INDIAN INSTITUTE OF TECHNOLOGY INDORE

UNDERGRADUATE THESIS

Data-Driven Model Reduction for Slowing Climate Change

Authors:
ARYAN VERMA
RISHABH YADAV

Supervisor:
DR. KAPIL AHUJA

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Declaration of Authorship

We declare that this thesis titled, "Data-Driven Model Reduction for Slowing Climate Change" and the work presented in it are our own. I confirm that:

- This work was done wholly or mainly while in candidature for the BTP project at IIT Indore.
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- Where we have consulted the published work of others, this is always clearly attributed.
- Where we have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- We have acknowledged all main sources of help.
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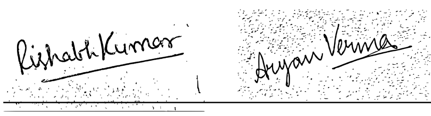

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Supervisor

Dr. KAPIL AHUJA

Professor,

Indian Institute of Technology

Indore

Date:

"For only in their dreams can men be truly free. 'Twas always thus, and always thus will be. "

Tom Schulman

INDIAN INSTITUTE OF TECHNOLOGY INDORE

Abstract

Department of Computer Science and Engineering

Bachelor of Technology

Data-Driven Model Reduction for Slowing Climate Change

Climate and weather models anticipate future climate and weather based on numerical solutions of a set of time-dependent partial differential equations given beginning circumstances that describe meteorological field values of the atmosphere at a certain point in time. Even the most powerful computers can't keep up with the computing needs of today's models. Models are requiring more calculations as they grow to represent a more thorough knowledge of atmospheric dynamics. Concerns regarding the viability of a desired solution for a given model on a certain hardware platform stem from the high expenses of both model creation and computational resources.

The goal of this project is to reduce these vast, complex models to smaller, simpler models that can properly describe the behaviour of the original process under a range of operating situations. Model Order Reduction (MOR) is a time-saving, systematic technique for creating a dynamical system that evolves in a much smaller area while maintaining similar response characteristics to the original system. Reduced order models might be utilised as efficient surrogates for the original model in bigger simulations, substituting it as a component. The Loewner framework for model reduction of linear systems will be used in this regard.

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Contents

List of Abbreviations

MOR **M**odel **O**rder **R**eduction

SVD **S**ingular **V**alue **D**ecomposition

POD **P**roper **O**rthogonal **D**ecomposition

RB **R**educed **B**asis

DAE **D**ifferential **A**legbraic **E**quation

CSRM **C**loud-**S**ystem-**R**esolving **S**ystem

LTI **L**inear-**T**ime **I**nvariant

Dedicated to all first graders, on the shores of an unexplored continent, just getting introduced to mathematics.

Chapter 1

Introduction

1.1 Background

The problem of Slowing Climate Change is the problem of predicting future climate and weather conditions or detecting the cause of any sudden aberrations in climate or weather patterns based on a set of initial conditions that represent meteorological field values of the atmosphere. Some of these conditions include evaporation, friction, precipitation, radiation, advection, pressure forces, etc. These values act as inputs to various state-of-the-art climate and weather models used worldwide by meteorological agencies.

Even the most powerful computers can't keep up with the computing needs of today's climate and weather models. Models are requiring more calculations as they grow to represent a more thorough knowledge of atmospheric dynamics. Concerns regarding the viability of a desired solution for a given model on a certain hardware platform stem from the high expenses of both model creation and computational resources [1].

To cope with this problem, Model Order Reduction (MOR) is introduced in this field. To simulate and regulate complicated physical processes, Model Order Reduction is often utilised. In such instances, the systems that eventually emerge are frequently too complicated to fulfil the time constraints of interactive design, optimization, or real-time control [2]. MOR was devised a time-saving, systematic technique for creating a dynamical system that evolves in a much smaller area while maintaining similar response characteristics to the original system. Reduced order models might be utilised as efficient surrogates for the original model in bigger simulations, substituting it as a component [3].

Singular Value Decomposition (SVD) based (e.g. balanced truncation), Krylov-based or moment matching methods, proper orthogonal decomposition (POD), and reduced basis (RB) approaches are all examples of model reduction methods. The majority of these approaches belong to the projection-based method family, in which the internal state variable is approximated by a projected variable into a certain subspace [3]. Here we are going to focus on a particular Data-Driven Model Order Reduction method for our task as in that case, the model essentials functions as a black box and all that is required are a set of data or interpolation points.

Data-Driven MOR involves constructing generalized state space representations of interpolants matching tangential interpolation data [4]. In simpler words, we construct models that fit (interpolate) given sets of data using rational interpolation. The Loewner and shifted Loewner matrices are the key tools for this approach.

The Loewner matrix is a flexible approach for data-driven model reduction that was originally created for rational interpolation but has now been expanded to the Loewner framework. Its main characteristic is that it allows for a trade-off between model complexity and fit precision. Furthermore, building models from the data is a natural process.

Chapter 2

Literature Survey

The following chapter discusses literature pertaining to linear systems and Model Order Reduction methods. It describes the data-driven MOR with focus on Loewner framework in detail. Some context on the climate equation used for our work is also provided.

2.1 Model Reduction of Linear Systems

2.1.1 Linear Systems

A linear time-invariant dynamical system Σ with m inputs, n internal variables, and p outputs in descriptor-form representation is given by a set of differential algebraic equations (DAEs):

$$\Sigma : \ \mathbf{E}\frac{d}{dt}x(t) = \mathbf{A}x(t) + \mathbf{B}u(t), \quad y(t) = \mathbf{C}x(t) + \mathbf{D}u(t) \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the internal variable; $x(t_0) = x_0$, where t is the time and x_0 is the initial state; $u(t) \in \mathbb{R}^m$ is the input function, and $y(t) \in \mathbb{R}^p$ is the output function; and:

$$\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times m}, \mathbf{C} \in \mathbb{R}^{p\times n}, \mathbf{D} \in \mathbb{R}^{p\times m}$$

are constant matrices. If the matrix $\mathbf{A} - \lambda\mathbf{E}$ is non-singular for some finite $\lambda \in \mathbb{C}$, the matrix pencil ($\mathbf{A}$,$\mathbf{E}$) is said to be *regular*.

These set of differential equations, called as linear differential equations, are ordinary differential equations that are linear in their derivatives with respect to time, linear in the dependent variables, and linear in the input function or control.

