Voronoi Bound for a Generalized Divisor Function

M.Sc. Thesis

by

Diksha Rani

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

MAY 2022

Voronoi Bound For A Generalized Divisor Function

A THESIS

Submitted in partial fulfilment of the

requirements for the award of the degree

of

Master of Science

by

Diksha Rani

(Roll No. **2003141006**)

Under the guidance of

Dr. Bibekananda Maji

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

MAY 2022

INDIAN INSTITUTE OF TECHNOLOGY INDORE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**Voronoi Bound for a Generalized Divisor Function**" in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF MATHEMATICS**, **Indian Institute of Technology Indore**, is an authentic record of my work carried out during the period from July 2021 to May 2022 under the supervision of **Dr. Bibekananda Maji**, Assistant Professor, Department of Mathematics, IIT Indore.

The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

Diksha 28/05/2022

Signature of the student with date

(**Diksha Rani**)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Bibekananda Maji 28/05/2022

Signature of Thesis Supervisor with date

(**Dr. Bibekananda Maji**)

Diksha Rani has successfully given her M.Sc. Oral Examination held on 26th May, 2022.

Bibekananda Maji

Signature of supervisor of M.Sc Thesis

Date: 28/05/2022

Signature of Convener, DPGC

Date: 28/05/2022

Signature of PSPC Member

Dr. Debajyoti Sarkar

Date: 28/05/22

V.K. Sohani

Signature of PSPC Member

Dr. Vijay Kumar Sohani

Date: 28/05/2022

Acknowledgements

I am incredibly grateful to my supervisor Dr. Bibekananda Maji for his assistance and guidance throughout this course of two years. He gave his most profound concern for my overall well-being in these challenging times. His friendly nature, enthusiasm, and supportive attitude encouraged me to successfully do my work and complete this thesis. He has always shown his confidence and trust in me without any prejudgements, that always helped me to have faith in myself even in the times when I was stuck. Also, he conveyed the spirit of adventure and pushed me forward towards my area of interest concerning research. I feel very lucky and honoured to have him as my guide. I thank the almighty for showing his blessings upon me.

Sincere gratitude is also extended to DPGC Convener, Dr. Sk. Safique Ahmad, Discipline of Mathematics and the PSPC members Dr. Debajyoti Sarkar and Dr. Vijay Kumar Sohani for their valuable remarks and kind suggestions. I am grateful to Dr. Niraj Kumar Shukla, Head-Discipline of Mathematics, for his academic support and lab facilities.

I am very grateful to all our teachers as well for their kind and valuable suggestions. I am very fortunate to have all of them and even my most profound gratitude is not enough. This acknowledgement note would be irrelevant without giving a thanks to my family members. They gave me strength to continue my work with perseverance and hard work. And lastly, I want to thank all my friends who always got my back. I feel very grateful to have been provided by the universe, each basic facility to live such a beautiful life and work with so much support from everyone around me.

Abstract

Let $d(n)$ be the well-known divisor function. Using hyperbola method, Dirichlet, in 1849, proved that

$$\sum_{n\leq x} d(n) = x\log x + (2\gamma - 1)x + \frac{1}{4} + E(x), \quad \text{as } x \to \infty,$$

with $E(x) = O(\sqrt{x})$. After a long period of time, in 1904, Voronoi used the method of contour integration to improve the error term as $O\left(x^{\frac{1}{3}+\epsilon}\right)$, for any positive ϵ. Recently, Gupta and Maji studied a generalized form of $d(n)$ given by

$$D_{k,r}(n) := \sum_{d^k|n} \left(\frac{n}{d^k}\right)^r,$$

where $k \in \mathbb{N}$ and $r \in \mathbb{Z}$. In this thesis, we study the summatory function of $D_{k,r}(n)$ and establish a Voronoi-type bound for the error term. Moreover, we recover the Voronoi's error bound for the summartory function of $d(n)$.

Contents

List of Symbols

Symbol	Description
γ	Euler's constant
$\mathrm{Re}(s)$	Real part of a complex number s
$\mathrm{Im}(s)$	Imaginary part of a complex number s
$f_1(x) = O(f_2(x))$	$\lvert f_1(x)\rvert \leq K f_2(x)$, for $x \geq x_1$, where K is some positive constant
$f_1(x) \ll f_2(x)$	$f_1(x) = O(f_2(x))$

Chapter 1

Introduction

First, we define an important arithmetic function, namely, divisor function $d(n)$, which will be one of the main objects of study of this thesis.

1.0.1 Divisor function

Definition 1.1 (Divisor function)**.** *The divisor function $d(n)$ is an arithmetic function that counts the number of distinct positive divisors of a positive integer n.*

We often write $d(n)$ in the following manner:

$$d(n) := \sum_{d|n} 1.$$

The theory of the divisor function $d(n)$ has a pivotal role in the progress of analytic number theory. It is a non-monotonic multiplicative function. Over the times, this function has been analysed by many mathematicians. Ramanujan [9, 10] gave many interesting identities associated with the divisor function. There are many generalizations of $d(n)$ in the literature. In this thesis, we will be studying the average order of one of these generalizations.

