HIGGS INFLATION
M.Sc. Thesis

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HIGGS INFLATION

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OUSEPH C.J.

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled HIGGS INFLATION in the partial fulfillment of the requirements for the award of the degree of MASTER OF SCIENCE and submitted in the DISCIPLINE OF PHYSICS, Indian Institute of Technology Indore, is an authentic record of my own work carried out during the time period from July-2016 to June-2018 under the supervision of Dr. Subhendu Rakshit, Professor, IIT Indore.
The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

OUSEPH C.J.

....

This is to certify that the above statement made by the candidate is correct to the best of my/our knowledge.

Dr. Subhendu Rakshit

....

OUSEPH C.J. has successfully given his/her M.Sc. Oral Examination held on 21-June-2018.

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ABSTRACT

In this thesis, we have investigated the possibility that the Standard Model Higgs boson playing the role of inflaton. A standard $\lambda\phi^4$ potential requires an unphysically small coupling constant ($\lambda \sim 10^{-13}$) to obtain density perturbations in agreement with observational data. A large coupling between the scalar field and the Ricci curvature scalar relaxes this condition, and the field ϕ might be identified with the Standard Model Higgs field. In such a model, the predicted values of the spectral index and the tensor-to-scalar ratio are also in agreement with current observational data. However, quantum corrections seem to break the theory down at the cut-off scale $\Lambda \sim M_{\rm Pl}/\xi$, which is below the energy scale where inflation takes place. By the extension of standard model, we solved the unitarity problem associated with the Higgs Inflation. We examine the possibility of minimal Higgs Inflation via the modification of potential.

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Chapter 1

Introduction

The idea that all matter around us consists of indivisible particles dates back to the ancient Greeks, although it should be said that this principle has always been more based on abstract reasoning or just pure speculation rather than empirical grounds. This was changed at the beginning of the 18th century when experimental observations led to the development of atomic theory. Later it turned out that also atoms consist of smaller subparticles. The first known particle still seen as elementary today is the electron, discovered by J.J. Thomson in 1897. In the middle of the 20th century, with particle accelerators reaching higher energies, more and more exotic particles were found. A careful analysis of all these experiments resulted in the conclusion that all matter is made up of three generations of quarks and leptons. The Standard Model, which was more or less completed in the mid 1970's, successfully describes the dynamics of these elementary particles. A crucial ingredient is the Higgs mechanism, which is the way particles obtain their masses in this model. This predicts the existence of the Higgs boson, which was finally discovered at the Large Hadron Collider in 2012. Unfortunately, the Standard Model fails to describe gravitation. Although gravity is negligibly small compared to the other forces in the Standard Model, at large mass/energy scales it becomes the dominant force. The currently accepted description of gravitation, that suffice at least at large scales, is Einstein's General Theory of Relativity. The Einstein's equations, which are part of this theory, allow the possibility of a non-static universe. Indeed, in 1929 it was shown that we live in an expanding universe and in 1998 it turned out that it is even accelerating. Reversing this picture, we see the universe must originate from a singular point in space-time, known as the Big Bang. According to this theory, the universe was once much hotter than it is now. The 'thermal history' of our universe after $t \sim 10^{-10}$ seconds is quite well understood by a combination of different disciplines in physics and cosmology. Going back even further in time, the energy density was so high that the Standard Model can no longer be trusted. Interestingly, the very early universe has left its mark on the Cosmic Microwave Background. Thus a careful study of the CMB might tell us more about physics beyond the Standard Model. The standard theory of an expanding universe has several shortcomings as was realized in the 1970's. First of all, the Cosmic Microwave Background looks almost exactly the same in every direction. However, at the time of decoupling only regions of the CMB observed over an angle of about one degree were in causal contact. Another problem is that the energy density of the universe seems to have an extremely fine tuned value. If it was only slightly bigger or smaller, the universe would immediately have collapsed or ripped apart, yet we know the universe is more than 13 billion years old. This is of course related to the anthropic principle, but this does not give a satisfactory answer. In 1981, A. Guth proposed an inflationary epoch right after the Big Bang as a solution to these problems. An important confirmation for inflation was the observed scale invariance of the density perturbations. Although we do not know what caused the inflationary epoch, it is well known that a scalar field slowly rolling down a potential can lead to inflation. This mechanism is known as slow-roll inflation. In this thesis, we will investigate the possibility that inflation is caused by the only scalar field in the Standard Model, namely the Higgs field. As we will see, a crucial ingredient will be a coupling between the Ricci scalar and the Higgs field. But before discussing Higgs inflation, we give an introduction to cosmology and particle physics in the first two chapters.

Chapter 2

Cosmology

2.1 Homogeneous and Isotropic Universe

The Cosmological Principle states that our universe is homogeneous and isotropic [1]. A homogeneous universe is one that is translation invariant, while isotropy means that there is no preferred direction. This may seem strange, since this does certainly not hold for the universe as we see it. The Earth is clearly inhomogeneous and anisotropic and there is also structure on larger scales, e.g. planets, stars, galaxies etc. However, on very large scales the cosmological principle does hold and then the universe can be regarded as some kind of cosmic fluid, the dynamics of it described by the laws of gravity.

- Homogeneous
  The cosmos on a big scale seems to be homogeneous. Via that we mean, the universe, on a sufficiently massive scale, appears the identical, independent of the position of the observer. The statistical distribution of the energy-matter and cosmic structures (galaxies) in our universe seems to be the same, independent of the point of observation.

- Isotropic
  Isotropic implies direction independent. There is no most preferred direction in our universe. The universe, looks alike whichever direction we see. Symmetry of the cosmos supported by the observation Cosmic Microwave Background Radiation (CMBR).

2.2 FRW Cosmology

2.2.1 Space-Time Geometry

Geometry of a space-time is entirely described by the metric tensor. It turns observer-dependent coordinates $X^\mu = (t, x^i)$ into the invariant line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{1}$$

In special relativity, the Minkowski metric is the same everywhere in space and time,

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1). \tag{2}$$

In general relativity, the metric will take care of the entire information of the space-time coordinate,

$$g_{\mu\nu}(t, \vec{x}). \tag{3}$$

The space-time dependence of the metric incorporates the effects of gravity. How the metric depends on the position in space-time is determined by the distribution of matter and energy in the universe. For an arbitrary matter distribution, it is impossible to find the metric from the Einstein equations.

2.2.2 Einstein field equation and FRW Cosmology

Einstein's field equation [1, 2] determines the behavior of space-time in the presence of mass-energy density is given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \tag{4}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ is the energy momentum tensor and $G_{\mu\nu} =$

$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is called the Einstein tensor. The energy momentum tensor can be represented as,

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \tag{5}$$

Here P and ρ represent the pressure and energy density respectively.

2.2.3 Robertson-Walker Metric and Frieddman Solution

According to Einstein's Theory of General Relativity, gravity is not a force but occurs due to the fact that space-time is curved, the source of the curvature being the stress-energy tensor. The geometry of space-time is described by the metric, which determines the gravitational field. It can be shown that the only metric consistent with the cosmological principle is the Robertson-Walker metric [3]. This metric is an exact solution of the Einstein field equations and can be obtained by multiplying the spatial part of the metric for static space by a time-dependent scale factor $a(t)$. In terms of the spherical coordinates (r, θ, ϕ) it has the following form:

$$ds^2 = -c^2 dt^2 + a^2(t)\Big[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big]. \tag{6}$$

The curvature of the space is determined by the value of k. FRW metric with $k = 0$ corresponding to flat universe. $k = +1$ coresponding to open universe and $k = -1$ for that of closed universe, where a is called the scaling factor. The metric tensor $g_{\mu\nu}$ is defined as

$$g_{\mu\nu} = \begin{pmatrix} a^2 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2r^2 & 0 \\ 0 & 0 & 0 & a^2r^2sin^2\theta \end{pmatrix}. \tag{7}$$

- Friedmann Equations
  Using the metric tensor (eq. 7) and the energy-momentum tensor (eq. 5) in the Einstein's field equation, (eq. 4) gives the following set of equations:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G\rho}{3}, \tag{8}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} = -8\pi GP. \tag{9}$$

  Here ρ and P should be understood as the sum of all contributions to the energy density and pressure in the universe, $\frac{\dot{a}}{a}$ is called the Hubble parameter $H(t)$.

2.3 Non-static Models of Universe

In this section, we will consider the time evolution of $a(t)$ for a matter dominated universe with spatial curvature parameter $k = 0, -1$ and $+1$. In a matter dominated universe, the energy density is dominated by that of non-relativistic particles and pressure $P = 0$. Frieddman equations for a matter dominated universe take the form[1, 3],

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G\rho}{3}, \tag{10}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} = 0. \tag{11}$$

Using eq. 11 and eq. 12 we get,

$$2\frac{\ddot{a}}{a} = -\frac{8\pi G\rho}{3}. \tag{12}$$

We can write eq. 10 as

$$H^2 + \frac{k}{a^2} = \frac{8\pi G\rho}{3}, \tag{13}$$

or

$$\begin{aligned} \frac{k}{a^2} &= \frac{8\pi G}{3c^2}\big[\rho - \frac{3H^2}{8\pi G}\big], \\ &= \frac{8\pi G}{3c^2}\big[\rho - \rho_c\big], \end{aligned} \tag{14}$$

$$\rho_c = \frac{3H^2}{8\pi G}. \tag{15}$$

Here ρ_c is called the critical density of the universe. Another useful quantity, the deceleration parameter is defined as

$$q(t) = -\frac{\ddot{a}(t)a(t)}{\dot{a}(t)^2}. \tag{16}$$

From eq. 13 and eq. 16

$$q(t) = \frac{8\pi G\rho_0 a_0^3}{3H^2(t)a^3(t)}. \tag{17}$$

For the present time, we have

$$q_0 = \frac{8\pi G\rho_0 a_0^3}{6H_0^2(t)a_0^3} = \frac{\rho_0}{2\rho_c}. \tag{18}$$

- Case 1: Closed Universe
  For a closed universe $k = 1, \rho > \rho_c$.
  For $k = 1, q > \frac{1}{2}$ eq. 10 gives

$$\frac{\dot{a}^2}{a^2} + \frac{1}{a^2} = \frac{8\pi G\rho_0 a_0^3}{3a^3}, \tag{19}$$

$$\dot{a}^2 + 1 = \frac{8\pi G\rho_0 a_0^3}{3a}, \tag{20}$$

$$\dot{a}^2 + 1 = \frac{C}{a}, \tag{21}$$

where we define $C = \frac{8\pi G\rho_0 a_0^3}{3}$. Now eq. 21 becomes,

$$\frac{da}{dt} = \sqrt{\frac{C-a}{a}}, \tag{22}$$

$$\implies \int_0^t dt = \int_0^a \sqrt{\frac{a}{C-a}} da. \tag{23}$$

Using the angular parameter θ, we write

$$a = C\sin^2\frac{\theta}{2} = \frac{C}{2}(1-\cos\theta) \tag{24}$$

$$\implies da = C\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta. \tag{25}$$

By simplifying eq. 23 gives

$$a = \frac{C}{2}(1-\cos\theta), \tag{26}$$

$$t = \frac{C}{2}(\theta - \sin\theta). \tag{27}$$

- Case 2: Flat Universe
  For a flat universe we have $k = 0, \rho = \rho_c$ and $q = \frac{1}{2}$ from eq. 11 with $k = 0$, we have

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho_0 R_0^3}{3R^3}. \tag{28}$$

From eq. 17

$$H^2 = \frac{8\pi G\rho_0}{3}, \tag{29}$$

eq. 28 becomes

$$\frac{\dot{a}^2}{a^2} = \frac{H^2 a_0^3}{a^3} \tag{30}$$

or

$$a\dot{a}^2 = H^2 a_0^3 \implies \sqrt{a}\dot{a} = (H^2 a_0^3)^{\frac{1}{2}}. \tag{31}$$

By integrating the above equation we get

$$a(t) = \left[\frac{3}{2}((H^2 a_0^3)^{\frac{1}{2}})\right]^{\frac{2}{3}} t^{\frac{2}{3}} \tag{32}$$

and

$$t = \frac{2}{3(H^2 a_0^3)^{\frac{1}{2}}} a^{\frac{3}{2}}(t). \tag{33}$$

For a universe dominated by relativistic particles, i.e., a radiation dominated universe,

$$a(t) \approx t^{\frac{1}{2}}.$$

- Case 3: Open Universe
  For an open universe $k = -1, \rho < \rho_c$ and $q < \frac{1}{2}$.
  For $k = -1$, eq. 11 gives

$$\dot{a}^2 - 1 = \frac{8\pi G \rho_0 a_0^3}{3a} = \frac{C}{a}, \tag{34}$$

$$\implies \frac{da}{dt} = \sqrt{\frac{C + a}{a}}, \tag{35}$$

$$\int_0^t dt = \int_0^a \sqrt{\frac{a}{C + a}} da, \tag{36}$$

$$a = C \sinh^2 \frac{\theta}{2} = \frac{C}{2}(1 - \cosh\theta). \tag{37}$$

Eq. 36 on simplification gives

$$a = \frac{C}{2}(\cosh\theta - 1), \tag{38}$$

$$t = \frac{C}{2}(\sinh\theta - \theta). \tag{39}$$

Chapter 3

Cosmological Inflation

Alan Guth proposed inflation as a solution of horizon and flatness problem. The inflation hypothesis states that the universe underwent a period of extremely rapid exponential expansion somewhere between 10^{-36} and 10^{-32} seconds after the Big Bang in which its volume increased by at least a factor 10^{78}. Up to now the mechanism behind inflation is unclear, but after more than thirty years the inflation model itself is still a working hypothesis about the very early universe.

