UNIVALENT FUNCTIONS AND AREA PROBLEMS

M.Sc. Thesis

by

SAI RASMI RANJAN MOHANTY

DISCIPLINE OF MATHEMATICS

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INDIAN INSTITUTE OF TECHNOLOGY INDORE
CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **UNIVALENT FUNCTIONS AND AREA PROBLEMS** in the partial fulfilment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DISCIPLINE OF MATHEMATICS, Indian Institute of Technology Indore,** is an authentic record of my own work carried out during the time period from July 2016 to MAY 2018 under the supervision of Dr. Swadesh Kumar Sahoo, Associate Professor, Indian Institute of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Signature of the student with date
(SAI RASMI RANJAN MOHANTY)

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Signature of Thesis Supervisor with date
(Dr. SWADESH KUMAR SAHOO)

- -

SAI RASMI RANJAN MOHANTY has successfully given his MSc Oral Examination held on **May 18, 2018**.

Signature Supervisor of MSc thesis
Date:

Convener, DPGC
Date:

Sign. of PSPC Member #1
Date:

Sign. of PSPC Member #2
Date:

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ABSTRACT

KEYWORDS: Analytic functions, univalent functions, convex functions, starlike functions, hypergeometric functions, area maxima, and length maxima

This thesis contains a survey of basic properties of univalent functions in the analytic function theory. Mostly we focuses on the class of univalent functions in the unit disk in which each of them has a Taylor's series expansion with a specific normalized form. This class of functions is preserved under certain elementary transformations. The well-known Bieberbach theorem, the growth theorem, the distortion theorem, the Koebe 1/4-theorem, area theorems are presented in this thesis. The classical subclasses of univalent functions, namely, the class of convex and starlike functions are also studied including their characterizations. As a part of applications of above and other related properties considered in this thesis, we compute areas of image domains of the unit disk and its subdisks under functions of some special types looking into the fact that the image domains are bounded. These are also examined through several examples of functions and their graphs. Finally, in the line of area of regions, we expect that a number of problems can be studied to maximize length of image of unit circle over the class of univalent functions. A few analysis on the latter part are covered in the concluding chapter.

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CHAPTER 1

INTRODUCTION

We focus on the family of univalent functions in the unit disk. A univalent function does not take the same value more than once for two distinct points in the unit disk. The choice unit disk as a domain in this context comes from Riemann Mapping Theorem, which gives us an one-one analytic function from a simply connected proper subdomain of $\mathbb{C}$ onto the unit disk. Thus, whatever properties of univalent functions in the unit disk easily transformed into the properties of simply connected domains.

If f is analytic and univalent in the unit disk, then f has a Taylor's series expansion of the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots ,$$

where a_n's are Taylor's coefficients. If f is univalent then $f(z) - a_0$ is also univalent. If $a_1 \neq 0$, then $(f(z) - a_0)/a_1$ is also univalent. By [9, Theorems 11.2-11.3, pp. 383] we see that $a_1 = f'(0) \neq 0$. Thus, to convert into normalized form, we can consider functions of the form

$$\frac{f(z) - f(0)}{f'(0)}$$

to study its univalency. We denote this class of functions by $\mathcal{S}$. Therefore, we consider such type of normalized analytic and univalent functions in this context. Univalent function theory is a part of geometric function theory. It started in the beginning of twentieth century. In 1907 Koebe proved that range of each functions in the class $\mathcal{S}$, contains a disk $|w| < \rho$, where $\rho > 0$ is an absolute constant. After few year Bieberbach gave the value of ρ as $1/4$. One of the well known theorems, Bieberbach Theorem gives a bound for the coefficient of z^2 of the family $\mathcal{S}$, i.e. $|a_2| \leq 2$. and this inequality can be converted into equality for the function $k(z) = z(1 - ze^{i\theta})^{-2}$, θ is real, which is also a member of the class $\mathcal{S}$. In 1916, he conjectured that for every $f \in \mathcal{S}, |a_n| \leq n$ for every n. This conjecture is known as Bieberbach Conjecture. Finally in 1985, de Branges proved this conjecture. In the meanwhile, the proof of Bieberbach's theorem was investigated by a number of authors by introducing some subclasses of the class of univalent functions. Therefore, the study of several subclasses of the class $\mathcal{S}$ is highlighted in the literature. Bieberbach Theorem have many applications. One of them is Koebe one-quarter Theorem.

In **Chapter 2**, we consider univalent functions which are the member of the class $\mathcal{S}$ and these functions bears some special properties. In formal, we can say that this class preserves some elementary transformations. Also, we discuss *Area Theorem* which gives us the area of image of the unit disk under f in $\mathcal{S}$. This area theorem provides many results related to univalent functions. We survey some elegant application of *Area Theorem* and then we end this chapter with *Growth and Distortion Theorem* that sharply estimates the lower and upper bounds of absolute value of the functions in $\mathcal{S}$ and their derivatives. We shall use these theorems in the proof of some of the results discussed in the subsequent chapters.

In **Chapter 3**, we move down into the subclasses of the class $\mathcal{S}$. We survey the characterizations of these subclasses, namely, convex functions and starlike functions. We also state a theorem known as *Alexander's Theorem* that gives a beautiful relation between these two functions and then we gently move into **Chapter 4**, which is all about area of image domains of the unit disk under some special type of functions that is constructed using the functions from that class and subclasses, which is an elegant application of above surveyed facts and properties. The maximum area attained by $k(z) = z(1-z)^{-2}$ which is unbounded in unit disk so that we choose some special type of function from the class $\mathcal{S}$ and its subclasses whose image domain is bounded and we discuss some examples with graphs related to areas of these special type of functions.

Finally, we get into **Chapter 5** in which we survey the length problems for functions in $\mathcal{S}$. It is expected that a number of problems can be studied to maximize length of image of unit circle under the class of univalent functions and its subclasses.

CHAPTER 2

UNIVALENT FUNCTIONS

In this chapter we consider the class of normalized analytic and univalent functions defined in the unit disk, and some of the elementary properties which say that this class is preserved under certain elementary transformations. Some important well-known results are also presented which are useful to main objective of this thesis.

A function f is said to be *univalent* in a domain $D \subset \mathbb{C}$ if it never take the same value twice i.e if $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$ for z_1 and z_2 in D. Note that a univalent function need not be analytic in a domain. For instance, the function $f(z) = z/(1 - 3z + z^2)$ is univalent in the unit disk $\mathbb{D} := \{z : |z| < 1\}$, but it is not analytic there as there is a pole at $z = (3 - \sqrt{5})/2$. Indeed, we have

$$f(z_1) = f(z_2) \iff (z_1 - z_2)(1 - z_1 z_2) = 0 \iff z_1 = z_2.$$

An analytic function f *locally univalent at* z_0 is equivalent to the condition $f'(z_0) \neq 0$.

2.1. The class $\mathcal{S}$

This section, primarily concerned with the class $\mathcal{S}$ consisting of functions f analytic and univalent in the unit disk $\mathbb{D}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, i.e.,

$$\mathcal{S} := \{f : f \text{ is analytic and univalent with } f(0) = 0 \text{ and } f'(0) = 1\}.$$

Each $f \in \mathcal{S}$ has Taylor's series expansion of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad |z| < 1. \tag{2.1}$$

An analytic univalent function in a domains can be treated as conformal because of its angle preserving property. Indeed, this is an application of Rouché's Theorem (see [9, p. 384]).

Example 2.1. Consider $f(z) = z$ for $z \in \mathbb{D}$. It is clear that f is analytic in the unit disk. If $f(z_1) = f(z_2)$ for $z_1, z_2 \in \mathbb{D}$, then $z_1 = z_2$. Therefore f is univalent in $\mathbb{D}$. Also $f(0) = 0$ and $f'(0) = 1$. Hence $f \in \mathcal{S}$.

Example 2.2. Let $f(z) = z/(1-z)$ so that f be analytic in $\mathbb{D}$ and if $f(z_1) = f(z_2)$ for $z_1, z_2 \in \mathbb{D}$, then

$$\frac{z_1}{1-z_1} = \frac{z_2}{1-z_2} \iff z_1 = z_2.$$

Therefore, f is univalent in the unit disk. Also $f(0) = 0$ and $f'(z) = 1/(1-z)^2$ which gives $f'(0) = 1$. Hence $f \in \mathcal{S}$.

Example 2.3. Let $f(z) = z/(1-z^2)$ so that f be analytic in $\mathbb{D}$. Now,

$$\frac{z_1}{1-z_1^2} = \frac{z_2}{1-z_2^2} \iff (z_1 - z_2)(1 + z_1 z_2) = 0 \iff z_1 = z_2$$

for $z_1, z_2 \in \mathbb{D}$, since $z_1 z_2 \neq -1$. Therefore f is univalent in $\mathbb{D}$. Also $f(0) = 0$ and $f'(0) = 1$. Hence $f \in \mathcal{S}$.

Example 2.4. Consider the Koebe function $k(z) = \dfrac{z}{(1-z)^2}$. Since the Koebe function can be written as

$$k(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4},$$

it can easily be seen that it maps the unit disk onto $\mathbb{C} \setminus (-\infty, -1/4]$. It is easy to check from the relations

$$\frac{z_1}{(1-z_1)^2} = \frac{z_2}{(1-z_2)^2} \iff (z_1 - z_2)(1 - z_1 z_2) = 0,$$

$k(0) = 0$ and $k'(0) = 1$ that $k \in \mathcal{S}$.

Note that the sum of two functions in the class $\mathcal{S}$ need not be in $\mathcal{S}$. For example, consider two functions of $\mathcal{S}$:

$$f(z) = \frac{z}{1-z} \quad \text{and} \quad g(z) = \frac{z}{1+iz}.$$

Now

$$h(z) = f(z) + g(z) = \frac{z}{1-z} + \frac{z}{1+iz}$$

and

$$h'(z) = \frac{1}{(1-z)^2} + \frac{1}{(1+iz)^2}.$$

At $z_0 = (1+i)/2, h'(z_0) = 0$. This shows that h is not locally univalent at z_0 and hence not univalent.

2.2. Elementary transformations

The class $\mathcal{S}$, preserved under some elementary transformations, see [3, pp. 27–28]. That helps to construct examples of functions belonging to the class $\mathcal{S}$.

(i) **Conjugation.** *If* $f \in \mathcal{S}$ and $g(z) = \overline{f(\bar{z})}$, *then* $g \in \mathcal{S}$.

Proof. Suppose $f \in \mathcal{S}$ and consider the function $h : \mathbb{D} \to \mathbb{D}$ defined as $g(z) = \bar{z}$ which is univalent and analytic in $\mathbb{D}$. Set $g(z) = \overline{f(\bar{z})} = (h \circ f \circ h)(z)$. Clearly, g is univalent and analytic, since composition of two univalent functions is univalent (same for analytic function). Also $g(0) = 0$ and $g'(0) = 1$. Hence $g \in \mathcal{S}$.

(ii) **Rotation.** *If* $f \in \mathcal{S}$ *and* $g(z) = e^{-i\theta} f(e^{i\theta} z)$, *then* $g \in \mathcal{S}$.

Proof. Suppose $f \in \mathcal{S}$. Consider the function $h : \mathbb{D} \to \mathbb{D}$ and $j : \mathbb{D} \to \mathbb{D}$ defined as $h(z) = e^{i\theta} z$ and $j(z) = e^{-i\theta} z$ which are univalent and analytic in $\mathbb{D}$. Set $g(z) = e^{-i\theta} f(e^{i\theta} z) = (j \circ f \circ h)(z)$. Clearly g is univalent and analytic, since composition of two univalent functions is univalent (same for analytic function). Therefore

$$g'(z) = e^{-i\theta} e^{i\theta} f'(e^{i\theta} z) = f'(e^{i\theta} z)$$

so that $g(0) = f(0) = 0$ and $g'(0) = f'(0) = 1$. Hence $g \in \mathcal{S}$.

(iii) **Dilation.** *If* $f \in S, 0 < r < 1$, *and* $g(z) = r^{-1} f(rz)$, *then* $g \in \mathcal{S}$.

Proof. Suppose $f \in \mathcal{S}$. Consider the function $h, j : \mathbb{D} \to \mathbb{D}$ defined as $h(z) = rz$ and $j(z) = z/r$ which are univalent and analytic. Let $g(z) = r^{-1} f(rz) = (j \circ f \circ h)(z)$, which is univalent and analytic, since composition of two univalent functions is univalent (same for analytic function). Therefore $g'(z) = f'(rz)$ so that $g(0) = 0$ and $g'(0) = f'(0) = 1$. Hence $g \in \mathcal{S}$.

(iv) **Disk Automorphism.** *If* $f \in \mathcal{S}$ *and*

$$g(z) = \frac{f\left(\dfrac{z + z_0}{1 + \bar{z}_0 z}\right) - f(z_0)}{(1 - |z_0|^2) f'(z_0)},$$

for any $|z_0| < 1$, *then* $g \in \mathcal{S}$.

