APPROXIMATE SOLUTION OF GRAVITATIONAL WAVES

A Thesis

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By

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "Approximate solution of Gravitational Waves" in the partial fulfillment of the requirements for the award of the degree of MASTER OF SCIENCE and submitted in the DISCIPLINE OF PHYSICS, **Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from August, 2021 to June, 2022 under the supervision of **Dr. Manavendra N. Mahato**, Associate Professor. The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Diya Bansal
05/06/2022

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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ABSTRACT

One of the most interesting topics of current physics is the study of gravitational waves. It promises a wealth of fresh and vital knowledge about the Universe as a relatively young observational field. Einstein proposed the concept of gravitational waves early on, but eventually concluded that they did not exist as a physical reality. A number of authors discovered exact solutions to Einstein's equations depicting waves, resulting in their recognition as part of physics. The main objective of this thesis is to study the solutions of gravitational waves using the linear perturbations in Friedmann–Lemaître–Robertson–Walker metric which describes a universe that is homogeneous, isotropic and expanding (or contracting).

Conventions

Certain mathematical conventions are used and a comprehension of Einstein's general theory of relativity is assumed.

- The majority of the metric signature will be positive, i.e. (-,+,+,+).
- The Gravitational constant (G) and the speed of light (c) are expressed in natural units, i.e. G=c=1.
- Spatial 3-vectors are denoted by Latin indices, such as x_i, whereas spacetime 4-vectors of the kind $(t,\vec{x})$ are denoted by Greek indices, such as x_μ. Unless indicated otherwise, the Einstein summation convention will be used.
- Conformal and cosmological time will be represented by η and t respectively.

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Chapter 1

Introduction

1.1 Gravitational Waves

Newtonian gravity was the basis of research for about 200 years, and it was acknowledged in its absolute perfection, but there was a fundamental concern regarding the concept of absoluteness of space, time, and simultaneity, among other things. Albert Einstein was the first to address such uncertainties with his special theory of relativity and general theory of relativity, the latter of which included gravity. Gravity is caused by the ”warping” of space and time, according to Einstein. On the premise of the equivalence principle, Einstein developed general relativity. Gravitational waves that travel at the speed of light are an unavoidable result of such a principle. These are oscillations in gravitational fields, or ”ripples” in spacetime, caused mostly by huge masses moving. A tidal gravitational force acts perpendicular to the wave's propagation direction on each body in its path; these forces modify the distance between points, and the magnitude of the changes is proportional to the distance between the points.

Gravitational waves are produced by coherent bulk motions of matter (such as the implosion of a star's core during a supernova explosion) or coherent oscillations in spacetime curvature, and serve as a probe of these processes. The focus of this thesis will be on cosmological gravitational waves from the early Universe.

1.2 Einstein's Equations

For general relativity, Einstein's field equation determines how the metric responds to energy and momentum and is written as:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \tag{1.1}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, $T_{\mu\nu}$ is the energy-momentum tensor ($T_{\mu\nu} = diag(\rho, p, p, p)$; ρ and p are density and pressure components respectively) for perfect fluids, $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar.
Finding a solution to Einstein's equations in a linearized regime is the best way to comprehend the nature of gravitational waves.

1.3 Linearized gravity

When the spacetime metric, $g_{\mu\nu}$, is viewed as only a slightly deviated from a flat metric, $\eta_{\mu\nu}$, linearized gravity is an adequate approximation to general relativity:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}; \quad \|h_{\mu\nu}\| \ll 1 \tag{1.2}$$

where $h_{\mu\nu}$ is a tensor expressing the variations caused in the spacetime metric, and $\eta_{\mu\nu}$ is defined as diag$(-1, 1, 1, 1)$.

The linearized expression of the components of Christoffel symbols are given by:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}\left(h_{\nu\lambda,\mu} + h_{\lambda\mu,\nu} - h_{\mu\nu;\lambda}\right) \tag{1.3}$$

The derivatives of the Γ' 's, not the Γ^2 terms, will be the only contribution to the Riemann tensor, because the connection coefficients are first-order quantities:

$$R^{\mu}_{\upsilon\rho\sigma} = \Gamma^{\mu}_{\nu\sigma,\rho} - \Gamma^{\mu}_{\nu\rho,\sigma} = \frac{1}{2}\left(h^{\mu}_{\sigma,\rho\nu} + \partial^{\mu}h_{\nu\rho,\sigma} - \partial^{\mu}h_{\nu\sigma,\rho} - h^{\mu}_{\rho,\sigma\nu}\right) \tag{1.4}$$

From this, Ricci tensor can be written as

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = \frac{1}{2}\left(h^{\rho}_{\mu,\rho\nu} + \partial^{\rho}h_{\nu\rho,\mu} - \Box h_{\mu\nu} - h_{,\mu\nu}\right) \tag{1.5}$$

where $h = h^{\mu}_{\mu}$ is the trace of the metric perturbation, and the d'Alembertian is simply the one from flat space, $\Box = \partial_{\rho}\partial^{\rho} = \nabla^2 - \partial^2_t$. Contracting again to obtain the Ricci scalar yields:

$$R = R^{\mu}_{\mu} = \eta^{\mu\upsilon}R_{\mu\upsilon} = \partial^{\mu}h^{\rho}_{\mu,\rho} - \Box h \tag{1.6}$$

Putting it all together in (1.1) we obtain:

$$\left(\; h^{\rho}_{\mu,\rho\nu} + \partial^{\rho}h_{\nu\rho,\mu} - \Box h_{\mu\nu} - h_{,\mu\nu} - \eta_{\mu\nu}\partial^{\mu}h^{\rho}_{\sigma;\rho} + \eta_{\mu\nu}\Box h\right) = 16\pi T_{\mu\nu} \tag{1.7}$$

Now, rather than working with the metric perturbation $h_{\mu\upsilon}$, we use the tracereversed perturbation $\left(\overline{h}^{\mu}_{\mu} = -h\right)$:

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{1.8}$$

Solving (1.7) for this new metric, we get:

$$\frac{1}{2}\left(\overline{h}^{\rho}_{\mu,\rho\nu} + \partial^{\rho}\overline{h}_{\nu\rho,\mu} - \Box\overline{h}_{\mu\nu} - \eta_{\mu\nu}\partial^{\mu}\overline{h}^{\rho}_{\sigma,\rho}\right) = 16\pi T_{\mu\nu} \tag{1.9}$$