Example 1. *Below are the examples how divisor function returns the values for different values of n:*

1. $d(2022) = 8$,

2. $d(p) = 2$, *for any prime p.*

Refer to the figure below for the graph of $d(n)$.

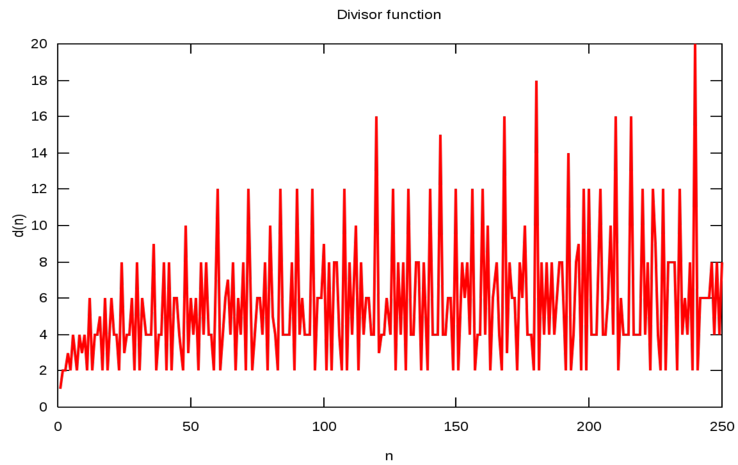


Figure: Graph of $d(n)$ with respect to n
Picture has been taken from www.wikipedia.com

All the values of n where the graph cuts the line $d(n) = 2$ are prime numbers and we can observe that the graph will cut this line infinitely many times, due to the infinitude of prime numbers. We can also see that the graph of the divisor function does not behave so nicely. So, we rather try to study the summatory function of $d(n)$, that is, $D(x) := \sum_{1 \leq n \leq x} d(n)$ for large positive real number x.

1.0.2 Work of Dirichlet and Voronoi

Here, we will see some different generalized forms of the divisor function $d(n)$. In 1849, Dirichlet, with the help of the hyperbola method, showed that, for $x > 0$,

$$\sum_{n \leq x}{}' d(n) = \frac{1}{4} + x \log(x) + (2\gamma - 1)x + E(x), \tag{1.1}$$

with the error term $E(x) = O(\sqrt{x})$ and the prime $'$ on the top of the summation means that the last term is $d(n)/2$ if x is an integer. Improving this error term had been considered as one of the challenging problems in analytic number theory. The first improvement was due to the Russian mathematician Georgy F. Voronoi [12] in 1904. He found an exact infinite series representation for the error term $E(x)$. Mainly, he proved that

$$E(x) = -\sqrt{x} \sum_{m=1}^{\infty} \frac{d(m)}{\sqrt{m}} \left(Y_1(4\pi\sqrt{mx} + \frac{2}{\pi} K_1(4\pi\sqrt{mx} \right), \tag{1.2}$$

where $Y_1(x)$ and $K_1(x)$ denote the Bessel function and the modified Bessel function of the 2nd kind, respectively, of order 1. Utilizing the available asymptotic properties of $Y_1(x)$ and $K_1(x)$, Voronoi was able to show that, for any positive ϵ,

$$E(x) = O\left(x^{1/3+\epsilon}\right). \tag{1.3}$$

This bound has been further improved by many mathematicians over the times. The current best bound is due to Huxley [3]. In 2003, he was able to show that

$$E(x) = O\left(x^{\frac{131}{416}+\epsilon}\right). \tag{1.4}$$

The problem of finding the smallest value of θ such that $E(x) = O(x^{\eta+\epsilon})$ is well-known as the *Dirichlet's divisor problem.* Hardy, in 1916, derived that $\inf(\eta) \geq 1/4$. It is broadly conjectured that $\inf(\eta)$ must be equals to $1/4$.