Proof. Suppose that $f \in \mathcal{S}$ and let $h(z) = (z + z_0)/(1 + \bar{z}_0 z)$ be the Möbious transformation which maps the unit disk $\mathbb{D}$ conformally onto itself with $h(0) = z_0$. Now set

$$g(z) = \frac{f(h(z)) - f(z_0)}{(1 - |z_0|^2)f'(z_0)},$$

which is univalent and analytic in $\mathbb{D}$ with $g(0) = 0$ and

$$g'(z) = \frac{h'(z)f'(h(z))}{(1 - |z_0|^2)f'(z_0)} = \frac{f'(h(z))}{(1 - \bar{z}_0 z)^2 f'(z_0)}$$

so that $g'(0) = 1$. Hence $g \in \mathcal{S}$.

(v) **Range Transformation.** *If $f \in \mathcal{S}$, $\phi : f(\mathbb{D}) \to \mathbb{C}$ is analytic and univalent on $f(\mathbb{D})$ and*

$$g(z) = \frac{(\phi \circ f)(z) - \phi(0)}{\phi'(0)},$$

then $g \in \mathcal{S}$.

Proof. Suppose that $f \in \mathcal{S}$ and let $\phi(z) : f(\mathbb{D}) \to \mathbb{C}$ be analytic and univalent in $f(\mathbb{D})$. Now set

$$g(z) = \frac{(\phi \circ f)(z) - \phi(0)}{\phi'(0)},$$

so that g is clearly univalent and analytic in $\mathbb{D}$. Furthermore,

$$g'(z) = \frac{f'(z)\phi(f(z))}{\phi'(0)},$$

$g(0) = 0$ and $g'(0) = 1$. Thus $g \in \mathcal{S}$.

(vi) **Omitted value Transformation.** *If $f \in \mathcal{S}$ with $f(z) \neq w$ and*

$$g(z) = \frac{wf(z)}{w - f(z)},$$

then $g \in \mathcal{S}$.

Proof. Suppose that $f \in \mathcal{S}$ and $f(z) \neq w$, let

$$g(z) = \frac{wf(z)}{w - f(z)}.$$

Now take $h(z) = w\xi/(w - \xi)$. It is clear that $h(z)$ is univalent and analytic, if $w \neq \xi$. Now set $g(z) = (hof)(z)$, which is univalent and analytic in $\mathbb{D}$. Therefore

$$g'(z) = \frac{w^2 f'(z)}{(w - f(z))^2},$$

since $w \neq f(z)$ with $g(0) = 0$ and $g'(0) = 1$. Thus $g \in \mathcal{S}$. $\square$

Theorem 2.5. *If f is analytic on $\mathbb{D}$ with $0 \notin f(\mathbb{D})$, then there exists an analytic function h on $\mathbb{D}$ with $h^2 = f$.*

Proof. For any other $w \in \mathbb{D}$, let

$$g(w) = g(0) + \int_0^w \frac{f'(z)}{f(z)} dz,$$

with $e^{g(0)} = f(0)$. By fundamental theorem of calculus, it can be written as

$$g'(z) = \frac{f'(z)}{f(z)}.$$

Now $f(z) \neq 0$ for $z \in \mathbb{D}$, so that g is analytic on $\mathbb{D}$. We can see that

$$(fe^{-g})'(w) = f'(w)e^{-g(w)} - g'(w)e^{-g(w)}f(w) = e^{-g(w)}(f'(w) - g'(w)f(w)) = 0.$$

Therefore, $f(w) = e^{g(w)}$. Let $h(z) = e^{g(z)/2}$, so that h is analytic on $\mathbb{D}$ with $h^2(z) = f(z)$ for every $z \in \mathbb{D}$. $\square$

Theorem 2.6. *If $f \in \mathcal{S}$, then there exist an odd function $h \in \mathcal{S}$ such that $h(z) = \sqrt{f(z^2)}$ for every $z \in \mathbb{D}$.*

Proof. If $f \in \mathcal{S}$, then

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots .$$

Rearranging this

$$\frac{f(z)}{z} = 1 + a_2 z + a_3 z^2 + \cdots ,$$

so $f(z)/z$ is non zero and analytic on $\mathbb{D}$. By Theorem 2.5, there exist an analytic function F on $\mathbb{D}$ with $F^2(z) = f(z)/z$. Let us define $h(z) = zF(z^2)$. It is clear that h is odd function with

$$h^2(z) = z^2 F^2(z^2) = \frac{z^2 f(z^2)}{z^2} = f(z^2),$$

also $h(0) = 0$ and $h'(0) = F(0) = 1$. Let $z_1, z_2 \in \mathbb{D}$, to show univalency we are assuming that $h(z_1) = h(z_2)$ i.e., $f(z_1^2) = f(z_2^2)$ so that $z_1 = z_2$ or $z_1 = -z_2$, since f is univalent. If $z_1 = z_2$, then there is nothing to show. If $z_1 = -z_2$, then $h(z_1) = -h(-z_1) = -h(z_2)$, since h is odd. This leads to a contradiction. Hence $h \in \mathcal{S}$. $\square$

(vii) **Square root Transformation.** *If $f \in \mathcal{S}$ and $g(z) = \sqrt{f(z^2)}$, then $g \in \mathcal{S}$. Here, the branch is chosen so that*

$$\sqrt{f(z^2)} = \exp\left(\frac{1}{2}\mathrm{Log}\, f(z^2)\right)$$

with a principal value of the logarithmic function.

Proof. Suppose that $f \in \mathcal{S}$ and $g(z) = \sqrt{f(z^2)}$. Now expanding

$$\begin{aligned} g(z) = \sqrt{f(z^2)} &= (z^2 + a_2 z^4 + a_3 z^6 + \cdots)^{\frac{1}{2}} \\ &= z(1 + a_2 z^2 + a_3 z^4 + \cdots)^{\frac{1}{2}} \\ &= z + b_3 z^3 + b_5 z^4 + \cdots , \end{aligned}$$

which is an odd function. By Theorem 2.6, g is univalent and analytic on $\mathbb{D}$ with $g(0) = 0$ and $g'(0) = 1$. Hence $g \in \mathcal{S}$.

2.3. Area Theorems

In this section we consider two versions of the area theorem, one of the versions gives the area of image of unit disk under a function $f \in \mathcal{S}$ in terms of the coefficients a_n in the Taylor expansion of f. By using the area theorem, there are many geometric properties of the class $\mathcal{S}$ proved in the literature. For instance, the Koebe one-quarter theorem says that the ranges of class $\mathcal{S}$ contain a disk $\{w : |w| < 1/4\}$. Bieberbach's Theorem for second coefficient of the class $\mathcal{S}$ says that $|a_2| \leq 2$. One of the famous conjectures i.e. Bieberbach conjecture states that $|a_n| \leq n$ for all $n \geq 2$ which was settled in 1985 by de Branges. Second version of area theorem provides the area problem for functions belonging to a class of meromorphic functions having a simple pole at ∞ with residue 1. Detailed information on the above problems is presented below.

First we are proving area theorem for analytic function, reader can refer [4].

Theorem 2.7. *Suppose that*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is analytic in a closed disk $\overline{E_r} : |z| \leq r$. *If* $A(r)$ *is area of* $f(\overline{E_r})$, *then*

$$A(r) = \pi \sum_{n=0}^{\infty} n |a_n|^2 r^{2n}.$$

Proof. We know from calculus that

$$A(r) = \int\int_{f(\overline{E_r})} du\, dv = \int\int_{\overline{E_r}} \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dx\, dy, \tag{2.2}$$

where $z = x + iy$ and $w = u + iv$. Now

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}.$$

Since f is analytic, the partial derivative are continuous and satisfying the Cauchy-Riemann equations. Therefore

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & -v_x \\ v_x & u_x \end{vmatrix} = u_x^2 + v_x^2 = |f'(z)|^2.$$

From equation (2.2), we have

$$A(r) = \int\int_{|z|\leq r} |f'(z)|^2 dxdy.$$

Changing to polar coordinate, we get

$$A(r) = \int_0^r \int_0^{2\pi} |f'(\rho e^{i\theta})|^2 \rho d\theta d\rho.$$

Now

$$f'(\rho e^{i\theta}) = a_1 + 2a_\rho e^{i\theta} + 3a + 3\rho^3 e^{2i\theta} + \cdots = \sum_{n=1}^{\infty} na_n(\rho e^{i\theta})^{n-1}.$$

Therefore,

$$\begin{aligned} A(r) &= \int_0^r \int_0^{2\pi} |f'(\rho e^{i\theta})|^2 \rho d\theta d\rho \\ &= \int_0^r \rho d\rho \int_0^{2\pi} f'(\rho e^{i\theta})\overline{f'(\rho e^{i\theta})} d\theta \\ &= \int_0^r \rho d\rho \int_0^{2\pi} \left(\sum_{n=1}^{\infty} na_n(\rho e^{i\theta})^{n-1}\right)\left(\sum_{m=1}^{\infty} ma_m(\rho e^{i\theta})^{m-1}\right) \\ &= \int_0^r 2\pi \sum_{n=1}^{\infty} n^2|a_n|^2 \rho^{2n-2} \rho d\rho \\ &= \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}, \end{aligned}$$

since the product are of the form $e^{i(m-n)\theta}, m \neq n$, are integrate to zero over range 0 to 2π. This complete the proof. □

Definition (Laurent series). If $f(z)$ is analytic in the annulus enclosed by the concentric circles C_1 and C_2 centred at $z = z_0$ and of radii r_1 and r_2 respectively and $r_2 < r_1$, then

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} b_n (z - z_0)^{-n},$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

and

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{-n+1}} d\zeta.$$

In particular, for center at origin

$$f(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^{-n},$$

Rearranging this series, we can write

$$\frac{1}{a_1}\left(f(z) - \sum_{n=2}^{\infty} a_n z^n\right) = z + \sum_{n=0}^{\infty} \frac{b_n}{a_1} z^{-n}.$$

Now we rewrite of this form

$$g(z) = z + \sum_{n=0}^{\infty} c_n z^{-n}, \quad c_n \in \mathbb{C},$$

let us denotes Σ, by the set of such functions. If $g \in \Sigma$ then g maps $\Delta := \{z \in \mathbb{C} : |z| > 1\}$ onto complement of a compact connected set E and $g(z)$ is analytic and univalent in the domain Δ exterior to $\mathbb{D}$, except for a simple pole at infinity with residue 1. Consider the subclass Σ' of functions $g \in \Sigma$ for which $g(z) \neq 0$ in Δ i.e $0 \in E$. The functions belongs to Σ' by adjustment of constant term. If $f \in \mathcal{S}$ and g is defined by inversion,

$$g(z) = (f(z^{-1}))^{-1} = z - a_2 + (a_2^2 - a_3)z^{-1} + \cdots, \quad |z| > 1,$$

then g belongs to Σ'. This transformation is called inversion. It establishes a one-one correspondence between $\mathcal{S}$ and Σ. We denote by $\overline{\Sigma}$, the subclass of all functions $g \in \Sigma$ whose omitted set E has two dimensional Lebesgue measure zero.

Theorem 2.8. (Area Theorem). *If $g \in \Sigma$, then*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1, \tag{2.3}$$

with equality if and only if $g \in \bar{\Sigma}$.

Proof. Let E be the set omitted by g. For $r > 1$, let C_r be the image under g of the circle $|z| = r$. Since g is univalent, C_r is a simple closed curve which encloses a domain $E_r \supset E$. By Green's theorem, for $w = g(z)$

$$\begin{aligned}
\mathrm{Area(E_r)} &= \frac{1}{2i}\int_{C_r} \bar{w}dw \\
&= \frac{1}{2i}\int_{|z|=r} \overline{g(z)}g'(z)dz \\
&= \frac{1}{2i}\int_{|z|=r} \left(\bar{z} + \sum_{n=0}^{\infty} \overline{b_n}(\overline{z})^{-n}\right)\left(1 - \sum_{m=1}^{\infty} mb_m z^{-m-1}\right) dz \\
&= \frac{1}{2i}\int_0^{2\pi} \left(re^{-i\theta} + \sum_{n=0}^{\infty} \overline{b_n}r^{-n}e^{in\theta}\right)\left(1 - \sum_{m=1}^{\infty} mb_m r^{-m-1}e^{-i(m+1)}\theta\right) ire^{i\theta}d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left(re^{-i\theta} + \sum_{n=0}^{\infty} \overline{b_n}r^{-n}e^{in\theta}\right)\left(re^{i\theta} - \sum_{m=1}^{\infty} mb_m r^{-m}e^{-im\theta}\right) d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right) d\theta \\
&= \pi\left(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right).
\end{aligned}$$

Area is non-negative so that

$$\pi\left(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right) \geq 0$$

that implies $r^2 \geq \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}$. Now letting $r \to 1$, we obtain

$$\mathrm{Area}(E) = \lim_{r\to 1}\mathrm{Area}(E_r) = \pi\left(1 - \sum_{n=1}^{\infty} n|b_n|^2\right).$$

This implies

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1,$$

since $\mathrm{Area}(E) \geq 0$. Equality holds iff $\mathrm{Area(E)} = 0$ i.e., $\sum_{n=0}^{\infty} n|b_n|^2 = 1$. □

Corollary 2.9. *If $g \in \Sigma$, then $|b_1| \leq 1$, with equality if and only if g has the form $g(z) = z + b_0 + b_1 z^{-1}$, where $|b_1| = 1$.*

Proof. We know from (2.3) that $n|b_n|^2 \leq \sum n|b_n|^2 \leq 1$. Now for $n = 1$, $|b_1| \leq 1$. If $g(z) = z + b_0 + b_1 z^{-1}$, then $|b_1| = 1$.