2.1.2 Transfer Function

The Transfer Function of a linear system works in the frequency domain. The transfer function for continuous-time input and output signals is the linear mapping of the input Laplace transform to the output Laplace transform. In the domains of communication theory, signal processing, and control theory, it is often utilised in single-input single-output systems. It connects one linear system input to one output.
The transfer function of Σ from Equation 2.1 is the $p \times m$ rational matrix function:

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{2.2}$$

The set of matrices **(E,A,B,C,D)** is called a *descriptor realization* of **H**(s).

2.1.3 Rank-Revealing Factorization

We eliminate **D** in Equation 2.1 by incorporating it in the remaining matrices. The reason for introducing descriptor realizations where the **D**-term is incorporated in the remaining matrices is that the Loewner framework yields precisely such descriptor realizations [2].
A rank-revealing factorization of a matrix $A \in \mathbb{R}^{mxn}$ is a factorization

$$A = XDY^T, X \in \mathbb{R}^{mxp}, D \in \mathbb{R}^{pxp}, Y \in \mathbb{R}^{nxp},$$

where $p <= min(m,n)$, D is a diagonal and non-singular matrix, and X and Y are well conditioned.

To achieve this, we have to allow the dimension of the realization to increase by rank **D**. Consider a rank-revealing factorization

$$\mathbf{D} = \mathbf{D}_1\mathbf{D}_2 \quad where \quad \mathbf{D}_1 \in \mathbb{R}^{p\times\rho}, \mathbf{D}_2 \in \mathbb{R}^{\rho\times m}, \tag{2.3}$$

and $\rho = rank\ \mathbf{D}$. It readily follows that

$$E_\delta = \begin{pmatrix} \mathbf{E} & 0 \\ 0 & O_{\rho\times\rho} \end{pmatrix}, A_\delta = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & -\mathbf{I}_\rho \end{pmatrix}, B_\delta = \begin{pmatrix} \mathbf{B} \\ \mathbf{D}_2 \end{pmatrix}, C_\delta = \begin{pmatrix} \mathbf{C} & \mathbf{D}_1 \end{pmatrix} \tag{2.4}$$

is a descriptor realization of the same system with no $\mathbf{D}$-term (i.e., $\mathbf{D}_\delta = 0$).

The *model reduction* problem consists of constructing reduced-order DAE systems of the form

$$\hat{\Sigma} : \hat{\mathbf{E}}\frac{d}{dt}\hat{x}(t) = \hat{\mathbf{A}}\hat{x}(t) + \hat{\mathbf{B}}u(t), \quad \hat{y}(t) = \hat{\mathbf{C}}\hat{x}(t) + \hat{\mathbf{D}}u(t), \tag{2.5}$$

where $\hat{x}(t) \in \mathbb{R}^r$ is the internal variable (the state if $\hat{\mathbf{E}}$ is invertible), $\hat{y}(t) \in \mathbb{R}^p$ is the output of $\hat{\Sigma}$ corresponding to the same input $u(t)$, and

$$\hat{\mathbf{E}}, \hat{\mathbf{A}} \in \mathbb{R}^{r\times r}, \hat{\mathbf{B}} \in \mathbb{R}^{r\times m}, \hat{\mathbf{C}} \in \mathbb{R}^{p\times r}, \hat{\mathbf{D}} \in \mathbb{R}^{p\times m}.$$

Thus, the number of inputs m and output p remains the same while $r \ll n$ [2].

2.1.4 Interpolatory reduction for linear systems

Consider the system Σ and its transfer function $H(s)$ defined by Equation 2.2. Given left interpolation points $\{\mu_j | 1 \le j \le q\} \subset \mathbb{C}$, left tangential directions $\{\boldsymbol{\ell}_j | 1 \le j \le q\} \subset \mathbb{C}^p$, right interpolation points $\{\lambda_i | 1 \le i \le k\} \subset \mathbb{C}$, right tangential directions $\{\mathbf{r}_i | 1 \le i \le k\} \subset \mathbb{C}^m$. We seek a reduced-order system $\hat{\Sigma}$ such that the associated transfer function $\hat{\mathbf{H}}(s)$ is a *tangential interpolant* to $\mathbf{H}(s)$:

$$\begin{array}{ccc} \hat{\mathbf{H}}(\lambda_i)\mathbf{r}_i = \mathbf{H}(\lambda_i)\mathbf{r}_i & \textbf{and} & \boldsymbol{\ell}_j^T\hat{\mathbf{H}}(\mu_j) = \boldsymbol{\ell}_j^T\mathbf{H}(\mu_j) \\ \text{for } i = 1, \cdots, k & & \text{for } j = 1, \cdots, q \end{array} \tag{2.6}$$

If we are given input/output data instead of descriptor-form data as in Equation 2.1, the resulting problem is slightly modified. Given a set of input-output response measurements specified by left driving frequencies $\{\mu_j | 1 \leq j \leq q\} \subset \mathbb{C}$, using left input or tangential directions $\{\boldsymbol{\ell}_j | 1 \leq j \leq q\} \subset \mathbb{C}^p$, producing left responses $\{\boldsymbol{v}_j | 1 \leq j \leq q\} \subset \mathbb{C}^m$, and right interpolation points $\{\lambda_i | 1 \leq i \leq k\} \subset \mathbb{C}$, using right input or tangential directions $\{\mathbf{r}_i | 1 \leq i \leq k\} \subset \mathbb{C}^m$, producing right responses $\{\mathbf{w}_i | 1 \leq i \leq k\} \subset \mathbb{C}^p$, find a low-order system $\hat{\Sigma}$ such that the resulting transfer function $\hat{\mathbf{H}}(s)$ is an *tangential interpolant* to the data:

$$\begin{array}{ccc} \hat{\mathbf{H}}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i & \textbf{and} & \boldsymbol{\ell}_j^T\hat{\mathbf{H}}(\mu_j) = \boldsymbol{v}_j^T \\ \text{for } i = 1, \cdots, k & & \text{for } j = 1, \cdots, q \end{array} \tag{2.7}$$

