Einstein Maxwell Dilaton Solution for (2+1) dimensions

M.Sc. Thesis

By
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of
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When forced to summarise the general theory of relativity in one sentence: Time and space and gravitation have no separate existence from matter.

Albert Einstein

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Abstract

de Sitter space is a solution of Einstein equation with positive cosmological constant that models the expanding universe. This thesis presents an elementary discussion on de Sitter space-time and its coordinates.The first two chapter describes the Einstein equation and its solution. A brief review of literature is given in the third chapter and work done is presented in next chapter

We've obtained and analysed an exact solution to Einstein Maxwell Dilaton gravity that contains a scalar field coupled to gravity in minimal way and a potential that depends solely on scalar field. The solution contains a charged black hole with regular horizon. The Komar integrals and various graphs are presented in order to give further insight of the work done. The project assumes familiarity with the basics of general relativity and differential geometry, we've tried our level best to present in lucid manner.

This thesis aims to give a detailed explanation of the motivation for the objective of the project, research done, challenges faced, novelty of the program used and the impressive results obtained.

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Chapter 1

Introduction

General Relativity - shining achievement of modern physics that describes gravity in terms of rigorous mathematics. It was formulated out of philosophical questions, like why freely falling observer doesn't experience gravity?, but within few years it gained enormous attention. Before stepping into one of the solution of Einstein equation, we briefly define the terms used in it.

1.1 Einstein Equation

Curvature of space time gives a very intuitive idea of what a force is. According to Einstein, space and time are inextricably linked together and can get distorted by massive objects. In layman's language "mass tells space time how to curve, and space time tells mass how to move."

$$\mathrm{R}_{\mu\nu} + Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.1}$$

It is a set of 10 non linear highly coupled differential equation that can be reduced to 6 by using Bianchi Identities. Here μ and ν are indices that run from 0 to 3. The term in the left is called Einstein Tensor and it describes the curvature of space-time whose effects we experience in our daily life and term on the right is Stress Energy Tensor that tells everything about massive object- mass, energy, momentum, pressure etc.(12)

1.1.1 Metric and Christoffel Symbol

Metric is covariant, second degree, symmetric tensor on a differentiable manifold. It captures all the geometric structure of spacetime and is given as a bilinear form on tangent space T_pM defined on a Manifold M. Mathematically, its a map from tangent space to a real number. If we have two tangent vectors u and v at a given point in the Manifold M, then metric is

$$g_{\mu\nu} : T_pM \times T_pM \rightarrow R$$

$g_{\mu\nu}$ is metric tensor that computes distance between any two points.

Christoffel symbol defines how the covariant vector changes from point to point. On parallel transporting a vector on curved spacetime, we come up with this additional factor.(11) It is given as:

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(g_{bd,c} + g_{cd,b} - g_{bc,d} \right)$$

With proper choice of coordinate transformation we can get rid of these factors, hence they are not tensors. Christoffel symbols provides a representation for the metric connection of Riemannian Geometry in terms of coordinates of the mainfold.

1.1.2 Riemann curvature tensor

It is one of the basic mathematical tool in theory of relativity that assigns tensor to every point, and measures how parallel translation of a vector differs if we go in direction 1 and then in direction 2 or vice versa.The magnitude of spacetime curvature is expressed in terms of Riemann curvature tensor.(11; 7)
It is given as:

$$\mathrm{R}^{\rho}_{\sigma\mu\nu} = \delta_{\mu}\Gamma^{\rho}_{\sigma\nu} - \delta_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{1.2}$$

where Γ is Christoffel symbol.

1.1.3 Ricci Tensor and Ricci Scalar

Contraction of Riemann tensor gives Ricci tensor.On further contracting Ricci tensor we get Ricci scalar.

$$R^{\rho}_{\mu\rho\nu} = R_{\mu\nu} \tag{1.3}$$

$$R = g^{\mu\nu} R_{\mu\nu} \tag{1.4}$$

If we parallel propagate along a geodesics, evolution of volume is measured by Ricci tensor, i.e. it measures how geodesics tends to get rarer or denser around a point in a given direction. Sphere has positive curvature because its geodesic converges.

Ricci scalar gives the amount by which a small volume deviates from the same amount of volume taken in Euclidean space. Being a scalar, it is coordinate invariant. If Ricci scalar is positive, then volume we are working with will be smaller than that in Euclidean space and curvature will decrease with increase in r, like in sphere.(11)

If Ricci Scalar is negative, curvature will increase with increase in r, like in bowl. If Ricci scalar is zero then we can have flat space time.

1.1.4 Stress Energy Tensor

A scalar can be associated to an element of n-dimensional volume with the help of density.If normal four vector n gives the orientation of surface in space time then element of three volume is **n**Δ**V**. So we conclude that mass density transforms as a 00-component of rank 2 tensor. Energy and momentum density are sources of distortion in same time.To associate a four vector Δp^{α} to nΔV, we need two indices $T^{\alpha\beta}$, such that

$$\Delta \, \mathrm{p}^{\alpha} = T^{\alpha\beta} n_{\beta} \Delta \mathrm{V}$$

The second rank tensor $T^{\alpha\beta}$ is called Stress Energy tensor.(7;

11) In order to have a physical interpretation, lets consider an inertial frame in flat space time and three dimensional volume ΔV, along time like slice.

If n_α = (1,0,0,0) is the normal along three dimensional volume which is at rest then components of stress energy tensor are:

T^{00} is energy density
T^{0i} is the energy flux in the i-direction;
T^{i0} is the momentum density in the i-direction;
T^{ij} is force per unit area with normal in j direction(7).

So, Einstein equation tells mass warps the space and warped space acts like a moving mass.

The left hand side of Einstein equation tells how the curvature is changing while travelling along a vector and term containing $g_{\mu\nu}$ tells how measurements are affected. Right hand side of the EFE tells what physical quantities govern those changes.

Chapter 2

Solution of Einstein Equation

Einstein Field Equation (EFE) depends on dynamics of matter and energy which is related to gravitational field. It depends upon on us what type of solution one is looking for. For a weak field limit dynamics of matter can be evaluated using Newtonian laws and result of stress energy tensor can be plugged in the EFE. For an exact solution stress energy tensor and metric should go side by side along with continuity equation.For (3+1) dimension, these equations are not enough as we have to calculate 20 unknowns from 14 equation. One heads more into seas of problem if there are internal degrees of freedom. In order to simplify the problem, we usually take approximation like:

$$\begin{aligned} T_{\alpha\beta} &= 0 && \text{for vaccum} \\ T_{\alpha\beta} &= \rho u_a u_b && \text{for non interacting dust} \end{aligned}$$

2.1 Schwarzschild Solution to Einstein's Equation

It is one of the simplest solution of EFE with no matter and most number of symmetries.This solution holds for empty space around a spherically symmetric source of curvature, with assumption that cosmological constant, electric charge, angular momentum all are zero.(14) The line element for the Schwarzchild geometry is given by:

$$ds^2 = -(1-2GM/c^2r)c^2dt^2 + (1-2GM/c^2r)^{-1}dr^2$$

$$+ r^2(d\theta^2 + sin^2\theta d\phi^2) \quad (2.1)$$

Schwarzchild's metric has following properties(11) :

1. **Time independent** : metric is independent of t thus there is a killing vector associated with it.

$$\xi^\alpha = (1,0,0,0)$$

2. **Spherically symmetric** : metric is independent of ϕ. For S^2 it will have three killing vectors that preserves angular momentum L_x, L_y and L_z, which is given as:

$$\eta^\alpha = (0,0,0,1)$$

Killing vector tells there's conservation law associated with these parameters –

ξ^α gives law of conservation of energy .

