] & & \mathbb{C} \\
& (\mathbb{Z}/c\mathbb{Z})^* \arrow[ur, "\tilde{\chi}"'] &
\end{tikzcd}$$

We say a character is *primitive* if its conductor is equal to its period. From any Dirichlet character $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}$ of conductor c we can produce an associated primitive Dirichlet character of period and conductor c by taking the character $\tilde{\chi} : (\mathbb{Z}/c\mathbb{Z}) \to \mathbb{C}$ in the above diagram. A character χ is called *even* or *odd* if $\chi(-1) = 1$ or $\chi(-1) = -1$, respectively.

Table 2.1: A character modulo 14 and its associated primitive character modulo 7.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	period	conductor	parity
χ	0	1	0	ζ	0	ζ^5	0	0	0	ζ^2	0	ζ^4	0	-1	14	7	odd
χ	0	1	ζ^2	ζ	ζ^4	ζ^5	-1	0	1	ζ^2	ζ	ζ^4	ζ^5	-1	7	7	odd

Definition 2. (Bernoulli polynomials) The generating function for Bernoulli polynomial $B_n(x)$ is defined as

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

The *nth Bernoulli number* is given by $B_n(0) = B_n$. Also, Bernoulli numbers have many properties and one can easily see that $B_{2n+1} = 0$ for all $n \in \mathbb{N}$.

Table 2.2: The first few Bernoulli numbers

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$

Definition 3. (Generalized Bernoulli numbers) Let χ be any primitive character modulo q. Then we define the generalized Bernoulli numbers $B_{n,\chi}$ by

$$\sum_{k=1}^{q} \chi(k) \frac{te^{kt}}{e^{qt} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}. \tag{2.2}$$

In fact, we can replace q by any multiple of it in (2.2), using the identity

$$\sum_{k=0}^{r-1} \frac{x^k}{x^r - 1} = \frac{1}{x - 1}.$$

Note that if χ is odd, $B_{n,\chi} = 0$ for every even n. In general, $B_{n,\chi} = 0$ for $n \neq 0$ modulo 2, with the single exception of $B_{1,1} = \frac{1}{2}$.

Table 2.3: The first few Bernoulli numbers associated to the primitive character χ

n	0	1	2	3	4	5	6	7
$B_{n,\chi}$	0	$-\frac{4}{7} - \frac{2\sqrt{3}}{7}i$	0	$3+3\sqrt{3}i$	0	$-\frac{445}{7} - \frac{565\sqrt{3}}{7}i$	0	$\frac{22249}{7} + \frac{30049\sqrt{3}}{7}i$

Definition 4. (Gamma function) Let $s \in \mathbb{C}$ with $\Re(s) > 0$. The classical Gamma function $\Gamma(s)$ is given by

$$\Gamma(s) := \int_0^\infty x^{s-1} \exp(-x) \mathrm{d}x.$$

One can easily check that $\Gamma(s+1) = s\Gamma(s)$ and it can be analytically extended to a meromorphic function on the whole $\mathbb{C}$ except at non-positive integers.

Lemma 2.0.1. *For any $z \in \mathbb{C}$ with $\Re(z) > 0$ and $c > 0$, we have*

$$\exp(-z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) z^{-s} \mathrm{d}s. \tag{2.3}$$

Proposition 2.0.2. The function $\Gamma(s)$ is never zero for all $s \in \mathbb{C}$.

Proposition 2.0.3. The set of non-positive integers are the only simple poles of $\Gamma(s)$. For $m \in \mathbb{N} \cup \{0\}$, the residue at $-m$ is $(-1)^m/m!$.

Lemma 2.0.4. *The function $\Gamma(s)$ has the following expansion around $s = 0$:*

$$\Gamma(s) = \frac{1}{s} - \gamma + \frac{1}{2}\left(\gamma^2 + \frac{\pi^2}{6}\right) s - \frac{1}{6}\left(\gamma^3 + \frac{\gamma\pi^2}{2} + 2\zeta(3)\right) s^2 + O(s^3). \tag{2.4}$$

Lemma 2.0.5 (Stirling's bound for $\Gamma(s)$)**.** *[13, p. 151] Let $s = \sigma + iR$. For $p \leq \sigma \leq q$, one has*

$$\Gamma(\sigma + iR) \ll |R|^{\sigma - \frac{1}{2}} \exp\left(-\frac{\pi |R|}{2}\right), \tag{2.5}$$

as $|R| \to \infty$.

Lemma 2.0.6. *For $s \in \mathbb{C} \backslash \mathbb{Z}$,*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}. \tag{2.6}$$

Some trigonometric identities

The following two lemmas will serve a major role in proving the main identities.

Lemma 2.0.7. *Suppose $z \in \mathbb{C}$. Then for any $m \in \mathbb{N}$,*

$$\frac{\sin(mz)}{\sin(z)} = \sum_{j=-(m-1)}^{(m-1)}{}'' \exp(izj), \tag{2.7}$$

where $''$ indicates that the summation runs through $j = -(m-1), -(m-3), \cdots, (m-3), (m-1)$. Thus, for m even,

$$\frac{\sin(mz)}{\cos(z)} = (-1)^{\frac{m}{2}} \sum_{j=-(m-1)}^{(m-1)}{}'' i^j \exp(izj), \tag{2.8}$$

and for m odd,

$$\frac{\cos(mz)}{\cos(z)} = (-1)^{\frac{m-1}{2}} \sum_{j=-(m-1)}^{(m-1)}{}'' i^j \exp(-izj). \tag{2.9}$$

Lemma 2.0.8. *[10, p. 12, Lemma 3.1] Suppose α, β, γ are three real numbers. Then we have*

$$2\Re\left(\frac{e^{i\alpha\beta}}{\exp(\gamma e^{-i\alpha}) - 1}\right) = \frac{\cos(\gamma\sin(\alpha) + \alpha\beta) - e^{-\gamma\cos(\alpha)}\cos(\alpha\beta)}{\cosh(\gamma\cos(\alpha)) - \cos(\gamma\sin(\alpha))}.$$

Chapter 3

The Functions $\zeta(s)$ and $L(s,\chi)$

In this chapter, we shall define the Riemann zeta function and discuss one of its generalization.

3.0.1 Riemann zeta function and its generalization

Let $s = \sigma + it$ be any complex number. The Riemann zeta function is denoted by $\zeta(s)$ and defined as follows:

$$\zeta(s) := \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \text{for} \quad \Re(\text{s}) = \sigma > 1. \tag{3.1}$$

The Dirichlet series (3.1) converges uniformly and absolutely if $\sigma \geq \sigma_0 > 1$. By Weierstrass' theorem, $\zeta(s)$ is holomorphic for $\sigma > 1$. Using product representations, Euler gave many interesting identities in number theory. For example, Euler used

$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}, \Re(s) > 1, \tag{3.2}$$

to show that the series $\sum_p \frac{1}{p} = \infty$. Here both product and sum are taken over all primes p. The identity (3.2) gives a connection between $\zeta(s)$ and primes.

There are numerous ways to generalize the Riemann zeta function. We shall only discuss one of these generalizations, which will be used throughout the thesis.

Let $\chi(m)$ be a character modulo q, $q \geq 1$. Then, for $\Re(s) > 1$, the Dirichlet $L-$function is defined by

$$L(s,\chi) := \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}. \tag{3.3}$$

Since, the character $\chi(m)$ is bounded by 1, the above $L-$series converges absolutely and uniformly if $\sigma \geq \sigma_0 > 1$. Therefore, $L(s,\chi)$ is holomorphic for $\sigma > 1$. Moreover, analogous to the Euler product for $\zeta(s)$, since $\chi(m)$ is multiplicative, $L(s,\chi)$ satisfies the Euler product:

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \qquad \sigma = \Re(s) > 1.$$

One can see the following relation by assuming that $\chi(m)$ is a principal character(2.1) modulo q:

$$L(s,\chi_0) = \sum_{m=1}^{\infty} \frac{\chi_0(m)}{n^s} = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

The above identity shows that $L(s,\chi)$ differs from $\zeta(s)$ only by the finite factor

$$\prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

3.1 Functional equation of $\zeta(s)$ and $L(s,\chi)$

Riemann [23] established the analytic continuation of $\zeta(s)$ in $\mathbb{C}$ except at $s = 1$, and it satisfies a beautiful functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin\left(\frac{\pi s}{2}\right) \zeta(1-s). \tag{3.1}$$

The above functional equation can be written in the following symmetric form:

$$\nabla(s) = \nabla(1-s), \tag{3.2}$$

where

$$\nabla(s) := \frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

Now to see the functional equation for $L(s,\chi)$, we need the Gauss sum. The

Gauss sum corresponding to a character χ modulo q is defined as

$$\mathcal{G}(\chi) := \sum_{r=1}^{q} \chi(r) e^{2\pi i r/q}. \tag{3.3}$$

Let

$$a := \frac{1-\chi(-1)}{2} = \begin{cases} 0, & \text{if } \chi \text{ is even,} \\ 1, & \text{if } \chi \text{ is odd.} \end{cases} \tag{3.4}$$

Now, we assume that χ is any primitive character modulo q. Then for every $s \in \mathbb{C}$,

$$L(s,\chi) = \varepsilon_\chi 2^s \pi^{s-1} q^{\frac{1}{2}-s} \sin\left(\frac{\pi}{2}(s+a)\right) \Gamma(1-s) L(1-s,\bar{\chi}), \tag{3.5}$$

where

$$\varepsilon_\chi = \frac{\mathcal{G}(\chi)}{i^a \sqrt{q}}$$

is an algebraic number of absolute value 1.