Interpolation points and tangential directions are determined by the problem. A specific case of this is the systems with a single input and a single output $(m = p = 1)$, in other words, the SISO systems. Here, left and right directions can be taken equal to one $(\ell_j = 1, r_i = 1)$. So, conditions 2.6 become

$$\hat{\mathbf{H}}(\mu_j) = \mathbf{H}(\mu_j),\ j = 1, \cdots, q, \quad \hat{\mathbf{H}}(\lambda_i) = \mathbf{H}(\lambda_i),\ i = 1, \cdots, k \tag{2.8}$$

while conditions 2.7 become

$$\hat{\mathbf{H}}(\mu_j) = \mathbf{v}_j,\ j = 1, \cdots, q, \quad \hat{\mathbf{H}}(\lambda_i) = \mathbf{w}_i,\ i = 1, \cdots, k \tag{2.9}$$

2.2 The Loewner framework for Linear Systems

Given a row array of pairs of complex numbers $(\mu_j, \mathbf{v}_j), j = 1, ..., q$, and a column array of pairs of complex numbers $(\lambda_i, \mathbf{w}_i), i = 1, ..., k$, with λ_i, μ_j distinct, the associated Loewner or, also known as, the divided-differences matrix is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k} \tag{2.10}$$

The key characteristic of the Loewner matrix is that its rank conveys information about the minimal admissible complexity of the solutions of the interpolation problem. When dealing with measured data, determining the numerical rank of an appropriate Loewner matrix or a Loewner pencil is required.

2.2.1 The Loewner Pencil

We will formulate the results for the more general tangential interpolation problem. We are given the right data, $(\lambda_i; \mathbf{r}_i, \mathbf{w}_i), i = 1, \cdots, k$, and the left data, $(\mu_j; \boldsymbol{\ell}_j^T, \mathbf{v}_j^T), j = 1, \cdots, q$. The data can be organized as follows:

The right data are:

$$\mathbf{\Lambda} = \text{diag}\left[\lambda_1, \lambda_2, \cdots, \lambda_k\right] \in \mathbb{C}^{k \times k},$$

$$\mathbf{R} = [\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_k] \in \mathbb{C}^{m \times k},$$

$$\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k] \in \mathbb{C}^{p \times k},$$

and the left data are:

$$\mathbf{M} = \text{diag}\left[\mu_1, \mu_2, \cdots, \mu_q\right] \in \mathbb{C}^{q \times q},$$

$$\mathbf{L}^T = \left[\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \cdots, \boldsymbol{\ell}_q\right] \in \mathbb{C}^{q \times p},$$

$$\mathbf{V}^T = \left[\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q\right] \in \mathbb{C}^{q \times m}$$

The Loewner matrix $\mathbb{L} \in \mathbb{C}^{q \times k}$ is then defined as:

$$\mathbb{L} = \begin{bmatrix} \dfrac{\mathbf{v}_1^T \mathbf{r}_1 - \boldsymbol{\ell}_1^T \mathbf{w}_1}{\mu_1 - \lambda_1} & \dots & \dfrac{\mathbf{v}_1^T \mathbf{r}_k - \boldsymbol{\ell}_1^T \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \dfrac{\mathbf{v}_q^T \mathbf{r}_1 - \boldsymbol{\ell}_q^T \mathbf{w}_1}{\mu_q - \lambda_1} & \dots & \dfrac{\mathbf{v}_q^T \mathbf{r}_k - \boldsymbol{\ell}_q^T \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \tag{2.11}$$

The shifted Loewner matrix $\mathbb{L}_s \in \mathbb{C}^{q \times k}$ is then defined as:

$$\mathbb{L}_s = \begin{bmatrix} \dfrac{\mu_1 \mathbf{v}_1^T \mathbf{r}_1 - \boldsymbol{\ell}_1^T \mathbf{w}_1 \lambda_1}{\mu_1 - \lambda_1} & \dots & \dfrac{\mu_1 \mathbf{v}_1^T \mathbf{r}_k - \boldsymbol{\ell}_1^T \mathbf{w}_k \lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \dfrac{\mu_q \mathbf{v}_q^T \mathbf{r}_1 - \boldsymbol{\ell}_q^T \mathbf{w}_1 \lambda_1}{\mu_q - \lambda_1} & \dots & \dfrac{\mu_q \mathbf{v}_q^T \mathbf{r}_k - \boldsymbol{\ell}_q^T \mathbf{w}_k \lambda_k}{\mu_q - \lambda_k} \end{bmatrix} \tag{2.12}$$

The shifted Loewner Matrix $\mathbb{L}_s$, along with the Loewner Matrix $\mathbb{L}$ forms the Loewner pencil which constitutes a high-order realization of the underlying $(\mathbf{A}, \mathbf{E})$ pencil.

2.3 Construction of Reduced Order models

Model Order Reduction has several important applications such as autonomous soft robotic fishtail, commercial aircraft design, Euler-Bernoulli beam, spring-mass-damper systems, baroclinic instabilities described by Eady's model, cantilever Timoshenko beam, and many other real-life use cases [5].

2.3.1 Singular Value Decomposition

Singular Value Decomposition (SVD) is a factorization of an mxn real or complex matrix into three matrices. It is given by:

$$M = \mathbf{U}\Sigma\mathbf{V}^T, \tag{2.13}$$

where, $\mathbf{U}$ is an mxn matrix of the orthonormal eigenvectors of $\mathbf{AA^T}$,
Σ is an nxn diagonal matrix of the singular values which are the square roots of the eigenvalues of $\mathbf{AA^T}$,
$\mathbf{V}$ is an nxn matrix containing the orthonormal eigenvectors of $\mathbf{A^TA}$.

SVD has various applications including computing the matrix approximation, pseudoinverse, and determining the null space, range, and rank of a matrix.

2.3.2 MOR for right amount of data

Assume that $k = q$, and let $(\mathbb{L}_s, \mathbb{L})$ be a regular pencil such that none of the interpolation points λ_i, μ_j are its eigenvalues. Then

$$\mathbf{E} = -\mathbb{L},\ \mathbf{A} = -\mathbb{L}_s,\ \mathbf{B} = \mathbf{V},\ \mathbf{C} = \mathbf{W} \tag{2.14}$$

is a minimal realization of an interpolant of the data, i.e., the rational function $\mathbf{H}(s) = \mathbf{W}(\mathbb{L}_s - s\mathbb{L})^{-1}\mathbf{V}$ interpolates the data.