η^α gives law of conservation of angular momentum

3. **Mass M and coordinate r** : Coordinate r is not radius but is related to the area of 2-dimensional sphere by:

$$r = (A/4\pi)^{1/2} \quad (2.2)$$

Mass M is total mass of source of curvature. The geometry away is independent of how the mass is radially distributed.

4. The metric blows off at r = $2GM/c^2$ and r = 0. These two points act as singular points. We can find true singularity by using Ricci scalar because it being scalar is coordinate invariant . On solving Schwarzschild metric we get

$$R^{abcd}R_{abcd} = \frac{48G^2M^2}{r^6} \quad (2.3)$$

This gives that r = 0 is the true singularity and the other one $r_s = 2GM/c^2$ arises because of choice of the coordinate.r_s is known as Schwarzschild radius which is the characteristic length scale for curvature in Schwarzschild geometry.(16; 7; 4) With mere change of coordinates one can get rid of coordinate singularity.

2.2 Reissner-Nordstrom Solution to Einstein's equation

Reissner Nordstrom (R.N.) solution of Einstein field equation describes a geometry around a charged, spherically symmetric black hole. Even though charged black hole can never exist because it will be neutralise as soon as it comes in the vicinity of matter, it has been a fascinating topic among physicist.(4; 5)

For no charge metric should turn to Schwarzschild metric and the behaviour at asymptotically far away should be that of flat space. Both of these condition are followed by the solution given by Reissner and Nordstorm.

Consider an Einstein-Maxwell action:(Hartman)

$$\int d^4x\sqrt{-g}[R - F^{\mu\nu}F_{\mu\nu}] \tag{2.4}$$

Equation of motion are;

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -\tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + F_{\mu\rho}F^{\rho}$$

$$\nabla_\mu F^{\mu\nu} = 0$$

The solution for the above set of equation is given by-

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_2 \tag{2.5}$$

where $$f(r) = 1 - \tfrac{2M}{r} + \tfrac{Q^2}{r^2}$$

which is obtained by assuming that the vector potential is a function of radial coordinate. Using this the electric field component in the radial direction is given as;

$$F_{tr} = \tfrac{Q}{r^2}$$

Komar Mass and Komar Charge have been discussed in the upcoming chapters. On using the same calculation one can show that M and Q are mass and charge of the black hole respectively.(16)
The key features of RN metric is obtained by comparing Q and M.
Roots of f(r) are

$$r\pm = \mathrm{M} \pm \sqrt{M^2 - Q^2}$$

- For Q≤ M, f(r) will be positive for r≥ r_+ and r≤ r_- and negative in between the two roots. This swapping nature will give us a coordinate horizon. One can get rid of this horizon by choosing proper coordinate.

- For Q≥ M, f(r) will always be positive that is t remains timelike and r remains spacelike. At r = 0 metric blows off, and since we don't have any event horizon the singularity is not hidden. Hence at r = 0 we have a naked singularity.(7)

r_+ is called event horizon and r_- is called Cauchy horizon. Physicist for studying black hole thermodynamics and other black hole related event always take consider;

$$M > Q > 0$$

Two reason related to this choice are;

- In order to follow the Cosmic Censorship principle given Penrose which states that all singularities need to be hidden from an observer at infinity by the event horizon of black hole.

- Not much is known about the naked singularity, and its hard to predict a model in a day or two.

Second, if there were a naked singularity, then physics outside the black hole depends on the UV (since the naked singularity can spit out visible very heavy particles), and we should not trust our effective theory anyway. We can divide the metric in three regions;

$$Region1 : r_+ < r < \infty$$

$$Region 2 : r_- < r < r_+$$

$$Region 3 : 0 < r < r_-$$

For a particle crossing the event horizon from region 1 to region 2, an observer far away from the geometry of black hole would think him to be redshifted. However the falling observer takes finite time to reach the horizon.

Inside region 2 particles move in the direction of decreasing r. On reaching the region 3 r switches back to spacelike coordinate and hence we are saved from being hit by the singularity. One can continue moving in the direction of singularity or can move in the direction of increasing r. Again on reaching region 2 r swaps its nature but with reversed direction.(7)

Chapter 3

Literature Survey

3.0.1 What is de Sitter Space

de Sitter space is the maximally symmetric vacuum solution of Einstein's field equations with a positive cosmological constant λ. It may be realised as a hyper surface described by the equation;(15; 3)

$$-X_0^2 + X_1^2 + X_2^2 + ... + X_d^2 = r^2 \tag{3.1}$$

in flat $(d+1)$ Minkowski Space. Here r is de Sitter radius. The isometry group of de Sitter space is the Lorentz group O(1, n), and such embedding preserves the isometry group.(9) For an empty space with positive cosmological constant, Einstein equation is given as;

$$G_{ab} + \lambda g_{ab} = 0 \tag{3.2}$$

$$\lambda = \frac{(d-2)(d-1)}{2r^2} \tag{3.3}$$

The solution of this equation describes the de sitter space for an empty universe.

3.1 Coordinates in de Sitter Space

Various coordinate system helps in giving different insight to the geometry of de Sitter Space. The choice of coordinate system depends on the problem we are working at.(15; 9; 12)

3.1.1 Global coordinate

It covers entire dS space. The spatial section starts at $\tau \to \infty$, then shrinks to finite size at τ =0 and then grows again to finite size as $\tau \to \infty$. It is given by;

$$X_0 = sinh\tau$$
$$X_i = w_i cosh\tau$$

where;

$$\omega_1 = cos\theta_1$$
$$\omega_2 = sin\theta_1 cos\theta_2$$
$$.$$
$$.$$
$$.$$
$$\omega_d = sin\theta_1 ... sin\theta_{d-2} sin\theta_{d-1}$$

such that line element becomes;

$$ds_2^2 = -d\tau^2 + (cosh^2\tau)d\Omega_{d-1}^2$$

Using global coordinates one can prove that de Sitter space time is geodesically complete, that is the affine parameter of any geodesic passing through any point can be extended to reach arbitrary value. This implies de Sitter space is a non-stationary space time.

3.1.2 Conformal Coordinates

It is related to global coordinate via;

$$cosh\tau = \frac{1}{cosT}$$

This coordinate system helps in visualising that de Sitter space is conformally flat space time with line element;

$$ds_2^2 = \frac{1}{cos^2 T}\left(-dT^2 + d\Omega_{d-1}^2\right)$$

This coordinate system tells no single observer can access the entire space time. A classical observer sitting on the south pole will never be able to observe anything beyond the diagonal line stretching from the north pole at I- to the south pole at I+, and he can only send message to a part stretching from south pole at I- to north pole at I+. The highlighted portion represents a casual diamond where an observer sitting at south can observe as well as send message.

