A VARIANT OF NEWTON'S METHOD BASED ON QUADRATURE FORMULA

M.Sc. Thesis

By

DEEPIKA GOYAL

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CANDIDATE'S DECLARATION

I hereby certify that the work presented in the thesis entitled **A variant of Newton's Method Based on Quadrature formula** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DISCIPLINE OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY INDORE**, is an authentic record of my own work carried out during the time period from July, 2016 to June, 2018 under the supervision of **DR. ANTONY VIJESH**, Associate Professor, Discipline of Mathematics, IIT Indore.

The matter presented in this thesis by me has not been submitted for the award of any other degree in this institute or any other institute.

Signature of the student with date

(Deepika Goyal)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Signature of Thesis Supervisor with date

(Dr. Antony Vijesh)

Deepika Goyal has successfully given her M.Sc. Oral Examination held on 6^{th} July, 2018.

Signature of Thesis Supervisor

Date:

Convener, DPGC

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Sign. of PSPC Member #1

Date:

Sign. of PSPC Member #2

Date:

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Deepika Goyal
1603141004
Discipline of Mathematics
Indian Institute of Technology Indore

Abstract

KEYWORDS : Bisection method, Successive approximation, Secant method, Halley's method, Newton's method, Quadrature rule.

In this dissertation, the two chapters propose an accelerated iteration method for finding the zeroes of a nonlinear function $f : \mathbb{R} \to \mathbb{R}$. While accelerated iterative method for finding the zeros of a nonlinear function have been studied already by Weerakoon Fernando and many others, the proposed technique is an efficient alternative for the existing methods. Moreover this thesis provides a semilocal convergence theorem for variation of Newton's method based on various quadrature rules.

Chapter 1 provides a short bird view on various iterative methods for finding the zeros of a nonlinear real valued function.

Chapter 2 proposes a variant of Newton's method based on Weddle's rule. A local convergence theorem is provided for the proposed iteration. Theoretically a cubic order of convergence is obtained. This section also provides a semilocal convergence theorem for a variant of Newton's method based on various quadrature rules. A comparative numerical study is also provided to support the theory.

Contents

Notations

$\mathbb{R}$	the set of all real numbers
$\mathbb{N}$	the set of all natural numbers
$\mathbb{R}^n$	n dimensional real coordinate space
$C^k[a, b]$	$\{f : [a, b] \to \mathbb{R};\ f$ has continuous k^{th} derivative$\}$

Chapter 1

Introduction

One of the oldest problems in mathematics is finding the zeroes of a given function. Though finding the zeroes of a real valued function is easy, when it is linear, the same problem become difficult when the function is nonlinear. For example, the simplest nonlinear functions are polynomials of degree greater than or equal to two. Consider the quadratic polynomial $ax^2 + bx + c$. One can easily locate the zeroes of this polynomial by applying the formula

$$zero_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad zero_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The well known result([8, p. 256]) from algebra due to Abel, polynomial of degree greater than or equal to five can not be solved by radicals, ensure that these kind of technique may not be extended for higher degree polynomials.

This problem will be further complicated if the function involves transcendental function. For example, finding the zeroes of the following functions $x - \tan x$, $x - a\sin x - b$ (arising from Kepler's equation), $k - Ae^{-E_a/k_b t}$ (arising from Arrhenius equation) play a crucial role in the theory of diffraction of light [9], in the calculation of planetary orbits [9] and the theory of chemical reactions respectively. Due to the application of zeroes of nonlinear function arising from real life problems, this urges us to develop efficient methods to find the zeroes of nonlinear functions. Recent literature supports that this is still an active area of research. The aim of this section is to provide a bird eye of certain basic methods to find the zeroes of a real valued function.

1.1 Bisection method

One of the basic technique to find the zeroes of a real valued continuous nonlinear function is the use of Bisection method. This method produces zeroes of nonlinear function as a limit of a convergent sequence. The construction of the sequence is based on the following well known intermediate value theorem.

Theorem 1.1.1. [1, p. 133]
Let I be an interval and let $f : I \to \mathbb{R}$ be a continuous on I. If $a, b \in I$, and if $f(a) < k < f(b)$, then there exists a point $c \in I$ between a and b such that $f(c) = k$.

The construction of the sequence is described as follows:
Step 1. Find an interval $[a, b]$ such that $f(a)f(b) < 0$. Set $a_0 = a$ and $b_0 = b$.
Step 2. Define $c = \frac{a_0+b_0}{2}$.

- If $f(a_0)f(c) < 0$ then set $a_1 = a_0$ and $b_1 = c$.
- If $f(b_0)f(c) < 0$ then set $a_1 = c$ and $b_1 = b_0$.

Step 3. Repeat the above steps.
The following theorem guarantees the convergence of the two sequences (a_n) and (b_n) to the zero of the real valued function.

Theorem 1.1.2. [9, p. 78-79]
If $[a_0, b_0], [a_1, b_1], \cdots, [a_n, b_n], \cdots$ denote the intervals in the bisection method, then limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, that are equal, and represent a zero of f. If $r = \lim_{n\to\infty} c_n$ and $c_n = \frac{1}{2}(a_n + b_n)$, then

$$|r - c_n| \le 2^{-(n+1)}(b_0 - a_0) \tag{1.1}$$

Proof: Denote the successive intervals that arise from the Bisection method by $[a_0, b_0], [a_1, b_1]$, and so on. From the construction of the subintervals it is easy to conclude that

$$a_0 \leq a_1 \leq a_2 \leq \cdots \leq b_0 \tag{1.2}$$

$$b_0 \geq b_1 \geq b_2 \geq \cdots \geq a_0 \tag{1.3}$$

$$b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n) \quad (n \geq 0). \tag{1.4}$$

By Monotone Convergence theorem both (a_n) and (b_n) converge. Moreover, it is easy to verify that a_n and b_n are satisfying the following relation

$$b_n - a_n = 2^{-n}(b_0 - a_0). \tag{1.5}$$

Hence, $\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{-n}(b_0 - a_0) = 0$. Consequently both the sequences (a_n) and (b_n) converge to the same limit r(say). In other words

$$r = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Note that from the construction of (a_n) and (b_n), we have $f(a_n)f(b_n) \leq 0$. Using the property of continuity one can conclude that, $[f(r)]^2 \leq 0$. Thus $f(r) = 0$. Define $c_n = (a_n + b_n)/2$. Then $r, \quad c_n \in [a_n, b_n]$. Thus,

$$|r - c_n| \leq \frac{1}{2}(b_n - a_n) \leq 2^{-(n+1)}(b_0 - a_0).$$

Remark The bisection method is also known as the method of interval halving. The order of convergence of Bisection method is 1.

Definition Let (x_n) be a sequence of real numbers that converges to x^*. If $\exists$ positive constant c and α and an integer $n_0 \in \mathbb{N}$ such that

$$|x_{n+1} - x^*| \leq c|x_n - x^*|^\alpha \quad \forall \quad n \geq n_0$$

then we say that the rate of convergence is of the order α at least.

