Indian Institute of Technology Indore

Generate the Page curve for the Kerr black hole near the horizon limit

By

Ankush Semwal

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

Master of Science in Physics

Simrol, Indore, India

may, 2022

SUPERVISOR

Dr Manavendra N Mahato

Acknowledgement

It is a great pleasure for me to undertake this project. I feel high about doing the project entitled-Generate the page curve for Kerr Blackhole near the horizon limit.

I am grateful to my supervisor Dr. Manavendra N Mahato and my PSPC faculty members Prof. Subhendu Rakshit and Dr. Debajyoti Sarkar.

This project would not have been completed without their enormous support and worthy experience. Whenever I was in need, they were there behind me. Although, this report has been prepared with utmost care and deep-rooted interest. Even then I accept respondents and imperfections.

DECLARATION

I, **ANKUSH SEMWAL (Roll No: 2003151005)**, hereby declare that this thesis entitled **Generate the Page curve for the Kerr black hole near the horizon limit** submitted to Indian Institute of Technology Indore towards partial requirement of **Masters of Science** in **Department Of Physics** is an original work carried out by me under the supervision of **Dr. Manavendra N Mahato** and has not formed the basis for the award of any degree or diploma, in this or any other institution or university. I have sincerely tried to uphold academic ethics and honesty. Whenever a piece of external information or statement or result is used then, that has been duly acknowledged and cited.

The matter presented in this thesis has not been submitted by me for the award of any other degrees at this or any other institute.

Ankush Semwal

. . . .

.

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Dr. Manavendra N Mahato

. . . .

Ankush Semwal, has successfully given his M.Sc. oral examination held on 3rd June 2022.

Thesis Supervisor
Dr. Manavendra N Mahato

Convener, DPGC
Dr. Manavendra N Mahato

PSPC Member
Prof. Subhendu Rakshit

PSPC Member
Dr. Debajyoti Sarkar

ABSTRACT

We study the Page curve and the information paradox for the rotating black hole. For this purpose, we apply the island prescription to the Kerr black hole near the horizon limit and analyze the behavior of the entanglement entropy at early and late times. The results demonstrate that the Page curve is consistent with the unitarity principle: without the island, the entanglement entropy grows linearly in time, and in the presence of the island, it is saturated twice the Bekenstein Hawking entropy. We observe the Page time is universal for all different models studied by our method: $t_{Page} = \frac{3S_{BH}}{\pi T_H c}$. However, for extremal rotating black holes, the Page time becomes divergent or vanishing, which the semi-classical theory needs further investigation.

Contents

List of Figures

Chapter 1

Introduction

From the Einstein equations, one intuitively searches for the solutions with the maximum radial symmetry. Less symmetric solutions to the equations were later found. However, some physical problems have occurred as a result of these solutions, such as singularities, which were difficult to understand and we got a new object known as a black hole by John Archibald Wheeler [24]. Initially, Einstein, like many other scientists, suspected that the emergence of black holes from his theory was due to an incomplete physical description. Nonetheless, our interpretation of the black hole solution is much more comprehensive today, and the presence of black holes is widely recognized. Furthermore, we now have stronger insight into how black holes form. Above all, from Einstein's theory of general relativity, the existence of black holes is one of the most exciting predictions.

Many years later, in the 1970s Jacob Bekenstein [5] introduced that a black hole has finite entropy that is proportional to the area of its event horizon. Not long after, in 1974, Stephen Hawking made a remarkable discovery: black holes behave as thermal objects, emitting thermal radiation known as "Hawking radiation." As a result, black holes are not completely "black." Hawking's discovery of black hole evaporation has raised numerous questions in general relativity, quantum mechanics, and, most notably, quantum information theory. According to Hawking's argument, black hole evaporation appears to violate unitarity, a fundamental property of quantum mechanics that states that quantum information in a system is preserved over time. Black holes may not store information, unlike quantum and classical systems. Undoubtedly, Hawking initially concluded that information from black holes would be lost. Many physicists, however, were not pleased with the idea of information loss, and much research has been devoted to finding the solution to the paradox due to Hawking's discovery. To solve the problem, it was quickly proposed that a full theory of quantum gravity, combining general relativity and quantum mechanics, be developed. Nonetheless, some scientists speculated that, as Hawking proposed [12], this theory must be non-unitary. Many ideas and solutions to the problem have been proposed in the years after Hawking's discovery. These ideas initially provided some new insights, but they did not fully solve the problem, researchers were still sticking with many unanswered questions. For example, the concept of black hole complementarity provided radical insights. Although this study provided a better knowledge of the paradox, all scientists were not convinced.

In the 1990s, results from string theory with quantum gravity suggested that information must indeed escape. The AdS/CFT correspondence discovered by Juan Maldacena resulted in significant progress. The entropy of a black hole in d+2-dimensional anti-de Sitter (AdS) spacetime can be computed using AdS/CFT, which involves a dual d+1-dimensional conformal field theory (CFT) on the AdS boundary. In particular, unitarity on the CFT boundary implies that information is preserved. As a result, Maldacena demonstrated that information can escape the black hole. However, boundary unitarity is insufficient to resolve the paradox; a wider understanding is required. After all, recent research [1] [22] suggests that we have discovered a definite, more general solution to the information paradox, which is based on AdS/CFT findings. The new understanding, however, is much broader: the results apply to asymptotically flat Minkowski space, and anti-de Sitter spacetime is not required.

According to the new proposal, Hawking did not use the correct formula to calculate the black hole entropy. The correct formula is the gravitational fine-grained entropy, which was first studied in AdS/CFT by Ryu and Takayanagi [25] but has now been extended and generalized by new research. The new formula produces spatially disconnected regions in the black hole's interior, which we refer to as islands. The formula is commonly referred to as the island formula, and it can be obtained using a mathematical method called the replica trick [21]. The island formula eventually yields a unitary Page curve [19], which denotes the black hole's unitary behavior. The fact that unitarity is preserved by the computation of the Page curve, however, is only one aspect of the paradox because it would like to know how information ended up in the outgoing Hawking radiation.

In this thesis, My goal is to draw the Page curve for the Kerr metric near the horizon limit. Before that, we will start with some theoretical background that needs to be mentioned. Firstly, we will discuss Hawking radiation, evaporation of black holes, and Page curves, and afterward, discuss AdS/CFT correspondence in chapter 2. In chapter 3, we encounter gravitational entropy where we argue that the gravitational fine-grained entropy, which was originally studied by Ryu and Takayanagi [14] in AdS/CFT. In chapter 4, we describe the entanglement entropy, and island prescription for general black holes and draw the Page curve for that. In chapter 5, we discover the island prescription and calculate the generalized entropy for the Kerr black hole near the horizon limit. Furthermore, we calculate the Page time and scrambling time for our case.

Chapter 2

Preliminaries

2.1 Hawking Radiation

Hawking's seminal paper [13] led to the obvious conclusion that black holes are inherently thermal objects: The QFT vacuum is unstable, resulting in entangled modes of radiation at the horizon, implying that black holes should evaporate and have a temperature. Consider the simplistic description of a particle/antiparticle pair being formed, with one ingoing and the other outgoing. The one that comes in is absorbed by the black hole, which reduces its energy, mass, and size, which is defined by the Schwarzschild radius of $r_s = 2G_N M$. The emitted particle, on the other hand, leaves the black hole and can be recorded as radiation by a viewer. Hawking provided a great formula for the temperature of a black hole summarizing the unification of the concepts it relates to gravity, thermodynamics, and quantum mechanics in a very comprehensible way. It is given by

$$T_H = \frac{\hbar c^3}{8\pi G_N M k_B} \tag{2.1}$$

All units are natural. We meaningfully reintroduced the universal constants to make this unity manifest. The radiation emitted is identical to the radiation emitted by a black body at the same temperature. Surprisingly, the higher the mass, the lower the temperature according to this formula. This is the temperature as measured by a distant observer. The temperature rises dramatically as one gets closer to the black hole. In QFT, we assume that information is defined as being located in a region of space. Quantum gravity, on the other hand, maybe different, and information may be available somewhere, non-locally, around the black hole.

2.2 Entropy

There are two types of entropy that we use generally in physics. The fine-grained entropy of a system, which is sometimes known as Von-Neumann entropy, describes the system's specific microstate. Apart from that other one is the coarse-grained entropy, which describes the system's macrostate and can be calculated by maximizing the fine-grained entropy over all possible density matrices that produce the same expectation values for the observables of interest [2]. The coarse-grained entropy always increases with time for an isolated system that obeys the second law of thermodynamics.