3.2 Some essential properties of $\zeta(s)$ and $L(s,\chi)$

The function $\zeta(s)$ and $L(s,\chi)$ have many similar basic properties.

Theorem 3.2.1. *For $\Re(s) \geq 1$, $\zeta(s)$ is never zero.*

Proposition 3.2.2. (Euler) Let $B_{2\ell}$ be the 2ℓth Bernoulli number. For any positive integer ℓ,

$$\zeta(2\ell) = (-1)^{\ell+1} \frac{(2\pi)^{2\ell} B_{2\ell}}{2(2\ell)!}.$$

Proposition 3.2.3. For any $\ell \in \mathbb{N} \cup \{0\}$,

$$\zeta(-\ell) = (-1)^\ell \frac{B_{\ell+1}}{\ell+1}.$$

This implies that $\zeta(s)$ vanishes at -2ℓ because $B_{2\ell+1} = 0$, for all $\ell \in \mathbb{N}$. These zeros are known as the trivial zeros of $\zeta(s)$.

Proposition 3.2.4 (Zeros of $L-$function)**.** Assume that χ is a primitive character modulo q, with $q > 1$. When $\Re(s) > 1$, there are no zeros of $L(s,\chi)$ and for the case $\Re(s) \leq 0$, there are zeros at certain negative integers.

(i) If χ is an even primitive character, the only zeros of $L(s, \chi)$ are simple zeros at $0, -2, -4, -6, \cdots$

(ii) If χ is an odd primitive character, the only zeros of $L(s, \chi)$ are simple zeros at $-1, -3, -5, -7 \cdots$.

Proposition 3.2.5. Let χ be a primitive character. Then for every integers $k \geq 0$, we have

$$L(-k, \chi) = -\frac{B_{k+1,\chi}}{k+1}.$$

One can find the proof in [8, p. 186].

Proposition 3.2.6. [8, p. 188] Suppose χ is a character modulo q. Then

$$L(0, \chi) = \begin{cases} 0 & \text{if } \chi \text{ is even and } \ q > 1, \\ -\frac{1}{2} & \text{if } \ q = 1, \\ -\frac{1}{q} \sum_{r=1}^{q-1} \chi(r) r & \text{if } \chi \text{ is odd.} \end{cases}$$

Proposition 3.2.7. [8, p. 189] Suppose χ is a character modulo q. Then

$$L'(0, \chi) = \begin{cases} -\frac{1}{2} \sum_{r=1}^{q-1} \chi(r) \log \left(\sin \left(\frac{r\pi}{q} \right) \right) & \text{if } \chi \ \textit{is} \text{ even and non-principle,} \\ \sum_{r=1}^{q-1} \chi(r) \log \left(\Gamma \left(\frac{r}{q} \right) \right) - \log(q) L(0, \chi) & \text{if } \chi \text{ is odd,} \\ -\frac{1}{2} \Lambda(q) & \text{if } \chi \text{ is principle and } q > 1, \\ -\frac{1}{2} \log(2\pi) & \text{if } q = 1. \end{cases}$$

Hurwitz zeta function

In 1882, Hurwitz provided one of the many zeta functions, the "shifted" zeta function, $\zeta(s; x)$ by the series

$$\zeta(s; x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s},$$

for any x such that $0 < x \leq 1$. The Hurwitz zeta function initially defined for $\Re(s) > 1$ and also, it can be extended analytically to the whole complex plane, except at $s = 1$. In his research, Hurwitz was inspired by the problem of analytic continuation of Dirichlet L-functions. Then we may write the following proposition.

Proposition 3.2.8. For any character χ modulo q, we have

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} = q^{-s} \sum_{r=1}^{q} \chi(r)\zeta\left(s, \frac{q}{r}\right).$$

One can find the proof of this relation in [1].

Lemma 3.2.9. *Suppose $s = \sigma + iR$ be a complex number. Then for any $\sigma \geq \sigma_0$, $\exists$ a constant $M(\sigma_0)$, such that*

$$|\zeta(s)| \ll |R|^{M(\sigma_0)} \tag{3.1}$$

as $|R| \to \infty$.

Proof. One can see the proof in [25, p. 95]. □

Lemma 3.2.10. *Let $s = \sigma + iR \in \mathbb{C}$ and χ be any character modulo q. Then for any $\sigma_0 \leq \sigma \leq b$, $\exists$ a constant $A(\sigma_0)$, such that*

$$|L(s,\chi)| \ll |R|^{A(\sigma_0)} \tag{3.2}$$

as $|R| \to \infty$.

Proof. The proof can be found in [13, p. 97, Lemma 5.2]. □

Lemma 3.2.11. *[8, p. 206] The Laurent series expansion of $\zeta(s)$ around $s = 1$ is given by*

$$\zeta(s) = \frac{1}{s-1} + \sum_{k\geq 0} (-1)^k \frac{\gamma_k}{k!}(s-1)^k = \frac{1}{s-1} + \gamma + O(s-1), \tag{3.3}$$

where the constants γ_k are called Stieltjes constants. Since, $\zeta(s)$ is analytic at $s = 0$. The Taylor series of $\zeta(s)$ around $s = 0$ is given by

$$\zeta(s) = -\frac{1}{2} - \frac{1}{2}\log(2\pi)s + \frac{1}{48}(24\gamma_1 + 12\gamma^2 - \pi^2 - 12(\log(2\pi))^2)s^2 + O(s^3). \tag{3.4}$$

Chapter 4

Main Results

All the main results of this thesis are stated in this chapter.

Theorem 4.0.1. *Let $x \in \mathbb{R}^+, q \in \mathbb{N}$, and χ be a primitive character modulo q. Let $N \in \mathbb{N}$ and $h \in \mathbb{Z}$ with $N - 2h \neq -1$. Let us define*

$$F(2h-N,x,\chi) := \sum_{r=1}^{q}\sum_{n=1}^{\infty} \frac{\chi(r)n^{N-2h}\exp\left(-\frac{r}{q}n^N x\right)}{1-\exp(-n^N x)},$$

and

$$G_j\left(\frac{2h-1}{N},x,\chi\right) := \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2h-1}{N}}} \frac{\chi(n)}{\exp\left[2\pi\left(nx\right)^{1/N}\exp\left(-\frac{i\pi j}{2N}\right)\right]-1}.$$

Then, we have

$$\begin{aligned} F(2h-N,x,\chi) &= \zeta(-N+2h)L(0,\chi) + \mathcal{R}_1(x) \\ &\quad + \frac{1}{N}\Gamma\left(\frac{N-2h+1}{N}\right)L\left(\frac{N-2h+1}{N},\chi\right)\left(\frac{x}{q}\right)^{\frac{2h-N-1}{N}} \\ &\quad + \sum_{j=1}^{\lfloor\frac{2h}{N}\rfloor-1} \frac{(-1)^{j+1}}{(j+1)!}B_{j+1,\chi}\,\zeta(-Nj-N+2h)\left(\frac{x}{q}\right)^{j} \\ &\quad + \mathcal{J}_\chi(x), \end{aligned} \tag{4.1}$$

where

$$\mathcal{R}_1(x) := \begin{cases} \frac{\zeta(2h)}{x}, & \text{if } q = 1, \\ 0, & \text{if } q > 1, \end{cases} \tag{4.2}$$

and

$$\mathcal{J}_\chi(x) := (-1)^{h+1}\frac{\mathcal{G}(\chi)}{N}\left(\frac{2\pi}{x}\right)^{\frac{N-2h+1}{N}} \sideset{}{''}\sum_{j=-(N-1)}^{N-1} \upsilon_{N,\chi}(j)\exp\left(\frac{i\pi j(2h-1)}{2N}\right) \times G_j\left(\frac{2h-1}{N}, \frac{2\pi}{x}, \bar{\chi}\right), \tag{4.3}$$

where $''$ *indicates that the sum over* j *takes the values* $j = -(N-1), -(N-3), \cdots, (N-3), (N-1)$. *and*

$$\upsilon_{N,\chi}(j) = \begin{cases} 1, & \text{if } \chi \text{ is even, } N \in \mathbb{N}, \\ (-1)^{\frac{N}{2}} i^{j-1}, & \text{if } \chi \text{ is odd, } N \in 2\mathbb{N}. \end{cases} \tag{4.4}$$

Remark 1. *The above theorem is not valid for* $N - 2h = -1$.