2.3.3 MOR for redundant amount of data

If $(\mathbb{L}_s, \mathbb{L})$ is a singular pencil, then we're dealing with the case of more realistic redundant amount of data. In this case, if the assumption

$$rank(\gamma\mathbb{L} - \mathbb{L}_s) = rank\begin{pmatrix} \mathbb{L} \\ \mathbb{L}_s \end{pmatrix} = rank\begin{pmatrix} \mathbb{L} & \mathbb{L}_s \end{pmatrix} = r \leq k \tag{2.15}$$

is satisfied for all $\gamma \in \{\lambda_i | 1 \leq i \leq k\} \bigcup \{\mu_j | 1 \leq j \leq q\}$, we consider the following SVD factorizations:

$$\begin{pmatrix} \mathbb{L} & \mathbb{L}_s \end{pmatrix} = \mathbf{Y}^{(1)}\mathbf{S}^{(1)}(\mathbf{X}^{(1)})^{\mathbf{T}}, \begin{pmatrix} \mathbb{L} \\ \mathbb{L}_s \end{pmatrix} = \mathbf{Y}^{(2)}\mathbf{S}^{(2)}(\mathbf{X}^{(2)})^{\mathbf{T}} \tag{2.16}$$

where $\mathbf{Y}^{(1)}, \mathbf{X}^{(2)} \in \mathbb{C}^{k \times k}$. The projection matrices $\mathbf{Y} \in \mathbb{C}^{k \times r}$ and $\mathbf{X} \in \mathbb{C}^{k \times r}$ are obtained by selecting the first r columns of the matrices $\mathbf{Y}^{(1)}$ and $\mathbf{X}^{(2)}$, respectively.

A realization $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$ of an approximate interpolant is given by the system matrices:

$$\mathbf{E} = -\mathbf{Y}^{\mathbf{T}}\mathbb{L}\mathbf{X},\ \mathbf{A} = -\mathbf{Y}^{\mathbf{T}}\mathbb{L}_s\mathbf{X},\ \mathbf{B} = \mathbf{Y}^{\mathbf{T}}\mathbf{V},\ \mathbf{C} = \mathbf{W}\mathbf{X}. \tag{2.17}$$

Hence, the transfer function $\mathbf{H}(s) = \mathbf{W}\mathbf{X}(\mathbf{Y}^{\mathbf{T}}\mathbb{L}_s\mathbf{X} - s\mathbf{Y}^{\mathbf{T}}\mathbb{L}_s\mathbf{X})^{-1}\mathbf{Y}^{\mathbf{T}}\mathbf{V}$ approximately matches the data.

Thus, if we have more data than necessary, we can consider $(\mathbb{L}_s, \mathbb{L}, \mathbf{V}, \mathbf{W})$ as a singular model of the data. An appropriate projection then yields a reduced system of order k.
As a direct result of the singular values of $\mathbb{L}$, the Loewner framework provides a trade-off between reduced order system accuracy and complexity.

2.4 Climate Change Equation

Coming to the theory of the second half of our research topic, we introduce an atmospheric equation concerning the movement of individual or a set of clouds in their entire lifetime, or through multiple lifetimes. Movement of clouds play an important role in determining the temperature and humidity conditions of any place. Accurate prediction of the cloud quality of any particular area can help in determining the cause of any sudden weather or climate change of that place. This also helps in tacking the problem of slowing climate change in different parts of the world.

2.4.1 Cloud System-Resolving Model

Since the mid-1908s, Cloud System-Resolving Models (CSRMs) have been used to evaluate cloud parameterizations. Yamasaki (1975), Krueger (1988), Arakawa, and Xu were among the first to use CSRMs to the parameterization problem in the mid-1980s. CSRMs may now be found in hundreds of locations throughout the world. CSRMs were formerly confined to two dimensions (2D) in order to save computing costs, but with today's computers, 3D CSRMs are now feasible for a wide range of applications [6].
CSRM is a numerical model with fine enough grid spacing to allow precise simulations of individual clouds during their whole life cycle or a portion of it. Large-scale circulations in the homogeneous direction can form spontaneously in CSRM simulations with horizontally or zonally homogenous border and driving conditions [7].
Linear Response Function of a CSRM encapsulates the macroscopic behavior of moist convection. It is useful for understanding the linear stability of moist convecting atmosphere as represented by the CSRM. Thunderstorms are formed by moist convection and are frequently responsible for severe weather across the world. Hail, downbursts, and tornadoes are all potential risks from thunderstorms. So, moist convection is responsible for so-called instability in the climatic conditions, that is, severe or unexpected climate anywhere in the world.

2.4.2 Linear Stability Problem

When moist convection is connected to 2D linear gravity waves, the linear stability problem is now investigated. The essential characteristics of the interaction between convection and largescale dynamics are captured in this prototype problem, to which effects such as those from an equatorial β plane can be incorporated. [7].

For each horizontal wavenumber k, the system can be written as

$$\frac{d}{dt}\begin{pmatrix}\mathbf{x}\\ \mathbf{w}\end{pmatrix}=\begin{pmatrix}\mathbf{M} & \mathbf{A}\\ k^2\mathbf{C} & \mathbf{D}\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{w}\end{pmatrix} \tag{2.18}$$

where, $\mathbf{x}$ is a vector containing the vertical profiles of temperature and specific humidity,
$\mathbf{M}$ is the linear response function derived from the CSRM,
$\mathbf{A}$ represents the effect of vertical temperature and moisture advection on $\mathbf{x}$,
$k^2\mathbf{C}$ is the effect of temperature and specific humidity on the vertical velocity profile $\mathbf{x}$,
$\mathbf{D}$ represents momentum damping.

Gravity waves are formed when gravity or buoyancy seeks to reestablish balance in a fluid medium or at the interface between two mediums. The contact between the atmosphere and the water, which causes wind waves, is an example of such an interaction. These wind waves are responsible for the movement of clouds all across the globe, and thus play an important role in our equation concerning the movement of individual clouds.

Chapter 3

Analysis and Objectives

Model Order Reduction has been used for quite some time now in various different fields like aeronautics, cantilever beams, and so on. There have been few recent advances in Model Order Reduction in the climate change context too. Chen, Chen, et al. ([8]) did several empirical model reduction framework experiments for the advancement of the understanding of diversity, nonlinearity, seasonality, and memory effect in ENSO simulation and prediction. Kuang et al. ([7]) did model order reduction for the linear stability problem concerning the climate change equation involving the movement of individual clouds using the algorithm of Safanov and Chiang (1989) ([9]). It used Balanced-Truncation method for model order reduction. This, along with all other model reduction methods, use the original model for its reduction, so the model does not act as a black box as such. Loewner framework, on the other hand, is used for data-driven model order reduction as it only uses input and output data to reduce the original model.