Conversely, if $|b_1| = 1$ and $g \in \Sigma$ then to prove $g(z) = z + b_0 + b_1 z^{-1}$, take $b_1 = e^{i\theta}$. Now by Theorem 2.8, we obtain

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$$

i.e. $|b_1|^2 + 2|b_2|^2 + 3|b_3|^2 + \cdots \leq 1$. Putting the value of b_1, we obtain

$$1 + 2|b_2|^2 + 3|b_3|^2 + \cdots \leq 1.$$

That implies $b_2 = 0,\ b_3 = 0,\ b_3 = 0$ and so on. Hence $g(z) = z + b_0 + b_1 z^{-1}$. □

Corollary 2.9 is most important to prove Bieberbach's theorem which estimates the second coefficient of function in the class $\mathcal{S}$. This theorem gives the basic result for Biererbach conjecture.

Theorem 2.10. (Bieberbach's Theorem). *If $f \in \mathcal{S}$ and it is of the form* (2.1)*, then $|a_2| \leq 2$ with equality if and only if f is a rotation of the Koebe function.*

Proof. Let $f \in \mathcal{S}$. The class $\mathcal{S}$ preserved square root transformation so that for $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, $g(z) = (f(z^2))^{1/2} \in \mathcal{S}$. Now applying inversion transformation to g, we obtain $(g(1/z))^{-1} \in \Sigma$ i.e. $f((1/z^2))^{-1/2} \in \Sigma$. But

$$f((1/z^2))^{-1/2} = z - \frac{a_2}{2} z^{-1} + \left(\frac{3a_2^2}{8} - \frac{a_3^3}{2}\right) z^{-3} + \cdots .$$

From Corollary 2.9, it is clear that $\left|-\frac{a_2}{2}\right| = \left|\frac{a_2}{2}\right| \leq 1$ that implies $|a_2| \leq 2$. □

One of the applications of Bieberbach's theorem is covering theorem which introduce by Koebe called Koebe one-quarter theorem. Each function $f \in \mathcal{S}$ satisfies the condition $f(0) = 0$ so that range of f contains some disk center at origin. Koebe first introduce the ranges of all class $\mathcal{S}$ functions contains a common disk $\{w : |w| < \rho\}$, where ρ is an absolutely constant. The Koebe function $k(z) = z/(1-z)^2$ shows that $\rho \leq 1/4$. Bieberbach later proved that ρ may be taken $1/4$.

Theorem 2.11. (Koebe one-quarter Theorem). *The range of every function of class $\mathcal{S}$ contains the disk* $\{w : |w| < 1/4\}$.

Proof. If a function $f \in \mathcal{S}$ omits the value of $w \in \mathbb{C}$ i.e $f(z) \neq w$ i.e $w \notin f(\mathbb{D})$, then

$$g(z) = \frac{wf(z)}{w - f(z)} = b_0 + b_1 z + b_2 z^2 + \cdots ,$$

is well defined, analytic and univalent in $\mathbb{D}$. Equivalently, $g(z)(w - f(z)) = wf(z)$. It can be written as

$$(b_0 + b_1 z + b_2 z^2 + \cdots)(w - (z + a_2 z^2 + a_3 z^3 + \cdots)) = w(z + a_2 z^2 + a_3 z^3 + \cdots).$$

Taking Cauchy product and comparing coefficients of z^n, we obtain

$$b_0 = 0$$

$$wb_1 = w$$

$$wb_2 - b_1 = wa_2$$

$$wb_3 - b_1 a_2 - b_2 = wa_2, \ldots$$

Therefore, $b_1 = 1, b_2 = \dfrac{1}{w}(wa_2 + 1) = a_2 + \dfrac{1}{w}$ and so on. Thus

$$g(z) = z + (a_2 + \frac{1}{w})z^2 + \cdots .$$

From Bieberbach's theorem (Theorem 2.10), $|a_2 + \frac{1}{w}| \leq 2$, since $g \in \mathcal{S}$ and also $|a_2| \leq 2$, since $g \in \mathcal{S}$. Therefore

$$\left|\frac{1}{w}\right| - |a_2| \leq \left|\frac{1}{w} + a_2\right| \leq 2.$$

Simplifying this, we obtain $|\frac{1}{w}| \leq 4$ i.e. $|w| \geq \frac{1}{4}$. Thus every omitted value must lie outside the disk $|w| < \frac{1}{4}$.

If $\left|\dfrac{1}{w}\right| = 4$, we can write

$$\left|\frac{1}{w}\right| - |a_2| \leq 2 \text{ i.e. } \left|\frac{1}{w}\right| \leq 2 + |a_2| \leq 4.$$

That implies $|a_2| = 2$. Hence f is some rotation of the Koebe function. □

2.4. Growth and Distortion Theorems

This section provides the sharp upper and lower bounds for $|f'(z)|$ for all $f \in \mathcal{S}$, which is important consequence of the Koebe distortion theorem. Distortion theorem is one of the applications of Bieberbach's theorem $|a_2| \leq 2$. By using distortion theorem, growth theorem is proved, which provides the sharp upper and lower bounds of $|f(z)|$ for all $f \in \mathcal{S}$. To prove the distortion theorem, we use the following theorem.

Theorem 2.12. *For each $f \in \mathcal{S}$,*

$$\left| \frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2} \right| \leq \frac{4r}{1-r^2}, \quad |z| = r < 1.$$

This equality holds for a rotation of the Koebe function in $\mathbb{D}$.

Proof. Given $f \in \mathcal{S}$ and fix $\zeta \in \mathbb{D}$, by disk of automorphism we can write

$$F(z) = \frac{f\left(\dfrac{z+\zeta}{1+\overline{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)}.$$

Now take $w = (z+\zeta)/(1+\overline{\zeta}z)$ and the Taylor series expansion of f at $w = \zeta$ given by

$$f(w) = f(\zeta) + (w-\zeta)f'(\zeta) + \frac{(w-\zeta)^2}{2!}f''(\zeta) + \cdots .$$

Putting the value of w in the above equation, we rewrite this as

$$f(w) = f(\zeta) + \left(\frac{z+\zeta}{1+\overline{\zeta}z} - \zeta\right) f'(\zeta) + \frac{\left(\dfrac{z+\zeta}{1+\overline{\zeta}z} - \zeta\right)^2}{2!} f''(\zeta) + \cdots .$$

By simple calculation, we get

$$f(w) = f(\zeta) + z\left(\frac{1-|\zeta|^2}{1+\overline{\zeta}z}\right) f'(\zeta) + \frac{z^2}{2}\frac{(1-|\zeta|^2)^2}{(1+\overline{\zeta}z)^2} f''(\zeta) + \cdots .$$

Now

$$\begin{aligned}
F(z) &= \frac{f(w)-f(\zeta)}{(1-|\zeta|^2)f'(\zeta)} \\
&= \frac{1}{(1-|\zeta|^2)f'(\zeta)}\left(f(\zeta) + z\left(\frac{1-|\zeta|^2}{1+\overline{\zeta}z}\right) f'(\zeta) + \frac{z^2}{2}\frac{(1-|\zeta|^2)^2}{(1+\overline{\zeta}z)^2} f''(\zeta) + \cdots - f(\zeta)\right) \\
&= z(1+\overline{\zeta}z)^{-1} + \frac{z^2}{2}(1-|\zeta|^2)(1+\overline{\zeta}z)^{-2}\frac{f''(\zeta)}{f'(\zeta)} + \cdots \\
&= z(1-\overline{\zeta}z+(\overline{\zeta}z)^2+\cdots) + \frac{z^2}{2}(1-|\zeta|^2)(1-2\overline{\zeta}z+3(\overline{\zeta}z)^2-\cdots)\frac{f''(\zeta)}{f'(\zeta)} + \cdots \\
&= z + z^2\left(\frac{(1-|\zeta|^2)}{2}\frac{f''(\zeta)}{f'(\zeta)} - \overline{\zeta}\right) + \cdots .
\end{aligned}$$

Since $F(z) \in \mathcal{S}$, by Bieberbach's theorem (Theorem 2.10), we obtain

$$\left| \frac{(1-|\zeta|^2)}{2}\frac{f''(\zeta)}{f'(\zeta)} - \overline{\zeta} \right| \leq 2,$$

multiplying by $2|\zeta|/(1-|\zeta|^2)$ both sides, we obtain

$$\left| |\zeta|\frac{f''(\zeta)}{f'(\zeta)} - \frac{2|\zeta|^2}{1-|\zeta|^2} \right| \leq \frac{4|\zeta|}{1-|\zeta|^2}.$$

Now replacing ζ by z, above inequality can be written as

$$\left| z\frac{f''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2} \right| \leq \frac{4|z|}{1-|z|^2}.$$

Now the Koebe function $k(z) = \dfrac{z}{(1-z)^2}$ gives $k'(z) = \dfrac{1+z}{(1-z)^3}$ and hence

$$\frac{k''(z)}{k'(z)} = \frac{2(2+z)}{1-z^2}.$$

Therefore

$$\left| \frac{zk''(z)}{k'(z)} - \frac{2|z|^2}{1-|z|^2} \right| = \left| \frac{2(2+z)z}{1-z^2} - \frac{2z^2}{1-z^2} \right| = \left| \frac{4z}{1-z^2} \right|.$$

This complete the proof. $\square$

Theorem 2.13. (Distortion Theorem). *For each $f \in \mathcal{S}$,*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \quad |z| = r < 1.$$

For each $z \in \mathbb{D}$, equality holds if and only if f is a rotation of the Koebe function.

Proof. From Theorem 2.12, we can write

$$\frac{-4r}{1-r^2} \leq \operatorname{Re}\left(\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2} \right) \leq \frac{4r}{1-r^2},$$

since $|\alpha| \leq c$ implies that $-c \leq \operatorname{Re}(\alpha) \leq c$. It can be written as

$$\frac{-4r}{1-r^2} + \frac{2r^2}{1-r^2} \leq \operatorname{Re}\left(\frac{zf''(z)}{f'(z)} \right) \leq \frac{4r}{1-r^2} + \frac{2r^2}{1-r^2}$$

or

$$\frac{2r(r-2)}{1-r^2} \leq \operatorname{Re}\left(\frac{zf''(z)}{f'(z)} \right) \leq \frac{2r(r+2)}{1-r^2}. \tag{2.4}$$

Since $f'(z) \neq 0$ and $f'(0) = 1$, we can choose a single valued branch of $\log f'(z)$ which vanishes at origin i.e. $\log f'(0) = 0$. Now

$$r\frac{\partial}{\partial r}\operatorname{Re}\left(\log f'(z)\right) = r\frac{\partial}{\partial r}\operatorname{Re}\left(\log|f'(z)| + i\arg f'(z)\right) = r\frac{\partial}{\partial r}\log|f'(z)|$$

and

$$r\frac{\partial}{\partial r}\log(f'(z)) = r\frac{\partial}{\partial z}\log f'(z)\frac{\partial z}{\partial r} = r\frac{f''(z)}{f'(z)}e^{i\theta} = z\frac{f''(z)}{f'(z)}, \quad z = re^{i\theta}.$$

It is obvious that

$$\operatorname{Re}\left(r\frac{\partial}{\partial r}\log(f'(z)) \right) = \operatorname{Re}\left(z\frac{f''(z)}{f'(z)} \right).$$

But

$$\operatorname{Re}\left(r\frac{\partial}{\partial r}\log(f'(z))\right) = \operatorname{Re}\left(r\frac{\partial}{\partial r}(\log|f'(z)| + i\arg f'(z))\right) = r\frac{\partial}{\partial r}\log|f'(z)|.$$

Therefore

$$\operatorname{Re}\left(z\frac{f''(z)}{f'(z)}\right) = r\frac{\partial}{\partial r}\log|f'(z)|. \tag{2.5}$$

Now from (2.4), it is clear that

$$\frac{2r(r-2)}{1-r^2} \leq r\frac{\partial}{\partial r}\log|f'(z)| \leq \frac{2r(r+2)}{1-r^2}.$$