2.2.1 Von Neumann Entropy and coarse-grained entropy

Let us consider a quantum state $|\psi\rangle$ which is pure at zero temperature. Then density matrix is defined as [19]

$$\rho = \langle\psi|\psi\rangle$$

Now if we have the density matrix for quantum systems that follow the von Neumann equation. Then we can define von Neumann entropy which obeys $S \geq 0$ only for the pure quantum state, von Neumann is also known as the fine-grained entropy written as [26]

$$S = -Tr(\rho \ln \rho) \tag{2.2}$$

Obtaining the von Neumann entropy is practically unachievable, but we have a set of system observables. The idea is to find a set of density matrices that all produce the same set of macroscopic observables, and then choose the density matrix that maximizes von Neumann entropy. The thermodynamic entropy is then calculated using the coarse-grained entropy. This also ensures that the fine-grained entropy is always less than or equal to the coarse-grained entropy. When the temperature is non-zero, the coarse-grained entropy which is also known as thermodynamic entropy, will always be in a mixed state.Then we again define density matrix [18] for mixed state as

$$\rho = \frac{e^{-\beta H}}{Tr(e^{-\beta H})} \tag{2.3}$$

Where H is the Hamiltonian of the system. to find a pure state again we just take $T \to 0$ or $\beta \to \infty$. The second law of thermodynamics holds to coarse-grained entropy. It increases during unitary time evolution.

2.3 Bekenstein-Hawking entropy

Hawking demonstrated that distant observers computing the thermodynamic entropy of radiation would exceed the classical bound for entropy, known as the Bekenstein bound and can be written as [27]

$$S_{BH} = \frac{\kappa_B c^3 A}{4 G_N \hbar} \tag{2.4}$$

Where A is the area of the horizon. 1/4 factor comes from Hawking's calculation. This formula gives us a suitable relation between the entropy and the area of the horizon. According to the second law of thermodynamics, $S_{BH} - S > 0$, where S_{BH} is the entropy of a black hole and S is the sum of the ordinary entropy.

For a holographic description of gravity, we consider a sphere in flat space, and its volume is denoted as V. We distribute the interior space into very small boxes, each the size of a Planck length denoted as l_p. There is a fermionic oscillator field in each of these boxes, resulting in two degrees of freedom. As a result, the total number of states associated with this system is

$$N = 2^{\frac{V}{l_P^3}}$$

So maximum entropy of the system obtain as

$$S_{maximum} = \frac{V}{l_P^3} \ln 2$$

We can excite the quantum oscillators of the fermionic field in the boxes by adding energy to the confined system. With enough energy, the system's Schwarzschild radius will equal the radius of the sphere, and the entire system will collapse into a black hole. On the other hand, by eq.(2.4) we can see entropy is proportional to the area of the sphere (horizon area). Because the degrees of freedom required to describe a region of space are proportional to the area rather than the volume which is an indication for a holographic description of gravity.

2.4 Unitarity and Page curve

The Page curve is the curve of the von Neumann entropy of the Hawking radiation [20]. Page claimed that the entropy of a black hole must follow a curve known as the Page curve [3]. However, according to Hawking, the entropy of a black hole increases until it completely evaporates. But on the other hand, Page argued that the maximum entropy is obtained at the Page time. This period occurs when approximately half of the total radiation is emitted. The entropy decreases to zero after the Page time. We consider the black hole and radiation two subsystems as a bipartite system. According to the Schmidt decomposition, the entanglement entropy of the radiation equals the entanglement entropy of the black hole because we begin with a pure state. Radiation and black holes must produce a pure state, we know that the fine-grained entanglement entropy must be less than the Bekenstein-Hawking entropy, the coarse-grained entropy. This is also why, after the Page time, the Page curve should bend down. The Page time is precisely at $S_{BH} = S_{Rad}$ and $S_{blackhole} < S_{BH}$ [2].

If the black hole entropy follows the Hawking curve [12], we will encounter a contradiction because entropy can be thought of as the number of microstates in comparison to the black hole macrostate, the coarse-grained entropy must be greater than the entanglement entropy. We expect that the black hole should follow Hawking's predicted curve before the Page time. After the Page time, the curve must bend down and go to zero because the coarse-grained entropy is less than the radiation entropy. This must occur to achieve a pure state.

A key question in developing a solution is how to bend the curve down while maintaining unitarity. This is sometimes referred to as the true information puzzle because it is difficult to prove that entropy follows this curve. After all such discussions, there have been recent studies that claim to solve the paradox which is known as the "island formula".

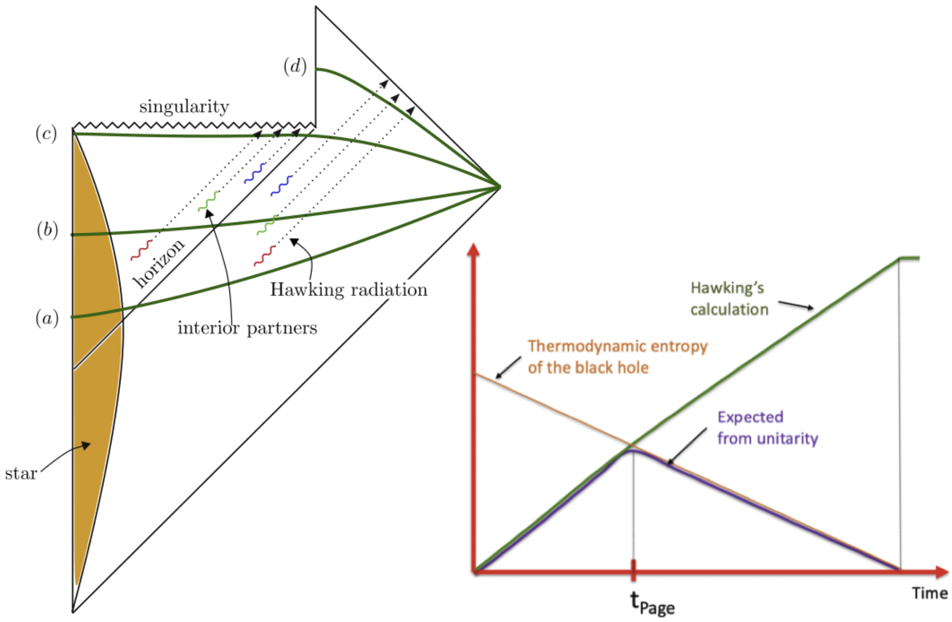

Figure 2.1: Left side figure show the Penrose diagram for the formation and evaporation of a black hole. Right side figure represent the Page curve for Hawking radiation.

2.5 Conformal field theory

In the study of the AdS/CFT-duality, the conformal field theory (CFT) plays a crucial role. A CFT is defined as a relativistic quantum field theory that is invariant under a large set of Poincare transformation-generated spacetime transformations [10]. This is accomplished through the use of a coordinate transformation

$$x'^{\mu} = \lambda x^{\mu} \tag{2.5}$$

$$x'^{\mu} = \frac{x^{\mu} + a^{\mu} x^2}{1 + 2x_{\nu} a^{\nu} + a^2 x^2} \tag{2.6}$$

Angles in the conformal group are preserved due to this transformation. The conformal group for d dimensional theory is isomorphic to the group $SO(d, 2)$. CFT's are often studied on a cylinder ($R \times S^{d-1}$). CFTs have many nice properties, but some are more important than others. First, for all CFTs, a set of primary operators that transform under conformal transformations with a conformal dimension Δ of the primary operator $\mathcal{O}$ can be found as

$$\mathcal{O}'(x') = \lambda^{-\Delta}\mathcal{O}(x) \tag{2.7}$$

This quantity obeys $d \geq \frac{d-2}{2}$ if it is real and positive, and $\mathcal{O}$ is a scalar field. Such primary operators usually lack complex correlation functions. For instance for a CFT,a scalar primary $\mathcal{O}$ with dimension Δ has a correlation function

$$\langle\psi|T\mathcal{O}(x,t)\mathcal{O}(0,0)|\psi\rangle = \frac{1}{[x^2 - t^2 + i\epsilon]^\Delta} \tag{2.8}$$

Holographic principle

Originally Susskind and t'Hooft [16] gave the ideas about holography which is inspired by black hole entropy. According to the principle, degrees of freedom in a region of space is proportional to the area of its boundary rather than to its volume. In a quantum gravity theory, a copy of all the information available on a Cauchy slice is also available near the Cauchy slice's boundary.

