Homotopy Perturbation Method To Solve Non-Linear Differential Equations

M.Sc. Thesis

By

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DISCIPLINE OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

MAY 2018

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A THESIS

Submitted in partial fulfillment of the requirements for the award of the degree

of

Master of Science

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Acknowledgements

It is an honour for me to express my deep sense of respect and gratitude to my supervisors Dr. Santanu Manna and Dr. Sk. Safique Ahmad, Discipline of Mathematics, Indian Institute of Technology Indore for their support and excellent guidance throughout in my M. Sc. project. Their valuable guidance, untiring efforts, timely suggestions and enthusiasm to always learn and explore different scientific areas, their patience and persistence to truth in research have been an excellent example for me, both in research and life. I will never forget their supporting nature and the facilities provided during my project.

Also, I extend my thanks to Dr. Vijay Kumar Sohani, Assistant Professor, Discipline of Mathematics IIT Indore, for his help and active co-operation during my stint in IIT Indore. My sincere thanks also go to my PSPC members Dr. Niraj Kumar Shukla, Dr. Shanmugam Dhinakaran and Dr. Aquil Khan (DPGC Convener) IIT Indore for a series of insightful comments and constructive ideas at different stages of my project.

I deeply acknowledge Indian Institute of Technology Indore, for providing me all the necessary facilities during the course of my project work. Lastly and most importantly, I would also like to express my deepest gratitude to my parents Mr. Pramod Kumar and Smt. Machala Devi who showered their blessings and love throughout in my life.

Date: (Rahul Kumar)

Synopsis

The thesis contains a survey of the perturbation method [cf. Nayfeh, 1981] and generalization of the perturbation method for the solution of linear and Nonlinear differential equations. The generalization of perturbation method has been first introduced by Ji-Huan He (1999) called *Homotopy Perturbation Method (HPM)*. According to Cheniguel and Reghioua (2013), Biazar and Eslami (2011), Mechee et al. (2017), etc., HPM is one of the new and excellent methods for solving the nonlinear differential equation. It is well known that the perturbation theory is based on an assumption of an equation (in the form of power series) with a small parameter. A perfect choice of small parameter leads to the excellent result. However, if the choice of the small parameter is not suitable then the solution is going to be a bad asset. In such cases, the HPM can find the accurate approximate solution of the differential equation. This method does not depend on the small parameter in the assumed equation. The HPM is a combination of homotopy and perturbation method which provides an advantageous way to obtain an analytical or approximate solution of the differential equations. **Chapter 1**, contains the literature survey of perturbation, homotopy and generalized homotopy perturbation method. In **Chapter 2**, we have discussed the basic theory of perturbation method and its application from the books of Nayfeh (1981) and Liao (1995). The **Chapter 3**, started with the basic idea of homotopy [cf. Ji-Huan He, 1999] and extended to the study of the HPM [He, 2005, 2006; Hemeda, 2012]. In this chapter, we have discussed Inviscid Burgers Nonlinear problem and nonhomogeneous Advection Nonlinear problem using the HPM. In **Chapter 4**, we survey the literature of the generalization of the homotopy perturbation method (GHPM) from the article of Hector, (2014). The last Chapter contains the conclusions and future plan.

Keywords: Homotopy, Homotopy perturbation method, Linear, Nonhomogeneous, Nonlinear, ODEs, PDEs, Perturbation method, Power series, Topology.

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Contents

Chapter 1

Introduction

The homotopy perturbation method (HPM) is an efficient and powerful technique to find the approximate solutions of nonlinear ordinary and partial differential equations which describe different fields of Science and Engineering. In this project report, we plan to focus on the survey of perturbation method, homotopy perturbation method and the generalization of homotopy perturbation method to solve linear and nonlinear differential equation.

The differential equations can model many physical and engineering problems with the specified initial/boundary conditions. However, sometimes it is difficult to obtain closed-form solutions for them. In most of the cases, only approximate solutions can be expected either in analytical form or numerical form. The perturbation method is one of the well-known methods for solving nonlinear problems analytically. But perturbation methods have some limitations, all perturbation methods require the presence of a small parameter in the linear or nonlinear equation and approximate solution of the equation containing this parameter is expressed as series expansions in the small parameter. The homotopy perturbation technique does not depend upon a small parameter in the equation. This method is a combination of homotopy and perturbation techniques, provides a convenient way to obtain an analytic or approximate solution to a wide variety of problems arising in the different field.

Ji-Huan He in 1999 first introduced the Homotopy perturbation technique. In his paper, Ji-Huan studied few problems with or without small parameters with the homotopy perturbation technique and concluded that the proposed method does not require small parameters in the equations, so the limitations of the traditional perturbation methods can be eliminated. He (2006) also studied the homotopy perturbation method in a very effective, simple, and convenient way to solve nonlinear boundary value problems. After He's work in 1999, many researchers used the homotopy perturbation method to approximate the solutions of differential equations and integral equations. Biazar et al. (2009) discussed the application of the homotopy perturbation method to Zakharov-Kuznetsov equations. Later on, a new homotopy perturbation method for solving systems of partial differential equations was introduced by Biazar and Eslami (2011). Recently, many researchers like, Cheniguel and Reghioua (2013), Hector (2014), Mechee et al. (2017), etc. studied the homotopy perturbation method for solving some initial boundary value problems. The solution of delay differential equations via a homotopy perturbation method was also discussed by Shakeri and Dehghan (2008).

Definition 1.0.1 (Topology): *A topology on a set* $\mathbf{X}$ *is a collection* τ *of subset of* $\mathbf{X}$ *having the following properties:*

1. ϕ *and* $\mathbf{X}$ *are in* τ.

2. *The union of the element of any sub-collection of* τ *is in* τ.

3. *The intersection of the element of any finite sub-collection of* τ *is in* τ.

A set $\mathbf{X}$ *for which a topology* τ *has been specified is called a topological space. If* $\mathbf{X}$ *is a topological space with topology* τ, *we say that a subset* V *of* $\mathbf{X}$ *is an* open set *of* $\mathbf{X}$ *if* V *belong to the collection* τ. *Using this terminology, one can say that a topological space is a set* $\mathbf{X}$ *together with a collection of subset of* $\mathbf{X}$, *called* open sets *such that* ϕ *and* $\mathbf{X}$ *are both open, and such that arbitrary union and finite intersection of open sets are open.*

Definition 1.0.2 (Continuity of a function): *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be topological spaces. A function* $f : \mathbf{X} \longrightarrow \mathbf{Y}$ *is said to be* continuous *if for each open subset* V *of* $\mathbf{Y}$, *the set* $f^{-1}(V)$ *is an open subset of* $\mathbf{X}$.

Definition 1.0.3 (Homotopy): *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two topological spaces and* f,g *are continuous functions from* $\mathbf{X}$ *to* $\mathbf{Y}$. *If there exists a continuous function, say,*

$$\mathscr{S} : \mathbf{X} \times [0,1] \longrightarrow \mathbf{Y}$$

such that $\mathscr{S}(x,0) = f(x)$ *and* $\mathscr{S}(x,1) = g(x)$ *for every* $x \in \mathbf{X}$*, then we say that* f *is homotopic to* g *and* $\mathscr{S}$ *is called a homotopy between* f *and* g.

Example 1.0.1 *If we consider two real valued functions* $\cos(\pi\, x)$ *and* $8x^2(x-1)$ *defined on the interval* $[0,1]$, *then* $\mathscr{S} : [0,1] \times [0,1] \to \mathbb{R}$ *given by*

$$\mathscr{S}(x,p) = (1-p)\cos(\pi\, x) + p[8x^2(1-x)], \tag{1.0.1}$$

will be a homotopy between $\cos(\pi x)$ *and* $8x^2(x-1)$.

Let $\mathbb{C}[a,b]$ be a set of all continuous real valued functions defined on the interval [a,b]. For given continuous functions $f,g \in \mathbb{C}[a,b]$, we consider a homotopy

$$\mathscr{S} : [a,b] \times [0,1] \longrightarrow \mathbb{R}$$

between f and g, defined by

$$\mathscr{S}(x,p) = (1-p)\ f(x) + p\ g(x), \quad x \in [a,b], \quad p \in [0,1], \tag{1.0.2}$$

where p is called the homotopy parameter.

Note: We can construct several different homotopy between two continuous functions $f,g \in \mathbb{C}[a,b]$. For example, for every pair of positive integers (n,m), the mapping

$$\mathscr{S}_{n,m} : [a,b] \times [0,1] \longrightarrow \mathbb{R}$$

given by

$$\mathscr{S}_{n,m}(x,p) = (1-p^n)\ f(x) + p^m\ g(x)$$

will be a homotopy between f and g.