This can be written as

$$\frac{2(r-2)}{1-r^2} \leq \frac{\partial}{\partial r}\log|f'(z)| \leq \frac{2(r+2)}{1-r^2}. \tag{2.6}$$

Now fix θ and integrate (2.6) with respect to r from 0 to R, we obtain

$$\int_0^R \frac{2(r-2)}{1-r^2}dr \leq \int_0^R \frac{\partial}{\partial r}\log|f'(z)|dr \leq \int_0^R \frac{2(r+2)}{1-r^2}dr. \tag{2.7}$$

Now

$$\begin{aligned}
\int_0^R \frac{2(r-2)}{1-r^2}dr &= 2\int_0^R \left(\frac{r-1}{1-r^2} - \frac{1}{1-r^2}\right)dr \\
&= -2\int_0^R \frac{1}{1+r}dr - 2\int_0^R \frac{1}{1-r^2}dr \\
&= \left[-2\log(1+r) - \log\left(\frac{1+r}{1-r}\right)\right]_0^R \\
&= \log\left(\frac{1-R}{(1+R)^3}\right).
\end{aligned}$$

Similarly,

$$\int_0^R \frac{2(r+2)}{1-r^2}dr = \log\left(\frac{1+R}{(1-R)^3}\right).$$

Hence from (2.7), we obtain

$$\log\left(\frac{1-R}{(1+R)^3}\right) \leq \log|f'(Re^{i\theta})| \leq \log\left(\frac{1+R}{(1-R)^3}\right).$$

For $0 \leq r \leq R$,

$$\log\left(\frac{1-r}{(1+r)^3}\right) \leq \log|f'(z)| \leq \log\left(\frac{1+r}{(1-r)^3}\right).$$

For the Koebe function $k(z) = z/(1-z)^2$, equality holds for $z \in \mathbb{D}$. Because $k'(z) = (1+z)/(1-z)^3$ and for $z = r$,

$$k'(z) = \frac{1+r}{(1-r)^3}.$$

Also for $z = -r$,

$$k'(z) = \frac{1-r}{(1+r)^3}.$$

Hence

$$\frac{1-r}{(1+r)^3} = |k'(z)| = \frac{1+r}{(1-r)^3}.$$

Conversely, if equality holds, we need to show f is a rotation of the Koebe function. Now

$$\frac{1-r}{(1+r)^3} = |k'(z)| = \frac{1+r}{(1-r)^3}.$$

From the proof of the distortion theorem, above equality is equivalent to

$$\frac{2(r-2)}{1-r^2} = \frac{\partial}{\partial r} \log |(f'(z))| = \frac{2(r+2)}{1-r^2}, \quad 0 \leq r \leq R.$$

But from (2.5), we clearly see that

$$\frac{\partial}{\partial r} \log |(f'(z))| = \frac{1}{r} \mathrm{Re} \left(z \frac{f''(z)}{f'(z)} \right),$$

this is equivalent to

$$\frac{\partial}{\partial r} \log |(f'(re^{i\theta})| = \mathrm{Re} \left(e^{i\theta} \frac{f''(z)}{f'(z)} \right).$$

In particular,

$$-4 = \mathrm{Re} \left(e^{i\theta} \frac{f''(0)}{f'(0)} \right) = 4$$

i.e. $\mathrm{Re} \left(e^{i\theta} \dfrac{f''(0)}{f'(0)} \right) = \pm 4.$

Therefore,

$$\mathrm{Re} \left(e^{i\theta} \left(\frac{2a_2 + 6a_3 z^2 + \cdots}{1 + 2a_2 + \cdots} \right) \right) = \pm 4,$$

so that $|e^{i\theta}(2a_2)| = 4$ i.e. $|a_2| = 2$.

Hence by Bieberbach's theorem f must be a rotation of the Koebe function. This completes the proof. □

Theorem 2.14. (Growth Theorem). *For each $f \in \mathcal{S}$,*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \quad |z| = r < 1.$$

For each $z \in \mathbb{D}$, equality holds if and only if f is a rotation of the Koebe function.

Proof. For each $f \in \mathcal{S}$, fix $z = re^{i\theta}$ with $0 < r < 1$. It is clear that

$$f(z) = \int_0^r f'(\rho e^{i\theta}) e^{i\theta} d\rho,$$

since $f(0) = 0$.

Now by the distortion theorem

$$|f(z)| \leq \int_0^r |f'(\rho e^{i\theta})| d\rho \leq \int_0^r \frac{1+\rho}{(1-\rho)^3} d\rho. \tag{2.8}$$

But

$$\int_0^r \frac{1+\rho}{(1-\rho)^3} d\rho = \int_0^r \frac{1+\rho-1+1}{(1-\rho)^3} d\rho = \int_0^r \left(-\frac{1-\rho}{(1-\rho)^3} + \frac{2}{(1-\rho)^3} \right) d\rho.$$

By simple integration over 0 to r, we obtain

$$\int_0^r \frac{1+\rho}{(1-\rho)^3} d\rho = \frac{r}{(1-r)^2}.$$

From (2.8), we get

$$|f(z)| \leq \frac{r}{(1-r)^2}.$$

Now for $0 < r < 1$, $r/(1+r)^2 < 1/4$. So the inequality holds if $|f(z)| \geq 1/4$. If $|f(z)| < 1/4$, then the radial segment from 0 to $f(z)$ lies entirely in the range of f by the Koebe one-quarter theorem.

Since f is univalent, the preimage C of the segment $[0, f(z)]$ is an arc inside $\{z : |z| \leq r\}$ and $f(z) = \int_C f'(\zeta) d\zeta$. But $f'(\zeta) d\zeta$ has constant signum along C, so by the distortion theorem

$$|f(z)| = \int_C |f'(\zeta)||d\zeta| \geq \int_0^r \frac{1-\rho}{(1+\rho)^3} d\rho.$$

But

$$\int_0^r \frac{1-\rho}{(1+\rho)^3} d\rho = \frac{r}{(1+r)^2}.$$

Therefore

$$|f(z)| \geq \frac{r}{(1+r)^2}.$$

Hence we obtain

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \quad |z| = r < 1.$$

Now if equality holds, we can write

$$\frac{r}{(1+r)^2} = |f(z)| = \frac{r}{(1-r)^2}, \quad |z| = r < 1.$$

Taking derivative with respect to r both sides, we obtain

$$\frac{1-r}{(1+r)^3} = |f'(z)| = \frac{1+r}{(1-r)^3}.$$

Hence from the distortion theorem $f(z)$ is a rotation of the Koebe function. $\square$

Corollary 2.15. *If $f \in \mathcal{S}$ and $z \in \mathbb{D}$, then*

$$\frac{1-|z|}{1+|z|} \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+|z|}{1-|z|}.$$

Proof. Suppose $f \in \mathcal{S}$, we obtained from F by a disk automorphism for a fix $\zeta \in \mathbb{D}$

$$F(z) = \frac{f\left(\dfrac{z+\zeta}{1+\overline{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)}.$$

By the growth theorem, we can write

$$\frac{|\zeta|}{(1+|\zeta|)^2} \leq |F(-\zeta)| \leq \frac{|\zeta|}{(1-|\zeta|)^2}, \tag{2.9}$$

but

$$F(-\zeta) = \frac{f(0)-f(\zeta)}{(1-|\zeta|^2)f'(\zeta)} = \frac{-f(\zeta)}{(1-|\zeta|^2)f'(\zeta)}.$$

Therefore, from (2.9) we have

$$\frac{|\zeta|}{(1+|\zeta|)^2} \leq \left|\frac{-f(\zeta)}{(1-|\zeta|^2)f'(\zeta)}\right| \leq \frac{|\zeta|}{(1-|\zeta|)^2}.$$

Now replacing ζ by z, we obtain

$$\frac{|z|}{(1+|z|)^2} \leq \left|\frac{-f(z)}{(1-|z|^2)f'(z)}\right| \leq \frac{|z|}{(1-|z|)^2},$$

it can be written as

$$\frac{(1+|z|)^2}{1-|z|^2} \geq \left|\frac{zf'(z)}{f(z)}\right| \geq \frac{(1-|z|)^2}{1-|z|^2}.$$

Rearranging above inequalities, we get the required inequality. $\square$

CHAPTER 3

CONVEX AND STARLIKE FUNCTIONS

3.1. Introduction

In this section we discuss certain subclass of $\mathcal{S}$. A set $E \subset \mathbb{C}$ is said to be *starlike with respect to a point* $w_0 \in E$ if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E. The set E is said to be *convex* if it is starlike with respect to each of its points i.e., if the linear segment joining any two points of E lies entirely in E. A function maps the unit disk conformally into a convex domain is called *convex function.* A function maps the unit disk conformally onto a domain starlike with respect to the origin is called *starlike function.* The subclass of $\mathcal{S}$ consisting of convex functions is denoted by C, and S^* denotes the class of starlike functions. In notationally

$$S^* := \{f : f \in \mathcal{S}, f \text{ is a starlike function}\}$$

and

$$C := \{f : f \in \mathcal{S}, f \text{ is a convex function}\}.$$

The inclusion relations $C \subset S^* \subset \mathcal{S}$ are proper. Note that the Koebe function is starlike but not convex.

We define the class P by the set of all functions ϕ which are analytic and having positive real part in $\mathbb{D}$, with $\phi(0) = 1$ i.e.

$$P := \{\phi : \phi \text{ analytic in } \mathbb{D}, \phi(0) = 1, \mathrm{Re}(\phi(z)) > 0\}.$$

We now discuss some important relations among the classes C, S^* and P.

3.2. Characterization of starlike functions

In this section, we discuss an analytic characterization of starlike functions. This helps us to check whether a given function is starlike in the unit disk or not.

Theorem 3.1. *Let f be analytic in $\mathbb{D}$, with $f(0) = 0$ and $f'(0) = 1$. Then $f \in S^*$ if and only if $zf'(z)/f(z) \in P$.*

Proof. Let $f \in S^*$. We claim that f is starlike in each subdisk i.e. each subdisk $|z| < \rho$ maps onto a starlike domain. It is equivalent to saying $g(z) = f(\rho z)$ is starlike in $\mathbb{D}$. In other words, we must show that for each t $(0 < t < 1)$ and for each $z \in \mathbb{D}$, $tg(z)$ is in the range of g. Consider

$$w(z) = f^{-1}(tf(z)).$$

Then $w : \mathbb{D} \to \mathbb{D}$ and $w(0) = 0$, w is holomorphic. By using an application of Schwarz lemma we obtain $|w(z)| \leq |z|$.

Suppose $w_0 \in f(|z| < \rho)$, then there exists z_0 such that $f(z_0) = w_0$ that implies that $|z_0| = |f^{-1}(w_0)| < \rho$ and also

$$|f^{-1}(tf(z_0))| = |f^{-1}(tw_0)| = |w(z_0)| \leq |z_0| < \rho.$$

This is also written as $tw_0 \in f(|z| < \rho)$.

The function f maps each circle $|z| = \rho < 1$ onto a curve C_ρ that bounds starlike domain and f is conformal. That implies

$$\frac{\partial}{\partial\theta} arg(f(re^{i\theta})) \geq 0,$$

as z moves around in positive direction. Now

$$\begin{aligned}\frac{\partial}{\partial\theta} arg(f(re^{i\theta})) &= \mathrm{Im}\left(\frac{\partial}{\partial\theta}\log(f(re^{i\theta})\right)\\ &= \mathrm{Im}\frac{if'(z)z}{f(z)} = \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq 0\end{aligned}$$

and also $zf'(z)/f(z) = 1$ at $z = 0$.

Hence by the maximum modulus principle for harmonic functions,

$$\frac{zf'(z)}{f(z)} \in P.$$

Conversely, suppose f is analytic in $\mathbb{D}$, with $f(0) = 0$, $f'(0) = 1$ and $zf'(z)/f(z) \in P$. Now $f(0) = 0$, $f'(0) = 1$ i.e. f has a simple zero at origin.

If $f(c) = 0$ $(c \neq 0)$, $zf'(z)/f(z)$ is analytic for all $z \in \mathbb{D}$. But $cf'(c)/f(c)$ has a pole at $z = c$, a contradiction. Therefore f has no other zero elsewhere in the disk.

Given $\mathrm{Re}(zf'(z)/f(z)) \geq 0$, that implies that for each $\rho < 1$,

$$\frac{\partial}{\partial\theta} arg(f(re^{i\theta})) \geq 0, \quad 0 \leq \theta \leq 2\pi.$$

Thus as z moves around the circle $|z| = \rho$ in the counter-clockwise direction, the point $f(z)$ moves a closed curve C_ρ with increasing argument.