2.5.1 AdS/CFT correspondence statement

Now we introduce the most successful part or can say the realization of the holographic principle which is known as the anti-de Sitter / conformal field theory (AdS/CFT) correspondence. The AdS/CFT correspondence, in summary, demonstrates a close relationship between theories of gravity in D-dimensional asymptotically AdS spacetimes and the CFTs living far in the D-1 dimensional conformal boundaries of spacetime [14]. The boundary theory is a conformal field theory, which means that it is invariant under Lorentz symmetries, dilatation symmetry, and special conformal transformations which are examples of conformal transformations. There are no propagating gravitational degrees of freedom at the boundary. Gravity emerges along with the AdS space's radial direction.

In the framework of black holes, it strongly suggests that any black hole living in such AdS spacetimes would have to obey unitarity, as its behavior could be directly mapped to a QFT that is manifestly unitary. However, the glory of the AdS/CFT correspondence is that by analyzing the toy models, one can derive general conclusions about gravity theories that are not dependent on such environments, as has been the case with gravitational entropy formulas. By exciting degrees of freedom associated with the quantum fields that live on the boundary, a black hole can be created within AdS. Because the boundary theory has a unitary evolution, information must be preserved. To preserve information, the quantum state of the black hole must evolve unitarity.

Chapter 3

Gravitational Entropy

Bekenstein Hawking (BH) entropy has performed a fundamentally important role in understanding degrees of freedom and unitarity in black holes; however, it must be reconciled with the von-Neumann entropy prescription to be fully understood. There are substantial corrections to the BH entropy that result from the full quantum treatment that leads to the fine-grained version.

This is an important consideration to take into account in a unitary treatment of black holes that would solve the information problem. At face value, the BH entropy expression cannot account for the appearance degrees of freedom arising from Hawking radiation to external observers. The general expression of the entropy is obtained from AdS/CFT. By the key application of AdS/CFT, we can see the entanglement entropy of a pure CFT region corresponds to the entanglement entropy of a corresponding region in the AdS bulk.

3.1 Ryu-Takayanagi Formula

Ryu and Takayanagi [25] surprised the world with their hypothesized relationship between entanglement in a CFT and the minimal surface in the dual bulk. Surprisingly, the formula is identical to Bekenstein's entropy expression, with the exception that the area of interest is not always a black hole horizon. Later on, this hypothesis was eventually proved. One was to apply its generalized formula to black holes and gradually be able to reproduce the Page curve for an evaporating black hole. Briefly, we can take a Cauchy slice of the cylinder (which has a disk-like topology) and divide it into two regions, then compute the von Neumann entropy of one region for an observer in the other. Consider AdS/CFT and the BH entropy-area relationship. It proposed the definition of entropy. Now if we consider a system A in a d-dimensional Cauchy slice of AdS_{d+1} whose entropy is defined by that of a (d-1) dimensional boundary section δA of the $R \times S^{d-1}$ conformal boundary. Then the formula for entropy should be written as [25] [**?**]

$$S_A = \frac{\text{Area of}\gamma_A}{4G_N^{d+1}} \tag{3.1}$$

Here, A is the static minimal surface for d-1 dimensions that traverses the Cauchy slice and shares its boundary with A. We can measure the fine-grained or von-Neumann entropy by an extremal surface in the bulk using the Ryu-Takayanagi entropy formula. If the extremal surface is a co-dimensional two extremal area, which means it has two dimensions less than the entire spacetime. Furthermore, the surface must follow $\delta\gamma_A = \delta A$. If there are more than two such surfaces, choose the smallest.

Figure 3.1 well shows the correspondence between AdS_3 and CFT_2 which is most simpler example for AdS/CFT duality.

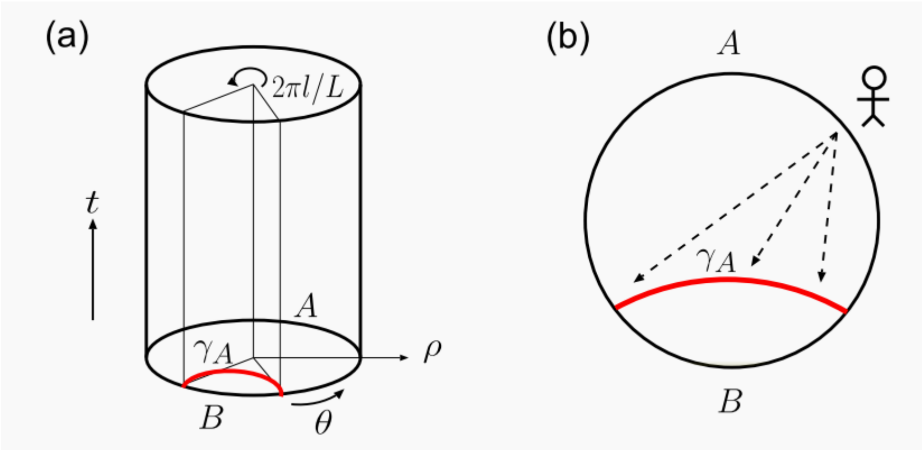

Figure 3.1: Left side figure show AdS_3 and CFT_2 living on its boundary. Right side figure represent a geodesics γ_A as a holographic screen.

3.2 Quantum Extremal Surface

The QES explanation is a generalized version of the Ryu-Takayanagi definition, which, as previously stated, takes into account the von Neumann entropy of a subsystem in holographic quantum field theory [25] [17]. When computing the gravitational fine-grained entropy, one can come across two QES. one is a new QES which is a Ryu-Takayanagi surface that Hawking did not include in his calculations, and an already accepted QES. Thus, in comparison to Hawking's calculations, a significant finding is the existence of the new QES, which is found close to the shrinking black hole horizon. Furthermore, the calculated entropy will behave similarly to the Bekenstein-Hawking entropy and the area of the evaporating black hole under the new QES. After the black hole completely evaporates, the entropy reaches a final value of zero and we see the final state as a pure state. Furthermore, the QES connects entanglement to the area, a geometrical property that may hint at the properties of a quantum gravity theory.

According to the new research, the Page curve can be computed using the island formalism, which analyses a region inside the black hole defined by the QES. The replica trick, a mathematical tool, can be used to derive the island formula. We will study island formulation in the next chapter.

Chapter 4

Islands formula

The information paradox is a big challenge in quantum gravity. Because Hawking radiation reacts like thermal radiation and the entanglement entropy outside the black hole appears to increase indefinitely. A key problem in the information problem is how to obtain the Page curve from the Hawking radiation's entanglement entropy. It has been discovered from recent developments, techniques from AdS/CFT, and holography that the entropy of a black hole or its Hawking radiation, can be calculated using Quantum Extremal Surfaces (QES), and this explanation incorporates the general relativity and quantum field theory. The Page curve has been suggested to be estimated using the island formalism, which analyses a region inside of the black hole described by the QES. The replica trick [23] which is a mathematical tool, can be used to obtain the island formula.

4.1 Gravitational fine-grained entropy

In recent years, researchers have gained extensive knowledge of a gravitational version of the von-Neumann entropy. The structure relies on a surface area (not the horizon). The recent explanation of the von-Neumann entropy in gravitational systems is more abstract than the Ryu-Takayanagi formula [25]. The most interesting fact is that there is no need for anti-de Sitter space or holography in this description. It is a broad formula that yields a fine-grained entropy formula for the quantum system linked to gravity. The gravitational von-Neumann entropy is made up of a generalized entropy that describes the black hole area as well as the entropy of fields that lie outside the black hole. Somehow the complete formula looks complex because we want a surface that minimizes the generalized entropy in the spatial direction for the gravitational interpretation of entropy. It should, however, maximize the generalized entropy in the time direction. We search for specific extremal surfaces by moving the surfaces in space and time. When there are multiple extremal surfaces, we should take the global minimum. So one can write the generalized entropy as

$$S_{gen} = \frac{Area(\gamma)}{4G_N} + S_{semi-classical}(\Sigma_\gamma) \tag{4.1}$$

Therefore, the complete formula for entropy can write as [2]

$$S = min_\gamma \left[ext_\gamma \left(\frac{Area(\gamma)}{4G_N} + S_{semi-classical}(\Sigma_\gamma) \right) \right] \tag{4.2}$$

where Σ_γ is a region bounded by γ and a cut-off surface. $S_{semi-classical}(\Sigma_\gamma)$ is the von-Neumann entropy of the quantum fields in Σ_γ which is derived from the semi-classical description.