1.0.3 Some generalized divisor functions

One of the well-known generalizations of the divisor function is given by,

$$\sigma_r(n) = \sum_{d|n} d^r, \tag{1.5}$$

where r is any complex number. One can clearly see that $\sigma_0(n) = d(n)$. Like divisor function, the summatory function of $\sigma_r(n)$ has been studied by a few mathematicians. Readers can see [1, Sections 6] for a Voronoi-type formula for $\sigma_r(n)$. Let k be a natural number. In 1925, Wigert studied a restricted divisor function, namely, $d^{(k)}(n)$, which counts the number of k-full divisors of n. We write

$$d^{(k)}(n) := \sum_{d^k|n} 1. \tag{1.6}$$

One can clearly see that $d^{(k)}(n) = d(n)$ when $k = 1$. Wigert studied a transformation formula for the following Lambert series

$$L_k(w) := \sum_{n=1}^{\infty} d^{(k)}(n) e^{-nw}, \tag{1.7}$$

where $\mathrm{Re}(w) > 0$, to derive a formula $\zeta\left(\frac{1}{k}\right)$ for $k \in 2\mathbb{N}$. In 1929, Koshliakov [6] established a Voronoi-type exact formula corresponding to $d^{(k)}(n)$. Although, he did not give any explicit bound for the error term while studying the summatory function $\sum_{n\leq x} d^{(k)}(n)$. Recently, Gupta and Maji [2, p. 4] studied a new generalization $D_{k,r}(n)$ of the divisor function $d(n)$ while obtaining a new extension of Ramanujan's famous formula for odd zeta values. The divisor function $D_{k,r}(n)$ is defined as

$$D_{k,r}(n) = \sum_{d^k|n} \left(\frac{n}{d^k}\right)^r, \tag{1.8}$$

where $k \in \mathbb{N}$ and $r \in \mathbb{Z}$. We can easily see that $D_{1,0}(n) = d(n)$, $D_{k,0}(n) = d^{(k)}(n)$, $D_{1,r}(n) = \sigma_r(n)$. One of our main objectives of this thesis is to study the summatory function of $D_{k,r}(n)$, that is,

$$\sum_{n\le x} D_{k,r}(n), \quad \text{as } x\to\infty.$$

Chapter 2

Main Results

In this chapter, we state the main identities of this thesis.

Theorem 2.1. *Let r be a non-negative real number and k be a natural number such that $k < \frac{2r+3}{(r+1)(2r+1)}$. Let $D_{k,r}(n)$ be the arithmetic function defined as in (1.8). Then, we have*

$$\sum_{n\le x}{}' D_{k,r}(n) = M_{k,r}(x) + E_{k,r}(x), \tag{2.1}$$

where the main term $M_{k,r}(x)$ is given by

$$M_{k,r}(x) = \begin{cases} -\frac{\zeta(-r)}{2} + \zeta\left(\frac{1}{k} - r\right) x^{\frac{1}{k}} + \frac{\zeta(k(1+r))x^{1+r}}{(1+r)}, & \text{if } (k,r) \neq (1,0), \\ \frac{1}{4} + x\log x + (2\gamma - 1)x, & \text{if } (k,r) = (1,0). \end{cases} \tag{2.2}$$

and the error term $E_{k,r}(x)$ is

$$E_{k,r}(x) = O_{k,r,\epsilon}\left(x^{\frac{(r+1)(2r+1)}{(2r+3)}+\epsilon}\right), \tag{2.3}$$

for any positive ϵ.

Letting $(k,r) = (1,0)$ in the above identity, we can immediately recover the main term and the error term of Voronoi's bound for the summatory function of $d(n)$. Moreover, for $r = 0$, one can check that the only possibilities of k are 1 and 2. Thus, for $(k,r) = (2,0)$, we obtain the below identity.

Corollary 2.2. *For any positive $\epsilon > 0$, we have*

$$\sum_{n\le x}{}' d^{(2)}(n) = \frac{1}{4} + \zeta\left(\frac{1}{2}\right) x^{\frac{1}{2}} + \frac{\pi^2}{6}x + O_\epsilon\left(x^{\frac{1}{3}+\epsilon}\right).$$

Remark 1. *If $k > \frac{2r+3}{(r+1)(2r+1)}$ in Theorem 2.1, then the error term will dominate the main term $\zeta\left(\frac{1}{k} - r\right) x^{\frac{1}{k}}$. Due to this fact, for $k \ge 3$, we will have*

$$\sum_{n\le x}{}' d^{(k)}(n) = \frac{1}{4} + \zeta(k)x + O_\epsilon\left(x^{\frac{1}{3}+\epsilon}\right). \tag{2.4}$$

Again, considering $k = 1$ in Theorem 2.1, we obtain the following bound.

Corollary 2.3. *Let r be any real number lying in the interval $\left[0, \frac{\sqrt{17}-1}{4}\right)$. Then, we have*

$$\sum_{n \leq x}{}' \sigma_r(n) = M_r(x) + E_r(x), \tag{2.5}$$

where

$$M_r(x) = \begin{cases} -\frac{\zeta(-r)}{2} + \zeta\left(1-r\right)x + \frac{\zeta(1+r)x^{1+r}}{(1+r)}, & \textit{if } r \neq 0, \\ \frac{1}{4} + x\log x + (2\gamma - 1)x, & \textit{if} r = 0. \end{cases} \tag{2.6}$$

and the error term $E_{k,r}(x)$ is

$$E_r(x) = O_{r,\epsilon}\left(x^{\frac{(r+1)(2r+1)}{(2r+3)}+\epsilon}\right). \tag{2.7}$$

