An Infinite Series Associated to the Rankin-Selberg L-Function

M.Sc. Thesis

by

Pritam Naskar

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

MAY 2022

An Infinite Series Associated to the Rankin-Selberg L-Function

A THESIS

Submitted in partial fulfillment of the requirements for the award of the degree of

Master of Science

by

Pritam Naskar

(Roll No.2003141016)

Under the guidance of

Dr. Bibekananda Maji

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

MAY 2022

INDIAN INSTITUTE OF TECHNOLOGY INDORE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"An infinite series associated to the Rankin-Selberg L-function"** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF MATHEMATICS**, **Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2021 to May 2022 under the supervision of **Dr. Bibekananda Maji**, Assistant Professor, Department of Mathematics, IIT Indore. The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

Pritam Naskar 28/05/2022

Signature of the student with date

(**Pritam Naskar**)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Bibekananda Maji 28/05/2022

Signature of Thesis Supervisor with date

(**Dr. Bibekananda Maji**)

Pritam Naskar has successfully given his M.Sc. Oral Examination held on 26th May, 2022.

Bibekananda Maji

Signature of supervisor of M.Sc Thesis

Date: 28/05/2022

Signature of Convener,DPGC

Date: 28/05/2022

Anand Prakash

Signature of PSPC Member

Dr. Anand Prakash

Date: 28-05-2022

V.K. Sohani

Signature of PSPC Member

Dr. Vijay Kumar Sohani

Date: 28/05/2022

Dedicated to my Family

We are what our thoughts have made us;

So take care about what you think.

Thoughts live; They travel far.

–***Swami Vivekananda***

Acknowledgements

I am incredibly grateful to my supervisor Dr. Bibekananda Maji for his assistance and guidance throughout my whole M.Sc. course. He gave his most profound concern for my overall well-being in these challenging times. His friendly nature, enthusiasm, and supportive attitude encourage me to successfully complete this thesis. His confidence and trust in me always push me to do better in every direction. Also, he conveyed the spirit of adventure and pushed me forward toward my area of interest concerning research. I believe that it is an honor for me to work with him. I thank the Almighty for bestowing his blessings upon me. I am very thankful to my PSPC members Dr. Anand Prakash, Dr. Vijay Kumar Sohani as well as DPGC convener Dr. Safique Ahmed and HOD Dr. Niraj Kumar Shukla for their continous support. I am also thankful to all our teachers for their kind and valuable suggestion. I am too fortunate to have all these teachers that even my most profound gratitude is not enough. This acknowledgment note would be irrelevant without being thankful to my family members. Their support and faith gradually and firmly boost my confidence from inside. They gave me the strength to pursue my interests even in difficult times. And lastly, I thank my

friends for always being there with me. My heart is full of favours received from every single person. I remain indebted and grateful to every one of you.

Abstract

Around four decades ago, Don Zagier speculated that the constant term of an automorphic form associated to the Ramanujan delta function has an asymptotic expansion. Moreover, he observed that it has a connection with the complex zeros of $\zeta(s)$. This speculation was finally proved by Hafner and Stopple in 2000. Later in 2017, Chakraborty, Kanemitsu and Maji protracted this observation by taking any cusp form over $SL_2(\mathbb{Z})$. This thesis examine a similar infinite sum, namely,

$$\sum_{n=1}^{\infty} c_f^2(n) n^{\nu/2} K_\nu(\sqrt{nx}),$$

where $c_f(n)$ represents nth Fourier coefficient of a cusp form $f(z)$ and K_ν represents the modified Bessel function of second kind with order ν. Interestingly, we also observe that this series has a connection with the complex zeros of $\zeta(s)$.

Contents

CHAPTER 1

Basics of Modular Forms

In this chapter, our main intention is to develop the theory of modular forms of level one on $\Gamma := SL_2(\mathbb{Z})$ in such a way that makes a building block for this work. Later part of this chapter, we discuss the Fourier series expansion of modular forms.

Definition 1.1. *We say $f(z)$ is a modular form of weight k if the following conditions are true:*
(a) $f(z)$ is analytic in $\mathcal{H}$.
(b) $f(\gamma z) = f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$.
(c) $f(z)$ has a Fourier series as $f(z) = \sum_{n=0}^{\infty} c_f(n) e^{2\pi i n z}$.

We say the nth coefficient $c_f(n)$ as the nth Fourier coefficient of $f(z)$. Note that, here k is an integer. Also, in the property (b) we are minimising the restriction on $f(z)$ that it need not be invariant under the unimodular transformations as in the case of modular functions. Taking $x = e^{2\pi i z}$, the Fourier series of $f(z)$

represent the Laurent series of $f(z)$ near $x = 0$, and then the property (c) tells us that $f(z)$ is analytic everywhere in $\mathcal{H}$ as well as at $i\infty$, since the Laurent series has no negative power terms. The constant term $c_f(0)$ is the value of $f(z)$ at $i\infty$. If a modular form vanishes at $i\infty$, i.e. if $c_f(0) = 0$ then it is called a cusp form.

Example 1.1 (Ramanujan's cusp form)**.** *The discriminant function $\Delta(z)$ of Ramanujan is an example of a cusp form of weight 12, which is defined as*

$$\Delta(z) := e^{2\pi iz} \prod_{n=1}^{\infty} \left(1 - e^{2\pi inz}\right)^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi inz}, \tag{1.1}$$

where $\tau(n)$ is popularly known as Ramanujan's tau function.

Ramanujan [20] conjectured the following facts about this function:

(i) $\tau(n)$ is completely multiplicative,

(ii) $\tau(p^{\ell+1}) = \tau(p)\tau(p^{\ell}) - p^{11}\tau(P^{\ell-1})$, for any $\ell \in \mathbb{N}$,

$(iii) |\tau(p)| \leq 2p^{11/2}$, for any prime p.

In the next year, Mordell proved the first two properties, and the 3^{rd} property was proved by Pierre Deligne in 1974. Ramanujan [20] also gave many interesting congruence properties for $\tau(n)$. There is a famous conjecture, due to Lehmer, which asserts that $\tau(n) \neq 0$ for all n.

CHAPTER 2

Modular Forms and L-function

2.1 Dirichlet series and its properties

Before starting with the main topic of this chapter, first we shall study Dirichlet series in more general setting. Consider the series $\sum_{n=1}^{\infty} a(n)e^{-\alpha_n s}$, where $a(n)$'s and s are complex numbers and $\{\alpha_n\}$ is a monotonically increasing sequence of positive real numbers. Let us take $\alpha_n = \log n$. Then the series will be called the Dirichlet series and it is of the form

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} \tag{2.1}$$

which has the most essence in this chapter. It was Euler who first studied the series with $a(n) = 1$ and taking the variable s to be real. But later in 1859, Riemann generalized the variable s to be complex. He considered the Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{where } Re(s) > 1, \tag{2.2}$$

which is now popularly known as the Riemann zeta function. Absolute convergence of the series follows whenever $Re(s) > 1$. To see this, take $s = \sigma + it$, then

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} < \infty$$

only for $\sigma > 1$. The infinite sum (2.2) can be written as an infinite product over primes:

$$\prod_{p:prime} \frac{1}{1 - \frac{1}{p^s}} \text{ for } Re(s) > 1. \tag{2.3}$$