1.2 Successive iteration method

Let $f : \mathbb{R} \to \mathbb{R}$ be a nonlinear function. Similar to the Bisection method, Successive iteration method is also a useful tool to find the zeroes of f. This method also produces zeroes of the nonlinear function f as the limit of a convergent sequence. To apply this method first rewrite the equation $f(x) = 0$ in the form of $x = g(x)$. In other words, express $f(x)$ as $x - g(x)$. Now the construction of the sequence is as follows:

Step 1. Choose x_0 as an initial guess to the equation $f(x) = 0$.

Step 2. Generate the sequence (x_n) by the recursive formula $x_{n+1} = g(x_n), \quad n = 0, 1, 2, \cdots$.

It is interesting to note that the construction of sequence in the Successive iteration method is much more simpler than the Bisection method. The following theorem ensures the convergence of the Successive iteration scheme.

Theorem 1.2.1. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$|g(x) - g(y)| \leq k|x - y| \quad \forall x, y \in \mathbb{R}. \tag{1.6}$$

If $k < 1$ then the successive iteration scheme will converge to the unique solution of $x = g(x)$. In other words, the successive iterative scheme converges to the unique zero of $f(x)$.

Proof: Let (x_n) be the sequence of iterations obtained in Step 2 in section 1.2. First, we will prove that (x_n) is a Cauchy sequence. Consider

$$\begin{aligned} |x_{n+1} - x_n| &= |g(x_n) - g(x_{n-1})| \\ &\leq k|x_n - x_{n-1}| \leq k^2|x_{n-1} - x_{n-2}| \\ |x_{n+1} - x_n| &\leq k^n|x_1 - x_0|. \end{aligned}$$

Thus for $n > m$,

$$\begin{aligned}
|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - \cdots - x_{m+1} + x_{m+1} - x_m| \\
&\leq |x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m| \\
&\leq k^{n-1}|x_1 - x_0| + \cdots + k^m|x_1 - x_0| \\
&\leq (k^{n-1} + k^{n-2} + \cdots + k^m)|x_1 - x_0| \\
&\leq k^m(1 + k + k^2 + \cdots + k^{n-1-m})|x_1 - x_0| \\
&\leq k^m(1 + k + k^2 + \cdots + k^{n-1-m} + \cdots)|x_1 - x_0| \\
|x_n - x_m| &\leq \frac{k^m}{1-k}|x_1 - x_0|.
\end{aligned}$$

Since $0 < k < 1$, we have $\lim_{m\to\infty} |x_n - x_m| = 0$. Thus, (x_n) is a Cauchy sequence. Consequently, (x_n) will converge to some $x^* \in \mathbb{R}$. Using the property of continuity one can conclude that $g(x^*) = x^*$. Let x^* and y^* be two solutions of $x = g(x)$.

$$|x^* - y^*| = |g(x^*) - g(y^*)| \leq k|x^* - y^*| < |x^* - y^*|.$$

This is a contradiction that leads to uniqueness.

Remark

1. The condition (1.6) is known as Contraction condition. In other words, g satisfies Lipschitz condition with Lipschitz constant k.

2. Sufficient condition for (1.6) to hold true is $g \in C^1(\mathbb{R})$ and $\sup_{x\in\mathbb{R}} |g'(x)| \leq k < 1$.

3. If x^* is a zero of $f(x)$, then the following error estimate hold:

$$|x_n - x^*| \leq \frac{k^n}{1-k}|x_1 - x_0|, \quad \forall \quad n \geq 1.$$

The local version of the above theorem can be stated as follows:

Theorem 1.2.2. [2, p. 62]

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all x in $[a, b]$. Suppose, in addition, that g' exists on (a,b) and a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \quad \forall\, x \in (a,b). \tag{1.7}$$

Then, for any number x_0 in $[a,b]$, the sequence (x_n) defined by

$$x_{n+1} = g(x_n), \quad n \geq 1$$

converges to the unique fixed point r in $[a,b]$.

Remark

1. From the error estimate one can conclude that the order of convergence of the successive iterative scheme is one.

2. In the above Theorem 1.2.2., the interval $[a,b]$ can be replaced by any interval which is complete.

Note

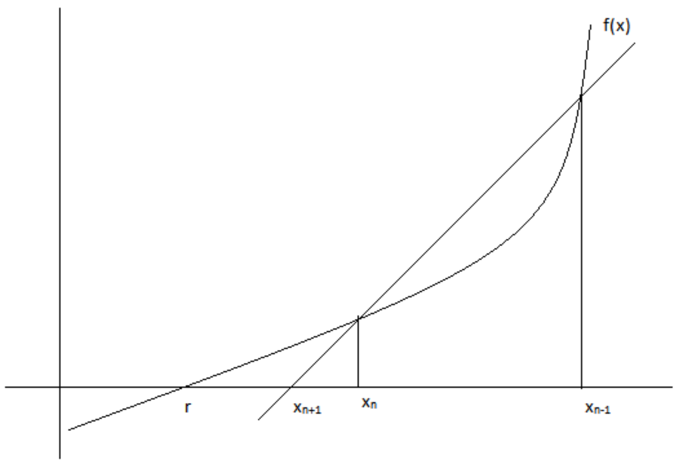
The above theorems suggest that one should be careful when rewriting the equation $f(x) = 0$ into $x = g(x)$. Consider the example of finding the zeroes of the function $f(x) = x^3 - x - 2$ discussed in [10, p. 42]. To find the zero of $f(x)$ it can be written as $x = x^3 - 2$. Here $g(x) = x^3 - 2$. Note that $g(1) = -1 < 1$, $g(2) = 6 > 2$. Thus $g : [1,2] \to \mathbb{R}$ and by intermediate value theorem g has a zero between 1 and 2 but we can not apply Theorem 1.2.1. as well as Theorem 1.2.2., as $g(x)$ fails to satisfy the hypothesis. Rewrite $f(x) = 0$ as $x = (x+2)^{1/3}$. Here $g(x) = (x+2)^{1/3}$. It is easy to see that $\sup_{x\geq 1} |g'(x)| \leq \frac{1}{3\times 9^{1/3}}$. Thus g is a contraction in $[1,\infty)$. Thus the successive iteration $x_{n+1} = g(x_n)$ where $g(x) = (x+2)^{1/3}$ converge to the zero of the polynomial $x^3 - x - 2$.

1.3 Secant method

Though the Bisection method and Successive iteration scheme are able to approximate the zero of the nonlinear function, their order of convergence is linear. To overcome this issue Secant method is used. This method also known as the method of chords. In the Secant method the sequences are generated using the recurrence relation

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \quad n = 1, 2, \cdot.$$

In contrast, to the successive iteration, secant method requires two initial guess.



Graphical reprsentation of Secant method

The following theorem ensures the convergence of the sequence produced by Secant method.