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Chapter 2

Perturbation Method

The perturbation method is a classical method which has been used over a century to obtain approximate analytical solutions of a mathematical problem by exploiting the presence of a small parameter. It is the study of the effects of small disturbances. If the effects are small, the perturbations or disturbances are said to be regular perturbation, otherwise, they are singular perturbation. The technique has been successfully applied to the differential equations, integro-differential equations, and algebraic equations. One major question may arises in this method that "what is the advantage to use the perturbation method, as the numerical method also gives the approximation solution ?" To answer this question, we can say [cf. Simmonds and Mann, 1986] that perturbation methods produce analytic approximations that often reveal the essential dependence of the exact solution on the parameters in a more satisfactory way than does a numerical solution.

In this chapter, we discuss the singular perturbation theory and regular perturbation theory. We refer the book of Nayfeh (1981) for details. If we assume a problem $f(x)=0$ and suppose the problem is perturbed by a parameter (ε), then we have problem $(f(x,\varepsilon)=0)$. The solution of these type of perturbed problem are complex in a sense that they are dependent on parameter. Perturbation theory gives us approximate solution in the form of power series. In this theory the solution of perturbed problem $f(x,\varepsilon)=0$ has been considered as

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots, \tag{2.0.1}$$

where x_0 is the solution for un-perturbed problem and $x_1, x_2, \ldots$ are higher order terms. The ε has been taken in such way that we can neglect higher order terms. The above series (2.0.1) is called perturbed solution series. The perturbation problem can be classified into two categories, namely regular perturbation and singular perturbation.

Theorem 2.0.1 (The Fundamental Theorem of Perturbation Theory): *If*

$$x_0 + x_1\varepsilon + x_2\varepsilon^2 + x_3\varepsilon^3 + \cdots + x_n\varepsilon^n + \bigcirc(\varepsilon^{n+1}) = 0, \tag{2.0.2}$$

for ε sufficiently small and if the coefficients $x_0, x_1, x_2, \cdots$ are independent of ε, then

$$x_0 = x_1 = x_2 = \cdots = x_n = 0.$$

Definition 2.0.4 (Big '$\bigcirc$')(Order of Convergence): *Suppose $f(x)$ and $g(x)$ are two functions defined on some subset of the real numbers, then*

$$f(x) = \bigcirc(g(x))$$

if and only if there exist constants N and K such that $|f(x)| \leq K|g(x)|$ for all $x \geq N$. Intuitively, this means that $f(x)$ does not grow faster than $g(x)$.

2.1 Regular and Singular Perturbation

The solution of perturbed problem is called *perturbed solution* and the solution of unperturbed problem is called an *exact solution*. If every solution of the perturbed problem $f(x,\varepsilon) = 0$ converges to some solution of the unperturbed problem $f(x) = 0$ as $\varepsilon \to 0$, then such perturbation problem is called the regular perturbation problem, otherwise singular perturbation problem.

Theorem 2.1.1 (cf. Shivamoggi, 2003): *Consider an implicit equation*

$$f(x,\varepsilon) = 0.$$

If there exists $x = x_0$ such that

$$f(x_0, \varepsilon) = 0$$

and if $f_x(x_0, \varepsilon)$ is an invertible linear map, then there exists a unique solution of the given equation in the neighbourhood of $\varepsilon = 0$ given by

$$x = g(\varepsilon).$$

Example 2.1.1 (Regular Perturbation) *Consider the algebraic equation*

$$y^2 + 2y + \varepsilon = 0, \quad \varepsilon << 1. \tag{2.1.1}$$

This equation has two roots given by

$$y_1 = -1 - \sqrt{1-\varepsilon}$$
$$y_2 = -1 + \sqrt{1-\varepsilon}.$$

Expanding $\sqrt{1-\varepsilon}$ as a Taylor series

$$\sqrt{1-\varepsilon} = 1 - \frac{1}{2}\,\varepsilon - \frac{1}{8}\,\varepsilon^2 - \cdots, \tag{2.1.2}$$

we obtain

$$y_1 = -2 + \frac{1}{2}\,\varepsilon + \frac{1}{8}\,\varepsilon^2 + \cdots$$
$$y_2 = -\frac{1}{2}\,\varepsilon - \frac{1}{8}\,\varepsilon^2 - \cdots.$$

Let us assume that the solution of the equation (2.1.1) *is of the form*

$$x(\varepsilon) = x_0 + \varepsilon x_1 + x_2\varepsilon^2 + \cdots + x_n\varepsilon^n + \bigcirc(\varepsilon^{n+1}). \tag{2.1.3}$$

Substituting equation (2.1.3) *into the equation* (2.1.1), *we obtain*

$$\left[x_0 + \varepsilon x_1 + x_2\varepsilon^2 + \cdots + x_n\varepsilon^n + \bigcirc(\varepsilon^{n+1})\right]^2 + 2\Big[x_0 + \varepsilon x_1 + x_2\varepsilon^2 + \cdots + x_n\varepsilon^n + \bigcirc(\varepsilon^{n+1})\Big] + \varepsilon = 0, \tag{2.1.4}$$

from which

$$\left[x_0^2 + \varepsilon\, 2x_0x_1 + \varepsilon^2\,(x_1^2 + 2x_0x_2) + \bigcirc(\varepsilon^3)\right] + 2\left[x_0 + \varepsilon x_1 + x_2\varepsilon^2 + \bigcirc(\varepsilon^3)\right] + \varepsilon = 0. \tag{2.1.5}$$

Rearranging the above equation, we obtain

$$(x_0^2+2x_0)+\varepsilon(2x_0x_1+2x_1+1)+\varepsilon^2(x_1^2+2x_0x_2+2x_2)+\bigcirc(\varepsilon^3)=0. \tag{2.1.6}$$

Equating the coefficients of like power of ε to zero, we obtain a system of equations

$$\left.\begin{aligned} &x_0^2+2x_0 &&=0\\ &2x_0x_1+2x_1+1 &&=0\\ &x_1^2+2x_0x_2+2x_2 &&=0\\ &\vdots &&,\end{aligned}\right\} \tag{2.1.7}$$

which is to be solved recursively.

Thus,

$$x_0=0 \quad or \quad x_0=-2.$$

For $x_0=0$, we obtain

$$x_1=-\frac{1}{2},\; x_2=-\frac{1}{8},$$

which gives the second root of the given equation (2.1.1)

$$y_2=-\frac{1}{2}\,\varepsilon-\frac{1}{8}\,\varepsilon^2-\cdots.$$

For $x_0=-2$, we obtain

$$x_1=\frac{1}{2},\; x_2=\frac{1}{8},$$

which gives the first root of the given equation (2.1.1)

$$y_1=-2+\frac{1}{2}\,\varepsilon+\frac{1}{8}\,\varepsilon^2+\cdots.$$

Example 2.1.2 (Regular Perturbation) *Let us consider a differential equation with the initial condition*

$$\frac{d^2y}{dx^2}+\varepsilon\frac{dy}{dx}+1=0, \quad y(0)=0, \quad \frac{dy}{dx}(0)=1. \tag{2.1.8}$$

Assume that the solution of the equation (2.1.8) *is of the form*

$$y(x)=y_0(x)+\varepsilon y_1(x)+\varepsilon^2y_2(x)+\bigcirc(\varepsilon^3). \tag{2.1.9}$$

Substituting equation (2.1.9) *in* (2.1.8), *we have*

$$\frac{d^2}{dx^2}(y_0(x)+\varepsilon y_1(x)+\varepsilon^2y_2(x)+\bigcirc(\varepsilon^3))+\varepsilon\frac{d}{dx}(y_0(x)+\varepsilon y_1(x)+\varepsilon^2y_2(x)+\bigcirc(\varepsilon^3))+1=0$$

$$\Rightarrow \frac{d^2y_0}{dx^2}+1+\varepsilon\left(\frac{d^2y_1}{dx^2}+\frac{dy_0}{dx}\right)+\varepsilon^2\left(\frac{d^2y_2}{dx^2}+\frac{dy_1}{dx}\right)+\bigcirc(\varepsilon^3)=0.$$

Now, equating the coefficient of like power of ε, we get

$$\left.\begin{array}{lll}\frac{d^2y_0}{dx^2}+1 &= 0, \quad y_0(0)=0, & \frac{dy_0}{dx}(0)=1\\ \frac{d^2y_1}{dx^2}+\frac{dy_0}{dx} &= 0, \quad y_1(0)=0, & \frac{dy_1}{dx}(0)=0\\ \frac{d^2y_2}{dx^2}+\frac{dy_1}{dx} &= 0, \quad y_2(0)=0, & \frac{dy_2}{dx}(0)=0.\end{array}\right\} \tag{2.1.10}$$

By solving the system of equations (2.1.10)*, we have*

$$\left.\begin{array}{ll} y_0(x) &= x-\frac{x^2}{2}\\ y_1(x) &= -\frac{x^2}{2}+\frac{x^3}{6}\\ y_2(x) &= \frac{x^3}{6}-\frac{x^4}{24}.\end{array}\right\} \tag{2.1.11}$$

By putting equation (2.1.11) *in equation* (2.1.9)*, the approximate solution of the differential equation* (2.1.8) *is*

$$y(x)=x-\frac{x^2}{2}+\varepsilon\left(\frac{-x^2}{2}+\frac{x^3}{6}\right)+\varepsilon^2\left(\frac{x^3}{6}-\frac{x^4}{24}\right)+\bigcirc(\varepsilon^3),$$

which is known as perturbation solution.

