HOPFION SOLUTION IN NON-ABELIAN GAUGE THEORY

M.Sc. Thesis

By

THOUNAOJAM VICKY SINGH

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **HOPFION SOLUTION IN NON-ABELIAN GAUGE THEORY** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DISCIPLINE OF PHYSICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July, 2016 to June, 2018 under the supervision of Dr. Manavendra N. Mahato, Associate Professor, Indian Institute of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Signature of the student (with date)
Thounaojam Vicky Singh

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Signature of the thesis Supervisor
(with date)
Dr. Manavendra N. Mahato

THOUNAOJAM VICKY SINGH has successfully given his M.Sc. Oral Examination held on **22nd June, 2018**.

Signature of Supervisor of MSc thesis
Date:

Convener, DPGC
Date:

Signature of PSPC Member
Dr. Ankhi Roy
Date:

Signature of PSPC Member
Dr. Swadesh Kumar Sahoo
Date:

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Vicky Thounaojam
1603151024
Discipline of Physics
Indian Institute of Technology Indore

Dedication

This is dedicated to to my siblings Annie Thounaojam, Khemraj Thounaojam and Bonita Thounaojam.

Abstract

The purpose of this project is to study the development of a model of topological electromagnetism in empty space and to extend it further to non-Abelian theories. This topological model, which we are looking for, is based on Hopf map, a many-to-one continuous function (or "map") from the S^3 onto the S^2, discovered by a German mathematician, Hein Hopf in 1931. In the literature survey, we have studied how such solutions (Hopfion solution) were constructed via Bateman's and Ranada's construction. Some attempts have been made to construct Hopfion solutions in SU(2) Yang-Mills theory. An ansatz has been made by taking a analogy similar to Ranada's construction. This approach has shown to be a promising candidate for such type of solution in SU(2) Yang-Mills theory.

Contents

Chapter 1

Introduction

"Topology will play a very important role in future field theory."-A.F. Ranada

Since 1931, Dirac's monopole solution opens up to more challenging questions about the existence of solutions of similar type in field theories. Some already known solutions of field theory include instanton, vortex, domain walls etc. One interesting thing about these solutions is the fact that they are classified by various homotopy classes, each characterized by a topological invariant. In fact, Antonio F Ranada of Complutense University of Madrid, Spain have proposed a model of electromagnetism [4] in which he has shown that any pair of magnetic field lines are exactly linked once and so are any pair of electric field lines. Their corresponding helicity is a topological constant of the motion separately. It allows for the classification into various homotopy classes as expected. The solution is based on Hopf fibration, introduced by a German mathematician Heinz Hopf in 1931.

Chapter 2 discusses some topological solutions of gauge theories. For electromagnetic Hopfion solution, two constructions have been found in the literature survey and are known by the name- Bateman's construction and Ranada's construction. Riemann-Silverstein complex vector is introduced in Bateman's construction while in Ranada's construction, a properly normalized area 2-form is used to define Faraday 2-form from which the entire topological model is developed. The helicity defined for this model is closely related with the charge of the Abelian Chern-Simons current. Such a realization naturally leads to the generalization in non-Abelian gauge theories. As a matter of fact, Bateman's construction have been shown to be mapped to non-Abelian Yang-Mills theory via quaternionic formulation, to the vanishing field strength tensor [2] . In SU(2) Yang-Mills theory, there exist a non-perturbative solution called instanton solution based on the mapping $S^3 \to S^3$ [5], characterized by a unit winding number. On the boundary of real space S^3, the corresponding curvature tensor vanishes, which is one of the important feature of such solutions. However these two solutions are very much different in their behaviour because Hopfions are 3D solitons. Despite the differences between them, an attempt to construct SU(2) Yang-Mills Hopfion is discussed in Chapter 3.

Chapter 2

Toplogical solutions

The following literature survey will discuss more about the two types of topological solutions in gauge theory. One is the well known instanton solution in SU(2) Yang-Mills theory. This solution is based on the mapping $S^3 \to S^3$. While the other one is based on a non-trivial mapping $S^3 \to S^2$, also known as Hopf map.

2.1 Hopfion solutions

Since the original formulation of the laws of electrodynamics with the famous Maxwell's equations, many methods have been proposed to determine the evolution of $\vec{E}$ and $\vec{B}$ field in space and time. The method that we are dealing with is the one that solves the simplest configuration of the fields but, at the same time, admits a non-trivial topology. These are known by the name Hopfion solutions. In U(1) gauge theory, two constructions have been proposed to obtain Hopfion solutions. They are

1) Bateman's construction

2) Ranada's construction

2.1.1 Bateman's construction

This construction is based on the grouping of electric and magnetic field to form a complex vector field $\vec{F}$ called Riemann-Silberstein vector. It is expressed in terms of two complex functions α and β [1] as

$$\vec{F} = \vec{E} + i\vec{B} = \vec{\nabla}\alpha \times \vec{\nabla}\beta \tag{2.1}$$

In the absence of charge matter, Maxwell's equations reduce to

$$\vec{\nabla} \cdot \vec{F} = 0, \vec{\nabla} \times \vec{F} = i\partial_t \vec{F} \tag{2.2}$$

In terms of α and β, the dynamical equation becomes

$$\vec{\nabla}\alpha \times \vec{\nabla}\beta = i(\partial_t\alpha\vec{\nabla}\beta - \partial_t\beta\vec{\nabla}\alpha) \tag{2.3}$$

The divergenceless condition of $\vec{F}$ is obvious with the choice of the ansatz $\vec{F} = \vec{\nabla}\alpha \times \vec{\nabla}\beta$ and the dynamical equation and the norm of the field $\vec{F}$ give the following constraint to $\vec{E}$ and $\vec{B}$ as

$$\begin{aligned} \vec{E}^2 - \vec{B}^2 = 0 \\ \vec{E} \cdot \vec{B} = 0 \end{aligned} \tag{2.4}$$

and hence they are known as null solutions. The condition $\vec{\nabla} \cdot \vec{F} = 0$ implies the existence of a vector potential function as

$$F = \vec{\nabla} \times \vec{V} = \vec{\nabla} \times (\vec{C} + i\vec{A})$$

so that we can express $\vec{V}$ in terms of α and β. Upto gauge transformations,

$$\vec{V} = \alpha \vec{\nabla} \beta$$

Then,

$$\vec{C} = Re(\alpha \vec{\nabla} \beta), \vec{A} = Im(\alpha \vec{\nabla} \beta)$$

To characterize the non-trivial topology of electromagnetic fields, a common set of observables are the helicities. Helicity is a generalization of the topological concept of linking number to differential quantities required to describe the electromagnetic field (for our case). The electric and magnetic helicities are defined as

$$\begin{aligned} h_{ee} &= \int d^3x \vec{C} \cdot \vec{E} \\ h_{mm} &= \int d^3x \vec{A} \cdot \vec{B} \end{aligned} \tag{2.5}$$