The main contributions of this project are as follows:

- We first implement the Loewner framework for model order reduction on right amount of data using Equation 2.14. This code will have further usage in implementing MOR for the climate change case.
- We compare the climate change equation with the Linear Time-Invariant Control System equation and try to analyse if we can apply the Loewner framework for

linear systems onto the linear stability equation.

- We conduct two experiments for applying Loewner framework to the climate change equation:
  1. First, we take the matrix **M** value as 0 and compare our results of the interpolation points to the original values.
  2. Next we apply rank-revealing facorization of **D** matrix to convert all other matrices into a descriptor realization of the same system with no **D**-term.

The rest of the report is organized as follows. In Section 4, we describe the pseudocode and examples used to implement the Loewner framework. We also propose the comparison of the climate change equation with the climate change equation. Along with that, we propose our strategy for applying the Loewner framework to the climate change equation. Section 5 describes our experimental setup and evaluates the results obtained from our tests. The report concludes in Section 6.

Chapter 4

Design Proposal

4.1 Interpolation using Loewner Framework

We could not find any code for implementing the interpolation using the Loewner framework. So the pseudocode as well as the code are written by us from scratch.

4.1.1 Pseudocode

Algorithm 1 INTERPOLATE($\Lambda, \mathbf{R}, \mathbf{W}, \mathbf{M}, \mathbf{L}, \mathbf{V}$)

$m, k = \mathbf{R}.shape$
$q, p = \mathbf{L}.shape$
Initialise $\mathbb{L}$ and $\mathbb{L}_s$ as matrices of dimension $q \times k$.
for $i \leftarrow 1$ to q **do**
  for $j \leftarrow 1$ to k **do**
    $\mathbb{L}[i, j] \leftarrow \dfrac{\mathbf{v_i}^T \mathbf{r_j} - \boldsymbol{\ell}_i^T \mathbf{w_j}}{\mu_i - \lambda_j}$
    $\mathbb{L}_s[i, j] \leftarrow \dfrac{\mu_i \mathbf{v_i}^T \mathbf{r_j} - \boldsymbol{\ell}_i^T \mathbf{w_j} \lambda_j}{\mu_i - \lambda_j}$
Set $\mathbf{E} = -\mathbb{L}, \mathbf{A} = -\mathbb{L}_s, \mathbf{B} = \mathbf{V}, \mathbf{C} = \mathbf{W}$
Initialise $\hat{\mathbf{V}}^T = \left[\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \cdots, \hat{\mathbf{v}}_q\right]$ and $\hat{\mathbf{W}} = [\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2, \cdots, \hat{\mathbf{w}}_k]$, with dimensions same as $\mathbf{V}$ and $\mathbf{W}$ respectively.
for $i \leftarrow 1$ to k **do**
  $\hat{\mathbf{w}}_i \leftarrow \mathbf{C}(\lambda_i \mathbf{E} - \mathbf{A})^{-1} \mathbf{B} \mathbf{r}_i$
for $i \leftarrow 1$ to q **do**
  $\hat{\mathbf{v}}_i^T \leftarrow \boldsymbol{\ell}_i^T \mathbf{C}(\mu_i \mathbf{E} - \mathbf{A})^{-1} \mathbf{B}$

4.1.2 Examples

We test our implementation of the Loewner framework for interpolation through several examples of varying dimensions. Some of those are shown here.

Example 1

Left data:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{L}^T = \begin{bmatrix} i & 1-i \\ -i & 2 \end{bmatrix}, \mathbf{V}^T = \begin{bmatrix} -1 & 1+i \\ 2 & -1 \end{bmatrix}$$

Right data:

$$\mathbf{\Lambda} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \mathbf{R} = \begin{bmatrix} 1+i & 0 \\ i & 1-i \end{bmatrix}, \mathbf{W} = \begin{bmatrix} 1 & -1+i \\ i & -1-i \end{bmatrix}$$

Using Steps 6 and 7 of the above algorithm, we obtain the Loewner and Shifted Loewner matrices:

$$\mathbb{L} = \begin{bmatrix} -1-i & 1-3i \\ 0.5-0.5i & -i \end{bmatrix}, \mathbb{L}_s = \begin{bmatrix} 0 & -1-3i \\ 0.5+1.5i & 0 \end{bmatrix}$$

Using Step 8, we obtain the $\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}$ matrices which are used to obtain $\hat{\mathbf{W}}$ and $\hat{\mathbf{V}}^{\mathbf{T}}$ using Steps 10-13:

$$\hat{\mathbf{W}} = \begin{bmatrix} 1 & -1+i \\ i & -1-i \end{bmatrix}, \hat{\mathbf{V}}^{\mathbf{T}} = \begin{bmatrix} -1 & 1+i \\ 2 & -1 \end{bmatrix}$$

As we can see, $\hat{\mathbf{W}}$ comes out to be exactly equal to $\mathbf{W}$ and $\hat{\mathbf{V}}^{\mathbf{T}}$ comes out to be exactly equal to $\mathbf{V}^{\mathbf{T}}$.

4.2 Comparison with Linear Systems

Climate change equation, that is, Equation 2.18 can be expressed as two ordinary differential equations as follows:

$$\frac{d\mathbf{w}}{dt} = \mathbf{D}\mathbf{w} + k^2\mathbf{C}\mathbf{x}, \quad \frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{w} + \mathbf{M}\mathbf{x} \tag{4.1}$$

where, $\mathbf{w}$ is the internal state variable,

$\mathbf{x}$ is the input, and

$\frac{d\mathbf{w}}{dt}$ is the output.