Letting $q = 1$ i.e., χ being a trivial character in Theorem 4.0.1 and upon simplification, one can recover the main identity of Dixit and Maji [10, Theorem 1.1]. Again, substituting $N = 1$ and $q = 1$ in Theorem 4.0.1, we can derive Ramanujan's identity (1.2).

Now considering $q > 1$ and letting $N = 1$ in Theorem 4.0.1, we obtain the below identity.

Corollary 4.0.2. Let χ be a primitive even character modular q. Then for $x > 0$ and $h \neq 0$, we have

$$\begin{aligned} F(2h-1, x, \chi) = {} & \zeta(2h-1)L(0,\chi) + \mathcal{R}_1(x) \\ & + \Gamma(2-2h)\,L(2-2h,\chi)\left(\frac{x}{q}\right)^{2h-2} \\ & + \sum_{j=1}^{\lfloor 2h\rfloor-1} \frac{(-1)^{j+1}}{(j+1)!}B_{j+1,\chi}\,\zeta(-j-1+2h)\left(\frac{x}{q}\right)^{j} \\ & + (-1)^{h+1}\mathcal{G}(\chi)\left(\frac{2\pi}{x}\right)^{2-2h} G_0\left(2h-1, \frac{2\pi}{x}, \bar{\chi}\right). \end{aligned} \tag{4.5}$$

The next lemma gives a new formula for $L(1/3,\chi_5)$.

Corollary 4.0.3. Let χ_5 be an odd character modulo 5. For any $x>0$, we have

$$\sum_{r=1}^{5}\sum_{n=1}^{\infty}\chi_5(r)\frac{n\exp\left(-\frac{r}{5}n^3x\right)}{1-\exp\left(-n^3x\right)}=\frac{\mathcal{G}(\chi_5)}{3}\left(\frac{2\pi}{x}\right)^{\frac{2}{3}}\left\{L\left(\frac{1}{3},\chi_5\right)\right.$$
$$\left.+\sum_{n=1}^{\infty}\frac{\chi_5(n)}{n^{\frac{1}{3}}}\left(\frac{e^u}{2\sinh(u)}+\frac{\cosh\left(\sqrt{3}u+\frac{\pi}{3}\right)-\frac{e^u}{2}}{\cosh(u)-\cos\left(\sqrt{3}u\right)}\right)\right\},$$

where

$$u:=u(x)=\pi\left(\frac{2\pi n}{x}\right)^{1/3}. \tag{4.6}$$

Theorem 4.0.4. *Let χ be a primitive Dirichlet character modulo q, $q\geq 1$ and $x>0$ be any real number. Let h be a integer. Then for any positive integer N with $h=(N+1)/2$, wh have*

$$\sum_{r=1}^{q}\sum_{n=1}^{\infty}\frac{\chi(r)}{n}\frac{\exp\left(-\frac{r}{q}n^Nx\right)}{1-\exp(-n^Nx)}=\mathcal{R}_0+\mathcal{R}_1+\sum\mathcal{R}_{-j}+\mathcal{K}_{\chi,N}, \tag{4.7}$$

where

$$\mathcal{R}_0=\begin{cases}\frac{1}{2N}\left(\gamma(1-N)-\log(2\pi)+\log(x)\right), & \text{if } \chi \text{ is even and } q=1,\\ -\frac{1}{2N}\sum_{r=1}^{q-1}\chi(r)\log\left(\sin\left(\frac{r\pi}{q}\right)\right), & \text{if } \chi \text{ is even and } q>1,\\ \frac{1}{N}\left[\frac{1}{q}\left(\log(x)-\gamma(N-1)\right)\sum_{r=1}^{q-1}\chi(r)r\right. & \\ \left.\quad+\sum_{r=1}^{q-1}\chi(r)\log\left(\Gamma\left(\frac{r}{q}\right)\right)\right], & \text{if } \chi \text{ is odd,}\end{cases} \tag{4.8}$$

$$\mathcal{R}_1=\begin{cases}\frac{\zeta(N+1)}{x}, & \text{if } q=1,\\ 0, & \text{if } q>1,\end{cases} \tag{4.9}$$

$$\sum\mathcal{R}_{-j}=\begin{cases}\frac{x}{2q}L(-1,\chi), & \text{if } N=1,\\ 0, & \text{if } N>1,\end{cases} \tag{4.10}$$

and

$$\mathcal{K}_\chi(x)=(-1)^{\frac{N-1}{2}}\frac{\mathcal{G}(\chi)}{N}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)}{n}\sideset{}{''}\sum_{j=-(N-1)}^{N-1}v_{N,\chi}(j)\frac{\exp\left(\frac{i\pi j}{2}\right)}{\exp\left(2\pi\left(\frac{2\pi n}{x}\right)^{(1/N)}\exp\left(\frac{-i\pi j}{2N}\right)\right)-1}, \tag{4.11}$$

where

$$v_{N,\chi}(j) = \begin{cases} 1, & \text{if } \chi \text{ is even } N \in \mathbb{N}, \\ (-1)^{\frac{N}{2}} i^{j-1}, & \text{if } \chi \text{ is odd}, N \in 2\mathbb{N}. \end{cases} \tag{4.12}$$

Chapter 5

Proof of Main Results

This chapter contains the proofs of all the main identities of this thesis. Before going to the proof of the main Theorem 4.0.1, we state the following lemma.

Lemma 5.0.1. *Let $x > 0$, $N \in \mathbb{N}$ and χ be any Dirichlet character modulo q, $q \in \mathbb{N}$. Then for any $h \in \mathbb{Z}$, we have*

$$\sum_{r=1}^{q} \sum_{n=1}^{\infty} \frac{\chi(r) n^{N-2h} \exp\left(-\frac{r}{q} n^N x\right)}{1 - \exp(-n^N x)} = \frac{1}{2\pi i} \int_{(c_0)} \Gamma(s) L(s, \chi) \zeta(Ns-(N-2h)) \left(\frac{q}{x}\right)^s \mathrm{d}s,$$

where $c_0 = \Re(s) > \max\{1, (N-2h+1)/N\}$.