The condition $\vec{E}.\vec{B} = 0$ guarantees the conservation of electric and magnetic helicity (when integrated over the whole space) as can be shown explicitly as

$$
\begin{aligned}
\partial_t h_{mm} &= \int d^3x(\partial_t\vec{A}\cdot\vec{B}+\vec{A}\cdot\partial_t\vec{B}) \\
&= -\int d^3x((\vec{E}+\vec{\nabla}\phi)\cdot\vec{B}+\vec{A}\cdot(\vec{\nabla}\times\vec{E})) \\
&= -\int d^3x(\vec{E}\cdot\vec{B}+\vec{\nabla}\cdot(\phi\vec{B})-\phi\vec{\nabla}\cdot\vec{B}+\vec{E}\cdot(\vec{\nabla}\times\vec{A}) \\
&-\vec{\nabla}\cdot(\vec{A}\times\vec{E})) \\
&= -2\int d^3x\vec{E}\cdot\vec{B} \\
&= 0
\end{aligned}
$$

where we have used Maxwell's equation and the fact that the divergence term vanishes when integrated over the whole space. Similar calculation shows the same result for electric helicity. Even the cross helicities $h_{em} = \int d^3xC\cdot B$ and $h_{me} = \int d^3xA\cdot E$ can also be shown to be conserved by the condition $\vec{E}^2-\vec{B}^2=0$

In particular, we have an interesting result of the helicities defined above using Bateman's construction. The cross helicities turn out to be equal in magnitude but opposite in sign while the electric and the magnetic helicity have the same value. Explicitly,

$$\vec{V}\cdot\vec{F}=\alpha\vec{\nabla}\beta\cdot(\vec{\nabla}\alpha\times\vec{\nabla}\beta)=0$$

$$(\vec{C}+i\vec{A})\cdot(\vec{E}+i\vec{B})=0$$

$$\vec{C}\cdot\vec{E}-\vec{A}\cdot\vec{B}=0,\ \text{and}\ \vec{C}\cdot\vec{B}+\vec{A}\cdot\vec{E}=0.$$

As a matter of fact, the solution of topologically non-trivial nature must also be characterized by Noether charges corresponding to space-time translations and rotations for which we have the following three charges.

- Energy density: $\mathcal{E}=\frac{1}{2}(E^2+B^2)$
- Momentum density: $\vec{p}=(\vec{E}\times\vec{B})$
- Angular momentum density: $\vec{l}=(\vec{p}\times\vec{x})$

A topologically non-trivial solution must be characterized by the finite value of the above charges and the helicities. In fact, such a solution does exist and is based on the Hopf map $S^3\rightarrow S^2$. Hence, they are known by the name "Hopfion" solution. In addition, there exists 4 more charges owing to the invariance of the Maxwell's equation under conformal transformations. We will not be dealing with such charges for the moment since the topological nature of the Hopfion solutions is revealed by topological invariants (helicity or Hopf index in our case) but not by Noether charges. Having mentioned some features of a Hopfion solution, an explicit solution is worth mentioning here. The solution is found in [1], [2]

$$\begin{aligned}\alpha&=\frac{l-t^2-2+2iz}{l-t^2+2it},\\ \beta&=\frac{2(x-iy)}{l-t^2+2it}\end{aligned}\tag{2.6}$$

where $l = 1 + r^2 = 1 + x^2 + y^2 + z^2$.

The explicit form of the Riemann-Silverstein vector is given in Appendix A. There, it is easy to see that the solution has finite energy E, momentum $\vec{p} = (0, 0, -\frac{E}{2})$ and angular momentum $\vec{l} = (0, 0, \frac{E}{2})$. Furthermore, $h_{ee} = h_{mm} = \frac{E}{2}$ and $h_{em} = h_{me} = 0$ which implies topologically non-trivial nature in the sense that electric and magnetic field lines are linked with each other as shown in the following figure (2.1) (image is shown for illustration purpose only, source [2]).

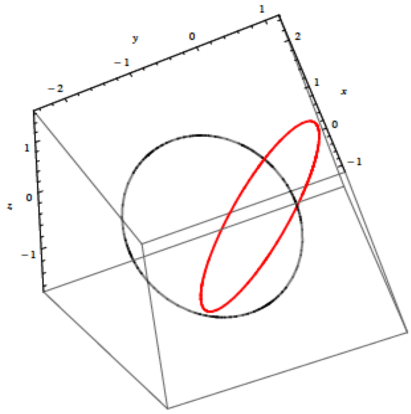

FIGURE 2.1: Electric field line (black) and magnetic field line (red) at t=0

2.1.1.1 Covariant formulation

Since we have the expression of $\vec{E}$ and $\vec{B}$ field in terms of α and β, we can recast them in a covariant form. So we begin with the component form of the ansatz (2.1) as

$$E^i + iB^i = \epsilon^{ijk}\partial_i\alpha\partial_j\beta \tag{2.7}$$

Before we move on using the electromagnetic field strength tensor, we have to adopt a proper convention so that the following chapters do not have any inconsistencies. The conventions are

$$\begin{aligned} E^i &= F^{0i} \\ B^i &= \frac{1}{2}\epsilon^{ijk}F_{jk} \\ \epsilon^{ijk} &= \epsilon_{ijk} \\ \epsilon_{0ijk} &= \epsilon_{ijk} \\ \epsilon^{0ijk} &= -\epsilon_{ijk} \end{aligned} \tag{2.8}$$

In covariant form,

$$F^{\mu\nu} - i\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = -\epsilon^{\mu\nu\lambda\gamma}\partial_\lambda\alpha\partial_\gamma\beta \tag{2.9}$$

and they are found to be imaginary anti-self dual [2]. If we define a complex potential function as

$$H_\mu = \frac{1}{2}(\alpha\partial_\mu\beta - \beta\partial_\mu\alpha) \tag{2.10}$$

then for $A_\mu = Im(H_\mu)$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = Im(\partial_\mu\alpha\partial_\nu\beta - \partial_\nu\alpha\partial_\mu\beta) \tag{2.11}$$

consistent with the expression derived from the covariant formulation. Similarly, the real part of H_μ i.e. $C = Re(H_\mu)$ gives the corresponding dual

$$G^{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}.$$

Since we are working out the U(1) gauge theory, so the helicity which we defined above is nothing but the charge of Abelian Chern-Simons current κ_μ defined as

$$\kappa^\mu = \epsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}, \tag{2.12}$$

so that for the null solutions

$$\partial_\mu \kappa^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \propto \vec{E}\cdot\vec{B} = 0. \tag{2.13}$$

Then the charge defines the helicity as shown below

$$\begin{aligned} -\frac{1}{2}\int d^3x\kappa^0 &= \frac{1}{2}\epsilon^{ijk}A_iF_{jk} \\ &= \int d^3x\vec{A}\cdot\vec{B} \\ &= h_{mm}. \end{aligned} \tag{2.14}$$

2.1.2 Ranada's construction

In this construction, we start by defining a complex scalar field $\phi(\vec{r})$ in $\mathbb{R}^3$. Let the field be well defined as $r \to \infty$ i.e. the map does not depend on the direction. Identifying $S^3 \equiv \mathbb{R}^3 \cup \{\infty\}$ and $S^2 \equiv \mathbb{C}^1 \cup \{\infty\}$ via stereographic projection, it is easy to see that the above map ϕ is indeed a Hopf map $\phi : S^3 \to S^2$.