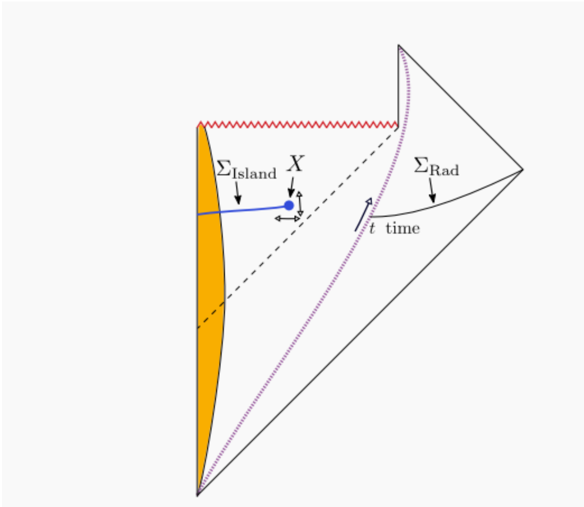

Figure 4.1: Figure shows two different region which are island and radiation region

4.2 The fine-grained entropy of Hawking radiation

Now, we shall shift our gaze to areas of space outside the black hole thus outside the cut-off surface. The gravitational entropy formula is also applicable for radiation. The semi-classical entropy occurrence of radiation can be reduced by including the black hole interior in an extra area term. This extra space inside the black hole is referred to as an island. Because the disconnected island region is included within the black hole, the island eventually causes the entropy to decrease [2] [11].

The gravitational fine-grained entropy approach for radiation does not require complete knowledge of the radiation's quantum state: the formula does not provide a full quantum explanation of the state. The fine-grained entropy for radiation can write as

$$S_{Rad} = min_{\gamma}\left[ext_{\gamma}\left(\frac{Area(\gamma)}{4G_N} + S_{semi-classical}(\Sigma_{Rad} \cup \Sigma_{Island})\right)\right] \tag{4.3}$$

The min/ext function is based on the island's location and appearance. In the semi-classical description, the term $\Sigma_{Rad} \cup \Sigma_{Island}$ represents the von-Neumann entropy of the entire radiation as well as the island state, which differs from the exact quantum state of radiation. The 'island formula', which is used to calculate the entropy of radiation, is simply a more generalized form for the gravitational entropy of black holes. The formula has the significant advantage of not requiring any complicated theories from holography or higher-dimensional AdS spacetime. we do not require many ideas which come from AdS/CFT. To calculate the entropy, we extremize the equation, which is dependent on γ's position. Following that, the equation is minimized for each available extreme position and probable island.

4.3 Island prescription for general black hole

We can define an asymptotically flat spherically symmetric eternal black hole metric as

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\ \mathrm{d}\Omega^2 \tag{4.4}$$

Here, $f(r)$ is a function with two horizons $r_+$and r_-. The metric in Kruskal coordinates can be obtained as

$$\mathrm{d}s^2 = -e^{2\rho}\mathrm{d}U\mathrm{d}V + r^2\ \mathrm{d}\Omega^2$$

where $\kappa U = +e^{\kappa(t+r_*)}$, $\kappa V = -e^{-\kappa(t-r_*)}$ and r_* is defined as radial tortoise coordinate which can be written as

$$r_* = \int^r (1/f(r))\mathrm{d}r \tag{4.5}$$

and,

$$e^{2\rho} = f(r)/\left(-\kappa^2 UV\right)$$

The Hawking temperature is define as

$$T_H = \beta_H^{-1} = \frac{\kappa_H}{2\pi} \tag{4.6}$$

where κ_H is surface gravity and T_H is the on shell temperature.

The von-Neumann entropy of subsystem with an interval of length ℓ was obtained some time ago [15], in the framework of black hole physics, where it was referred to as 'geometric' entropy. Using methods of conformal field theory, based in part on earlier work of Cardy and Peschel[6], it was found that if an infinite system consists of two semi-infinite pieces at a certain point then we ensure that entanglement entropy should be finite and obtain the universal formula with any constant α, as [7] [9]

$$S_{\text{matter}} = \frac{c}{3}\log\left(\frac{\ell}{\alpha}\right) \tag{4.7}$$

where c is the central charge for a 2-D massless scalar field.

Without an island, one can write approximately the generalized entropy for the matter part. it is represented by the entanglement region between the boundaries b_+ and b_- as

$$S_{\text{matter}} = \frac{c}{3}\log d\left(b_+, b_-\right) \tag{4.8}$$

where $b_+ = (t_b, b)$ and $b_- = (-t_b - i\pi/\kappa_+, b)$ denotes the cut-off surface of the left and right wedges. where d is define as

$$d^2(x,y) = (U(x) - U(y))\,(V(y) - V(x))\, e^{\rho(x)} e^{\rho(y)}$$

Now, with the island we have the set of disjoint intervals represented as $R \cup I = R_+ \cup I \cup R_-$, where the finite boundaries of R_- and R_+ are defined as b_- and b_+, respectively. The island is located between two different radiation wedges with boundaries of a_- and a_+. If we construct a single island then the expression for the entanglement entropy of matter part with a single island is defined as [9]

$$S_{\text{matter}} = \frac{c}{3} \log \left[\frac{d(a_+, a_-)\, d(b_+, b_-)\, d(a_+, b_+)\, d(a_-, b_-)}{d(a_+, b_-)\, d(a_-, b_+)} \right] \tag{4.9}$$

where $a_+ = (t_a, a)$ and $a_- = (-t_a - i\pi/\kappa_+, a)$ is the boundary of the island. If, once S_{matter} calculated then to determine the fine-grained entropy of Hawking radiation, we use the so-called "island formula" by eq.(4.2)

$$S = \min \{\text{ext}\, S_{\text{gen}}\} = \min \left\{ \text{ext} \left[\frac{\mathcal{A}(\partial I)}{4G_N} + S_{\text{matter}}(R \cup I) \right] \right\} \tag{4.10}$$

Here, the first term of generalized entropy will vanish for the case without constructing an island only contribution will come from the second term.

D.Page made a simple argument that, early on, when the subsystem is significantly smaller than the total system, the entanglement entropy can be estimated by the subsystem's thermal entropy. Using this logic, we should expect fine-grained entropy to increase linearly at the start of the radiation. We can use this statement again later in the evaporation process, with the small subsystem replaced by the black hole. The black hole's fine-grained entropy should then decrease linearly.(2,12)
Now, one can calculate the location of the island by extremizing the eq.(4.9) for a.

Chapter 5

Recovering the Page curve for Kerr black hole

5.1 Kerr spacetime

The Kerr metric, in Boyer-Lindquist coordinates discribe by

$$ds^2 = -\frac{\rho^2}{\Sigma^2}\Delta dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2}{\rho^2}sin^2\theta(d\phi - \omega dt)^2, \omega = \frac{2aMr}{\Sigma^2}, \quad (5.1)$$

$$\Delta = r^2 - 2Mr + a^2, \rho^2 = r^2 + a^2 cos\theta > 0,$$

$$\Sigma^2 = (r^2 + a^2)\rho^2 + 2Ma^2 r sin^2\theta > 0,$$

The total mass is M, and the angular momentum is J = Ma which we will assume is positive.
The extremal limit corresponds to $a^2 = M^2$, so $\Delta = (r - M)^2$ and the event horizon is at r = M. The area of the extremal horizon is

$$A = 8\pi M^2 = 8\pi J$$

The value of ω at the horizon is called the angular velocity of the horizon and, in the extremal case, is simply $\omega = \frac{1}{2M}$ Since $g_{rr} = \frac{\rho^2}{\Delta}$, In a constant t surface, the spatial distance to the extremal horizon is clearly infinite. As one moves down this throat, we want to extract the limiting geometry in the same way that extreme charged black holes do.