Proof. Note that for $x > 0, n \in \mathbb{N}$, one has $|\exp(-n^N x)| < 1$. We write

$$\frac{1}{1 - \exp(-n^N x)} = \sum_{k=0}^{\infty} \exp(-n^N x k). \tag{5.1}$$

Now, using inverse Mellin transform (2.3) for $\exp(-z)$, for $c_0 > 0$, one has

$$\begin{aligned} \exp\left(-n^N x \left(k + \frac{r}{q}\right)\right) &= \frac{1}{2\pi i} \int_{(c_0)} \Gamma(s) \left(n^N x \left(k + \frac{r}{q}\right)\right)^{-s} \mathrm{d}s \\ &= \frac{1}{2\pi i} \int_{(c_0)} \frac{\Gamma(s) x^{-s} \left(k + \frac{r}{q}\right)^{-s}}{n^{Ns}} \mathrm{d}s. \end{aligned} \tag{5.2}$$

Now, use (5.1) and (5.2) to see that

$$\begin{aligned}\sum_{r=1}^{q}\sum_{n=1}^{\infty}\frac{\chi(r)n^{N-2h}\exp\left(-\frac{r}{q}n^Nx\right)}{1-\exp(-n^Nx)} &= \sum_{r=1}^{q}\sum_{n=1}^{\infty}\chi(r)n^{N-2h}\sum_{k=0}^{\infty}\frac{1}{2\pi i}\int_{(c_0)}\frac{\Gamma(s)x^{-s}\left(k+\frac{r}{q}\right)^{-s}}{n^{Ns}}\mathrm{d}s\\ &= \frac{1}{2\pi i}\int_{(c_0)}\Gamma(s)\sum_{r=1}^{q}\chi(r)\sum_{n=1}^{\infty}\frac{1}{n^{Ns-N+2h}}\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{r}{q}\right)^s}\frac{\mathrm{d}s}{x^s}\\ &= \frac{1}{2\pi i}\int_{(c_0)}\Gamma(s)\zeta(Ns-(N-2h))\sum_{r=1}^{q}\chi(r)\zeta\left(s,\frac{r}{q}\right)\frac{\mathrm{d}s}{x^s}\\ &= \frac{1}{2\pi i}\int_{(c_0)}\Gamma(s)\zeta(Ns-(N-2h))L(s,\chi)\left(\frac{x}{q}\right)^{-s}\mathrm{d}s,\end{aligned}$$

for $c_0 > \max\{1, (N-2h+1)/N\}$. In the last line we have used the Proposition 3.2.8. □

5.1 Proof of Theorem 4.0.1

Proof. From Lemma 5.0.1, for $\Re(s) = c_0 > \max\{1, (N-2h+1)/N\}$, we have seen that

$$\sum_{r=1}^{q}\sum_{n=1}^{\infty}\frac{\chi(r)n^{N-2h}\exp\left(-\frac{r}{q}n^Nx\right)}{1-\exp(-n^Nx)} = \frac{1}{2\pi i}\int_{(c_0)}\Gamma(s)L(s,\chi)\zeta(Ns-(N-2h))\left(\frac{x}{q}\right)^{-s}\mathrm{d}s. \tag{5.1}$$

To simplify the above integral, we move the line integration from $\Re(s) = c_0$ to $\Re(s) = d_0$, where $d_0 < -2h/N + 1$. Later, we shall explain the purpose of considering this upper bound for d_0.

First, we shall analyse the singularities of the integrand. Note that the only singularities of $\Gamma(s)$ are non-positive integers and all of them are simple pole. Again, we know, $s = 1$ is the only simple pole of $\zeta(s)$. Thus, $s = (N-2h+1)/N$ is the simple pole of $\zeta(Ns-(N-2h))$. Note that for any $j \in \mathbb{N}$, $s = (N-2h-2j)/N$ are the trivial zeros of $\zeta(Ns-(N-2h))$.

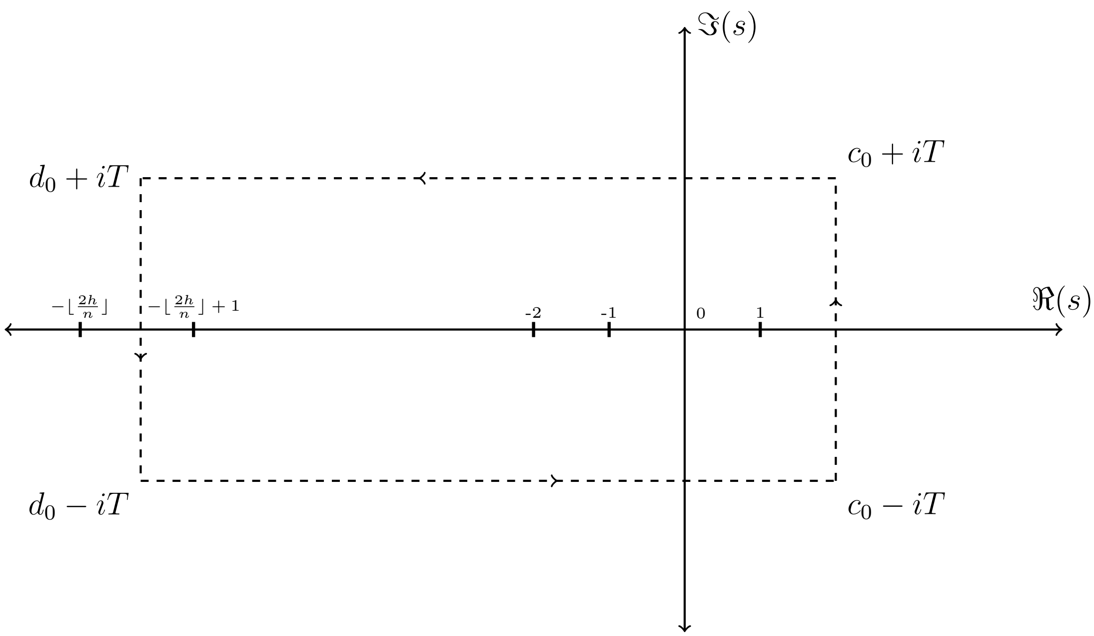
When, $q > 1$, one knows that $L(s,\chi)$ is analytic in $\mathbb{C}$ and on the other hand, if $q = 1$, it has a simple pole at $s = 1$. Now, our aim is to determine which poles of the integrand function in (5.1) contribute. Mainly, we shall try to see which poles of $\Gamma(s)$ are neutralized by the trivial zeros of $\zeta(Ns-(N-2h))$. Suppose, k is a positive integer such that the pole of $\Gamma(s)$ at $s=-k$ is getting neutralized by some trivial zeros of $\zeta(Ns-(N-2h))$. Therefore, for some $j \in \mathbb{N}$, we must have $-k=(N-2h-2j)/N$, which suggests that $N+Nk-2h=2j>0 \Leftrightarrow -k<-2h/N+1$. This indicates that every negative integer that are less than $-2h/N+1$ are by some real zeros of $\zeta(Ns-(N-2h))$.

Now, we observe that

$$\left\lfloor -\frac{2h}{N} \right\rfloor = \begin{cases} -\lfloor \frac{2h}{N} \rfloor, & \text{if } \frac{2h}{N} \in \mathbb{Z}, \\ -\lfloor \frac{2h}{N} \rfloor - 1, & \text{if } \frac{2h}{N} \notin \mathbb{Z}. \end{cases}$$

This shows that we must consider the contribution of the poles at $s=-j$ of $\Gamma(s)$ with $1 \leq j \leq \lfloor 2h/N \rfloor - 1$. We shall also take the contribution of the pole at $s=0$ of $\Gamma(s)$. One can verify that the all poles of integrand, namely, at $s=0,1,(N-2h+1)/N,-j$, for $1 \leq j \leq \lfloor 2h/N \rfloor - 1$, are all simple.

We consider a contour $\mathcal{C}$ with the line segments $[c_0-iT, c_0+iT]$, $[c_0+iT, d_0+iT]$, $[d_0+iT, d_0-iT]$, and $[d_0-iT, c_0-iT]$.



Now, making use of Cauchy's residue theorem, we arrive at

$$\frac{1}{2\pi i}\int_{\mathcal{C}}\Gamma(s)L(s,\chi)\zeta(Ns-(N-2h))\left(\frac{q}{x}\right)^{s}\mathrm{d}s=\mathcal{R}_0+\mathcal{R}_1+\mathcal{R}_{\frac{N-2h+1}{N}} \tag{5.2}$$

$$+\sum_{j=1}^{\lfloor\frac{2h}{N}\rfloor-1}\mathcal{R}_{-j}, \tag{5.3}$$

where $\mathcal{R}_\alpha$ represents the residual term at $s=\alpha$. Letting $T\to\infty$, and using Stirling's bound (2.0.5), and together with known bounds on $\zeta(s)$ and $L(s,\chi)$, one can show that the horizontal integrals vanish. Therefore, from (5.2), we get

$$\frac{1}{2\pi i}\left[\int_{(c_0)}-\int_{(d_0)}\right]\Gamma(s)L(s,\chi)\zeta(Ns-(N-2h))\left(\frac{q}{x}\right)^{s}\mathrm{d}s$$

$$=\mathcal{R}_0+\mathcal{R}_1+\mathcal{R}_{\frac{N-2h+1}{N}}+\sum_{j=1}^{\lfloor\frac{2h}{N}\rfloor-1}\mathcal{R}_{-j}. \tag{5.4}$$