Theorem 1.3.1. [5, p. 104]
Let $f(x)$ be twice continuously differentiable on the interval $[a, b]$ and let the following conditions be satisfied:

1. $f(a)f(b) < 0$
2. $f'(x) \neq 0, \quad x \in [a, b]$
3. $f''(x)$ is either $\geq$ or $\leq 0 \quad \forall x \in [a, b]$

4. At the end points a, b

$$\frac{|f(a)|}{|f'(a)|} < b - a, \quad \frac{|f(b)|}{|f'(b)|} < b - a.$$

Then Secant method converges to the unique solution r of $f(x) = 0$ in $[a, b]$ for any choice of $x_0, x_1 \in [a, b]$.
It is interesting to note that Conditions 1 and 2 ensure that f has a unique zero in $[a, b]$. Condition 3 guarantees that the graph of $f(x)$ is either convex or concave. Moreover, condition 2 and 3 together provided that $f'(x)$ is monotone on $[a, b]$. Condition 4 states that the tangent to the curve at either endpoint intersects the x-axis within the interval $[a, b]$.

Note
The convergence of the Secant method is much better than the bisection method and the successive iteration method. If x^* is the root of the equation $f(x) = 0$ then under sufficient conditions one can show that

$$|x_{n+1} - x^*| \leq C|x_n - x^*|^\alpha, \quad \alpha = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

1.4 Newton's method

Newton's method / Newton-Raphson method can be thought of an improvement of the secant method. This method is systematically introduced three hundred years back for solving algebraic equation by Newton in 1669 and later by Raphson in 1690 [13]. In 1600, Vieta developed a procedure to solve monic polynomials of the form $p(x) = N$, N being a constant. This method is similar to the Secant method. Newton improved the procedure and used it to find the zeroes of the cubic polynomial $f(x) = x^3 - 2x - 5 = 0$. His procedure is described as follows:

1. Let $x_0 = 2$ be an initial guess.

2. Replace x by $p+2$ in $f(x)$ and obtain the new polynomial,

$$g(p) = p^3 + 6p^2 + 10p - 1. \tag{1.8}$$

3. Obtain the linear equation $10p - 1 = 0$ by neglecting the higher terms from $g(p)$. Thus, $p = 0.1$.

4. Replace p by $q + 0.1$ in $g(p)$ and obtain the new polynomial

$$h(q) = q^3 + 6.3q^2 + 11.23q + 0.061.$$

5. Obtain the linear equation $11.23q + 0.061 = 0$ from $h(q)$ by neglecting the higher terms in $h(q)$. Hence, $q = -0.0054$.

By repeating the above procedure in the third step, Newton obtain $r \approx 0.00004853$. Now the approximation to the zero of $f(x)$ was provided by adding $x_0 + p + q + r \approx 2.09455147$. Though this idea does not explicitly contain the derivative concept, one can easily see that p and q satisfy

$$\begin{aligned} p &= x_1 - x_0 = -f(x_0)/f'(x_0) \\ q &= x_2 - x_1 = -f(x_1)/f'(x_1). \end{aligned}$$

In general, Newton's method to solve $f(x) = 0$ with a solution x^* is defined as follows:

1. Let x_k be an approximation to x^*

2. Solve the linear equation

$$f(x_k) + f'(x_k)h = 0 \tag{1.9}$$

with respect to h, provided $f'(x_k)$ is non-zero.

3. Set $x_{k+1} = x_k + h$, expecting for it to be an improvement to x_k, where $k = 0, 1, 2, ...$

In 1690, Raphson independently gave the iterative scheme,

$$x_{k+1} = x_k - f(x_k)/f'(x_k)$$

for the equation $x^3 - ax - b = 0$. However it can be seen that both methods are equivalent to find the solution of $f(x) = 0$. It is worth mentioning that the usage of this technique had been in use prior to Newton's time. For example, to calculate the square root of a positive number, a Greek engineer and architect Heron(lived between 100 B.C. and 10 A.D. [9]) used the formula $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$. This procedure for finding the square root was used in Mesopotamia before 1500 B.C. [1]

1.4.1 Graphical interpretation of Newton's method

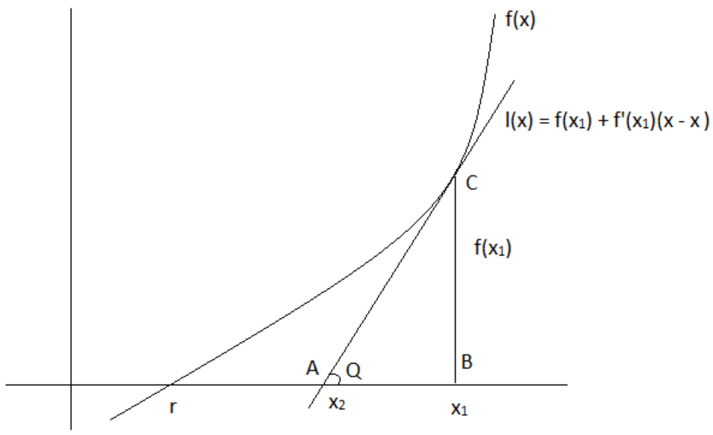


Fig 2. Graphical interpretation of Newton's method

Consider $\triangle$ ABC in Fig 2.

$$tanQ = \frac{f(x_1)}{x_1 - x_2}$$

We know, $\tan Q = f'(x_1)$. Hence $f'(x_1) = \frac{f(x)}{x_1 - x_2}$. Consequently, $x_2 = x_1 - \frac{f(x)}{f'(x_1)}$. Similarly, we draw tangent at x_2 and proceed in the similar way and we obtain the Newton's iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Similarly, one can derive the Newton's method from Taylor's theorem. Let x be an approximation to the root r. If $f''(x)$ exists and is continuous, then by Taylor's theorem

$$0 = f(r) = f(x+h) = f(x) + hf'(x) + O(h^2)$$

where $h = r - x$. If h is small neglect $O(h^2)$ and solve the remaining equation for h. Therefore we have, $h = f'(x)^{-1} f(x)$ Newton's method begins with an estimate x_0 of r and then defines inductively

$$x_{n+1} = x_n - f'(x_n)^{-1} f(x_n). \tag{1.10}$$

Basically there are two types of convergence theorem available in the literature for the above iterative scheme 1.10;

1. By assuming the existence of solution for the equation $f(x) = 0$ and proving the convergence of the iterative scheme (1.10). This type of convergence theorem is known as local convergence theorem.

2. By assuming sufficient condition on the initial guess x_0 and proving the convergence of the iterative scheme as well as the existence of the solution to $f(x) = 0$. This type of the convergence theorem is known as semi-local convergence theorem.