5.2 Kerr Metric near horizon limit

To describe this near horizon geometry, we set

$r = M + \lambda r$, $t = \frac{t}{\lambda}$, $\phi = \phi + \frac{t}{2M\lambda}$

and takes the limit $\lambda \to 0$, the shifting of ϕ makes $\frac{\partial}{\partial t}$ tangent to the horizon. In simple way, the coordinates corotate with the horizon. The matric is [4]

$$ds^2 = \left(\frac{1 + cos^2\theta}{2}\right)\left[\frac{-r^2}{2M^2}dt^2 + \frac{2M^2}{r^2}dr^2 + 2M^2 d\theta^2\right] + \frac{4M^2 sin^2\theta}{1 + cos^2\theta}\left(d\phi + \frac{r}{2M^2}dt\right)^2 \quad (5.2)$$

This spacetime is no longer asymptotically flat. It is similar to $AdS_2 \times S^2$ in many respects. Likewise for $\theta = 0$ and π. one can see that the spacetime along the axis is precisely AdS_2. The factor of $2M^2$ appears as an overall factor in front of the metric while doing rescaling of t. eq.(5.2) is invariant under $r \to \alpha r$, $t \to t/\alpha$ with any constant α. So the dilation symmetry of AdS_2 is present in the above metric. There is also less obvious asymmetry, but still true, called the global time translation in AdS_2.

For product spacetimes like $AdS_2 \times S^2$, one can omit the angular direction, and conformally rescale AdS_2 to view infinity as a finite boundary. one can remove the angular directions, and conformally rescale AdS_2 to view infinity as a finite boundary. Then the Killing fields of AdS_2 become conformal symmetries of the boundary. To bring infinity to finite distance the good way is to rescale the whole metric, although the conformal metric is no longer smooth at the boundary. therefore we start with the metric in Poincare coordinates (5.2) and set $2M^2 = 1$. Multiplying by $1/r^2$and considering $x = 1/r$ the metric becomes [4]

$$ds^2 = \left(\frac{1 + cos^2\theta}{2}\right)[-dt^2 + dx^2 + x^2 d\theta^2] + \frac{2sin^2\theta}{1 + cos^2\theta}(xd\phi + dt)^2 \quad (5.3)$$

5.2.1 Entanglement entropy

By (5.3) the metric is given by

$$ds^2 = \left(\frac{1 + cos^2\theta}{2}\right)[-dt^2 + dx^2 + x^2 d\theta^2] + \frac{2sin^2\theta}{1 + cos^2\theta}(xd\phi + dt)^2 \quad (5.4)$$

Define the Kruskal coordinates U and V

$$\begin{aligned} U &= t - x, V = t + x \\ dU &= dt - dx, dV = dt + dx \end{aligned} \quad (5.5)$$

Under the Kruskal transformation, the metric becomes

$$ds^2 = -\left(\frac{1 + cos^2\theta}{2}\right)dUdV + \left(\frac{V - U}{2}\right)^2 \left(\frac{1 + cos^2\theta}{2}\right)d\theta^2$$
$$+\frac{2sin^2\theta}{1 + cos^2\theta}\left[\left(\frac{V - U}{2}\right)d\phi + \frac{dU + dV}{2}\right]^2 \quad (5.6)$$

Now we define the conformal factor as

$$g(\theta) = \left[\frac{1 + cos^2\theta}{2}\right]^{1/2} \quad (5.7)$$

one can define the Kruskal coordinate

$$\begin{aligned} \text{Right Wedge} : U &= -e^{-\kappa(t-r^*)}, \quad V = e^{+\kappa(t+r^*)}, \\ \text{Left Wedge} : U &= e^{-\kappa(t-r^*)}, \quad V = -e^{+\kappa(t+r^*)}, \end{aligned} \tag{5.8}$$

where κ_+ is the surface gravity of the event horizon and $r^*(a,b)$ denotes the tortoise coordinate as

$$\begin{aligned} r^*(r) &= \int f^{-1}(r)dr \\ f(r) &= g(\theta)r^2 \end{aligned} \tag{5.9}$$

f(r) is given by metric (5.2).

Entanglement without island

The geodesic distance $d(x_1, x_2)$ between two points in the whole spacetime can be written as follows

$$d(x,y) = \sqrt{g(x)\,g(y)\,[U(y) - U(x)]\,[V(x) - V(y)]} \tag{5.10}$$

In four or high dimensions, the entanglement entropy is normally unknown. we assume at an initial time, the system is a pure quantum state. Therefore the entanglement entropy of region $[b_+, b_-]$ is the same as the entanglement entropy of the radiation. The matter entropy per unit area in two- dimensions is given by

$$S_R(\text{without island}) = S_{matter} = \frac{c}{3}\log\left[d\left(b_+, b_-\right)\right] \tag{5.11}$$

where c is centeral charge in CFT and $d(b_+, b_-)$ is the distance between b_+ and b_- in Kerr geometry. the spacetime coordinates are defined as $b_\pm = (\pm t_b, b)$. Therefore, following the conformal mapping and by using eq.(5.3), the matter entropy in Kerr geometry is obtained to be

$$S_R(\text{without island}) = \frac{c}{6}\log[g^2(b)2e^{2\kappa_+ b}(1 + \cosh(2\kappa_+ t_b)] \tag{5.12}$$

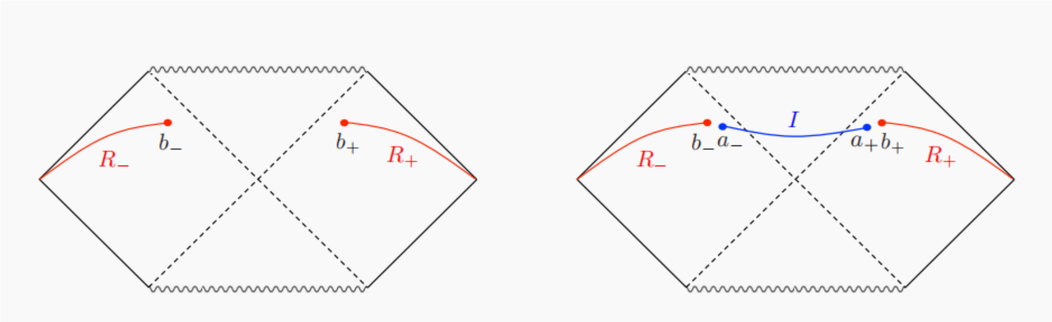

Figure 5.1: Left side figure show the Penrose diagram without island. Right side Penrose with island.

At late times, we assume $t_b >> b$, we can write

$$\cosh(\kappa_+ t_b) \simeq \frac{1}{2} e^{\kappa_+ t_b}$$

Therefore S_{matter} is reduced to

$$S_R(\text{without island}) \sim \frac{c}{6} \log[e^{2\kappa_+ t_b}] \simeq \frac{c}{3} \kappa_+ t_b \tag{5.13}$$

Entropy varies linearly with time and eventually becomes infinite. Instead, we expect that the growth of entanglement entropy will finish in a finite time. As a result, in the absence of an island, there is no Page curve and no information runs away from the black hole. The entanglement entropy will be vastly higher than the black hole's Bekenstein-Hawking entropy, which contradicts unitarity. In the following section, we will calculate the entanglement entropy of construction with an island, and the unitary Page curve will be reproduced as long as this construction is considered.

Entropy with island

Now, we construct the island and set the boundary of the island located at $a_\pm = (\pm t_a, a)$, We highlight two mathematical tricks before getting into specific calculations.