Theorem 2.1.2 (cf. Simmonds and Mann, 1986): *If we consider a polynomial*

$$\begin{aligned} P_\varepsilon(x) &= (1+b_0\varepsilon+c_0\varepsilon^2+\cdots)\\ &+ A_1\varepsilon^{\alpha_1}(1+b_1\varepsilon+c_1\varepsilon^2+\cdots)x\\ &\vdots\\ &+ A_n\varepsilon^{\alpha_n}(1+b_n\varepsilon+c_n\varepsilon^2+\cdots)x^n, \end{aligned} \tag{2.1.12}$$

where the α_i's are rational numbers, the b_i's, c_i's, $\cdots$ are constants, A_k are non-zero ε-factors and each of the factors $1+b_k\varepsilon+\cdots$, $k=0,1,\cdots n$ is assumed to be regular, that is, to have an expansion of the form (2.1.3)*, then each root of the polynomial* (2.1.12) *is of the form*

$$x(\varepsilon)=\varepsilon^n w(\varepsilon), \;\; w(0)\neq 0, \; n>0, \tag{2.1.13}$$

where, $w(\varepsilon)$ is a continuous function of ε, which is sufficiently small.

Example 2.1.3 (Singular Perturbation) *Consider the algebraic equation*

$$\varepsilon y^2 + 2y + 1 = 0, \quad \varepsilon << 1. \tag{2.1.14}$$

In the limit $\varepsilon \longrightarrow 0$, *the degree of the equation drops from 2 to 1. So this problem cannot be treated adequately by a regular perturbation expansion. This is a singular perturbation problem.*

Roots of the equation (2.1.14) *are given by*

$$y_1 = \frac{-1 + \sqrt{1-\varepsilon}}{\varepsilon}$$
$$y_2 = \frac{-1 - \sqrt{1-\varepsilon}}{\varepsilon}.$$

Expanding $\sqrt{1-\varepsilon}$ *as a Taylor series*

$$\sqrt{1-\varepsilon} = 1 - \frac{1}{2}\,\varepsilon - \frac{1}{8}\,\varepsilon^2 - \cdots, \tag{2.1.15}$$

we obtain

$$y_1 = -\frac{1}{2} - \frac{1}{8}\,\varepsilon - \cdots$$
$$y_2 = \frac{-2}{\varepsilon} + \frac{1}{2} + \frac{1}{8}\,\varepsilon - \cdots.$$

Let us assume that the solution of the equation (2.1.14) *is of the form,*

$$x(\varepsilon) = x_0 + \varepsilon x_1 + x_2\varepsilon^2 + \cdots + x_n\varepsilon^n + \bigcirc(\varepsilon^{n+1}). \tag{2.1.16}$$

Suppose that εy^2 *is small compared with* $2y+1$ *in the given equation* (2.1.14)*.*

On substituting the equation (2.1.16) *into equation* (2.1.14)*, we obtain*

$$\varepsilon\Big[x_0 + \varepsilon x_1 + x_2\varepsilon^2 + \cdots + x_n\varepsilon^n + \bigcirc(\varepsilon^{n+1})\Big]^2 + 2\Big[x_0 + \varepsilon x_1 + x_2\varepsilon^2 + \cdots + x_n\varepsilon^n + \bigcirc(\varepsilon^{n+1})\Big] + 1 = 0, \tag{2.1.17}$$

from which

$$\varepsilon\Big[x_0^2 + \varepsilon\, 2x_0x_1 + \bigcirc(\varepsilon^2)\Big] + 2\Big[x_0 + \varepsilon x_1 + \bigcirc(\varepsilon^2)\Big] + 1 = 0. \tag{2.1.18}$$

Rearranging the above equation, we obtain

$$(2x_0 + 1) + \varepsilon(x_0^2 + 2x_1) + \bigcirc(\varepsilon^2) = 0. \tag{2.1.19}$$

Equating the coefficients of like power of ε to zero, we obtain a system of equations

$$\left.\begin{aligned} 2x_0+1 &= 0 \\ x_0^2+2x_1 &= 0 \\ \vdots \quad & \quad , \end{aligned}\right\} \tag{2.1.20}$$

which is to be solved recursively.

Thus,

$$x_0 = -\frac{1}{2},\ x_1 = -\frac{1}{8},$$

which gives the first root of the given equation (2.1.14)

$$y_1 = -\frac{1}{2} - \frac{1}{8}\,\varepsilon - \cdots .$$

Now, in order to find the second root y_2, *we may write*

$$\begin{aligned} \varepsilon y^2 + 2y + 1 &= \varepsilon(y-y_1)(y-y_2) \\ &= \varepsilon y^2 - \varepsilon(y_1+y_2)y + \varepsilon y_1 y_2, \end{aligned}$$

from which

$$\varepsilon y_1 y_2 = 1 \ \ or \ \ y_2 \sim \frac{1}{\varepsilon},$$

so that, for the root y_2, εy^2 *is small compared with* $2y+1$, *in the limit as* $\varepsilon \longrightarrow 0$, *so the previous regular perturbation expansion does not set up the right dominant balance for* y_2, *and therefore not suited for recovering* y_2. *In order to obtain* y_2, *we use the theorem (2.1.2) as,*

$$\varepsilon y(\varepsilon) = w(\varepsilon), \quad w(0) \neq 0,$$

so that the given equation (2.1.14) *becomes*

$$w^2 + 2w + \varepsilon = 0. \tag{2.1.21}$$

It is observed that to find the second root of the given equation (2.1.14), *we can convert the singular problem to the regular problem by changing the variable from* y *to* w. *The roots of the equation* (2.1.21) *can be found using a regular perturbation expansion*

$$w(\varepsilon) = b_0 + \varepsilon b_1 + b_2\varepsilon^2 + \bigcirc(\varepsilon^3). \tag{2.1.22}$$

Substituting equation (2.1.22) *into equation* (2.1.21), *we obtain*

$$(b_0^2+2b_0)+\varepsilon(2b_0b_1+2b_1+1)+\bigcirc(\varepsilon^3)=0. \tag{2.1.23}$$

Equating the coefficients of like power of ε *to zero, we obtain a system of equations*

$$\left.\begin{aligned} b_0^2+2b_0 &= 0 \\ 2b_0b_1+2b_1+1 &= 0 \\ \vdots \qquad &, \end{aligned}\right\} \tag{2.1.24}$$

which is to be solved recursively.

Thus,

$$b_0=-2, \quad b_1=\frac{1}{2}.$$

Note that the other root for $b_0=0$*, has to be discarded as* $w(0)\neq 0$*. Thus, we have*

$$w(\varepsilon)=-2+\frac{1}{2}\varepsilon+\bigcirc(\varepsilon^2),$$

from which, we get for the second root,

$$y_2=-\frac{2}{\varepsilon}+\frac{1}{2}+\bigcirc(\varepsilon).$$

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Chapter 3

Homotopy Perturbation Method

The homotopy perturbation method was first proposed by J. Huan He, (1999) for obtaining the approximate analytical solutions of linear and nonlinear differential equation. The essential idea of this method is to introduce a homotopy parameter, say p, which takes values from 0 to 1 (cf. Shakeri and Dehghan, 2008). Generally, when $p = 0$, the system of equations reduces to an allogeneic simplified form, which normally allows getting a quite simple solution. But when p is slowly increased to 1, the system goes through a sequence of deformations and the solution for each of which is near to, that at the earlier stage of deformation. Eventually, at $p = 1$, the system takes the original form of the equation and the final stage of deformation gives the desired solution. One of the excellent features of the homotopy perturbation method is that usually, some few perturbation terms are sufficient for obtaining a reasonably accurate solution. This method has been employed to solve a large variety of linear and nonlinear problems.

3.1 Basic of Homotopy

In Chapter 1, we have discussed the homotopy between two continuous functions and in the following example, we discuss the homotopy of equation between two equations.