3.1 The Horizon problem

The Hubble law shows that the universe was once much denser than it is now. If we go back in time, we see that all space-time collapses into a single point. This is the Big Bang scenario. Conventional Big Bang theories fails to address problems in the early universe cosmology.

3.1.1 Light and Horizon

The size of a causal patch of space is determined by how far light can travel in a certain amount of time. In an expanding space-time, the propagation of light (photons) is best studied using conformal time (τ) [5, 7] via the relation

$$d\tau = \frac{dt}{a(t)}. \tag{40}$$

The FRW metric factorizes into a static Minkowski metric $\eta_{\mu\nu}$ multiplied by a time dependent conformal factor

$$ds^2 = a^2(t)(-d\tau^2 + dr^2 + d\Omega^2). \tag{41}$$

Since the space-time is isotropic, we can always define the co-ordinate system so that the light travels purely in the radial direction (i.e., $\theta = \phi =$ constant). The evolution is then determined by a two-dimensional line element

$$ds^2 = a^2(t)(-d\tau^2 + dr^2) \tag{42}$$

for a photon $ds^2 = 0$ (photons are traveling along null geodesic) i.e., $d\tau^2 = dr^2$.

- Particle horizon
  The causal or particle horizon is equal to the maximum distance a light ray coming from a particle could have travelled. In co-moving coordinates the co-moving particle horizon τ or r is given by,

$$\Delta r = \Delta\tau = \tau_1 - \tau_0 = \int_{t_i}^{t} \frac{dt}{a(t)} = \int (aH)^{-1} d\ln(a), \tag{43}$$

  where $(aH)^{-1}$ is the Hubble radius.

- Event horizon
  In co-moving coordinates, the greatest distance from which an observer at time t_f will receive signals emitted at any time later than t is given by,

$$\Delta r = \tau_f - \tau = \int_{t}^{t_f} \frac{dt}{a(t)}. \tag{44}$$

  This is called the Event Horizon.

3.1.2 The Growing Hubble Sphere

It is the particle horizon that is relevant for the horizon problem of the standard Big Bang cosmology. The above equation shows that the elapsed conformal time depends on the evolution of the co-moving Hubble radius $(aH)^{-1}$. For example, for a universe dominated by a fluid with equation of state $w = \frac{P}{\rho}$, we find that this evolves as

$$(aH)^{-1} \propto a^{\frac{(1+3w)}{2}}. \tag{45}$$

Note the dependence of the exponent on the combination $1+3w$. All familiar matter sources satisfy the [4] strong energy condition (SEC), $1 + 3w < 0$. Hence it was reasonable for post-Hubble physicists to assume that the co-moving Hubble radius increases as the universe expands. Performing the integral in (eq. 43) gives,

$$\tau \propto \frac{2}{(1+3w)} a^{\frac{1+3w}{2}} \tag{46}$$

up to an irrelevant integration constant. For conventional matter sources the initial singularity therefore is at $\tau_i = 0$:

$$\tau_i \propto a^{\frac{1+3w}{2}} = 0;\ w > \frac{1}{3} \tag{47}$$

and the co-moving horizon is finite;

$$\Delta r \propto a^{\frac{1+3w}{2}};\ w > \frac{1}{3}. \tag{48}$$

3.1.3 Why is the CMB so uniform?

The CMB is a nearly perfect black-body spectrum, anisotropies are only at the 10^{-5} level. Thus two photons coming from opposite directions on the sky must have been in thermal equilibrium in the past. However at the time of last scattering the universe we observe today consisted of a large number of causally disconnected regions. Only regions within angle of about one degree were causally connected at the time of recombination. This is illustrated by the space-time diagram in figure 1.

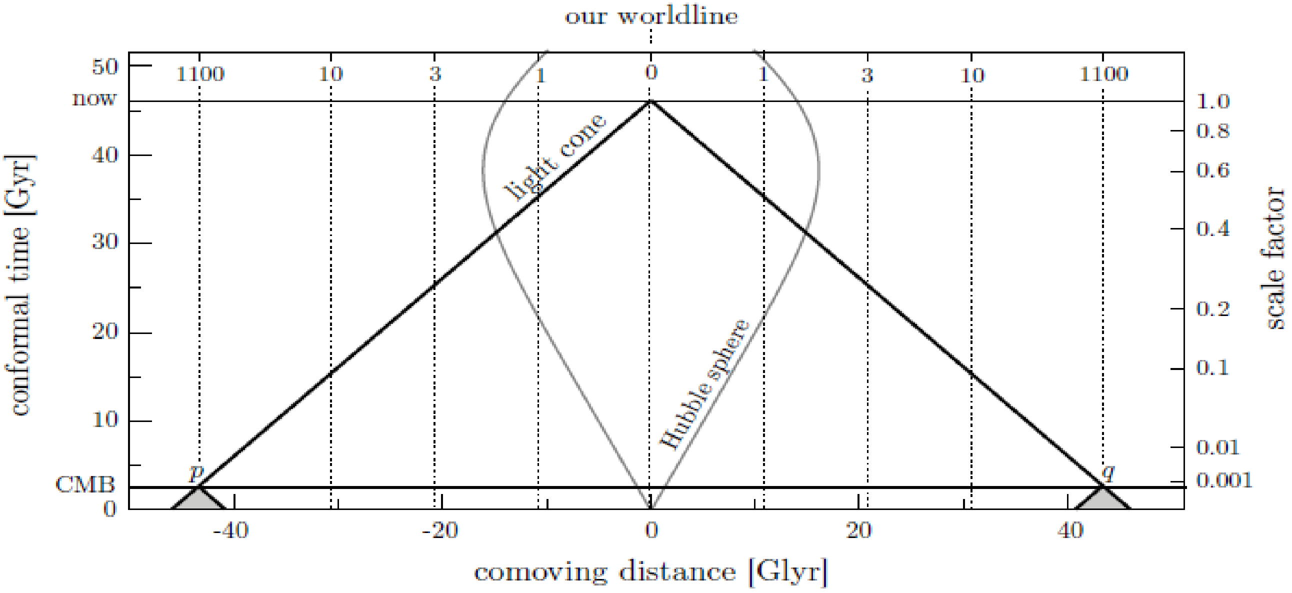

Figure 1: World line of early universe showing horizon problem.

3.14 A Shrinking Hubble sphere

During inflation the co-moving Hubble length, which is the characteristic length scale of the universe, decreases with time. In co-moving coordinates the observable universe becomes smaller.

$$\frac{d(aH)^{-1}}{dt} < 0 \tag{49}$$

The shrinking Hubble sphere requires a SEC-violating fluid. Now the big-bang singularity is pushed to negative conformal time. This is illustrated by the space-time diagram in figure 2.

$$\tau_i \propto \frac{a_i^{\frac{1+3w}{2}}}{1+3w} \xrightarrow{a_i \to 0, w<1/3} -\infty \tag{50}$$

3.2 Flatness Problem

The closure parameter was defined as $\Omega = \frac{\rho}{\rho_c}$. The critical density is the density that would close the universe. From the

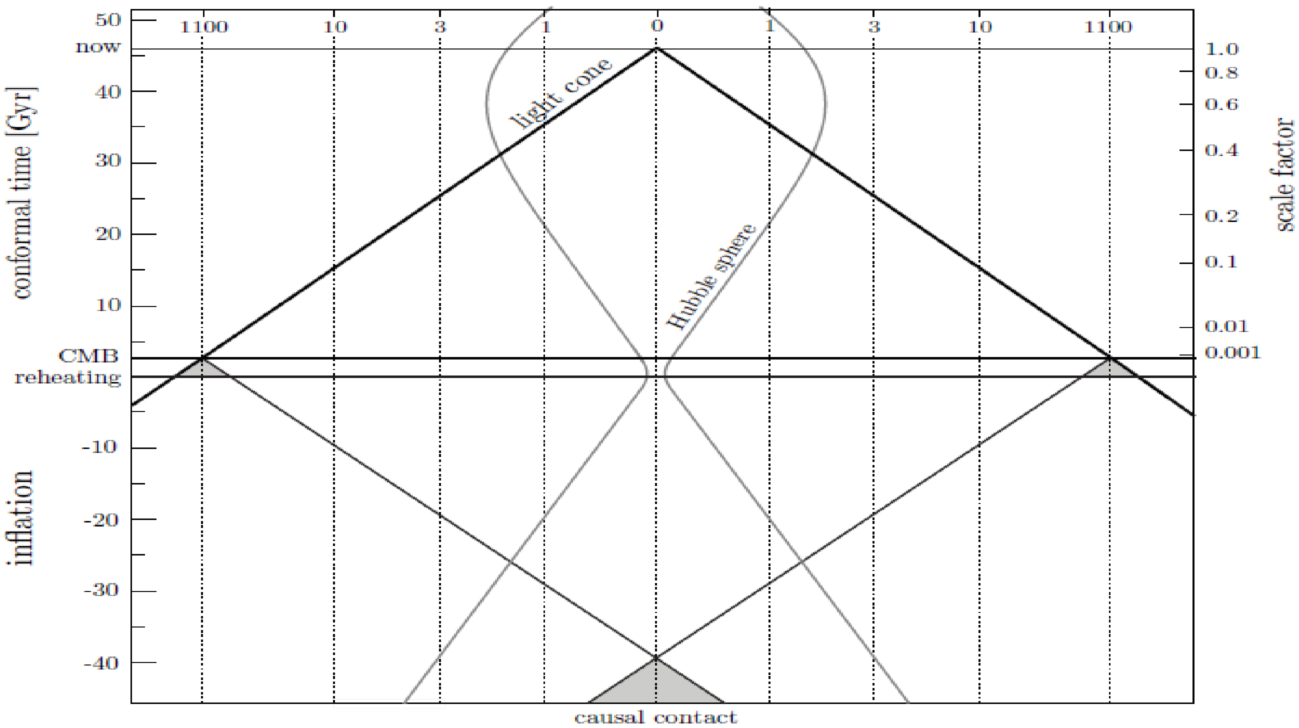

Figure 2: World line of early universe with inflation.

Friedmann equation

$$H^2 = \frac{\rho}{3M_{\rm Pl}^2} - \frac{k}{a^2}$$

and $\rho_c = 3M_{\rm Pl}^2 H^2$, then,

$$\Omega = \frac{k}{(aH)^2}.$$

A flat universe leads the value of $\Omega = 1$. In standard cosmology the co-moving Hubble radius increases with time, thus $\mid \Omega - 1 \mid$ diverges with time. Experiments show that Ω is very close to 0, the observed value is $\Omega_{\rm obs} = 1.00 \pm 0.01$. To have Ω still of order unity today, Ω must have been extremely fine-tuned to the value $\Omega_0 = 1$ in the past. This brings up the question why our universe is so flat or equivalently why is our universe is so old?

A universe that would not immediately collapse or end up in a big crunch requires Ω to be extremely fine-tuned to zero.

3.2.1 Solution of Flatness problem

Flatness problem is solved by inflation, because during in inflation

$$|\ \Omega(a) - 1\ |= \frac{k}{(aH)^2}$$

is driven to zero. During the conventional expansion of the universe the co-moving Hubble radius $(aH)^{-1}$ increases with time and the above equation get diverges. In contrast, in an inflationary era $(aH)^{-1}$ decreases by definition. Physically this means that inflation flattens the curvature of the universe.

3.3 Inflation-A Shrinking Hubble sphere

Inflation is an era of accelerated expansion, the shrinking Hubble sphere consider as the definition of inflation since it relates directly to the horizon problem and is also key for the inflationary mechanism of generating fluctuations. This definition of inflation is equivalent with the following ones

- **Accelerated expansion**:

  From the relation

  $$\frac{d}{dt}(aH)^{-1} = \frac{d}{dt}(\dot{a})^{-1} = -\frac{\ddot{a}}{\dot{a}}^2, \tag{51}$$

  shrinking co-moving Hubble radius implies accelerated expansion

  $$\ddot{a} > 0. \tag{52}$$

  This explains why inflation is often defined as an era of accelerated expansion.

- **Slowly varying Hubble parameter**:
  Alternatively, we may write

$$\frac{d}{dt}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1-\epsilon), \tag{53}$$

  where $\epsilon = -\frac{\dot{H}}{H^2}$.
  The shrinking Hubble sphere implies

$$\epsilon = -\frac{\dot{H}}{H^2} < 1. \tag{54}$$

- **Quasi-de Sitter**:
  For perfect inflation, $\epsilon = 0$, the space-time becomes de Sitter space.

$$ds^2 = dt^2 - e^{2Ht}dx^2 \tag{55}$$

  where $H = \partial_t \ln a =$ constant. Inflation has to end, so it should not correspond to perfect de Sitter space. However, for small but finite $\epsilon \neq 0$, the line element (eq. 55) is still a good approximation to the inflationary background. This is why we will often refer to inflation as a quasi-de Sitter period.

- **Negative pressure**: What form of stress-energy tensor gives accelerated expansion? Let us consider a perfect fluid with pressure P and density ρ. The Friedmann equation, $H^2 = \frac{\rho}{6M_{\rm Pl}^2}$, and the continuity equation, $\dot{\rho} = -3H(P+\rho)$, together imply

$$\dot{H} + H^2 = -\frac{1}{6M_{\rm Pl}^2}(\rho + 3p) = -\frac{H^2}{2}\big(1 + \frac{3P}{\rho}\big). \tag{56}$$

  From the above equation,

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2}\Big(1 + \frac{P}{\rho}\Big) < 1 \iff w = \frac{P}{\rho} < -\frac{1}{3}, \tag{57}$$

inflation requires negative pressure or a violation of the strong energy condition.