To further simplify the above equation, Lorentz gauge condition is assumed:

$$\overline{h}_{\mu\nu} = 0 \tag{1.10}$$

Thus, in Lorentz gauges, (1.9) simply reduces to:

$$\Box\overline{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{1.11}$$

In vacuum, (1.11) reduces to:

$$\Box\overline{h}_{\mu\nu} = 0 \tag{1.12}$$

which is a three-dimensional wave equation and has a class of homogeneous solutions which are superposition of plane waves:

$$\overline{h}_{\mu\nu}(\boldsymbol{x}, t) = \mathrm{Re}\int d^3k A_{\mu\nu}(\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} \tag{1.13}$$

Here, $\omega = |\boldsymbol{k}|$ and $A_{\mu\nu}(\boldsymbol{k})$ is the polarization tensor which contains the information about the polarization of the wave.

By applying Hilbert gauge condition, we obtain:

$$k_{\mu}k^{\mu} = 0 \tag{1.14}$$

$$k_{\mu}A^{\mu\nu} = 0 \tag{1.15}$$

with $k^{\mu} = (\omega, \boldsymbol{k})$.

If it is assumed that the wave moves in z-direction, and the gauge conditions (1.14) and (1.15) are imposed, then:

$$A_{0z} = A_{xz} = A_{yz} = A_{zz} = 0 \tag{1.16}$$

The equation (1.15) for the linear regime can be expressed as:

$$A_{\mu\nu}\eta^{\mu\nu} = 0 \tag{1.17}$$

which implies:

$$A_{xx} = -A_{yy} \tag{1.18}$$

From (1.16) and (1.18), the polarization tensor can be written as:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.19}$$

In transverse traceless gauge, the polarisation tensor (1.19) suggests that we reduce to only two degrees of freedom, which are intrinsically two (plus and cross) modes of gravitational wave, i.e.

$$A_{+} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.20}$$

$$A_{\times} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.21}$$

Every half period, the orientation of the field lines changes, resulting in the distortions shown in Figure 1.1. Any point accelerates in the

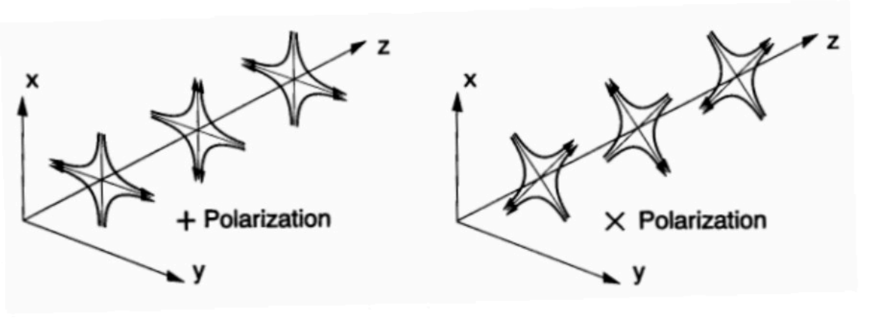

Figure 1.1: Force lines for a entirely *plus* GWs, and for a entirely *cross* GWs[6].

directions of the arrows; the stronger the acceleration, the denser the lines. The force lines become denser as one moves away from the origin because acceleration is proportional to distance from the centre of mass.

Outline

- In chapter 2, there is a very quick review of the stable solutions to Einstein field equations for spherically symmetrical Schwarzschild metric subjected to linear and odd type of perturbations.

- Schwarzschild's original result is for a simple system close to a large mass, generally a star, which is spherical, static, and vacuum. In chapter 3, The Einstein equation will be used to develop a model of the universe as a homogeneous, isotropic, perfect fluid composed of particles that are galaxies (or galactic clusters or superclusters).

- In chapter 4, we have adapted Regge and Wheeler's method to obtain an axial mode propagation equation in Friedman-Lemaitre-Robertson-Walker (FLRW) spacetimes.

Chapter 2

Literature Review

2.1 Stability of a Schwarzchild Singularity

Karl Schwarzschild discovered the solution to Einstein's equations for the metric centred on a definite spherically symmetrical centre of mass in spherical coordinates (t, r, θ, ϕ):

$$ds^2 = -\left(1 - 2m/r\right) dT^2 + \left(1 - 2m/r\right)^{-1} dr^2 \\ + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) = g_{\mu\nu} dx^\mu dx^\nu, \tag{2.1}$$

This explains the Schwarzschild Black Hole as a non-rotating spherically symmetric black hole. In the beginning, these metrics were not recognized as black holes, but rather as mathematical singularities. It was not known whether these characteristics were physical or if they would be stable enough to be found in nature. As a result, prominent physicist John Wheeler established the framework of stability research for the Schwarzschild metric.

The equations are linear as in every other type of stability problem in physics, and it is possible to break down the disturbance into proper modes and determine the frequency, real or imaginary, for each. Because imaginary frequencies would necessitate an unrealistic spatial behaviour of initial perturbation, it is found that the Schwarzschild singularity is mainly stable. As a result [12], we conclude that a typical disturbance from the equilibrium configuration will oscillate around equilibrium rather than growing with time.

2.2 Polar differential equations for small first-order deviations from the Schwarzschild metric

2.2.1 General equations

Stability is determined by introducing a small perturbation into the metric and computing the variation of the Einstein equation in vacuum. The spacetime metric is denoted as $g_{\mu\nu}$, and a small perturbation in it can be denoted as $h_{\mu\nu}$. Hence, if the new metric is $g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, then $R'_{\mu\nu} = R_{\mu\nu} + \delta R_{\mu\nu}$. From Eisenhart's calculation:

$$\delta R_{\mu\nu} = \delta\Gamma^{\beta}_{\mu\nu;\beta} - \delta\Gamma^{\beta}_{\mu\beta;\nu} \tag{2.2}$$

where the semicolons refer to the covariant differentiation and where we have used:

$$\delta\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\nu} \left(h_{\beta\nu;\gamma} + h_{\gamma\nu;\beta} - h_{\beta\gamma;\nu} \right) \tag{2.3}$$

Imposing $\delta R_{\mu\nu} = 0$ implies that the perturbed metric also lacks any distributed mass or energy (vacuum).