Example 3.1.1 *Consider a family of algebraic equations,*

$$\mathscr{E}(p) : (1+3p)x^2 + y^2/(1+3p) = 1, \quad p \in [0,1], \tag{3.1.1}$$

where $p \in [0,1]$ is the homotopy parameter. In the literature homotopy parameter is defined in the case of homotopy between two functions as well as homotopy of equation between two equations.
If $p = 0$ in equation (3.1.1), then we have

$$\mathscr{E}_0 : x^2 + y^2 = 1,$$

where $y = \pm\sqrt{1-x^2}$ are the solutions of the $\mathscr{E}_0$. Putting $p = 1$ in equation (3.1.1), we get the ellipse equation

$$\mathscr{E}_1 : 4x^2 + y^2/4 = 1,$$

whose solutions are $y = \pm 2\sqrt{1-4x^2}$. The solutions y are depending on x as well as p ∈[0,1], and we can rewrite the expression (3.1.1) in the form

$$\mathscr{E}(p) : (1+3p)x^2 + \frac{y^2(x,p)}{(1+3p)} = 1, \quad p \in [0,1], \tag{3.1.2}$$

which is called a homotopy of equation between $\mathscr{E}_0$ and $\mathscr{E}_1$ and is denoted by

$$\mathscr{E}(p) : \mathscr{E}_0 \sim \mathscr{E}_1.$$

If $y(x,p)$ is a nonnegative solution of the equation (3.1.2), then $-y(x,p)$ is also a solution of the equation (3.1.2). Therefore, we can say that $y(x,p)$ is a homotopy between $\sqrt{1-x^2}$ to $2\sqrt{1-4x^2}$ and $-y(x,p)$ is a homotopy between $-\sqrt{1-x^2}$ to $-2\sqrt{1-4x^2}$.

Definition 3.1.1 (Zeroth Order Deformation [Liao, 1995]): *Let $\mathscr{E}_0$ be an initial equation whose solution u_0 is known. Given an equation denoted by $\mathscr{E}_1$, which has at least one solution u. If one can construct a homotopy of equation $\mathscr{E}(p) : \mathscr{E}_0 \sim \mathscr{E}_1$ such that, as the homotopy parameter $p \in [0,1]$ increases from 0 to 1, $\mathscr{E}(p)$ deforms continuously from the the initial equation $\mathscr{E}_0$ to the the original equation $\mathscr{E}_1$, while its solution varies continuously from the known solution u_0 of $\mathscr{E}_0$ to the unknown solution u of $\mathscr{E}_1$, then this kind of homotopy of equations is called the zeroth-order deformation equation.*

Definition 3.1.2 (Mth-order Homotopy Approximation [Liao, 1995]): *If the solution of the zeroth-order deformation equation $\mathscr{E}(p) : \mathscr{E}_0 \sim \mathscr{E}_1$ exists and is a real analytic about*

$p \in [0,1]$, *then the homotopy-series solution of the original equation* $\mathscr{E}_1$ *can be written as*

$$u(z,t) = u_0(z,t) + \sum_{n=1}^{+\infty} u_n(z,t).$$

And the Mth-order homotopy-approximation of $u(z,t)$ *is*

$$u(z,t) \approx u_0(z,t) + \sum_{n=1}^{+M} u_n(z,t),$$

where $z,\ t$ *are the independent variables.*

3.2 Analysis of the Homotopy Perturbation Method

Let us consider the nonlinear differential equation

$$A(u) - f(x) = 0, \quad x \in \Omega, \tag{3.2.1}$$

with the boundary condition

$$B(u, \frac{\partial u}{\partial r}) = 0, \quad r \in \Gamma, \tag{3.2.2}$$

where A is a general differential operator and B is a boundary operator. Boundary of the domain Ω is Γ and $f(x)$ is a known real analytic function. The operator A can be divided into nonlinear (N) and linear (L) parts. From equation (3.2.1) we get

$$L(u) + N(u) - f(x) = 0. \tag{3.2.3}$$

By the basic of homotopy, we construct a homotopy function such that

$$\mu : \Omega \times [0,1] \longrightarrow \mathbb{R},$$

which satisfies

$$\mathscr{S}(\mu, p) = (1-p)[L(\mu) - L(u_0)] + p[A(\mu) - f(x)] = 0, \quad p \in [0,1], \quad x \in \Omega \tag{3.2.4}$$

or

$$\mathscr{S}(\mu, p) = L(\mu) - L(u_0) + pL(u_0) + p[N(\mu) - f(x)] = 0, \tag{3.2.5}$$

where u_0 is an initial approximation of equation (3.2.1), which satisfies the boundary condition. Therefore equation (3.2.4) represents

$$\mathscr{S}(\mu,0) = L(\mu) - L(u_0) = 0 \tag{3.2.6}$$

$$\mathscr{S}(\mu,1) = A(\mu) - f(x) = 0. \tag{3.2.7}$$

Let us assume that the solution of equation (3.2.4) can be written as a power series of p

$$\mu = \mu_0 + p\mu_1 + p^2\mu_2 + p^3\mu_3 + \cdots, \tag{3.2.8}$$

setting $p = 1$ in equation (3.2.8), we have

$$u = \lim_{p\to 1} \mu = \mu_0 + \mu_1 + \mu_2 + \mu_3 + \cdots. \tag{3.2.9}$$

The power series (3.2.8) is convergent for the most cases, however the convergent rate depends upon the nonlinear operator A. Let us apply the concept of HPM to solve some nonlinear differential equations.

Example 3.2.1 *Consider a nonlinear differential equation*

$$\frac{dy}{dx} + y^2 = 0, \quad x \geq 0, \quad x \in \Omega, \quad y(0) = 1, \tag{3.2.10}$$

with an exact solution $\frac{1}{(1+x)}$. *We will find the approximate solution by using the homotopy perturbation method.*

We can construct the following homotopy $Y : \Omega \times [0,1] \longrightarrow \mathbb{R}$ *between the constant function* 1 *(which is initial approximation* $y_0 = 1$*) and* $Y(x,1)$ *(which is the solution of the above differential equation), which satisfies*

$$(1-p)\Big(\frac{\partial Y}{\partial x} - \frac{dy_0}{dx}\Big) + p\Big(\frac{\partial Y}{\partial x} + Y^2\Big) = 0, \quad p \in [0,1], \quad x \in \Omega. \tag{3.2.11}$$

Suppose the solution of equation (3.2.10) *is of the form*

$$Y = Y_0 + pY_1 + p^2Y_2 + \cdots. \tag{3.2.12}$$

Substituting equation (3.2.12) *into equation* (3.2.11), *we get*

$$\left.\begin{array}{ll} p^0 & : \frac{dY_0}{dx} = \frac{dy_0}{dx} \\ p^1 & : \frac{dY_1}{dx} + \frac{dy_0}{dx} + Y_0^2 = 0, \quad Y_1(0) = 0 \\ p^2 & : \frac{dY_2}{dx} + 2Y_0Y_1 = 0, \quad Y_2(0) = 0. \end{array}\right\} \tag{3.2.13}$$

For simplicity we always set $Y_0 = y_0 = 1$. *Accordingly we have* $Y_1 = -x$, *and* $Y_2 = x^2$, *thus the second order of approximation of equation* (3.2.10) *can be written as*

$$y_2 = Y_0 + Y_1 + Y_2 = 1 - x + x^2. \tag{3.2.14}$$

3.3 Application of the Homotopy Perturbation Method

For some special partial differential equations like, Inviscid Burger's problem, Advection problem, etc., sometimes it is difficult to get the solution by analytical method, due to the presence of high nonlinearity in the equation. In such cases HPM will help to get the approximate solution easily. We explain this theory in some well known examples.

Example 3.3.1 (Inviscid Burger's Equation) *Let us consider the Inviscid Burger's equation*

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = 0, \quad w(x,0) = x + 2, \tag{3.3.1}$$

and a given the exact solution of the equation (3.3.1) *is* $w(x,t) = \frac{x+2}{t+1}$. *Our aim is to find the approximate solution by using the homotopy perturbation method.*
Applying the homotopy perturbation method (see cf. (3.2.5)*) in the equation* (3.3.1), *we obtain*

$$\frac{\partial w}{\partial t} - \frac{\partial w_0}{\partial t} = p\Big[-w\frac{\partial w}{\partial x} - \frac{\partial w_0}{\partial t}\Big], \tag{3.3.2}$$

where $w_0(x,t) = w(x,0) = x + 2$ *is an initial approximation.*
Now, we assume that the solution of the problem (3.3.1) *is of the form*

$$w = w_0 + pw_1 + p^2w_2 + \cdots. \tag{3.3.3}$$

Substituting equation (3.3.3) *into the equation* (3.3.2) *and equating the coefficient of like power of* p, *we get the set of differential equations*

$$\left.\begin{array}{ll} p^0 & : \frac{\partial w_0}{\partial t} - \frac{\partial w_0}{\partial t} = 0 \\ p^1 & : \frac{\partial w_1}{\partial t} = -\left(w_0 \frac{\partial w_0}{\partial x} + \frac{\partial w_0}{\partial x}\right) \\ p^2 & : \frac{\partial w_2}{\partial t} = -\left(w_0 \frac{\partial w_1}{\partial x} + w_1 \frac{\partial w_0}{\partial x}\right) \\ p^3 & : \frac{\partial w_3}{\partial t} = -\left(w_0 \frac{\partial w_2}{\partial x} + w_1 \frac{\partial w_1}{\partial x} + w_2 \frac{\partial w_0}{\partial x}\right) \\ \vdots & \qquad . \end{array}\right\} \tag{3.3.4}$$

Solving the above equations, we get

$$\left.\begin{array}{ll} w_1 & = -(x+2)t \\ w_2 & = (x+2)t^2 \\ w_3 & = -(x+2)t^3. \end{array}\right\} \tag{3.3.5}$$

Therefore the approximate solution of equation (3.3.1) *becomes*

$$\begin{aligned} w(x,t) &= w_0 + w_1 + w_2 + w_3 + \cdots, \\ &= (x+2) - (x+2)t + (x+2)t^2 - (x+2)t^3 + \cdots. \end{aligned} \tag{3.3.6}$$

The absolute error between the exact solution and approximation solution has been shown using graphical representation. Higher accuracy can be obtained by introducing some more components of the approximate solution. Figure 3.2 depicts the approximate solution whereas Figure 3.1 depicts the exact solution.