But f has exactly one zero inside the circle $|z| = \rho$, by argument principle C_ρ surrounds the origin exactly once. Now C_ρ has no self-intersections, since C_ρ winds about the origin only once with increasing argument. Thus C_ρ is a simple closed curve which bounds a starlike domain D_ρ, and f assumes each value $w \in D_\rho$ exactly once in the disk $|z| < \rho$. Since this is true for every $\rho < 1$, it follows that f is univalent and starlike in $\mathbb{D}$. Hence this concludes the proof. □

3.3. Characterization of convex functions

In this section, we discuss an analytic characterization of convex functions similar to that of the class of starlike functions. This helps us to check whether a given function is convex in the unit disk or not.

Theorem 3.2. *Let f be analytic in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. Then $f \in C$ iff $1 + zf''/f' \in P$.*

Proof. Suppose $f \in C$. We claim that f must map each subdisk $|z| < r$ onto a convex domain. Choose $z_1, z_2 \in \mathbb{D}_r$, $(|z_1| \leq |z_2| < r)$. Let $w_1 = f(z_1)$ and $w_2 = f(z_2)$. Now let

$$w_0 = tw_1 + (1-t)w_2, \quad 0 < t < 1.$$

But f is convex, that implies there is a unique $z_0 \in \mathbb{D}$ such that $f(z_0) = w_0$. It is enough to show that $|z_0| < r$. Let

$$g(z) = tf\left(\frac{z_1 z}{z_2}\right) + (1-t)f(z)$$

is analytic in $\mathbb{D}$ with $g(0) = 0$ and $g(z_2) = w_0$.

Now $tf(z_1 z/z_2) \in$ Range of f and $(1-t)f(z) \in$ Range of f. Therefore

$$g(z) = tf\left(\frac{z_1 z}{z_2}\right) + (1-t)f(z) \in \text{ Range of } f.$$

Now $h(z) = f^{-1}(g(z))$ is well defined and $h(0) = f^{-1}(g(0)) = f^{-1}(0) = 0$ with $|h(z)| \leq 1$. Then from Schwarz lemma we obtain $|h(z)| \leq |z|$.

Therefore

$$|z_0| = |h(z_2)| \leq |z_2| < r.$$

Hence each circle $|z| = r < 1$ maps onto the circle C_r which bounds a convex domain.

Since f is convex in this subdisk. That implies that the argument of tangent to C_r is nondecreasing as the curve is moving in the positive direction. This can be written as

$$\frac{\partial}{\partial\theta}\left(arg\left(\frac{\partial}{\partial\theta}f(re^{i\theta})\right)\right) \geq 0, \quad 0 \leq \theta \leq 2\pi.$$

Now

$$\begin{aligned}
& \frac{\partial}{\partial\theta}\left(arg\left(\frac{\partial}{\partial\theta}f(re^{i\theta})\right)\right) \geq 0 \\
or, \quad & \frac{\partial}{\partial\theta}\left(arg\left(f'(re^{i\theta})re^{i\theta}i\right)\right) \geq 0 \\
or, \quad & \operatorname{Im}\left(\frac{\partial}{\partial\theta}\log(ire^{i\theta}f'(re^{i\theta}))\right) \geq 0 \\
or, \quad & \operatorname{Im} i\left(\frac{ire^{i\theta}f'(re^{i\theta}) + ir^2e^{2i\theta}f''(re^{i\theta})}{ire^{i\theta}f'(re^{i\theta})}\right) \geq 0 \\
or, \quad & \operatorname{Im} i\left(1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right) \geq 0 \\
or, \quad & \operatorname{Im} i\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq 0 \\
or, \quad & \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq 0, \quad |z| = r.
\end{aligned}$$

Thus by the maximum modulus principle for harmonic functions,

$$1 + \frac{zf''(z)}{f'(z)} \in P.$$

Conversely, suppose f is a analytic function with $f(0) = 0$ and $f'(0) = 1$. Also $1 + zf''(z)/f'(z) \in P$.

Now

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq 0.$$

That is

$$\frac{\partial}{\partial\theta}\left(arg\left(\frac{\partial}{\partial\theta}f(re^{i\theta})\right)\right) \geq 0.$$

This show that the slope of the tangent to the curve C_r increases monotonically. But as a point makes a complete circuit of C_r, the argument of tangent vector has a net change

$$\begin{aligned}\int_0^{2\pi} \frac{\partial}{\partial\theta}\left(arg\left(\frac{\partial}{\partial\theta} f(re^{i\theta})\right)\right) d\theta &= \int_0^{2\pi} \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) d\theta \\ &= \operatorname{Re}\left[\int_{|z|=r}\left(1+\frac{zf''(z)}{f'(z)}\right)\frac{dz}{iz}\right] \\ &= 2\pi, \quad z = re^{i\theta}.\end{aligned}$$

This shows that C_r is a simple curve bounding a convex domain. Thus for arbitrary $r < 1$ implies that f is univalent with convex range. □

Now we discuss an important relationship between convex functions and starlike functions. This was first introduce by Alexander in 1915. This helps us to construct a starlike function if a convex function is given, and vice versa.

Theorem 3.3. (Alexander's Theorem). *Let f be analytic in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. Then $f \in C$ iff $zf' \in S^*$.*

Proof. Consider $g(z) = zf'(z)$. Taking logarithmic derivative, we obtain

$$\frac{zg'(z)}{g(z)} = 1 + \frac{zf''(z)}{f'(z)}.$$

Let $f \in C$. Then from Theorem 3.2, we have

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geq 0,$$

which is equivalent to

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) \geq 0.$$

Thus $g \in S^*$ i.e. $zf'(z) \in S^*$.

Conversely, let $g(z) = zf'(z) \in S^*$. Then from Theorem 3.1, we have

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) \geq 0$$

or

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geq 0.$$

That implies $f \in C$. This completes the proof. □

CHAPTER 4

AREA PROBLEMS

For any analytic function g in $\mathbb{D}$, we denote $\triangle(r,g)$ by the area of the image of the disk $|z|<r$ under g. In integral form, we can write

$$\triangle(r,g)=\int\int_{|z|<r}|g'(z)|^2dx\,dy,\quad 0<r<1,$$

where $z=x+i\,y$. We say that the function g is Dirichlet finite if $\triangle(1,g)<\infty$.

Let $f\in\mathcal{S}$ and set a new function

$$F_f(z)=\frac{f(z)}{z}=1+a_2z+a_3z^2+\cdots.$$

From area theorem, we obtain

$$\triangle(r,F_f)=\pi\sum_{n=1}^{\infty}n|a_{n+1}|^2r^{2n}.$$

Also by de Brange's theorem, we have $|a_n|\leq n\,(n\geq 2)$ for $f\in\mathcal{S}$. Therefore

$$\triangle(r,F_f)\leq\pi\sum_{n=1}^{\infty}n(n+1)^2r^{2n}.$$

Now $F_k=k(z)/z=1+2z+3z^2+\cdots$. From area theorem, we obtain

$$\triangle(r,F_k)=\pi\sum_{n=1}^{\infty}n|a_n|^2r^{2n}=\pi\sum_{n=1}^{\infty}n(n+1)^2r^{2n}.$$

Therefore $\triangle(r,F_f)\leq\triangle(r,F_k)$. Hence

$$\max_{f\in\mathcal{S}}\triangle(r,F_f)=\triangle(r,F_k).$$

Yamashita in [11] proved that the area $\triangle(r,1/F_f)$ of $F_f(\mathbb{D}_r)$ is bounded for $f\in\mathcal{S}$, with the help of the area theorem (Theorem 2.8).

Theorem 4.1. [11] *We have* $\max_{f\in\mathcal{S}}\triangle(r,1/F_f)=2\pi r^2(r^2+2)$ *for* $0<r<1$. *For each* $r, 0<r\leq 1$, *the maximum is attained only by* k_θ'*s.*

Proof. For $f \in \mathcal{S}$, we know that $(f(1/z))^{-1} \in \Sigma$ and we can write it as

$$(f(1/z))^{-1} = z - a_2 + (a_2^2 - a_3)z^{-1} + \cdots = z - a_2 + \sum_{n=1}^{\infty} b_n z^{-n}, \quad |z| > 1.$$

From Theorem 2.8, we have

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1. \tag{4.1}$$

Now

$$\frac{1}{F_f(z)} = \frac{z}{f(z)} = 1 - a_2 z + (a_2^2 - a_3)z^2 + \cdots = 1 - a_2 z + \sum_{n=1}^{\infty} b_n z^{n+1}$$

and from Theorem 2.7, we have

$$\begin{aligned} \pi^{-1} \triangle (r, 1/F_f) &= |a_2|^2 + \sum_{n=1}^{\infty} (n+1)|b_n|^2 r^{2r+2} \\ &= |a_2|^2 r^2 + 2r^4 \sum_{n=1}^{\infty} \frac{1}{2}(n+1)|b_n|^2 r^{2n-2} \end{aligned}$$

Since $f \in \mathcal{S}$ that implies $|a_2| \leq 2$, it follows that

$$\pi^{-1} \triangle (r, 1/F_f) \leq 4r^2 + 2r^4 \sum_{n=1}^{\infty} n|b_n|^2.$$

From (4.1), we get

$$\pi^{-1} \triangle (r, 1/F_f) \leq 2r^2(r^2 + 2).$$

Therefore,

$$\max_{f \in \mathcal{S}} \triangle(r, 1/F_f) \leq 2\pi r^2(r^2 + 2).$$

Next to find $\triangle(r, 1/F_k)$, where k is the Koebe function.

Now

$$\frac{1}{F_k} = \frac{z}{k(z)} = 1 - 2z + z^2.$$

Then

$$\triangle(r, 1/F_k) = \pi(1 \cdot (-2)^2 r^2 + 2 \cdot 1 \cdot r^4) = \pi(4r^2 + 2r^2) = 2\pi r^2(2 + r^2).$$

If maximum is attained by any function $f_1(z)$, then

$$\pi^{-1} \triangle (r, f_1) = 2r^2(2 + r^2) = 4r^2 + 2r^4.$$

But

$$\pi^{-1} \triangle (r, 1/F_{f_1}) = |a_2|^2 r^2 + \cdots .$$

Now compare the coefficient of r^2, we obtain $|a_2|^2 = 4$ i.e. $|a_2| = 2$. Hence $f_1 = k$. The maximum is attained only by a rotation of the Koebe function. $\square$

Now F_f is analytic and convex function for each $f \in C$. We find area of image domains under F_f functions. For $0 < r \leq 1$,

$$\triangle(r, F_f) = \int\int_{|z|<r} |F_f'(z)| dxdy = \pi \sum_{n=1}^{\infty} n|a_{n+1}|^2 r^{2n}.$$

Since F_f is convex function, $|a_n| \leq 1$,

$$\begin{aligned}
\triangle(r, F_f) &\leq \pi \sum_{n=1}^{\infty} n r^{2n} \\
&= \pi(r^2 + 2r^4 + 3r^6 + \cdots) \\
&= \pi r^2(1 + 2r^2 + 3r^4 + \cdots) \\
&= \pi r^2(1 + 2(r^2) + 3(r^2)^2 + \cdots) \\
&\leq \pi r^2(1 + 2r + 3^2 \cdots) \\
&= \pi r(r + 2r^2 + 3r^3 + \cdots) \\
&= \pi r \cdot \frac{r}{(1-r)^2} = \frac{\pi r^2}{(1-r)^2},
\end{aligned}$$

where first inequality holds since $r < 1$ that implies $r^2 < r$. Therefore

$$\max_{f \in C} \Delta(r, F_f) = \frac{\pi r^2}{(1-r)^2}, \quad 0 < r < 1.$$

The maximum is attained only by a rotation of $J(z) = z/(1-z), z \in \mathbb{D}$.

In [11], Yamashita conjectured the following:

Theorem 4.2 (Yamashita's Conjecture)**.** *We have*

$$\max_{f \in C} \Delta(r, 1/F_f) = \pi r^2, \quad 0 < r < 1,$$

where maximum is attained by a rotation of $J(z) = z/(1-z), z \in \mathbb{D}$.

This conjecture has been proved in [6]. Proof of this is also included in this chapter in the latter part, as it involves ideas of the order of starlikeness. Before this, we first investigate the conjecture by constructing some examples.