Chapter 3

Important Lemmas

In this chapter, we state a few essential results which will be crucial in the proof of the main results. First, we define the Riemann zeta function, for $\mathrm{Re}(s) > 1$,

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{3.1}$$

We can easily show that the Dirichlet series for $d(n)$ is $\zeta^2(s)$, that is,

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}.$$

Again, one can derive that,

$$\zeta(s)\zeta(s-r) = \sum_{n=1}^{\infty} \frac{\sigma_r(n)}{n^s}, \quad \text{for} \quad \mathrm{Re}(s) > \max\{1, 1+r\}.$$

The Dirichlet series for the Wigert's divisor function $d^{(k)}(n)$ is $\zeta(s)\zeta(ks)$ for $\mathrm{Re}(s) > 1$. Moreover, one can show that, for $\mathrm{Re}(s) > \max\{1/k, 1+r\}$, the Dirichlet series for the divisor function $D_{k,r}(n)$ is $\zeta(ks)\zeta(s-r)$. Here one can observe that the Dirichlet series for these divisor functions are all product of $\zeta(s)$ with different arguments. Thus, the theory of the Riemann zeta function will play an important role in the study of the summatory function of these divisor functions. The Riemann zeta function $\zeta(s)$ has an analytic continuation in the whole complex plane except at $s = 1$ and satisfies the following symmetric functional equation:

$$\Delta(s) = \Delta(1-s), \tag{3.2}$$

where $\Delta(s) = \frac{\Gamma(s/2)\zeta(s)}{\pi^{s/2}}$. Simplifying this functional equation [8], one can rewrite

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{3.3}$$

where

$$\chi(s) = \frac{(2\pi)^s}{\pi}\Gamma(1-s)\sin\left(\frac{\pi}{2}s\right). \tag{3.4}$$

The next lemma provides a bound for $\chi(s)$.

Lemma 3.1. *For $a > 0$ and $1 \leq t \leq T$, we have*

$$\chi(-a+it) = C\exp\left(-it\log t + it\log(2\pi) + it\right)t^{a+\frac{1}{2}} + O\left(t^{a-\frac{1}{2}}\right). \tag{3.5}$$

To prove this lemma we need to make use of the Stirling's formula for $\Gamma(s)$, which is stated below.

Lemma 3.2. *For $z \to \infty$, we have*

$$\Gamma(z) = \left(\frac{2\pi}{z}\right)^{\frac{1}{2}}\left(\frac{z}{e}\right)^{z}\left(1 + O\left(\frac{1}{z}\right)\right). \tag{3.6}$$

Proof. This lemma can be found in [7, p. 92]. □

The next lemma plays an important role in proving the main identity.

Lemma 3.3. *[4, p. 486] Let a_n be an arithmetical function such that*

$$a_n = O(\Phi(n)),$$

where $\Phi(x)$ is an increasing function for $x \geq x_0$. Suppose the Dirichlet series

$$A(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

is absolutely convergent for $\mathrm{Re}(s) > c_0$ *and if*

$$\sum_{n=1}^{\infty}\frac{a_n}{n^{\sigma}} \ll \frac{1}{(\sigma - c_0)^{\alpha}} \quad \text{if} \quad \sigma \to c_0^{+},$$

for some $\alpha > 0$. Then for any $c > c_0$, one has

$$\sum_{n\leq x}{}' a_n = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} A(s)\frac{x^s}{s}ds + O\left(\frac{x^c}{T}\frac{1}{(c-c_0)^{\alpha}}\right) + O\left(\frac{\Phi(2x)}{T}x\log 2x\right) + O(\Phi(2x)).$$

The below result gives a bound for a finite integral with an exponential function in the integrand.

Lemma 3.4. *Let $F(x)$ be a real valued function, $F(x) \in C^2[a,b]$ and*

$$F''(x) \geq L > 0, \quad \text{or} \quad F''(x) \leq -L < 0,$$

for some $L \in \mathbb{R}^{+}$, $\forall x \in [a,b]$, $\frac{G(x)}{F'(x)}$ is monotonic and $|G(x)| \leq P$, then

$$\left|\int_a^b G(x)e^{iF(x)}dx\right| \leq \frac{8P}{\sqrt{L}}.$$

Proof. Proof of this lemma can be found in [11, p. 72]. □

Chapter 4

Proof of Main Results

In this chapter, we provide proofs of the main results of this thesis.