The area 2-form [B.4] in S^2 is defined to be

$$\mathcal{F} = \frac{1}{2\pi i}\frac{d\overline{\phi} \wedge d\phi}{(1+\overline{\phi}\phi)^2}, \tag{2.15}$$

where the bar over ϕ denotes the complex conjugation. Since $\mathcal{F}$ is closed in S^3, then it must also be exact i.e., there exists a one-form $\mathcal{A}$ such that $\mathcal{F} = d\mathcal{A}$. Then the Hopf index, defined in the context of Bateman's construction, becomes

$$n = \int \mathcal{A} \wedge \mathcal{F} \tag{2.16}$$

The pull-back of $\mathcal{F}$ in $\mathbb{R}^3$ is given by

$$\mathcal{F} = \frac{1}{2}F_{ij}dx^i \wedge dx^j = \frac{1}{4\pi i}\frac{\partial_i\overline{\phi}\partial_j\phi - \partial_j\overline{\phi}\partial_i\phi}{(1+\overline{\phi}\phi)^2}dx^i \wedge dx^j \tag{2.17}$$

Now we can define a vector field out of F_{jk} as

$$\begin{aligned} B^i = B_i &= \frac{1}{2}\epsilon^{ijk}F_{jk} \\ &= \frac{1}{2}\epsilon^{ijk}\frac{1}{2\pi i}\frac{\partial_j\overline{\phi}\partial_k\phi - \partial_k\overline{\phi}\partial_j\phi}{(1+\overline{\phi}\phi)^2} \\ &= \frac{1}{2}\epsilon^{ijk}\frac{1}{2\pi i}\frac{2\partial_j\overline{\phi}\partial_k\phi}{(1+\overline{\phi}\phi)^2} \\ &= \frac{1}{2\pi i}\frac{(\vec{\nabla}\overline{\phi} \times \vec{\nabla}\phi)_i}{(1+\overline{\phi}\phi)^2} \end{aligned}$$

Thus

$$\vec{B} = g(\overline{\phi}\phi)\vec{\nabla}\overline{\phi} \times \vec{\nabla}\phi \tag{2.18}$$

The reason why we choose the letter $\vec{B}$ to define the vector field is because it satisfies $\vec{\nabla} \cdot \vec{B} = 0$ (as can be explicitly shown by employing the usual vector calculus identities). Indeed, the vector $\vec{B} = B_i dx^i = W(\phi)$ is the Whitehead vector of the map ϕ [9]. In addition, g is a well behaved function that depends on $(\vec{r}, t)$ through $\overline{\phi}$ and ϕ where we have included t for a more realistic model and will be discussed later at the end of this section.
It is easy to see from the above definition of $\vec{B}$ that [2.24]

$$\vec{B} \cdot \vec{\nabla}\phi = 0 \tag{2.19}$$

then the magnetic lines are seen to be the level curves of $\phi(X, Y, Z, T)$ which define a line at any point for each value of time. Here, X, Y, Z and T are the dimensionless coordinates.

Remarks: Here we have mentioned a realistic model without having to worry about the dimension of the magnetic field we just defined. For the moment, we will just continue using the same model by pretending that we employ the dimensionless coordinates. For the sake of brevity, we introduce a constant a [10] in the form below

$$F_{\mu\nu} = \frac{\sqrt{a}}{2\pi i} \frac{\partial_\mu \overline{\phi} \partial_\nu \phi - \partial_\nu \overline{\phi} \partial_\mu \phi}{(1 + \overline{\phi}\phi)^2}$$

$$\star F_{\mu\nu} = \frac{\sqrt{a}c}{2\pi i} \frac{\partial_\mu \overline{\theta} \partial_\nu \theta - \partial_\nu \overline{\theta} \partial_\mu \theta}{(1 + \overline{\theta}\theta)^2}$$

so that the electric and the magnetic fields have got the correct dimensions and c is the speed of light. The above two choices are the ansatz for the topological model of EM in Ranada's construction, to be discussed in a moment in the next subsection.

We can do the same analysis to come up with a suitable object that defines the electric field by introducing another map θ (precisely a Hopf map θ) so that $\vec{E}$ is the Whitehead vector $W(\theta)$ and takes the form

$$\vec{E} = g(\bar{\theta}\theta)\vec{\nabla}\bar{\theta} \times \vec{\nabla}\theta \tag{2.20}$$

Then the electric lines are seen to be the level curves of $\theta(X, Y, Z, T)$ since $\vec{E}.\vec{\nabla}\theta = 0$. Note that the same function g appears in both $\vec{E}$ and $\vec{B}$. This shows that the electromagnetic field of the model is built out of two scalars. Despite the appearance, θ can be derived from ϕ through a duality relation as shown in (2.22).

2.1.2.1 An ansatz for the topological model of the EM field

The above Hopf theory and the Maxwell's equation has one big difference for the role 'time' plays in both theories. In the Hopf theory, 'time' plays no role while it is inevitable in Maxwell's theory. But the Hopf theory gives us a promising, yet incomplete, attempt to describe the topological nature of the electromagnetic field. Nevertheless, the apparent contradiction can be resolved by assuming that the solutions obtained in Hopf theory are the initial value at t_0 of some time-dependent functions.