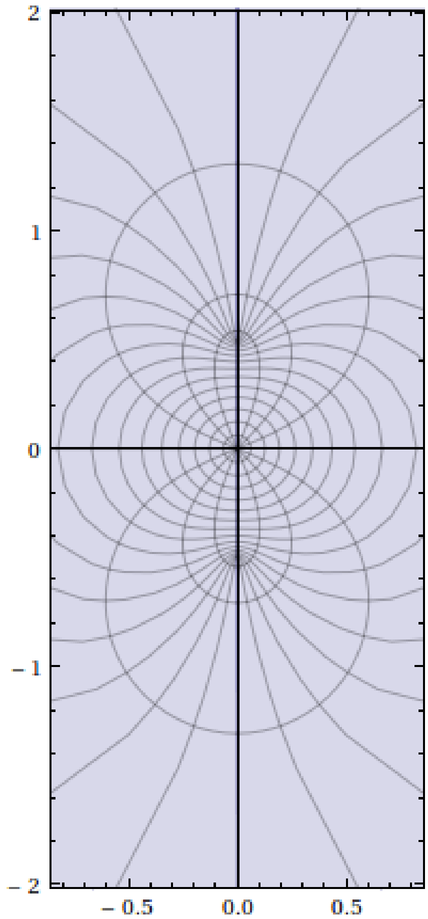

FIGURE 4.1. Graph of image of the unit disk under $f(z) = z/(1-z^2)$

Example 4.3. Consider the function

$$f(z) = \frac{z}{1-z^2}.$$

It has been proved in Chapter 2 that $f \in \mathcal{S}$. From Figure 4.1 it is clear that f is unbounded in $\mathbb{D}$. Now

$$\frac{z}{f(z)} = 1 - z^2.$$

Now

$$\triangle(r, z/f(z)) = \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n} = \pi(1 \cdot 0 \cdot r^2 + 2 \cdot 1 \cdot r^4) = 2\pi r^4.$$

This shows that area of image of $\mathbb{D}$ under $z/f(z)$ is bounded for all r, $0 \le r < 1$. Clearly,

$$\triangle(r, z/f(z)) = 2\pi r^4 < 2\pi r^2(2 + r^2) = \triangle(r, z/k(z)).$$

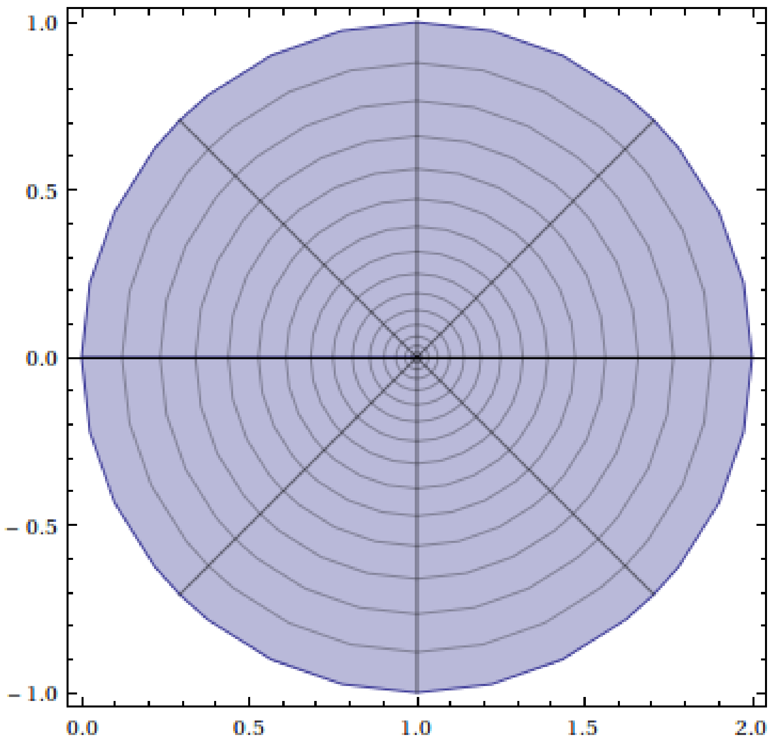

FIGURE 4.2. Graph of image of the unit disk under $z/f(z) = 1 - z^2$

Example 4.4. Let $f(z) = z/(1-z)$ and $f'(z) = 1/(1-z)^2$. Now the disk automorphism g of f is given by

$$g(z) = \frac{f\left(\dfrac{z+\alpha}{1+\bar{\alpha}z}\right) - f(\alpha)}{(1-|\alpha|^2)f'(\alpha)}, \quad z \in \mathbb{D},$$

where $|\alpha| < 1$. That implies

$$g(z) = \frac{\dfrac{\dfrac{z+\alpha}{1+\bar{\alpha}z}}{1-\dfrac{z+\alpha}{1+\bar{\alpha}z}} - \dfrac{\alpha}{1-\alpha}}{(1-|\alpha|^2)\dfrac{1}{(1-\alpha)^2}}.$$

Simplifying this, we obtain

$$g(z) = \frac{z}{1-\left(\dfrac{1+\bar{\alpha}}{1-\alpha}\right)z}.$$

Now

$$\frac{z}{g(z)} = 1 - \left(\frac{1+\bar{\alpha}}{1-\alpha}\right)z.$$

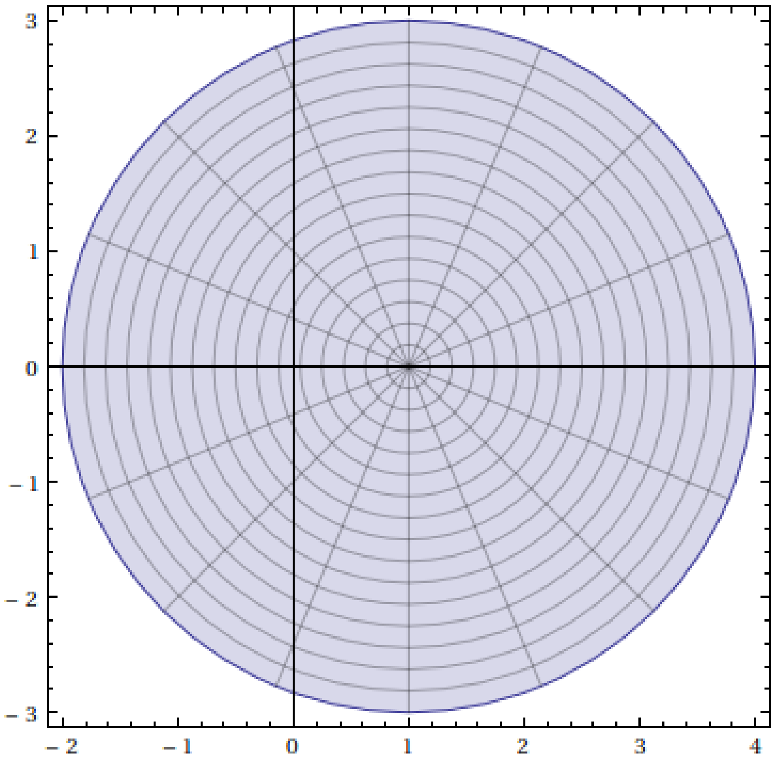

FIGURE 4.3. Graph of image of the unit disk under $z/g(z) = 1 - ((1+\bar{\alpha})(1-\alpha))\, z$

Therefore, area is

$$\triangle(r, z/g(z)) = \pi \left| \frac{1+\bar{\alpha}}{1-\alpha} \right|^2 r^2 = \pi r^2.$$

This shows that $\triangle(r, z/g(z))$ is bounded for all r, $0 \leq r < 1$. Clearly,

$$\triangle(r, z/g(z)) = \pi r^2 < 2\pi r^2(2 + r^2) = \triangle(r, z/k(z)).$$

In particular for $\alpha = 0.5$, we see from Figure 4.3, that $\triangle(r, z/g(z))$ is bounded.

Note. It is observed that area of image domain of $\mathbb{D}_r$ under $z/f(z)$ and under $z/g(z)$ are the same.

Example 4.5. Let $f(z) = z/(1-z)^2$ and $\phi(z) = z/(1-z), z \in \mathbb{D}$.

From range transformation, we know that, if $f \in \mathcal{S}, \phi$ is a function which is analytic and univalent on the range of f with $\phi(0) = 0$ and $\phi'(0) = 1$, then $g = \phi \circ f \in \mathcal{S}$.

Now

$$g(z) = \phi \circ f(z) = \frac{\dfrac{z}{(1-z)^2}}{1 + \dfrac{z}{(1-z)^2}} = \frac{z}{1 - z + z^2}.$$

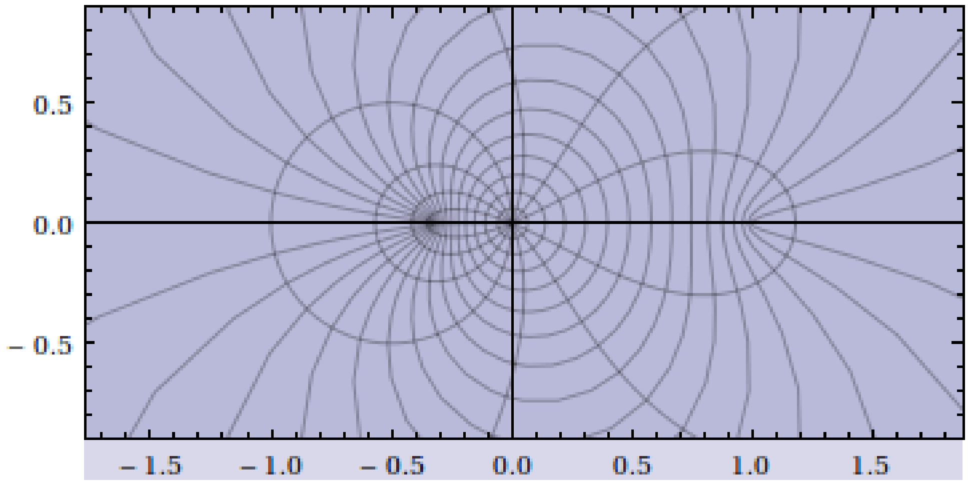

FIGURE 4.4. Graph of image of the unit disk under $g(z) = z/(1 - z + z^2)$

We see in Figure 4.4 that g in unbounded in $\mathbb{D}$. Now

$$\frac{z}{g(z)} = 1 - z + z^2.$$

Therefore

$$\triangle(r, z/g(z)) = \pi \sum_{n=0}^{\infty} n|a_n|^2 r^{2n} = \pi(r^2 + 2r^4).$$

It is clear that area is bounded for all r, $0 \le r < 1$. The figure is given below. Clearly,

$$\triangle(r, z/g(z)) = \pi(r^2 + 2r^4) < 2\pi r^2(2 + r^2) = \triangle(r, z/k(z)).$$

Now using omitted-value transformation under functions in the class $\mathcal{S}$, we discuss the following example:

Example 4.6. Let $f(z) = z/(1 - z^2) \in \mathcal{S}$ and $g(z) = wf/(w - f)$.

Here $f(z) \neq i = w$ i.e. w is omitted by f. Now

$$g(z) = \frac{if}{i - f} = \frac{iz}{i(1 - z^2) - z} = \frac{z}{1 + iz - z^2}.$$

It is clear from Figure 4.6 that g is unbounded in $\mathbb{D}$ and

$$\frac{z}{g(z)} = 1 + iz - z^2.$$

Therefore

$$\triangle(r, z/g(z)) = \pi \sum_{n=0}^{\infty} n|a_n|^2 r^{2n} = \pi(r^2 + 2r^4).$$

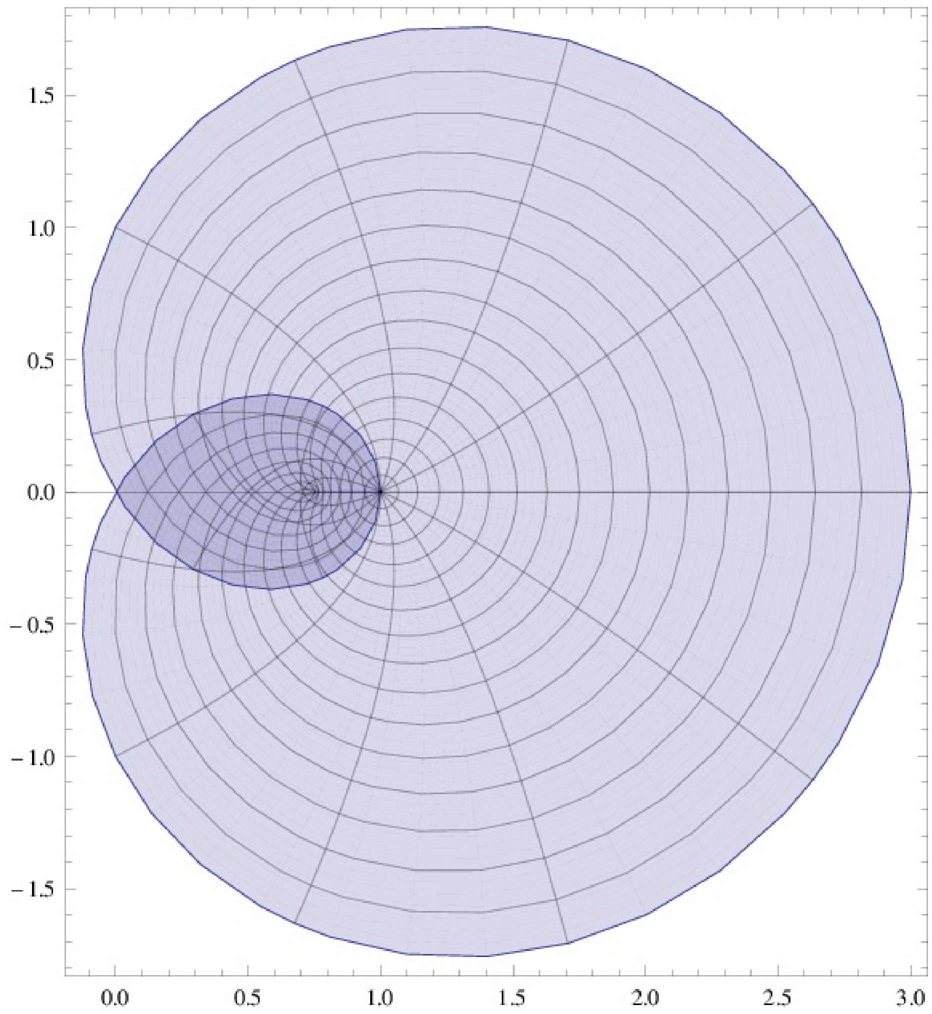

FIGURE 4.5. Graph of image of the unit disk under $z/g(z) = 1 - z + z^2$

It is clear that area is bounded for all r, $0 \leq r < 1$. Clearly,

$$\triangle(r, z/g(z)) = \pi(r^2 + 2r^4) < 2\pi r^2(2 + r^2) = \triangle(r, z/k(z)).$$