More importantly, Riemann showed that $\zeta(s)$ has an analytic extension in $\mathbb{C}$ with only one exception at $s = 1$. He also showed that $\zeta(s)$ satisfies the following equation:

$$\nabla(s) = \nabla(1 - s), \tag{2.4}$$

where $\nabla(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$. This is known as the functional equation

2.2 L-functions linked to modular forms

In the first chapter, we have already seen what it means by the Fourier expansion of a cusp form. Now we shall study some properties of it related to the Dirichlet series. Let f be a cusp form over Γ and its Fourier expansion at $i\infty$ is given by

$$f(z) = \sum_{n=1}^{\infty} c_f(n) e^{2\pi i n z}. \tag{2.5}$$

We consider the Fourier coefficients $c_f(n)$ of the modular form $f(z)$ in place of the complex numbers $a(n)$ in the series (2.1) and define the L-function:

$$L(f, s) := \sum_{n=1}^{\infty} \frac{c_f(n)}{n^s}. \tag{2.6}$$

Hecke showed that $L(f, s)$ admits an analytic extension and and it can be continued to an entire function which satisfies the following functional equation:

$$\nabla(s) = i^k \nabla(k - s), \tag{2.7}$$

where $\nabla(s) = (2\pi)^{-s}\Gamma(s)L(f, s)$.

In particular, let us consider the Dirichlet series associated to the Ramanujan's function $\tau(n)$, given by $L(s,\Delta)=\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s}$, where $Re(s)>13/2$, which satisfies the functional equation

$$\nabla(s)=\nabla(12-s),$$

where $\nabla(s)=(2\pi)^{-s}\Gamma(s)L(s,\Delta)$. In a recent work, Berndt et al. [6] showed that, for $Re(\nu), Re(s)>0$,

$$\sum_{n=1}^{\infty}\tau(n)n^{(\nu+1)/2}K_{\nu+1}(s\sqrt{n})=2^{36+\nu}s^{\nu+1}\pi^{12}\Gamma(13+\nu)\sum_{n=1}^{\infty}\frac{\tau(n)}{(s^2+16\pi^2 n)^{\nu+13}}. \quad (2.8)$$

Now, for $\nu=1/2$ one can use (3.7) to obtain $K_{1/2}(s\sqrt{n})=\sqrt{\frac{s}{2s\sqrt{n}}}e^{-s\sqrt{n}}$. Utilizing this and taking $\nu=-1/2$, in the above equation they obtained the following identity as a corollary due to Chandrasekharan and Narasimhan [11, p. 16, Eq.(56)]:

$$\sum_{n=1}^{\infty}\tau(n)e^{-s\sqrt{n}}=2^{36}\pi^{23/2}\Gamma\left(\frac{25}{2}\right)\sum_{n=1}^{\infty}\frac{s\tau(n)}{(s^2+16\pi^2 n)^{25/2}}. \quad (2.9)$$

Theorem 2.1 (Delinge)**.** *For any cusp form $f(z)$ the Fourier coefficient satisfies the following bound:*

$$c_f(n)=O\left(n^{\frac{k-1}{2}}+\epsilon\right),\ \forall\,\epsilon>0.$$

Using the above bound, it can be easily observed that the definition (2.6) of the L-function is valid only when $Re(s)>\frac{k+1}{2}$. It is a well known result that the below product representation of $L(f,s)$ is valid for $Re(s)>\frac{k+1}{2}$ and given by

$$L(f,s)=\prod_{p:prime}(1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1},$$

where α_p and β_p are complex conjugates satisfying $\alpha_p+\beta_p=c_f(p)$ and $\alpha_p\beta_p=p^{k-1}$. Making use of these two complex numbers α_p and β_p, Shimura [23] deeply studied the below L-function.

Definition 2.1. *Let f be a Hecke eigenform defined on Γ. Then the symmetric square L-function is defined by the following Euler product formula:*

$$L(Sym^2(f),w,\psi):=\left(1-\psi(p)\frac{\alpha_p^2}{p^w}\right)^{-1}\left(1-\psi(p)\frac{\alpha_p\beta_p}{p^w}\right)^{-1}\left(1-\psi(p)\frac{\beta_p^2}{p^w}\right)^{-1}, \quad (2.10)$$

where ψ is a primitive Dirichlet character. Here we only concentrate on the trivial Dirichlet character. Then $D(f,w)$ becomes

$$L(Sym^2(f),w) = \left(1-\frac{\alpha_p^2}{p^w}\right)^{-1}\left(1-\frac{\alpha_p\beta_p}{p^w}\right)^{-1}\left(1-\frac{\beta_p^2}{p^w}\right)^{-1}. \tag{2.11}$$

Shimura, in the same paper, studied the functional equation and analytic extension for $L(Sym^2(f),w)$.

Theorem 2.2 (Shimura)**.** *Take*

$$\mathscr{R}(f,w) := \pi^{-3w/2}\Gamma\left((w+1)/2\right)\Gamma(w-k+2-\lambda_0)/2)L(Sym^2(f),w),$$

where λ_0 is defined as 1 or 0 if $\psi(-1)=-1$ or 1, respectively. Then $\mathscr{R}(f,w)$ has a meromorphic continuation with simple pole at the point $w=k$ or $w=k-1$ and it satisfies

$$\mathscr{R}(f,w) = \mathscr{R}(f,2k-w-1). \tag{2.12}$$

Definition 2.2. *The Rankin-Selberg L-function associated to the eigenform $f(z)$ is defined by, for $\Re(w)>k$,*

$$L(f\otimes f,w) := \sum_{n=1}^{\infty}\frac{c_f^2(n)}{n^w}. \tag{2.13}$$

This L-function was independently investigated by Rankin [21] and Selberg [22] in 1940.

Theorem 2.3. *The function defined above has the following properties:*

(a) The absolutely convergence of $L(f\otimes f,w)$ follows whenever $Re(w)>k$,

(b) It has a meromorphic extension in $\mathbb{C}$ with a pole of order 1 at $w=k$,

(c) The functional equation of $L(f\otimes f,w)$ is given by:

$$\mathscr{R}^*(f,w) = \mathscr{R}^*(f,2k-w-1), \tag{2.14}$$

where

$$\mathscr{R}^*(f,s) := \frac{1}{(2\pi)^{2s}}\Gamma(s)\Gamma(s-k+1)\zeta(2s-2k+2)L(f\otimes f,s). \tag{2.15}$$

The following relationship was discovered by Shimura [23], to connect $L(Sym^2(f),w)$

and $L(f \otimes f, w)$:

$$L\left(w-k+1, \psi\chi\right) L(Sym^2(f), w, \psi) = L\left(2w-2k+2, \psi^2\chi^2\right) L(f \otimes f, w, \psi). \tag{2.16}$$

Taking both ψ and χ as trivial character, we will have

$$\zeta\left(2w-2k+2\right) L(f \otimes f, w) = \zeta\left(w-k+1\right) L(Sym^2(f), w). \tag{2.17}$$

In view of (2.14) and (2.15), one can see that

$$(4\pi^2)^{2k-1}\Gamma(w)L(f \otimes f, w) = (4\pi^2)^{2w}\frac{\Gamma(2k-w-1)\Gamma(k-w)}{\Gamma(s-k+1)} \tag{2.18}$$

$$\times \frac{\zeta(2k-2w)}{\zeta(2w-2k+2)}L(f \otimes f, 2k-w-1). \tag{2.19}$$

This identity will be crucial in the proof of our main identity.