An ansatz to handle this situation is to extend (2.17) in 4 dimensional space-time so that

$$F_{\mu\nu} = \frac{1}{2\pi i}\frac{\partial_\mu\overline{\phi}\partial_\nu\phi - \partial_\nu\overline{\phi}\partial_\mu\phi}{(1+\overline{\phi}\phi)^2} \tag{2.21}$$

The corresponding dual is

$$\begin{aligned} M_{\mu\nu} = \star F_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \\ &= \frac{1}{2\pi i}\frac{\partial_\mu\overline{\theta}\partial_\nu\theta - \partial_\nu\overline{\theta}\partial_\mu\theta}{(1+\overline{\theta}\theta)^2}. \end{aligned} \tag{2.22}$$

It is now easy to see that the so formed topological electromagnetic fields are actually the radiation fields (except for the dimensions of the electric and magnetic fields). That means the electric and magnetic field satisfy the condition $\vec{E}.\vec{B} = 0$ as they evolve. Moreover, $E^2 - B^2 = 0$. These two conditions are found in [9], where

$$\begin{aligned} E_i = F_{i0} = \frac{1}{2}\epsilon_{ijk}M^{jk} = W(\theta), \\ B_i = M_{0i} = \frac{1}{2}\epsilon_{ijk}F^{jk} = W(\phi) \end{aligned} \tag{2.23}$$

Since $\vec{E}.\vec{B} = 0$, the electric and magnetic helicity, defined out of $\vec{E}$ and $\vec{B}$ in the same way we did within the context of Bateman's construction, are separately conserved. In addition, the level curve of ϕ and θ are orthogonal to each other. To see this, we calculate

$$\begin{aligned}\vec{B} \cdot \vec{\nabla}\phi &= B_i \partial_i \phi \\ &= \partial_i \phi \frac{1}{2} \epsilon_{ijk} F_{jk} \\ &= \frac{1}{4\pi i} \epsilon_{ijk} \partial_i \phi (\partial_j \bar{\phi} \partial_k \phi - \partial_k \bar{\phi} \partial_j \phi) \\ &= 0\end{aligned} \tag{2.24}$$

owing to the anti-symmetric nature of ϵ_{ijk} and symmetric nature of $\partial_i \phi \partial_j \phi$ thus giving 0. Similarly,

$$\vec{E} \cdot \vec{\nabla}\theta = 0$$

and by the condition of radiation fields $\vec{E} \cdot \vec{B} = 0$, we see that the two level curves of ϕ and θ are orthogonal to each other.

2.1.2.2 Hopf index

Since helicity is a common set of observable that defines the topological nature of the solution, it will be quite proper to give a treatment to this topological number. For our case, we have the corresponding Hopf fibration $S^3 \to S^2$, and the term for the topological number is called Hopf index. Within the context of Ranada's construction, the Hopf index n as shown in 2.15 is

$$n = \int \mathcal{A} \wedge \mathcal{F}$$

In order to find the 1-form $\mathcal{A}$, we must rewrite the Clebsch representation of the area 2-form. So we parametrize the map ϕ as

$$\phi = R \exp^{2\pi i \psi} \tag{2.25}$$

where R and ψ are real maps $\mathbb{R}^3 \to \mathbb{R}$. Then

$$\begin{aligned} \mathcal{F} &= d\rho \wedge d\psi \\ \mathcal{A} &= d\psi + d\xi \end{aligned} \tag{2.26}$$

where ρ and ξ are another real map $\mathbb{R}^3 \to \mathbb{R}$ and ρ takes the form

$$\rho = -\frac{1}{1+R^2}$$

Thus the Hopf index becomes

$$\begin{aligned} n &= \int d\xi \wedge d\rho \wedge d\psi \\ &= \int d(\xi d\rho \wedge d\psi) \end{aligned} \tag{2.27}$$

and is zero by Gauss' theorem unless ψ or ρ is multiple valued. We will assume ψ to be multiple valued, and $\vec{\nabla}\psi$ to be a single valued function. Since $\vec{B}$ is the Whitehead vector of the map ϕ, so the Hopf index for the map ϕ must correspond to the magnetic helicity. In fact, it is found in [1] that

$$n(\phi) = h_{mm}$$

.

2.2 Instanton

Instantons are localized objects in four-dimensional (Euclidean) space-time. Originally, it was called "pseudo-particles" by Polyakov in the first paper [5] of these type of solutions. The name "instantons" was suggested by 't Hooft and we will stick to this name for the very reason that follows. In the quasi-classical approximation they describe the least action trajectory (in Euclidean time) that connects two distinct energy-degenerate states in the space of fields. The initial point of the instanton trajectory at $t = -\infty$ is one such state, while the final point at $t = +\infty$ is another such state. Naturally, instantons are present only in those theories in which energy-degenerate states in the space of fields exist. They minimize the (Euclidean) action under the given boundary conditions. Therefore, instantons present classical solutions of the Euclidean equations of motion.
In non-Abelian gauge theories, they were discovered by Belavin, Polyakov, Schwarz and Tyupkin [5] and are usually referred to as BPST instantons. Our main focus is the **SU(2) Yang-Mills theory** for which we

have the corresponding Lagrangian

$$L = -\frac{1}{4} G^a_{\mu\nu} G_{a\mu\nu}$$

where $G^a_{\mu\nu}$ is the corresponding component of the gauge field strength tensor

$$G_{\mu\nu} = g G^a_{\mu\nu} \frac{\tau^a}{2}$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu$$

The corresponding gauge field is

$$A_\mu = g A^a_\mu \frac{\tau^a}{2}$$

where 'g' is the coupling constant, τ^a are the Pauli matrices for $a = 1, 2, 3$. They satisfy the SU(2) Lie algebra

$$\left[\frac{\tau^a}{2}, \frac{\tau^b}{2}\right] = \imath\epsilon^{abc} \frac{\tau^c}{2}$$

and $a, b, c = 1, 2, 3$

Instanton solutions are the solutions for which the corresponding gauge fields A_μ minimizes the **Euclidean action**. Therefore the gauge potential should tend to 0 as $x \to \infty$ or equivalently the gauge transformation of $A_\mu = 0$ i.e.

$$A_\mu = \imath S(x)\partial_\mu S(x)^\dagger$$

where $S(x)$ is a matrix belonging to SU(2) that depends on x. This boundary condition also implies the vanishing gauge field strength as $x \to \infty$. In fact, $S(x)$ can be realized as a continuous mapping $S^3(space - time) \to S^3(SU(2))$ and all such mappings are classified by **Homotopy theory** into various classes $S_n(x)$ characterized by winding number (the number of times the group space sphere S_3 is swept when the coordinate x sweeps the sphere in coordinate space once). Mathematically, this is expressed by the formula

$$\pi_3(SU(2)) \approx \pi_3(S^3) = \mathbb{Z}$$

The unit matrix $\mathbb{I}$ represents the class $S_0(x)$. For a unit winding number, the simplest choice of $S_1(x)$, not homotopic to $\mathbb{I}$, can be taken as

$$S_1(x) = \frac{x_4 + i\vec{x}\cdot\vec{\tau}}{\sqrt{x^2}} \tag{2.28}$$

It corresponds to the unit topological charge q (winding number). In fact, we can see that the Euclidean action in a given sector, labelled by q, is bounded. Explicitly, we can show by rewriting the instanton action by introducing the corresponding dual $\star G_{\mu\nu}(=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}$, where $\epsilon_{\mu\nu\rho\sigma}$ is fully-antisymmetric rank 4 tensor) as