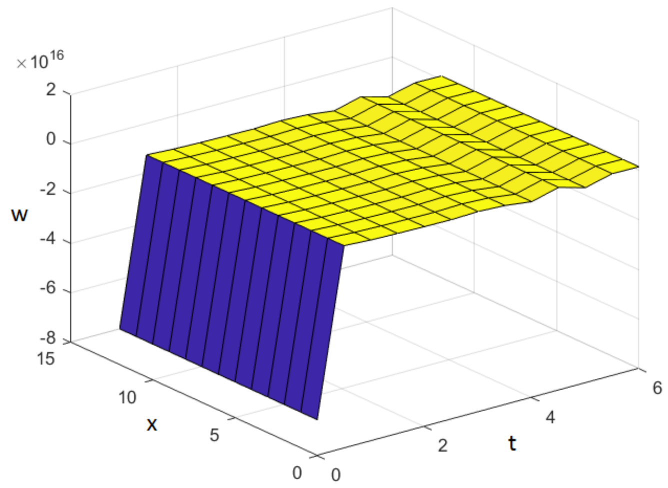

Figure 3.1: Graph of exact solution $w(x,t) = (x+2)/(t+1)$.

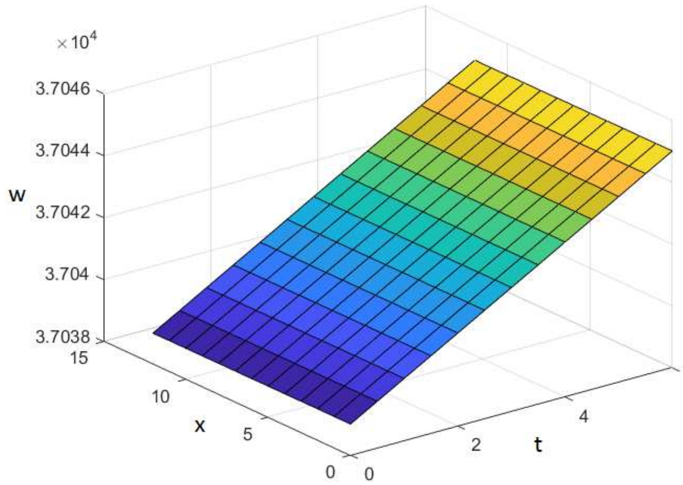

Figure 3.2: Graph of approximate solution $w(x,t) = (x+2) - (x+2)t + (x+2)t^2 + \cdots$.

Example 3.3.2 (Advection Problem) *We consider the nonhomogeneous Advection problem*

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=x, \quad v(x,0)=2, \tag{3.3.7}$$

and given solution is $v(x,t)=2\ sech(t)+x\ tanh(t)$ Applying the homotopy perturbation method (see cf. (3.2.5)*) in the equation* (3.3.7), *we obtain*

$$\frac{\partial v}{\partial t}-\frac{\partial v_0}{\partial t}=p\Big[-v\frac{\partial v}{\partial x}+x-\frac{\partial v_0}{\partial t}\Big]. \tag{3.3.8}$$

We assume that the solution of the problem (3.3.7), *takes the form*

$$v=v_0+pv_1+p^2v_2+\cdots. \tag{3.3.9}$$

Substituting equation (3.3.9) *into the equation* (3.3.8) *and equating the coefficient of like power of p, we obtain the set of differential equations*

$$\left.\begin{array}{ll}
p^0 & : \frac{\partial v_0}{\partial t}-\frac{\partial v_0}{\partial t}=0 \\
p^1 & : \frac{\partial v_1}{\partial t}=-v_0\frac{\partial v_0}{\partial x}+x-\frac{\partial v_0}{\partial x} \\
p^2 & : \frac{\partial v_2}{\partial t}=-\Big(v_0\frac{\partial v_1}{\partial x}+v_1\frac{\partial v_0}{\partial x}\Big) \\
p^3 & : \frac{\partial v_3}{\partial t}=-\Big(v_0\frac{\partial v_2}{\partial x}+v_1\frac{\partial v_1}{\partial x}+v_2\frac{\partial v_0}{\partial x}\Big) \\
p^4 & : \frac{\partial v_4}{\partial t}=-\Big(v_0\frac{\partial v_3}{\partial x}+v_1\frac{\partial v_2}{\partial x}+v_2\frac{\partial v_1}{\partial x}+v_3\frac{\partial v_0}{\partial x}\Big) \\
\vdots & \qquad\qquad\qquad\qquad\qquad\qquad .
\end{array}\right\} \tag{3.3.10}$$

Solving the above equations, we obtain

$$\left.\begin{array}{ll}
v_1 & =x\,t \\
v_2 & =-t^2 \\
v_3 & =-x\frac{t^3}{3} \\
v_4 & =\frac{5}{12}t^4.
\end{array}\right\} \tag{3.3.11}$$

Therefore, the approximate solution of the equation (3.3.7) *is*

$$\begin{aligned}
v(x,t) &= v_0+v_1+v_2+v_3+v_4+\cdots, \\
&= 2+xt-t^2-x\frac{t^3}{3}+\frac{5}{12}t^4+\ldots, \\
&= 2-t^2+\frac{5}{12}t^4+x\Big(t-\frac{t^3}{3}\Big)+\ldots \\
&= 2\Big(1-\frac{t^2}{2!}+\frac{5}{4!}t^4-\ldots\Big)+x\Big(t-\frac{1}{3}t^3+\ldots\Big).
\end{aligned} \tag{3.3.12}$$

Figure 3.4 depicts the approximate solution whereas Figure 3.3 depicts the exact solution.

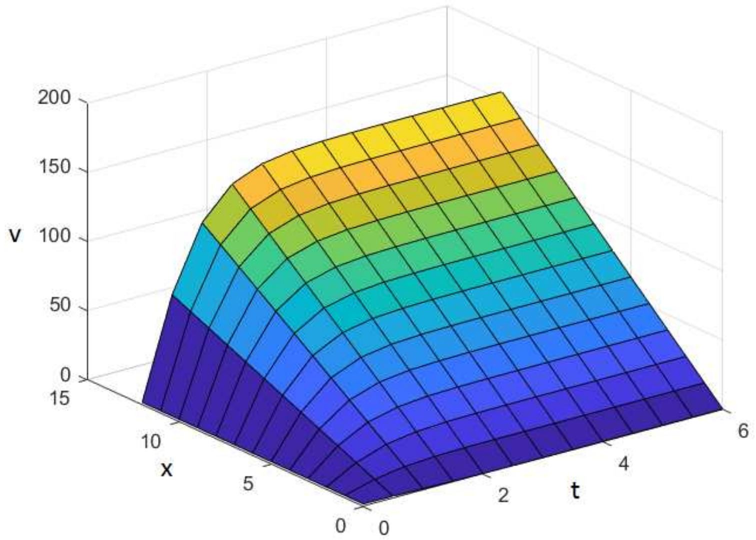

Figure 3.3: Graph of exact solution $v(x,t) = 2\ sech(t) + x\ tanh(t)$.