In 1829, Cauchy first proved a convergence theorem which does not assume any existence of a solution, i.e., semi - local convergence theorem which is stated as follows

Theorem 1.4.1. [13]

Let $X = \mathbb{R}$, $F = f \in C^2, x_0 \in X, f'(x_0) \neq 0, \sigma_0 = -\frac{f(x_0)}{f'(x_0)}, \eta = |\sigma_0|$,

$$I = \big\langle x_0, x_0 + 2\sigma_0 \big\rangle \equiv \begin{cases} [x_0, x_0 + 2\sigma_0] & if \sigma_0 \geq 0, \\ [x_0 + 2\sigma_0, x_0] & if \sigma_0 < 0 \end{cases}$$

and $|f''(x)| \leq K$ in I . Then the following results hold: If $2K\eta < |f'(x_0)|$, then $f(x) = 0$ has a unique solution x^* in I. Also if $|f'(x)| \geq m$ in I and $2K\eta < m$, then the Newton's sequence x_k starting from x_0 satisfies the following:

$$|x_{k+1} - x_k| \leq \frac{K}{2m}|x_k - x_{k-1}^2|, \quad k \geq 1$$

and

$$x^* \in \big\langle x_k, x_k + 2\sigma_k \big\rangle,$$

where, $\sigma_k = -f(x_k)/f'(x_k) = x_{k+1} - x_k$, so that

$$|x^* - x_k| \leq 2\eta\Big(\frac{K\eta}{2m}\Big)^{2^k - 1} \quad (k \geq 0).$$

Ostrowski improved Cauchy's theorem [1.4.1] by replacing $2K\eta < m$ by $2K\eta \leq |f'(x_0)|$ and showed

$$|x_{k+1} - x_k| \leq \frac{K}{2|f'(x_0)|}|x_k - x_{k-1}|^2.$$

Later, Kantorovich extended the Cauchy's Theorem 1.4.1. to operator equations in Banach spaces. The one dimensional version of Kantorovich result can be stated as follows

Theorem 1.4.2. [4, p. 127]

Let $f : \mathbb{R} \to \mathbb{R}$. Let $x_0 \in \mathbb{R}$ such that $f'(x_0) \neq 0$. Define

1. $a_0 = \frac{|f(x_0)|}{|f'(x_0)|}$

2. $b_0 = |f'(x_0)^{-1}|$
3. $S = \{x \in \mathbb{R} : |x - x_0| \leq 2a_0\}$
4. $k = 2\sup\left\{\frac{|f'(u)-f'(v)|}{|u-v|} : u, v \in S, u \neq v\right\}$

If f is differentiable in S and if $a_0b_0k \leq \frac{1}{2}$, then f has a unique zero in S. Moreover the Newton's iteration 1.10 started at x_0 converges quadratically to the zero.

The following theorem on Newton's method is a sample for local convergence theorem.

Theorem 1.4.3. [9]
Let f'' be a continuous and r be a simple zero of f. Then there is a neighbourhood of r and a constant C such that if Newton's method is started in that neighbourhood, then the iterative method (1.10) converges and satisfy

$$|x_{n+1} - r| \leq C|x_n - r|^2.$$

In contrast to the above theorems, the following theorem provides a greater flexibility to choose the initial condition when the function is concave.

Theorem 1.4.4. [9]
Let f belongs to $C^2(\mathbb{R})$. If f is strictly increasing, strictly convex and has a zero then the Newton's iteration will converge from any starting point to the unique zero of $f(x)$.

1.4.2 System of nonlinear equations

In this section we outline the derivation of Newton's method for a system of nonlinear equations based on the presentation [9]. For a system of equation in the Newton's iteration 1.10 is still valid whereas $f'(x)$ denotes the Jacobian of f at x. Here is an illustration where we have a pair of equations with two variables:

$$\begin{cases} g_1(r_1, r_2) = 0 \\ g_2(r_1, r_2) = 0. \end{cases} \tag{1.11}$$

If (r_1, r_2) is an approximate solution of (1.11), let us find the corrections h_1 and h_2 so that $(r_1 + h_1, r_2 + h_2)$ will be a better approximation solution. We will use linear terms in the Taylor's expansion in two variables , we have

$$\begin{cases} 0 = g_1(r_1 + h_1, r_2 + h_2) \approx g_1(r_1, r_2) + h_1 \frac{\partial g_1}{\partial r_2} + h_2 \frac{\partial g_1}{\partial r_2} \\ 0 = g_2(r_1 + h_1, r_2 + h_2) \approx g_2(r_1, r_2) + h_1 \frac{\partial g_2}{\partial r_2} + h_2 \frac{\partial g_2}{\partial r_2}. \end{cases} \tag{1.12}$$

In the above equations, the value of the partial derivatives needs to be evaluated at (r_1, r_2). The coefficient matrix of equation (1.12) to find h_1 and h_2 is the Jacobian matrix of g_1 and g_2,

$$J = \begin{bmatrix} \frac{\partial g_1}{\partial r_1} & \frac{\partial g_1}{\partial r_2} \\ \frac{\partial g_2}{\partial r_1} & \frac{\partial g_2}{\partial r_2} \end{bmatrix}.$$

J has to be nonsingular in order to solve equation (1.12). In that case, the solution is

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = -J^{-1} \begin{bmatrix} g_1(r_1, r_2) \\ g_2(r_1, r_2) \end{bmatrix}.$$

Hence , Newton's method for two nonlinear equations in two variables is

$$\begin{bmatrix} r_1^{(n+1)} \\ r_2^{(n+1)} \end{bmatrix} = \begin{bmatrix} r_1^{(n)} \\ r_2^{(n)} \end{bmatrix} + \begin{bmatrix} h_1^{(n)} \\ h_2^{(n)} \end{bmatrix}.$$

The following theorem ensures the semilocal convergence theorem for system of non linear equations

Theorem 1.4.5. [6]

Let $f_i(x_1, x_2, ..., x_n)$, $(i = 1, 2, ..., n)$, be a system of real functions of the real variables $x_1, x_2, ..., x_n$ which have continuous first and second derivatives in the region R, $(x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)})$ be a set of values of $x_1, x_2, ..., x_n$ belonging to this region, and $\xi_1, \xi_2, ..., \xi_n$ the set of numbers determined by the equations.

$$f_i(x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)}) + \sum_{k=1}^{n} \frac{\partial f_i}{\partial x_k^{(0)}} \xi_k = 0 \quad (i = 1, 2, ..., n)$$

and let S denote the interval, circle, sphere, or hypersphere whose centre is $(x_1^{(0)} + \xi_1, ..., x_n^{(0)} + \xi_n)$ and whose radius is $\rho = (\sum_{k=1}^{n} \xi_k^2)^{1/2}$, S_ρ being supposed belong to R.

Suppose also that in S the functional determinant F of the functions f_i does not vanish, $\mu(< \infty)$ is the upper bound of the absolute values of the fractions whose denominators are F and whose numerators are the several first minors of F, and $v(< \infty)$ is the upper bound of the absolute values of the second derivatives of the functions f_i.

Then, if

$$[\sum_{i=1}^{n} (f_i(x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)}))^2]^{1/2} < \frac{1}{n^{7/2} \mu^2 \nu} \tag{1.13}$$

the equations $f_i = 0$ have one and but one solution in S, and an approximate value of this solution, as close as may be desired, will be obtained by determinations first of $\xi_1^{(j)}, \xi_2^{(j)}, ..., \xi_n^{(j)}$ and then of $x_1^{(j+1)}, x_2^{(j+1)}, ..., x_n^{(j+1)}$, for $j = 0, 1, 2, ...$ successively, by the equations

$$f_i(x_1^{(j)}, x_2^{(j)}, ..., x_n^{(j)}) + \sum_{k=1}^{n} \frac{\partial f_i}{\partial x_k^{(j)}} \xi_k^{(j)} = 0 \quad (i = 1, 2, ..., n) \tag{1.14}$$

$$x_k^{(j+1)} = x_k^{(j)} + \xi_k^{(j)}, \quad (k = 1, 2, ..., n) \tag{1.15}$$

(where $\xi_k^{(0)} = \xi_k$), the solution being $\lim_{j=\infty}(\xi_1^{(j)}, \xi_2^{(j)}, ..., \xi_n^{(j)})$.