3.4 Physics of Inflation

The condition for the cosmic acceleration is given by,

$$\epsilon = -\frac{\dot{H}}{H^2} = -\frac{d\ln H}{dN} < 1. \tag{58}$$

Here $dN = d\ln a = Hdt$ which measures the number of 'e-folds' N of inflationary expansion. In order to solve the Horizon problem, we want inflation to last a for a sufficiently long time. To achieve this requires ϵ to remain small for a sufficiently large number of Hubble times. This condition is measured by a second parameter,

$$\eta = \frac{d\ln\epsilon}{dN} = \frac{\dot{\epsilon}}{H\epsilon}. \tag{59}$$

For $|\eta| < 1$, the fractional change of ϵ per Hubble time is small and inflation persists.

3.5 Reheating

Reheating is considered as the last phase of inflation. During the time of inflation, the potential energy dominates over the kinetic energy of the inflaton. During the reheating phase, inflaton field picks up the kinetic energy. This energy transferred to the SM particles.

- Scalar field oscillation
  During inflation, inflaton slowly roll down through the potential and it reaches the minima of the potential and starts to oscillate about the minima of potential. Consider a potential of the form $V(\phi) = \frac{1}{2}m^2\phi^2$, where the amplitude of

the ϕ is small. The equation of motion is given by

$$\ddot{\phi} + 3H\dot{\phi} = -m^2\phi^2.$$

We can then neglect the friction term, and the field undergoes oscillations with frequency m. We can write the energy continuity equation as

$$\dot{\rho}_\phi + 3H\rho_\phi = -3HP_\phi = -\frac{3}{2}H(m^2\phi^2 - \dot{\phi}^2).$$

The R.H.S. averages to zero over one oscillation period. The oscillating field therefore behaves like pressure-less matter, with $\rho_\phi \propto a^{-3}$. The fall in the energy density is reflected in a decrease of the oscillation amplitude.

- Inflaton decay
  Inflaton gets coupled with the SM field and then decays into standard model particle. If the decay is slow (which is the case if the inflaton can only decay into fermions) the inflaton energy density follows the equation

  $$\dot{\rho}_\phi + 3H\rho_\phi = \Gamma_\phi\rho_\phi$$

  where Γ_ϕ gives the decay rate of inflaton. Decay of inflaton into boson is a rapid process, the involving mechanism is called the parametric resonance. This type of rapid decay of inflaton is called the preheating, since the bosons thus created are far from thermal equilibrium.

- Thermalisation
  The produced particle by the decay of inflaton can create other particle. The resulting soup will reach a thermal equilibrium with a temperature $T_{\rm rh}$. The reheating temperature is determined by the the energy density at the reheating epoch $\rho_{\rm rh}$. If $\rho_{\rm rh} << \rho_{\phi,E}$ ($\rho_{\phi,E}$ is the inflaton energy density at the end of inflation), reheating takes a longer time.

Chapter 4

Inflationary Dynamics

4.1 Scalar Field Dynamics

As a straightforward toy model for inflation, we take into consideration of an inflaton field $\phi(x,t)$. As indicated by the notation, the value of the field can depend on time t and the position in space x. Associated with every field value is a potential energy density $V(\phi)$ (see figure 3). If ϕ is dynamic (i.e., changes with time) then it also carries kinetic energy density. If the stress-energy associated with the scalar field dominates the universe, it give rise to the evolution of the FRW background.

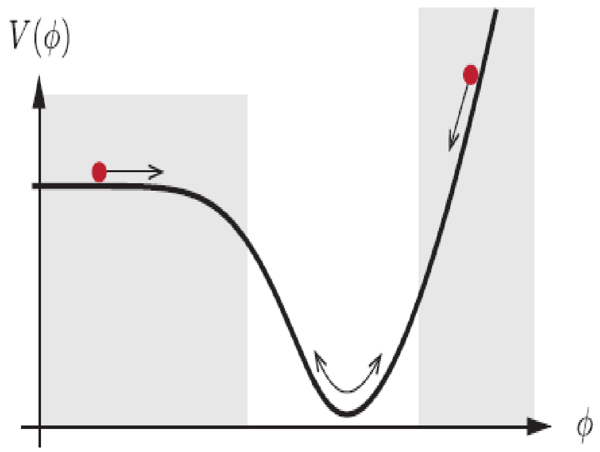

Figure 3: Example of a slow-roll potential. Inflation occurs in the shaded parts of the potential.

The stress-energy [9] tensor of the scalar field is given by,

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\Big(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\Big). \qquad (60)$$

What is needed for inflation is some dynamical vacuum-like state. This can be obtained if we assume that the universe was once dominated by one or more scalar fields. Suppose the universe is dominated by the scalar field ϕ, then the energy density of the universe is $\rho \approx \rho_\phi$. By comparing the stress-energy tensor of the scalar field with that of a perfect fluid (eq. 60), the energy density and pressure are found to be

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{61}$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{62}$$

We see that a field configuration leads to inflation, $P_\phi < -\frac{1}{3}\rho_\phi$, if the potential energy dominates over the kinetic energy. Substituting the value of ρ_ϕ into Friedman equation (eq. 11) with $k = 0$ gives,

$$H^2 = \frac{1}{3M_{\rm Pl}^2}\Big[\frac{1}{2}\dot{\phi}^2 + V(\phi)\Big]. \tag{63}$$

Taking time derivative, we find

$$2H\dot{H} = \frac{1}{3M_{\rm Pl}^2}\Big[\dot{\phi}\ddot{\phi} + V'\dot{\phi}\Big], \tag{64}$$

where, $V' = \frac{dV}{d\phi}$. Substituting eq. 63 and eq. 64 in eq. 56 we get

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_{\rm Pl}^2}. \tag{65}$$

Using eq. 64 and eq. 65 we are arrives at the [7, 9] Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{66}$$

This is the evolution equation for the scalar field. Notice that the potential acts like a force, V'. While the expansion of the universe adds friction, $H\dot{\phi}$.

4.2 Slow-roll Inflation

Substituting eq. 65 in eq. 58 we get the value of ϵ as,

$$\epsilon = \frac{\frac{1}{2}\dot{\phi}^2}{M_{\rm Pl}^2 H^2}. \tag{67}$$

Inflation therefore occurs if the kinetic energy makes a small contribution to the total energy, $\rho_\phi = 3M_{\rm Pl}^2 H^2$. This is called slow-roll inflation. In order for this condition to persist, the acceleration of the scalar field has to be small. To study this, it is useful to define the dimensionless acceleration per Hubble time

$$\delta = -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{68}$$

Taking the time derivative of (eq. 69)

$$\dot{\epsilon} = \frac{\dot{\phi}\ddot{\phi}}{M_{\rm Pl}^2 H^2} = -\frac{\dot{\phi}^2 \dot{H}}{M_{\rm Pl}^2 H^3} \tag{69}$$

and comparing to (eq. 61) we will get

$$\eta = \frac{\dot{\epsilon}}{H\epsilon} = 2\frac{\ddot{\phi}}{H\dot{\phi}} - 2\frac{\dot{H}}{H^2} = 2(\epsilon - \delta). \tag{70}$$

Inflation persists and occurs in the region $\epsilon, \eta < 1$. This is the condition used to simplify the equation of motion. This is called the slow-roll approximation. The condition $\epsilon << 1$ implies $\frac{1}{2}\dot{\phi}^2 << V$ and hence leads to the following simplification of the Friedmann equation

$$H^2 = \frac{V}{3M_{\rm Pl}^2}. \tag{71}$$

Slow-roll condition simplifies the Klein-Gordon equation (eq. 58) to

$$3H\dot{\phi} \approx -V'. \tag{72}$$

Using the above two equations the slow-roll parameter ϵ is defined in terms of potential as

$$\epsilon \approx \frac{M_{\rm Pl}^2}{2}\left(\frac{V'}{V}\right)^2. \tag{73}$$

Furthermore, taking the time derivative of Klein-Gordon equation,

$$3\dot{H}\dot{\phi} + 3H\ddot{\phi} = -V''\dot{\phi}. \tag{74}$$

From the above equation the other slow-roll parameter defined as,

$$\eta \approx M_{\rm Pl}^2 \frac{V''}{V}. \tag{75}$$

The number of 'e-folds' is a measure of the amount of inflation.

$$N_{\rm tot} = \int_{a_i}^{a_E} d\ln a = \int_{t_i}^{t_E} H(t)dt, \tag{76}$$

where t_I and t_E are defined as the $\epsilon(t_I) = \epsilon(t_E) = 1$. In the slow-roll regime, we can use

$$H dt = \frac{H}{\dot{\phi}} d\phi \approx \frac{1}{\sqrt{2\epsilon}} \frac{\mid d\phi \mid}{M_{\rm Pl}}. \tag{77}$$

Using the above equation, the number of 'e-folds' can expressed as,

$$N_{\rm tot} = \int_{\phi_I}^{\phi_E} \frac{\mid d\phi \mid}{M_{\rm Pl}}. \tag{78}$$

Where ϕ_I and ϕ_E are defined as the boundaries of the interval where $\epsilon < 1$. The largest scales observed in the CMB are produced about 60 'e-folds' before the end of inflation

$$N_{\rm CMB} = \int_{\phi_{\rm CMB}}^{\phi_E} \frac{\mid d\phi \mid}{M_{\rm Pl}} \approx 60. \tag{79}$$

A successful solution to the horizon problem requires $N_{\rm tot} > N_{\rm CMB}$.

Chapter 5

Quantum Fluctuations During Inflation

During inflation, all classical inhomogeneities are wiped out and at the end of inflation classically we would deal with a perfectly homogeneous universe. But when we look into the universe, we see a lot of structures on different scales. Why are there small temperature fluctuations ΔT in the CMB, i.e., $\frac{\Delta T}{T} \sim 10^{-5}$. The answer is through quantum fluctuations during inflation. Fluctuations in the inflaton field lead to a local time delay, when inflation ends (figure 4). So different regions inflate by a different amount, which leads to density perturbations and ultimately to the temperature fluctuations in the CMB. Hence inflation on the one side explains why the universe is extremely homogeneous, since Inflation wipes away all initial classical inhomogeneities, but on the other side it explains the origin of small inhomogeneities via quantum fluctuations.

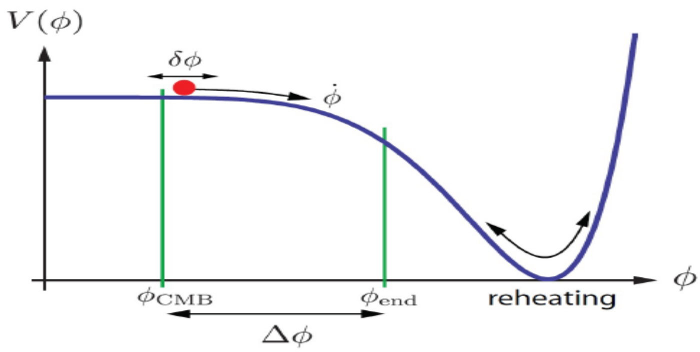

Figure 4: Perturbations in the inflaton field lead to time delays for the end of inflation.

5.1 Cosmological Perturbation Theory

The inhomogeneities in the CMB are at the 10^{-5} level so the strategy is to split all quantities $X(t,x)$ (the metric $g_{\mu\nu}$ and the matter quantities ϕ, ρ, and P) into a homogeneous background $\tilde{X}(t)$ and an inhomogeneous perturbation

$$\delta X(t,x) = X(t,x) - \tilde{X}(t). \tag{80}$$

The evolution of Einstein field equation is give by

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}. \tag{81}$$

The SVT (scalar, vector, tensor) decomposition of the perturbed metric [5] is given by,

$$ds^2 = -(1+2\phi)dt^2 + 2aB_i dx^i dt + a^2[(1-2\psi)\delta_{ij} + E_{ij}]dx^i dx^j \tag{82}$$

where a is the scaling factor and $B_i = \partial_i B - S_i$ with $\partial^i S_i = 0$ and $E_{ij} = 2\partial_{ij}E + 2\partial_{(i}F_{j)} + h_{ij}$, where $\partial_{(i}F_{j)} = \frac{1}{2}(\partial_i E_j + \partial_j E_i)$ with $\partial^i F_i = 0$, $h^i_i = \partial^i h_{ij} = 0$. The quantity h_{ij} looks similar to gravitational waves, which will turn out to be true. The tensor perturbations h_{ij} are responsible for the production of primordial gravitational waves. The degrees of freedom associated with the whole system is counted by,

- 5 scalar modes: ϕ, B, ψ, E and the inflaton perturbations $\delta\phi$.
- 4 vector modes: We have two vector perturbations S_i and F_i but both are constrained by $\partial^i S_i$ and $\partial^i F_i$ ($\partial^i S_i = 0$, $\partial^i F_i = 0$).
- 2 tensor modes: h_{ij} is symmetric and has four constraints $h^i_i = 0 = \partial^i h_{ij} \implies 6 - 4 = 2$ polarization modes.

5.2 Remark on gauge choice

Consider a completely homogeneous universe. Now we can make a gauge transformation [7], e.g.,

$$t \to t + \tau(t, x). \tag{83}$$

Then the field ϕ transforms as:

$$\phi(t) \to \phi(t) + \delta\phi(t, x). \tag{84}$$

So it seems like we have an inhomogeneity, although we assumed an homogeneous universe. This is an artifact of coordinate system. So our interpretation of fluctuations seems to depend on the choice of gauge. Hence it is necessary to introduce gauge invariant measures for inhomogeneities. One such quantity is the co-moving curvature perturbation.

$$\mathcal{R} = \psi - \frac{H}{\bar{\rho} + \bar{p}} \delta q, \tag{85}$$

which measures geometrically the spatial curvature of co-moving (or constant-ϕ) hyper-surfaces. The 3-momentum density δq is defined via $T_i^0 = \partial_i \delta q$. During inflation we have ($T_i^0 = -\dot{\bar{\phi}} \partial_i \delta\phi$):

$$\mathcal{R} = \psi + \frac{H}{\dot{\bar{\phi}}} \delta\phi. \tag{86}$$

The tensor perturbations turn out to be gauge invariant as well. A gauge choice eliminates two scalar and two vector degrees of freedom. Corresponding to these gauge choices there are four constraint equations from the Einstein equations, that eliminate another two scalar and two vector modes (constraint equations are equations without second derivatives, which do not describe dynamics). So in the end we are left with one scalar and two tensor degrees of freedom. The scalar modes are responsible for the density perturbations we observe in the CMB and the tensor modes lead to the production of primordial gravitational waves.