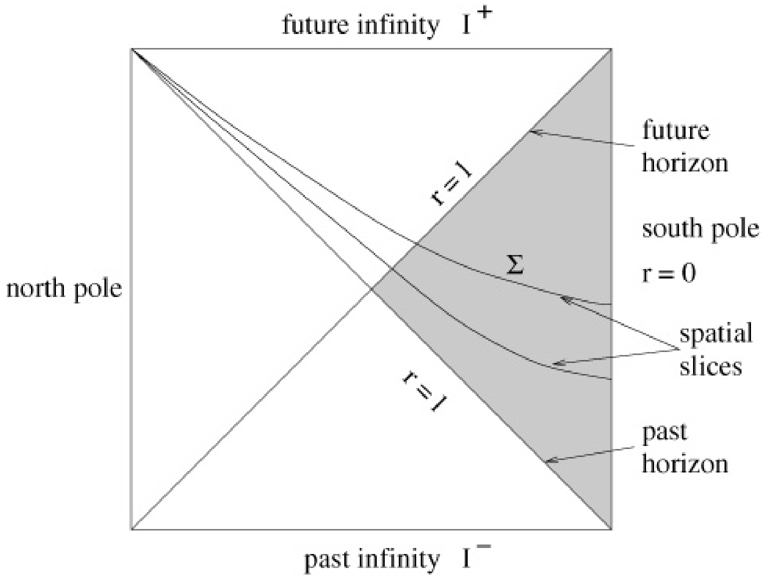

Figure 3.1: Penrose diagram for dS_d.(15)
North pole and South pole are time like lines and each point in interior represents a S^{d-2}

3.1.3 Static Coordinate

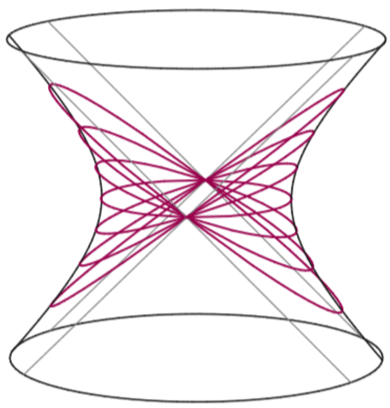

Figure 3.2: Spatial slicing in Static coordinate(12)

The spatial sections of static coordinates represents equator in some global coordinate system. It covers two separate parts of dS and the grey line represents the boundaries of static coordinate. The coordinate system is generated by taking;

$$X_0 = \sqrt{1-r^2} sinht$$
$$X_i = r\omega_i \qquad \text{where} \qquad 1 \le i \le d-1$$
$$X_d = \sqrt{1-r^2} cosht$$

whre ω_i is the same as defined in Global coordinate such that line element;

$$ds^2 = -(1-r^2)dt^2 + \frac{dr^2}{(1-r^2)} + r^2 d\Omega_{d-2}^2$$

It has axial as well as time translational symmetries. The timelike killing vector can only be used to define a sensible time evolution in the southern diamond of de Sitter space as in northern diamond it points towards past infinity. Since we can't boost the past the like cone, static coordinate covers only half of de sitter space. And since all the spatial section meet at a point we expect a coordinate singularity at that point.

3.2 Embedding in Minkowski space

de Sitter space can be viewed as a sub manifold of generalised Minkowski space of one higher dimension.(12; 9)

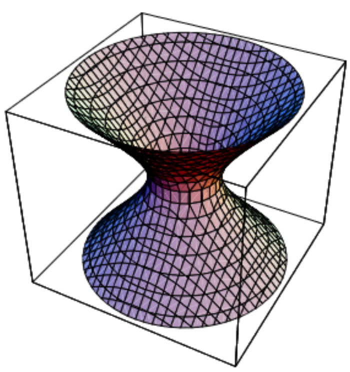

Figure 3.3: dS_2 embedded in M_3(9)

Let us consider a Minkowski space in 4+1 dimensional space, such that line element is given as;

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = l^2$$

If we take the form of X_i's as;

$$\mathrm{X}_0 = \sqrt{l^2 - r^2}sinh(\tfrac{t}{l})$$
$$\mathrm{X}_1 = \sqrt{l^2 - r^2}cosh(\tfrac{t}{l})$$
$$\mathrm{X}_2 = rcos\theta$$
$$\mathrm{X}_3 = rcos\phi sin\theta$$
$$\mathrm{X}_4 = rsin\phi sin\theta$$

If we take the derivative of following transformation and substitute in (4+1) dimensional Minkowski space, we get;

$$\mathrm{ds}^2 = -(1 - \tfrac{r^2}{l^2})dt^2 + \frac{dr^2}{(1-\frac{r^2}{l^2})} + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right)$$

which is (3+1) dimensional de Sitter space time in static coordinate. Hence dS is a hyperboloid embedded in one higher dimension Minkowski space.

3.3 Direction of time in de Sitter Space

From conformal coordinates it was clear that a quadrant space is available to an observer sitting at south pole to receive and observe something.(9) Let us suppose we are working on a slice of global coordinate (elliptic de sitter space), denoted by dS/A, where A is a map containing identity and antipodal transformation (XA → -XA). An observer not constrained with speed of light can appreciate the entire de Sitter space time.

Let us consider the part of dS which was not observable to south pole observer is observable to an antipodal observer. If we start at one end of equator by assigning a forward time vector at that point, the antipodal observer will reverse the time direction vector. On going around the equator, we reach a point where time orientation can't be defined. It can go in either forward or backward direction, hence we can't talk about global time coordinate.

Consider two observers; Antipodal observer " superman" and another normal observable "common man". If normal observable sends a video which superman is receiving, superman will first watch it first being played in reverse direction and then in forward direction. This can even help in showing that assigning local coordinate can violate causality.

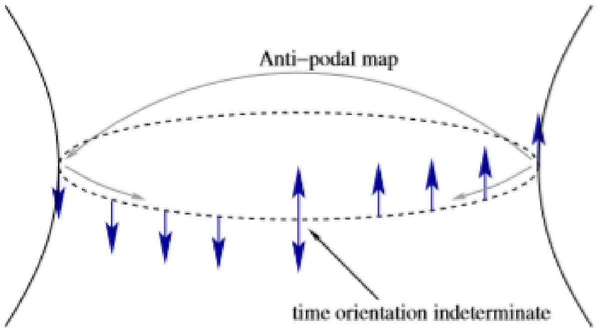

Figure 3.4: Situation on assigning global time coordinate

3.4 Einstein Maxwell Dilaton gravity

A particle of a scalar field , that couples to the gravity is known as dilaton. Einstein Maxwell dilaton theory is the simplest low effective theory that arises when scalar field couples to gauge field. It generalises the Einstein Maxwell theory by adding a kinetic term from dilaton and also few coupling terms.(6; 17; 2) The presence of dilaton has given a new insight in this field. Dilaton has enriched the physics of the solutions to the field equations compared to the Reissner-Nordström solution, which is the charged black hole solution of the Einstein-Maxwell theory.