Moreover the equations (1.14) and (1.15) will yield a similar sequence of approximations to this solution if instead of $(x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)})$ any other point in S be chosen as the point of departure.

Remark

Though the order of convergence of Newton's method is bigger than the secant's method, Newton's method requires evaluation of two functions f and f' at each step whereas secant method requires the evaluation of f alone. Consequently secant method may produce the solution faster than the Newton's method occasionally.

Definition

Let r be zero of a function f. r is said to be zero of $f(x)$ of order m if $f(x)$ can be expressed as $f(x) = (x-r)^m h(x), \quad h(r) \neq 0$.

Till now, all the theorems related to the convergence of Newton's method presented above have been based on the assumption that the zero r of f is of the order 1. If r is zero of the order $m > 1$, then the Newton's method fails to possess the quadratic convergence properly.

One can express the Newton's method as a successive approximation method in the following way

$$x_{n+1} = g(x_n), \quad g(x) = x - \frac{f(x)}{f'(x)}. \tag{1.16}$$

Note that if r is zero of order m then $f(x) = (x-r)^m h(x)$. In this case g can be expressed as $g(x) = x - \frac{(x-r)h(x)}{mh(x)+(x-r)h'(x)}$. Consequently,

$$g'(r) = 1 - \frac{1}{m} = \frac{m-1}{m}$$

For $m > 1$, this is nonzero. One can easily conclude that

$$r - x_{n+1} \approx \lambda(r - x_n), \quad \lambda = \frac{m-1}{m}.$$

Thus, Newton's method converges linearly.

If one knows the order of zero in advance a small modification in classical Newton's method will ensure the quadratic convergence. For example, if r is zero of order m then generate the sequence as follows:

$$x_{n+1} = g(x_n), \quad g(x) = x - m\frac{f(x)}{f'(x)}. \tag{1.17}$$

It is interesting to note that $g'(r) = 0$.

1.4.3 Halley's method

Considerable efforts have been made to improve the order of convergence of the Newton's iteration. To accelerate the Newton's method, Halley introduced a modification in Newton's method and obtained a cubic order of convergence. This method was developed based on the assumption that the nonlinear function f is twice differentiable. The following recurrence formula is used to generate the sequence in Halley's method.

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}, \quad n = 1, 2, \cdots . \tag{1.18}$$

with x_0 being an initial guess.
By assuming the function to be three times continuously differentiable, bounded and with suitable choice of the initial guess, the Halley's method satisy the following inequality

$$|x_{n+1} - r| \leq K|x_n - r|^3, \quad \text{for some} \quad K > 0.$$

The following alternative formulation shows the similarities between Halley's method and Newton's method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 - \frac{f(x_n)}{f'(x_n)}\frac{f''(x_n)}{2f'(x_n)}\right]^{-1}.$$

It is interesting to note that when the second derivative is very close to zero, the Halley's method iteration is almost the same as the Newton's method iteration.

1.4.4 Variants of Newton's method

In 2000, Weerakoon and Fernando proposed modification in Newton's method and obtain a cubic order iterative scheme to find the zeros of a real valued nonlinear function. More specifically, the following iterative scheme is proposed

$$x_{n+1} = x_n - \frac{2f(x_n)}{[f'(x_n) + f'(x^*_{n+1})]}, \quad n = 0, 1, 2, \cdots \quad \text{where} \quad x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1.19}$$

The approximation of the integral term by trapezoidal rule in the relation

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(\lambda) d\lambda \tag{1.20}$$

is the key step in deriving the above iterative scheme. For convergence analysis, the following theorem has been provided:

Theorem 1.4.6. [12]

Let $f : D \to \mathbb{R}$ and D be an open interval in $\mathbb{R}$. Assume that f has first, second, and third derivatives in the interval D. If $f(x)$ has a simple root $\alpha \in D$ and x_0 is sufficiently close to α, then the new method defined by (1.19) satisfies the following error estimates:

$$e_{n+1} = (C_2^2 + \frac{1}{2}C_3)e_n^3 + O(e_n^4),$$

where,

$$e_n = x_n - \alpha, \ \textit{and}\ C_j = \frac{1}{j!}\frac{f^j(\alpha)}{f^1(\alpha)}, \ j = \mathit{1,2,3}, \cdots .$$

Similar attempts have been made by Hansanov, Ivanov and Nedjibov [7] in 2002, where they approximated the integral by simpson's 1/3rd rule. Hasanov, Ivanov, Nedjibov proposed the following variant of Newton's method

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x^*_{n+1}) + 4f'(x^{**}_{n+1}) + f'(x_n)}, \tag{1.21}$$

where

$$x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad x^{**}_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n)}. \tag{1.22}$$

For the convergence analysis, they gave the following theorem

Theorem 1.4.7. [7]

Let $f : D \to \mathbb{R}$ for an open interval D. Assume that $f \in C^3(D)$. Assume that α is the simple root of $f(x) = 0$ and x_0 is sufficiently close to α, then the method defined by (1.21) and (1.22) has cubic convergence.

Recently Chen, Kincaid and Lin [3] in 2018 approximated the integral by simpson's 3/8th rule. More specifically they proposed the following iterative scheme

$$x_{n+1} = x_n - \frac{8f(x_n)}{f'(x_n) + 3f'(x_n - \frac{1}{3}h(x_n)) + 3f'(x_n - \frac{2}{3}h(x_n)) + f'(x_n - h(x_n))}. \tag{1.23}$$

The following local convergence theorem ensures the cubic order of the iterative scheme 1.23

Theorem 1.4.8. [3]
Let $f : D \to \mathbb{R}$ for an open interval D. Assume that $f \in C^3(D)$ and α is the simple root of $f(x) = 0$ and x_0 is sufficiently close to α, then the method defined by (1.23) has cubic convergence.

Chapter 2

A variant of Newton's method based on Weddle's rule

In this chapter we derive a variant of Newton's method to find the zeros of a nonlinear function $f : \mathbb{R} \to \mathbb{R}$ using Weddle's rule. Theoretically, cubic order of convergence is obtained for the proposed iteration. A unified semilocal convergence theorem for variant of Newton's method based on quadrature rule is obtained in this chapter. An interesting comparative study is done numerically.

2.1 A variant of Newton's method based on Weddle's rule

Weddle's rule is one of the Quadrature formula which is used to approximate the definite integral. One can get Weddle's rule from the Newton-Cotes formula for the choice $n = 6$. More specifically, the formula for Weddle's rule is given by

$$I = \int_a^b f(x)dx \cong \frac{3h}{10}[f_0 + 5f_1 + f_2 + 6f_3 + f_4 + 5f_5 + f_6],$$

where $h = \frac{b-a}{6}$ and $f_j = f(x_j)$, $x_0 = a$, $x_6 = b$, $x_{i+1} = x_i + h$.