5.3 Quantum fluctuations

Now we start with the quantization. The action for the inflaton field is:

$$S = \int d^4x\sqrt{-g}(R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2V(\phi)). \tag{87}$$

If we expand this action [8] using the metric in our chosen co-moving gauge to second order in $\mathcal{R}$ we get:

$$S^2 = \frac{1}{2}\int d^4xa^3\frac{\dot{\phi}^2}{H^2}[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i\mathcal{R})^2]. \tag{88}$$

We define, $v \equiv z\mathcal{R}$ where, $z^2 = a^2\frac{\dot{\phi}^2}{H^2}$. Rewriting the action using the conformal time τ

$$S^2 = \frac{1}{2}\int d\tau d^3x\Big[(v')^2 - (\partial_i v)^2 + \frac{z''}{z}v^2\Big] = \frac{1}{2}\int d\tau d^3xL. \tag{89}$$

By expressing eq. 88 in Fourier modes v_k, i.e.,

$$v(\tau, x) = \int \frac{d^3k}{(2\pi)^3}v_k(\tau)e^{ikx}. \tag{90}$$

The equation of motion for v follows from the Euler-Lagrange equation

$$\frac{\partial}{\partial X^\mu}\left(\frac{\partial L}{\partial(\partial_\mu v)}\right) = \frac{\partial L}{\partial v},$$

where the Lagrangian L was defined in eq. 89. This gives the Mukhanov-Sasaki equation

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0. \tag{91}$$

Observe that eq. 91 is the equation of a simple harmonic oscillator with time-dependent frequency $f(\tau) = k^2 - \frac{z''}{z}$.

5.4 Quantization of the Mode Functions

The mode function v_k promoted to operator $\hat{v}_k$,

$$\hat{v}_k = v_k(\tau)\hat{a}_k + v^*_{-k}(\tau)a^\dagger_{-k} \tag{92}$$

with creation and annihilation operators $a^\dagger_{-k}$ and $\hat{a}_k$, respectively, satisfying,

$$\hat{a}_k = \frac{W[v^*_k, \hat{v}_k]}{W[v^*_k, v_k]} \tag{93}$$

and

$$[\hat{a}_k, \hat{a}^\dagger_{k'}] = (2\pi)^3\delta^{(3)}(k - k') \iff W[v_k, v_k] = 1, \tag{94}$$

where $W[v, w] = \frac{i}{h}(v^*w' - v'^*w)$.

5.5 Non-Uniqueness of Vacuum State

- Mode functions $v_k(\tau)$ and $v^*_k(\tau)$ in the operator description eq. 92 are linear independent solutions of the Mukhanov-Sasaki equation. Linear combination of $v_k(\tau)$ and $v^*_k(\tau)$, $\chi_k(\tau) = \alpha_k v_k(\tau) + \beta_k v^*_k(\tau)$ is solution of eq. 91.
- If the operator $\hat{v}_k$ is constructed with operators $\hat{a}_k$ and mode functions $v_k(\tau)$, then using a different set of mode functions, e.g., $\chi_k(\tau)$ has to be constructed with operators $\hat{b}_k$ according to eq. 93:

$$\hat{v}_k = \chi_k(\tau)\hat{b}_k + \chi^*_{-k}(\tau)\hat{b}^\dagger_{-k}. \tag{95}$$

- Non-uniqueness of the vacuum state:
  The b-vacuum state, defined by $\hat{b}_k|0\rangle_b = 0$, contains particles created from the a-vacuum state $\hat{a}^\dagger|0\rangle_a$: $\langle 0|\hat{a}^\dagger_k\hat{a}_k|0\rangle = |\beta_k|^2\delta(0)$.

5.6 Bunch-Davies Mode Functions v_k

- vacuum state for the fluctuations of $v_k(\tau)$:
  The vacuum state is chosen to be the Minkowski vacuum state $\hat{a}|0\rangle = 0$ observed for $\tau \to -\infty$.

- Boundary conditions:
  For $z^2 = 2a^2\epsilon$ the following equation holds

$$\frac{z''}{z} = (aH)^2\left[2 - \epsilon + \frac{3}{2}\eta + \frac{1}{2}\epsilon\eta + \frac{1}{4}\eta^2\eta k\right] \tag{96}$$

with $\epsilon = -\frac{\dot{H}}{H^2}, \eta = \frac{\dot{\epsilon}}{H\epsilon}, k = \frac{\dot{\eta}}{H\eta}$ [7]. In the de Sitter limit, i.e., $\epsilon \to 0$ (eq. 96) simplifies to

$$\frac{z''}{z} = 2(aH)^2 = \frac{2}{\tau^2} \tag{97}$$

with $a(\tau) = -\frac{1}{H\tau}$. In the chosen sub-horizon limit $\tau \to -\infty$ (eq. 91) reads

$$v_k'' + k^2 v_k = 0 \tag{98}$$

which has an oscillating solution $v_k = \frac{e^{\pm ik\tau}}{\sqrt{2k}}$. The vacuum state $|0\rangle$ is the state with minimum energy for the solution $v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}$. The initial condition for all modes:

$$\lim_{\tau \to -\infty} v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}. \tag{99}$$

In de Sitter limit eq. 91 become

$$v_k'' + \left(k^2 - \frac{2}{\tau^2}\right)v_k = 0, \tag{100}$$

which has the general solution

$$v_k = \alpha\frac{e^{-ik\tau}}{\sqrt{2}}\left(1 - \frac{i}{k\tau}\right) + \beta\frac{e^{ik\tau}}{\sqrt{2}}\left(1 + \frac{i}{K\tau}\right). \tag{101}$$

Observe that α and β are free parameters owing to the non-uniqueness of the mode functions. However, the sub-horizon limit (eq. 99) sets $\beta = 0$ and the normalization condition (eq. 94) set $\alpha = 1$. The unique Bunch-Davies mode functions result:

$$\boxed{v_k = \frac{e^{-ik\tau}}{\sqrt{2}}\Big(1 - \frac{i}{k\tau}\Big).} \tag{102}$$

With super-horizon limit

$$\lim_{k\tau\to 0} v_k = \frac{1}{i\sqrt{2}}\frac{1}{k^{\frac{3}{2}}\tau}. \tag{103}$$

5.7 Power Spectrum $\mathcal{P}_{\mathcal{R}}(\mathbf{k})$ for Scalar Perturbations from Quantum Fluctuations

Power spectrum $\mathcal{P}_v(k)$:
Using eq. 92 and eq. 94 we can calculate power spectrum as,

$$\begin{aligned}
\big\langle \hat{v}_k, \hat{v}'_k \big\rangle =& \big\langle 0|\hat{v}_k, \hat{v}'_k|0\big\rangle \\
=& \big\langle 0|(v_k(\tau)\hat{a}_k + v^*_{-k}(\tau)\hat{a}^\dagger_{-k})(v'_k(\tau)\hat{a})k' + v^*_{-k'}(\tau)\hat{a}^\dagger_{-k'})|0\big\rangle \\
=& |v_k|^2\big\langle 0|[\hat{a}_k, \hat{a}^\dagger_{-k'}]|0\big\rangle \\
=& |v_k|^2\delta(k+k') \\
=& \mathcal{P}_v(k)\delta(k+k').
\end{aligned}$$

The quantum zero-point fluctuations $\big\langle \hat{v}_k, \hat{v}_{k'} \big\rangle$ are created on sub-horizon scales and freeze on super-horizon scales because the co-moving curvature perturbation $\mathcal{R}$ is constant on super-horizon scales. Since the power spectrum $\mathcal{P}_R(k)$ is [4] calculated at horizon crossing the super-horizon limit (eq. 108) for the mode functions is used yielding $\mathcal{P}_v(k) = \frac{1}{2k^3}\frac{1}{\tau^2} = \frac{1}{2k^3}(aH)^2$.

Power spectrum $\mathcal{P}_R(k)$:
Using $v = z\mathcal{R}$

$$\mathcal{P}_R = \frac{1}{z^2}\mathcal{P}_v$$

$$\implies \mathcal{P}_R(k) = \frac{1}{2k^3}\frac{H_*^4}{\dot{\phi}_*^2},$$

with the relation $\epsilon = \frac{1}{2}\frac{\dot{\phi}^2}{H^2}$ for a scalar field with action (eq. 87). Quantities with lower index '$*$' are evaluated with the time of horizon crossing. Most of the time we will work with the dimensionless power spectrum, which is defined as:

$$\Delta_{\mathcal{R}}^2(k) = \frac{k^3}{2\pi^2}\mathcal{P}_R(k) = \frac{H_*^4}{(2\pi)^2\dot{\phi}_*^2} = \Delta_s^2(k), \tag{104}$$

where the subscript s denotes scalar perturbations. In an analogous calculation we get for the tensor fluctuations:

$$\Delta_t^2(k) = 2\Delta_h^2(k) = \frac{2}{\pi^2}\frac{H_*^2}{M_{\rm Pl}^2}. \tag{105}$$

5.7.1 Power spectra in terms of slow-roll parameters

Now we introduce the spectral indices n_s and n_t as well as the tensor-to-scalar ratio r, which measures the amount of primordial gravitational waves, and express them in terms of the slow-roll parameters:

$$n_s - 1 = \frac{d\ln\Delta_s^2}{d\ln k} = 2\eta - 4\epsilon, \tag{106}$$

$$n_t = \frac{d\ln\Delta_t^2}{d\ln k} = -2\epsilon, \tag{107}$$

$$r = \frac{\Delta_t^2}{\Delta_s^2} = -8n_t = 16\epsilon. \tag{108}$$

The parameters n_s, n_t and r can be measured with the CMB. These three parameters are not independent and the equation

$r = -8n_t$ is therefore called consistency relation. Usually n_s and r are measured.

Chapter 6

Inflation Models

6.1 Modified Gravity

The corresponding action for the Einstein equation in vacuum (i.e., $T_{\mu\nu}$) is the Einstein-Hilbert action,

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{M_{\mathrm{Pl}}^2}R\Big] \tag{109}$$

where $M_{\mathrm{Pl}} = \frac{1}{\sqrt{8\pi G_N}} = 2.44 \times 10^{18}\mathrm{GeV}$ is the reduced Planck mass and R is the Ricci Scalar. The factor $\sqrt{-g}$ with g defined as the determinant of $g_{\mu\nu}$ is included to make the action invariant under general coordinate transformations. The simplest extension to this model would be the addition of some scalar matter fields to the vacuum:

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{M_{\mathrm{Pl}}^2}R + \mathcal{L}_{\mathrm{mat}}\Big]. \tag{110}$$

In this action, we assumed that there is no coupling between the Ricci Scalar and the field in the $\mathcal{L}_{\mathrm{mat}}$. In the quantum theory the Lagrangian can contain any term not forbidden by some symmetry. This could be an indication that a non-minimal coupling exists. Moreover, a non-minimal coupling is required for renormalization purposes in theories of interacting scalar fields in curved space-time. Consider the case where $\mathcal{L}_{\mathrm{mat}}$ contains the field ϕ that couples to R. The potential of this field is supposed to be of the form

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2 \tag{111}$$

with $v = \langle\phi\rangle$ the vacuum expectation value of the field ϕ. If we choose to separate ϕ from $\mathcal{L}_{\mathrm{mat}}$, the action can be written as

$$S = \int d^4x\sqrt{-g}\Big[f(\phi)R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + \mathcal{L}_{\mathrm{mat}}\Big] \tag{112}$$

with $f(\phi) = \frac{1}{2}(M^2 + \xi\phi^2)$ and $M_{\rm Pl}^2 = M^2 + \xi v^2$.