CHAPTER 3

Background of this thesis

3.1 Zagier's conjecture

We write the automorphic form $\mathcal{F}(z) := y^{12}|\Delta(z)|^2$, with $Im(z) = y$, in the following way:

$$\begin{aligned}\mathcal{F}(z) =& y^{12}|\Delta(z)|^2\\ =& y^{12}\Delta(z)\overline{\Delta(z)}\\ =& y^{12}\sum_{n=1}^{\infty}\tau(n)e^{2\pi i n z}\sum_{m=1}^{\infty}\tau(m)e^{-2\pi i m\bar{z}}\\ =& y^{12}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\tau(n)\tau(m)e^{2\pi i(n-m)x}e^{-2\pi y(n+m)}.\end{aligned}$$

Considering $n = m$, in the above equation, one can obtain the constant term of the Fourier expansion of $\mathcal{F}(z)$ as

$$a_0(y) = y^{12}\sum_{n=1}^{\infty}\tau(n)^2 e^{-4\pi n y}, \tag{3.1}$$

.

In 1981, Zagier [25, p. 417], [26, p. 271] observed that the infinite series (3.1) has a connection with the complex zeros of $\zeta(s)$ whenever $y \to 0^+$. In that paper, he mainly mentioned that:

$$a_0(y) \sim \mathcal{C} + \sum_{\rho} A_\rho y^{1-\frac{\rho}{2}}$$

as $y \to 0^+$, where C denotes the average value of $F(z)$ in $\mathbb{H}$ and the sum over ρ runs over the complex zeros of $\zeta(s)$. Also, if we assume that all the complex zeroes of $\zeta(s)$ lie on $Re(s) = 1/2$ i.e., $\rho = \frac{1}{2} \pm i\gamma_n$, where γ_n is real, the above mentioned expression takes the shape as

$$a_0(y) \sim \mathcal{C} + y^{3/4} \sum_{n=1}^{\infty} a_n \cos\left(\frac{1}{2}\gamma_n \log y + \phi_n\right), \text{as } y \to 0^+,$$

where a_m and ϕ_n are some real constants. As the right side contains a cosine part, Zagier noticed that the behaviour of the graph of $a_0(y)$ is oscillatory as $y \to 0^+$. In the year 2000, Hafner and Stopple [12] showed the asymptotic behaviour of $a_0(y)$ as $y \to 0^+$ and also the oscillatory characteristic, assuming the Riemann hypothesis. A few years back, Chakraborty et al. [9] generalized this observation for any cusp form $f(z)$ of weight k over Γ. They mainly demonstrated that the series of the form

$$y^k \sum_{n=1}^{\infty} c_f^2(n) \exp(-ny)$$

can also be represented in terms of the complex zeros of $\zeta(s)$. Further, this phenomenon corresponding to cusp forms for congruence subgroups was generalized by Chakraborty et al. [10]. Banerjee and Chakraborty examined the asymptotic growth of a similar infinite series related with Maass cusp forms in their paper [7]. Recently, Agnihotri [3] has studied the same problem for Hilbert modular forms. Juyal, Maji, and Satyanarayana [16] were inspired by these studies and examined a Lambert series connected to a cusp form and Möbius function, resulting in an exact formula. Interested readers can see [16, 17] to know related works.

3.2 Bessel functions

In both mathematics and physics, Bessel functions are the most significant special functions. The theory of Bessel functions has had a considerable influence on the theory of differential equation, integral equation, and on analytic number theory also. The Bessel functions are the solution of the following 2^{nd} order linear differential equation:

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \nu^2)y = 0, \tag{3.2}$$

where ν is called the order of the Bessel function. When ν is not an integer or half-integer, then by Frobenius method, two linearly independent solutions can be found. The Bessel function of first kind is defined as

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}, \ \forall \nu \in \mathbb{C}.$$

From Frobenius method, J_ν and $J_{-\nu}$ are two linearly independent solutions whenever $\nu \notin \mathbb{Z}, \mathbb{Z} + \frac{1}{2}$. But when $\nu = -m$, $m \in \mathbb{Z}$, then it is easy to see that

$$J_{-m}(z) = (-1)^m J_m(z),$$

which shows that J_ν and $J_{-\nu}$ are linearly dependent. In this case to have two linearly independent solutions let us define

$$Y_\nu(z) := \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} \ \text{ for } \nu \notin \mathbb{Z},$$

and for $\nu = m \in \mathbb{Z}$ define

$$Y_m(z) := \lim_{\nu \to m} Y_\nu(z).$$

Then Y_ν is also a solution of the Bessel differential equation (3.2). Also J_ν and Y_ν represents two linearly independent solution of the Bessel differential equation (3.2) for any $\nu \in \mathbb{C}$.

For the complex argument also, one can define the same differential equation i.e., replacing z by iz in (3.2), we obtain

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} - (z^2 + \nu^2)y = 0. \tag{3.3}$$

The above equation is commonly known as the modified Bessel differential equation and its solutions are usually referred to modified Bessel functions. Again solving this equation, we can obtain

$$I_\nu(z) := \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu} \quad \text{for any } \nu \in \mathbb{C}. \tag{3.4}$$

Here also $I_\nu(z)$ and $I_{-\nu}(z)$ are linearly independent whenever $\nu \notin \mathbb{Z}$. But when $\nu \in \mathbb{Z}$, to obtain an independent solution, we define

$$K_\nu(z) := \frac{\pi}{2} \cdot \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)} \text{ for } \nu \notin \mathbb{Z}, \tag{3.5}$$

and for $\nu = m \in \mathbb{Z}$ define

$$K_m(z) := \lim_{\nu \to m} K_\nu(z).$$

Then K_ν represents a solution of the equation (3.3) and for any $\nu \in \mathbb{C}$, I_ν and K_ν are linearly independent, known as the modified Bessel function of first kind and second kind respectively.

Now let us record some results related to the modified Bessel function of second kind.

For a fixed ν and for large values of z, satisfying $|\arg(z)| < \frac{3\pi}{2}$, from [1, p. 378, 9.7.2], we have

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}. \tag{3.6}$$

It is interesting to note that, in view of $I_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sinh z$ and $I_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cosh z$ we have

$$K_{1/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}. \tag{3.7}$$

Lemma 3.1. *Let $\nu \in \mathbb{C}$ and K_ν be the modified Bessel function of second kind. Then for any c satisfying $c > \max\{0, -Re(\nu)\}$, the following holds*

$$t^{\nu/2} K_\nu(a\sqrt{tx}) = \frac{1}{2}\left(\frac{2}{a\sqrt{x}}\right)^{\nu} \frac{1}{2\pi i} \int_{c} \Gamma(s)\Gamma(s+\nu) \left(\frac{4}{a^2 x}\right)^{s} t^{-s} ds \tag{3.8}$$

Proof. From the Mellin transform [18, p. 24] of $K_\nu(at)$ is given by

$$\int_0^\infty t^{s-1} K_\nu(at) dt = \frac{2^{s-2}}{a^s} \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu}{2}\right).$$

whenever $Re(a) > 0$ and $Re(s) > |Re(\nu)|$.