To obtain the variant of Newton's method based on Weddle's rule first approximate the integral $\int_{x_m}^{x} f'(\lambda)d\lambda$ by Weddle's rule. Hence

$$\begin{aligned}\int_{x_m}^{x} f'(\lambda)d\lambda \;&\approx\; \frac{(x-x_m)}{20}[f'(x_m) + 5f'(x_m+h) + f'(x_m+2h) + 6f'(x_m+3h)\\ &\quad +f'(x_m+4h) + 5f'(x_m+5h) + f'(x)],\end{aligned}$$

where $h = \frac{x-x_m}{6}$. Note that if α is a root of $f(x)$, then $f(\alpha)$ can be written as

$$f(\alpha) \;=\; f(x_n) + \int_{x_n}^{\alpha} f'(\lambda)d\lambda \approx f(x_n) + \frac{(\alpha - x_n)}{20}[P(x_n)],$$

where

$$\begin{aligned}P(x_n) \;&=\; f'(x_n) + \;5f'(x_n + \frac{\alpha - x_n}{6}) + f'(x_n + 2.\frac{\alpha - x_n}{6})\\ &\quad + 6f'(x_n + 3.\frac{\alpha - x_n}{6}) + f'(x_n + 4.\frac{\alpha - x_n}{6})\\ &\quad + 5f'(x_n + 5.\frac{\alpha - x_n}{6}) + f'(\alpha).\end{aligned}$$

Setting $\alpha = x_{n+1}$ in $P(x_n)$, we have

$$x_{n+1} - x_n = \frac{-20f(x_n)}{P(x_n)} \Longrightarrow x_{n+1} = x_n - \frac{20f(x_n)}{P(x_n)}.$$

Since x_{n+1} is implicit in both sides, it is difficult to solve x_{n+1}. Instead, replace x_{n+1} by ${x^*}_{n+1}$ in the right hand side obtained from the Newton's method

$$x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Consequently one can get the following iterative scheme

$$x_{n+1} = x_n - \frac{20f(x_n)}{T(x_n)}, \tag{2.1}$$

where, $T(x) = f'(x) + 5f'(x - \frac{1}{6}h(x)) + f'(x - \frac{2}{6}h(x)) + 6f'(x - \frac{3}{6}h(x)) + f'(x - \frac{4}{6}h(x)) + 5f'(x - \frac{5}{6}h(x)) + f'(x - h(x))$ and $h(x) = -\frac{f(x)}{f'(x)}$.

2.2 Convergence analysis

In this section a semi-local convergence theorem as well as local convergence theorem for the above iterative scheme 2.1 is proved. From recent literature it is evident that only local convergence version theorem only available for variants of Newton's method discussed in [3], [7]. In [11], the convergence analysis has been done for the following iterative scheme $x_{n+1} = x_n - (\lambda f'(x_n) + (1-\lambda)f'(z_n))^{-1}f(x_n)$. for solving the abstract operator equation $f(x) = 0$. From a simple observation, one can get the semi-local convergence theorem for variants of Newton's method [12] from the work of [11]. In this section a more general semi-local convergence theorem is proved for variants of Newton's method obtained from quadrarture rules. More specifically a semi-local convergence theorem for the following iterative scheme is proved

$$x_{n+1} = x_n - (\sum_{i=1}^{m} \lambda_i f'(x_n^{(i)}))^{-1} f(x_n) \quad n = 0, 1, 2, \cdots . \tag{2.2}$$

where $x_n^{(i)} = x_n - \lambda_i \frac{f(x_n)}{f'(x_n)}$ and $\sum_{i=1}^{m} \lambda_i = 1$. One of the main theorem of this section given below

Theorem 2.2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Let $\lambda_i \in [0, 1]$, $1 \leq i \leq m$ such that $\sum_{i=1}^{m} \lambda_i = 1$. Assume further that

1. $f'(x_0) \neq 0$;
2. for some $\eta > 0$,$|f'(x_0)^{-1}f(x_0)| \leq \eta$;
3. $|(f'(x_0)^{-1}(f'(x_0) - f'(x))| < \epsilon$ whenever $x \in [x_0 - 2r, x_0 + 2r]$. Set $c_0 = \frac{(\lambda_1 + \sum_{i=2}^{m} 2\lambda_i)\epsilon}{1-\epsilon}$, $c = \frac{2\epsilon}{1-\epsilon}$ such that $(1 + \frac{c_0}{1-c})\eta < r$ and $0 < 3\epsilon < 1$;
4. for some $x_0^{(i)} \in [x_0 - r, x_0 + r]$,$|\sum_{i=1}^{m} (\lambda_i f'(x_0^{(i)}))^{-1} f(x_0)| \leq \eta$.

Then, the sequence of iterates (x_n) generated by

$$x_{n+1} = x_n - (\sum_{i=1}^{m} \lambda_i f'(x_n^{(i)}))^{-1} f(x_n), \quad x_n^{(i)} = x_n - \lambda_i \frac{f(x_n)}{f'(x_n)}$$

is well-defined, remains in $[x_0-r, x_0+r] \quad \forall n \geq 0$ and converges to a unique solution $x^* \in [x_0 - r, x_0 + r]$ of the equation $f(x) = 0$. Moreover for $n \geq 2$, the following error-estimates hold

$$|x_{n+1} - x_n| \leq c^{n-1} c_0 \eta,$$
$$|x_n - x^*| \leq \frac{c^{n-1} c_0 \eta}{1-c}.$$

Proof: Let $L_n = \sum_{i=1}^{m} \lambda_i f'(x_n^{(i)})$.Here $f'(x_0^{(1)}) = f'(x_0)$. For $n = 0, \quad |x_1 - x_0| = |(L_0)^{-1} f(x_0)| \leq \eta < r$. Therefore $x_1 \in [x_0 - r, x_0 + r]$ and

$$\begin{aligned}
|1 - (f'(x_0))^{-1} L_1| &= |(f'(x_0))^{-1}(f'(x_0) - L_1)| \\
&= |(f'(x_0))^{-1}(\sum_{i=1}^{n} \lambda_i f'(x_0) - \sum_{i=1}^{m} \lambda_i f'(x_1^{(i)}))| \\
&\leq \sum_{i=1}^{m} \lambda_i |(f'(x_0))^{-1}(f'(x_0) - f'(x_1^{(i)}))| \\
|1 - (f'(x_0))^{-1} L_1| &\leq \sum_{i=1}^{m} \lambda_i \epsilon = \epsilon < 1. \quad [by(3)]
\end{aligned}$$