6.2 Minimally Coupled Scalar Field as an Inflaton

In this case the parameter ξ is set to zero and the system is said to be minimally coupled [10, 13]. Can this give rise to inflation? First assume that the field ϕ is large with respect to its vacuum expectation value, so that the potential (eq. 111) becomes

$$V(\phi) = \frac{\lambda}{4}\phi^4. \tag{113}$$

Using eq. 73 we see that slow-roll ends when

$$\epsilon = \frac{M_{\rm Pl}^2}{2}(4\phi^{-1})^2 \simeq 1. \tag{114}$$

Assuming slow-roll inflation, so H is constant during inflation, eq. 78 becomes

$$N = \frac{1}{M_{\rm Pl}^2}\int_{\phi_{\rm end}}^{\phi} \frac{V}{V'} d\phi. \tag{115}$$

Thus $\phi_{\rm end} = \sqrt{8}M_{\rm Pl}$. The number of 'e-folds' between ϕ_0 and $\phi_{\rm end}$ is

$$N \simeq \frac{1}{M_{\rm Pl}^2}\int_{\phi_0}^{\sqrt{8}M_{\rm Pl}} \frac{1}{4}\phi d\phi. \tag{116}$$

Thus the value of the field ϕ at N 'e-folds' before the end of inflation is

$$\phi = \sqrt{8(N+1)}M_{\rm Pl}. \tag{117}$$

Now we have seen at which energy scales inflation takes place, we will analyze the cosmological implications of this model. According to the WMAP normalization $\frac{V}{\epsilon} = (0.027M_{\rm Pl})^4$ at 62 'e-folds' before the end of inflation. Evaluating $V(\phi)$ and $V'(\phi)$ at ϕ_{62} we get,

$$V(\phi_{62}) = \frac{\lambda}{4}(\sqrt{8(62+1)}M_{\rm Pl})^4 = \lambda \times 63504 \times M_{\rm Pl}^4, \tag{118}$$

$$V'(\phi_{62}) = \lambda(\sqrt{8(62+1)}M_{\rm Pl})^4 = \lambda \times 504^{\frac{3}{2}} \times M_{\rm Pl}^3. \tag{119}$$

Thus we obtain

$$\epsilon = \frac{M_{\rm Pl}^2}{2}\Big(\frac{V'(\phi_{62})}{V(\phi_{62})}\Big) \approx 0.016 \tag{120}$$

and

$$\frac{V}{\epsilon} = 2\frac{(\lambda \times 63504)^3}{(\lambda \times 504^{\frac{3}{2}})^2}M_{\rm Pl}^4 = (0.027M_{\rm Pl})^4. \tag{121}$$

From the above equation $\lambda \approx 1.33 \times 10^{-13}$. Such an extremely fine-tuned coupling constant seems very unnatural. The tensor-to-scalar ratio is [19] $r = 16\epsilon \approx 0.26$ which is also in conflict with the observed value of r.

6.3 Induced Gravity

Induced Gravity is an another version of minimally coupled system. Here the parameter M is set to zero and it is assumed that the Planck scale is generated by the field ϕ, analogous to the Higgs field generating the electroweak scale in the Standard Model. $f = \frac{1}{2}\xi\phi^2$ and $M_{\rm Pl}^2 = \xi v^2$ The Planck mass is completely generated by the vacuum expectation value of the field ϕ. It starts to run at energies above v.

6.4 Variable Planck mass

In a variable Planck mass theory, the Planck scale is more or less set by choosing M, but a small part of it is still 'induced' by the field ϕ. This corresponds to

$$f = \frac{1}{2}(M + \xi\phi^2), \qquad M_{\rm Pl}^2 = M^2 + \xi v^2. \tag{122}$$

For $1 << \sqrt{\xi} << \frac{M_{\rm Pl}}{v}$ the potential

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2. \tag{123}$$

Chapter 7

Higgs Inflation

In 2008, F.L. Bezrukov and M. Shaposhnikov proposed that Higgs boson could act as an inflaton [11]. They shown that non-minimal coupling between the Higgs field and gravity can lead to inflation with cosmological implications in agreement with WMAP data. The non-minimal coupling means by adding a term of the form $\xi H^\dagger HR$ to the Einstein-Hilbert action [10]. A potential problem is that a large ξ is unlikely from a particle physics point of view. The action that play crucial role in this chapter is

$$S_J = \int d^4x\sqrt{-g}\Big[\frac{1}{2}M_{\rm Pl}^2 R + \xi H^\dagger HR + L_{\rm SM}\Big]. \tag{124}$$

Here we are using the unitarity gauge $H = \frac{h}{\sqrt{2}}$, with h the real neutral component of Higgs doublet being the only real degree of freedom left after the Higgs mechanism. Here we will see that a non-minimal coupling of Higgs field leads to inflation because the potential becomes flat for large field values $h >> \frac{M_{\rm Pl}}{\sqrt{\xi}}$. In this regime, there is running of effective Planck mass in the Jordan frame and the parameter $\frac{\lambda}{\xi^2}$ determines the size of CMB fluctuation. By setting $\xi \sim 10^4$, the predicted values of the spectral index and the tensor-to-scalar ratio are in agreement with WMAP data.

7.1 Standard Model Higgs as an Inflaton

Gravity looks quite different for high field values in the case of a nonzero coupling ξ, However, with a conformal transformation the action can be rewritten in Einstein frame. It is assumed that the only degree of freedom during inflation is the Higgs field, so in what follows all remaining Standard Model terms are dropped. The action written in the Jordan frame is,

$$S_J = \int \sqrt{-g}\Big[f(h)R - \frac{1}{2}\partial_\mu h \partial^\mu h - V(h)\Big] \tag{125}$$

and the potential is given by $V(h) = \frac{\lambda}{2}(h^2 - v^2)^2$ and the function in front of the Ricci scalar given by

$$f(h) = \frac{1}{2}(M^2 + \xi h^2). \tag{126}$$

The whole story of inflationary dynamics is studied in the Einstein frame. Einstein frame can be obtained by conformal transformation from Jordan frame

$$g^E_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{127}$$

with

$$\Omega^2 = 1 + \frac{\xi h^2}{M^2_{\rm Pl}}. \tag{128}$$

Action in the Einstein frame becomes

$$S_E = \int d^4x \sqrt{-g}\Bigg[\frac{M^2_{\rm Pl}}{2}R_E - \frac{3M^2_{\rm Pl}}{4f(h)^2} g^{\mu\nu}_E \partial_\mu f(h) \partial_\nu f(h) - \frac{M^2_{\rm Pl}}{4f(h)} g^{\mu\nu}_E \partial_\mu h \partial_\nu h - V_E(h)\Bigg]. \tag{129}$$

The potential in the Einstein frame is

$$V_E = \frac{V(h)}{\Omega^2} = \frac{\lambda}{4}\frac{(h^2 - v^2)^2}{(1 + \frac{\xi h^2}{M^2_{\rm Pl}})^2}. \tag{130}$$

Potential becomes flat for $h >> \frac{M_{\rm Pl}}{\sqrt{\xi}}$ and to make the kinetic term canonical we introduced a new field χ defined by

$$-\frac{1}{2} g^{\mu\nu}_E \partial_\mu \chi \partial_\nu \chi = -\frac{3M^2_{\rm Pl}}{4f(h)^2} g^{\mu\nu}_E \partial_\mu f(h) \partial_\nu f(h) - \frac{M^2_{\rm Pl}}{4f(h)} g^{\mu\nu}_E \partial_\mu h \partial_\nu h \tag{131}$$

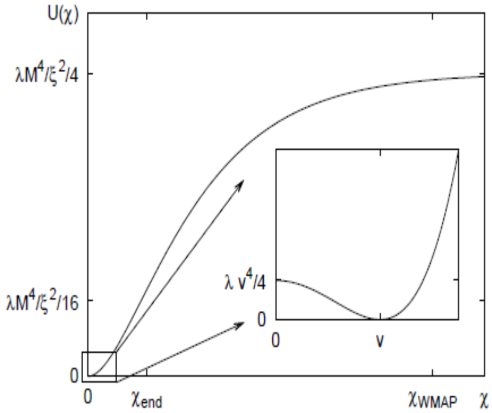

Figure 5: The potential of the Higgs field in the Einstein frame [11].

or it can be written as

$$\Big(\frac{d\chi}{dh}\Big) = M_{\rm Pl}\sqrt{\frac{f(h)+3f(h)'^2}{2(f(h))^2}}. \qquad (132)$$

Action in Einstein frame is defined by

$$S_E = \int d^4x\sqrt{-g}\Big[\frac{1}{2}M_{\rm Pl}^2 R_E - \frac{1}{2}(\partial_E\chi)^2 - V_E(\chi)\Big]. \qquad (133)$$

In the inflationary region, there is a simple analytic relation between the fields h and χ,

$$1+\frac{\xi h^2}{M_{\rm Pl}^2} \approx \exp\Big(\frac{2\chi}{\sqrt{6}M_{\rm Pl}}\Big). \qquad (134)$$

Substituting this into the potential (eq. 130), we will get,

$$V_E = \frac{\lambda M_{\rm Pl}^2}{4\xi^2}\Big(1-\exp\big(-\frac{2\chi}{\sqrt{6}M_{\rm Pl}}\big)\Big)^2. \qquad (135)$$

During inflation all slow-roll parameter are calculated as

$$\epsilon = \frac{M_{\rm Pl}^2}{2}\Big(\frac{dV/d\chi}{V}\Big)^2 \approx \frac{4M_{\rm Pl}^2}{3\xi^2 h^4}, \qquad (136)$$

$$\eta = M_{\rm Pl}^2 \frac{d^2V/d\chi^2}{V} \approx -\frac{4M_{\rm Pl}^2}{3\xi h^2}, \tag{137}$$

$$\zeta^2 = M_{\rm Pl}^2 \frac{(d^3V/d\chi^3)dV/d\chi}{V^2}. \tag{138}$$

Slow-roll inflation ends when $\chi \approx 1$, which enables us to find the Higgs field at the end of inflation which is given by,

$$h_{end} = (4/3)^{1/4} \frac{M_{\rm Pl}}{\sqrt{\xi}}. \tag{139}$$

Using eq. 115 we can calculate the number of 'e-folds'

$$\begin{aligned} N &= \frac{1}{M_{\rm Pl}^2} \int_{\chi_{end}}^{\chi_0} \frac{V}{dV/d\chi} d\chi = \frac{1}{M_{\rm Pl}^2} \int_{h_{end}}^{h_0} \frac{V}{dV/dh} \Big(\frac{d\chi}{dh}\Big)^2 dh \\ &\approx \frac{6}{8} \frac{h_0^2 - h_{end}^2}{M_{\rm Pl}^2/\xi}. \end{aligned} \tag{140}$$

This gives us the value of the Higgs field $N_{WMAP} \approx 62$ 'e-folds' before the end of inflation

$$h_{62} \approx 9.4 M_{\rm Pl}/\sqrt{\xi}. \tag{141}$$

The WMAP normalization [19] constrains $V/\epsilon = (0.027 M_{\rm Pl})^4$ at 62 'e-folds' before the end of inflation. Now the value of the coupling ξ can be determines $\xi \approx 49000\sqrt{\lambda}$.

The values of scalar spectral index and tensor-to-scalar ratio calculated as,

$$n = 1 - 6\epsilon + 2\eta \approx 0.97, \tag{142}$$

$$r = 16\epsilon \approx 0.0033. \tag{143}$$

7.2 Unitarity of Higgs Inflation

The non-minimal coupling between Higgs field and gravity causes the running of effective Planck mass at energies $h \sim \frac{M_{\rm Pl}}{\sqrt{\xi}}$. The potential in the eq. 137 has a plateau for $h >> M_{\rm Pl}/\sqrt{\xi}$.

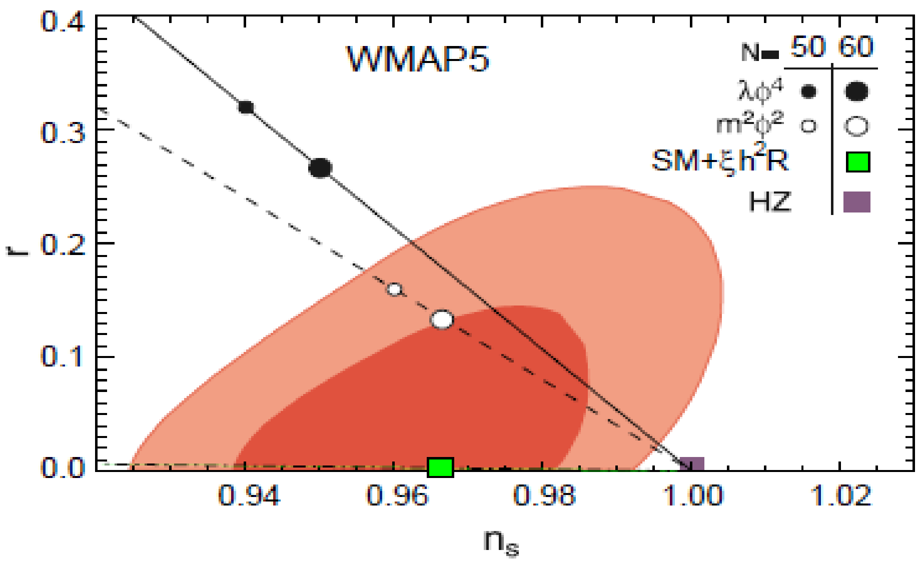

Figure 6: WMAP constraints and predicted values of n_s and r [12].

7.2.1 Single Field

Assume that the only degree of freedom is h and others are absorbed by gauge fields [13]. Consider the non-minimal coupling term in $\frac{1}{2}\xi H^2 R$ in the Jordan frame. By making an expansion around at space, the metric can be decomposed as

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{\gamma_{\mu\nu}}{M_{\rm Pl}}, \tag{144}$$

where $\eta_{\mu\nu}$ is the Minkowski tensor and the graviton is represented by the perturbation $\gamma_{\mu\nu}$. And the Ricci Scalar is given by,

$$R \sim \frac{\partial_\mu \partial^\mu \gamma^\nu_\nu - \partial_\mu \partial_\nu \gamma^{\mu\nu}}{M_{\rm Pl}} + \mathcal{O}(\gamma^2) \tag{145}$$

and,

$$\frac{\xi h^2 R}{2} \to \frac{\xi h^2}{2M_{\rm Pl}} [\partial_\mu \partial^\mu \gamma^\nu_\nu - \partial_\mu \partial_\nu \gamma^{\mu\nu}]. \tag{146}$$

The leading order term is the dimension 5 operator

$$\frac{\xi h^2}{2M_{\rm Pl}}\eta^{\mu\nu}\partial^2\gamma_{\mu\nu}. \tag{147}$$

It seems that this operator substantially contributes for $E \sim \frac{M_{\rm Pl}}{\xi}$ so it is tempting to say that it has a cut-off at $\Lambda = M_{\rm Pl}/\xi$. However, this is incorrect as can be seen from the scattering process $2h \to 2h$. At high energies the mass of h can be neglected and the tree-level process corresponds to the exchange of a single graviton (see figure 7). This gives the scattering amplitude

$$\mathcal{M}_c(2h \to 2h) \sim \frac{\xi^2 E^2}{M_{\rm Pl}^2}, \tag{148}$$

This appears to confirm that the cut-off is indeed $\Lambda = M_{\rm Pl}/\xi$, but this conclusion is premature. Let us explain why. The index c on $\mathcal{M}$ stands for "channel"; there are s, t, and u-channels, which all scale similarly. When we sum over all three channels to get $\mathcal{M}_{\rm tot}$ and put the external particles on-shell, an amusing thing happens: they cancel. So the leading term in powers of ξ vanishes. The first non-zero piece is

$$\mathcal{M}_{\rm tot}(2h \to 2h) \sim \frac{E^2}{M_{\rm Pl}^2}. \tag{149}$$

So the true cut-off for this process is at the Planck scale.