In higher-dimensional spacetime, the entanglement entropy has an area-like UV divergent that varies on the cut-off ϵ and can be absorbed by renormalizing the Newton constant $G_N^{(r)}$.

$$\begin{aligned} S_{\text{field}}\,(R\cup I) &= \frac{\text{Area}(\partial I)}{\epsilon} + S_{\text{field}}^{(\text{fin}\,)}(R\cup I), \\ \frac{1}{4G_N^{(r)}} &= \frac{1}{4G_N} + \frac{1}{\epsilon}. \end{aligned}$$

In disconnected intervals, we can calculate the entanglement entropy of the QFT. In two-dimensional geometry, there is a reduced QFT of massless fermions. In the theory of free Dirac fermions, the formula of non-universal entropy is written by [8]

$$S_{\text{field}}\,(R\cup I) = \frac{c}{3}\log\left[\frac{d\,(a_-,a_+)\,d\,(b_-,b_+)\,d\,(a_-,b_-)\,d\,(a_+,b_+)}{d\,(a_-,b_+)\,d\,(a_+,b_-)}\right], \tag{5.14}$$

One can find the value of $d\,(x_-,x_+)$ by using eq.(5.11) and we can obtain

$$\begin{aligned} S_{\text{field}}\,(R\cup I) &= \frac{c}{6}\log\left[g^4(\theta)16e^{2\kappa_+(a+b)}\cosh^2(\kappa_+t_a)\cosh^2(\kappa_+t_b)\right] \\ &+ \frac{c}{3}\log\left[\frac{e^{2\kappa_+a}+e^{2\kappa_+b}-e^{\kappa_+(a+b)}\cosh[\kappa_+(t_a-t_b)]}{e^{2\kappa_+a}+e^{2\kappa_+b}+e^{\kappa_+(a+b)}\cosh[\kappa_+(t_a+t_b)]}\right]. \end{aligned} \tag{5.15}$$

One can define the generalized entropy by eq(4.1) as [28]

$$S_{\text{gen}} = \frac{\pi a}{G_N} + S_{\text{matter}}\,(R\cup I) \tag{5.16}$$

Where first term corresponds to the contribution from the area of island. Bekenstein-Hawking entropy of the Kerr black hole read as [28]

$$S_{BH} = \frac{\pi r}{2G_N} \tag{5.17}$$

We can obtain the full expression of generalized entropy by using (5.15) and (5.16) as

$$\begin{aligned} S_{\text{gen}} =& \frac{\pi a}{G_N} + \frac{c}{6} \log \Bigg[g^4(\theta) 16 e^{2\kappa_+(a+b)} \cosh^2(\kappa_+ t_a) \cosh^2(\kappa_+ t_b) \\ &+ \frac{c}{3} \log \left[\frac{\cosh\left[\kappa_+(a-b)\right] - \cosh[\kappa_+(t_a - t_b)]}{\cosh\left[\kappa_+(a-b)\right] + \cosh[\kappa_+(t_a + t_b)]} \right] \Bigg]. \end{aligned} \tag{5.18}$$

The fine-grained entropy of Hawking radiation can be found by extremizing generalized entropy all over the extremal surface and selecting the minimal value. In the next section, we will see the behaviors of entanglement entropy at early and late time limits.

5.2.2 Island absent at early times

Our last expression of generalized entropy eq.(5.17) looks complex and few appropriate approximations are needed. At early times, we assume $t_a, t_b \ll a, b$ and choose the boundaries of radiation regions beyond the event horizon. Therefore, the last term of eq.(5.17) can be neglected, i.e.

$$\frac{c}{3} \log \left[\frac{\cosh\left[\kappa_+(a-b)\right] - \cosh[\kappa_+(t_a - t_b)]}{\cosh\left[\kappa_+(a-b)\right] + \cosh[\kappa_+(t_a + t_b)]} \right] \to 0$$

And we can write as

$$\cosh^2\left[\kappa_+\left(t_a, t_b\right)\right] \simeq 1$$

Therefore, expression can be recast as

$$\begin{aligned} S_{\text{gen}}(\text{ early times }) &\simeq \frac{\pi a}{G_N} + \frac{c}{6} \log\left[g^4(\theta)\right] + \frac{2c}{3} \log\left[2 e^{2\kappa_+(a+b)}\right] \\ &= \frac{\pi a}{G_N} + \frac{2c}{3} \log\left[2 g^4(\theta) e^{2\kappa_+(a+b)}\right] \end{aligned} \tag{5.19}$$

We can see $g(\theta)$ is only for θ. We extremize the eq.(5.18) for a. This equation has no real extreme point which means the quantum extremal surface is nonvanishing which makes the generalized entropy its extremal value. Therefore, the island is absent at early times and all contributions came from radiation. Finally, it increases linearly with time and follows eq (5.13).

5.2.3 Island present at late times

Now, we shift our attention to the behavior of the entropy at late times. Because of the growing amount of radiation at the end stage of evaporation, the entanglement entropy comes from the matter part also goes up. A phase transition occurs when the second term in eq.(5.16) grows older to $\mathcal{O}\left(G_N^{-1}\right)$ (same as the order of the first term in eq.(5.16)). As a result, the fine-grained entropy of Hawking radiation started to decrease, and the behavior of the entanglement entropy, as expected, is constrained by unitarity. At late times, we draw the structure with an island and assume $t_a, t_b \gg b$). Firstly we focus on the time component of eq.(5.17)

$$S_{\text{gen}}\text{ (time)}=\frac{c}{3}\log\left\{\cosh\left(\kappa_+ t_a\right)\cosh\left(\kappa_+ t_b\right)\frac{\cosh[\kappa_+(a-b)]-\cosh[\kappa_+(t_a-t_b)]}{\cosh[\kappa_+(a-b)]+\cosh[\kappa_+(t_a+t_b)]}\right\} \tag{5.20}$$

we can use some approximations

$$\cosh\kappa_+ t_a\simeq\frac{1}{2}e^{\kappa_+ t_a},\quad \cosh\kappa_+ t_b\simeq\frac{1}{2}e^{\kappa_+ t_b} \tag{5.21}$$

and

$$\cosh[\kappa_+(t_a+t_b)]\gg\cosh\left[\kappa_+(a-b)\right] \tag{5.22}$$

Then expression of Eq.(5.16) tends to

$$S_{gen}(time)\simeq\frac{c}{3}\log\{\cosh\left[\kappa_+(a-b)\right]-\cosh\left[\kappa_+(t_a-t_b)\right)\} \tag{5.23}$$

we can notice that at $t_b = t_a$ last term reach at its minimum value. If we set the value of $t_b = t_a = t$ then eq.(5.20) can be reduced to

$$S_{gen}(time) \simeq \frac{c}{3} \log\{\cosh\left[\kappa_+(a-b)\right] - 1\} \tag{5.24}$$

We noticed here that the entanglement entropy does not depend on time and it converges at late time. Now, we calculate the expression of the generalized entropy at late time and also taking the approximation

$$\cosh\left[\kappa_+(a-b)\right] \simeq \frac{1}{2} e^{\kappa+(b-a)} \tag{5.25}$$

Therefore, the generalized entropy by eq.(5.17) is obtained as

$$\begin{aligned} S_{\text{gen}} &= \frac{\pi a}{G_N} + \frac{c}{3} \log\left\{g^2(\theta) 4\cosh(\kappa_+ t_a)\cosh(\kappa_+ t_b)\right\} + \frac{c}{3}\kappa_+(a+b) \\ &\quad + \frac{c}{3}\log\left[\frac{\cosh\left[\kappa_+(a-b)\right] - \cosh[\kappa_+(t_a - t_b)]}{\cosh\left[\kappa_+(a-b)\right] + \cosh[\kappa_+(t_a + t_b)]}\right]. \end{aligned}$$

by taking $t_a = t_b = t$ and using eq.(5.20), (5.21), (5.24). one can obtain

$$\begin{aligned} S_{\text{gen}} \simeq& \frac{\pi a}{G_N} + \frac{c}{3} \log\left\{g^2(\theta) e^{2\kappa_+ t}\right\} + \frac{c}{3}\kappa_+(a+b) \\ &+ \frac{c}{3} \log \frac{\frac{1}{2} e^{\kappa+(b-a)} - 1}{\frac{1}{2} e^{\kappa_+(b-a)} + \frac{1}{2} e^{2\kappa_+ t}} \end{aligned}$$

$$\begin{aligned} S_{\text{gen}} &= \frac{\pi a}{G_N} + \frac{c}{3} \log\left\{g^2(\theta) e^{2\kappa_+ t}\right\} + \frac{c}{3}\kappa_+(a+b) + \frac{c}{3} \log \frac{[e^{\kappa_+(b-a)} - 2] e^{-2\kappa_+ t}}{e^{\kappa_+[b-a-2t]} + 1} \\ &= \frac{\pi a}{G_N} + \frac{c}{3} \log[g^2(\theta)] + \frac{2c}{3}\kappa_+ b + \frac{c}{3} \log \frac{1 - 2e^{\kappa_+(a-b)}}{1 + e^{\kappa_+[b-a-2t]}} \end{aligned}$$

$$= \frac{\pi a}{G_N} + \frac{c}{3} \log[g^2(\theta)] + \frac{2c}{3}\kappa_+ b + \frac{c}{3} \log[1 - 2e^{\kappa_+(a-b)}] + \frac{c}{3} \log[1 + e^{\kappa_+(b-a-2t)}]$$