Proof of Theorem 2.1. First, we shall make of use of Lemma 3.3. Recall that the Dirichlet series for $D_{k,r}(n)$ is $A(s) = \zeta(ks)\zeta(s-r)$, when $\mathrm{Re}(s) > \max\{\frac{1}{k}, 1+r\} = 1+r$ as $r \geq 0$. To apply Lemma 3.3, we let

$$a_n = D_{k,r}(n).$$

One can check that, for any $\epsilon > 0$,

$$D_{k,r}(n) = O(n^{r+\epsilon}), \quad \text{as} \quad r \geq 0.$$

By taking $\Phi(x) = x^{r+\epsilon}$ and using the fact that $\log(x) = O(x^\epsilon)$ in Lemma 3.3, for any $c > 1+r$, we have

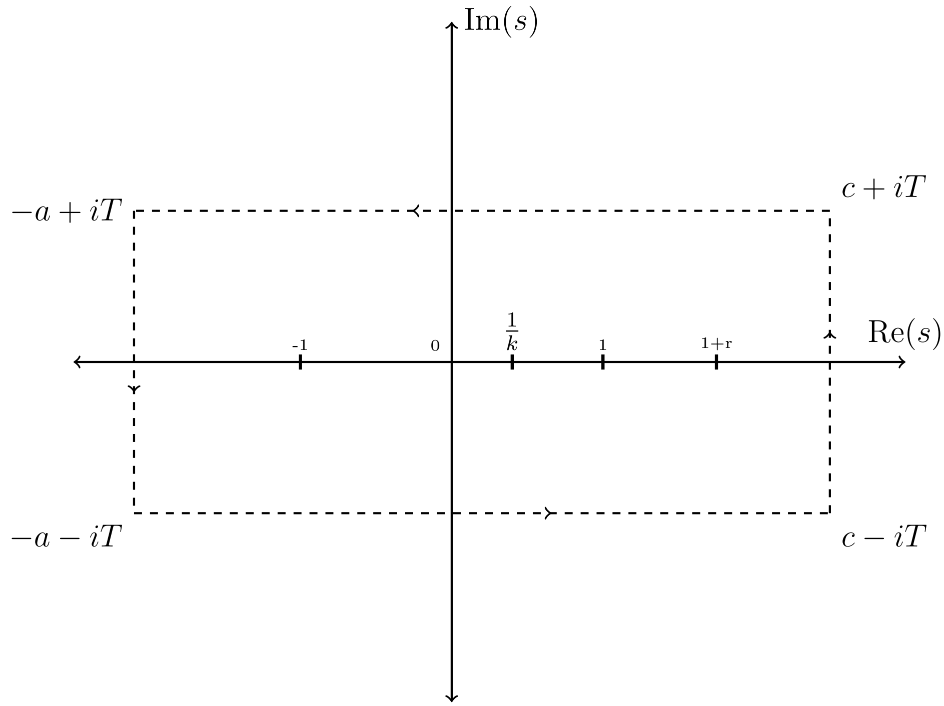
$$\begin{aligned}\sideset{}{'}\sum_{n \leq x} D_{k,r}(n) =& \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\zeta(ks)\zeta(s-r)x^s}{s} ds + O\left(\frac{x^c}{T(c-(1+r))}\right) \\ &+ O_{r,\epsilon}\left(\frac{x^{1+r+\epsilon}}{T}\right),\end{aligned} \tag{4.1}$$

where T is some large positive number. We will now evaluate the following vertical integral:

$$V_{k,r}^{(1)}(x;T) := \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\zeta(ks)\zeta(s-r)x^s}{s} ds. \tag{4.2}$$

To simplify the above line integral, we construct a rectangular path $U_{a,c}$ in such a way so that it contains all the singularities of the integrand. Thus, we choose the contour $U_{a,c}$ containing the line segments $[c-iT, c+iT], [c+iT, -a+iT], [-a+iT, -a-iT]$, and $[-a-iT, c-iT]$, where $c > 1+r$ and $a > 0$.

We first inspect the singularities of the integrand. Note that $\frac{x^s}{s}$ has a simple pole at $s = 0$, due to presence of the factor $\frac{1}{s}$. We know $\zeta(s)$ has a simple pole at $s = 1$. Hence, $\zeta(ks)$ has a simple pole at $s = \frac{1}{k}$ and $\zeta(s-r)$ has a simple pole at $s = 1+r$. Thus, the integrand function has simple poles at $s = 0, \frac{1}{k}, 1+r$.