So $(L_1)^{-1}$ exists and $|(f'(x_0)^{-1} L_1)^{-1}| \leq \frac{1}{1-\epsilon}$. Now

$$\begin{aligned}
x_2 - x_1 &= -(L_1)^{-1} f(x_1) \\
&= -(L_1)^{-1}(f(x_1) + f(x_0) - f(x_0)) \\
&= -(L_1)^{-1}(f'(c_1)(x_1 - x_0) + f(x_0)), \quad where \quad c_1 \in (x_0, x_1) \\
&= -(f'(x_0)^{-1} L_1)^{-1}(f'(x_0)^{-1} f'(c_1)(x_1 - x_0) - L_0 f'(x_0)^{-1}(x_1 - x_0)) \\
&= -(f'(x_0)^{-1} L_1)^{-1}(f'(x_0)^{-1}(f'(c_1) - L_0)(x_1 - x_0)) \\
&= -(f'(x_0)^{-1} L_1)^{-1}(f'(x_0)^{-1}(f'(c_1) - \sum_{i=1}^{m} \lambda_i f'(x_0^{(i)}))(x_1 - x_0)) \\
x_2 - x_1 &= -(f'(x_0)^{-1} L_1)^{-1}(\sum_{i=1}^{m} \lambda_i f'(x_0)^{-1}(f'(c_1) - f'(x_0^{(i)}))(x_1 - x_0)).
\end{aligned}$$

So

$$|x_2 - x_1| \leq \frac{1}{1-\epsilon}[\lambda_1 + 2(\sum_{i=2}^{m} \lambda_i)]|x_1 - x_0| \leq c_0\eta \quad [by(3)].$$

Consequently $|x_2 - x_0| \leq |x_2 - x_1| + |x_1 - x_0| \leq c_0\eta + \eta < r$. Thus, $x_2 \in [x_0 - r, x_0 + r]$. Assume that $x_k \in [x_0 - r, x_0 + r]$ and $|x_{k+1} - x_k| \leq c^{k-1}c_0\eta$ for $k = 2, 3, ..., n-1$. In view of hypothesis (3.), $(L_k)^{-1}$ exists and $|(f'(x_0))^{-1}L_k)^{-1}| \leq \frac{1}{1-\epsilon}$ for $k = 2, 3, ..., n-1$. From the definition of x_{n+1},

$$\begin{aligned}
x_{n+1} - x_n &= -L_n^{-1}f(x_n) \\
&= -(L_n)^{-1}(f(x_n) + f(x_{n-1}) - f(x_{n-1})) \\
&= -(L_n)^{-1}(f'(c_n)(x_n - x_{n-1}) + f(x_{n-1})), \quad \text{where} \quad c_n \in (x_{n-1}, x_n) \\
&= -(f'(x_0)^{-1}L_n)^{-1}(f'(x_0)^{-1}f'(c_n)(x_n - x_{n-1}) - L_{n-1}f'(x_0)^{-1}(x_n - x_{n-1})) \\
&= -(f'(x_0)^{-1}L_n)^{-1}(f'(x_0)^{-1}(f'(c_n) - L_{n-1})(x_n - x_{n-1})) \\
&= -(f'(x_0)^{-1}L_1)^{-1}(f'(x_0)^{-1}(f'(c_n) - \sum_{i=1}^{m} \lambda_i f'(x_{n-1}^{(i)}))(x_n - x_{n-1})) \\
x_{n+1} - x_n &= -(f'(x_0)^{-1}L_1)^{-1}(\sum_{i=1}^{m} \lambda_i f'(x_0)^{-1}(f'(c_n) - f'(x_{n-1}^{(i)}))(x_n - x_{n-1})) \\
|x_{n+1} - x_n| &\leq \frac{2\epsilon}{1-\epsilon}|x_{n+1} - x_n| \leq c^{n-1}c_0\eta, \quad \text{by} \quad \text{induction} \quad \text{hypothesis.}
\end{aligned}$$

Thus, for all $n \geq 2, |x_{n+1} - x_n| \leq c^{n-1}c_0\eta$. Since

$$\begin{aligned}
|x_{n+1} - x_0| &\leq |x_{n+1} - x_n| + ... + |x_1 - x_0| \\
&\leq c^{n-1}c_0\eta + c^{n-2}c_0\eta + ... + cc_0\eta + c_0\eta + \eta \\
|x_{n+1} - x_0| &\leq \eta(1 + \frac{c_0}{1-c}) < r,
\end{aligned}$$

$x_{n+1} \in [x_0 - r, x_0 + r]$. Let $k \geq 2$ and $m \in \mathbb{N}$. Then

$$\begin{aligned}
|x_{k+m} - x_k| &\leq |x_{k+m} - x_{k+m-1}| + ... + |x_{k+1} - x_k| \\
&\leq c^{k+m-2}c_0\eta + c^{k+m-3}c_0\eta + ... + c^{k-1}c_0\eta \\
|x_{k+m} - x_k| &\leq \frac{1-c^m}{1-c}c^{k-1}c_0\eta \leq \frac{c^{k-1}c_0\eta}{1-c}.
\end{aligned}$$

Since $0 < c < 1$, it follows that (x_n) is a Cauchy sequence in $[x_0 - r, x_0 + r]$ and hence converges to an element x^* in $[x_0 - r, x_0 + r]$. From hypothesis (3.) using triangle inequality $|L_n| \leq M$, where $M = (\frac{\epsilon}{|f'(x_0)^{-1}| + |f'(x_0)|})$. Since $x_{n+1} = x_n - (L_n)^{-1} f(x_n)$, $f(x_n) = -L_n(x_{n+1} - x_n)$. So,

$$|f(x_n)| \leq |L_n||x_{n+1} - x_n| \leq M|x_{n+1} - x_n|. \tag{2.3}$$

Proceeding to the limit in (2.1) as n tends to infinity and using the continuity of f, it follows from the convergence of (x_n) to x^*, that $f(x^*) = 0$. Suppose x^* and y^* are two solutions of $f(x) = 0$, in $[x_0 - r, x_0 + r]$. Then $0 = f(x^*) - f(y^*) = f'(c^*)(x^* - y^*)$, $c^* \in (x^*, y^*)$. Thus, $c^* \in [x_0 - r, x_0 + r]$. Consequently, $c^* \in [x_0 - 2r, x_0 + 2r]$. Then,

$$\begin{aligned} |1 - f'(c^*)(f'(x_0))^{-1}| &= |(f'(x_0))^{-1} f'(x_0) - f'(c^*)(f'(x_0))^{-1}| \\ &= |(f'(x_0))^{-1}(f'(x_0) - f'(c^*))| \\ &\leq \epsilon < 1. \end{aligned}$$

So, $f'(c^*)(f'(x_0))^{-1} \neq 0$ and hence $f'(c^*) \neq 0$. Thus, $x^* = y^*$.

Remark

1. For the choices of λ_i will be $\lambda_1 = \lambda_2 = 1/2$, $i = 1, 2$ in the above theorem one can get the semi-local convergence theorem for the iterative scheme proposed by Weerakoon and Fernando [12].

2. For the choices of λ_i will be $\lambda_1 = 1/6, \quad \lambda_2 = 4/6, \quad \lambda_3 = 1/6$, $i = 1, 2, 3$ in the above theorem one can get the semi-local convergence theorem for the iterative scheme proposed by Hasanov, Ivanov, Nedjibov [7].

3. For the choices of λ_i will be $\lambda_1 = 1/8, \quad \lambda_2 = 3/8, \quad \lambda_3 = 3/8, \quad \lambda_4 = 1/8$, $i = 1, 2, 3, 4$ in the above theorem one can get the semi-local convergence theorem for the iterative scheme proposed by Chen, Kincaid, Lin [3].