Now, changing the variable t by $\sqrt{tx}$, we obtain

$$\int_0^\infty t^{\frac{s}{2}-1}x^{\frac{s}{2}}K_\nu(a\sqrt{tx})dt = \frac{1}{2}\left(\frac{2}{a}\right)^s \Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu}{2}\right),$$

Again replacing s by 2s, to get

$$\int_0^\infty t^{s-1}K_\nu(a\sqrt{tx})dt = \frac{1}{2}\left(\frac{2}{a\sqrt{x}}\right)^{2s} \Gamma\left(s-\frac{\nu}{2}\right)\Gamma\left(s+\frac{\nu}{2}\right),$$

which is valid for $Re(s) > |Re(\nu)|/2$. Now another change of variable, s by $s+\nu/2$, will give

$$\int_0^\infty t^{s-1}t^{\nu/2}K_\nu(a\sqrt{tx})dt = \frac{1}{2}\left(\frac{2}{a\sqrt{x}}\right)^{2s+\nu} \Gamma(s)\Gamma(s+\nu),$$

which is valid for $Re(s) > 0$, when $Re(\nu) \geq 0$ and $Re(s) > |Re(\nu)|$, when $Re(\nu) < 0$. Hence the Mellin transform of the function $f(t) = t^{\nu/2}K_\nu(a\sqrt{ax})$ is

$$F(s) := \mathcal{M}(f(s)) := \frac{1}{2}\left(\frac{2}{a\sqrt{x}}\right)^{2s+\nu} \Gamma(s)\Gamma(s+\nu). \tag{3.9}$$

Now, using the inverse Mellin transform we will get the result of this lemma. $\square$

CHAPTER 4

Main Results

In the previous chapter, we have already introduced the background of this project work. Now we are ready to mention the main results of this thesis.

Theorem 4.1. *Let $f(z)$ be a cusp form of weight k over $SL_2(\mathbb{Z})$ with (2.5). Let us suppose that all the complex zeros of $\zeta(s)$ are simple. Let K_ν be the modified Bessel function of second kind of order ν. Then, for $Re(\nu) > 1-k$ and $x > 0$, we have*

$$\sum_{n=1}^{\infty} c_f^2(n) n^{\nu/2} K_\nu(\sqrt{nx}) = \frac{1}{2}\left(\frac{4}{x}\right)^{\frac{\nu}{2}} \Bigg[\mathcal{R}_{f,\nu}(x) + \frac{6}{\pi^2}\Gamma(k)\Gamma(k+\nu) L(Sym^2(f), k)\left(\frac{4}{x}\right)^{k} + \frac{x^\nu (16\pi)^{2k-1}}{\sqrt{\pi}} \frac{\Gamma(2k+\nu-1)\Gamma(k+\nu)}{\Gamma(k+\nu-\frac{1}{2})} \sum_{n=1}^{\infty} \frac{A^*(f\otimes f, n)}{(64n\pi^2)^{2k-1+\nu}} \times {}_2F_1\left(2k+\nu-1, k+\nu; k+\nu-\frac{1}{2}; -\frac{x}{64n\pi^2}\right)\Bigg],$$

where

$$\mathcal{R}_{f,\nu}(x) = \sum_{\rho} \frac{\Gamma(\frac{\rho}{2}+k-1)\Gamma(\frac{\rho}{2}+k+\nu-1) L(Sym^2(f), \frac{\rho}{2}+k-1)\zeta(\frac{\rho}{2})}{\zeta'(\rho)} \left(\frac{4}{x}\right)^{\frac{\rho}{2}+k-1},$$

and ρ varies over all the complex zeros of $\zeta(s)$ which lies on $Re(s) = 1/2$.

We now draw two immediate consequences of the above theorem just taking two special values of ν. Letting $\nu = 1/2$, one can obtain the following result.

Corollary 4.2. *Let us suppose that all the hypotheses as in Theorem 4.1. Then*

$$\sum_{n=1}^{\infty} c_f^2(n) e^{-\sqrt{nx}} = \left(\frac{x^{1/4}}{\sqrt{\pi}}\right)\Bigg[\sum_{\rho} \frac{\Gamma\left(\frac{\rho}{2}+k-1\right)\Gamma\left(\frac{\rho}{2}+k-\frac{1}{2}\right) D\left(\frac{\rho}{2}+k-1\right)\zeta(\frac{\rho}{2})}{\zeta'(\rho)}$$
$$\times \left(\frac{4}{x}\right)^{\frac{\rho}{2}+k-1} + \frac{6}{\pi^2}\Gamma(k) L(Sym^2(f), k)\Gamma\left(k+\frac{1}{2}\right)\left(\frac{4}{x}\right)^{k}$$
$$+ \frac{x^{1/2}(16\pi)^{2k-1}}{\sqrt{\pi}} \frac{\Gamma(2k-\frac{1}{2})\Gamma(k+\frac{1}{2})}{\Gamma(k)} \sum_{n=1}^{\infty} \frac{A^*(f\otimes f, n)}{(64n\pi^2)^{2k-\frac{1}{2}}}$$
$$\times {}_2F_1\left(2k-\frac{1}{2}, k+\frac{1}{2}; k; -\left(\frac{x}{64n\pi^2}\right)\right)\Bigg].$$

Corollary 4.3. *Considering $f(z) = \Delta(z)$, the above corollary implies the following:*

$$\sum_{n=1}^{\infty} \tau^2(n) e^{-\sqrt{nx}} = \left(\frac{x}{\pi^2}\right)^{1/4}\Bigg[\sum_{\rho} \frac{\Gamma\left(\frac{\rho}{2}+11\right)\Gamma\left(\frac{\rho}{2}+\frac{23}{2}\right) D\left(\frac{\rho}{2}+11\right)\zeta(\frac{\rho}{2})}{\zeta'(\rho)}$$
$$\times \left(\frac{4}{x}\right)^{\frac{\rho}{2}+11} + \frac{6\cdot 11!}{\pi^2} L(Sym^2(f), 12)\Gamma\left(\frac{25}{2}\right)\left(\frac{4}{x}\right)^{12}$$
$$+ \frac{x^{1/2}(16\pi)^{23}}{\sqrt{\pi}} \frac{\Gamma(\frac{47}{2})\Gamma(\frac{25}{2})}{11!} \sum_{n=1}^{\infty} \frac{A^*(\tau\otimes\tau, n)}{(64n\pi^2)^{\frac{47}{2}}}$$
$$\times {}_2F_1\left(\frac{47}{2}, \frac{25}{2}; 12; -\left(\frac{x}{64n\pi^2}\right)\right)\Bigg].$$

CHAPTER 5

Some Well-known Results

In this chapter, we shall gather a few customary results those will be used frequently in the proof of our main results mentioned in the previous chapter. First, we define an important special function, namely, Meijer G-function.