Now, we assume the first order expansion, i.e $\lim_{x\to 0} \log(1+x) \simeq x$. We can obtain

$$S_{\text{gen}} \simeq \frac{\pi a}{G_N} + \frac{c}{3} \log[g^2(\theta)] + \frac{2c}{3}\kappa_+ b - \frac{2c}{3} e^{\kappa_+(a-b)} + \frac{c}{3} e^{\kappa_+(b-a-2t)} \tag{5.26}$$

Here one interesting thing is that the last term of eq.(5.25) is time dependence. However, with time it decays too fast because it is a subleading term. Hence at late time the location of the island is constant and independent from time. To find the location of the island we extremize eq.(5.25) with respect to a.

$$\frac{\partial S_{gen}}{\partial a} = 0$$

Because time dependent term decays quickly so we neglect it and near horizon limit $b \to r$ (The event horizon). We can obtain

$$a \simeq r + \frac{1}{\kappa_+} \log Bigg[\frac{3\pi}{2cG_N \kappa_+} \Bigg] \tag{5.27}$$

one can directly see that boundary of the island lies outside the event horizon. Now we put this value of a in our general formula for the generalized entropy i.e. eq.(5.18), the real fine grained entropy for Hawking radiation is obtained as

$$S_{EE} = \frac{\pi r}{G_N} + \frac{c}{6} \log \mathcal{O}(c) + ... \simeq 2S_{BH} \tag{5.28}$$

The Bekenstein-Hawking entropy is a leading order term, which results from the island's contribution. The quantum effects from matter fields are represented by the subleading order and other terms, which can be omitted in comparison to the first term. As a result, the entropy of radiation approaches an asymptotic constant, implying that the island construction is correct.

Now we are going to discuss the behavior of generalized entropy. There is no island in the beginning; the generalized entropy is driven by the contribution from the matter part and increases over time. We are now considering the construction of an island outside the event horizon. The presence of the island is required for generalized entropy to achieve the minimal value. The generalized entropy is now saturated and asymptotically becomes a constant, where the Bekenstein-Hawking entropy is the leading order.

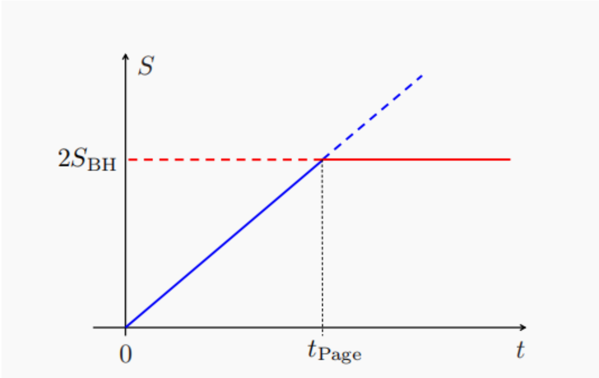

Figure 5.2: Red dash line shows EE entropy without island, blue solid line represents the saturation value for EE entropy with island

5.2.4 Page time and scrambling time

Page time is the time when the entropy of the radiation approaches its maximum. In an evaporating black hole, when the entropy of the radiation begins to decrease and the black hole has lost nearly half of its initial mass. After the Page time, the entropy of an eternal black hole will be constant. By eq.(5.13) and eq.(5.27), the Page time can be obtained as

$$t_{Page} = \frac{3S_{BH}}{\pi T_{H}c} \tag{5.29}$$

where T_H is Hawking temperature which is define as $\kappa_+ = 2\pi T_H$. The scrambling time is defined as the shortest time during which the information can be recovered from Hawking radiation by the Hayden-Preskill experiment [60]. Because the degree of freedom of the island corresponds to radiation, the scrambling time corresponds to the time for information to enter the island. At time t_1,let us consider that an observer on the cut off surface ($r = b$) sends a light signal. It reaches the island ($r = a$) at time t_2. Then distance in the null direction is defined as

$$V(t_1, b) - V(t_2), a = (t_1 + r^*(b)) - (t_2 + r^*(a))$$

$$t_2 - t_1 = [r^*(b) - r^*(a)] + [V(t_2, a) - V(t_1, b)]$$

For shortest time

$$[V(t_2, a) - V(t_1, b)] \to 0$$

Therefore, the scrambling time can be written as

$$t_{scr} \equiv r^*(b) - r^*(a) \tag{5.30}$$

By eq.(5.9), we can write

$$r^*(r) = -\frac{1}{g(\theta)r} \tag{5.31}$$

Finally, we obtain

$$t_{scr} \simeq \frac{1}{g(\theta)}\left(\frac{b-a}{ab}\right) \tag{5.32}$$

Here, we can see the scrambling time depends on the conformal factor which depends on θ.

Chapter 6

Discussion

we have studied the information paradox and later we generated the Page curve for the Kerr black hole near the horizon limit. If we start with a pure state at the initial time so Hawking radiation goes up to infinity at the end-stage for evaporation. At the end stage of evaporation entropy of Hawking radiation should be less than BH entropy and follow the unitarity. Due to this new research, the Page curve can now be reproduced. As a result, unitarity is preserved, and the paradox appears to be resolved. On the other hand, the fact that the island formula's derivation is dependent on the Euclidean path integral is a source of contention. We also used a cutoff surface to derive the island formula. There are significant arguments that appear to support the use of the cutoff surface.

We have talked about results that only involve gravity. String theory and holography have been critical in ensuring that these results are correct. The island formula research provides important insights into how fundamental quantum degrees of freedom can be used to construct the geometry of spacetime. The island formula has been developed as a result of recent research. Though the researchers attempted to present a comprehensive solution, it is unclear whether the findings are correct.

Chapter 7

Conclusion

In this thesis, we calculated the Page time and the scrambling time for the Kerr metric near the horizon limit and reproduced the corresponding Page curve. We see that near the horizon limit our original Kerr metric tends to be a little bit simpler metric with some symmetry. As a result of that, we can check our result with some old calculations which have been done already. we have calculated the Page time $t_{Page} = \frac{3S_{BH}}{\pi T_H c}$ and later the scrambling time $t_{scr} \simeq \frac{1}{g(\theta)} \frac{(b-a)}{ab}$. Where it depends somehow on θ but for the fixed value of θ our results tend to be independent from θ.

We analyze that early the entanglement entropy of a system is contributed from and no island is formed but at a late time, we construct an island outside the event horizon. Its consequence is that the entanglement entropy is dominated by the island area. As a result, we got a Page curve where at starting the entanglement entropy follows Hawking radiation, and after Page time entanglement entropy goes to zero as expected for a pure state.

Appendix A

Null geodesic of Kerr metric

Kerr metric near event horizon given by eq.(5.4)

$$ds^2 = \left(\frac{1 + cos^2\theta}{2}\right)[-dt^2 + dx^2 + x^2 d\theta^2] + \frac{2sin^2\theta}{1 + cos^2\theta}(xd\phi + dt)^2$$

Now, we see different case for θ for θ= 0, $d\theta^2 \to 0$, our metric can be obtain as

$$ds^2 = -dt^2 + dx^2 \tag{A.1}$$

For $\theta = 0$, metric tends to

$$ds^2 = \frac{1}{2}(-3dt^2 + dx^2) + x^2 d\phi^2 + 2xd\phi dt \tag{A.2}$$

Here, we can see in eq.(A.1) our resulting metric is quite simpler. Now in order to calculate null geodesic we firstly take original Kerr metric by eq.(5.1) with metric component $g_{\mu\nu}$

$$g_{tt} = \left[1 - \frac{2Mr}{\rho^2}\right]$$

$$g_{t\phi} = g_{\phi t} = -\frac{2Marsin^2\theta}{\rho^2}$$

$$g_{rr} = \frac{\rho^2}{\Delta}$$

$$g_{\theta\theta} = \rho^2$$

$$g_{\phi\phi} = \left[r^2 + a^2 + \frac{2Ma^2rsin^2\theta}{\rho^2}\right]\sin^2\theta$$

where $\Delta = r^2 - 2Mr + a^2$, $\rho^2 = r^2 + a^2\cos^2\theta$ Hence, for null geodesic

$$\ddot{x^\mu} + {\Gamma^\mu}_{\rho\sigma}\dot{x^\rho}\dot{x^\sigma} = 0$$

For $\mu = 1, 2, 3$ equation can obtained as

$$\ddot{r} + {\Gamma^r}_{\rho\sigma}\dot{x^\rho}\dot{x^\sigma} = 0 \tag{A.3}$$

$$\ddot{\theta} + {\Gamma^\theta}_{\rho\sigma}\dot{x^\rho}\dot{x^\sigma} = 0 \tag{A.4}$$

$$\ddot{\phi} + {\Gamma^\phi}_{\rho\sigma}\dot{x^\rho}\dot{x^\sigma} = 0 \tag{A.5}$$