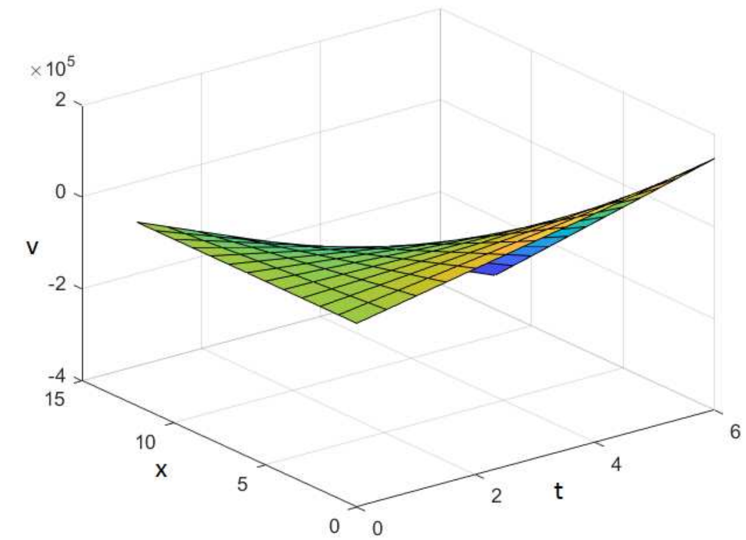

Figure 3.4: Graph of approximate solution $v(x,t) = 2(1 - \frac{t^2}{2!} + \dots) + x(t - \frac{1}{3}t^3 + \dots)$

Example 3.3.3 *Let us consider a nonhomogeneous problem*

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \cos x, \quad v(x,0) = \sin x, \tag{3.3.13}$$

and for this problem, the exact solution is $v(x,t) = \sin x\, e^{-t} + \cos x\ (1 - e^{-t})$. *By using the homotopy perturbation method (see cf.* (3.2.5)*), equation* (3.3.13) *can be written as*

$$\frac{\partial v}{\partial t} - \frac{\partial v_0}{\partial t} = p\Big[\frac{\partial^2 v}{\partial x^2} + \cos x - \frac{\partial v_0}{\partial t}\Big]. \tag{3.3.14}$$

We assume that the solution of the problem (3.3.13) *is of the form,*

$$v = v_0 + p v_1 + p^2 v_2 + \cdots. \tag{3.3.15}$$

Substituting equation (3.3.15) *into equation* (3.3.14) *and equating the coefficient of like power of p, we have the set of differential equations*

$$\left.\begin{array}{ll} p^0 & : \frac{\partial v_0}{\partial t} - \frac{\partial v_0}{\partial t} = 0 \\ p^1 & : \frac{\partial v_1}{\partial t} = \left[\frac{\partial^2 v_0}{\partial x^2} + \cos x - \frac{\partial v_0}{\partial t}\right] \\ p^2 & : \frac{\partial v_2}{\partial t} = \frac{\partial^2 v_1}{\partial x^2} \\ p^3 & : \frac{\partial v_3}{\partial t} = \frac{\partial^2 v_2}{\partial x^2} \\ \vdots & \qquad\qquad\qquad\qquad . \end{array}\right\} \tag{3.3.16}$$

Solving the above equations in (3.3.16)*, we have*

$$\left.\begin{array}{ll} v_1 & = (\cos x - \sin x)t \\ v_2 & = (\cos x - \sin x)\frac{t^2}{2!} \\ v_3 & = (\cos x - \sin x)\frac{t^3}{3!}. \end{array}\right\} \tag{3.3.17}$$

Therefore, the approximate solution of the equation (3.3.13) *is*

$$\begin{aligned} v(x,t) &= v_0 + v_1 + v_2 + \cdots \\ &= \sin x + (\cos x - \sin x)t + (\cos x - \sin x)\frac{t^2}{2!} + (\cos x - \sin x)\frac{t^3}{3!} + \cdots \\ &= \sin x\left(1 - t + \frac{t^2}{2!} - \cdots\right) + \cos x\left(t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right). \end{aligned} \tag{3.3.18}$$

Example 3.3.4 (Application of HPM for Solving the System of PDEs) *Consider the nonlinear system of equations with initial conditions as*

$$\left.\begin{array}{ll} \frac{\partial u}{\partial t}-v\frac{\partial u}{\partial x}-\frac{\partial v}{\partial t}\,\frac{\partial u}{\partial y} & =1-x+y+t \\ \frac{\partial v}{\partial t}-u\frac{\partial v}{\partial x}-\frac{\partial u}{\partial t}\frac{\partial v}{\partial y} & =1-x-y-t \\ u(x,y,0) & =x+y-1 \\ v(x,y,0) & =x-y+1. \end{array}\right\} \tag{3.3.19}$$

Using the homotopy perturbation method (see cf. (3.2.5)*), in the equation* (3.3.19)*, we get*

$$\left.\begin{array}{ll} \frac{\partial u}{\partial t}-\frac{\partial u_0}{\partial t} & =p\Big[v\frac{\partial u}{\partial x}+\frac{\partial v}{\partial t}\frac{\partial u}{\partial y}-\frac{\partial u_0}{\partial t}+1-x+y+t\Big] \\ \frac{\partial v}{\partial t}-\frac{\partial v_0}{\partial t} & =p\Big[u\frac{\partial v}{\partial x}+\frac{\partial u}{\partial t}\frac{\partial v}{\partial y}-\frac{\partial v_0}{\partial t}+1-x-y-t\Big]. \end{array}\right\} \tag{3.3.20}$$

Assume that the solution of the system (3.3.19) *is of the form*

$$\left.\begin{array}{ll} u(x,t) & =u_0+pu_1+p^2u_2+p^3v_3+p^4u_4+\cdots \\ v(x,t) & =v_0+pv_1+p^2v_2+p^3v_3+p^4v_4+\cdots. \end{array}\right\} \tag{3.3.21}$$

Substituting equation (3.3.21) *into equation* (3.3.20) *and equating the coefficient of the like power of* p*, we have the set of differential equations*

$$\left.\begin{array}{ll} p_0 & :\frac{\partial u_0}{\partial t}-\frac{\partial u_0}{\partial t}=0 \\ p_1 & :\frac{\partial u_1}{\partial t}=v_0\,\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial t}\frac{\partial u_0}{\partial y}-\frac{\partial u_0}{\partial t}+1-x+y+t \\ p^2 & :\frac{\partial u_2}{\partial t}=v_1\frac{\partial u_0}{\partial x}+v_0\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial t}\,\frac{\partial u_0}{\partial y}+\frac{\partial u_1}{\partial y}\,\frac{\partial v_0}{\partial t} \\ \vdots & \qquad . \end{array}\right\} \tag{3.3.22}$$

Further the coefficient of the like power of p can be written as

$$\left.\begin{array}{ll} p^0 & :\frac{\partial v_0}{\partial t}-\frac{\partial v_0}{\partial t}=0 \\ p^1 & :\frac{\partial v_1}{\partial t}=u_0\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial t}\,\frac{\partial v_0}{\partial y}-\frac{\partial v_0}{\partial t}+1-x-y-t \\ p^2 & :\frac{\partial v_2}{\partial t}=u_1\frac{\partial u_0}{\partial x}+u_0\frac{\partial v_1}{\partial x}+\frac{\partial u_0}{\partial t}\,\frac{\partial v_1}{\partial y}+\frac{\partial u_1}{\partial t}\,\frac{\partial v_0}{\partial y} \\ \vdots & \qquad . \end{array}\right\} \tag{3.3.23}$$

Solving equations (3.3.22) *and* (3.3.23)*, we get*

$$\left.\begin{array}{ll} u_0 & =x+y-1 \\ u_1 & =2t+\frac{t^2}{2} \\ u_2 & =-\frac{t^3}{6}-\frac{t^2}{2}. \end{array}\right\} \tag{3.3.24}$$

Similarly,

$$\left.\begin{aligned} v_0 &= x-y+1 \\ v_1 &= -\tfrac{t^2}{2} \\ v_2 &= \tfrac{t^3}{6}+\tfrac{t^2}{2}-2t. \end{aligned}\right\} \tag{3.3.25}$$

Substituting the values of v's and u's in the equation (3.3.21), *as* $p \to 1$

$$\begin{aligned} u(x,t) &= u_0+u_1+u_2+u_3+u_4+\cdots \\ &= x+y-1+2t-\frac{t^3}{6}+\cdots \end{aligned} \tag{3.3.26}$$

and

$$\begin{aligned} v(x,t) &= v_0+v_1+v_2+v_3+v_4+\cdots \\ &= x-y+1+\frac{t^3}{6}-2t+\cdots. \end{aligned} \tag{3.3.27}$$

This is the approximate solution of the system of partial differential equation using by HPM.

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Chapter 4

Generalized Homotopy Perturbation Method

The homotopy perturbation method is based on the power series of the homotopy parameter, which transforms the original nonlinear differential equations into a series of linear differential equations. The concept of generalized homotopy perturbation method is proposed by Hector in 2014. In generalized homotopy perturbation method instate of using power series of homotopy parameter in homotopy perturbation method, Taylor transform of a series of trial function (see cf. (4.1.1)) in term of the homotopy parameter is used. The generalized homotopy perturbation method transform the nonlinear differential equation into a series of linear differential equation, generating high precision expression with few algebraic term.