The Linear-Time Invariant control system equation is of the form:

$$\bar{\mathbf{E}}\frac{d\bar{\mathbf{x}}}{dt} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{B}}\mathbf{u}, \quad \bar{y} = \bar{\mathbf{C}}\bar{\mathbf{x}} + \bar{\mathbf{D}}\mathbf{u} \tag{4.2}$$

We can convert the simplified linear stability equation into LTI control system form as follows:

$$\begin{aligned} &\bar{\mathbf{x}} = \mathbf{w}, \quad \bar{\mathbf{A}} = \mathbf{D}, \\ &\bar{\mathbf{B}} = k^2\mathbf{C}, \quad \bar{\mathbf{C}} = \mathbf{A}, \\ &\bar{\mathbf{D}} = \mathbf{M}, \quad \bar{\mathbf{E}} = I \;\; (Identity\ matrix), \\ &\mathbf{u} = \mathbf{x}, \quad \bar{\mathbf{y}} = \frac{d\mathbf{x}}{dt} \end{aligned} \tag{4.3}$$

This means that we can now try if we can apply the Loewner Framework for linear systems onto the linear stability equation.

4.3 Applying Loewner to Climate Change Equation

We first describe here the algorithm used for applying Loewner framework. Then, Section 5 presents some examples to verify the results obtained from our method. The steps of the algorithm are as follows:

- For applying the Loewner Framework to the simplified Climate Change Equation described above, we first take the most common values of $\mathbf{A}, \mathbf{C}, \mathbf{D}$ matrices.
- We take the wavenumber k as 1, and $\bar{\mathbf{E}}$ as identity matrix.
- As the Loewner framework does not have any **D**-term, but it is present in the simplified form of the climate change equation, we need to accomodate **D**-term when applying the loewner framework here. So, we consider two cases for the matrix **D**:
  1. We don't consider the **D**-term. So, we get the matrix **M** as 0.
  2. We compute the rank-revealing factorization of the matrix **D**. This results in two matrices as described in Equation 2.3. This converts the $\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}$ matrices into a descriptor system which does not contain any **D**-term as described by the Equation 2.4.
- Next, we use the $\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}$ values obtained from the previous step to obtain the transfer function $\mathbf{H}(\mathbf{s})$.
- This transfer function is used to obtain the values of $\mathbf{V}, \mathbf{W}$ matrices.
- To use $\mathbb{L}$, $\mathbb{L}_s$ V, W in Equation 2.17, we also need Y and X. They can be obtained by Singular Value Decomposition (SVD) factorization of Loewner ($\mathbb{L}$) and Shifted Loewner ($\mathbb{L}_s$) matrices.
- Now we use the Equation 2.17 to obtain the final $\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}$ values.

4.3.1 Pseudocode

We present the algorithm for the above process here:

Algorithm 2 INTERPOLATE($\boldsymbol{\Lambda}, \boldsymbol{R}, \boldsymbol{W}, \boldsymbol{M}, \boldsymbol{L}, \boldsymbol{V}$)

$m, k = \mathbf{R}.shape$

$q, p = \mathbf{L}.shape$

Initialise $\mathbb{L}$ and $\mathbb{L}_s$ as matrices of dimension $q \times k$.

for $i \leftarrow 1$ to q **do**

  for $j \leftarrow 1$ to k **do**

    $\mathbb{L}[i, j] \leftarrow \dfrac{\mathbf{v_i}^T \mathbf{r_j} - \boldsymbol{\ell}_i^T \mathbf{w_j}}{\mu_i - \lambda_j}$

    $\mathbb{L}_s[i, j] \leftarrow \dfrac{\mu_4 i \mathbf{v_i}^T \mathbf{r_j} - \boldsymbol{\ell}_i^T \mathbf{w_j} \lambda_j}{\mu_i - \lambda_j}$

Let $L_1 = \begin{bmatrix} \mathbb{L} & \mathbb{L}_s \end{bmatrix}$, $L_2 = \begin{bmatrix} \mathbb{L} \\ \mathbb{L}_s \end{bmatrix}$, $r = rank(L_1) = rank(L_2)$

Compute the SVD factorizations $L_1 = \mathbf{Y}^{(1)} \mathbf{S}^{(1)} (\mathbf{X}^{(1)})^T$ and $L_2 = \mathbf{Y}^{(2)} \mathbf{S}^{(2)} (\mathbf{X}^{(2)})^T$

Define $\mathbf{Y}, \mathbf{X} \in \mathbb{C}^{k \times r}$ by selecting the first r columns of $\mathbf{Y}^{(1)}$ and $\mathbf{X}^{(2)}$ respectively.

Set $\mathbf{E} = -\mathbf{Y}^T \mathbb{L} \mathbf{X},\ \mathbf{A} = -\mathbf{Y}^T \mathbb{L}_s \mathbb{X},\ \mathbf{B} = \mathbf{Y}\mathbf{V},\ \mathbf{C} = \mathbf{W}\mathbf{X}$

Initialise $\hat{\mathbf{V}}^T = \left[\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \cdots, \hat{\mathbf{v}}_q\right]$ and $\hat{\mathbf{W}} = [\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2, \cdots, \hat{\mathbf{w}}_k]$, with dimensions same as $\mathbf{V}$ and $\mathbf{W}$ respectively.

for $i \leftarrow 1$ to k **do**

  $\hat{\mathbf{w}}_i \leftarrow \mathbf{C}(\lambda_i \mathbf{E} - \mathbf{A})^{-1} \mathbf{B} \mathbf{r}_i$

for $i \leftarrow 1$ to q **do**

  $\hat{\mathbf{v}}_i^T \leftarrow \boldsymbol{\ell}_i^T \mathbf{C}(\mu_i \mathbf{E} - \mathbf{A})^{-1} \mathbf{B}$

Chapter 5

Experiments

5.1 Datasets

We use real-life climatic data from various sources for all our experiments. These data are used by meteorological agencies all around the world to predict future climatic and weather conditions.

5.2 Experimental Setup

All the experiments[1] have been conducted on Python 3.0 in Linux (Ubuntu) environment. Multi-dimensional Numpy arrays are used to define all input matrix data.

5.3 Experiment: Applying Loewner Framework to Climate Change Equation

We consider both cases mentioned in Chapter 4 of accomodating the **D**-term here.

5.3.1 Case 1: Not considering the D-term

As the **D**-term is not considered, the matrix **M** will be a null matrix.