5.1 Meijer G-function

Let m, n, p, q be integers satisfying $0 \leq m \leq q$, $0 \leq n \leq p$. Let us consider the complex numbers $a_1, \cdots, a_p$ and $b_1, \cdots, b_q$ with the property $a_i - b_j$ is not a natural number, for all $i \in [1, n]$, $j \in [1, m]$. The following line integral is known as the Meijer G-function [19, p. 415]:

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix} \middle| z\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1 - a_j + s)z^s}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + s)\prod_{j=n+1}^{p}\Gamma(a_j - s)}\mathrm{d}s. \tag{5.1}$$

The above definition is valid whenever the following properties are satisfied. All the poles of the factors $\Gamma(b_j - s)$ and $\Gamma(1 - a_j + s)$ are isolated by the line of integration L, whose imaginary part varies from $-i\infty$ to $+i\infty$. If $2(m+n) > p+q$ and $(2m+2n-p-q)\pi > 2|\arg(z)|$ both are satisfying then the integral converges. To express the Meijer G-function in connection with the generalized hypergeometric function, we need the following theorem.

Theorem 5.1 (Slater's identity)**.** *For $1 \leq j \neq k \leq m$, suppose $b_j - b_k \notin \mathbb{Z}$. Let us define*

$$A^{m,n}_{p,q,k}(z) := \frac{z^{b_k} \prod_{j=1, j\neq k}^{m} \Gamma(b_j - b_k) \prod_{j=1}^{n} \Gamma(1 + b_k - a_j)}{\prod_{j=m+1}^{q} \Gamma(1 + b_k - b_j) \prod_{j=n+1}^{p} \Gamma(a_j - b_k)}.$$

Then for $p \leq q$, one has

$$G^{m,n}_{p,q}\left(\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix} \middle| z\right) = \sum_{k=1}^{m} A^{m,n}_{p,q,k}(z) \tag{5.2}$$

$$\times {}_pF_{q-1}\left(\begin{matrix} 1 + b_k - a_1, \cdots, 1 + b_k - a_p \\ 1 + b_k - b_1, \cdots, \#, \cdots, 1 + b_k - b_q \end{matrix} \middle| (-1)^{p-m-n} z\right),$$

here # means that the term $1 + b_k - b_k$ is excluded.

5.2 Bounds for various functions

Lemma 5.2 (Stirling's bound for $\Gamma(z)$)**.** *[14, p. 151, A.4.] Let $z = \sigma + i\mathcal{T}$ with $p \leq \sigma \leq q$, we have*

$$|\Gamma(\sigma + i\mathcal{T})| \ll \sqrt{2\pi} |\mathcal{T}|^{\sigma - 1/2} e^{-\frac{1}{2}\pi|\mathcal{T}|}, \quad \text{as} \quad |\mathcal{T}| \to \infty. \tag{5.3}$$

Lemma 5.3. *Consider a vertical strip $\sigma_0 \leq \sigma \leq d$, we have*

$$|L(f \otimes f, \sigma + i\mathcal{T})| = O(|\mathcal{T}|^{A(\sigma_0)}), \quad \text{as } \mathcal{T} \to \infty,$$

where $A(\sigma_0)$ is a constant depending on σ_0.

Proof. The proof is available in [14, p. 97, Lemma 5.2]. □

CHAPTER 6

Proof of the main results

Proof of Theorem 4.1. Substituting $a = 1$ and replacing t by n in Lemma 3.1, one can see that

$$n^{\nu/2}K_\nu(\sqrt{nx}) = \frac{1}{2}\left(\frac{2}{\sqrt{x}}\right)^\nu \frac{1}{2\pi i}\int_{(\alpha)} \Gamma(s)\Gamma(s+\nu)\left(\frac{4}{xn}\right)^s ds,$$

whenever $\alpha > \max\{0, -Re(\nu)\}$. Throughout the proof, we denote the line integral $\alpha - i\infty$ to $\alpha + i\infty$ by (α) only. Now, we use the above the line integral representation for $K_\nu(\sqrt{nx})$ to write the left hand side infinite series in Theorem 4.1 as

$$\begin{aligned}\sum_{n=1}^{\infty} c_f^2(n)n^{\nu/2}K_\nu(\sqrt{nx}) =& \sum_{n=1}^{\infty} c_f^2(n)\frac{1}{2}\left(\frac{2}{\sqrt{x}}\right)^\nu \frac{1}{2\pi i}\int_{(\alpha)} \Gamma(s+\nu)\Gamma(s)\left(\frac{4}{xn}\right)^s ds \\ =& \frac{1}{2}\left(\frac{2}{\sqrt{x}}\right)^\nu \frac{1}{2\pi i}\int_{(\alpha)} \Gamma(s+\nu)\Gamma(s)\left(\frac{4}{x}\right)^s \sum_{n=1}^{\infty}\frac{c_f^2(n)}{n^s} ds.\end{aligned} \tag{6.1}$$

Here the changing in the order of summation and integration is possible only when we take $Re(s) = \alpha > \max\{k, -Re(\nu)\}$ to have the uniform convergence of the infinite series. Now, using the definition (2.13) of the function $L(f \otimes f, s)$, the above equation changes to

$$\sum_{n=1}^{\infty} c_f^2(n) n^{\nu/2} K_\nu(\sqrt{nx}) = \frac{1}{2}\left(\frac{4}{x}\right)^{\frac{\nu}{2}} I_{f,\nu}(x), \tag{6.2}$$