$$\begin{aligned}
S_E = S_{inst} &= \frac{1}{2g^2}\int d^4xTr(G_{\mu\nu}G^{\mu\nu}) \\
&= \frac{1}{4g^2}\int d^4xTr(G_{\mu\nu}\mp\star G^{\mu\nu})^2 \pm 2Tr(G_{\mu\nu}\star G^{\mu\nu}) \\
&\geq \pm\frac{1}{2g^2}\int d^4xTr(G_{\mu\nu}\star G^{\mu\nu}) \\
&\geq \pm\frac{1}{2g^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}\partial_\mu K_\mu \\
&\geq \frac{8\pi^2}{g^2}|q|
\end{aligned}$$

where we have used the Chern-Simon current $K_\mu = 2\epsilon_{\mu\nu\rho\sigma}(A^a_\nu\partial_\rho A^a_\sigma + \frac{g}{3}\epsilon^{abc}A^a_\nu A^b_\rho A^c_\sigma)]$ to define the topological charge $q = \frac{g^2}{32\pi^2}\int d^4x\partial_\mu K_\mu$. The equality holds true only if

$$\star G_{\mu\nu} = \pm G_{\mu\nu},$$

where $'\pm'$ corresponds to self-dual and anti-self-dual condition which also corresponds to $q > 0$ and $q < 0$ respectively. In fact, instanton solutions are the one that satisfy the Euclidean equation of motion $D_\mu G_{\mu\nu} = 0$ and also at the same time minimizes the action S_{inst}. The above duality condition holds true only in the Euclidean space but not in the Minkowski space and so this is the reason why we begin with the Euclidean space. An explicit way of showing why the above duality condition with $\pm$ sign do not work in Minkowski space can be found in the Appendix C.

It is easy to see that

$$D \star G_{\mu\nu} = 0, \tag{2.29}$$

which is nothing but the Bianchi identity, which, taking the above duality condition into account, further implies that they also satisfy the equation of motion.

Now that we have a choice for $S_1(x)$ for which the topological charge is unity. The BPST instanton solution can be constructed in an explicit form by using all the properties that we defined above. But before we do that, we need to introduce 't Hooft symbols that links the generators of $SU(2)$ subgroup of the higher group $SO(4)(= SU(2) \times SU(2))$, the group of rotations in 4D Euclidean space, and has the defining condition

$$\eta_{a\mu\nu} = \begin{cases} \epsilon_{a\mu\nu} & \mu, \nu = 1, 2, 3 \\ -\delta_{a\nu} & \mu = 4 \\ \delta_{a\mu} & \nu = 4 \\ 0 & \mu = \nu = 4 \end{cases}$$

so that

$$\begin{aligned} I_1^a &= \frac{1}{4}\eta_{a\mu\nu}M_{\mu\nu} \\ I_2^a &= \frac{1}{4}\bar{\eta}_{a\mu\nu}M_{\mu\nu} \end{aligned} \tag{2.30}$$

for $a = 1, 2, 3$ and $\bar{\eta}_{a\mu\nu}$ differs from $\eta_{a\mu\nu}$ in the sign of δ. Here, I_1^a and I_2^a are the generators of the two subgroups of $SO(4)$, which has $M_{\mu\nu}$ as its generators. Now we consider an element of $SU(2)$ as

$$x_4 + i\vec{x} \cdot \vec{\tau} = i\tau_\mu^+ x_\mu \tag{2.31}$$

where we define $\tau_\mu^\pm$ matrices, the Euclidean analogs of the Minkowski matrices σ^μ and $\bar{\sigma}^\mu$ as

$$\tau_\mu^\pm = (\vec{\tau}, \mp i) \tag{2.32}$$

The transformations of the combination of $\tau_\mu^\pm$ is shown for completeness as

$$\begin{aligned} \tau_\mu^+ \tau_\nu^- &= \delta_{\mu\nu} + i\eta_{a\mu\nu}\tau^a \\ \tau_\mu^- \tau_\nu^+ &= \delta_{\mu\nu} + i\bar{\eta}_{a\mu\nu}\tau^a \end{aligned} \tag{2.33}$$

The corresponding explicit form of the Instanton (Self-Dual) solution with its centre at the point x_0 and with size ρ [8] is given by

$$\begin{aligned} A_\mu^a &= \frac{2}{g}\eta_{a\mu\nu}\frac{(x - x_0)_\nu}{(x - x_0)^2 + \rho^2}, \\ G_{\mu\nu}^a &= -\frac{4}{g}\eta_{a\mu\nu}\frac{\rho^2}{[(x - x_0)^2 + \rho^2]^2}. \end{aligned} \tag{2.34}$$

Allowing the self-dual and anti-self-dual nature of $\eta_{\mu\nu}$ and $\bar{\eta}_{\mu\nu}$ respectively, we can define a rank two self-dual and anti-self-dual tensor as

$$\begin{aligned} \Sigma_{\mu\nu} &= \eta_{a\mu\nu}\frac{\tau^a}{2} \\ \bar{\Sigma}_{\mu\nu} &= \bar{\eta}_{a\mu\nu}\frac{\tau^a}{2} \end{aligned} \tag{2.35}$$

so that the self-dual instanton solution becomes

$$\begin{aligned} A_\mu &= \Sigma_{\mu\nu}\frac{(x-x_0)_\nu}{(x-x_0)^2+\rho^2}, \\ G_{\mu\nu} &= -2\Sigma_{\mu\nu}\frac{\rho^2}{[(x-x_0)^2+\rho^2]^2}. \end{aligned} \tag{2.36}$$

The instanton action for the above solution is $\frac{8\pi^2}{g^2}$ which means the topological charge is 1. The anti-instanton (anti-self-dual) solution is obtained by the substitution $\eta_{a\mu\nu} \to \bar{\eta}_{a\mu\nu}$.

Chapter 3

Hopfion solution in SU(2) Yang-Mills theory

Before we begin discussing some of the novel attempts to construct Hopfion solutions in SU(2) Yang-Mills theory, we need to introduce a bit of the mathematics of 'Quaternion'. This particular branch of mathematics forms a bridge between Abelian gauge theory to that of non-Abelian gauge theory. With this realization, it is worthwhile to try to understand some of the basics of Quaternionic algebra.

Quaternions are number systems and serves as an extensions of the complex numbers. It is written generally as

$$q = a + bi + cj + dk$$

and the conjugate quaternion $\bar{q}$,

$$\bar{q} = a - bi - cj - dk$$

where i, j, k are the fundamental quaternionic units having the property

$$i^2 = j^2 = k^2 = -1$$

$$ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$$

and a, b, c, d are the real numbers. A general quaternion [2] can also be expressed in terms of two complex numbers(functions) as

$$q = (\alpha + \beta j) \tag{3.1}$$

where $\alpha = a+bi$ and $\beta = c+di$. Here a, b, c, d are numbers (functions) depending on the context where they are being used. In conjunction with α and β complex numbers(functions), the quaternionic conjugate is defined by

$$\bar{q} = (\bar{\alpha} - \beta j) \tag{3.2}$$

so that $q\bar{q} = |\alpha|^2 + |\beta|^2$

In particular, $q\bar{q} = 1$ represents the hyper-sphere of unit radius in 4D space, which is usually denoted by S^3.