3.4.1 General form of Einstein Maxwell Dilaton

The general form of Einstein Maxwell dilaton action for (2+1)dimensions is given as(13);

$$S = \int d^3x\sqrt{-g}\left(R - 2\partial_\mu\phi\partial^\mu\phi - W(\phi)F_{\mu\nu}F^{\mu\nu} - V(\phi)\right)$$

which is for some dilaton ϕ. $W(\phi)$ is usually an exponential function of the dilaton field and $V(\phi)$ is self interacting potential. The form of the field strength is given as;

$$F\mu\nu = \partial_\mu A_\nu - \partial_\nu A_\mu$$

We are taking $16\pi G = 1$. The equation of motion obtained after perturbing the action are as follows:

for metric

$$R_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}W(\phi)F_{\rho\sigma}F^{\rho\sigma} + 2\mathrm{W}(\phi)F^{\mu\rho}F^{\rho}_{\nu} + \tfrac{1}{2}g_{\mu\nu}V(\phi)$$

for dilaton

$$\nabla_\mu(\partial^\mu\phi) - \frac{1}{4}\frac{W(\phi)}{\phi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\frac{\partial V}{\partial\phi} = 0$$

for gauge field

$$\nabla_\mu\big(W(\phi)F^{\mu\nu} = 0$$

The first Einstein Maxwell dilaton solution was presented by Gary Gibbons in 1982 for electric as well as magnetic charges, and was known as dyonic solution. Later a series of solutions came for (2+1) and other higher dimensions.

3.4.2 Why (2+1) dimensions ?

Study of black hole physics in lower dimension is easier and can give us a deeper insight in framing physics for higher dimension. A/dS CFT correspondence states that there lies a dual between quantum gravity on A(dS) space and a Euclidean conformal field theory for lower dimensional spacetimes (2; 6).

The first study of three dimensional black hole, as a result of Einstein's theory of relativity was done by Banados, Teitelboim and Zanelli, and the solution is commonly known as BTZ black hole. After the discovery of first BTZ black hole there is a flood of different types of black holes for (2+1)dimensions.

3.4.3 Centaur Geometry

Centaur geometry is one of the solution of Einstein Maxwell dilaton gravity with a specific class of potential for dilaton. It can be considered as thermal state in putative quantum mechanics dual to AdS that evolved with global Hamiltonian.

Centaur geometry is asymptotically AdS but flows in interior to static patch of de Sitter space.(1) Such type of geometry is essential because:

1. Applying Holographic principle to dS is not possible beacuse static patch in de sitter doesn't have aysmptotically spatial boundary.

2. AdS has its own distinguished feature like " a symmetric box with a boundary" that will help in removing IR effects.

With the help of such geometries we are able to deal with the interior of static patch and interpret them in terms of putative quantum mechanical dual to asymptotically AdS geometry.

Chapter 4

Charged Black Hole with scalar hair in (2+1) dimensions

In this thesis, we aim to study black hole solution of an Einstein Maxwell gravity with minimally coupled scalar field in (2+1) dimensions. Black hole solution comprising gravity coupled to a scalar field is known as hairy black hole. We start with an action in which scalar couples to gravity in a minimal way and it also couples to itself via a self interacting potential V(ϕ). We started with the following action:

$$S = \int \sqrt{-g} d^3x (R - g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - 2V(\phi) - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}) \quad (4.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

On perturbing it with respect to $g_{\mu\nu}$ and using the trace reversal technique we get;

$$R_{\mu\nu} - g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi + 2\nabla_\mu\phi\nabla_\nu\phi - 2g_{\mu\nu}V(\phi) -$$

$$\frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + F_{\mu\rho}F^{\rho}_{\nu} = 0 \quad (4.2)$$

On perturbing it with respect to ϕ and A_μ, we get;

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) - \frac{\partial V}{\partial\phi} = 0 \quad (4.3)$$

$$\partial_\nu(\sqrt{-g}F^{\mu\nu}) = 0 \tag{4.4}$$

We aim to obtain a static, symmetric solution so we take ansatz of the following form;

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + y^2(r)dz^2 \tag{4.5}$$

where coordinate ranges from $-\infty \leq$ t $\leq \infty$, r$\geq$0 and $-\pi \leq$ z $\leq \pi$ Let us consider that $A_\mu$and ϕ are function of radial coordinate only. Using equation (4.4) and above assumption Maxwell's field can be given by choosing another ansatz,

$$A_\mu = F(r)dt \tag{4.6}$$

such that;

$$F_{tr} = -\frac{Q}{y(r)},$$

where Q is a constant that will be related to Komar charge in later section.

Using above ansatz we can write below the equation of motion by substituting the value of R$\mu\nu$ obtained using mathematica. Christoffel symbol having non zero values and R$\mu\nu$ are mentioned in the Appendix A.

$$\frac{ff'y'}{2y} + \frac{ff''}{2} + f^2(\phi')^2 + 2f(V) + \frac{fQ^2}{2y^2} = 0 \tag{4.7}$$

$$-\frac{f'y'}{2fy} - \frac{y''}{y} - \frac{f''}{2f} + (\phi')^2 - \frac{2V}{f} - \frac{Q^2}{2fy^2} = 0 \tag{4.8}$$

$$-\frac{f'y'}{y} - \frac{fy''}{y} - f(\phi')^2 - 2V + \frac{Q^2}{2y^2} = 0 \tag{4.9}$$

$$\frac{y'f\phi'}{y} + f'\phi' + f\phi'' - \frac{dV}{d\phi} = 0 \tag{4.10}$$

In the above set of equation f(r) has been written as f for simplicity. On adding (4.7) and (4.8)$*f^2(r)$, we get a relation between ϕ and y(r), which is given as

$$(\phi')^2 = \frac{y''}{2y} \tag{4.11}$$

On substituting (4.11) in (4.9) we get a constraint equation, i.e. equation independent of derivative of second order. Consistency of equation is checked by expressing the derivative of constraint equation as a linear combination of (4.7) and(4.8).

Since (4.10) contains derivative of V we can use constraint equation to evaluate $V(\phi)$. V can be replaced by using constraint equation and either of two equation (4.7) and (4.8) can be replaced by (4.11), such that we are left with two equations for three parameters viz. f(r), y(r) and ϕ. Therefore we have a choice of assuming a form of one out of three unknown parameters. In order to give a concrete form to scalar field we can solve differential equation (4.11) by assuming a form of y(r), which we chose to be;

$$y(r) = exp(\phi^2) \tag{4.12}$$

Most of the literature(6; 17; 2) works with the linear form of y(r), we thought of taking it in given form in order to know the extent till which solution can vary and obtained results were quite good. Solving for ϕ then gives;

$$\phi = -\mathrm{i}^*\mathrm{erf}^{-1}\left(\frac{2i}{\sqrt{\pi}}(c_1 r + c_1 c_2)\right)$$

Using the power series expansion of error function and choosing $c_2 = 0$, we get;

$$\phi = c_1 r - \frac{c_1^3 r^3}{6} \tag{4.13}$$

To obtain the form of f(r) and V(ϕ) we use the four equation of motion. Substituting the value of ϕ and y(r) and on doing the required mathematics we get a differential equation for f(r) which is given as,

$$3f''\phi\phi' + 11f'(\phi')^2 + 3f'\phi\phi'' + \frac{f'''}{2} + 12f\phi'\phi'' + 8f\phi(\phi')^3 = 0 \tag{4.14}$$

The equation for f(r) turned out to be third order, non homogeneous differential equation. From (4.13) it is clear that kinetic term will keep on increasing with the increase in r. Kinetic part will shoot to infinity at r $\rightarrow\infty$, if and only if is is heavily influenced by external source. To avoid any external source we solved f(r) in steps first for small r and then for large r. Only the form of f(r) is shown for large r, further section deals with small r only.