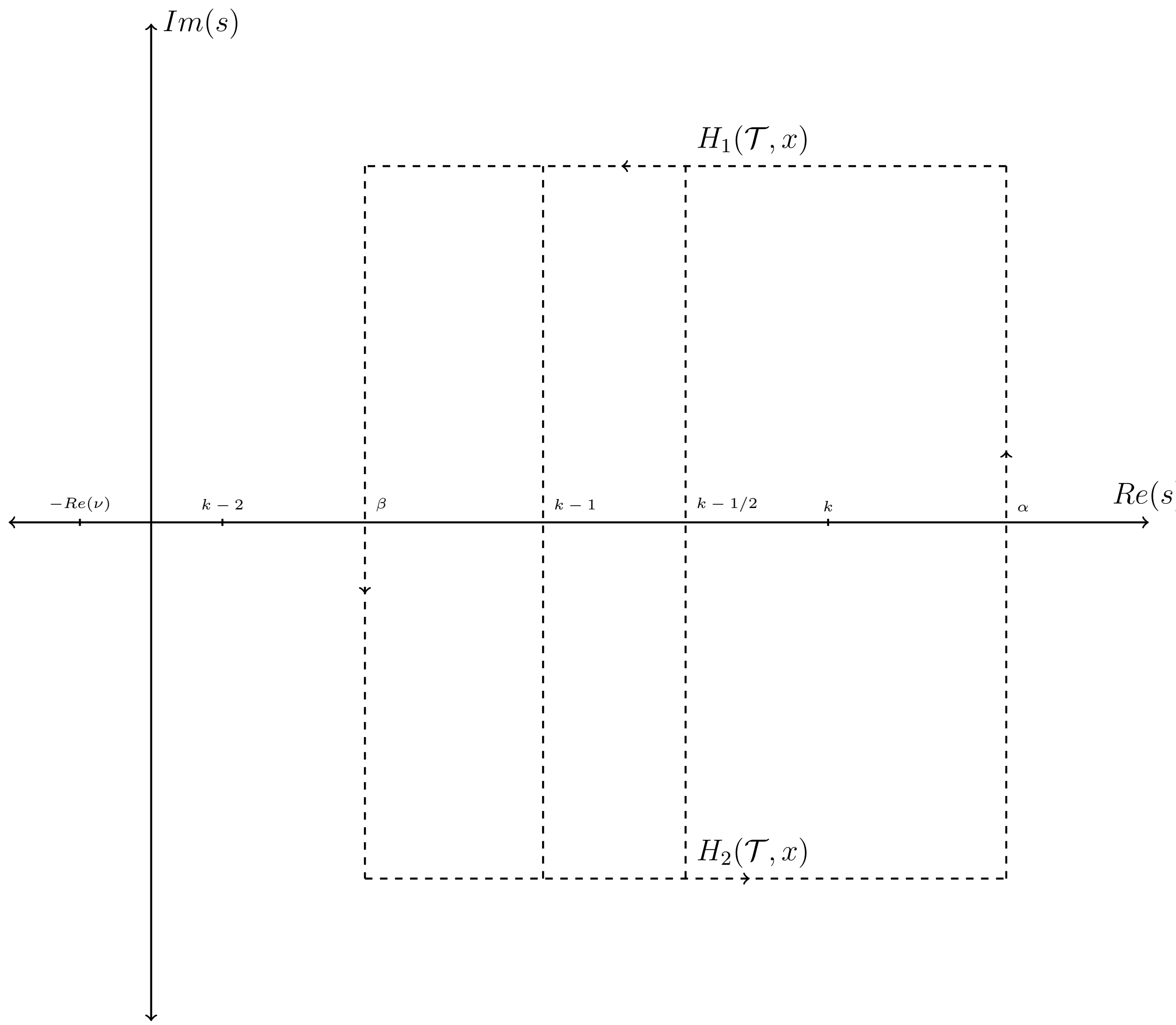
where

$$I_{f,\nu}(x) := \frac{1}{2\pi i}\int_{(\alpha)} \Gamma(s)\Gamma(s+\nu) L(f\otimes f, s)\left(\frac{4}{x}\right)^{s} ds, \tag{6.3}$$

for $Re(s) = \alpha > \max\{k, -Re(\nu)\}$. Now our main goal is simplify this line integral. Let us denote the integrand by

$$F_{f,\nu,x}(s) := \Gamma(s)\Gamma(s+\nu)L(f\otimes f, s)\left(\frac{4}{x}\right)^{s}.$$

First, we shall examine all the poles of the integrand $F_{f,\nu,x}(s)$. Theorem 2.3, tells us that $s = k$ is a simple of $L(f\otimes f, s)$. Moreover, utilizing the identity (2.17), one can observe that $L(f \otimes f, s)$, inside the critical strip $k-1 < Re(s) < k-\frac{1}{2}$, has infinite number of poles due to the complex zeros of $\zeta(2s-2k+2)$. Assuming Riemann hypothesis, one can say that these poles will lie on the line $Re(s) = k-\frac{3}{4}$. Further, note that $\Gamma(s)\Gamma(n+\nu)$ has infinitely many poles at $s = -n, -n-\nu$ for $n \in \mathbb{N}\cup\{0\}$. We now consider a rectangular contour $\mathcal{C}$ consisting of vertices $[\alpha - i\mathcal{T}, \alpha + i\mathcal{T}, \beta + i\mathcal{T}, \beta - i\mathcal{T}]$, where $\max\{-Re(\nu), k-2\} < \beta < k-1$ and $\mathcal{T}$ is some large positive real number. The interval for β will be explained in the later stage of proof. One thing we can immediately observe that the poles of $\Gamma(s)\Gamma(n+\nu)$ are not lying inside the contour $\mathcal{C}$.



Now we use Cauchy's residue theorem, to obtain

$$\frac{1}{2\pi i}\int_{\mathcal{C}}\Gamma(s)\Gamma(s+\nu)L(f\otimes f,s)\left(\frac{4}{x}\right)^s ds = \mathrm{Res}_{s=k}F_{f,\nu,x}(s)+\mathcal{R}_{f,\nu}(x), \tag{6.4}$$

where $\mathcal{R}_{f,\nu}(x)$ denotes the infinite residual term due to infinitely many zeros of $\zeta(2s-2k+2)$. One can easily calculate the residue at $s=k$. Mainly, we will have

$$\mathrm{Res}_{s=k}F_{f,\nu,x}(s)=\frac{3}{\pi}\left(\frac{16\pi}{x}\right)^k\Gamma(k+\nu)\langle f,f\rangle, \tag{6.5}$$

where $\langle f,f\rangle$ is the Petersson inner product of f with itself. Here we have used the fact that $\mathrm{Res}_{s=k}L(f\otimes f,s)=\frac{3}{\pi}\frac{(4\pi)^k}{\Gamma(k)}\langle f,f\rangle$. In another way, using (2.17), one can see that

$$\begin{aligned}\mathrm{Res}_{s=k}F_{f,\nu,x}(s) =& \lim_{s\to k}\frac{(s-k)\Gamma(s)\Gamma(s+\nu)L(Sym^2(f),s)\zeta(s-k+1)}{\zeta(2s-2k+2)}\left(\frac{4}{x}\right)^s\\ =&\frac{\Gamma(k)\Gamma(k+\nu)L(Sym^2(f),k)}{\zeta(2)}\left(\frac{4}{x}\right)^k\lim_{s\to k}(s-k)\zeta(s-k+1)\\ =&\frac{6}{\pi^2}\left(\frac{4}{x}\right)^k\Gamma(k)\Gamma(k+\nu)L(Sym^2(f),k). \end{aligned}\tag{6.6}$$

Now, let us calculate the infinite residual term $\mathcal{R}_{f,\nu}(x)$. Using (2.17) and assuming all the zeros of $\zeta(s)$ on the line $Re(s)=1/2$ are simple, we obtain

$$\begin{aligned}\mathcal{R}_{f,\nu}(x) =& \sum_{\rho}\lim_{s\to\frac{\rho}{2}+k-1}\frac{\left(s-\left(\frac{\rho}{2}+k-1\right)\right)\Gamma(s)\Gamma(s+\nu)L(Sym^2(f),s)\zeta(s-k+1)}{\zeta(2s-2k+2)}\left(\frac{4}{x}\right)^s\\ =&\sum_{\rho}\frac{\Gamma(\frac{\rho}{2}+k-1)\Gamma(\frac{\rho}{2}+k+\nu-1)L(Sym^2(f),\frac{\rho}{2}+k-1)\zeta(\frac{\rho}{2})}{\zeta'(\rho)}\left(\frac{4}{x}\right)^{\frac{\rho}{2}+k-1}.\end{aligned}\tag{6.7}$$

Here we emphasize that the convergence of this series is quite delicate since the nature of the lower bound for $\zeta'(\rho)$ is unknown to us. These kind of series appeared in the work of Hardy and Littlewood [13, p. 156], while correcting an identity of Ramanujan. To know more about this identity of Hardy and Littlewood, readers are encouraged to see [2], [5, p. 470].