To prove the Yamashita conjecture, we need to recall Parseval's Identity as follows:

Theorem 4.7 (Parseval's Identity). *If f is an analytic function defined on the unit disk with power series $f(z) = \sum_{n=1}^{\infty} a_n z^n$ then*

$$\sum_{n=1}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta.$$

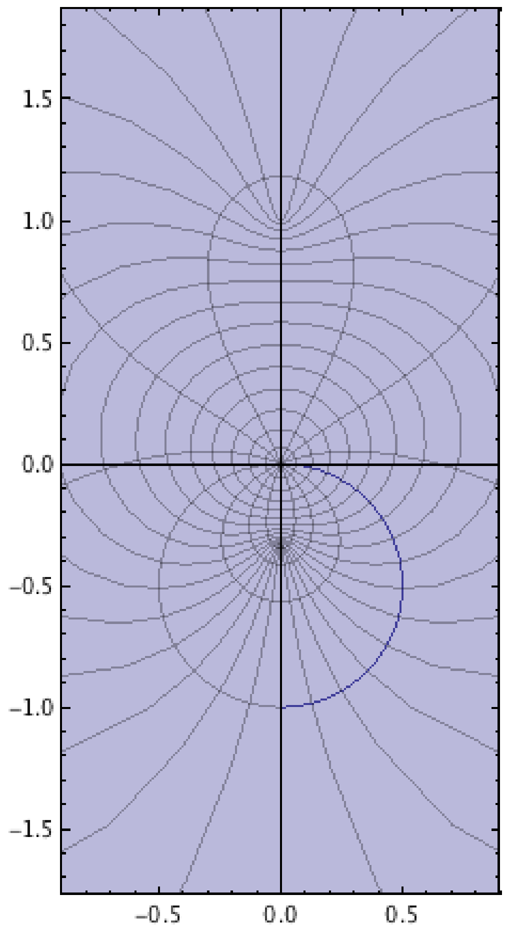

FIGURE 4.6. Graph of image of the unit disk under $g(z) = z/(1 + iz - z^2)$.

Proof. We compute

$$\begin{aligned}
\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta &= \int_0^{2\pi} f(re^{i\theta})\overline{f(re^{i\theta})} d\theta \\
&= \int_0^{2\pi} \sum_{m=1}^{\infty} a_m r^m e^{im\theta} \sum_{n=1}^{\infty} \overline{a_n} r^n e^{-in\theta} d\theta \\
&= \int_0^{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m \overline{a_n} r^{m+n} e^{i(m-n)\theta} d\theta \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m \overline{a_n} 2\pi r^{m+n} \\
&= 2\pi \sum_{n=1}^{\infty} |a_n|^2 r^{2n}.
\end{aligned}$$

Hence it proves Parseval's Identity. $\square$

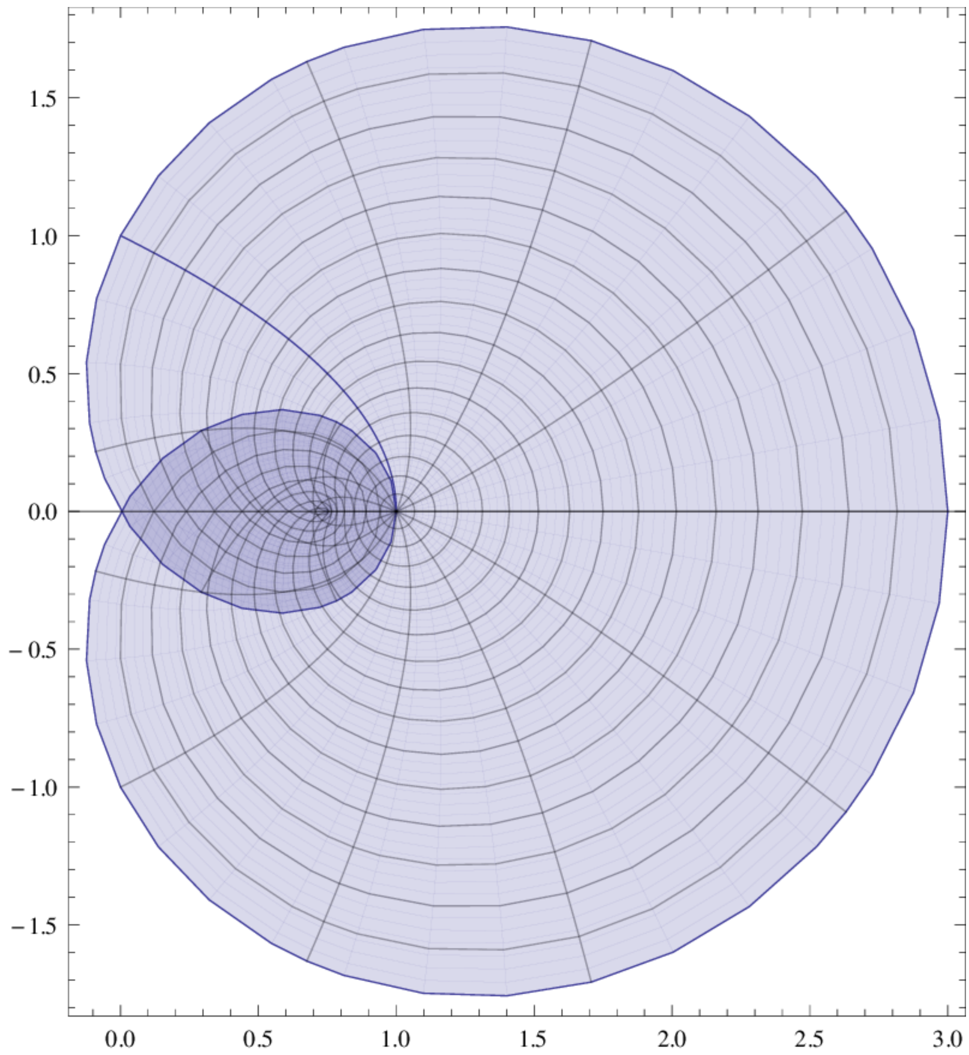

FIGURE 4.7. Graph of image of the unit disk under $z/g(z) = 1 + iz - z^2$

Definition 4.8 (Subordination). Let $f(z)$ and $g(z)$ be analytic in the unit disk. Then g is called subordinate to f if there exits an analytic function $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$ such that

$$f(w(z)) = g(z), \quad |z| < 1.$$

Symbolically, it is written as $g \prec f$.

If f is univalent in $\mathbb{D}$, then $g \prec f$ is equivalent to the conditions that $g(0) = f(0)$ and $g(\mathbb{D}) \subset f(\mathbb{D})$; see [7, Lemma 2.1, p. 36]. For instance, we can easily show that

$$g(z) = \frac{1}{1-z} \prec \frac{1+z}{1-z} = f(z)$$

for all $z \in \mathbb{D}$. Indeed, if we use the definition, then $w(z) = z/(2-z)$.

Theorem 4.9 (Littlewood's Subordination Theorem [3]). *Let f and g be analytic in the unit disk $\mathbb{D}$ and suppose $g \prec f$. Then $0 < p < \infty$*

$$M_p(r, g) \leq M_p(r, f), \quad 0 \leq r < 1,$$

where $M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{\infty} |f(re^{i\theta})|^p d\theta \right)^{1/p}$.

Theorem 4.10 (Rogosinski's Theorem). *Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ be analytic in $\mathbb{D}$, and suppose $g \prec f$. Then*

$$\sum_{n=1}^{\infty} |b_n|^2 \leq \sum_{n=1}^{\infty} |a_n|^2, \quad n = 1, 2, \ldots.$$

Proof. Let $s_n(z) = \sum_{k=1}^{n} a_k z^k$ and $f(z) = s_n(z) + r_n(z)$. Also let $t_n(z) = \sum_{k=1}^{n} b_k z^k$. Then

$$g(z) = f(w(z)) = s_n(w(z)) + r_n(w(z)).$$

Since $w(0) = 0$, it follows that

$$s_n(w(z)) = t_n(z) + \sum_{k+1}^{\infty} c_k z^k$$

for some c_k.Therefore by parseval's relation, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} |s_n(w(re^{i\theta}))|^2 d\theta = \sum_{k=1}^{n} |b_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |c_k|^2 r^{2k}. \tag{4.2}$$

But $s_n \circ w \prec s_n$, so by Littlewood's subordination theorem

$$\frac{1}{2\pi} \int_0^{2\pi} |s_n(w(re^{i\theta}))|^2 d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |s_n(re^{i\theta})|^2 d\theta = \sum_{k=1}^{n} |a_k|^2 r^{2k}.$$

From (4.2) we obtain

$$\sum_{k=1}^{n} |b_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |c_k|^2 r^{2k} \leq \sum_{k=1}^{n} |a_k|^2 r^{2k}.$$

Also this inequality written as

$$\sum_{k=1}^{n} |b_k|^2 r^{2k} \leq \sum_{k=1}^{n} |a_k|^2 r^{2k}.$$

For $r \to 1$, it follows the theorem. □

The method of the proofs of Theorem 4.7 and Theorem 4.10 is adopted by Clunie in [2]. The same has been quoted in the proof of Lemma 4.11.

For $\beta \in [0,1)$, let $\mathcal{S}^*(\beta)$ denote the usual normalized class of all (univalent) starlike functions of order β. Analytically, a function $f \in \mathcal{S}$ is said to belong to the class $\mathcal{S}^*(\beta)$ if f satisfies the condition

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z \in \mathbb{D}.$$

It is well-known that $C \subset \mathcal{S}^*(1/2)$; see [5, Theorem 2.6a, p. 57]. Also, $\mathcal{S}^* := \mathcal{S}^*(0)$ is the usual class of starlike functions i.e., $f \in \mathcal{S}$ such that $f(\mathbb{D})$ is starlike with respect to the origin.

Recently Obradovic, Ponnusamy and Wriths [6] proved that the area $\Delta(r, z/f(z)) = \pi r^2$ for convex functions f that was conjectured by Yamashita in [11]. To present its proof, the following lemma is useful.

Lemma 4.11. *Let $f \in \mathcal{S}^*(\beta)$ and let $z/f(z)$ have the following expansion near $z = 0$,*

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots .$$

Then the following coefficients inequality holds:

$$\sum_{n=1}^{\infty} (n - (1-\beta))|b_n|^2 \leq 1 - \beta.$$

Proof. Let $g(z) = z/f(z)$ and $f \in \mathcal{S}^*(\beta)$. Since

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z \in \mathbb{D}$$

and by taking logarithmic derivative, we get

$$\frac{zg'(z)}{g(z)} = 1 - \frac{zf'(z)}{f(z)}, \quad z \in \mathbb{D},$$

there exits a function $w : \mathbb{D} \to \bar{\mathbb{D}}$ analytic in the unit disk such that

$$\frac{zg'(z)}{g(z)} = 1 - \frac{1-(1-2\beta)zw(z)}{1+zw(z)} = \frac{2(1-\beta)zw(z)}{1+zw(z)}, \quad z \in \mathbb{D}$$

equivalently $g'(z) = (2(1-\beta)g(z) - zg'(z))w(z)$.