Topological solutions:

Given the above information, we wish to check if there exists topological solutions of the SU(2) Yang-Mills theory of the Hopfion type. So we start with the pure gauge condition $A_\mu \to \imath S_1(x)\partial_\mu S_1(x)^\dagger$. Using the representation of 1, i, j and k as

$$1 = \mathbb{I}, i = -\tau^1, j = -i\tau^2, k = -i\tau^3$$

It is easy to realize that $S_1(x)$ serves as a quaternion. So $S_1(x)\partial_\mu S_1(x)^\dagger$ serves as a quaternionic valued potential of the general form [2]

$$Q_\mu = q\,(\partial_\mu \bar{q}) \tag{3.3}$$

It is worth to pause for a moment and study some of the properties of Q_μ.

Properties:

1) Q_μ is purely quaternionic since $Q_\mu = -\bar{Q}_\mu$

2) If $Q_{\mu\nu} = \partial_\mu Q_\nu - \partial_\nu Q_\mu$, then

$$Q_{\mu\nu} + [Q_\mu, Q_\nu] = 0 \tag{3.4}$$

Since by property 1, Q_μ is expressed in terms of i, j, k and so can be expressed explicitly in terms of Pauli matrices or SU(2) generators. Property 2 also implies the fact that the corresponding non-Abelian gauge field strength vanishes, irrespective of the choice of q which defines Q_μ.

Our strategy is to express Q_μ in terms of the SU(2) generators as

$$Q_\mu = Q_\mu^a \frac{\tau^a}{2i}$$

where the components Q_μ^a are related to the Hopf map since we seek solutions of the Hopfion type and the complex i is introduced for if the gauge potential is expressed in anti-Hermitian form, then the corresponding gauge field strength becomes the LHS of 3.4. Putting the above expression in property 2 yields the following constraint

$$\partial_\mu Q_\nu^a - \partial_\nu Q_\mu^a = -\epsilon^{abc} Q_\mu^b Q_\nu^c \tag{3.5}$$

Imposing the Lorentz gauge condition $\partial_\mu A^\mu = 0$, we can deal with the 4th component by only working with the spatial parts. So the above constraint reduces to

$$\partial_j Q_k^a - \partial_k Q_j^a = -\epsilon^{abc} Q_j^b Q_k^c$$

Multiplying on both sides by ϵ^{ijk}, we have

$$\vec{\nabla} \times \vec{Q}^a = -\frac{\epsilon^{abc}}{2} \vec{Q}^b \times \vec{Q}^c \tag{3.6}$$

where i, j, k =1,2,3

To obtain a solution using Hopf fibration map, $\phi : S^3 \to S^2$, one can pullback a volume form on S^2 to get 2-form on S^3 and such 2-form on S^3 can be written as an exact differential of a 1-form. By using stereographic projection. we can associate with a vector field on S^3, determined by the 1-form, a vector field $\vec{A}$ in $\mathbb{R}^3$. It is found in [7]

$$\begin{aligned} A_1 &= \frac{xz-y}{2l^2} \\ A_2 &= \frac{yz+x}{2l^2} \\ A_3 &= \frac{2z^2+2-l}{4l^2} \end{aligned} \tag{3.7}$$

for $l = 1 + x^2 + y^2 + z^2$. The above vector field $A = A_1, A_2, A_3$ has the following identities

$$\vec{\nabla} \times \vec{A} = 4\frac{\vec{A}}{l}$$

$$\vec{r} \times \vec{A} = \vec{A} + \frac{\vec{v}}{4l}$$

where $\vec{r} = (x, y, z)$ is the position vector and $\vec{v} = (-y, x, 1)$.

We intend to take Q_i^a proportional to A_i for $i = 1, 2, 3$ with unknown functions so as to satisfy $\vec{\nabla} \times \vec{Q}^a = -\frac{\epsilon^{abc}}{2}\vec{Q}^b \times \vec{Q}^c$. We further plan to work in this direction to develop a partial Hopfion solution of SU(2) Yang-Mills theory.

In fact, a general SU(2) gauge theory need not have a vanishing field strength tensor for the existence of a solution of Hopfion type. The 2nd property (3.4) needs to be modified so that the pre defined quaternionic valued potential (derived from two complex Hopf maps) do not map to the vanishing field strength tensor.

Some other attempts to use quaternionic formulation to map to non-zero field strength tensor have also been tried out. But the necessary modifications of the quaternionic valued potential (3.3) to obtain a non-vanishing field strength tensor have not been observed yet.

Alternatively, seeking a solution of Hopfion type can also be analyzed starting from the the general expression of non-Abelian gauge field strength. For this purpose, we employ the differential form notation for SU(2) Yang-Mills theory. In this notation, the gauge field strength is seen to be

$$F = dA + A \wedge A \tag{3.8}$$

The Bianchi identity tells us that $dF = 0$ which implies

$$dF = dA \wedge A = 0$$

since $d^2A = 0$ and if $A = A^a \frac{\tau^a}{2}$ for $a, b, c = 1, 2, 3$, then

$$\begin{aligned} dF^a &= d[\frac{i}{2}\epsilon_{abc}(dA^b \wedge A^c - A^b \wedge dA^c)] = 0 \\ &= i\epsilon_{abc} dA^b \wedge A^c = 0 \end{aligned} \tag{3.9}$$

If we now write $A^a = A^a_\mu \frac{\tau^a}{2}$, then (3.9) becomes

$$\begin{aligned} dF^a &= i\epsilon^{abc} dA^b \wedge A^c \\ &= i\epsilon^{abc}\left[\frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)^b A^c_\rho dx^\mu \wedge dx^\nu \wedge dx^\rho\right] \\ &= i\epsilon^{abc}\frac{1}{2}A^b_{\mu\nu}A^c_\rho dx^\mu \wedge dx^\nu \wedge dx^\rho \\ &= \frac{i}{3!}\epsilon^{abc}A^b_{[\mu\nu}A^c_{\rho]} dx^\mu \wedge dx^\nu \wedge dx^\rho \end{aligned}$$

where $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the last step is achieved by anti-symmetrizing the component in μ, ν, ρ indices. So the Bianchi identity 3.9 reduces to

$$\epsilon^{abc}A^b_{[\mu\nu}A^c_{\rho]} = 0 \tag{3.10}$$

So a general expression, if it has to satisfy the equation of motion, must also satisfy the above identity. In particular, our ansatz for generating an SU(2) Hopfion solution must hold the above identity (3.10). So we take the ansatz similar to the one found in Ranada's construction

$$\begin{aligned} A^1_\mu &= \frac{\phi\partial_\mu\bar{\phi}}{(1+\phi\bar{\phi}+\theta\bar{\theta}+\psi\bar{\psi})^2}, \\ A^2_\mu &= \frac{\theta\partial_\mu\bar{\theta}}{(1+\phi\bar{\phi}+\theta\bar{\theta}+\psi\bar{\psi})^2}, \\ A^3_\mu &= \frac{\psi\partial_\mu\bar{\psi}}{(1+\phi\bar{\phi}+\theta\bar{\theta}+\psi\bar{\psi})^2}. \end{aligned} \tag{3.11}$$