For small r;

Equation (4.14) is bit tedious to solve by ordinary differential equation solving method. To ease our problem we use series solution method and assume the solution of f(r) as;

$$f(r) = \sum_{n=0}^{6} a_n r^n + (\sum_{n=0}^{6} b_n r^n) logr \tag{4.15}$$

Using DSolve in mathematica we find that all b's turn out to be zero and truncating the result up to sixth order, we get;

$$f(r) = a_0 + a_1 r + a_2 r^2 - \frac{11}{3} c_1^2 a_1 r^3 + (\frac{1}{3} c_1^4 a_0 - \frac{7}{3} c_1^2 a_2) r^4 +$$

$$\frac{41}{6} c_1^4 a_1 r^5 + \frac{(c_1^4 a_2 - 80 c_1^6 a_0 + 560 c_1^4 a_2) r^6}{180} \tag{4.16}$$

and potential V(ϕ) is given by substituting the value of f(r), ϕ and y(r) in (4.10) to get,

$$V(\phi) = \frac{1}{1440}\Big(-4a_2\Big(90 - 360c_1^2 r^2 + 930 c_1^4 r^4 + 10003 c_1^6 r^6 -$$

$$5764 c_1^8 r^8 + 835 c_1^{10} r^{10}\Big) + 5c_1^2\Big(-3a_1 r\Big(-96 +$$

$$352 c_1^2 r^2 + 4026 c_1^4 r^4 - 2450 c_1^6 r^6 + 369 c_1^8 r^8\Big)$$

$$+8a_0\Big(-36 + 18c_1^2 r^2 + 3c_1^4 r^4 + 124 c_1^6 r^6 - 70 c_1^8 r^8 + 10 c_1^{10} r^{10}\Big)\Big)\Big) \tag{4.17}$$

Obtained expression for V looks little complicated, in order to simplify it we take $a_0 = a_2 = c_1 = 1$ and $a_1 = 0$, and then a plot of V(ϕ) is drawn w.r.t r. We will see in further section that why we chose $a_1 = 0$.

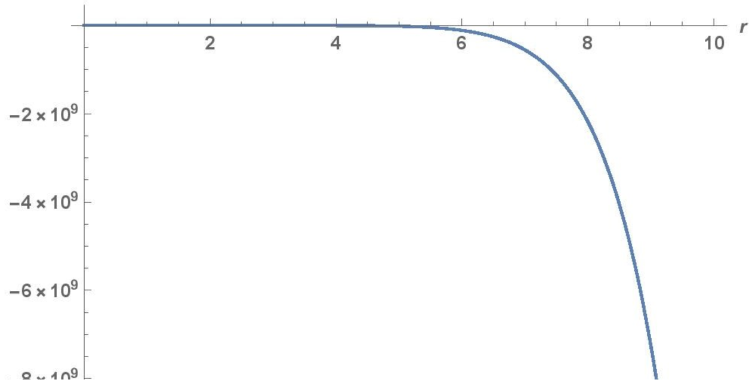

Figure 4.1: Plot of V(ϕ) vs r

For large r;

We take a new coordinate that is related to old coordinate via,

$$u = \frac{1}{r}$$

This helps us in interpreting the same case for small u. Substituting this in (4.14) leaves us in a big trouble since u starts to appear in denominator as well as. We define a new parameter "h" which is related to f(r) via;

$$f(u)=e^{h(u)}$$

With further more changes we obtained an ugly looking form of f, which is given as

$$f(r) = exp\left(\frac{1}{48c_1^5}(-\frac{8232c_1}{r} - \frac{1680c_1^3}{r^2} - 792c_1^5 log\frac{1}{r})\right)$$

$$cos\left(\frac{1}{48c_1^5}\left(764718 + 1261815c_1^2 r^2 - 1200c_1^4 r^4 - 4800c_1^6 r^6 + 80c_1^8 r^8\right)\right) \quad (4.18)$$

The set of equation defined from (4.12)-(4.18) forms an approximate solution to the system defined by the action in (4.1), which is different from the literature survey done on (2+1) dimensional black hole.

Stress tensor should vanish at boundary for physical solution hence we take r $\rightarrow 0$ as going towards boundary. For a very trivial case with $a_0 = a_1 = a_2 = c_1 = 0$, we get;

$$f(r) = 1 \qquad\qquad y(r) = 1$$

i.e. an asymptotically flat space solution. We tried to obtain an expression for horizon but the behaviour of ϕ at large r didn't help us. To get a value of horizon radius, we solved the problem numerically using NDSolve technique in mathematica. Largest root of f(r) for which curvature invariants viz Kretschmann scalar $R_{abcd}R^{abcd}$, Ricci scalar and $R_{ab}R^{ab}$ give finite result is interpreted as horizon. To check whether the horizon is coordinate singularity , naked or a real singularity, we find numerically the values of curvature invariant at horizon. For our case we have finite results that means a coordinate singularity.
NDSolve technique is based on Runge Kutta Method, that gives an iterative solution when initial condition is known at one end but has to be evaluated at the other boundary.

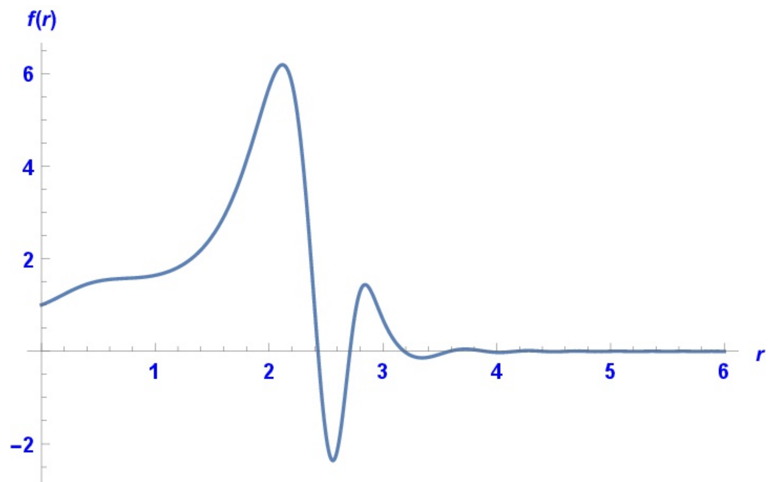

Figure 4.2: Plot of f(r) vs r

We confined our calculation to small r and in order to solve numerically the upper range of r was set to 10. On solving it we got a horizon at 9.9876, where above described parameter gave finite results.

The solution has 4 parameter viz a_0, a_1, a_2 and c_1. Each of these parameters are independent and can be interpreted in terms of physically measurable quantities.