2.2.2 Spherical Harmonics Analysis

We can analyze the Einstein equation in vacuum to obtain a differential wave equation on h if we consider a linear perturbation to the Schwarzschild metric to be a superposition of spherical harmonics. This analysis begins with variable separation in polar coordinates.

The first step is to write the perturbation as the sum of spherical harmonic modes, Y_L^M, where L is the angular momentum and M is the projection of angular momentum on z-axis. Rotations on the two-dimensional manifold $x^0 = T = constant$ and $x^1 = r = constant$ are used to analyse angular momentum. We can use partial and covariant derivatives to contract the spherical harmonics and create quantities that transform like scalars, vectors, and tensors, and then divide these constructions into even parity $(-)^L$ and odd parity $(-)^{L+1}$, as defined by the symmetry when reflected through the origin.

Scalar function can be formed as:

$$\phi_L^M = \text{const}\, Y_L^M\,(x_2, x_3) = \text{const}\, Y_L^M(\theta, \varphi), \quad \text{parity } (-)^L \tag{2.4}$$

Vector function:

$$\begin{aligned} \psi_{L^M,\mu} =&\ \text{const}\ \frac{\partial}{\partial x^\mu} Y_L^M\,(\theta,\varphi)\,, \quad \text{parity } (-)^L \\ \phi_L{}^M, \mu =&\ \text{const}\ \epsilon_\mu{}^\nu \frac{\partial}{\partial x_\nu} Y_L{}^M\,(\theta,\varphi)\,, \quad \text{parity } (-)^{L+1} \end{aligned} \tag{2.5}$$

Tensors:

$$\begin{aligned} \psi_{L\ \mu\nu}^M &= \text{const}\, Y_{L\ ;\mu\nu}^M, \quad \text{parity } (-)^L \\ \phi_{L\ \mu\nu}^M &= \text{const}\, \gamma_{\mu\nu} Y_L^M, \quad \text{parity } (-)^L \\ \chi_{L\ \mu\nu}^M &= \frac{1}{2}\,\text{const}[\epsilon_\mu{}^\lambda \psi_{L\lambda\nu}^{\ M} + \epsilon_\nu{}^\lambda \psi_{L\lambda\mu}^{\ M}] \end{aligned} \tag{2.6}$$

In these equations, $\epsilon_\mu{}^\nu$ represent the quantities $\epsilon_2{}^2 = \epsilon_3{}^3 = 0; \epsilon_2{}^3 = -1/\sin\theta, \epsilon_3{}^2 = \sin\theta$ and $\gamma_{\mu\nu} = g_{\mu\nu}/r^2$ represent the quantities $\gamma_{22} = 1, \gamma_{23} = 0 = \gamma_{32}; \gamma_{33} = \sin^2\theta$.

Because our metric is spherically symmetric, $h_{\mu\nu}^{odd}$ and $h_{\mu\nu}^{even}$ can be constructed and examined independently because even and odd parity do not combine in Einstein equations.

$$h_{\mu\nu}^{odd} =$$
$$\begin{bmatrix} 0 & 0 & -h_0(T,r)(\partial_\varphi/\sin\theta)Y_L^M & h_0(T,r)(\sin\theta\partial_\theta)Y_{L^M} \\ 0 & 0 & -h_1(T,r)(\partial_\varphi/\sin\theta)Y_L^M & h_1(T,r)(\sin\theta\partial_\theta)Y_L^M \\ \text{Sym} & \text{Sym} & \begin{array}{c} h_2(T,r)(\partial_\varphi\partial_\theta/\sin\theta \\ -\cos\theta\partial_\varphi/\sin^2\theta)Y_L^M \end{array} & \text{Sym} \\ \text{Sym} & \text{Sym} & \begin{array}{c} \frac{1}{2}h_2(T,r)(\partial_\varphi\partial_\theta/\sin\theta \\ +\cos\theta\partial_\theta - \sin\theta\partial_\theta^2)Y_L^M \end{array} & \begin{array}{c} -h_2(T,r)(\sin\theta\partial_\varphi\partial_\theta \\ -\cos\theta\partial_\varphi)Y_L^M \end{array} \end{bmatrix} \tag{2.7}$$

Similar equation can be found for even perturbation. The angular components and the radial-temporal components have now been separated in perturbation matrix. Now, because our original metric is radially symmetric and time independent, the dispersion relation $\omega = kc$ can be imposed, and thus $h_{\mu\nu}$ must have the dependence $\exp(-i\omega t) = \exp(-ikT)$ for $T = ct$. Furthermore, L and M being constants of motion, $M = 0$ can be chosen, keeping the constant radial dependence. This will make φ vanish entirely from calculations.

2.2.3 Gauge or Coordinate Transformations

We consider an infinitesimal coordinate transformation:

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}; \quad \xi^{\alpha} \ll x^{\alpha} \tag{2.8}$$

where ξ^{α} are transformed as a vector. In new frame:

$$g_{\mu\nu}{}' + h'_{\mu\nu} = g_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu} + h_{\mu\nu} \tag{2.9}$$

where

$$h'_{\mu\nu} = h_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu} \tag{2.10}$$

This significant condition can now be used to simplify the perturbation and make it unique.

2.2.4 Imposing Regge-Wheeler Gauge

h_0, h_1, and h_2 are three unknown radial functions in equation (2.7). The Regge-Wheeler Gauge can be used to simplify these to an only one radial wave equation. The transformation, known as the Regge-Wheeler Gauge, was developed by Tullio Regge and John A. Wheeler in 1957 and for odd wave, it is written as:

$$\begin{aligned} \xi^0 = 0, \quad \xi^1 = 0, \quad \xi^{\mu} = \Lambda(T,r)\epsilon^{\mu\nu}\,(\partial/\partial x^{\nu})\,Y_L^M(\theta,\varphi), \\ (\mu,\nu = 2,3) \end{aligned} \tag{2.11}$$

where Λ is used to reduce the h_2 factor. The following is the canonical form for an odd wave with total angular momentum L and projection $M = 0$:

$$h_{\mu\nu} = \exp(-ikT)(\sin\theta\partial_\theta)P_L(\cos\theta)$$

$$\times \begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ Sym & Sym & 0 & 0 \end{bmatrix} \tag{2.12}$$

Similar transformations and the canonical form for an even wave can be constructed where the seven unknown radial functions to a single radial wave equation can be reduced.