A similar result can be obtained in the Einstein frame. The kinetic sector in this frame is

$$-\frac{1}{2}\frac{1}{1+\xi h^2/M_{\rm Pl}^2}(\partial_\mu h)^2 - \frac{3\xi^2}{M_{\rm Pl}^2}\frac{h^2}{(1+\xi h^2/M_{\rm Pl}^2)^2}(\partial_\mu h)^2. \tag{150}$$

Expanding for small h this gives

$$-\frac{1}{2}(\partial_\mu h)^2 - \frac{3\xi^2 h^2}{M_{\rm Pl}^2}(\partial_\mu h)^2. \tag{151}$$

The second term looks like an operator with cut-off $\Lambda = M_{\text{Pl}}/\xi$. Indeed if we compute the contribution of the 4-point vertex to the tree-level process, we find (see figure 7)

$$\mathcal{M} \sim \frac{\xi^2 E^2}{M_{\text{Pl}}^2}. \tag{152}$$

However, when the external particles are on-shell the scatter-

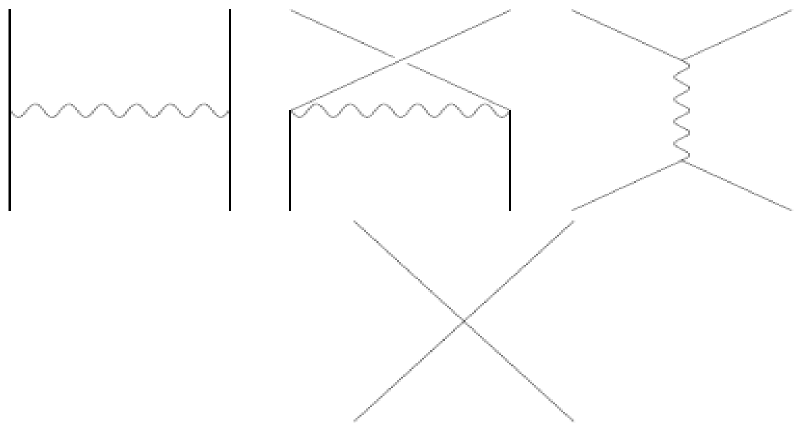

Figure 7: Tree level diagrams of the scattering process $2h \to 2h$. The upper panel shows graviton exchange through t, u, and s-channels in the Jordan frame. In the Einstein frame this is equivalent to a single 4-point vertex, as seen in the lower panel.[14]

ing amplitude vanishes. The reason for this is that for a non-minimally coupled system, we can make a field redefinition resulting in a minimally coupled system with a canonical kinetic term and modified potential. In this theory quantum corrections coming from kinetic part are suppressed by the Planck scale, resulting in the cut-off $\Lambda = M_{\text{Pl}}/\xi$.

Of course, since we have shifted to the Einstein frame, we have to deal with a modified potential. For small field values the Higgs field h can be expressed in terms of the redefined field χ by

$$h = \xi[1 - (\xi\chi/M_{\text{Pl}})^2] + ... \qquad . \tag{153}$$

Plugging this in the potential, we find the dimension 6 operator

$$-\frac{\lambda\xi^2}{M_{\mathrm{Pl}}^2}\chi^6 \tag{154}$$

and this is much smaller than the Higgs expectation value at N e-foldings during inflation, $h \sim \frac{M_{\mathrm{Pl}}}{\sqrt{\xi}}$. Hence we have to conclude that the theory breaks down at $\Lambda \sim M_{\mathrm{Pl}}/\xi$.

Chapter 8

Higgs Portal Inflation

In this chapter, we are reviewing the work done by Oleg Lebedev and Hyun Min Lee [16]. Here we study how the simple scalar extension of standard model play the role in Higgs-portal inflation and its role in ameliorating the unitarity issue associated with the Higgs inflation. For this model also, we will consider the scenario where both Higgs and the singlet are coupled non-minimally to gravity.

8.1 Inflation with Higgs-Singlet Combination

The Lagrangian in the Jordan frame for the Higgs-Singlet inflation with unitarity gauge is given by,

$$L_J/\sqrt{-g} = -\frac{M_{\rm Pl}^2}{2}R - \frac{\xi_h}{2}h^2R - \frac{\xi_s}{2}s^2R + \frac{1}{2}(\partial_\mu h)^2 + \frac{1}{2}(\partial_\mu s)^2 - V(h,s) \tag{155}$$

where,

$$V(h,s) = \frac{m_h^2}{2}h^2 + \frac{m_s^2}{2}s^2 + \frac{\lambda_h}{4}h^4 + \frac{\lambda_s}{4}s^4 + \frac{\lambda_{hs}}{4}h^2s^2. \tag{156}$$

Here ξ_h and ξ_s are the non-minimal coupling of Higgs and the singlet with gravity respectively and they are assumed to be large. Using conformal transformation we can move from Jordan frame to Einstein frame. The transformations are given by

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \qquad \Omega^2 = 1 + \frac{\xi_h h^2 + \xi_s s^2}{M_{\rm Pl}^2}. \tag{157}$$

As the conformal transformation makes the Lagrangian simpler it will help us to get rid of the non-minimal couplings.

Now consider the limit

$$\xi_h h^2 + \xi_s s^2 >> M_{\rm Pl}^2, \tag{158}$$

and we set $M_{\rm Pl}=1$. Then we have $\Omega^2 \simeq \xi_h h^2+\xi_s s^2$, and the kinetic and potential terms in the Lagrangian in Einstein frame are represented as

$$L_{\rm kin}=\frac{3}{4}(\partial_\mu \log(\xi_h h^2+\xi_s s^2))^2+\frac{1}{2(\xi_h h^2+\xi_s s^2)}((\partial_\mu h)^2+(\partial_\nu s)^2),$$
$$U=\frac{1}{(\xi_h h^2+\xi_s s^2)}V. \tag{159}$$

By the field redefinition [20] we can write as

$$\chi=\sqrt{\frac{3}{2}}\log(\xi_h h^2+\xi_s s^2), \tag{160}$$

$$\tau=\frac{h}{s}. \tag{161}$$

$$L_{\rm kin}=\frac{1}{2}(1+\frac{1}{6}\frac{\tau^2+1}{(\xi_h\tau^2+\xi_s)})(\partial_\mu\chi)^2+\frac{1}{\sqrt{6}}\frac{(\xi_h-\xi_s)\tau}{(\xi_h\tau^2+\xi_s)^2}(\partial_\mu\chi)(\partial^\mu\tau)$$
$$+\frac{\xi_h^2\tau^2+\xi_s^2}{2(\xi_h\tau^2+\xi_s)^3}(\partial_\mu\tau)^2. \tag{162}$$

Using (eq. 158) and ignoring the second term and simplify the first term in (eq. 162)

$$L_{kin}=\frac{1}{2}(\partial_\mu\chi)^2+\frac{\xi_h^2\tau^2+\xi_s^2}{2(\xi_h\tau^2+\xi_s)^3}(\partial_\mu\tau)^2,$$
$$U=\frac{\lambda_h\tau^4+\lambda_{hs}\tau^2+\lambda_s}{4(\xi_h\tau^2+\xi_s)^2}. \tag{163}$$

The minima of potential with different values of τ is given by

1. $2\lambda_h\xi_s-\lambda_{hs}\xi_h>0, \quad 2\lambda_s\xi_h-\lambda_{hs}\xi_s>0, \quad \tau=\sqrt{\frac{2\lambda_s\xi_h-\lambda_{hs}\xi_s}{2\lambda_h\xi_s-\lambda_{hs}\xi_h}}$ (164)

2. $2\lambda_h\xi_s-\lambda_{hs}\xi_h>0, \quad 2\lambda_s\xi_h-\lambda_{hs}\xi_s<0, \quad \tau=0$ (165)

$$3. \quad 2\lambda_h\xi_s - \lambda_{hs}\xi_h < 0, \quad 2\lambda_s\xi_h - \lambda_{hs}\xi_s > 0, \quad \tau = \infty \tag{166}$$

$$4. \quad 2\lambda_h\xi_s - \lambda_{hs}\xi_h < 0, \quad 2\lambda_s\xi_h - \lambda_{hs}\xi_s < 0, \quad \tau = \infty, 0 \tag{167}$$

In the first case Higgs field and the singlet will drive inflation. The values of potential (at minima) in the first three cases are given by,

$$\begin{aligned} U\mid_{(\min,(1))} &= \frac{1}{16}\frac{4\lambda_h\lambda_s - \lambda_{hs}^2}{\lambda_s\xi_h^2 + \lambda_h\xi_s^2 - \lambda_{hs}\xi_s\xi_h}, \\ U\mid_{(\min,(2))} &= \frac{\lambda_s}{4\xi_s^2}, \\ U\mid_{(\min,(3))} &= \frac{\lambda_h}{4\xi_h^2}. \end{aligned} \tag{168}$$

Inflaton potential for option 1 (minima of the potential with $\tau = \sqrt{\frac{2\lambda_s\xi_h - \lambda_{hs}\xi_s}{2\lambda_h\xi_s - \lambda_{hs}\xi_h}}$),

$$U(\chi) = \frac{\lambda_{\text{eff}}}{4\xi_h^2}\left(1 + \exp(\frac{-2\chi}{\sqrt{6}})\right)^{-2} \tag{169}$$

where

$$\lambda_{\text{eff}} = \frac{1}{4}\frac{4\lambda_h\lambda_s - \lambda_{hs}^2}{\lambda_s + \lambda_h x^2 - \lambda_{hs}x}, \tag{170}$$

$$x = \frac{\xi_s}{\xi_h}. \tag{171}$$

The whole story of inflation is described by the dimension-less parameter λ_{eff}, for Higgs inflation $\lambda_{\text{eff}} = \lambda_h$, for singlet inflation, $\lambda_{\text{eff}} = \lambda_s/x^2$ and for inflation assisted by singlet mixing with the Higgs, λ_{eff} as in the eq. 170. Using the potential in the eq. 169 we can calculate the inflationary parameters. For large values of χ, the exponential term is small and can be ignored. As χ takes smaller values the slow-roll parameter ϵ approaches to one and the inflation ends. ϵ is given by, $\epsilon = \frac{1}{2}\left(\frac{dU/d\chi}{U}\right)^2 \simeq \frac{4}{3\xi_h^2\tilde{h}^4}$, where $\tilde{h} = \frac{1}{\sqrt{\xi_h}}\exp(\chi/\sqrt{6})$.

The initial value of field for a given number of 'e-folds' N is $\tilde{h}_{\rm in} \simeq \sqrt{4N/3\xi_h}$ for the end of inflation by putting $\epsilon = 1$, $\tilde{h}_{\rm end} = (4/3)^{1/4}/\sqrt{\xi_h}$. By the COBE normalization $U/\epsilon = (0.027 \times M_{\rm Pl})^4$, we can fix the ξ_h in terms of $\lambda_{\rm eff}$.

$$\xi_h \simeq \sqrt{\frac{\lambda_{\rm eff}}{3}} \frac{N}{(0.027)^2}. \tag{172}$$

With N=60 and $\sqrt{\lambda_{\rm eff}} \sim 1$, that the non-minimal gravity coupling was about $\xi_h \sim 50000$, the spectral index is predicted to be $n = 1 - 6\epsilon + 2\eta \simeq 0.97 \simeq 1 - 2/N$, where ϵ, η while the tensor-to-scalar ratio is $r \simeq 12/N^2 \simeq 0.0033$.