The input data is as follows:

[1] All presented results in this report are reproducible. Codes can be produced on request.

$$\mathbf{M} = O_{5,5}, \quad k = 1$$

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & 9 & 7 & 5 \\ 5 & 9 & 5 & 4 & 5 \\ 1 & 4 & 5 & 2 & 6 \\ 3 & 8 & 6 & 2 & 2 \\ 3 & 4 & 7 & 9 & 5 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 2 & 9 & 7 & 3 & 9 \\ 2 & 3 & 5 & 2 & 2 \\ 1 & 8 & 5 & 8 & 3 \\ 9 & 6 & 3 & 4 & 8 \\ 3 & 5 & 2 & 4 & 8 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 6 & 2 & 7 & 4 & 3 \\ 3 & 1 & 1 & 3 & 6 \\ 5 & 1 & 6 & 9 & 5 \\ 2 & 3 & 1 & 6 & 4 \\ 5 & 5 & 8 & 3 & 7 \end{bmatrix}$$

$\bar{\mathbf{E}}$ is considered as an Identity matrix.

Setting $\bar{\mathbf{A}} = \mathbf{D}, \quad \bar{\mathbf{B}} = k^2\mathbf{C}, \quad \bar{\mathbf{C}} = \mathbf{A}, \quad \bar{\mathbf{D}} = \mathbf{M}$

The right data obtained is:

$$\mathbf{\Lambda} = \begin{bmatrix} 7.6 & & \\ & 11.4 & \\ & & 1.4 \end{bmatrix}, \mathbf{R} = \begin{bmatrix} 17 & 17.8 & 2 \\ 16.6 & 4 & 18.8 \\ 12.2 & 8.2 & 0.8 \\ 5 & 14.4 & 10.6 \\ 14 & 11.2 & 2.6 \end{bmatrix}, \mathbf{W} = \begin{bmatrix} -644.64 & -671.77 & 1460.21 \\ -758.3 & -733.24 & -1429.43 \\ -661.84 & -568.1 & 1417.57 \\ -546.51 & -538.11 & -624.93 \\ -588.54 & -614.99 & 1093.27 \end{bmatrix}$$

The left data obtained is:

$$\mathbf{M} = \begin{bmatrix} 6.6 & & \\ & 17.8 & \\ & & 7.4 \end{bmatrix}, \mathbf{L}^T = \begin{bmatrix} 0.2 & 17.4 & 5.8 \\ 16.4 & 17.2 & 11.6 \\ 6.2 & 2 & 1.6 \\ 1 & 7.6 & 19.6 \\ 6.4 & 15.8 & 4.4 \end{bmatrix}, \mathbf{V}^T = \begin{bmatrix} -281.73 & -1398.21 & -342.71 \\ -387.1 & -2466.95 & -501.57 \\ -204.44 & -1594.23 & -257.25 \\ -251.83 & -1686.61 & -326.33 \\ -412.44 & -2464.91 & -557.43 \end{bmatrix}$$

We compare the original values of v_j and w_i with their values that we just obtained. The error matrices obtained are:

$$\Delta \mathbf{W} = \mathbf{W} - \hat{\mathbf{W}} = 10^{-13} \times \begin{bmatrix} -160.30 & -10.230 & -97.770 \\ 112.55 & 10.230 & 95.500 \\ -159.16 & -12.510 & -40.930 \\ 42.060 & 5.6800 & 43.200 \\ -123.92 & -5.6800 & -75.030 \end{bmatrix},$$

$$\Delta \mathbf{V}^T = (\mathbf{V} - \hat{\mathbf{V}})^T = 10^{-14} \times \begin{bmatrix} 39.80 & 136.4 & 113.7 \\ 28.40 & 318.3 & 56.80 \\ 5.700 & 204.6 & 0 \\ 11.40 & 181.9 & 28.40 \\ 34.10 & 318.3 & 79.60 \end{bmatrix}$$

5.3.2 Case 2: Accommodating the D-term in Loewner Equations

The input data is given as follows:

$$\mathbf{M} = \begin{bmatrix} 67.77 & 83.18 & 72.22 & 63.35 & 60.05 \\ 32.78 & 13.05 & 46.88 & 39.53 & 85.61 \\ 3.9 & 23.78 & 62.33 & 11.37 & 24.44 \\ 61.07 & 94.65 & 42.63 & 47 & 15.3 \\ 30.7 & 56.72 & 64.91 & 62.46 & 64.78 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 74.73 & 40.3 & 19.89 & 91.26 & 87.95 \\ 28.78 & 53.46 & 23.47 & 23.39 & 53.76 \\ 0.55 & 64.37 & 33 & 38.24 & 31.92 \\ 72.83 & 92.92 & 99.67 & 18.06 & 7.84 \\ 47.65 & 15.87 & 65.27 & 90.93 & 0.14 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 34.82 & 16.27 & 48.19 & 47.82 & 89.09 \\ 18.1 & 57.55 & 49.21 & 68.34 & 10.01 \\ 36.22 & 31.58 & 17.33 & 3.43 & 25.24 \\ 6.26 & 35.01 & 91.1 & 93.11 & 2.94 \\ 86.04 & 48.26 & 86.87 & 95.59 & 8.9 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 26.25 & 61.43 & 24.43 & 84.97 & 69.76 \\ 98.77 & 80.6 & 75.02 & 31.83 & 33.53 \\ 17.33 & 81.98 & 76.97 & 64.88 & 73.02 \\ 4.8 & 97.21 & 53.01 & 69.19 & 39.78 \\ 2.63 & 38.59 & 82.89 & 13.91 & 73.06 \end{bmatrix},$$

$$k = 1$$

$$\bar{\mathbf{A}} = \mathbf{D}, \bar{\mathbf{B}} = \mathbf{C}, \bar{\mathbf{C}} = \mathbf{A}, \bar{\mathbf{D}} = \mathbf{M}$$

To accommodate the **D**-term, we do rank-revealing factorization:

$$\mathbf{D} = \begin{bmatrix} 67.77 & 83.18 & 72.22 & 63.35 & 60.05 \\ 32.78 & 13.05 & 46.88 & 39.53 & 85.61 \\ 3.9 & 23.78 & 62.33 & 11.37 & 24.44 \\ 61.07 & 94.65 & 42.63 & 47 & 15.3 \\ 30.7 & 56.72 & 64.91 & 62.46 & 64.78 \end{bmatrix} \tag{5.1}$$

$$= \begin{bmatrix} 67.77 & 83.18 & 72.22 & 63.35 & 60.05 \\ 32.78 & 13.05 & 46.88 & 39.53 & 85.61 \\ 3.9 & 23.78 & 62.33 & 11.37 & 24.44 \\ 61.07 & 94.65 & 42.63 & 47 & 15.3 \\ 30.7 & 56.72 & 64.91 & 62.46 & 64.78 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.2}$$

$$= \mathbf{D}_1\mathbf{D}_2 \tag{5.3}$$

This converts $\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}$ matrices into the following form:

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 26.25 & 61.43 & 24.43 & 84.97 & 69.76 & 0 & 0 & 0 & 0 & 0 \\ 98.77 & 80.6 & 75.02 & 31.83 & 33.53 & 0 & 0 & 0 & 0 & 0 \\ 17.33 & 81.98 & 76.97 & 64.88 & 73.02 & 0 & 0 & 0 & 0 & 0 \\ 4.8 & 97.21 & 53.01 & 69.19 & 39.78 & 0 & 0 & 0 & 0 & 0 \\ 2.63 & 38.59 & 82.89 & 13.91 & 73.06 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 34.82 & 16.27 & 48.19 & 47.82 & 89.09 \\ 18.1 & 57.55 & 49.21 & 68.34 & 10.01 \\ 36.22 & 31.58 & 17.33 & 3.43 & 25.24 \\ 6.26 & 35.01 & 91.1 & 93.11 & 2.94 \\ 86.04 & 48.26 & 86.87 & 95.59 & 8.9 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 74.73 & 40.3 & 19.89 & 91.26 & 87.95 & 67.77 & 83.18 & 72.22 & 63.35 & 60.05 \\ 28.78 & 53.46 & 23.47 & 23.39 & 53.76 & 32.78 & 13.05 & 46.88 & 39.53 & 85.61 \\ 0.55 & 64.37 & 33 & 38.24 & 31.92 & 3.9 & 23.78 & 62.33 & 11.37 & 24.44 \\ 72.83 & 92.92 & 99.67 & 18.06 & 7.84 & 61.07 & 94.65 & 42.63 & 47 & 15.3 \\ 47.65 & 15.87 & 65.27 & 90.93 & 0.14 & 30.7 & 56.72 & 64.91 & 62.46 & 64.78 \end{bmatrix},$$

The right data obtained is:

$$\mathbf{\Lambda} = \begin{bmatrix} 2.25 & & \\ & 9.21 & \\ & & 9.91 \end{bmatrix}, \mathbf{R} = \begin{bmatrix} 0.38, 0.95, 0.84 \\ 0.96, 0.61, 0.45 \\ 0.78, 0.37, 0.49 \\ 0.86, 0.22, 0.68 \\ 0.35, 0.63, 0.22 \end{bmatrix}, \mathbf{W} = \begin{bmatrix} 286.25 & 276.70 & 408.28 \\ -1759.57 & 1841.77 & 2058.04 \\ -521.40 & 300.58 & 289.27 \\ 644.00 & -575.03 & -741.88 \\ 5428.31 & -5017.30 & -5639.16 \end{bmatrix}$$

The left data obtained is:

$$\mathbf{M} = \begin{bmatrix} 8.51 & & \\ & 2.64 & \\ & & 10.44 \end{bmatrix}, \mathbf{L}^T = \begin{bmatrix} 0.72 & 0.4 & 0.98 \\ 0.15 & 0.84 & 0.58 \\ 0.36 & 0.99 & 0.42 \\ 0.6 & 0.31 & 0.91 \\ 0.56 & 0.56 & 0.28 \end{bmatrix}, \mathbf{V}^T = \begin{bmatrix} -778.11 & 183.72 & 4.05 \\ -483.22 & 160.38 & 41.48 \\ -3513.62 & 681.10 & -275.92 \\ -4481.97 & 764.56 & -418.68 \\ -827.83 & 142.34 & -58.44 \end{bmatrix}$$

Again comparing the original values of v_j and w_i with their values that we just obtained, we get the following error matrices:

$$\Delta\mathbf{W} = \mathbf{W} - \hat{\mathbf{W}} = 10^{-13} \times \begin{bmatrix} 68.78 & -25.58 & 5.12 \\ -43.20 & 4.55 & -4.55 \\ -42.06 & 14.78 & -3.41 \\ -37.52 & 19.33 & -7.96 \\ 136.42 & -9.09 & 0 \end{bmatrix},$$

$$\Delta\mathbf{V}^T = (\mathbf{V} - \hat{\mathbf{V}})^T = 10^{-13} \times \begin{bmatrix} 34.11 & -6.54 & 1.42 \\ -23.31 & 2.84 & -4.41 \\ -113.69 & 20.46 & -13.64 \\ 0 & -4.5500 & 0 \\ 45.47 & -7.67 & 4.26 \end{bmatrix}$$

5.4 Results and Discussion:

In order to verify our results for applying the Loewner framework on the climate change equation, we perform a series of experiments. The resulting error matrices are shown at the end of each experiment. As can be observed from the error matrices, the error obtained between the values that we obtained and the original values comes out in the range of 10^-14 to 10^-12. The accuracy of our implementation comes out to be quite high. Hence, we can say that our experiments are successful.

Chapter 6

Conclusion and Future Work

The objectives of this project were to

- Introduce a form of data-driven model order reduction to the climate change context.
- Verify its correctness
- Evaluate its results through various experiments with real-life data of different sizes.

In this project, we proposed a new application of data-driven model order reduction by introducing it to the field of climate change. We used Loewner framework for this task. Previous non-empirical methods used for model order reduction in climate change context had their strengths and shortcomings, which are explained in the previous sections of the report, and are addressed by the proposed empirical method.
We also proposed the conversion of the climate change equation into Linear-Time Invariant Control System equation form. This helped in the usage of Loewner framework for linear systems in this context.
The results are discussed for several experiments where real-life climatic data obtained from various sources is taken into consideration. The data comprised of different dimensions of matrices and the model reduction method performed exceptionally well on that.
This verifies that:

- Climate change equation based on linear stability problem can be efficiently converted to a set of Ordinary Differential Equations (ODEs) of the Linear System form by removing one variable (**D**-term) using Rank-Revealing factorization.
- The data-driven model reduction procedure for the above climate change equation can be successfully performed using Loewner framework with negligible error.

Future work in this field can be in the following directions:

- Using Loewner framework for other climate-based equations.
- Using Loewner framework in other fields where computationally expensive models are used.
- Using other data-driven Model Order Reduction methods in climate context.

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