So the Bianchi identity must be satisfied by the above choice of A^a for $a = 1, 2, 3$. So we take the component $a = 1$, then (3.10) reduces to

$$A^2_{[\mu\nu}A^3_{\rho]} = A^3_{[\mu\nu}A^2_{\rho]} \tag{3.12}$$

On expanding, we have

$$
\begin{aligned}
LHS &= A^2_{\mu\nu}A^3_\rho + A^2_{\nu\rho}A^3_\mu + A^2_{\rho\mu}A^3_\nu, \\
RHS &= A^3_{\mu\nu}A^2_\rho + A^3_{\nu\rho}A^2_\mu + A^3_{\rho\mu}A^2_\nu
\end{aligned}
$$

where $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as defined above. Then the LHS is evaluated to be

$$
LHS = \frac{(\partial_\mu\theta\partial_\nu\bar{\theta} - \partial_\nu\theta\partial_\mu\bar{\theta})\psi\partial_\rho\bar{\psi} + (\mu \to \nu \to \rho) + (\nu \to \rho \to \mu)}{(1+\mathcal{M}^2)^2}
$$

where we define $\mathcal{M}^2 = \phi\bar{\phi} + \theta\bar{\theta} + \psi\bar{\psi}$ such that $\partial_\mu\mathcal{M}^2 = 0$. Using $\partial_\mu\mathcal{M}^2 = 0$, we have

$$
(\partial_\mu\phi)\bar{\phi} + \phi\partial_\nu\bar{\phi} + (\partial_\mu\theta)\bar{\theta} + \theta\partial_\nu\bar{\theta} + (\partial_\mu\psi)\bar{\psi} + \psi\partial_\nu\bar{\psi} = 0 \tag{3.13}
$$

So using (3.13), the LHS can be rearranged to become

$$
\begin{aligned}
LHS =& RHS + \partial_\mu[(\theta\partial_\nu\bar{\theta})(\psi\partial_\rho\bar{\psi})] - \partial_\nu[(\theta\partial_\mu\bar{\theta})(\psi\partial_\rho\bar{\psi})] \\
&+ (\mu \to \nu \to \rho) + (\nu \to \rho \to \mu)
\end{aligned}
$$

By (3.12), we have

$$
\partial_\mu[(\theta\partial_\nu\bar{\theta})(\psi\partial_\rho\bar{\psi})] - \partial_\nu[(\theta\partial_\mu\bar{\theta})(\psi\partial_\rho\bar{\psi})] + (\mu \to \nu \to \rho) + (\nu \to \rho \to \mu) = 0 \tag{3.14}
$$

So our task now is to consider some functions of ϕ, θ and ψ that will satisfy both (3.13) and (3.14). So the obvious choice is

$$\begin{aligned} \phi &= c_1 e^{if(r)}, \\ \theta &= c_2 e^{ig(r)}, \\ \psi &= c_3 e^{ih(r)}. \end{aligned} \tag{3.15}$$

where c_1, c_2 and c_3 are constants and $f(r), g(r)$ and $h(r)$ for $r^2 = x^2 + y^2 + z^2$ are real functions from $\mathbb{R}^3$ to $\mathbb{R}$. This particular choice of the functions $f(r), g(r)$ and $h(r)$ still gives the vanishing field strength tensor. In this case, the equation of motion is trivially satisfied. So, we intend to work further by making a slight modification to (3.15) in such a way that the Bianchi identity is still satisfied but also at the same time gives non-vanishing field strength tensor.

Chapter 4

Conclusion and future scopes

As it turns out, there have been hardly any satisfactory results for the problem within the time limit given for the project. The problem becomes more complicated as we probe further with the ansatz. We have adopted three different ways of constructing Hopfion solutions, but none seem to have a simplified situation for the moment. But there is a greater hope that one of the above three attempts will spit out a desired solution, given some time and effort.

So, the 1st attempt is just a particular case where the gauge field strength simply vanishes. If there exist a solution that describes the behaviour of the corresponding gauge field using the Hopf map, then this solution could, in principle, be extended so that the new gauge field do not possess a vanishing field strength tensor. This motivation comes from the analysis of instanton solution wherein the pure gauge

condition at far away points is taken care for any points including infinity [5], [8].
2nd attempt is also related to the 1st attempt, but this time, we try not to solve the particular case. But, indeed, we try to modify the quaternionic valued potential in such a way that the corresponding gauge field do not vanish. Unfortunately, many modifications have been tried but none fruitful. We hope we would be able to find one good modification in the near future.
The last one is what we have been doing very recently. This time we started with the Bianchi identity. The reason is that, no matter what choice of the ansatz we consider, the solutions have to satisfy the Bianchi identity. So the ansatz might satisfy the Bianchi identity but may satisfy the equation of motion with or without source term. The ansatz we adopted here is very much similar to the one we found in Ranada's construction for U(1) gauge theory. Even here, the problem is complicated, so we make certain assumptions as we go along to simplify the problem. Fortunately, the Bianchi identity is satisfied this time. But the vanishing field strength tensor will be reviewed by making a slight modification to the choice of $f(r), g(r)$ and $h(r)$ in (3.15).

Future scopes:

If such a solution is constructed, we will try to study some of the properties and its significance with the other already known solutions.

One good example is the instanton solution. As seen in the paper [2], the Abelian Hopfion solution can be mapped to the non-Abelian theories and their corresponding current are also closely related. So, there is a possibility that the instanton solution can be described using the Hopf fibration. In fact, there is a paper on 'Hopf instantons in Chern-Simons theory' [11]. Since SU(2) is the simplest non-trivial group we have, getting some topological solutions like Hopfion type can lead us to gain some insight into the theory of higher non-Abelian group. For example, the gluonic congifuration in QCD can also be studied with the solutions since SU(2) is a subgroup of SU(3). Such type of solutions can also be studied within the context of gravity.