Now we denote the horizontal integrals in the contour $\mathcal{C}$ by $H_1(x,\mathcal{T})$ and $H_2(x,\mathcal{T})$ respectively. We shall show that these integral will vanish as $|\mathcal{T}|\to\infty$. Letting $s=\sigma+i\mathcal{T}$, we have

$$\begin{aligned}H_1(x,\mathcal{T}) =& \frac{1}{2\pi i}\int_{\alpha+i\mathcal{T}}^{\beta+i\mathcal{T}}\Gamma(s)L(f\otimes f,s)\Gamma(s+\nu)\left(\frac{4}{x}\right)^s ds\\ =&\frac{1}{2\pi i}\int_c^d\Gamma(\sigma+i\mathcal{T})L(f\otimes f,\sigma+i\mathcal{T})\Gamma(\sigma+i\mathcal{T}+\nu)\left(\frac{4}{x}\right)^{\sigma+i\mathcal{T}}d\sigma.\end{aligned}$$

Now, using the bounds of $\Gamma(\sigma+i\mathcal{T})$ and $L(f\otimes f,\sigma+i\mathcal{T})$ i.e. Lemma 5.2 and 5.3 respectively, we arrive at

$$|H_1(x,\mathcal{T})|\leq\frac{1}{2\pi}\int_c^d|\mathcal{T}|^{\sigma-1/2}e^{-\frac{\pi}{2}|\mathcal{T}|}|\mathcal{T}|^{\sigma+Re(\nu-1/2)}e^{-\frac{\pi}{2}|\mathcal{T}+Im(\nu)|}\left|\frac{4}{x}\right|^{\sigma}d\sigma,$$

which shows that $|H_1(x,\mathcal{T})|\to 0$ as $|\mathcal{T}|\to\infty$. In a similar way, it can be shown that $|H_2(x,\mathcal{T})|\to 0$ as $\mathcal{T}\to\infty$. Now our only aim is to simplify the left vertical integral which we write as

$$V(x) := \frac{1}{2\pi i}\int_{(\beta)} \Gamma(s)L(f\otimes f,s)\Gamma(s+\nu)\left(\frac{4}{x}\right)^{s} ds, \tag{6.8}$$

where $\max\{-Re(\nu), k-2\} < \beta < k-1$. First, we employ the functional equation (2.18) of the function $L(f\otimes f,s)$ to get

$$V(x) = \frac{1}{2\pi i}\int_{(\beta)} \frac{\Gamma(s+\nu)\Gamma(2k-s-1)\Gamma(k-s)}{\Gamma(s-k+1)}\frac{\zeta(2k-2s)}{\zeta(2s-2k+2)} \times L(f\otimes f, 2k-s-1)\left(\frac{4}{x}\right)^{s}(4\pi^2)^{2s-2k+1} ds. \tag{6.9}$$

To simplify further, we make a change of variable, namely, replace $2k-1-s$ by w. Then, $V(x)$ takes the following shape:

$$V(x) = \left(\frac{16\pi^2}{x}\right)^{2k-1}\frac{1}{2\pi i}\int_{(\beta_1)} \frac{\Gamma(2k-w+\nu-1)\Gamma(w)\Gamma(w-k+1)}{\Gamma(k-w)} \times \frac{\zeta(2w-2k+2)}{\zeta(2k-2w)}L(f\otimes f,w)\left(\frac{64\pi^4}{x}\right)^{-w} dw, \tag{6.10}$$

where $Re(w) = \beta_1 = 2k-1-\beta$. Replace s by $2k-2w$ in (2.4) to see that

$$\Gamma(k-w)\zeta(2k-2w) = \pi^{2k-2w-\frac{1}{2}}\Gamma\left(w-k+\frac{1}{2}\right)\zeta(2w-2k+1). \tag{6.11}$$

Making use of (6.11) in (6.10) and simplifying further, we arrive at

$$V(x) = \frac{1}{\sqrt{\pi}}\left(\frac{16\pi}{x}\right)^{2k-1}\frac{1}{2\pi i}\int_{(\beta_1)} \frac{\Gamma(2k-w+\nu-1)\Gamma(w)\Gamma(w-k+1)}{\Gamma(w-k+\frac{1}{2})} \times \frac{\zeta(2w-2k+2)L(f\otimes f,w)}{\zeta(2w-2k+1)}\left(\frac{64\pi^2}{x}\right)^{-w} dw. \tag{6.12}$$

Here we note that β_1 will lie in $(k, k+1)$ since $\beta \in (k-2, k-1)$. One can easily check that zeta functions and $L(f\otimes f, w)$ are absolutely and uniformly convergent in any compact subset of this strip. Using series expansions, we write

$$\frac{\zeta(2w-2k+2)}{\zeta(2w-2k+1)} = \sum_{n=1}^{\infty}\frac{n^{2k-2}}{n^{2w}}\sum_{m=1}^{\infty}\frac{\mu(m)m^{2k-1}}{m^{2w}} = \sum_{n=1}^{\infty}\frac{a(n)}{n^w}\sum_{n=1}^{\infty}\frac{b(n)}{n^w} = \sum_{n=1}^{\infty}\frac{(a*b)(n)}{n^w},$$

where $a(n) = \begin{cases} m^{2k-2}, & \text{if } n = m^2, \\ 0, & \text{otherwise,}\end{cases}$ and $b(n) = \begin{cases} \mu(m)m^{2k-1}, & \text{if } n = m^2, \\ 0, & \text{otherwise.}\end{cases}$

where $\mu(n)$ denotes the Möbious function and $(a*b)(n)$ is the Dirichlet convolution between the arithmetic functions $a(n)$ and $b(n)$. Again, using the series definition of $L(f \otimes f, w)$, one can obtain

$$\frac{\zeta(2w-2k+2)L(f \otimes f, w)}{\zeta(2w-2k+1)} = \sum_{n=1}^{\infty} \frac{A^*(f \otimes f, n)}{n^w}, \tag{6.13}$$

where

$$A^*(f \otimes f, n) := c_f^2(n) * (a * b)(n). \tag{6.14}$$

Now, plugging the series expansion (6.13) in (6.12) and swapping summation and integration, one can obtain

$$V(x) = \frac{1}{\sqrt{\pi}} \left(\frac{16\pi}{x}\right)^{2k-1} \sum_{n=1}^{\infty} \frac{A^*(f \otimes f, n)}{2\pi i} \int_{(\beta_1)} \frac{\Gamma(2k-w+\nu-1)\Gamma(w)}{\Gamma\left(w-k+\frac{1}{2}\right)} \times \Gamma(w-k+1) \left(\frac{x}{64n\pi^2}\right)^w dw. \tag{6.15}$$

At this situation, our main target is to simplify the above integral involving gamma factors and for that we need the help of the Meijer G-function. First, we note that the poles of $\Gamma(w)\Gamma(w-k+1)$ are at $-n, k-1-n$, for any $n \in \mathbb{N} \cup \{0\}$. Again, the poles of $\Gamma(2k-w+\nu-1)$ are at $2k+\nu-1+n$, for $n \in \mathbb{N} \cup \{0\}$. Therefore, the line $Re(w) = \beta_1 = 2k-1-\beta$ will separate the poles of $\Gamma(w)\Gamma(w-k+1)$ from the poles of $\Gamma(2k-w+\nu-1)$ since we have chosen $\max\{-Re(\nu), k-2\} < \beta < k-1$. Thus, it is permissible to use the definition (5.1). Letting $m = 1, n = 2, p = q = 2$, and $a_1 = 1, a_2 = k, b_1 = 2k+\nu-1, b_2 = k+\frac{1}{2}$ and checking all the requirements, it can be seen easily that

$$\begin{aligned}\frac{1}{2\pi i} \int_{(\beta_1)} & \frac{\Gamma(w)\Gamma(2k-w+\nu-1)\Gamma(w-k+1)}{\Gamma(w-k+\frac{1}{2})} \left(\frac{x}{64n\pi^2}\right)^w dw \\ =& G_{2,2}^{1,2}\left(\begin{matrix} 1, & k \\ 2k+\nu-1, & k+\frac{1}{2} \end{matrix} \Bigg| \frac{x}{64n\pi^2}\right).\end{aligned}$$