8.2 Improving the unitarity issue by means of the singlet help

The unitarity issue arose in Higgs inflation because of the high value of non-minimal coupling of Higgs field to gravity. In this section, we will see how the addition of real scalar ameliorating the unitarity issue [17]. Lagrangian of the model in Jordan frame is written as,

$$\begin{aligned} L_J/\sqrt{-g} =& \frac{1}{2}\Big(M^2 + \xi\sigma^2 + 2\zeta H^\dagger H\Big)R - \frac{1}{2}(\partial_\mu\sigma)^2 - \mid D_\mu H \mid^2 - \\ & \frac{1}{4}\lambda_\sigma\Big(\sigma^2 - u^2 + 2\frac{\lambda_{H\sigma}}{\lambda_\sigma}H^\dagger H\Big)^2 - \Big(\lambda_H - \frac{\lambda_{H\sigma}^2}{\lambda_\sigma}\Big)\Big(H^\dagger H - \frac{v^2}{2}\Big)^2. \end{aligned} \tag{173}$$

Here M,u and v are mass parameters and ξ, ζ are non-minimal coupling constants with $v << M$, $\xi >> \zeta$. The effective quartic coupling of the real singlet scalar and the Higgs doublet is $\lambda = \Big(\lambda_H - \frac{\lambda_{H\sigma}^2}{\lambda_\sigma}\Big)$. We will work on the large nonzero vacuum expectation value of σ, $\big\langle\sigma\big\rangle \simeq u$ and the unitarity cut-off find as

$$\Lambda_{UV} = (1 + 6r\xi)\frac{M_{\rm Pl}}{\xi}. \tag{174}$$

where $M_{\rm Pl}^2 = M^2 + \xi u^2$. Here $r = \xi u^2/M_{\rm Pl}^2$ and r measures the contribution of σ vev. It takes the value between 0 and 1, for small values of vev i.e., $r \to 0$, the cut off is $M_{\rm Pl}/\xi$ for moderate values of r it pushs up to $rM_{\rm Pl}$. Because of the huge non-minimal coupling of gravity with σ field it dominates inflation, while the Higgs field simply follows the σ field along the flat direction, with $\lambda_{H\sigma} < 0$. We can calculate the mass of the σ field from the above Lagrangian (eq. 174), for this we assume that tree-level Einstein term and the non-minimal coupling for the Higgs doublet is absent, $M = 0$ and $\eta = 0$, in Jordan frame. Mass of the σ field calculated as

$$M_\sigma^2 = \lambda_\sigma \frac{2rM_{\rm Pl}^2}{(1+6r\xi)\xi} \simeq \lambda_\sigma \frac{M_{\rm Pl}^2}{3\xi^2}. \tag{175}$$

With COBE constraint, we obtain the mass σ field to be $M_\sigma = 10^{13}$GeV. Using conformal transformation we can write the Lagrangian in Einstein frame as,

$$\begin{aligned} L_E/\sqrt{-g} =& \frac{M_{\rm Pl}^2}{2}R - \frac{1}{2}\Big(\frac{u}{\sigma}\Big)^2[(1+6\xi)(\partial_\mu\sigma) + (\partial_\nu h)^2] - \frac{1}{4}\Big(\frac{u}{\sigma}\Big)^4 \\ & - \frac{1}{4}(\sigma^2 - u^2 + 2\frac{\lambda_{H\sigma}}{\lambda_\sigma}h^2)^2 - (\lambda_H - \frac{(\lambda_{H\sigma})^2}{\lambda_\sigma})(h^2 - \frac{v^2}{2})^2. \end{aligned} \tag{176}$$

Redefinition of field gives $\sigma = u\exp(\chi/\sqrt{6}M_{\rm Pl})$ and $\tilde{h} = uh/\sigma$. By putting the field redefinition in the above Lagrangian becomes:

$$\begin{aligned} L_E/\sqrt{-g_E} =& \frac{M_{\rm Pl}^2}{2} - \frac{1}{2}\Big(1 + \frac{1}{6\xi} + \frac{\tilde{h}^2}{6M_{\rm Pl}^2}\Big)(\partial_\mu\chi)^2 - \frac{1}{2}(\partial_\mu\tilde{h})^2 \\ & - \frac{1}{\sqrt{6}}\frac{\tilde{h}}{M_{\rm Pl}}(\partial_\mu\chi)(\partial^\mu\tilde{h}) - \frac{1}{4}u^4\lambda_\sigma(1 - \exp(-2\chi/\sqrt{6}M_{\rm Pl}) \\ & + \frac{\lambda_{H\sigma}}{\lambda_\sigma}\frac{\tilde{h}^2}{u^2})^2 + \frac{1}{4}(\lambda_H - \frac{\lambda_{H\sigma}^2}{\lambda_\sigma})^2(\tilde{h}^2 - v^2\exp(-2\chi/\sqrt{6}M_{\rm Pl}))^2. \end{aligned} \tag{177}$$

Potential in the Einstein frame identified as

$$V_E \simeq \frac{1}{4}(\lambda_\sigma u^4 + 2\lambda_{H\sigma} u^2 \tilde{h}^2 + \lambda_H \tilde{h}^4). \tag{178}$$

By expanding around the minima of the potential we can get,

$$V_E = V_0(1 - \exp(-2\chi/\sqrt{6}M_{\rm Pl})) \tag{179}$$

with

$$V_0 = \frac{u^4}{4}(\lambda_\sigma - \frac{\lambda_{H\sigma}^2}{\lambda_H}). \tag{180}$$

The given potential is flat and is capable to give correct density perturbation that we are observing from the CMB data today.

In this chapter we discussed the Higgs portal inflation using the singlet extension model. Using the singlet extension of σ model we were able to address the unitarity issue. It is remarkable how a simplest extension of the SM enunciates and tackles difficult problems that is not possible within the frame work of SM.

Chapter 9

Inert Doublet as an Inflaton

In this model we are using the inert doublet which is coupled non-minimally to gravity as an inflaton. Here there is an extra doublet ϕ_2 other than the Higgs doublet ϕ_1. ϕ_2 is inert in the sense that, it does not have any Yukawa coupling because of the inherent $\mathbf{Z}_2$ symmetry under which doublet is odd($\phi_2 \to -\phi_2$) and Higgs and other SM particles are even ($\phi_1, \psi \to \phi_1, \psi$) where ψ denotes the other SM particles.

9.1 Inflationary Model

The action for the model [18] is given by,

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big[-\frac{1}{2}M_{\text{Pl}}^2 R - D_\mu\phi_1 D^\mu\phi_1^\dagger - D_\mu\phi_2 D^\mu\phi_2^\dagger \\ - V(\phi_1\phi_2) - \xi_1\phi_1^2 R - \xi_2\phi_2^2 R\Big]. \end{aligned} \tag{181}$$

Here D is the covariant derivative which contains the couplings with gauge bosons. During inflation, there are no fields other than the inflaton so that the covariant derivative will reduce to the normal derivative $D_\mu \to \partial_\mu$. Here ξ_1 and ξ_2 are the couplings to gravity. The potential is given by,

$$\begin{aligned} V = m_1 \mid \phi_1 \mid^2 + m_2 \mid \phi_2 \mid^2 + \lambda_1(\mid \phi_1 \mid^2)^2 + \lambda_2(\mid \phi_2 \mid^2)^2 \\ + \lambda_3 \mid \phi_1 \mid^2 \mid \phi_2 \mid^2 + \lambda_4(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) + \frac{1}{2}\lambda_5[(\phi_1^\dagger\phi_2)^2 + \text{c.c.}], \end{aligned} \tag{182}$$

and ϕ_1 and ϕ_2 are given by:

$$\phi_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}\chi \\ h\end{bmatrix}, \quad \phi_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}q \\ xe^{i\theta}\end{bmatrix}. \tag{183}$$

The action in the Einstein frame is obtained after the conformal transformation,

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \Omega^2 = 1 + \frac{\xi_1}{M_{\rm Pl}^2}(\chi^2 + h^2) + \frac{\xi_2}{M_{\rm Pl}^2}(q^2 + x^2). \tag{184}$$

The action in Einstein frame is given by,

$$S = \int d^4x\sqrt{-g}\Big[-\frac{1}{2}M_{\rm Pl}^2 - \frac{1}{2}G_{ij}\partial_\mu\phi_i\partial^\mu\phi_j - V_E(h,q,x,\theta)\Big] \tag{185}$$

ϕ is defining by $= (\chi, h, q, x, \theta)$ where

$$G_{ij} = \frac{1}{\Omega^2}\delta_{ij} + \frac{3}{2}\frac{M_{\rm Pl}^2}{\Omega^4}\frac{\partial\Omega^2}{\partial\phi_i}\frac{\partial\Omega^2}{\partial\phi_j}, \tag{186}$$

and

$$V_E = \frac{V}{\Omega^4}. \tag{187}$$

Expansion of the pre-factor G of the kinetic terms is given by

$$G = \begin{bmatrix} \frac{\Omega^2+6\xi_1^2\chi^2/M_{\rm Pl}^2}{\Omega^4} & 6\frac{\xi_1^2}{M_{\rm Pl}^2\Omega^2}\chi h & \frac{6\xi_1\xi_2}{M_{\rm Pl}^2\Omega^4}\chi q & \frac{6\xi_1\xi_2}{M_{\rm Pl}^2\Omega^4}\chi x & 0 \\ 6\frac{\xi_1^2}{M_{\rm Pl}^2\Omega^2}\chi h & \frac{\Omega^2+6\xi_1^2h^2/M_{\rm Pl}^2}{\Omega^4} & \frac{6\xi_1\xi_2}{M_{\rm Pl}^2\Omega^4}hq & \frac{6\xi_1\xi_2}{M_{\rm Pl}^2\Omega^4}hx & 0 \\ \frac{6\xi_1\xi_2}{M_{\rm Pl}^2\Omega^4}\chi q & \frac{6\xi_1\xi_2}{M_{\rm Pl}^2\Omega^4}hq & \frac{\Omega^2+6\xi_2^2q^2/M_{\rm Pl}^2}{\Omega^4} & \frac{6\xi_2^2}{M_{\rm Pl}^2\Omega^4}qx & 0 \\ \frac{6\xi_1\xi_2}{M_{\rm Pl}^2\Omega^4}\chi x & \frac{6\xi_1\xi_2}{M_{\rm Pl}^2\Omega^4}hx & \frac{6\xi_2^2}{M_{\rm Pl}^2\Omega^4}qx & \frac{\Omega^2+6\xi_2^2x^2/M_{\rm Pl}^2}{\Omega^4} & 0 \\ 0 & 0 & 0 & 0 & \frac{x^2}{\Omega^2} \end{bmatrix}. \tag{188}$$

At the time of inflation, only the inert doublet is present and other components give no contribution. The factor Ω^2 can be modified by excluding the components of other doublet than the inert doublet. The scaling factor is given by, $\Omega^2 = \frac{\xi_2}{M_{\rm Pl}^2}(\chi^2 + h^2)$.

The simplified G matrix is given as:

$$G = \begin{bmatrix} \frac{1}{\Omega^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\Omega^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\Omega^2+6\xi_1^2h^2/M_{\rm Pl}^2}{\Omega^4} & \frac{6\xi_1\xi_2}{M_{\rm Pl}^2\Omega^4}hx & 0 \\ 0 & 0 & \frac{6\xi_2^2}{M_{\rm Pl}^2\Omega^4}qx & \frac{\Omega^2+6\xi_2^2x^2/M_{\rm Pl}^2}{\Omega^4} & 0 \\ 0 & 0 & 0 & 0 & \frac{x^2}{\Omega^2} \end{bmatrix}. \tag{189}$$

By the field redefinition we can written as

$$A = \sqrt{\frac{3}{2}} M_{\rm Pl} \log(\Omega^2), \tag{190}$$

$$B = M_{\rm Pl}\frac{x}{q}. \tag{191}$$

The potential in the Einstein frame is given by,

$$V_E \simeq \frac{\lambda_2 M_{\rm Pl}^4}{4\xi_2^2}\Big[1 - \exp\Big(-\sqrt{\frac{2}{3}}\frac{A}{M_{\rm Pl}}\Big)\Big]^2. \tag{192}$$

9.2 Inflationary parameters

The slow-roll parameter of the model can be calculated as

$$\begin{aligned} \epsilon &= \frac{1}{2}M_{\rm Pl}^2\Big(\frac{1}{V_E}\frac{dV_E}{dA}\Big)^2 = \frac{4}{3}\Bigg[-1+\exp\Big(\sqrt{\frac{2}{3}}\frac{A}{M_{\rm Pl}})\Big)\Bigg]^{-2}, \\ \eta &= M_{\rm Pl}^2\frac{1}{V_E}\frac{d^2V_E}{dA^2} = \frac{4}{3}\frac{\Big[2-\exp\Big(\sqrt{\frac{2}{3}}\frac{A}{M_{\rm Pl}})\Big]}{\Big[-1+\exp\Big(\sqrt{\frac{2}{3}}\frac{A}{M_{\rm Pl}}\Big)\Big]^2}. \end{aligned} \tag{193}$$

For the field values $A >> M_{\rm Pl}$, both ϵ, $\eta << 1$. For the calculation of power spectra we need to estimate the value of A at the beginning and at the the end of inflation. $A_{\rm end}$ can be calculated by putting the value of $\epsilon = 1$. $A_{\rm ini}$ can be evaluated using the number of 'e-folds'. $\epsilon = 1$ gives

$$\exp\left(\sqrt{\frac{2}{3}}\frac{A_{\rm end}}{M_{\rm Pl}}\right) \simeq 2.15. \tag{194}$$

Using eq. 115 the number of 'e-folds' calculated as,

$$N = \frac{3}{4}\Big[\exp\left(\sqrt{\frac{3}{4}}\frac{A_{\rm ini}}{M_{\rm Pl}}\right) - \exp\left(\sqrt{\frac{3}{4}}\frac{A_{\rm end}}{M_{\rm Pl}}\right) - \sqrt{\frac{2}{3}}\frac{A_{\rm ini}}{M_{\rm Pl}} + \sqrt{\frac{2}{3}}\frac{A_{\rm end}}{M_{\rm Pl}}\Big]. \tag{195}$$

For $N = 60$, $A_{\rm ini}$ is given by

$$A_{ini} \approx \left(\sqrt{\frac{3}{2}}\right) 4.45 \times M_{\rm Pl}. \tag{196}$$

For a fixed number of 'e-folds' power spectra $\mathcal{P}_s$, the tensor-to-scalar ratio r and the spectral index n_s are calculated as

$$\begin{aligned} \mathcal{P}_s &= \frac{1}{12\pi^2}\frac{V_E^3}{M_{\rm Pl}^6 V_E'^2} = 5.57 \times \frac{\lambda_2}{\xi_2^2}, \\ r &= 16\epsilon = 0.0029, \\ n_s &= 1 - 6\epsilon + 2\eta = 0.9678. \end{aligned} \tag{197}$$

The values of r and n_s are well within the Planck bound [19] (figure 6) of $n_s = 0.9677 \pm 0.0060$ at 1σ level and $r < 0.11$ at 95% confidence level. WMAP constraints the value of $\mathcal{P}_s$. The required value of λ_2 and ξ_2 in order to meet the density perturbation is

$$\mathcal{P}_s = (2.430 \pm 0.091) \times 10^{-9} = 5.57\frac{\lambda_2}{\xi_2^2} \quad \Longrightarrow \xi_2 \sim 4.79 \times 10^4 \lambda_2^{1/2}. \tag{198}$$

In this model, SM Higgs is free from the unitarity problem.
A larger value of the non-minimal coupling of ϕ_2 with gravity is not favorable from the particle physics point of view.