Now employ (5.1) in (5.4) to see that

$$\sum_{r=1}^{q}\sum_{n=1}^{\infty}\frac{\chi(r)n^{N-2h}\exp\left(-\frac{r}{q}n^Nx\right)}{1-\exp(-n^Nx)}=V_{N,h}(x;\chi)+\sum_{j=1}^{\lfloor\frac{2h}{N}\rfloor-1}\mathcal{R}_{-j}$$

$$+\mathcal{R}_0+\mathcal{R}_1+\mathcal{R}_{\frac{N-2h+1}{N}}, \tag{5.5}$$

where

$$V_{N,h}(x;\chi):=\int_{(d_0)}\Gamma(s)L(s,\chi)\zeta(Ns-(N-2h))\left(\frac{x}{q}\right)^{-s}\mathrm{d}s. \tag{5.6}$$

The residue $\mathcal{R}_0$ is given by

$$\mathcal{R}_0=\lim_{s\to 0}s\;\Gamma(s)\zeta(Ns-(N-2h))L(s,\chi)\left(\frac{x}{q}\right)^{-s}=\zeta(-N+2h)L(0,\chi). \tag{5.7}$$

Here we point out that the residue at $s=1$ will depend on q. The residue at $s=1$ is given by

$$\mathcal{R}_1(x):=\mathcal{R}_1=\begin{cases}\frac{\zeta(2h)}{x}, & \text{if } q=1,\\ 0, & \text{if } q>1,\end{cases} \tag{5.8}$$

where $\mathcal{R}_1(x)$ is the same function defined in (4.2).

The residue $\mathcal{R}_{\frac{N-2h+1}{N}}$ can be calculated by the following way

$$\begin{aligned}\mathcal{R}_{\frac{N-2h+1}{N}} &= \lim_{s\to\frac{N-2h+1}{N}} \left(s-\frac{N-2h+1}{N}\right)\Gamma(s)\zeta(Ns-(N-2h))L(s,\chi)\left(\frac{x}{q}\right)^{-s} \\ &= \frac{1}{N}\Gamma\left(\frac{N-2h+1}{N}\right)L\left(\frac{N-2h+1}{N},\chi\right)\left(\frac{x}{q}\right)^{-\left(\frac{N-2h+1}{N}\right)}. \end{aligned} \tag{5.9}$$

Finally, the residue $\mathcal{R}_{-j}$ at $s=-j$, with $j\in\mathbb{N}$, is given by

$$\begin{aligned}\mathcal{R}_{-j} &= \lim_{s\to -j}(s+j)\Gamma(s)L(s,\chi)\zeta(Ns-(N-2h))\left(\frac{x}{q}\right)^{-s} \\ &= \frac{(-1)^j}{j!}L(-j,\chi)\zeta(-Nj-N+2h)\left(\frac{x}{q}\right)^{j} \\ &= \frac{(-1)^{j+1}}{(j+1)!}B_{j+1,\chi}\,\zeta(-Nj-N+2h)\left(\frac{x}{q}\right)^{j}, \end{aligned} \tag{5.10}$$

Now the only thing is left is to verify that the integral $V_{N,h}(x;\chi)$ is nothing but the expression $\mathcal{J}_\chi(x)$, where $\mathcal{J}_\chi(x)$ is defined as in (4.3).

To simplify the integral (5.6), we first employ asymmetric form of the functional equations of $\zeta(s)$ and $L(s,\chi)$ respectively, namely, the equations (3.1) and (3.5). Thus, upon simplification, we get

$$\begin{aligned}V_{N,h}(x;\chi) &= \frac{\mathcal{G}(\chi)}{\pi i^a}\frac{1}{2\pi i}\int_{(d_0)}\Gamma(s)\left(\frac{2\pi}{q}\right)^{s}\sin\left(\frac{\pi}{2}(s+a)\right)\Gamma(1-s)L(1-s,\bar{\chi}) \\ &\quad\times\frac{(2\pi)^{Ns}}{\pi(2\pi)^{N-2h}}\sin\left(\frac{\pi}{2}(Ns-(N-2h))\right)\Gamma(1-Ns+N-2h) \\ &\quad\times\zeta(1-Ns+N-2h)\left(\frac{x}{q}\right)^{-s}\mathrm{d}s \\ &= \left(\frac{1}{2\pi}\right)^{N-2h+1}\frac{\mathcal{G}(\chi)}{i^a}\frac{1}{2\pi i}\int_{(d_0)}\frac{2\sin\left(\frac{\pi}{2}(s+a)\right)}{\sin(\pi s)}\sin\left(\frac{\pi}{2}(Ns-(N-2h))\right) \\ &\quad\times L(1-s,\bar{\chi})\Gamma(1-Ns+N-2h)\zeta(1-Ns+N-2h)\left(\frac{(2\pi)^{N+1}}{x}\right)^{s}\mathrm{d}s \\ &= \left(\frac{1}{2\pi}\right)^{N-2h+1}\frac{\mathcal{G}(\chi)}{i^a}\frac{1}{2\pi i}\int_{(d_0)}\left[\frac{\cos(\frac{\pi}{2}a)}{\cos(\frac{\pi}{2}s)}+\frac{\sin(\frac{\pi}{2}a)}{\sin(\frac{\pi}{2}s)}\right]\sin\left(\frac{\pi}{2}(Ns-(N-2h))\right)\end{aligned}$$

$$\times L(1-s,\bar{\chi})\Gamma(1-Ns+N-2h)\zeta(1-Ns+N-2h)\left(\frac{(2\pi)^{N+1}}{x}\right)^{s}\mathrm{d}s. \tag{5.11}$$

Note that we have used the reflection identity (2.6) in the second step. Now, we would like to change the variable $1-Ns+N-2h \longrightarrow s_1$. In that case, $d_0 < -2h/N+1$ leads $\Re(s_1) = d_1 = 1+N-2h-Nd_0 > 1$. So, the equation (5.11) becomes

$$\begin{aligned}
V_{N,h}(x;\chi) &= \left(\frac{1}{2\pi}\right)^{N-2h+1}\frac{\mathcal{G}(\chi)}{i^a N}\frac{1}{2\pi i}\int_{(d_1)}\left[\frac{\cos(\frac{\pi}{2}a)}{\cos\left(\frac{\pi}{2}+\frac{\pi(1-s_1-2h)}{2N}\right)}+\frac{\sin(\frac{\pi}{2}a)}{\sin\left(\frac{\pi}{2}+\frac{\pi(1-s_1-2h)}{2N}\right)}\right] \\
&\quad \times \sin\left(\frac{\pi}{2}(1-s_1)\right)L\left(\frac{s_1+2h-1}{N},\bar{\chi}\right)\Gamma(s_1)\zeta(s_1)\left(\frac{(2\pi)^{N+1}}{x}\right)^{\frac{1-s_1+N-2h}{N}}\mathrm{d}s_1 \\
&=\frac{\mathcal{G}(\chi)}{i^a N}\left(\frac{2\pi}{x}\right)^{\frac{N-2h+1}{N}}\frac{1}{2\pi i}\int_{(d_1)}\left[\frac{\cos(\frac{\pi}{2}a)\cos(\frac{\pi}{2}s_1)}{\sin\left(\frac{\pi}{2}\left(\frac{1-s_1-2h}{N}\right)\right)}+\frac{\sin(\frac{\pi}{2}a)\cos(\frac{\pi}{2}s_1)}{\cos\left(\frac{\pi}{2}\left(\frac{1-s_1-2h}{N}\right)\right)}\right] \\
&\quad \times L\left(\frac{s_1+2h-1}{N},\bar{\chi}\right)\Gamma(s_1)\zeta(s_1)(X_N)^{-\frac{s_1}{N}}\mathrm{d}s_1, \\
&=\frac{\mathcal{G}(\chi)}{i^a N}\left(\frac{2\pi}{x}\right)^{\frac{N-2h+1}{N}}\left[\cos\left(\frac{\pi}{2}a\right)\mathcal{U}(X_N)+\sin\left(\frac{\pi}{2}a\right)\mathcal{V}(X_N)\right],
\end{aligned} \tag{5.12}$$

where $X_N = (2\pi)^{N+1}/x$, and

$$\mathcal{U}(X_N) := \frac{1}{2\pi i}\int_{(d_1)}\frac{\cos\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\left(\frac{1-s_1-2h}{N}\right)\right)}\Gamma(s_1)\zeta(s_1)L\left(\frac{s_1+2h-1}{N},\bar{\chi}\right)X_N^{-\frac{s_1}{N}}\mathrm{d}s_1, \tag{5.13}$$

$$\mathcal{V}(X_N) := \frac{1}{2\pi i}\int_{(d_1)}\frac{\cos\left(\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2}\left(\frac{1-s_1-2h}{N}\right)\right)}\Gamma(s_1)\zeta(s_1)L\left(\frac{s_1+2h-1}{N},\bar{\chi}\right)X_N^{-\frac{s_1}{N}}\mathrm{d}s_1. \tag{5.14}$$