Appendix A

Explicit form of the Riemann-Silverstein vector

The explicit form of the Riemann Silverstein vector of the Hopfion solution defined by 2.6 is

$$F = \frac{4}{(l - t^2 + 2it)^3} \begin{pmatrix} (t - x - z + i(y-1))(t + x - z - i(y+1)) \\ -i(t - y - z - i(x+1))(t + y - z + i(x-1)) \\ 2(x - iy)(t - z - i) \end{pmatrix}$$

Then the electric and the magnetic field is the real and the imaginary component of the above vector. Although the expression looks complicated, we can see the behaviour of the solution at $t = 0$ and at $\vec{r} = 0$. So, an expression for the x component of the F field is given below. At $t = 0$,

$$F_x = -\frac{4(2x^2 + 2 - l)}{l^3} + i\frac{4(yz + x)}{l^3}$$

while at $\vec{r} = 0$,

$$F_x = -4\frac{t^4 - 6t^2 + 1}{(1+t^2)^4} + i\frac{16t(1-t^2)}{(1+t^2)^4}$$

It is interesting to see that the components of electric and magnetic field at $t = 0$ are proportional to 3.7 or to cyclic permutation of some of them.

Appendix B

Area 2-form

The ordinary sphere of unit radius in $\mathbb{R}^3$ is defined by the equation

$$x^2 + y^2 + z^2 = 1$$

Realizing that $S^2 \approx \mathbb{R}^2 \cup \infty$ or $\mathbb{C}^1 \cup \infty$, we have the following coordinates of S^2 via the stereographic projection of S^2 from the north pole $(0, 0, 1)$

$$(x, y, z) \to \left(\frac{x}{1 - z}, \frac{y}{1 - z} \right)$$

Call it

$$(U, V) = \left(\frac{x}{1 - z}, \frac{y}{1 - z} \right)$$

Rewriting (x, y, z) in terms of (U, V), we have

$$(x, y, z) = \left(\frac{2U}{1 + U^2 + V^2}, \frac{2V}{1 + U^2 + V^2}, \frac{U^2 + V^2 - 1}{1 + U^2 + V^2} \right) \tag{B.1}$$

Let ω be an area 2-form in S^2. Now we express the area 2-form in terms of (U, V) by viewing the coordinates (x, y, z) as being parametrized by U and V, so that

$$\omega = ||\vec{T}_U \times \vec{T}_V|| dU \wedge dV \tag{B.2}$$

where $\vec{T}_U$ and $\vec{T}_V$ are the tangent vectors on the surface $U = \mathit{constant}$ and $V = \mathit{constant}$ respectively. Explicitly,

$$\begin{aligned}\vec{T}_U &= \left(\frac{\partial x}{\partial U}, \frac{\partial y}{\partial U}, \frac{\partial z}{\partial U}\right) \\ &= \frac{2}{1+U^2+V^2}\left(-U^2+V^2+1, -2UV, 2U\right)\end{aligned}$$

and

$$\begin{aligned}\vec{T}_V &= \left(\frac{\partial x}{\partial V}, \frac{\partial y}{\partial V}, \frac{\partial z}{\partial V}\right) \\ &= \frac{2}{1+U^2+V^2}\left(-2UV, U^2-V^2+1, 2V\right)\end{aligned}$$

so that B.2 becomes

$$\omega = \frac{4}{(1+U^2+V^2)^2} dU \wedge dV \tag{B.3}$$

Since a Hopf map ϕ is a surjective and many-to-one map from S^3 to S^2, we can visulaize S^3 as $\mathbb{R}^3 \cup \infty$ via stereographic projection. Now any point of three coordinates on S^3 will be mapped to a point of two coordinates on S^2 by the Hopf map ϕ. Thus

$$\phi = (U, V)$$

or

$$\phi = U + iV$$

allowing the complex mappping. Then

$$d\bar{\phi} \wedge d\phi = 2idU \wedge dV$$

Finally, the area 2-form B.2 can be written in terms of Hopf map as

$$\omega = \frac{4}{(1 + d\phi\bar{\phi})^2}\frac{\bar{\phi} \wedge d\phi}{2i}$$

Dividing by the total area 4π, we get the normalized area 2-form

$$\mathcal{F} = \frac{1}{2\pi i}\frac{d\bar{\phi} \wedge d\phi}{(1 + \phi\bar{\phi})^2} \tag{B.4}$$

Appendix C

Duality configurations in Euclidean space-time

In a 4D Euclidean spacetime, we have the following identity of the Levi-Civita symbol

$$\epsilon_{\rho\sigma\mu\nu}\epsilon^{\rho\sigma\alpha\beta} = 2!\delta^{\alpha\beta}_{\mu\nu}$$

where $\delta^{\alpha\beta}_{\mu\nu}$ is the generalized Krnonecker delta. However, in a 4D Minkowskian spacetime, the above identity becomes

$$\epsilon_{\rho\sigma\mu\nu}\epsilon^{\rho\sigma\alpha\beta} = -2!\delta^{\alpha\beta}_{\mu\nu}$$

So let's start by defining the dual of the dual of $G_{\mu\nu}$ as

$$\begin{aligned}\tilde{\tilde{G}}_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}\tilde{G}_{\rho\sigma} \\ &= \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\rho\sigma\alpha\beta}G_{\alpha\beta}\end{aligned} \tag{C.1}$$

In Euclidean space-time,

$$
\begin{aligned}
\tilde{\tilde{G}}_{\mu\nu} &= \frac{1}{4}\epsilon_{\rho\sigma\mu\nu}\epsilon^{\rho\sigma\alpha\beta}G_{\alpha\beta} \\
&= \frac{1}{4}2!\delta^{\alpha\beta}_{\mu\nu}G_{\alpha\beta} \\
&= \frac{1}{2}(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu})G_{\alpha\beta} \\
&= G_{\mu\nu}
\end{aligned}
\tag{C.2}
$$

Symbollically, we can write the above result as

$$
\tilde{\tilde{G}}_{\mu\nu} = \varepsilon^2 G_{\mu\nu} \tag{C.3}
$$

where $\varepsilon^2 = 1$ implies $\varepsilon = \pm 1$, which corresponds to self-dual and anti-self-dual configurations. While in 4D Minkowskian space-time, we have

$$
\begin{aligned}
\tilde{\tilde{G}}_{\mu\nu} &= \frac{1}{4}\epsilon_{\rho\sigma\mu\nu}\epsilon^{\rho\sigma\alpha\beta}G_{\alpha\beta} \\
&= -\frac{1}{4}2!\delta^{\alpha\beta}_{\mu\nu}G_{\alpha\beta} \\
&= -\frac{1}{2}(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu})G_{\alpha\beta} \\
&= -G_{\mu\nu}
\end{aligned}
\tag{C.4}
$$

for which we have $\varepsilon^2 = -1$, thus implying

$$
\varepsilon = \pm i
$$

which does not correspond to self-dual and anti-self-dual configurations [4].

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