Chapter 10

Minimal Higgs Inflation with Modified Potential

All the non-minimally coupled inflationary models can account the density perturbation associated with the CMB data with a very large non-minimal coupling constant. The large value of non-minimal coupling constant is unphysical from the particle physics point of view. In this chapter, we are checking the viability of a minimally coupled Higgs inflation model with a modified potential.

10.1 The Inflationary model

The model differs from the other models of inflation by the polynomial modification of potential. The action for a given model is defined by,

$$S_J = \int d^4x\sqrt{-g}\Big(M_{\rm Pl}^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu H\partial_\nu H^\dagger - V\Big). \qquad (199)$$

Here the potential V is defined as [22]

$$V = \frac{\lambda}{4}\frac{(H^\dagger H - v^2)^2}{1+\frac{(H^\dagger H)^2}{\alpha^4}}, \qquad (200)$$

where H is the SU(2) doublet, the modification of the potential is done such a way that it can produce large field value at the inflationary scale $\Lambda = \lambda^{1/4}\alpha$. We choose the real component of the doublet to address the inflation problem. So that,

$$V = \frac{\lambda}{4}\Big(\frac{h^4}{1+\frac{h^4}{\alpha^4}}\Big) \qquad (201)$$

We can evaluate the slow-roll parameters from the inflaton potential. At the end of inflation value of ϵ tends to unity. Using this condition, we can evaluate the value of inflaton field at the end of inflation. Initial field value of the inflaton can calculated with a quantity called number of 'e-fold' N.

$$N = \frac{\alpha^2}{4M_{\rm Pl}^2}\left(\frac{(\tilde{h}^6 - \tilde{h}^6_{end})}{6} + \frac{(\tilde{h}^2 - \tilde{h}^2_{end})}{6}\right) \simeq \frac{\alpha^2}{24M_{\rm Pl}^2}\tilde{h}^6. \tag{202}$$

Here $\tilde{h} = h/\alpha$ and the suffix indicate the field value at the end of inflation.

10.2 Inflationary parameters of Minimally coupled model

In this section, we are reviewing [22]. The inflationary parameters listed as,

$$n_s = \quad 1 - 6\epsilon + 2\eta \simeq \tfrac{5}{3N}, \tag{203}$$

$$r = \quad 16\epsilon \simeq \tfrac{4}{3^{\frac{5}{3}}}\left(\tfrac{\alpha}{M_{\rm Pl}}\right)^{\frac{4}{3}}\tfrac{1}{N^{\frac{5}{3}}}. \tag{204}$$

Calculated value of n_s and r with $\lambda = 0.2$ and $\alpha \sim 10^{15}$.

$$n_s \simeq \begin{cases} 0.967, & N = 50 \\ 0.972, & N = 60 \end{cases}, \tag{205}$$

$$r \simeq \begin{cases} 2 \times 10^{-8}, & N = 50 \\ 1 \times 10^{-8}, & N = 60 \end{cases}. \tag{206}$$

The allowed range of r from WMAP data is, $r \geq 10^{-2}$. In order to get right size of the density perturbation, we have to modifying the parameter space associated with the model.

10.3 New Constraints over Minimally Coupled Inflation Model

We can fix the values of inflationary parameters by changing the values of quartic coupling λ and α. We can constrain these parameters by expressing the potential and its derivatives in terms of inflationary parameters. The inflationary potential and and its derivative in terms of power spectra, spectral index and tensor-to-scalar ratio are given by

$$\begin{aligned} V(\phi_0) &= \frac{3\pi^2}{2} M_{\rm Pl}^4 \mathcal{P}_s r, \\ V'(\phi_0) &= \frac{3\pi^2}{4\sqrt{2}} M_{\rm Pl}^3 \mathcal{P}_s r^{\frac{3}{2}}, \\ V''(\phi_0) &= \frac{3\pi^2}{4} M_{\rm Pl}^2 \mathcal{P}_s r \Big[\frac{3}{8} r + (n_s - 1)\Big], \end{aligned} \tag{207}$$

where ϕ_0 is the initial field value. For Higgs inflation $\phi_0 = h_0 = (24NM_{\rm Pl}^2\alpha^4)^{1/6}$. The Taylor expansion of potential about h_0 is given by,

$$\begin{aligned} V(h_0) =& \frac{\lambda}{4}\Big(\frac{h_0^4}{1+\frac{h_0^4}{\alpha^4}}\Big) + \lambda\left(\frac{h_0^3}{(1+\frac{h_0^4}{\alpha^4})^2}\right)(h-h_0) \\ &+ \frac{1}{2}\lambda\frac{(3h^2 - 5h^6/\alpha^4)}{(1+\frac{h^4}{\alpha^4})^3}(h-h_0)^2. \end{aligned} \tag{208}$$

By equating with the above equation eq. 207, we can constrain the values of α and λ, to get sensible density perturbation. With a value of $\alpha \sim 10^{17}$GeV and $\lambda \sim 10^{-2}$ the value of inflationary parameter are given by

$$n_s \simeq \begin{cases} 0.968, & N = 50 \\ 0.982, & N = 60 \end{cases}, \tag{209}$$

$$r \simeq \begin{cases} 10^{-2}, & N = 50 \\ 10^{-2}, & N = 60. \end{cases} \tag{210}$$

Minimally coupled SM Higgs with a polynomial modification of potential having a constrained values of α, λ can be a good inflation model and it can give rise to the right amount density perturbations.

Chapter 11

Conclusion

Cosmological inflation solves the Horizon and Flatness problem. This theory explained the scale invariance of the density perturbations found in the Cosmic Microwave Background(CMB). While analyzing the minimally coupled model of inflation, we have seen that a minimally coupled model requires a fine-tuned value of the self-coupling constant in order to meet the observed density perturbation of the CMB. Such a fine-tuned value is not favorable from the particle physics point of view.

In 2008, Bezrukov and Shaposhnikov came up with a new model of inflation. They proposed that non-minimally coupled Higgs boson with gravity can drive inflation. For this model, a large coupling (non-minimal coupling constant ξ) between Ricci scalar with SM Higgs is required for to get the right size of the density perturbations. All the predicted values for spectral index n_s and the tensor-to-scalar ratio r are well within the limits of WMAP data. Another advantage of this model is that the SM Higgs itself addresses inflation without the help of any additional scalar field. The potential problem associated with this model is the high value of the non-minimal coupling constant. The large coupling leads to unitarity violation. Another problem is that this model requires the SM is valid up to the inflationary scale $\sim 10^{15}$GeV, while it has tested up to energies about one TeV. We have seen that unitarity cut-off for the Higgs inflation model is M_{Pl}/ξ.

SM Higgs with a singlet scalar can also drive inflation and it can give sensible values of inflationary parameters. Using the assistance of the real scalar extension of the Standard model, we do ameliorate the unitarity issue associated with the SM Higgs and we rise the unitarity cut-off. The only problem associated with this model is the mass of the inflaton coming in the order of 10^{13}GeV, we can not probe such a huge mass in LHC. We have checked the viability of the inert doublet model as an inflation. We have checked against the most recent WMAP data. All the predicted inflationary parameter are well within the WMAP data and the unitarity issue associated with the SM Higgs is transferred to the inert doublet. Hence, the SM Higgs is free from the unitarity issue.

In order to relax the unitarity problem associated with the present models, we have introduced a minimally coupled inflation model with a modification of potential. This modified potential along with constrained values of parameter space can give rise to inflation.

Appendix A

The Conformal Transformation

For the case of non-minimal coupling of a scalar field ϕ, gravity looks quiet different. Ordinary gravity can be obtained by making the conformal transformation from Jordan frame to Einstein frame. This transformation [20, 21] changes the curvature of space-time which mixes up the scalar and tensor degrees of freedom. It reduces the action to that of a field in the at Minkowski space-time. Thus a field in a conformally flat space-time is completely decoupled from gravity. Here metric change like

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}. \tag{211}$$

In D-dimensional space-time, the metric is diag$(-1, 1, 1, 1, 1.........)$

$$\sqrt{-\tilde{g}} = \Omega^D \sqrt{-g}. \tag{212}$$

The Ricci scalar R changes with conformal transformation. The transformed Ricci scalar $\tilde{R}$ in the Einstein frame is,

$$\tilde{R} = \frac{1}{\Omega^2}\left[R - \frac{2(D-1)}{\Omega}\Box\Omega - \frac{(D-1)(D-4)}{\Omega^2} g^{\mu\nu}\nabla_\mu\Omega\nabla_\nu\Omega\right]. \tag{213}$$

The box operator defined as,

$$\Box\Omega = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Omega). \tag{214}$$

For a 4 dimensional space $\tilde{R} = \frac{1}{\Omega^2}\left[R - \frac{6}{\Omega}\Box\Omega\right]$, the action in the Jordan frame is given by

$$S_J = \int d^D x\sqrt{-g}\left[f(\phi)R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \tag{215}$$

where,

$$f(\phi) = \frac{1}{2}(M_0^{D-2} + \xi\phi^2), \tag{216}$$
$$M_{(D)}^{D-2} = M_0^{D-2} + \xi\phi^2. \tag{217}$$

The first term in the action transformed as follows

$$\int d^D x\sqrt{-g}f(\phi)R = \int d^D x\frac{\sqrt{-g}}{\Omega^D}f(\phi)\Bigg[\Omega^2\tilde{R} + \frac{2(D-1)}{\Omega}\Box\Omega + \frac{(D-1)(D-4)}{\Omega^2}g^{\mu\nu}\nabla_\mu\Omega\nabla_\nu\Omega\Bigg]. \tag{218}$$

In D-dimensional space-time, to have an ordinary gravity in Einstein space-time.

$$\Omega^{D-2} = \frac{2}{M_{(D)}^{D-2}}f(\phi), \tag{219}$$

Using the above relation, the second term in the action (eq. 218) written as,

$$\int d^D x\sqrt{-\tilde{g}}f(\phi)\frac{2(D-1)}{\Omega^(D+1)}\Box\Omega = \int d^D x\sqrt{-\tilde{g}}M_{(D)}^{D-2}\frac{(D-1)}{\Omega^2}\Box\Omega. \tag{220}$$

if we write the box operator in terms of original metric $g^{\mu\nu} = \Omega^2\tilde{g}^{\mu\nu}$ using partial integration this becomes,

$$\int d^D x\sqrt{-\tilde{g}}M_D^{D-2}(D-1)\Omega^{-3}\Big[\Omega^D\frac{1}{\sqrt{-\tilde{g}}}\partial_\mu(\Omega^{-D}\Omega^2\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}\partial_\nu\Omega)\Big]$$
$$= \int d^D x\sqrt{-\tilde{g}}M_D^{(D-2)}(D-1)\Omega^{D-3}\tilde{g}_{\mu\nu}\partial_\mu(\Omega^{2-D}\partial_\nu\Omega)$$
$$= -\int d^D x\sqrt{-\tilde{g}}M_D^{D-2}(D-1)(D-3)\Omega^{D-4}\tilde{g}_{\mu\nu}\partial_\mu\Omega\tilde{g}^{\mu\nu}\Omega^{2-D}\partial_\nu\Omega.$$
$$= -\int d^D x\sqrt{-\tilde{g}}M_D^{D-2}(D-1)(D-3)\Omega^{-2}\tilde{g}_{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega.$$

The potential in the Einstein frame is defined as

$$\tilde{V}(\phi) = \frac{V(\phi)}{\Omega^D}. \tag{221}$$

For a 4 dimensional space, action in the Einstein frame becomes

$$\int d^4x\sqrt{-g_E}\Big[\frac{M_{\rm Pl}^2}{2}R_E - \frac{3M_{\rm Pl}^2}{\Omega^2}g_E^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega - \Omega^{-2}(-\frac{1}{2}g_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) - V_E(\phi)\Big]. \tag{222}$$

This can be rewritten using eq. 219 in terms of $f(\phi)$

$$\int d^4x\sqrt{-g_E}\Big[\frac{M_{\rm Pl}^2}{2}R_E - \frac{3M_{\rm Pl}^2}{4f(\phi)^2}g_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi f(\phi) - \frac{M_{\rm Pl}^2}{4f(\phi)}g_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\Big]. \tag{223}$$

By the introduction of a new field, the kinetic term will become canonical, the new field χ defined as

$$-\frac{1}{2}g_E^{\mu\nu}\partial_\mu\chi\partial_\nu\chi = \frac{3M_{\rm Pl}^2}{4f(\phi)^2}g_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi f(\phi) - \frac{M_{\rm Pl}^2}{4f(\phi)}g_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{224}$$

The action in the Einstein frame is

$$S_E = \int d^4x\sqrt{-g}\Big[\frac{1}{2}M_{\rm Pl}^2R_E - \frac{1}{2}(\partial_E\chi)^2 - V_E(\chi)\Big]. \tag{225}$$

The field redefinition is given by

$$\Big(\frac{d\chi}{d\phi}\Big) = M_{\rm Pl}\sqrt{\frac{f(\phi)+3f(\phi)'^2}{2f(\phi)^2}}. \tag{226}$$

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