In order to calculate christoffel, we start with $\rho, \sigma = 0, 1, 2, 3$

$${\Gamma^r}_{tt} = -\frac{1}{2\Delta}\left[-2M + \frac{4Mr^2}{\rho^2}\right]$$

$${\Gamma^r}_{t\phi} = \frac{Ma\sin^2\theta}{\Delta}\left[1 - \frac{2r^2}{\rho^2}\right]$$

$${\Gamma^r}_{rr} = \frac{\rho^2}{\Delta^2}\left[r - \frac{\rho^2}{\Delta}(r - M)\right]$$

$${\Gamma^r}_{\theta\theta} = -\frac{\rho^2}{\Delta}r$$

$${\Gamma^r}_{\phi\phi} = -\frac{\sin^2\theta}{\Delta}\left[r\rho^2 - 2Ma^2\sin^2\theta\left(1 - \frac{2r^2}{\rho^2}\right)\right]$$

$${\Gamma^\theta}_{t\phi} = Mar\sin 2\theta\left(1 - \frac{2\sin^2\theta}{\rho^2}\right)$$

$${\Gamma^\theta}_{rr} = \frac{2\rho^2a^2\sin 2\theta}{\Delta}$$

$$\Gamma^{\theta}{}_{\theta\theta} = -\frac{\rho^2 a^2 \sin 2\theta}{2}$$

$$\Gamma^{\theta}{}_{\phi\phi} = -\frac{\rho^2 \sin 2\theta}{2}\left[r^2 + a^2 + \frac{2Ma^2 r \sin^2\theta}{\rho^2}\left(2 + \frac{a^2 \sin^2\theta}{\rho^2}\right)\right]$$

we obtain this using eq.(A.3), (A.4), and all values of Christoffel

$$\ddot{r} - \frac{1}{2\Delta}\left[-2M + \frac{4Mr^2}{\rho^2}\right] + \frac{2Ma\sin^2\theta}{\Delta}\left[1 - \frac{2r^2}{\rho^2}\right]\dot{\phi} + \frac{\rho^2}{\Delta^2}\left[r - \frac{\rho^2}{\Delta}(r - M)\right]\dot{r^2}$$
$$-\frac{\rho^2}{\Delta}r\dot{\theta^2} - \frac{\sin^2\theta}{\Delta}\left[r\rho^2 - 2Ma^2\sin^2\theta\left(1 - \frac{2r^2}{\rho^2}\right)\right]\dot{\phi^2} = 0 \quad \text{(A.6)}$$

$$\ddot{\theta} + 2Mar\sin 2\theta\left(1 - \frac{2\sin^2\theta}{\rho^2}\right)\dot{\phi} + \frac{2\rho^2 a^2 \sin 2\theta}{\Delta}\dot{r^2} - \frac{\rho^2 a^2 \sin 2\theta}{2}\dot{\theta^2}$$
$$-\frac{\rho^2 \sin 2\theta}{2}\left[r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\rho^2}\left(2 + \frac{a^2\sin^2\theta}{\rho^2}\right)\right]\dot{\phi^2} = 0 \quad \text{(A.7)}$$

and remaining equation can written as

$$\ddot{t} = 0, \ddot{\phi} = 0 \quad \text{(A.8)}$$

For the different case for θ and non-extremal limit, all equation tends to be Solvable.

Bibliography

[1] Ahmed Almheiri, Netta Engelhardt, Donald Marolf, and Henry Maxfield. The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole. *Journal of High Energy Physics*, 2019(12):63, Dec 2019.

[2] Ahmed Almheiri, Thomas Hartman, Juan Maldacena, Edgar Shaghoulian, and Amirhossein Tajdini. The entropy of hawking radiation. *Reviews of Modern Physics*, 93(3), jul 2021.

[3] Ahmed Almheiri, Raghu Mahajan, Juan Maldacena, and Ying Zhao. The page curve of hawking radiation from semiclassical geometry. *Journal of High Energy Physics*, 2020, 03 2020.

[4] James Bardeen and Gary T. Horowitz. Extreme kerr throat geometry: A vacuum analog Physical Review D. 60(10), oct 1999.

[5] Jacob D Bekenstein. Black holes and entropy. In *JACOB BEKENSTEIN: The Conservative Revolutionary*, pages 307–320. World Scientific, 2020.

[6] Pasquale Calabrese and John Cardy. Entanglement entropy and quantum field theory. *Journal of Statistical Mechanics: Theory and Experiment*, 2004(06):P06002, jun 2004.

[7] Pasquale Calabrese and John Cardy. Entanglement entropy and quantum field theory: a non-technical introduction. 2005.

[8] H Casini, C D Fosco, and M Huerta. Entanglement and alpha entropies for a massive dirac field in two dimensions. *Journal of Statistical Mechanics: Theory and Experiment*, 2005(07):P07007–P07007, jul 2005.

[9] Thomas M. Fiola, John Preskill, Andrew Strominger, and Sandip P. Trivedi. Black hole thermodynamics and information loss in two dimensions. *Physical Review D*, 50(6):3987–4014, sep 1994.

[10] Daniel Harlow. Jerusalem Lectures on Black Holes and Quantum Information. *Rev. Mod. Phys.*, 88:015002, 2016.

[11] Koji Hashimoto, Norihiro Iizuka, and Yoshinori Matsuo. Islands in schwarzschild black holes. *Journal of High Energy Physics*, 2020:85, 06 2020.

[12] S. W. Hawking. Particle creation by black holes. *Communications in Mathematical Physics*, 43(3):199–220, Aug 1975.

[13] S. W. Hawking. Particle Creation by Black Holes. *Commun. Math. Phys.*, 43:199–220, 1975. [Erratum: Commun.Math.Phys. 46, 206 (1976)].

[14] Idse Heemskerk, Joao Penedones, Joseph Polchinski, and James Sully. Holography from conformal field theory. *Journal of High Energy Physics*, 2009(10):079–079, oct 2009.

[15] Christoph Holzhey, Finn Larsen, and Frank Wilczek. Geometric and renormalized entropy in conformal field theory. *Nuclear Physics B*, 424(3):443–467, aug 1994.

[16] G. 't Hooft. Dimensional reduction in quantum gravity, 1993.

[17] Raghu Mahajan. Recent progress on the black hole information paradox. *Resonance*, 26(1):33–46, Jan 2021.

[18] Horaiu Năstase. *Introduction to the AdS/CFT Correspondence*. Cambridge University Press, 2015.

[19] Don N. Page. Information in black hole radiation. *Physical Review Letters*, 71(23):3743–3746, dec 1993.

[20] Don N Page. Time dependence of hawking radiation entropy. *Journal of Cosmology and Astroparticle Physics*, 2013(09):028–028, sep 2013.

[21] Geoff Penington, Stephen H. Shenker, Douglas Stanford, and Zhenbin Yang. Replica wormholes and the black hole interior, 2019.

[22] Geoff Penington, Stephen H. Shenker, Douglas Stanford, and Zhenbin Yang. Replica wormholes and the black hole interior. *Journal of High Energy Physics*, 2022(3):205, Mar 2022.

[23] Geoffrey Penington. Entanglement wedge reconstruction and the information paradox. *Journal of High Energy Physics*, 2020, 09 2020.

[24] Remo Ruffini and John A. Wheeler. Introducing the black hole. *Phys. Today*, 24(1):30, 1971.

[25] Shinsei Ryu and Tadashi Takayanagi. Holographic derivation of entanglement entropy from the anti–de sitter space/conformal field theory correspondence. *Phys. Rev. Lett.*, 96:181602, May 2006.

[26] C. E. Shannon. A mathematical theory of communication. *The Bell System Technical Journal*, 27(3):379–423, 1948.

[27] Andrew Strominger and Cumrun Vafa. Microscopic origin of the bekenstein-hawking entropy. *Physics Letters B*, 379(1-4):99–104, jun 1996.

[28] Ming-Hui Yu, Cheng-Yuan Lu, Xian-Hui Ge, and Sang-Jin Sin. Island, page curve, and superradiance of rotating BTZ black holes. *Physical Review D*, 105(6), mar 2022.