4.1 Basic Concept of Generalized Homotopy Perturbation Method

In generalized homotopy method, the solution for the equation (3.2.4) in Chapter 3, can be written as a series of trial functions in the general form

$$\mu = \mu_0 + g_1(\mu_1 p^1) + g_2(\mu_2 p^2) + g_3(\mu_3 p^3) + \dots \tag{4.1.1}$$

or

$$\mu = \mu_0 + g_1(\mu_1 p^1 + \mu_2 p^2 + \mu_3 p^3 + \ldots), \tag{4.1.2}$$

where $g_i(\cdot)(i = 1, 2, \ldots)$ are arbitrary trial functions and $\mu_0, \mu_1, \mu_2, \mu_3, \ldots$ are unknown functions to be determined by the generalized homotopy method.

Now, equation (4.1.2) can be expanded using the Taylor transform and takes the form

$$\mu = \mu_0 + p\ h_1(\mu_0, \mu_1) + p^2\ h_2(\mu_0, \mu_1, \mu_2) + p^3\ h_3(\mu_0, \mu_1, \mu_2, \mu_3) + \cdots. \tag{4.1.3}$$

After substituting equation (4.1.3) into equation (3.2.4) and equating terms having the same order of p, we obtain a set of equations, which leads to calculate $\mu_0, \mu_1, \mu_2, \cdots$. And finally the approximate solution of equation (3.2.3) can be obtained by taking $p \to 1$ and substituting the calculated values of $\mu_0, \mu_1, \mu_2, \cdots$ in equation (4.1.1) as

$$u = \lim_{p \to 1} \mu = \mu_0 + g_1(\mu_1) + g_2(\mu_2) + g_3(\mu_3) + \ldots \tag{4.1.4}$$

or

$$u = \mu_0 + g_1(\mu_1 + \mu_2 + \mu_3 + \ldots). \tag{4.1.5}$$

4.2 Convergence of Generalized Homotopy Perturbation Method

The convergence of generalized homotopy method [see cf. Hector, 2014] can be discussed by reconsidering the equation (3.2.4) as

$$\mathscr{S}(\mu, p) = L(u_0) + p[f(x) - N(\mu) - L(u_0)] = 0. \tag{4.2.1}$$

Applying the inverse operator, L^{-1} to both sides of the equation (4.2.1), we get

$$\mu = u_0 + p[L^{-1} f(x) - L^{-1} N(\mu) - u_0]. \tag{4.2.2}$$

The equation (4.1.2) can be written as

$$\mu = \mu_0 + \sum_{i=1}^{\infty} g_i(\mu_i p^i) \tag{4.2.3}$$

or

$$\mu = \mu_0 + g_i\Big(\sum_{i=1}^{\infty}(\mu_i p^i)\Big). \tag{4.2.4}$$

Substituting equation (4.2.4) into the right hand side of the equation (4.2.2), we get

$$\mu = u_0 + p\Big[L^{-1}f(r) - (L^{-1}N)\Big[\mu_0 + \sum_{i=1}^{\infty} g_i(\mu_i p^i)\Big] - u_0\Big] \tag{4.2.5}$$

or

$$\mu = u_0 + p\Big[L^{-1}f(r) - (L^{-1}N)\Big[\mu_0 + g_i\Big(\sum_{i=1}^{\infty}(\mu_i p^i)\Big)\Big] - u_0\Big]. \tag{4.2.6}$$

The exact solution of equation (3.2.3) is obtained by taking the limit as $p \to 1$ into equation (4.2.6), we have

$$\begin{aligned} u &= \lim_{p\to 1}\Big(pL^{-1}f(x) - p(L^{-1}N)\Big[\mu_0 + \sum_{i=1}^{\infty} g_i(\mu_i p^i)\Big] + u_0 - pu_0\Big) \\ &= L^{-1}f(x) - \Big[\sum_{i=1}^{\infty}(L^{-1}N)\Big(\mu_0 + \sum_{i=1}^{\infty} g_i(\mu_i)\Big)\Big] \end{aligned} \tag{4.2.7}$$

or

$$\begin{aligned} u &= \lim_{p\to 1}\Big(pL^{-1}f(x) - p(L^{-1}N)\Big[\mu_0 + g_i\Big(\sum_{i=1}^{\infty}(\mu_i p^i)\Big)\Big] + u_0 - pu_0\Big) \\ &= L^{-1}f(x) - \Big[\sum_{i=1}^{\infty}(L^{-1}N)\Big(\mu_0 + g_i\Big(\sum_{i=1}^{\infty}(\mu_i)\Big)\Big)\Big]. \end{aligned} \tag{4.2.8}$$

Definition 4.2.1 (Banach Space): *A Banach space is a vector space* **X** *over the field* **R** *of real numbers, or over the field* **C** *of complex numbers, which is equipped with a norm and which is complete with respect to that norm, that is, for every Cauchy sequence* $\mathbf{x_n}$ *in* **X**, *there exists an element* x *in* **X** *such that*

$$\lim_{n\to\infty} \mathbf{x_n} = \mathbf{x},$$

or equivalent

$$\lim_{n\to\infty} \|\mathbf{x_n} - \mathbf{x}\| = \mathbf{0}.$$

Theorem 4.2.1 (A Fixed Point Theorem): *Let* V *be an open set in a Banach space* **X**, *and let* G *be a differentiable map from* V *to* **X**. *Suppose that there is a closed ball* $B = B(x_0, r)$ *in* V *such that*

$$(i) k = sup\|G'(x)\| < 1,$$

$$(ii) \|G(x_0) - x_0\| < r(1-k),$$

then G has a unique fixed point in B.

Definition 4.2.2 (Sufficient Condition of Convergence [Hector, 2014]): *Suppose that* $\mathbf{X}$ *and* $\mathbf{Y}$ *are Banach spaces and* $N : \mathbf{X} \longrightarrow \mathbf{Y}$ *is a contractive nonlinear mapping then*

$$\forall\, x, x^* \in \mathbf{X}; \quad \|N(x) - N(x^*)\| \le m\|x - x^*\|; \quad 0 \le m < 1.$$

According to Banach fixed-point theorem, N has a unique fixed point u such that $N(u) = u$. *Assume that the sequence can be written as*

$$W_n = N(W_{n-1}), \quad W_{n-1} = v_0 + \sum_{i=1}^{n-1} g_i(v_i), \quad n = 1, 2, 3, 4, \cdots.$$

If we have $W_0 = v_0 \in B_r(u)$, *where* $B_r(u) = \{x^* \in \mathbf{X} : \|x^* - u\| \le r\}$, *then, it can be shown that*

$$(i) W_n \in B_r(u),$$

$$(ii) \lim_{n \to \infty} W_n = u.$$

Proof (i)

By using the induction method, for $n = 1$, *we have*

$$\|W_1 - u\| = \|N(W_0) - N(u)\| \le m\|x_0 - u\|.$$

Suppose that as induction hypothesis that $\|W_{n-1} - u\| \le m^{n-1}\|x_0 - u\|$, *then*

$$\|W_n - u\| = \|N(W_{n-1}) - N(u)\| \le m\|W_{n-1} - u\| \le m^n\|x_0 - u\|.$$

We have

$$\|W_n - u\| = \|N(W_{n-1}) - N(u)\| \le m\|W_{n-1} - u\| \le m^n\|w_0 - u\| \le m^n r \le r$$

$$\Rightarrow W_n \in B_r(u).$$

Proof (ii)

We have

$$\|W_n - u\| = \|N(W_{n-1}) - N(u)\| \leq m\|W_{n-1} - u\| \leq m^n\|x_0 - u\|$$

$$\Rightarrow \|W_n - u\| \leq m^n\|x_0 - u\|$$

$$\Rightarrow \lim_{n\to\infty} m^n = 0, \quad \lim_{n\to\infty} \|W_n - u\| = 0, \quad m \neq 1,$$

that is,

$$\lim_{n\to\infty} W_n = u.$$

4.3 Generalized Newtonian Iteration Formula

Consider a nonlinear algebraic equation

$$f(x) = 0,$$

where $f(x) \in \mathscr{C}^\infty[a,b]$ is a continuous real valued function. Assume that the above equation has at least one solution defined in the interval [a,b].
Let $x_0 \in [a,b]$ be an initial guess of the unknown solution x. Obviously, $f(x) - f(x_0) \in \mathscr{C}^\infty[a,b]$ can be deformed continuously to $f(x) \in \mathscr{C}^\infty[a,b]$, i.e., they are homotopy. Thus, we can construct such a homotopy of function

$$\mathscr{S}(x,p) = (1-p)[f(x) - f(x_0)] + pf(x), \tag{4.3.1}$$

where $p \in [0,1]$ is the homotopy-parameter. When $p = 0$ and $p = 1$, we get

$$\mathscr{S}(x,0) = f(x) - f(x_0), \quad \mathscr{S}(x,1) = f(x),$$

respectively.