2.2.5 Radial Wave Equations

In the variation of Einstein field equations, we substitute (2.12) for the first order perturbation:

$$\delta\Gamma^{\beta}_{\mu\nu;\beta} - \delta\Gamma^{\beta}_{\mu\beta;\nu} = 0 \tag{2.13}$$

$\delta\Gamma^{\beta}_{\mu\beta;\nu}$ is going to vanish for odd waves. Now, the variation of Ricci tensor can be analyzed under the above mentioned simplified expression of perturbation (2.12):

$$
\begin{aligned}
&(1-2m/r)^{-1}kh_0 + (d/dr)(1-2m/r)h_1 = 0, \\
&\text{for } \delta R_{23} = 0 \\
&(1-2m/r)^{-1}k(dh_0/dr - kh_1 - 2h_0/r) \\
&+ (L-1)(L+2)h_1/r^2 = 0, \quad \text{for } \delta R_{13} = 0 \\
&(d/dr)(kh_1 - dh_0/dr) + 2kh_1/r = r^{-2}(1-2m/r)^{-1} \\
&\times (4mh_0/r - L(L+1)h_0), \quad \text{for } \delta R_{03} = 0
\end{aligned}
\tag{2.14}
$$

Defining the new quantity:

$$Q = (1-2m/r)h_1/r, \tag{2.15}$$

we eliminate h_0 and solve for the second order wave equation of Q:

$$d^2Q/dr^{*2} + {k_{\text{eff}}}^2(r)Q = 0 \tag{2.16}$$

where

$$dr^* = \exp\left(\frac{1}{2}\lambda - \frac{1}{2}\nu\right) dr \tag{2.17}$$

and

$${k_{\text{eff}}}^2 = k^2 - L(L+1)e^\nu/r^2 + 6me^\nu/r^3 \tag{2.18}$$

Here, the new metric is defined in the terms of λ and ν:

$$ds^2 = -e^\nu dT^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{2.19}$$

Hence,

$$e^\nu = e^{-\lambda} = 1 - 2m/r \tag{2.20}$$

2.2.6 Dynamic Modes

For the dynamic variation to be physical ($k \neq 0$), solutions of (2.16) should have a regular behavior at both points of singularity, i.e. at $r = 2m$ and at infinity. Following is how the general solution behaves:

$$\begin{aligned} Q &\sim c_1 e^{i\delta}(r/2m - 1)^{2ikm} + c_1 e^{-i\delta}(r/2m-1)^{-2ikm} \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{for } r \to 2m \\ Q &\sim c_2 \sin(kr + \eta) \qquad\qquad\qquad\quad \text{for } r \to \infty \end{aligned} \tag{2.21}$$

Now, if we look at the radial wave equation in these limits, we can see three possibilities:

- The first case involves having a high frequency, $k >> 1/2m$, which remains constant through radius. For $r >> 2m$, these waves travel into the black hole.
- The second case involves having a low frequency near $r = 2m$, i.e. $k < 1/2m$ that decreases as $r \to \infty$. These waves are caught up by the effective potential as they propagate out of the black hole.
- The third case involves having a high frequency for $r >> 2m$ which decays as $r \to 2m$. These waves travel towards the black hole before reflecting off of the curved spacetime.

The analysis of such solutions' space dependence reveals that the real value of frequency is uniquely determined by the demand of not going to infinity at large r. Because it drops off for large r, the solution is adequate because it also drops off at the Schwarzschild radius. As a result, we can conclude that no unstable solutions for odd waves exist.

Chapter 3

FLRW Metric

3.1 Introduction

The Friedmann–Lemaître–Robertson–Walker (FLRW) cosmological model, also known as the hot big bang model, is the basis for our present understanding of the universe's evolution. It is the metric that describes a homogeneous, isotropic, expanding (or contracting) universe based on the exact solution of Einstein's general relativity field equations.

3.2 General Metric

The FLRW metric is based on the assumption of space homogeneity and isotropy. It also makes the assumption that the metric's spatial component is time-dependent. The following is a generic metric that meets these criteria (in spherical coordinates):

$$ds^2 = -dt^2 + a^2(t)[dr^2/(1 - \kappa r^2) + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2] \tag{3.1}$$

Here, $a(t)$ is the cosmic scale factor; the increasing scale factor implies that the universe is expanding. The curvature of the space, which can be elliptical, Euclidean or hyperbolic, is represented by the constant κ. It's values can be taken as +1, -1, or 0 for positive, negative and zero curvature respectively. r is unitless, while $a(t)$ has length units. When $\kappa = 1$, $a(t)$ is the space's radius of curvature, which can also be sometimes written as $R(t)$.

3.3 Curvature

3.3.1 Christoffel Symbols, Ricci tensor and Ricci Scalar

The surviving components of affine connection are:

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{jk}) \tag{3.2}$$

where i, j and k are spatial components.

$$\begin{aligned} \Gamma^t_{rr} &= \frac{1}{2} g^{tt} (\partial_\theta g_{tr} + \partial_r g_{t\theta} - \partial_t g_{r\theta}) \\ &= \frac{1}{2} \partial_t \frac{a^2}{1 - \kappa r^2} = \frac{a\dot{a}}{1 - \kappa r^2} = -\frac{\dot{a}}{a} g_{rr} \end{aligned}$$

$\Gamma^t_{\theta\theta}$ and $\Gamma^t_{\phi\phi}$ can similarly be calculated as:

$$\Gamma^t_{\theta\theta} = -\frac{\dot{a}}{a} g_{\theta\theta}, \quad \Gamma^t_{\phi\phi} = -\frac{\dot{a}}{a} g_{\phi\phi}$$