Now, our main aim is to evaluate the integrals $\mathcal{U}(X_N)$ and $\mathcal{V}(X_N)$. First, let us see the integral $\mathcal{U}(X_N)$. To evaluate this integral we would like to expand $\zeta(s_1)L\left(\frac{s_1+2h-1}{N},\bar{\chi}\right)$ into a Dirichlet series. We can do this because $\Re(s_1) > 1$ and $\Re(\frac{s_1+2h-1}{N}) = 1+\Re(s) > 1$ as $s_1 = 1-Ns+N-2h$. Now, we write

$$L\left(\frac{s_1+2h-1}{N},\bar{\chi}\right)\zeta(s_1) = \sum_{n=1}^{\infty}\frac{\bar{\chi}(n)}{n^{\frac{s_1+2h-1}{N}}}\sum_{m=1}^{\infty}\frac{1}{m^{s_1}}$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n^{\frac{2h-1}{N}}} \, (m^N n)^{-\frac{s_1}{N}}. \tag{5.15}$$

Again, one can check that

$$\cos\left(\frac{\pi}{2} s_1\right) = (-1)^{h+1} \sin\left(N\left(\frac{\pi}{2} \frac{s_1 + 2h - 1}{N}\right)\right). \tag{5.16}$$

Now, using (2.7), (5.15) and (5.16) the integral $\mathcal{U}(X_N)$ in (5.13) becomes

$$\begin{aligned}
\mathcal{U}(X_N) &= (-1)^{h+1} \frac{1}{2\pi i} \int_{(d_1)} \sideset{}{''}\sum_{j=-(N-1)}^{(N-1)} \exp\left(\frac{ij\pi(s_1 + 2h - 1)}{2N}\right) \Gamma(s_1) \\
&\qquad \times \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n^{\frac{2h-1}{N}}} \, (m^N n)^{-\frac{s_1}{N}} X_N^{-\frac{s_1}{N}} \mathrm{d}s_1 \\
&= (-1)^{h+1} \sideset{}{''}\sum_{j=-(N-1)}^{(N-1)} \exp\left(\frac{\pi}{2N}(2h-1)\right) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n^{\frac{2h-1}{N}}} \\
&\qquad \times \frac{1}{2\pi i} \int_{(d_1)} \Gamma(s_1) \left(X_N^{\frac{1}{N}} m n^{\frac{1}{N}} \exp\left(-\frac{i\pi j}{2N}\right)\right)^{-s_1} \mathrm{d}s_1.
\end{aligned} \tag{5.17}$$

Here, we can verify that $\Re\left(X_N^{\frac{1}{N}} m n^{\frac{1}{N}} \exp\left(-\frac{i\pi j}{2N}\right)\right) = X_N^{\frac{1}{N}} m n^{\frac{1}{N}} \cos\left(-\frac{i\pi j}{2N}\right) > 0$, since $X_N^{\frac{1}{N}} m n^{\frac{1}{N}}$ is a positive real number and $\cos\left(-\frac{i\pi j}{2N}\right) > 0$ as j lies in the interval $-(N-1) \leq j \leq (N-1)$. Again, using the inverse Mellin integral (2.3) in (5.17), it reduces to

$$\mathcal{U}(X_N) \tag{5.18}$$

$$=(-1)^{h+1} \sideset{}{''}\sum_{j=-(N-1)}^{(N-1)} \exp\left(\frac{\pi}{2N}(2h-1)\right) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n^{\frac{2h-1}{N}}} \exp\left(-X_N^{\frac{1}{N}} m n^{\frac{1}{N}} \exp\left(-\frac{i\pi j}{2N}\right)\right). \tag{5.19}$$

Now we simplify the inner double sum as

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n^{\frac{2h-1}{N}}} \sum_{m=1}^{\infty} \left(\exp\left(-X_N^{\frac{1}{N}} n^{\frac{1}{N}} \exp\left(-\frac{i\pi j}{2N}\right)\right)\right)^m \\
&= \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n^{\frac{2h-1}{N}}} \frac{\exp\left(-X_N^{\frac{1}{N}} n^{\frac{1}{N}} \, \exp\left(-\frac{i\pi j}{2N}\right)\right)}{1 - \exp\left(-X_N^{\frac{1}{N}} n^{\frac{1}{N}} \exp\left(-\frac{i\pi j}{2N}\right)\right)}
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n^{\frac{2h-1}{N}}} \frac{1}{\exp\left(X_N^{\frac{1}{N}} n^{\frac{1}{N}} \exp\left(-\frac{i\pi j}{2N}\right)\right) - 1}. \tag{5.20}$$

Here, we have used the identity $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$, whenever $|x| < 1$. Note that, one has $\left|\exp\left(-X_N^{\frac{1}{N}} n^{\frac{1}{N}} \exp\left(-\frac{i\pi j}{2N}\right)\right)\right| < 1$, as $-(N-1) \leq j \leq (N-1)$. Now substitute (5.20) in (5.19) to get

$$\mathcal{U}(X_N) =(-1)^{h+1} \sideset{}{''}\sum_{j=-(N-1)}^{(N-1)} \exp\left(\frac{\pi}{2N}(2h-1)\right) \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n^{\frac{2h-1}{N}}} \frac{1}{\exp\left(X_N^{\frac{1}{N}} n^{\frac{1}{N}} \exp\left(-\frac{i\pi j}{2N}\right)\right) - 1}. \tag{5.21}$$

Now we will look into the integral $\mathcal{V}(X_N)$, defined in (5.14). Again, using (2.8), (5.15) and (5.16) in (5.14), for $N \in 2\mathbb{N}$, we obtain

$$\begin{aligned}\mathcal{V}(X_N) =&(-1)^{h+1} \frac{1}{2\pi i} \int_{(d_1)} (-1)^{\frac{N}{2}} \sideset{}{''}\sum_{j=-(N-1)}^{(N-1)} i^j \exp\left(\frac{ij\pi(s_1+2h-1)}{2N}\right) \Gamma(s_1) \\ &\times \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n^{\frac{2h-1}{N}}} \left(m^N n\right)^{-\frac{s_1}{N}} X_N^{-\frac{s_1}{N}} \mathrm{d}s_1.\end{aligned}$$

Simplification of $\mathcal{V}(X_N)$ goes in a similar direction to $\mathcal{U}(X_N)$. Hence, we have

$$\begin{aligned}\mathcal{V}(X_N) = (-1)^{h+1} &\sideset{}{''}\sum_{j=-(N-1)}^{(N-1)} (-1)^{\frac{N}{2}} i^j \exp\left(\frac{\pi}{2N}(2h-1)\right) \\ &\times \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n^{\frac{2h-1}{N}}} \frac{1}{\exp\left(X_N^{\frac{1}{N}} n^{\frac{1}{N}} \exp\left(-\frac{i\pi j}{2N}\right)\right) - 1}.\end{aligned} \tag{5.22}$$

Note that, here the above expression for $\mathcal{V}(X_N)$ in (5.22) is true only for even N due to the fact (2.8). Now, apply (3.4) in (5.12) to see that

$$V_{N,h}(x;\chi) = \begin{cases} \frac{\mathcal{G}(\chi)}{N} \left(\frac{2\pi}{x}\right)^{\frac{N-2h+1}{N}} \mathcal{U}(X_N), & \text{if } \chi \text{ is even }, \\ \frac{1}{i}\frac{\mathcal{G}(\chi)}{N} \left(\frac{2\pi}{x}\right)^{\frac{N-2h+1}{N}} \mathcal{V}(X_N), & \text{if } \chi \text{ is odd }. \end{cases} \tag{5.23}$$

Now, combining (5.21), (5.14), and (5.23), we arrive at

$$\begin{aligned}V_{N,h}(x;\chi) =&(-1)^{h+1} \frac{\mathcal{G}(\chi)}{N} \left(\frac{2\pi}{x}\right)^{\frac{N-2h+1}{N}} \sideset{}{''}\sum_{j=-(N-1)}^{N-1} v_{N,\chi}(j) \exp\left(\frac{i\pi j(2h-1)}{2N}\right) \\ &\times G_j\left(\frac{2h-1}{N}, \frac{2\pi}{x}, \bar{\chi}\right),\end{aligned}$$

where $v_{n,\chi}(j)$ is defined as in (4.4). The above final expression of the vertical integral $V_{N,h}(x;\chi)$ is nothing but the expression $\mathcal{J}_\chi(x)$ defined in (4.3). This settles the proof of Theorem 4.0.1.