Thus as p increases from 0 to 1, $\mathscr{S}(x,p)$ varies continuously from $f(x) - f(x_0)$ to $f(x)$.
Now, if $\mathscr{S}(x,p) = 0$, then the equation (4.3.1) becomes

$$(1-p)[f(x) - f(x_0)] + pf(x) = 0, \quad p \in [0,1],$$

this is a parameter-family of algebraic equations. The solution of the above parameter family of algebraic equations is dependent upon the homotopy parameter p. Replacing x by $x(p)$ in the above equation, we can write more precisely

$$(1-p)[f(x(p))-f(x_0)]+pf(x(p))=0, \quad p\in[0,1]. \tag{4.3.2}$$

When $p=0$, the equation (4.3.2) takes the form

$$f(x(0))-f(x_0)=0$$

and whose solution is

$$x(0)=x_0.$$

Similarly when $p=1$, we have

$$f[x(1)]=0,$$

which is exactly the same as the original algebraic equation $f(x)=0$ and whose solution is $x(1)=x$.

Therefore, as the homotopy-parameter p increases from 0 to 1, $x(p)$ deforms from the initial guess x_0 to the solution x of the original equation $f(x)=0$. So, the equation (4.3.2) defines a homotopy of function $x(p):x_0\sim x$. For simplicity the parameter family of equations (4.3.2) is called the zeroth-order deformation equation because it define a continuous deformation from the initial guess x_0 to the solution x of the original equation $f(x)=0$.

Note that $x(p)$ is a function of the homotopy parameter p and $x(p)$ is real analytic at $p=0$, so that it can be expanded into a Maclaurin series with respect to the homotopy-parameter p, i.e.,

$$x(p)\sim x_0+\sum_{k=1}^{+\infty}x_kp^k, \tag{4.3.3}$$

where $x(0)=x_0$ is employed, and

$$x_k=\frac{1}{k!}\frac{d^kx(p)}{dp^k}=\mathscr{D}_k[x(p)]. \tag{4.3.4}$$

Here the series (4.3.3) is called homotopy-Maclaurin series and $\mathscr{D}_k[x(p)]$ is called the kth-order homotopy-derivative of $x(p)$ [Liao, 1995]. If we consider $p = 1$ in equation (4.3.3) then we have

$$x = x_0 + \sum_{k=1}^{+\infty} x_k. \tag{4.3.5}$$

Now, differentiating the zeroth-order deformation equation (4.3.2) with respect to the homotopy-parameter p, we have

$$f'[x(p)]\frac{dx(p)}{dp} + f(x_0) = 0. \tag{4.3.6}$$

Setting $p = 0$ in the above equation and using the relationship $x(0) = x_0$, we get

$$f'(x_0)\frac{dx(p)}{dp} + f(x_0) = 0 \tag{4.3.7}$$

and

$$\frac{dx(p)}{dp} = x_1.$$

More precisely, the above equation (4.3.7) can be written of the form

$$x_1 f'(x_0) + f(x_0) = 0, \tag{4.3.8}$$

which is called first order deformation equation and whose solution is

$$x_1 = -\frac{f(x_0)}{f'(x_0)}.$$

Similarly, differentiating the zeroth-order deformation equation (4.3.2) twice with respect to p and dividing both side by 2!, we have

$$\frac{1}{2!}f''(x)\left(\frac{dx}{dp}\right)^2 + f'(x)\left(\frac{1}{2!}\frac{d^2x}{dp^2}\right) = 0. \tag{4.3.9}$$

Again setting $p = 0$ and using $x(0) = x_0$, the above equation takes the form

$$\frac{1}{2}f''(x_0)\left(\frac{dx}{dp}\right)^2 + f'(x_0)\left(\frac{1}{2!}\frac{d^2x}{dp^2}\right) = 0.$$

Using the equation (4.3.4), we have the 2nd-order deformation equation

$$\frac{1}{2}x_1^2 f''(x_0) + x_2 f'(x_0) = 0$$

and whose solution is

$$x_2 = -\frac{x_1^2 f''(x_0)}{2f'(x_0)} = -\frac{f^2(x_0)f''(x_0)}{2[f'(x_0)]^3}.$$

Using equation (4.3.8), we have the 1st-order homotopy-approximation

$$x \approx x_0 + x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \tag{4.3.10}$$

and the second order homotopy-approximation

$$x \approx x_0 + x_1 + x_2 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{f^2(x_0)\, f''(x_0)}{2[f'(x_0)]^3}. \tag{4.3.11}$$

The equation (4.3.11) is called a Newton iteration formula given by Sir Isaac Newton.

4.4 Application of Generalized Homotopy Perturbation Method

Let us consider a nonlinear boundary value problem,

$$\frac{d^2y}{dx^2} + n \exp(y) = 0, \quad y(0), \quad y(1) = 0, \tag{4.4.1}$$

where n is known as Gelfand's parameter. Apply the GHP method, introduced by Hector (2014) to get the approximate solution of the problem. In order to facilitate the application of GHP method, we approximate the exponential term using a sixth-order Taylor expansion as,

$$y'' + n\Big(1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \frac{1}{24}y^4\Big) = 0, \quad y(0) = 0, \quad y(1) = 0. \tag{4.4.2}$$

Using the equation (4.4.2), the following homotopy equations can be constructed

$$(1-p)(y'' + ny + n) + p\Big(y'' + n\Big(1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \frac{1}{24}y^4\Big)\Big) = 0, \tag{4.4.3}$$

and

$$(1-p)(y'') + p\Big(y'' + n\Big(1 + y + \frac{1}{2}y^2p^2 + \frac{1}{6}y^3p^3 + \frac{1}{24}y^4p^4\Big)\Big) = 0. \tag{4.4.4}$$

The solutions for equation (4.4.3) and (4.4.4) can be obtained in the form

$$y = y_0 + \log(1 + y_1 p + y_2 p^2 + y_3 p^3 + \cdots), \tag{4.4.5}$$

and the Taylor transform of equation (4.4.5) is

$$y = y_0 + y_1 p + (y_2 - y_1^2/2)p^2 + (y_3 - y_1 y_2/3 + y_1(-2y_2 + y_1^2)/3)p^3 + \cdots. \qquad (4.4.6)$$

Now, substituting equation (4.4.6) into equation (4.4.4), and rearranging the terms of the same order of p, we get the coefficient of like power of p

$$\left.\begin{array}{ll} p^0 & : y_0'' + ny_0 + n = 0, \quad y_0(0) = 0, \quad y_0(1) = 0 \\ p^1 & : y_1'' + ny_1 - ny_0^2/2 + ny_0^3/6 = 0, \quad y_1(0) = 0, \quad y_1(1) = 0 \\ \vdots & \end{array}\right\} \qquad (4.4.7)$$

and

$$\left.\begin{array}{ll} p^0 & : y_0'' + n = 0, \quad y_0(0) = 0, \quad y_0(1) = 0 \\ p^1 & : y_1'' + ny_1 = 0, \quad y_1(0) = 0, \quad y_1(1) = 0 \\ \vdots & . \end{array}\right\} \qquad (4.4.8)$$

Solving the above equations in (4.4.8), we get

$$\left.\begin{array}{ll} y_0 & = -\frac{1}{2}nx(x-1), \\ y_1 & = \frac{1}{24}n^2x(x^3 - 2x^2 + 1). \end{array}\right\} \qquad (4.4.9)$$

Substituting the values of equations (4.4.9) in equation (4.4.5) and taking $p \to 1$, we obtain the solution approximation upto the sixth order

$$y = \lim_{p \to 1} y = y_0 + \log\left(1 + \sum_{i=1}^{6} y_i\right). \qquad (4.4.10)$$

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Chapter 5

Conclusions and Future plan

5.1 Conclusions

In this project report, we have discussed the application of homotopy perturbation method (He, 1999) and generalized homotopy perturbation method (Hector, 2014) for solving nonlinear differential equations. We have applied the homotopy perturbation method to solve the Inviscid Burger's nonlinear problem and the nonhomogeneous Advection problem. The result of the homotopy perturbation method gives us evidence that the solution is effective and have a high accuracy (cf. Ganji et al., 2009) to find the solutions for the problems.

5.2 Future plan

- We will apply homotopy perturbation method to analyze stability and error estimation of Lotka Voltera model problem in near future.

- Solving delay differential equations (cf. Shakeri and Dehghan, 2008) using the method of homotopy perturbation.

- We will apply HPM and GHPM for solving higher order nonlinear differential equations, occur in physical problems.

- Our plan is to solve systems of 2nd order PDEs using GHPM.
- We also have a plan to solve fractional differential equations using HPM and GHPM.

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