Now making use of Cauchy's residue theorem, one has

$$\frac{1}{2\pi i}\int_{U_{a,c}} \zeta(ks)\zeta(s-r)\frac{x^s}{s}ds = R_0 + R_{\frac{1}{k}} + R_{1+r}, \tag{4.3}$$

where R_a denotes the residue at $s = a$. Using (4.2) in (4.3) and separating all the line integrals, we get

$$\begin{aligned} V_{k,r}^{(1)}(x;T) =& R_0 + R_{\frac{1}{k}} + R_{1+r} \\ &+ \frac{1}{2\pi i}\left[\int_{-a+iT}^{c+iT} + \int_{-a-iT}^{-a+iT} + \int_{c-iT}^{-a-iT}\right]\zeta(ks)\zeta(s-r)\frac{x^s}{s}ds. \end{aligned} \tag{4.4}$$

Now we shall analyse the terms on the right hand side of (4.4) one by one. One can immediately see that

$$R_0 = \lim_{s\to 0}\frac{s\zeta(ks)\zeta(s-r)x^s}{s} = \zeta(0)\zeta(-r) = -\frac{\zeta(-r)}{2}. \tag{4.5}$$

For calculating $R_{\frac{1}{k}}$ and R_{1+r}, we would like to use the following limit for $\zeta(s)$:

$$\lim_{s\to 1}(s-1)\zeta(s) = 1.$$

Note that $1/k$ and $1+r$ are simple poles if $(k,r) \neq (1,0)$. Thus, we have

$$R_{\frac{1}{k}} = \lim_{s\to\frac{1}{k}}\frac{\left(s-\frac{1}{k}\right)\zeta(ks)\zeta(s-r)x^s}{s} = \zeta\left(\frac{1}{k}-r\right)x^{\frac{1}{k}}, \tag{4.6}$$

and

$$R_{1+r} = \lim_{s\to(1+r)}\frac{(s-(1+r))\zeta(ks)\zeta(s-r)x^s}{s} = \frac{\zeta(k(1+r))x^{1+r}}{(1+r)}. \tag{4.7}$$

Moreover, if $(k,r) = (1,0)$, then $s = 1$ is a pole of order 2. In this case, the integrand function becomes $\zeta^2(s)\frac{x^s}{s}$. Using the Laurent series expansion of $\zeta(s)$ at $s = 1$, one can show that, the residue at $s=1$ is $x\log(x)+(2\gamma-1)x$, We now concentrate on the integrals that are present on the right hand side of (4.4). Let us write the top horizontal integral as

$$H_{k,r}^{(1)}(x;T)=\int_{-a+iT}^{c+iT}\zeta(ks)\zeta(s-r)\frac{x^s}{s}ds. \tag{4.8}$$

Employing Phragmén-Lindelöf principle, for $s=\sigma+iT$ with $-a<\sigma<c$, one can deduce that

$$\zeta(s)=O\left(T^{\left(a+\frac{1}{2}\right)\left(\frac{c-\sigma}{a+c}\right)}\right). \tag{4.9}$$

Now making use of the above bound (4.9) for $\zeta(s)$ and upon simplification, we obtain

$$\zeta(s-r)=O\left(T^{\left(a+r+\frac{1}{2}\right)\left(\frac{c-\sigma}{a+c}\right)}\right),$$

$$\zeta(ks)=O\left(T^{\left(ka+\frac{1}{2}\right)\left(\frac{c-\sigma}{a+c}\right)}\right).$$

Utilizing the above bounds in (4.8), we arrive at

$$H_{k,r}^{(1)}(x;T)=O\left(\int_{-a}^{c}T^{((k+1)a+r+1)\left(\frac{c-\sigma}{a+c}\right)-1}x^\sigma d\sigma\right). \tag{4.10}$$

Note that the integrand function is of the form $\exp[(A\sigma+B)\log(T)+\sigma\log(x)]$ for some constants A and B. Thus the maximum of this integrand function will be attained at one of the end points as the integral. Therefore, we get

$$H_{k,r}^{(1)}(x;T)=O\left(\frac{T^{(k+1)a+r}}{x^a}\right)+O\left(\frac{x^c}{T}\right). \tag{4.11}$$

In a similar way, the same bound can be obtained for the below horizontal integral. Now we shall try to find a bound for the following left vertical integral:

$$V_{k,r}^{(2)}(x;T):=\frac{1}{2\pi i}\int_{-a-iT}^{-a+iT}\zeta(ks)\zeta(s-r)\frac{x^s}{s}ds. \tag{4.12}$$

Notice that the real part of the integral is $\mathrm{Re}(s)=-a<0$. So, we need to use the functional equation of $\zeta(s)$ to shift the integral to the right half plane. We shall employ the asymmetric functional equation (3.3) of $\zeta(s)$. Thus, employing (3.3), the left vertical integral takes the following shape:

$$V_{k,r}^{(2)}(x;T)=\frac{1}{2\pi i}\int_{-a-iT}^{-a+iT}\chi(ks)\chi(s-r)\zeta(1-ks)\zeta(1-s+r)\frac{x^s}{s}ds, \tag{4.13}$$

where $\chi(s)=\pi^{(s-1)}2^s\Gamma(1-s)\sin\left(\frac{\pi}{2}s\right)$. Now looking at the argument of the both of the Riemann zeta functions, we can conclude that they are absolutely convergent. Therefore, using the series expansions, we can write