$\square$

5.1.1 Proof of Corollary 4.0.3

Proof. Taking $h = 1$, $N = 3$ and $\chi = \chi_5$, an even primitive character modulo 5 in Theorem 4.0.1, we get

$$\sum_{r=1}^{5}\sum_{n=1}^{\infty}\chi_5(r)\frac{n\exp\left(-\frac{r}{5}n^3x\right)}{1-\exp\left(-n^3x\right)} = \zeta(-1)L(0,\chi_5) + \frac{1}{3}\Gamma\left(\frac{2}{3}\right)L\left(\frac{2}{3},\chi_5\right)\left(\frac{x}{5}\right)^{-\frac{2}{3}} + \frac{\mathcal{G}(\chi_5)}{3}\left(\frac{2\pi}{x}\right)^{\frac{2}{3}}\sum_{n=1}^{\infty}\frac{\bar{\chi_5}(n)}{n^{\frac{1}{3}}} \times {\sum_{j=-2}^{2}}'' \frac{\exp\left(\frac{i\pi j}{6}\right)}{\exp\left(2\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\exp\left(\frac{-i\pi j}{6}\right)\right)-1}. \tag{5.24}$$

Now, using the functional equation of $L(s,\chi)$, one can derive

$$L\left(\frac{2}{3},\chi_5\right) = \mathcal{G}(\chi_5)\left(\frac{2}{5}\right)^{\frac{2}{3}}\pi^{-\frac{1}{3}}\sin\left(\frac{\pi}{3}\right)\Gamma\left(\frac{1}{3}\right)L\left(\frac{1}{3},\chi_5\right). \tag{5.25}$$

Note χ_5 is a real character, thus, $\chi_5 = \bar{\chi_5}$. Using (5.25) and (2.6), we have

$$\frac{1}{3}\Gamma\left(\frac{2}{3}\right)L\left(\frac{2}{3},\chi_5\right)\left(\frac{x}{5}\right)^{-\frac{2}{3}} = \frac{\mathcal{G}(\chi_5)}{3}\left(\frac{2\pi}{x}\right)^{\frac{2}{3}}L\left(\frac{1}{3},\chi_5\right). \tag{5.26}$$

Upon simplification, one can see that

$$\begin{aligned}{\sum_{j=-2}^{2}}'' \frac{\exp\left(\frac{i\pi j}{6}\right)}{\exp\left(2\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\exp\left(\frac{-i\pi j}{6}\right)\right)-1} &= \frac{1}{\exp\left(2\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\right)-1} \\ &+ \frac{\exp\left(-\frac{i\pi}{3}\right)}{\exp\left(2\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\exp\left(\frac{i\pi}{3}\right)\right)-1} \\ &+ \frac{\exp\left(\frac{i\pi}{3}\right)}{\exp\left(2\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\exp\left(-\frac{i\pi}{3}\right)\right)-1}.\end{aligned} \tag{5.27}$$

It is clear that

$$\left[\frac{\exp\left(-\frac{i\pi}{3}\right)}{\exp\left(2\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\exp\left(\frac{i\pi}{3}\right)\right)-1}+\frac{\exp\left(\frac{i\pi}{3}\right)}{\exp\left(2\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\exp\left(-\frac{i\pi}{3}\right)\right)-1}\right]$$
$$=2\Re\left[\frac{\exp\left(\frac{i\pi}{3}\right)}{\exp\left(2\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\exp\left(-\frac{i\pi}{3}\right)\right)-1}\right] \tag{5.28}$$

Now, putting $a=2\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}$, $u=\frac{\pi}{3}$ and $v=1$ in Lemma 2.0.8, we get

$$2\Re\left(\frac{\exp\left(\frac{i\pi}{3}\right)}{\exp\left(2\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\exp\left(-\frac{i\pi}{3}\right)\right)-1}\right)=\left(\frac{\cosh\left(\sqrt{3}\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}+\frac{\pi}{3}\right)-\frac{\exp\left(-\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\right)}{2}}{\cosh\left(\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\right)-\cos\left(\sqrt{3}\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\right)}\right) \tag{5.29}$$

Again,

$$\frac{1}{\exp\left(2\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\right)-1}=\frac{\exp\left(-\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\right)}{\exp\left(\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\right)-\exp\left(-\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\right)}$$
$$=\frac{\exp\left(-\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\right)}{2\sinh\left(2\pi\left(\frac{2\pi n}{x}\right)^{\frac{1}{3}}\right)}. \tag{5.30}$$

Finally, using (5.26), (5.27), (5.28), (5.29), and (5.30) in (5.24), one can complete the proof.

□

5.2 Proof of Theorem 4.0.4

Proof. First, we can figure out that the Theorem 4.0.1 is not valid for $N-2h=-1$, due to the presence of $\zeta(Ns-(N-2h))$ in the right side of (5.1). The proof of this identity, corresponding to $N-2h=-1$, is almost same as in Theorem

4.0.1. For the case $N-2h=-1$, Lemma 5.0.1 gives, for $\Re(s)=c_0>1$,

$$\sum_{r=1}^{q}\sum_{n=1}^{\infty}\frac{\chi(r)}{n}\frac{\exp\left(-\frac{r}{q}n^N x\right)}{1-\exp(-n^N x)}=\frac{1}{2\pi i}\int_{(c_0)}\Gamma(s)L(s,\chi)\zeta(Ns+1)\left(\frac{x}{q}\right)^{-s}\mathrm{d}s. \quad (5.1)$$

Now as in the proof of (4.0.1), we need to move the line of integration from $\Re(s)=c_0$ to $\Re(s)=d_0$, with $d_0<-1/N$. As the proof goes in a similar direction as in Theorem 4.0.1, so we only highlight the locations where the proof is different from theorem 4.0.1.

In this case, the main difference is that $\zeta(Ns+1)$ has a pole at $s=0$, which gives three following cases:

Case 1: If χ be is even character modulo q with $q=1$. Then $s=0$ is a double pole of the integrand in (5.1).

Case 2: If χ is an even character modulo q with $q>1$. Then $s=0$ is a simple pole of the integrand in (5.1).

Case 3: If χ be an odd character modulo q. Then $s=0$ is a double pole of the integrand in (5.1).

To find the residue in each case, we shall use the following Laurent series expansions around $s=0$:

$$\begin{aligned}
\Gamma(s) &= \frac{1}{s}-\gamma+\frac{1}{2}\left(\gamma^2+\frac{\pi^2}{6}\right)s+O(s^2),\\
\zeta(s) &= \frac{1}{s-1}+\gamma+O(s-1),\\
\zeta(Ns+1) &= \frac{1}{Ns}+\gamma-\gamma_1 Ns+O(s^2),\\
L(s,\chi) &= L(0,\chi)+L'(0,\chi)s+O(s^2),\\
\left(\frac{x}{q}\right)^{-s} &= 1-\log\left(\frac{x}{q}\right)s+O(s^2),\\
x^{-s} &= 1-\log(x)s+\frac{(\log(x))^2}{2}s^2+O(s^3). \quad (5.2)
\end{aligned}$$

The above Laurent series of $L(s,\chi)$ around $s=0$ is valid for non-principal primitive character modulo q with $q>1$. Now, we will discuss the cases one by one and calculate the residue at $s=0$ for each case.