Hence,

$$\Gamma^0_{ij} = -\frac{\dot{a}}{a} g_{ij} \tag{3.3}$$

Also,

$$\Gamma^i_{0j} = \frac{\dot{a}}{a} \delta^i{}_j \tag{3.4}$$

The non-vanishing components of Ricci tensor can be calculated as:

$$R_{00} = -3\frac{\ddot{a}}{a} \tag{3.5}$$

$$R_{ij} = -\left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2\kappa}{a^2}\right] g_{ij} \tag{3.6}$$

and so, the Ricci scalar will come out be:

$$R = -6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2}\right] g_{ij} \tag{3.7}$$

3.4 Solutions

Einstein's field equations, as well as a method of calculating the density, $\rho(t)$, such as a cosmological equation of state, are required to determine the time evolution of $a(t)$. Friedmann equation results after we put (3.6) and (3.7) in the 0-0 component of (1.1):

$$\frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} - \frac{\Lambda}{3} = \frac{\rho}{3} \tag{3.8}$$

whereas, the i-i component will give:

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} - \Lambda = -p \tag{3.9}$$

When we subtract (3.8) from (3.9), we get the "acceleration equation":

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{1}{2}\left(p + \frac{\rho}{3}\right) \tag{3.10}$$

As intended, increasing the density slows the expansion in this equation. From (3.8), (3.9) and (3.10), "fluid equation" can be derived which explains how energy density changes as the universe expands:

$$\dot{\rho} = 3\frac{\dot{a}}{a}(\rho + p) \tag{3.11}$$

3.5 Change of coordinates

We will substitute in FLRW metric:

$$\frac{dt}{a} = d\eta \tag{3.12}$$

For flat spacetimes ($\kappa = 0$), FLRW metric can be written as:

$$ds^2 = a^2(\eta)(-d\eta^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) \tag{3.13}$$

We define the 'conformal scale factor' as:

$$C(\eta) = a^2(\eta) = e^{2x} \tag{3.14}$$

and solve for a case when:

$$C(\eta) = A + B \tanh \rho_0 \eta; \quad A \geq B \tag{3.15}$$

Here, A, B and ρ_0 are some constants. The spacetimes then become Minkowskian in the distant past and future, because:

$$C(\eta) \to A \pm B, \quad \eta \to \pm\infty$$

(Figure 3.1). We can write the fluid equation and the acceleration equation respectively in this new coordinate system as:

$$\partial_x \rho = -3(\rho + p) \tag{3.16}$$

$$\frac{\ddot{a}}{a} = \frac{1}{a^2}\partial_\eta^2 x \tag{3.17}$$

3.6 Interpretation of ρ and p

In terms of A, B and ρ_0, (3.17) can be solved as:

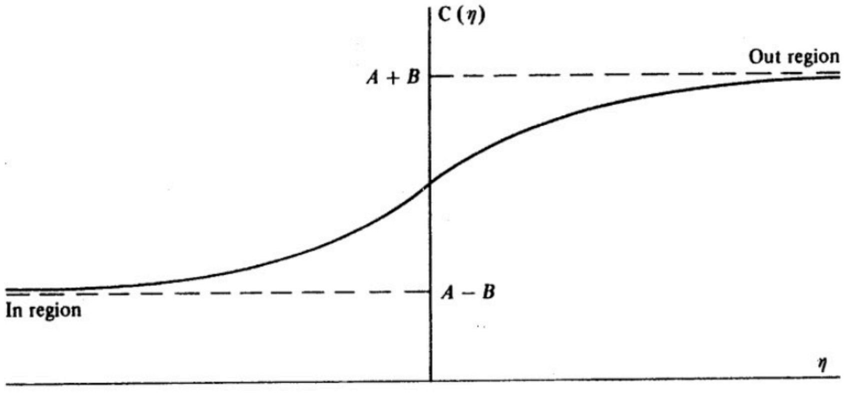

Figure 3.1: $C(\eta) = A + B\tanh\rho_0\eta$ represents a universe that is asymptotically static and expands smoothly.

$$\begin{aligned}
\frac{\ddot{a}}{a} &= \frac{-\rho_0^2}{B^2}[e^{-4x}(B^4 - 2A^2B^2 - 2AB^2 + A^4 + 2A^3) \\
&+ e^{-2x}(4AB^2 + 2B^2 - 4A^2 - 6A^2) \\
&+ e^0(-2B^2 + 6A^2 + 6A) \\
&+ e^{2x}(-4A - 2) \\
&+ e^{4x}(1)] \\
&= \alpha e^{-4x} + \beta e^{-2x} + \gamma e^0 + \delta e^{2x} + \epsilon e^{4x} \quad \text{(say)} \\
&= \frac{\Lambda}{3} - \frac{1}{6}(\rho + 3p)
\end{aligned} \tag{3.18}$$

(from (3.10)). On comparing the sides of (3.18), we can write ρ and p as:

$$\begin{aligned}
\rho &= \rho_1 e^{-4x} + \rho_2 e^{-2x} + \rho_3 e^0 + \rho_4 e^{2x} + \rho_5 e^{4x} \\
p &= p_1 e^{-4x} + p_2 e^{-2x} + p_3 e^0 + p_4 e^{2x} + p_5 e^{4x}
\end{aligned} \tag{3.19}$$

where $\rho = \rho(A, B, \rho_0, x)$ and $p = p(A, B, \rho_0, x)$. Again comparing the sides of (3.18), we can now write:

$$\begin{aligned}
&\alpha = -\frac{1}{6}(\rho_1 + 3p_1), \quad \beta = -\frac{1}{6}(\rho_2 + 3p_2) \\
&\gamma = -\frac{1}{6}(\rho_3 + 3p_3) + \frac{\Lambda}{3} \\
&\delta = -\frac{1}{6}(\rho_4 + 3p_4), \quad \epsilon = -\frac{1}{6}(\rho_5 + 3p_5)
\end{aligned} \tag{3.20}$$

From (3.16):

$$\begin{aligned}
&-4\rho_1 e^{-4x} - 2\rho_2 e^{-2x} + 2\rho_4 e^{2x} + 4\rho_5 e^{4x} = \\
&-3(\rho_1 + p_1)e^{-4x} - 3(\rho_2 + p_2)e^{-2x} - 3(\rho_3 + p_3)e^{0} \\
&-3(\rho_4 + p_4)e^{2x} - 3(\rho_5 + p_5)e^{4x}
\end{aligned} \tag{3.21}$$