Now expanding in series form

$$\sum_{k=1}^{\infty} kb_k z^{k-1} = \left(\sum_{k=1}^{\infty} kb_k z^k - 2(1-\beta)(1+\sum_{k=1}^{\infty} b_k z^k)\right) w(z)$$
$$= \left(\sum_{k=1}^{\infty} kb_k z^k - 2(1-\beta) - 2(1-\beta)\sum_{k=1}^{\infty} b_k z^k\right) w(z)$$
$$= \left(-2(1-\beta) + \sum_{k=1}^{\infty}(kb_k - 2(1-\beta)b_k)z^k\right) w(z).$$

Now rearrange this series expansion, we obtain

$$\sum_{k=1}^{n} kb_k z^{k-1} + \sum_{k=n+1}^{\infty} kb_k z^{k-1} = (-2(1-\beta) + \sum_{k=1}^{n-1}(kb_k - 2(1-\beta)b_k)z^k$$
$$+ \sum_{k=n}^{\infty}(kb_k - 2(1-\beta)b_k)z^k)w(z).$$

By Clunie's method, for any $n \in \mathbb{N}$ the inequality

$$\sum_{k=1}^{n-1} |b_k|^2 r^{2k-2}(k^2 - (k-2(1-\beta))^2) + |b_n|^2 r^{2n-2} n^2 \le 4(1-\beta)^2.$$

For $r = 1$,

$$\sum_{k=1}^{n-1} |b_k|^2 (k^2 - (k-2(1-\beta))^2) + |b_n|^2 n^2 \le 4(1-\beta)^2.$$

Simplifying this we obtain

$$\sum_{k=1}^{n-1} |b_k|^2 (k-(1-\beta))4(1-\beta) + |b_n|^2 n^2 \le 4(1-\beta)^2.$$

Also this inequality written as $\sum_{k=1}^{n-1} |b_k|^2 (k-(1-\beta))4(1-\beta) \le 4(1-\beta)^2$.

Hence we obtain the inequality

$$\sum_{k=1}^{\infty} |b_k|^2 (k-(1-\beta)) \le 1-\beta \text{ as } n \to \infty,$$

which completes the proof. □

Remark 4.12. For $\beta = 0$, this lemma is the well known area theorem for functions $f \in \mathcal{S}$.

Definition 4.13. The series

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, |z| < 1$$

is known as the hypergeometric series whereas the associate converging function ${}_2F_1(a, b; c; z)$ is called the hypergeometric function. Here $c \neq 0, -1, -2, \ldots$. If a or b or both 0 or negative, series terminates and $(a)_n$ denotes the Pochhammer symbol define $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ for $n \in \mathbb{N}$.

The following theorem is proved in [6].

Theorem 4.14. *We have*

$$\max_{f \in \mathcal{S}^*(1/2)} \triangle(r, z/f(z)) = \pi r^2, \quad 0 < r \leq 1,$$

where the maximum is attained by the rotations of the function $j(z) = z/(1-z)$.

Proof. Let $f \in \mathcal{S}^*(1/2)$. Since f is analytic and $f(z) \neq 0$. For $z \neq 0$, we can write f in the form

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots .$$

Now

$$\begin{aligned}
\pi^{-1} \triangle (r, z/f(z)) &= \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} \\
&\leq r^2 \sum_{n=1}^{\infty} n|b_n|^2 \\
&= r^2 \sum_{n=1}^{\infty} ((2n-1) - (n-1))|b_n|^2 \\
&= r^2 \left(\sum_{n=1}^{\infty} (2n-1)|b_n|^2 - \sum_{n=2}^{\infty} (n-1)|b_n|^2 \right),
\end{aligned}$$

where the first inequality holds since $r^{2n} \leq r^2$ for $0 < r \leq 1$. Since by lemma 4.11, for $\beta = 1/2$, we obtain

$$\sum_{n=1}^{\infty} (2n-1)|b_n|^2 \leq 1.$$

Therefore the inequality becomes

$$\pi^{-1} \triangle (r, z/f(z)) \leq r^2 \left(1 - \sum_{n=2}^{\infty} (n-1)|b_n|^2 \right) \leq r^2.$$

The equality holds clearly for $j(z) = z/(1-z)$ and for the rotation of $j(z)$, since $z/j(z) = 1 - z$ and by area theorem $\pi^{-1} \triangle (r, z/j(z)) = r^2$. $\square$

We now discuss a generalization of Theorem 4.14 for functions in $S^*(\beta)$, $0 \le \beta < 1$ in terms of hypergeometric functions of r.

For this, we consider $f_\beta(z) = \dfrac{z}{(1-z)^{2(1-\beta)}}$, where $0 \le \beta < 1$. It is easy to see that $f_\beta \in \mathcal{S}^*(\beta)$ is extremal for many extremal problems for full class $\mathcal{S}^*(\beta)$. We see that

$$\frac{z}{f_\beta(z)} = (1-z)^{2(1-\beta)} = F(1,\delta;1;z), \delta = -2(1-\beta).$$

But

$$F(1-\delta;1;z) = \sum_{k=1}^{\infty} \frac{(1)_k(\delta)_k}{(1)_k} \frac{z^k}{k\,!} \quad z \in \mathbb{D}.$$

Now we obtain

$$\triangle(r, z/f_\beta(z)) = \pi \sum_{n=1}^{\infty} n \left(\frac{(\delta)_n}{(1)_n} \right)^2 r^{2n}.$$

Simplifying this, we have

$$\begin{aligned}\triangle(r, z/f_\beta(z)) &= \pi\delta^2 r^2 \sum_{n=0}^{\infty} \frac{(\delta+1)_n(\delta+1)_n}{(1)_n(2)_n} r^{2n} \\ &= \pi\delta^2 r^2 F(\delta+1, \delta+1 : 2 : r^2) \\ &= 4\pi(1-\beta)^2 r^2 F(2\beta-1, 2\beta-1 : 2 : r^2) =: A_\beta(r).\end{aligned}$$

The above calculation leads to the following theorem, which is proved in [6].

Theorem 4.15. *Let $f \in \mathcal{S}^*(\beta)$ for some $0 \le \beta < 1$. Then we have*

$$\max_{f \in \mathcal{S}^*(\beta)} \triangle(r, z/f(z)) = A_\beta(r), \quad 0 < r \le 1,$$

where the maximum is attained by the rotations of $f_\beta(z) = z/(1-z)^{2(1-\beta)}$.

CHAPTER 5

CONCLUSION AND SCOPE FOR FUTURE WORK

5.1. Conclusion

In the last chapter, we discussed the problem of computing areas of image domains of the unit disk and its sub-disks under the class of univalent functions and functions in the convex and starlike families. It is a natural fact that computing areas of domains are meaningful when the domains are bounded. On the other hand, for those image domains, one can certainly ask the question of maximizing length of boundaries of such domains. This is partially discussed in the next section with an aim to widely investigate such problems for some subclasses of univalent functions in future.

5.2. Scope for future work

Length maxima problems

Recall that for $f \in \mathcal{S}$, we have

$$\triangle(r, 1/F_f) \leq 2\pi r^2(r^2+2).$$

In particular, for $r = 1$,

$$\triangle(1, 1/F_f) \leq 6\pi.$$

In the line of the above discussion, we investigate the size of arc length of the boundary curves that are nothing but images of the unit circle or subcircles inside the unit disk under $f \in \mathcal{S}$. If $f \in \mathcal{S}$, then f can be written as

$$f(z) = \frac{z}{z/f(z)} = \frac{z}{1/F_f(z)}.$$

Each function is the quotient of z and $1/F_f$. Each $f \in \mathcal{S}$, maps the unit circle onto a curve of length

$$L(r, f) = r\int_0^{2\pi} |f'(re^{i\theta})|d\theta, \quad 0 < r < 1.$$

Now

$$L(r,f) = r\int_0^{2\pi} |f'(z)|d\theta = \int_0^{2\pi} \left|\frac{zf'(z)}{f(z)}\right| |f(z)|d\theta.$$

From Corollary 2.15, we get

$$L(r,f) \le \int_0^{2\pi} \left(\frac{1+r}{1-r}\right) |f(z)|d\theta.$$

Therefore

$$L(r,f) \le \left(\frac{1+r}{1-r}\right) \int_0^{\infty} |f(z)|d\theta \le \left(\frac{1+r}{1-r}\right) \frac{2\pi r}{1-r}.$$

That is,

$$L(r,f) \le \frac{2\pi r(1+r)}{(1-r)^2}$$

and

$$L(r,k) > \frac{\pi r(1+r)}{2(1-r)^2}.$$

Also for $k \in \mathcal{S}$,

$$L(r,k) \le \sup_{f\in\mathcal{S}} L(r,f).$$

Hence

$$\frac{\pi r(1+r)}{2(1-r)^2} < L(r,k) \le \sup_{f\in\mathcal{S}} L(r,f). \tag{5.1}$$

We have

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots .$$

Therefore

$$\triangle(r,k) = \pi\sum_{n=1}^{\infty} n|a_n|^2 r^{2n} = \pi\sum_{n=1}^{\infty} n|n|^2 r^{2n} = \pi\sum_{n=1}^{\infty} n^3 r^{2n}. \tag{5.2}$$

But $k(z) = \sum_{n=1}^{\infty} nz^n$ and hence $k(r) = \sum_{n=1}^{\infty} nr^n$.

Now

$$k'(r) = \sum_{n=1}^{\infty} n^2 r^{n-1} = \frac{1+r}{(1-r)^3},$$

and

$$k''(r) = \frac{4+2r}{(1-r)^4}.$$

Now

$$r(rk'(r))' = r\left(r\sum_{n=1}^{\infty} n^2 r^{n-1}\right)' = r\left(\sum n^2 r^n\right)' = r\sum n^3 r^{n-1} = \sum n^3 r^n. \tag{5.3}$$

But

$$(rk'(r))' = rk''(r) + k'(r) = \frac{r^2+4r+1}{(1-r)^4}. \tag{5.4}$$

Therefore, from (5.3) and (5.4), we obtain

$$\sum n^3 r^n = \frac{r^2+4r+1}{(1-r)^4}. \tag{5.5}$$

Now replacing r by r^2 in (5.5) and from (5.2), we obtain

$$\triangle(r,k) = \frac{\pi(1+4r^2+r^4)r^2}{(1-r^2)^4} \to \infty \text{ as } r \to 1.$$

For each $f \in \mathcal{S}$, we obtain

$$\triangle(r,f) = \pi \sum n|a_n|^2 r^{2n} \leq \pi \sum n^3 r^{2n} = \triangle(r,k),$$

since by de Branges theorem $|a_n| \leq n, n \geq 2$. Therefore

$$\max \triangle(r,f) = \triangle(r,k) = \frac{\pi r^2(1+4r^2+r^4)}{(1-r^2)^4},$$

the maximum is attained only by a rotation of the Koebe function.

Theorem 5.1. *For each $0 < r < 1$, we have*

$$2\pi r(r^4+4r^2+1)^{1/2}(1-r^2)^{-2} \leq L(r,k) \leq \sup_{f\in\mathcal{S}} L(r,f).$$

Proof. We have

$$\max \triangle(r,f) = \triangle(r,k) = \frac{\pi r^2(1+4r^2+r^4)}{(1-r^2)^4} \tag{5.6}$$

Also from isoperimetric inequality,

$$A \leq \frac{L^2}{4\pi},$$

where L the length of a closed curve and A the area of the planar region that it encloses. Therefore

$$\triangle(r,k) \leq \frac{1}{4\pi} L^2(r,k).$$

From (5.6), we obtain

$$\frac{\pi r^2(1+4r^2+r^4)}{(1-r^2)^4} \leq \frac{L^2(r,k)}{4\pi}.$$

Simplifying above inequality and from (5.1), we obtain

$$\frac{2\pi r(1+4r^2+r^4)^{1/2}}{(1-r^2)^2} \leq L(r,k) \leq \sup_{f\in\mathcal{S}} L(r,f). \tag{5.7}$$

Thus the proof is complete. $\square$

Remark 5.2. The inequality (5.7) is better estimate than (5.1), possible only if

$$\frac{\pi r(1+r)}{2(1-r)^2} < \frac{2\pi r(1+4r^2+r^4)^{1/2}}{(1-r^2)^2}$$

equivalently,

$$\frac{1+r}{4(1-r)} < \frac{(1+4r^2+r^4)^{1/2}}{(1-r)^2(1+r)^2}$$

or,

$$\frac{1}{4} < \frac{(1+4r^2+r^4)^{1/2}}{(1+r)^3}.$$

Now take $g(r) = \dfrac{(1+4r^2+r^4)^{1/2}}{(1+r)^3}$, compute its derivative we have

$$g'(r) = \frac{(-r)^4+2r^3-8r^2+4r-3}{(1+r)^6(1+4r^2+r^4)(1/2)}.$$

But $(-r)^4+2r^3-8r^2+4r-3<0$, that implies $g'(x)<0$. Therefore $g(x)$ is decreasing. So $g(x)$ attains its infimum at $r=1$. For $r<1, g(r)<g(1)$ i.e.,

$$\frac{(1+4r^2+r^4)^{1/2}}{(1+r)^3} > \frac{1}{4}.$$

The problems of estimating the length of boundary of image domains are recently studied in [10]. In near future, I plan to read this paper and investigate some new problems in this direction.

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