Case 1: In this case, we assume χ is an even mod q with $q = 1$. Then the integrand becomes $\Gamma(s)\zeta(Ns+1)\zeta(s)x^{-s}$. Note, $s = 0$ is a double pole of the integrand due to $\Gamma(s)$ and $\zeta(Ns+1)$. Using the definition of residue, it will be

$$R_0(x) = \lim_{s\to 0} \frac{\mathrm{d}}{\mathrm{d}s} \left(s^2\Gamma(s)\zeta(Ns+1)\zeta(s)x^{-s} \right).$$

Using the above Laurent series expansions and after a small simplification, one can easily check that

$$\lim_{s\to 0} \left(\frac{\mathrm{d}}{\mathrm{d}s} \left(s^2\Gamma(s)\zeta(s)\zeta(Ns+1)x^{-s} \right) \right) = \frac{1}{2N} \left(\log(x) - \log(2\pi) - \gamma(N-1) \right).$$

Hence,

$$R_0(x) = \frac{1}{2N} \left(\log(x) - \log(2\pi) - \gamma(N-1) \right). \tag{5.3}$$

Case 2: In this case, we took χ as an even character modulo q with $q > 1$. It is clear that $s = 0$ is a simple zero of $L(s,\chi)$. So, one pole at $s = 0$ will get cancelled by this zero. Thus, $s = 0$ will remain as a simple pole of integrand. Hence, the residue at $s = 0$ is given by

$$\begin{aligned} R_0(x) =& \lim_{s\to 0} s\Gamma(s)\zeta(Ns+1)L(s,\chi)\left(x/q\right)^{-s} \\ =& \lim_{s\to 0} s\zeta(Ns+1)\frac{L(s,\chi)}{s}, \\ =& \frac{1}{N}\lim_{s\to 0} \frac{L(s,\chi)}{s} \\ =& \frac{L'(0,\chi)}{N}, \quad \text{using } L\text{−Hospital's rule.} \end{aligned}$$

In the second last step, we have used the fact that $\zeta(Ns+1)$ has a simple pole at $s = 0$ with residue $1/N$. Therefore, Proposition 3.2.7 gives

$$R_0(x) = -\frac{1}{2N}\sum_{r=1}^{q-1} \chi(r)\log\left(\sin\left(\frac{r\pi}{q}\right)\right). \tag{5.4}$$

Case 3: In this case, we have taken that χ is an odd character. So, $L(0,\chi) \neq 0$. Thus, $s = 0$ must be a double pole of the integrand. Again, applying the Laurent series expansions in (5.2) around $s = 0$, we reach

$$s^2\Gamma(s)L(s,\chi)\zeta(Ns+1)\left(\frac{x}{q}\right)^{-s}$$

$$= \frac{L(0,\chi)}{N} + \frac{1}{N}\left[L(0,\chi)\left(\gamma(N-1) - \log\left(\frac{x}{q}\right)\right) + L'(0,\chi)\right] s + O(s^2).$$

Thus,

$$\lim_{s\to 0} \frac{\mathrm{d}}{\mathrm{d}s}\left(s^2\Gamma(s)L(s,\chi)\zeta(Ns+1)\left(\frac{x}{q}\right)^{-s}\right) \tag{5.5}$$

$$= \frac{1}{N}\left[L(0,\chi)\left(\gamma(N-1) - \log\left(\frac{x}{q}\right)\right) + L'(0,\chi)\right]. \tag{5.6}$$

Hence, by using Propositions (3.2.6) and (3.2.7) in (5.5), the residue of the integrand takes the shape

$$R_0(x) = \frac{1}{N}\left[\frac{1}{q}\left(\log(x) - \gamma(N-1)\right)\sum_{r=1}^{q-1}\chi(r)r + \sum_{r=1}^{q-1}\chi(r)\log\left(\Gamma\left(\frac{r}{q}\right)\right)\right]. \tag{5.7}$$

Therefore, combining all the three cases, we can check that the residue at $s = 0$ is exactly same we defined in (4.8).

The residues at the remaining poles are same as in Theorem 4.0.1. For example, $\mathcal{R}_1$ is

$$\mathcal{R}_1 = \begin{cases} \frac{\zeta(N+1)}{x}, & \text{if } q = 1, \\ 0, & \text{if } q > 1, \end{cases} \tag{5.8}$$

and the residue at $s = -j$ is given by

$$R_{-j}(x) = \frac{(-1)^{j+1}}{(j+1)!}B_{j+1,\chi}\,\zeta(-Nj+1)\left(\frac{x}{q}\right)^j,$$

where $1 \leq j \leq \lfloor 1 + \frac{1}{N}\rfloor - 1$. In this case $N - 2h = -1$, so the only negative poles of integrand are $s = -1, -2, \ldots.., -\lfloor 1 + \frac{1}{N}\rfloor + 1$. Thus, the sum of these poles is given by

$$\sum \mathcal{R}_{-j} = \sum_{j=1}^{\lfloor 1+\frac{1}{N}\rfloor - 1} R_{-j}(x) = \begin{cases} \frac{x}{2q}L(-1,\chi), & \text{if } N = 1, \\ 0, & \text{if } N > 1. \end{cases} \tag{5.9}$$

The remaining proof is same as in theorem 4.0.1. Thus, in the proof of Theorem 4.0.1, we put $h = \frac{N+1}{2}$ in (4.3) to get

$$\mathcal{K}_\chi(x) = (-1)^{\frac{N-1}{2}}\frac{\mathcal{G}(\chi)}{N}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)}{n}\sideset{}{''}\sum_{j=-(N-1)}^{N-1} \upsilon_{N,\chi}(j)\frac{\exp\left(\frac{i\pi j}{2}\right)}{\exp\left(2\pi\left(\frac{2\pi n}{x}\right)^{(1/N)}\exp\left(\frac{-i\pi j}{2N}\right)\right) - 1}. \tag{5.10}$$

Hence, gathering all the residual terms and together with the above expression (5.10), we can conclude the proof of Theorem 4.0.4. $\square$

Chapter 6

Concluding Thoughts

Inspired from the work of Kanemitsu et al. [14], Dixit and Maji studied the below infinite series, for $N \in \mathbb{N}, h \in \mathbb{Z}$,

$$\sum_{m=1}^{\infty} \frac{m^{N-2h}}{\exp(m^N x) - 1} \tag{6.1}$$

Transformation formula for this series was crucial to find a new generalization for Ramanujan's identity for $\zeta(2m+1)$.

Later, Kanemitsu et al. further explored a character analogue of the series (1.3), namely, the following infinite series and its integral representation:

$$\sum_{r=1}^{q} \sum_{n=1}^{\infty} \frac{\chi(r) n^{N-2h} \exp\left(-\frac{r}{q} n^N x\right)}{1 - \exp(-n^N x)} = \frac{1}{2\pi i} \int_{(c_0)} \Gamma(s) L(s, \chi) \zeta(Ns - N + 2h) \left(\frac{x}{q}\right)^{-s} \mathrm{d}s, \tag{6.2}$$

where χ is a Dirichlet character modulo q, and for some large positive c_0. Although, they studied it for $N \in \mathbb{N}$ and $h \in \mathbb{Z}$ with some restriction on h. In this thesis, we studied the same series (6.2) for any $N \in \mathbb{N}$ and $h \in \mathbb{Z}$. This motivated us to find a new character analogue of Ramanujan's identity (1.2). For any $N \geq 1$ and $m \neq 0$, our generalization will give a relation between $\zeta(2Nm+1)$ and $L(2m+1, \chi)$. We were able to find a transformation formula for the series (6.2), for any $N \in \mathbb{N}$, when χ is even, whereas when χ is odd, our transformation is valid only for $N \in 2\mathbb{N}$. It would be interesting to find a formula for any $N \in \mathbb{N}$

when χ is any odd character.

In a forthcoming work, we are planning to study the following integral:

$$\frac{1}{2\pi i}\int_{(c_0)} \Gamma(s)L(s,\chi)L(Ns-N+2h,\psi)\left(\frac{x}{QR}\right)^{-s} \mathrm{d}s, \tag{6.3}$$

where χ and ψ are primitive characters modulo Q and R, respectively.

Verification of Theorem 4.0.1

Let $N \in \mathbb{N}$ and $h \in \mathbb{Z}$ with $N - 2h \neq -1$. Let x be a positive real number. We took the left-hand side and right-hand side sum over n considered only first 100 terms. This numerical data has been obtained using the Mathematica software.

Table 6.1: Verification of Theorem 4.0.1

N	h	x	q	Parity of χ	Left-hand side	Right-hand side
4	7	1.22	2	odd	0.315691	0.315691
4	7	1.22	5	even	0.0929631	0.0929631
6	10	π	5	odd	$0.472922 + 0.138771i$	$0.472921 + 0.13877i$
8	10	$\pi + 1$	5	odd	$0.406854 + 0.109186i$	$0.406807 + 0.109175i$
7	8	15	5	even	0.0471911	0.0471705
2	-3	e^2	7	even	$4.47864 - 0.107135i$	$4.47848 - 0.107115i$

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