Equating both sides of (3.21) gives:

$$\begin{aligned}
&p_1 = \frac{\rho_1}{3}, \quad p_2 = \frac{\rho_2}{3}, \quad p_1 = -\rho_3, \\
&p_4 = -\frac{5\rho_4}{3}, \quad p_5 = -\frac{7\rho_5}{3},
\end{aligned} \tag{3.22}$$

Putting this in (3.20), we find that:

$$\begin{aligned}
&\rho_1 = -3\alpha, \quad \rho_3 = 3\gamma - \Lambda \\
&\rho_4 = \frac{3\delta}{2}, \quad \rho_5 = \epsilon
\end{aligned} \tag{3.23}$$

Similarly,

$$\begin{aligned}
&p_1 = -\alpha, \quad p_3 = -(3\gamma - \Lambda) \\
&p_4 = -\frac{30\delta}{13}, \quad p_5 = -\frac{7\epsilon}{3}
\end{aligned} \tag{3.24}$$

ρ_2 and p_2 are arbitrary constants whose values can be taken to be zero.

Using (3.18) and (3.23), (3.19a) can be written as:

$$\rho = \frac{3\rho_0^2}{B^2}[(A^2 - B^2)(A^2 - B^2 - 2A)e^{-4x} - (6A^2 - 2B^2 + 6A + \frac{\Lambda B^2}{3\rho_0^2})e^0 + (2A+1)e^{2x} - \frac{1}{3}e^{4x}] \tag{3.25}$$

In the same way, using (3.18) and (3.24), we can write (3.19b) as:

$$p = \frac{3\rho_0^2}{B^2}[(A^2 - B^2)(A^2 - B^2 - 2A)e^{-4x} - (6A^2 - 2B^2 + 6A + \frac{\Lambda B^2}{3\rho_0^2})e^0 + \frac{20}{13}(2A+1)e^{2x} - \frac{7}{9}e^{4x}] \tag{3.26}$$

Finally, , since $A \geq B$, we observe that this system is physical because ρ and p satisfy certain conditions known as the "Energy Conditions", namely Null energy condition, Weak energy condition, Dominant energy condition and Strong energy condition (*Appendix A*). Hence, we can find the gravitational waves in this system.

Chapter 4

Axial gravitational waves in FLRW cosmology

4.1 Equations

We consider the metric defined in (3.13) and the conformal Hubble parameter as:

$$H(\eta) = \frac{\partial_\eta a}{a} \tag{4.1}$$

For convenience, we use $Y = Y(\theta) = Y_{l0}(\theta)$ and $Y' = \partial_\theta Y$, where Y_{lm} stands for spherical harmonics.

Considering the axial perturbation in the Regge-Wheeler guage, we write:

$$g_{\mu\nu} = a^2(\eta)\eta^{(0)}_{\mu\nu} + h_{\mu\nu} \tag{4.2}$$

where $\eta^{(0)}_{\mu\nu}$ is the Minkowski metric and the sole nonzero components of $h_{\mu\nu}$ can be given as:

$$h_{\eta\phi} = h_0 \sin\theta Y' \quad \text{and} \quad h_{r\phi} = h_1 \sin\theta Y' \tag{4.3}$$

where $h_0 = h_0(\eta, r)$ and $h_1 = h_1(\eta, r)$, unlike in Regge-Wheeler formalism in which these components were the functions of radius only. The stress-energy tensor that describes the material field is:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} - \Lambda g_{\mu\nu} \tag{4.4}$$

Just two components of the four-velocity of matter appear to be affected by axial modes of GWs. Up to terms linear in perturbations, we have:

$$\begin{aligned} u_\eta &= -a(\eta) \\ u_\phi &= \sin\theta \cdot u(\eta, r) Y' \end{aligned} \tag{4.5}$$

This guarantees that $u_\mu u^\mu = -1$ in linear terms of velocity:

$$\begin{aligned} g^{\eta\eta} u_\eta &= u^\eta = \frac{1}{a} \\ g^{\phi\phi} u_\phi &= u^\phi = \frac{u(\eta, r) Y'}{a^2 r^2 \sin\theta} \\ u_\mu u^\mu &= -1 + \frac{u^2 Y'^2}{a^2 r^2} \end{aligned}$$

Hence, the non-vanishing components of stress-energy tensor are:

$$\begin{aligned} T_{\eta\eta} &= -a^2(\rho + p) - a^2 p + a^2 \Lambda \\ T_{\eta\phi} &= -a \sin\theta Y' (\rho + p) u \end{aligned} \tag{4.6}$$

Using (4.1), (3.8) can be written as:

$$\rho = \frac{3}{a^2} H^2 - \Lambda \tag{4.7}$$

and (3.10) becomes:

$$p = \Lambda - \frac{H^2}{a^2} - \frac{2}{a^2} \frac{dH}{d\eta} \tag{4.8}$$

From (4.8) and (4.9), we can derive the relation:

$$H^2 + \frac{dH}{d\eta} = \frac{a^2}{2}\left(\frac{\rho}{3} - p + \frac{4\Lambda}{3}\right) \tag{4.9}$$

Again using (4.1), (3.11) gives:

$$\partial_\eta \rho = 3H(\rho + p) \tag{4.10}$$

The linearized Einstein equations for (3.13) with the application of (4.6) are:

$$\partial_r h_1 = \partial_\eta h_0 \tag{4.11}$$

$$\begin{aligned} &\partial_r \partial_\eta h_1 - \partial_r^2 h_0 - 2H\partial_r h_1 + \frac{2}{r}\partial_\eta h_1 \\ &- \frac{4H}{r}h_1 + \frac{l(l+1)}{r^2}h_0 = -2a^3(\rho + p)u \end{aligned} \tag{4.12}$$

$$\begin{aligned} &\partial_\eta^2 h_1 - \partial_r \partial_\eta h_1 - 2H\partial_\eta h_1 + \frac{2}{r}\partial_\eta h_0 \\ &- \frac{dH}{d\eta}h_1 + \frac{l(l+1) - 2}{r^2}h_1 = 0 \end{aligned} \tag{4.13}$$