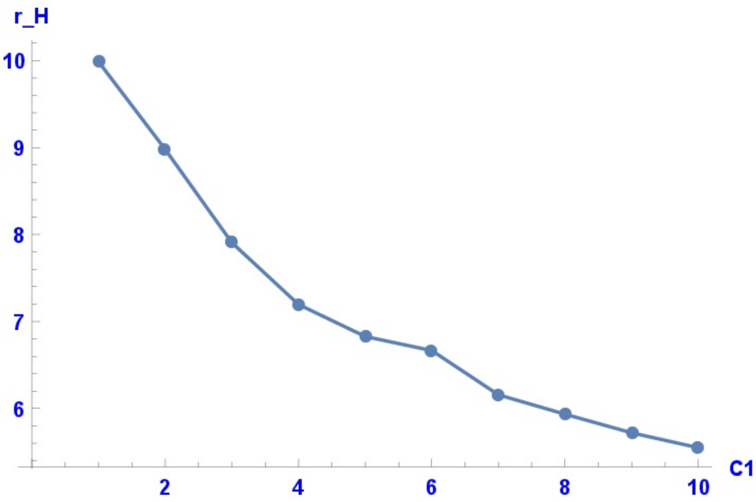

Figure 4.3: Plot of c1 Vs horizon Radius

Komar Charge

The electric current associated to the electromagnetic field tensor $F_{\mu\nu}$ is conserved, so we can associate an electric charges to spacelike surface Σby;

$$Q_{Komar} = -\frac{1}{4\pi}\int_{\partial\Sigma} d^{d-2}x\eta_{\mu}\sqrt{\alpha}\sigma_{\nu}F^{\mu\nu} \tag{4.19}$$

where $\partial\Sigma$ is the boundary of Σ at spatial infinity. α_{ij} is the induced metric on $\partial\Sigma$ and σ_{ν} is the outward pointing unit normal to $\partial\Sigma$.

For our case:

$$\eta^{\mu} = (\pi^{\frac{-1}{2}}, 0, 0)$$

$$\sigma^{\nu} = (0, \pi^{\frac{1}{2}}, 0)$$

$$\sqrt{\alpha} = y(r)$$

which gives the value of Komar charge as;

$$Q_{Komar} = \frac{Q}{2}$$

Komar Mass

Using the above analogy we can define Komar Mass since we have time like killing vector which can give rise to energy/ mass.(10; 16)

$$\begin{aligned} M_{Komar} &= \frac{1}{4\pi G}\int d^{d-2}x\eta_{\mu}\sigma_{\nu}\nabla^{\mu}\xi^{\nu}|_{r1\infty} \\ &= \frac{1}{4\pi G}\int_{-\pi}^{\pi} dz\frac{f''(r)}{2}y(r) \\ &= \frac{a_2}{2G} \end{aligned} \tag{4.20}$$

Thus a_2 can be interpreted in terms of mass.
From expression of ϕ and V(ϕ) we get;

$$\phi = c_1 r - \frac{c_1^3 r^3}{6}$$

$$\phi' = c_1 \text{ for r} \rightarrow 0$$

Thus c_1 can be interpreted in terms of first derivative of our scalar field.
Similarly for r→0 we get;

$$V(\phi) = \frac{1}{1440}(-360a_2 - 288a_0)$$

Out of four parameters, three are somehow related to mass, scalar field and potential but a_1 never comes into picture. In order to know whether a_1 is really a physical parameter we changed it, keeping other parameters fixed and found there is no change in graphs. So a_1 can be taken to zero and we are left with three parameters.