$$\zeta(1-ks)\zeta(1-s+r)=\sum_{n=1}^{\infty}\frac{E_{k,r}(n)}{n^{1-s}}, \tag{4.14}$$

where

$$E_{k,r}(n)=\sum_{d^k|n}\left(\frac{d^k}{n}\right)^r d^{k-1}.$$

Substituting (4.14) in (4.13) and then swapping the summation and integration, we reach

$$V_{k,r}^{(2)}(x;T) = \sum_{n=1}^{\infty} E_{k,r}(n) \frac{1}{2\pi i} \int_{-a-iT}^{-a+iT} \frac{\chi(ks)\chi(s-r)x^s}{n^{1-s}s} ds. \tag{4.15}$$

Replacing s by $-a+it$, the vertical integral becomes

$$V_{k,r}^{(2)}(x;T) = \frac{x^{-a}}{2\pi} \sum_{n=1}^{\infty} \frac{E_{k,r}(n)}{n^{1+a}} \int_{-T}^{T} \frac{\chi(-ka+ikt)\chi(-a-r+it)(xn)^{it}}{-a+it} dt. \tag{4.16}$$

In this situation, we must use the bound for $\chi(s)$. Thus, applying Lemma 3.1, we obtain, for $1 \leq t \leq T$,

$$\begin{aligned} \chi(-ka+ikt) &= C_1 \exp\left(i(-kt\log kt + kt\log 2\pi + kt)\right) t^{ka+\frac{1}{2}} + O\left(t^{ka-\frac{1}{2}}\right), \\ \chi(-(a+r)+it) &= C_2 \exp\left(i(-t\log t + t\log 2\pi + t)\right) t^{a+r+\frac{1}{2}} + O\left(t^{a+r-\frac{1}{2}}\right), \end{aligned} \tag{4.17}$$

where C_1 and C_2 are some constants. Note that $\frac{1}{-a+it} = \frac{1}{it} + O\left(\frac{1}{t^2}\right)$. Now utilizing these bounds and simplifying further, one can see that

$$\int_1^T \frac{\chi(-ka+ikt)\chi(-a-r+it)(xn)^{it}}{-a+it} dt \ll \int_1^T G(t)\exp(iF(t))dt, \tag{4.18}$$

where

$$F(t) = -t(k+1)\log t - kt\log k + t(k+1)\log(2\pi) + t(k+1) + t\log(nx), \tag{4.19}$$

$$G(t) = t^{(k+1)a+r}. \tag{4.20}$$

Note that $F'(t) = -(k+1)\log(t) - k\log k + (k+1)\log(2\pi) + \log(nx)$ and $F'(t)/G(t)$ is a monotonically decreasing function. Moreover, we have

$$F''(t) \leq -\frac{k+1}{T} \quad \text{and} \quad G(t) \leq T^{(k+1)a+r}.$$

Therefore, using these bounds and appealing to Lemma 3.4, we deduce that

$$\int_1^T G(t)\exp(iF(t))dt = O_k\left(T^{(k+1)a+r+\frac{1}{2}}\right). \tag{4.21}$$

In the interval $(-T,-1)$, one can obtain the same bound, whereas in the interval $(-1,1)$ the integrand is bounded. Now using (4.21) in (4.18), the final bound for the left vertical integral (4.16) reduces to

$$V_{k,r}^{(2)}(x;T) = O_{k,r}\left(\frac{T^{(k+1)a+r+\frac{1}{2}}}{x^a}\right). \tag{4.22}$$

To obtain the above bound, we have utilized the fact that the series $\sum_{n=1}^{\infty} E_{k,r}(n)n^{-s}$ is uniformly and absolutely convergent in $\mathrm{Re}(s) > 1$. Finally, taking into account all the residual terms (4.5)-(4.7) and the bounds for horizontal and vertical integrals, i.e., equations (4.11), (4.22), and in view of (4.1), (4.2), and (4.4), we get

$$\sum_{n\leq x}{}' D_{k,r}(n) - M_{k,r}(x) = E_{k,r}(x), \tag{4.23}$$

where $M_{k,r}$ is the sum of all residual terms and $E_{k,r}(x)$ is the error term given by

$$E_{k,r}(x) = O\left(\frac{x^c}{T(c-(1+r))}\right) + O\left(\frac{x^{1+\epsilon+r}}{T}\right) + O\left(\frac{T^{(k+1)a+r}}{x^a}\right) + O\left(\frac{x^c}{T}\right) + O_{k,r}\left(\frac{T^{(k+1)a+r+\frac{1}{2}}}{x^a}\right). \tag{4.24}$$