Putting (4.11) in (4.13):

$$\begin{aligned} &\partial_\eta^2 h_1 - \partial_r^2 h_1 - 2H\partial_\eta h_1 + \frac{2}{r}\partial_r h_1 \\ &- \frac{dH}{d\eta}h_1 + \frac{l(l+1) - 2}{r^2}h_1 = 0 \end{aligned} \tag{4.14}$$

We define the new term $Q(\eta, r)$, using which we can write h_1 as:

$$h_1(\eta, r) = ra(\eta)Q(\eta, r) \tag{4.15}$$

Finally, inserting (4.15) into (4.14) and using (4.9), one can arrive at so-called 'master equation':

$$\partial_\eta^2 Q - \partial_r^2 Q + \frac{l(l+1)}{r^2}Q - \frac{a^2}{2}\left(\frac{\rho}{3} - p + \frac{4\Lambda}{3}\right) = 0 \tag{4.16}$$

For our case (3.15), it becomes:

$$\partial_\eta^2 Q - \partial_r^2 Q + \frac{l(l+1)}{r^2}Q - \frac{A + B\tanh\rho_0\eta}{2}\left(\frac{\rho}{3} - p + \frac{4\Lambda}{3}\right) = 0 \quad (4.17)$$

4.2 Solving PDE

Let

$$f(\eta) = \frac{A + B\tanh\rho_0\eta}{2}\left(\frac{\rho}{3} - p + \frac{4\Lambda}{3}\right) \quad (4.18)$$

We focus on quadrupole ($l = 2$) modes only. Rearranging (4.17):

$$\partial_\eta^2 Q - f(\eta)Q = \partial_r^2 Q - \frac{6}{r^2}Q \quad (4.19)$$

By using the method of separation of variables:

$$Q = Q_1(r)Q_2(\eta) \quad (4.20)$$

Therefore, (4.19) becomes:

$$Q_1[\partial_\eta^2 Q_2 - f(\eta)Q_2] = Q_2\left[\partial_r^2 Q_1 - \frac{6}{r^2}Q_1\right]$$

$$\frac{\partial_\eta^2 Q_2 - f(\eta)Q_2}{Q_2(\eta)} = \frac{\partial_r^2 Q_1 - \frac{6}{r^2}Q_1}{Q_1(r)} = -K^2 \quad \text{(say)} \quad (4.21)$$

where K is some constant. (4.21) gives two wave equations as follows:

$$\partial_r^2 Q_1 - \frac{6}{r^2}Q_1 = -K^2 Q_1 \quad (4.22)$$

$$\partial_\eta^2 Q_2 - f(\eta)Q_2 = -K^2 Q_2 \quad (4.23)$$

where:

$$f(\eta) = -\frac{\rho_0^2 B^2 sech^2\rho_0\eta}{2(A + B\tanh\rho_0\eta)^2}\left(\frac{sech^2\rho_0\eta}{2} + 1\right) - \frac{2\tanh\rho_0\eta}{(A + B\tanh\rho_0\eta)^{3/2}} \quad (4.24)$$

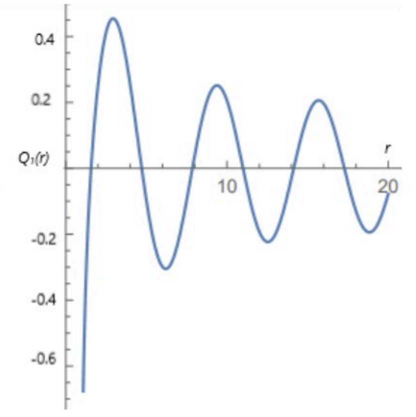

Figure 4.1: Variation of $Q_1(r)$ with r

We find the solution for (4.22) to be:

$$Q_1(r) = \sqrt{\frac{2}{\pi K}} c_1 \left(-\frac{3\cos Kr}{Kr} - \sin Kr + \frac{3\sin Kr}{K^2r^2} \right) + \sqrt{\frac{2}{\pi K}} c_2 \left(\cos Kr - \frac{3\cos Kr}{K^2r^2} - \frac{3\sin Kr}{Kr} \right) \tag{4.25}$$

where c_1 and c_2 are constants of integration. For outgoing real wave, we have:

$$Q_1(r) = \frac{e^{-\iota Kr}}{\sqrt{2\pi K}} c_1 \left(-\frac{3}{Kr} + \frac{2}{K^2r^2} - \frac{3}{K^4r^4} - \frac{1}{3} \right) \tag{4.26}$$

For a case where $A = 4$, $B = -2$, $\rho_0 = 10$ and $K = 1$ (say), $Q_1(r)$ can be plotted with respect to r coordinate as in Figure 4.1.

We now analyse (4.23):

$$\partial_\eta^2 Q_2 = [f(\eta) - K^2] Q_2 = -\tilde{\omega}^2 Q_2 \tag{4.27}$$

Therefore, we can assume:

$$Q_2(\eta) = e^{\iota \tilde{\omega} \eta} \tag{4.28}$$

where:

$$\tilde{\omega}^2 = -f(\eta) + K^2 \geq 0$$

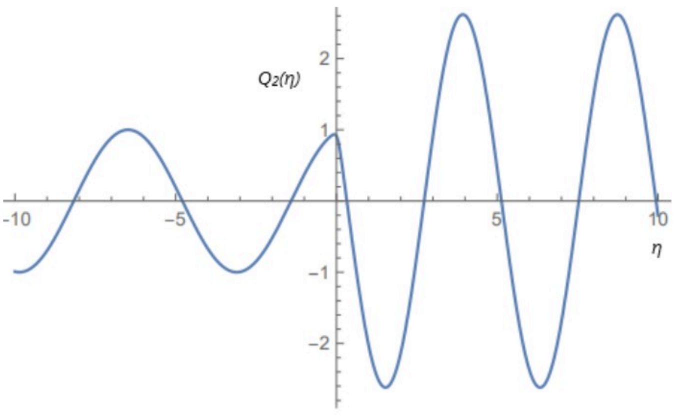

Figure 4.2: Variation of $Q_2(\eta)$ with η

Hence, the condition:

$$K^2 \geq f(\eta)|_{max} \tag{4.29}$$

should be satisfied. For the same case, where the values of A, B, ρ_0 and K were assumed earlier in this section, variation of $Q_2(\eta)$ with η can be plotted as in Figure 4.2.