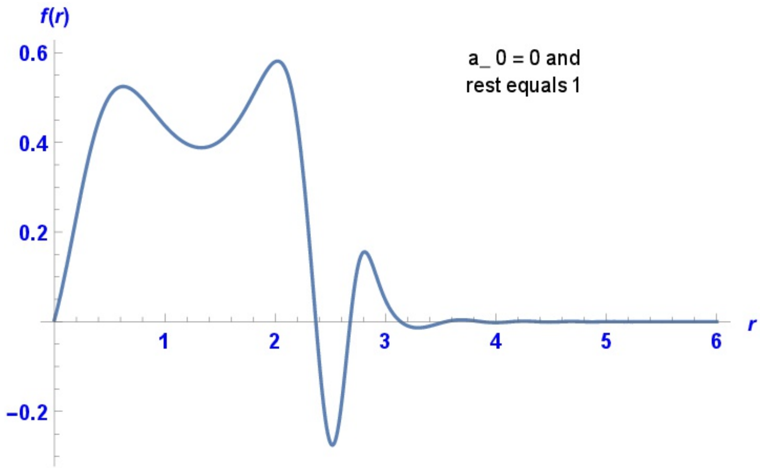

Figure 4.4: Plot of f(r) vs r for $a_0 = 0$

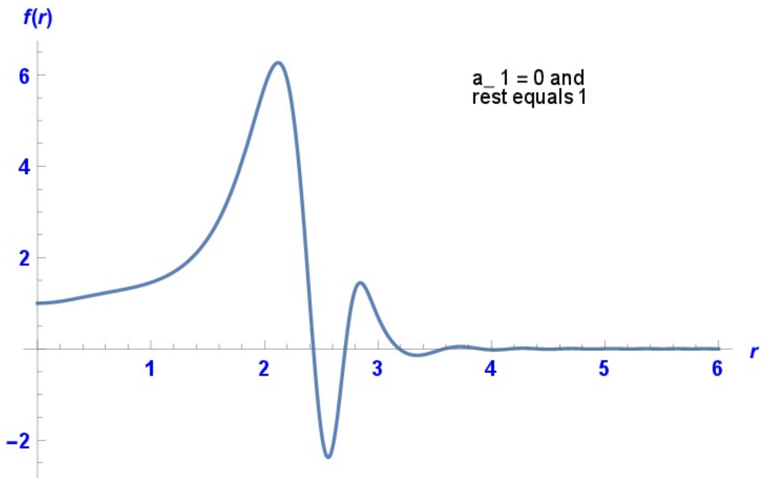

Figure 4.5: Plot of f(r) vs r for $a_1 = 0$

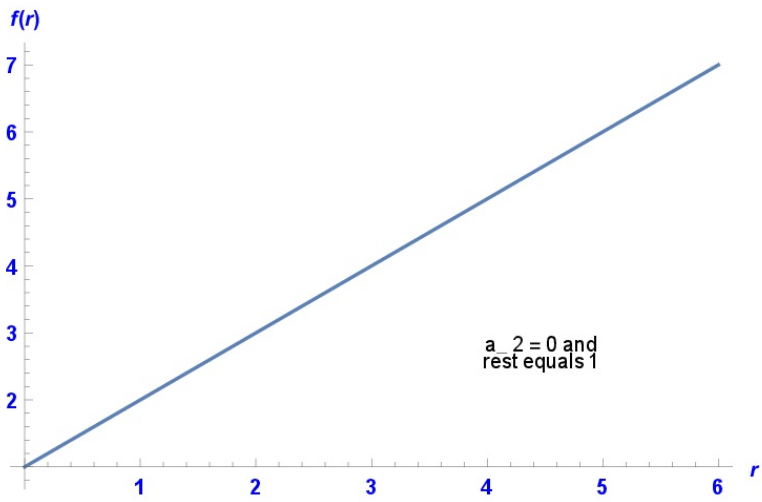

Figure 4.6: Plot of f(r) vs r for $a_2 = 0$

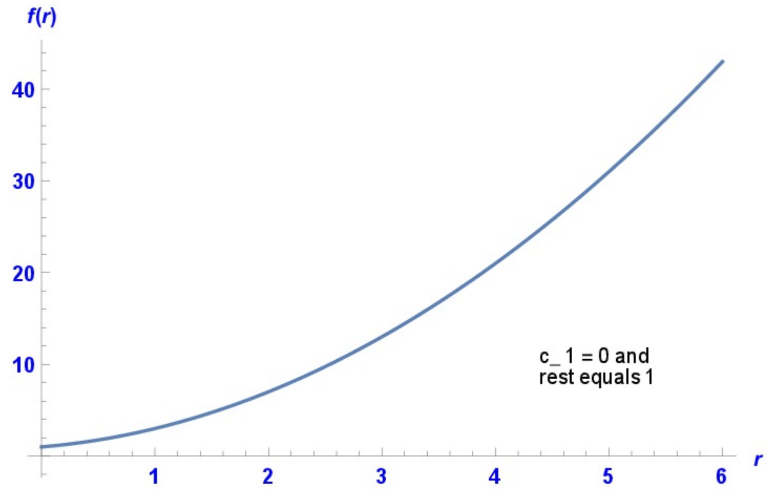

Figure 4.7: Plot of f(r) vs r for $c_1 = 0$

From above plot we see a_1 can be interpreted in terms of translation factor which can be absorbed by reparametrization of coordinate r. As changing a_1 doesn't lead us anywhere but a small change in rest of the parameters can end up giving a drastic change. Let us consider a form of V as;

$$V = V_0 + V_1 r^2 + V_2 r^4 + ...$$

It is symmetric about r. On varying r from r to $r + r_0$ we get linear terms in V.

$$V' = V_0 + V_1 r_0^2 + (2r_0 V_1) + V_1 r^2 + ...$$

We can choose value of r_0 such that linear term vanishes. Similarly for (4.17) we can choose value of a_1 in order to simplify the expression. For simplicity we have chosen $a_1 = 0$.

4.1 Results and Discussion

In this thesis we have obtained and analysed a solution for Einstein Maxwell Dilaton theory in (2+1) dimensions, in which scalar is coupled to gravity in minimal way and is also coupled to itself via a self interacting potential. The solution is static and spherically symmetric. From above calculation few results can be drawn which is listed below:

- In order to obtain a physical solution we worked for small r limit where stress tensor vanishes. For $r \to 0$ and $a_0 = a_1 = a_2 = c_1 = 0$ we obtained a flat space solution.

- With proper choice of parameter $a_0, a_1, a_2 and c_1$ for $r \to 0$ limit we can get a constant value of potential. Depending upon the sign of constant we can call our solution to be asymptotically de Sitter or asymptotically Anti de Sitter space solution.

- In order to characterise the solution asymptotically far away we used NDSolve technique based on Runge Kutta method in Mathematica. For largest root of f(r), curvature invariants viz. Kretschmann Scalar, Ricci Scalar and $R^{ab}R_{ab}$ were calculated and finite result saved us from violating the Cosmic Censorship principle.

- Charge and Mass associated with the black hole are calculated using the Komar integrals. Fortunately charge came out to be constant and out of four parameters a_2 could be physically interpreted as the mass of the black hole.

- In process of interpreting the remaining free parameters we came to know that a_1 is nothing but a translation factor. And for the sake of convenience we can take it to be zero. With this assumption a simplified form of V is obtained. Plot of V Vs r showed that we have an attracting potential.

- $c1^2$ can be interpreted as Kinetic energy of scalar field at boundary.

4.2 Conclusion and Future Scope

We started by coupling gravity to scalar and Maxwell field and found a static, spherically charged, massive black hole. The novelty of work lies in the value of V that can help in giving various solution depending upon the choice of the parameters. A positive value Of V can help us in studying the de Sitter universe whereas negative value of V can help us in studying certain gauge related theories.

It was shown that the horizon radius decays exponentially with c_1. We calculated the mass and charge of Einstein Maxwell using Komar integral and interpreted the four parameters a_0, a_1, a_2, c_1 in terms of physical quantities. We have used ND solve technique to find the numerical value of horizon in a given range. In future we can find a better way to know the expression of horizon radius and based on it we can work on Black hole thermodynamics.

We can extend our work for higher dimension and can study the dynamics of de sitter space or anti de sitter space time and study its vacuum properties and properties of wave propagating in it.

We can also analyse the stability of Einstein Maxwell black hole using the heat capacity and can find whether it undergoes any phase transition.

Chapter 5

Appendix

5.1 Appendix A

This is a mathematica notebook that one can use for evaluating Christoffel symbol, Ricci tensor, Riemann tensor and Einstein Tensor.
First dimension of spacetime is set.We are working in 2+1 dimension so we will take n=3 whenever required.
Inp: n = 3
Out: 3

Inp: coord = {t, r, x}
Out: {t,r,x}

Inp: metric = {{-f[r], 0, 0}, {0, $\frac{1}{f[r]}$, 0}, {0, 0, (y[r])^2}}
$Out: \{\{-f[r],0,0\}, \{0, \frac{1}{f[r]},0\}, \{0,0,y[r]^2\}\}$

Christoffel Symbols
$Inp : affine = Simplify[Table[(\frac{1}{2}) * Sum[(inversemetric[[i,s]])*(D[metric[[s,j]],coord[[k]]]+ D[metric[[s,k]],coord[[j]]]-D[metric[[j,k]],coord[[s]]]), \{s,1,n\}],\{i,1,n\},\{j,1,n\},\{k,1,n\}]]$

$Inp : listaffine = Table[If[UnsameQ[affine[[i,j,k]],0], \{ToString[\Gamma][i,j,k]],affine[[i,j,k]]\}],\{i,1,n\},\{j,1,n\},\{k,1,n\}]$

$Inp : TableForm[Partition[DeleteCases[Flatten[listaffine], Null],2],TableSpacing \rightarrow \{2,2\}]$

Here D is for partial derivative, Sum is for summing over dummy indices and Table is used to tabulate the result.