Now we let $c = 1+r+\epsilon$, since c is any number bigger than $1+r$, then the first two big-oh terms and the second last term are of the same order. Moreover, one can observe that the last term will dominate the third term. Therefore, simplifying, we arrive at

$$E_{k,r}(x) = O_\epsilon\left(\frac{x^{1+\epsilon+r}}{T}\right) + O_{k,r}\left(\frac{T^{(k+1)a+r+\frac{1}{2}}}{x^a}\right). \tag{4.25}$$

Again, to compare these two big-oh terms, we choose $a = \epsilon$ and let $\epsilon \to 0$ in the following identity:

$$\frac{x^{1+\epsilon+r}}{T} = \frac{T^{(k+1)\epsilon+r+\frac{1}{2}}}{x^\epsilon}. \tag{4.26}$$

One can easily determine that the best possible values for T is $x^{\frac{2r+2}{2r+3}}$. Hence putting this value of T in (4.25), the final error term becomes

$$E_{k,r}(x) = O_{k,r,\epsilon}\left(x^{\frac{(r+1)(2r+1)}{(2r+3)}+\epsilon}\right). \tag{4.27}$$

Note that the main terms are of the form Ax^{1+r} and $Bx^{1/k}$, for some positive constants A, B. We can clearly observe the exponent of x in the above error term is less than $1+r$, whereas to make it less than $1/k$ we need to take $1 \leq k < \frac{(2r+3)}{(r+1)(2r+1)}$. This concludes the proof of Theorem 2.1. □

Proof of Corollary 2.2. Letting $r = 0$ in Theorem 2.1, one can clearly observe that the possibilities of k are 1 and $k = 2$. Now corresponding to $k = 2$, the arithmetic function $D_{k,r}(n)$ is nothing but Wigert's function $d^{(2)}(n)$. Therefore, simplifying all the terms in Theorem 2.1 and together with the fact that $\zeta(2) = \pi^2/6$, we complete the proof. □

Proof of Corollary 2.3. Putting $k = 1$ in Theorem 2.1, one can see that the real number r satisfies the following inequality:

$$2r^2 + r - 2 < 0. \tag{4.28}$$

Therefore, r lies in the interval $\left(-\frac{1+\sqrt{17}}{4}, \frac{\sqrt{17}-1}{4}\right)$. Note that the arithmetic function $D_{k,r}(n) = \sigma_r(n)$ when $k = 1$. Upon simplification, one can obtain this corollary. □

Chapter 5

Concluding Thoughts

In this thesis, we have studied the summatory function of the divisor function $D_{k,r}(n)$, defined as in (1.8), and established a Voronoi-type bound for the error term. Moreover, we recover the Voronoi's error bound for the summatory function for $d(n)$. As an application, we also obtained a Voronoi-type bound for Wigert's divisor function $d^{(2)}(n)$, but our method does not give a good error bound for the summatory function of $d^{(k)}(n)$ when $k \geq 3$. So, it would be an interesting problem to study the summatory function of $d^{(k)}(n)$ for $k \geq 3$.

Bibliography

[1] B. C. Berndt, A. Dixit, A. Roy, and A. Zaharescu, *New pathways and connections in number theory and analysis motivated by two incorrect claims of Ramanujan*, Adv. Math. **304** (2017), 807–929.

[2] A. Gupta, B. Maji, *On Ramanujan's formula for* $\zeta(1/2)$ *and* $\zeta(2m+1)$, J. Math. Anal. Appl. **507** (2022) 125738.

[3] M. N. Huxley, *Exponential sums and lattice points* III. Proc. London Math. Soc. **87**, 591–609.

[4] A. Ivić, The Riemann Zeta-function: The Theory of the Riamann zeta-function with Applications, John Willey & Sons, 1985.

[5] H. Iwaniec, E. Kowalski, Analytic number theory, American Mathematical Society Colloquium Publications, vol. 53, 2004.

[6] N. S. Koshliakov, *Sur une intégrale définie et son application à la théorie des formules sommatoires*, J. Soc. Phys.-Math. Leningrad **2**, No. 2 (1929), 123–130.

[7] M. Ram Murty, Problems in Analytic Number Theory, Springer, 2nd Edition, 2007.

[8] B. Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsberichte der Berliner Akademie, 1859.

[9] S. Ramanujan, Notebooks (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957; second ed., 2012.

[10] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delhi, 1988.

[11] E. C. Titchmarsh, The Theory of the Riemann Zeta-function, Clarendon Press, Oxford, 1986.

[12] G. F. Voronoi, *Sur une fonction transcendante et ses applications à la sommation de quelques séries*, Ann. École Norm. Sup. (3) **21** (1904), 207–267, 459–533.

[13] S. Wigert, *Sur une extension de la série de Lambert*, Arkiv Mat. Astron. Fys. **19** (1925), 13 pp.

[14] S. Wigert, *Sur une nouvelle fonction entière et son application à la théorie des nombres*, Math. Ann. **96** No. 1 (1927), 420–429.