4. For the choices of λ_i will be $\lambda_1 = 1/20, \quad \lambda_2 = 5/20, \quad \lambda_3 = 1/20, \quad \lambda_4 = 6/20, \quad \lambda_5 = 1/20, \quad \lambda_6 = 5/20, \quad \lambda_7 = 1/20, i = 1,2,3,4,5,6,7$ in the above theorem one can get the semi-local convergence theorem for the proposed variant of Newton's iterative method based on Weddle's rule.

The following theorem guarantees the cubic order convergence of the proposed variant of Newton's method 2.1.

Theorem 2.2.2. Let $f : D \to \mathbb{R}$ for an open interval D. Assume that $f \in C^3(D)$ and α is the simple root of $f(x) = 0$ and x_0 is sufficiently close to α, then the method is defined by (2.1) and has cubic convergence.

Proof: Write the iterative scheme 2.1 as $x_{n+1} = \phi(x_n)$ where $\phi(x) = x - \frac{20f(x)}{K(x)}$, $K(x) = f'(x) + 5f'(x - \frac{1}{6}h(x)) + f'(x - \frac{2}{6}h(x)) + 6f'(x - \frac{3}{6}h(x)) + f'(x - \frac{4}{6}h(x)) + 5f'(x - \frac{5}{6}h(x)) + f'(x - h(x))$ and $h(x) = \frac{f(x)}{f'(x)}$. Note that

$$\begin{aligned}
\phi'(x) &= 1 - \frac{20f'(x)K(x)}{K^2(x)} + \frac{20f(x)K'(x)}{K^2(x)} = 1 - \frac{20f'(x)}{K(x)} + \frac{20f(x)K'(x)}{K^2(x)}. \\
\phi''(x) &= -\frac{20f''}{K} + \frac{20f'K'}{K^2} + \frac{20f'K'}{K^2} + \frac{20fK''}{K^2} - \frac{40f(K')^2}{K^3}.
\end{aligned}$$

Here,
$K'(x) = f''(x) + 5f''(x - \frac{1}{6}h(x))(1 - \frac{1}{6}h'(x)) + f''(x - \frac{2}{6}h(x))(1 - \frac{2}{6}h'(x)) + 6f''(x - \frac{3}{6}h(x))(1 - \frac{3}{6}h'(x)) + f''(x - \frac{4}{6}h(x))(1 - \frac{4}{6}h'(x)) + 5f''(x - \frac{5}{6}h(x))(1 - \frac{5}{6}h'(x)) + f''(x - h(x))(1 - h'(x))$ and $h'(x) = \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = 1 - \frac{ff''}{(f')^2}$.

Consequently
$K''(x) = f'''(x) + 5f'''(x - \frac{1}{6}h(x))(1 - \frac{1}{6}h'(x))^2 + 5f''(x - \frac{1}{6}h(x))(-\frac{1}{6}h''(x)) + f'''(x - \frac{2}{6}h(x))(1 - \frac{2}{6}h'(x))^2 + f''(x - \frac{1}{6}h(x))(-\frac{2}{6}h''(x)) + 6f'''(x - \frac{3}{6}h(x))(1 - \frac{3}{6}h'(x))^2 + 6f''(x - \frac{3}{6}h(x))(-\frac{3}{6}h''(x)) + f'''(x - \frac{4}{6}h(x))(1 - \frac{4}{6}h'(x))^2 + f''(x - \frac{4}{6}h(x))(-\frac{4}{6}h''(x)) + 5f'''(x - \frac{5}{6}h(x))(1 - \frac{5}{6}h'(x))^2 + 5f''(x - \frac{5}{6}h(x))(-\frac{5}{6}h''(x)) + f'''(x - h(x))(1 - h'(x))^2 + f''(x - h(x))(-h''(x))$ and $h''(x) = -\frac{f'f''}{(f')^2} - \frac{ff'''}{(f')^2} + \frac{2f(f'')^2}{(f')^3}$.
Since α is the simple root of $f(x)$,then $f(\alpha) = 0$. Note that $h'(\alpha) = 1$ and $K'(\alpha) =$

$10f''(\alpha)$, $h''(\alpha) = -\frac{f''(\alpha)}{f'(\alpha)}$ and $K''(\alpha) = \frac{20}{3}f'''(\alpha) - 10\frac{f''(\alpha)}{f'(\alpha)}$. Therefore

$$\begin{aligned}
\phi'(\alpha) &= (x - \frac{20f(x)}{K(x)})'|_{x=\alpha} \\
&= 1 - \frac{20f'(\alpha)}{K'(\alpha)} \\
\phi'(\alpha) &= 0.
\end{aligned}$$

Moreover

$$\begin{aligned}
\phi''(\alpha) &= (x - \frac{20f(x)}{K(x)})''|_{x=\alpha} \\
&= -\frac{20f''(\alpha)}{K(\alpha)} + \frac{400f'(\alpha).f''(\alpha)}{(20f'(\alpha))^2} \\
\phi''(\alpha) &= 0.
\end{aligned}$$

Define $e_n = x_n - \alpha$. Consequently

$$\begin{aligned}
x_{n+1} &= \phi(x_n) = \phi(\alpha + e_n) = \phi(\alpha) + e_n\phi'(\alpha) + \frac{e_n^2}{2!}\phi''(\alpha) + \frac{e_n^3}{3!}\phi'''(\alpha) \\
x_{n+1} &= \alpha + \frac{e_n^3}{3!}\phi'''(\alpha).
\end{aligned}$$

Thus $e_{n+1} = \frac{e_n^3}{3!}\phi'''(\alpha)$. Hence the theorem.
The proposed iterative method compared with various variants of Newton's method available in the literature in the following Table 2.1.

TABLE 2.1: Errors for different methods.

$f(x)$	x_0	NT	SR [7]	SL [3]	TP [12]	W
(1) $x^3 + x^2 - 2$	3	7	5	5	6	5
(2) $\cos x - x$	4	30	6	8	13	4
(3) $(x-1)^3 - 1$	-2	11	6	6	8	8
(4) $e^{x^2+7x-30} - 1$	3.5	12	8	8	9	8

NT - Newton's method

W - Proposed scheme

2.3 Sample MATLAB code for the proposed scheme

syms x

$FF = (x-1)^3 - 1$

f=inline(FF);

g= inline($diff(FF)$);

$noi = 0$;

$xx = -1$;

$yy = 0$;

$test1 = 1000$;

$test2 = f(xx)$

$eps = \sqrt{2.22 * 10^{-16}}$;

while $(test1 > eps || test2 > eps)$

AA=f(xx)/g(xx);

$denom = (g(xx) + 3 * g(xx - (1/3 * AA)) + 3 * g(xx - (2/3 * AA)) + g(xx - AA))$;

$nume = (8 * f(xx))$;

$yy = xx - (nume/denom)$;

$test1 = abs(yy - xx)$;

$test2 = abs(f(yy))$

$xx = yy$

$noi = noi + 1$

end

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