Out:

$\Gamma[1,1,2]$	$\frac{f'(r)}{2f(r)}$
$\Gamma[1,2,1]$	$\frac{f'(r)}{2f(r)}$
$\Gamma[2,1,1]$	$\frac{1}{2}f(r)f'(r)$
$\Gamma[2,2,2]$	$\frac{-f'(r)}{2f(r)}$
$\Gamma[3,2,3]$	$\frac{y'(r)}{y(r)}$
$\Gamma[3,3,2]$	$\frac{y'(r)}{y(r)}$
$\Gamma[2,3,3]$	$-f(r)y(r)y'(r)$

Riemann Tensor

$Inp : riemann = Simplify[Table[D[affine[[i,j,l]], coord[[k]]]-$

$D[affine[[i,j,k]], coord[[l]]]+Sum[affine[[s,j,l]]affine[[i,k,s]]-$

$affine[[s,j,k]]affine[[i,l,s]], \{s,1,n\}], \{i,1,n\}, \{j,1,n\}, \{k,1,n\},$
$\{l,1,n\}]]$

$Inp : listriemann := Table[If[UnsameQ[riemann[[i,j,k,l]],0],$
$\{ToString[R[i,j,k,l]], riemann[[i,j,k,l]]\}], \{i,1,n\}, \{j,1,n\}, \{k,1,n\},$
$\{l,1,k-1\}]$

$Inp : TableForm[Partition[DeleteCases[Flatten[listriemann],$
$Null],2], TableSpacing \rightarrow \{2,2\}]$

$Out:$

$R[1,2,2,1]$	$\frac{f''(r)}{2f(r)}$
$R[1,3,3,1]$	$\frac{1}{2}y(r)y'(r)f(r)$

$R[2,1,2,1]$ $\frac{1}{2}f(r)f''(r)$

$R[2,3,3,2]$ $\frac{1}{2}y(r)\left(f'(r)y'(r)+2f(r)y''(r)\right)$

$R[3,1,3,1]$ $\frac{f(r)f'(r)y'(r)}{2y(r)}$

$R[3,2,3,2]$ $-\frac{f'(r)y'(r)+2f(r)y''(r)}{2f(r)y(r)}$

Ricci Tensor

$Inp: ricci = Simplify[Table[Sum[riemann[[i,j,i,l]],$
$\{i,1,n\}],\{j,1,n\},\{l,1,n\}]]$

$Inp: listricci = Table[If[UnsameQ[ricci[[j,l]],0],$
$\{ToString[R[j,l]], ricci[[j,l]]\}],\{j,1,n\},\{l,1,j\}]$

$Inp: TableForm[Partition[DeleteCases[Flatten[listricci],$
$Null],2], TableSpacing \to \{2,2\}]$

$Out:$

$R[1,1]$ $\frac{f(r)}{2y(r)}\left(f'(r)y'(r)+y(r)f''(r)\right)$

$R[2,2]$ $-\frac{f'(r)y'(r)+y(r)f''(r)+2f(r)y''(r)}{2f(r)y(r)}$

$R[3,3]$ $-y(r)\left(f'(r)y'(r)+f(r)y''(r)\right)$

Ricci Scalar

$Inp: scalar = Simplify[Sum[inversemetric[[i,j]]$
$ricci[[i,j]],\{i,1,n\},\{j,1,n\}]]$

$Out:$ $-\frac{2f'(r)y'(r)+y(r)f''(r)+2f(r)y''(r)}{y(r)}$

Einstein Tensor

$Inp: einstein := einstein = Simplify[ricci-(1/2)scalar*$
$metric]$

$Inp: listeinstein = Table[If[UnsameQ[einstein[[j,l]],0],$
$\{ToString[G[j,l]], einstein[[j,l]]\}],\{j,1,n\},\{l,1,j\}]$

$Inp: TableForm[Partition[DeleteCases[Flatten[listeinstein],$
$Null],2], TableSpacing \to \{2,2\}]$

$Out:$

$G[1,1]$	$-\frac{f(r)\left(f'(r)y'(r)+2f(r)y''(r)\right)}{2y(r)}$
$G[2,2]$	$\frac{f'(r)y'(r)}{2f(r)y(r)}$
$G[3,3]$	$\frac{1}{2}y^2(r)f''(r)$

This program is written by Leonard Parker, University of Wisconsin.

5.2 Appendix B

This program was used by us to solve the differential equation using series solution method.

Inp: p[r] = $c_1 r - \frac{c_1^3 r^3}{6}$

$Out: c_1 r - \frac{c_1^3 r^3}{6}$

$Inp: fn = Sum[a[i] r^i, \{i, 0, 7\}] + Log[r] Sum[b[i] r^i, \{i, 0, 7\}]$

$Inp: f''[r] p[r] D[p[r], \{r, 1\}] + f'[r](D[p[r], \{r, 1\}])^2 +$
$f'[r] D[p[r], \{r, 2\}] p[r] + 4/3 f'[r](D[p[r], \{r, 1\}])^2 +$
$8/3 f[r] D[p[r], \{r, 1\}] D[p[r], \{r, 2\}] + f'''[r]/6 +$
$4/3(2p[r](D[p[r], \{r, 1\}])^3 f[r] + f'[r](D[p[r], \{r, 1\}])^2 +$
$f[r] D[p[r], \{r, 1\}] D[p[r], \{r, 2\}])$

$Out:$
Due to complicated result its better to check the answer in mathematica notebook itself. One can simplify the expression by using the //Simplify syntax

Inp:Solve[Coefficient[eqn, r^{-3}] == 0]

Out: {{b[0]→ 0}}

$Inp: Solution = \%[[1]]$
This will store the result in Append To

$Inp: Solve[Coefficient[eqn, r^{-2}] == 0]$

$Out: \{\{b[1] \to 0\}\}$

$Inp: AppendTo[Solution, \%[[1]][[1]]]$

$Out: \{b[0] \to 0, b[1] \to 0\}$
Same procedure is followed till r^{-1} and value of coefficient are stored in the Append To.

$Inp : eqn1 = Coefficient[eqn, Log[r]]//Simplify$
this will comment out the coefficient of log r
For result use mathematica notebook as the expression is little lengthy.
$Inp : Solve[\{eqn1/.\{r \to 0\}\} == 0]/.Solution$

$Out : \{\{b[3] \to 0\}\}$

$Inp : AppendTo[Solution, \%[[1]][[1]]]$

$Out : \{b[0] \to 0, b[1] \to 0, b[2] \to 0, b[3] \to 0\}$

$Inp : Solve[exp1 == 0]/.Solution$

$Inp : AppendTo[Solution, \%[[1]][[1]]]$

$Out : \{b[0] \to 0, b[1] \to 0, b[2] \to 0, b[3] \to 0, a[3] \to -\frac{11}{3}c1^2 a[1]\}$

Same procedure is followed with increasing power of r and every time solution is stored in Append To
$Inp : AppendTo[Solution, \%[[1]][[1]]]$

$Out : \{b[0] \to 0, b[1] \to 0, b[2] \to 0, b[3] \to 0,$
$a[3] \to -\frac{11}{3})c1^2 a[1], b[4] \to 0, 0 \to \frac{4}{13}(c1^4 a[0] -$
$7c1^2 a[2] - 3a[4]), b[5] \to 0, b[6] \to 0, b[7] \to 0,$
$0 \to 2/47(205c1^4 a[1] - 30a[5])\}$

The result is used in the chapter 4 for solving the differential equation for small r.

All b's turn out to zero. a[0], a[1] and a[2] are independent parameters and rest of the a's can be expressed as their linear combination.

$a[3] = -\frac{11}{3}c1^2 a[1]$

$a[4] = \frac{1}{3}\left(c1^4 a[0] - 7c1^2 a[2]\right)$

$a[5] = \frac{41}{6}c1^4 a[1]$

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