Note that in the above Meijer G-function we have $p = q = 2$ and $b_j - b_k \notin \mathbb{Z}$ for $1 \leq i \neq j \leq m = 1$, which is trivially true. Therefore, utilizing Slater's identity i.e., Theorem 5.1 we write

$$G_{2,2}^{1,2}\left(\begin{matrix} 1, & k \\ 2k+\nu-1, & k+\frac{1}{2}\end{matrix}\,\middle|\,\frac{x}{64n\pi^2}\right) = \left(\frac{x}{64n\pi^2}\right)^{2k+\nu-1}\frac{\Gamma(2k+\nu-1)\Gamma(k+\nu)}{\Gamma(k+\nu-\frac{1}{2})} \times {}_2F_1\left(\begin{matrix} 2k+\nu-1, \ k+\nu \\ k+\nu-\frac{1}{2}\end{matrix}\,\middle|\, -\left(\frac{x}{64n\pi^2}\right)\right). \tag{6.16}$$

Now, substituting (6.16) in (6.15) and upon simplification, the vertical integral $V(x)$ takes the shape

$$V(x) = \frac{x^{\nu}(16\pi)^{2k-1}}{\sqrt{\pi}}\frac{\Gamma(2k+\nu-1)\Gamma(k+\nu)}{\Gamma(k+\nu-\frac{1}{2})}\sum_{n=1}^{\infty}\frac{A^*(f\otimes f,n)}{(64n\pi^2)^{2k-1+\nu}} \times {}_2F_1\left(\begin{matrix} 2k+\nu-1, \ k+\nu \\ k+\nu-\frac{1}{2}\end{matrix}\,\middle|\, -\left(\frac{x}{64n\pi^2}\right)\right). \tag{6.17}$$

Ultimately, considering the above final expression (6.17) of the left vertical integral $V(x)$ and together with the residual terms (6.6) and (6.7) and in view of (6.2), (6.3) and (6.4), we can complete the proof of Theorem 4.1.

□

Proof of Corollary 4.2. Taking $\nu = 1/2$ and using (3.7) in (4.1), the result follows easily. □

Proof of Corollary 4.3. Letting $f(z) = \Delta(z)$ in Corollary 4.2 and upon simplification, one can get the identity.

□

CHAPTER 7

Concluding Remarks

In [9], Chakraborty et al. showed that the infinite series $y^k \sum_{n=1}^{\infty} c_f^2(n)e^{-ny}$, where $c_f(n)$ denotes the nth coefficient in the Fourier expansion of f, has a connection with the complex zeros of $\zeta(s)$. This motivated us to examine the series

$$\sum_{n=1}^{\infty} c_f^2(n)n^{\nu/2}K_\nu(\sqrt{nx}),$$

where $Re(\nu) > 1-k$. Interestingly, we found that the main term has a connection with the complex zeros of $\zeta(s)$ and the error terms can be evaluated using ${}_2F_1(a, b, c; d; z)$. It would be interesting to find a formula for the above series when $Re(\nu) \leq 1-k$.

Bibliography

[1] M. Abramowitz, I. A. Stegun, Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables, Dover Publications, 1970.

[2] A. Agarwal, M. Garg and B. Maji, Riesz-type criteria for the Riemann hypothesis, accepted, *Proc. Amer. Math. Soc.*, 2022.

[3] R. Agnihotri, Lambert series associated to Hilbert modular form, *Int. J. Number Theory*, 2022. DOI: https://doi.org/10.1142/S1793042122500671.

[4] T. M. Apostol, Modular Functions and Dirichlet Series in Number Theory, Springer, New York, Graduate Texts in Mathematics, 2012.

[5] B. C. Berndt, Ramanujan's Notebooks, Part V, Springer-Verlag, New York, 1998.

[6] B. C. Berndt, A. Dixit, R. Gupta, and A. Zaharescu, A Class of Identities Associated with Dirichlet Series Satisfying Hecke's Functional Equation, accepted, *Proc. Amer. Math. Soc.*, 2021. arXiv:2108.13991

[7] S. Banerjee and K. Chakraborty, Asymptotic behaviour of a Lambert series á la Zagier: Maass case, *Ramanujan J.* **48** (2019), 567–575.

[8] S. Bochner, Some properties of modular relations. *Ann. of Math.* (2) **53** (1951), 332–363.

[9] K. Chakraborty, S. Kanemitsu, and B. Maji, Modular-type relations associated to the Rankin-Selberg L-function, *Ramanujan J.* **42** (2017), 285–299.

[10] K. Chakraborty, A. Juyal, S. D. Kumar, and B. Maji, An asymptotic expansion of a Lambert series associated to cusp forms. *Int. J. Number Theory* **14** (2018), 289–299.

[11] K. Chandrasekharan and R. Narasimhan, Hecke's functional equation and arithmetical identities. *Ann. of Math.* **74** (1961), 1–23.

[12] J. Hafner and J. Stopple, A heat kernel associated to Ramanujan's tau function, *Ramanujan J.* **4** (2000), 123–128.

[13] G. H. Hardy and J. E. Littlewood, Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes, *Acta Math.* **41** (1916), 119–196.

[14] H. Iwaniec, E. Kowalski, Analytic number theory, American Mathematical Society Colloquium Publications, vol. 53, 2004.

[15] A. Juyal, B. Maji, and S. Satyanarayana, An exact formula for a Lambert series associated to a cusp form and the Möbious function, *Ramanujan J.* **57** (2022), 769–784.

[16] A. Juyal, B. Maji, and S. Satyanarayana, An asymptotic expansion for a Lambert series associated to the symmetric square L-function, submitted for publication, 2021.

[17] B. Maji, S. Sathyanarayana, and B. R. Shankar, An asymptotic expansion for a twisted Lambert series associated to a cusp form and the Mobius function: level aspect, accepted, *Results Maths.*, 2022.

[18] F. Oberhettinger, Table of Mellin Transforms, Springer-Verlag, New York-Heidelberg,1974.

[19] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, eds., NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge, 2010.

[20] S. Ramanujan, On certain arithmetical functions, Trans. Cambridge Philos. Soc. **22** (1916), no. 9, 159–184.

[21] R. A. Rankin, Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions, I , II, *Proc. Cambridge Philos. Soc.* **35** (1939), 351–372.

[22] A. Selberg, Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist, *Arch. Math. Naturvid.* **43** (1940), 47–50.

[23] G. Shimura, On the holomorphy of certain Dirichlet series, *Proc. London Math. Soc., Ser.* **31** (1975), 79–98.

[24] E. C. Titchmarsh, The Theory of the Riemann Zeta-function, Clarendon Press, Oxford, 1986.

[25] D. Zagier, The Rankin-Selberg method for automorphic functions which are not of rapid decay, *J. Fac. Sci. Univ. Tokyo IA Math.* **28** (1981), 415–437.

[26] D. Zagier, Introduction to Modular Forms: From Number Theory to Physics (Les Houches, 1989), Springer, 1992, 238–291.