For asymptotic and flat space ($\Lambda = 0$) case, $f(\eta)$ approximates to a constant as:

$$\begin{aligned} f(\eta) &= -\frac{2}{(A+B)^{3/2}}, \quad \text{for} \quad \eta \to \infty \\ f(\eta) &= \frac{2}{(A-B)^{3/2}}, \quad \text{for} \quad \eta \to -\infty \end{aligned} \tag{4.30}$$

Thus, from (4.29):

$$K^2 \geq \frac{2}{(A-B)^{3/2}} \tag{4.31}$$

Finally, the gravitational wave can be written as:

$$Q(\eta, r) = \frac{e^{\iota(\tilde{\omega}\eta - Kr)}}{\sqrt{2\pi K}} c_1 \left(-\frac{3}{Kr} + \frac{2}{K^2 r^2} - \frac{3}{K^4 r^4} - \frac{1}{3} \right) \tag{4.32}$$

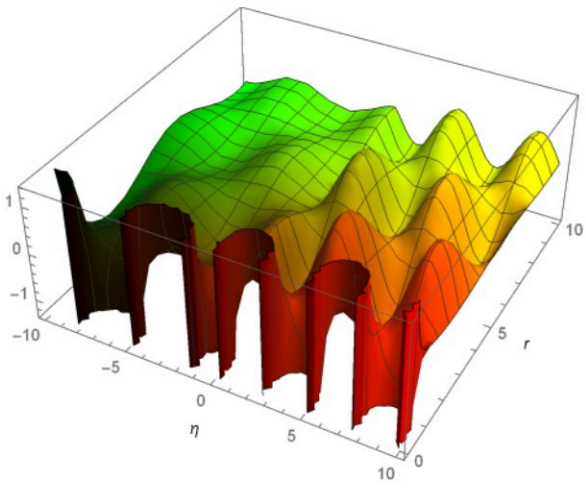

Figure 4.3: Variation of $Q(\eta, r)$ with η and r

for asymptotic limits for η, i.e. $\eta \to \pm\infty$ and the variation of $Q(\eta, r)$ with η and r (in accordance with Figure 4.1 and Figure 4.2) can be plotted as in Figure 4.3.

Chapter 5

Discussion

Wave vector is an important parameter of such wave which can be expressed as an integral over the stress energy tensor, $T_{\mu\nu}$:

$$h_{\mu\nu} = 4 \int \frac{T_{\mu\nu}(\overrightarrow{x'}, t - |\overrightarrow{x} - \overrightarrow{x'}|)}{|\overrightarrow{x} - \overrightarrow{x'}|} \, d^3x' \tag{5.1}$$

(Green's function).

Many sources do not require complete relativistic treatment. If they are moving slowly and the gravitational contribution to the total energy is modest, this expression can be reduced to the known quadrupole formula in the weak field limit:

$$h_{\mu\nu} = \frac{2}{r} \ddot{I}_{\mu\nu}(t - r) \tag{5.2}$$

where $\ddot{I}_{\mu\nu}$ is the *reduced (trace free) quadrupole moment tensor.* The power radiated in gravitational waves (luminosity) is given by:

$$\frac{dE_{GW}}{dt} = \frac{1}{5} \langle \ddot{I}_{\mu\nu} \ddot{I}^{\mu\nu} \rangle \tag{5.3}$$

For our case,

$$\begin{aligned}
\ddot{I}_{\mu\nu}\ddot{I}^{\mu\nu} &= \frac{h_{\mu\nu}h^{\mu\nu}}{(2/r)^2} \\
&= \frac{g^{00}g_{\phi\phi}h_{0\phi}h_{0\phi} + g^{rr}g_{\phi\phi}h_{r\phi}h_{r\phi}}{(2/r)^2} \\
&= \frac{(h_1^2 - h_0^2)Y'^2}{4(A + B\tanh\rho_0\eta)^2}
\end{aligned} \tag{5.4}$$

Hence, (5.3) is given as:

$$\frac{dE_{GW}}{dt} = \frac{1}{5}\left\langle \frac{(h_1^2 - h_0^2)Y'^2}{4(A + B\tanh\rho_0\eta)^2} \right\rangle \tag{5.5}$$

Chapter 6

Conclusion

Gravitational waves are produced by any heavy object that accelerates, for example, automobiles. However, the masses and accelerations of things on Earth are too weak to produce gravitational waves that our equipment can detect. So we need to look far beyond our solar system to locate large enough gravitational waves. It turns out that the Universe is packed with really huge objects that accelerate rapidly and, as a result, emit gravitational waves that can actually be detected.

Change in vacuum is one of the major sources of gravitational waves as we have seen in our intended case of $C = A + B \tanh \rho_0 \eta$, where the space is Minkowski for asymptotic limits but shows a sudden change in between. We then governed the expansion of space with the Friedmann equations in a material field for different values of ρ and p of an homogeneous and isotropic universe, and solve the solution of gravitational waves (frequency, $\tilde{\omega}$) for a linear axial mode perturbation, with the help of "master equation" (4.17), where the time and azimuthal components of 4-velocity, and T_{00} and $T_{0\phi}$ of stress-energy tensor are the only affected components due to such perturbation. Plotting of graphs for a particular case shows the behavior of such waves. Finally, we see that a physical quantities of such waves, like radiated power which is the time derivative of energy of gravitational waves, is calculable which relates to the average value of the relation of predefined quantities such as h_0, h_1, Y' and $C(\eta)$.

Appendix A

Energy Conditions

Energy condition in general relativity is a mathematical formulation that generalises the statement "a region of space cannot have a negative energy density".

For perfect fluids ($T^{\mu}{}_{\nu} = diag(-\rho, p, p, p)$), these can be stipulated in mathematical form as:

- **Null energy condition**: $\rho + p \geq 0$
- **Weak energy condition**: $\rho \geq 0$, $\rho + p \geq 0$
- **Dominant energy condition**: $\rho \geq |p|$
- **Strong energy condition**: $\rho + p \geq 0$, $\rho + 3p \geq 0$

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