ON THE ORTHOGONALITY OF FRAME PAIRS OVER LOCALLY COMPACT ABELIAN GROUPS

Ph.D. Thesis

By

ANUPAM GUMBER

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **ON THE ORTHOGONALITY OF FRAME PAIRS OVER LOCALLY COMPACT ABELIAN GROUPS** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DISCIPLINE OF MATHEMATICS, Indian Institute of Technology Indore,** is an authentic record of my own work carried out during the time period from January 2013 to December 2017 under the supervision of Dr. Niraj Kumar Shukla, Assistant Professor, Indian Institute of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Signature of the student with date

(ANUPAM GUMBER)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Signature of Thesis Supervisor with date

(Dr. NIRAJ KUMAR SHUKLA)

ANUPAM GUMBER has successfully given her Ph.D. Oral Examination held on **July 17, 2018**.

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Sign of PSPC Member #1 Date:	Sign of PSPC Member #2 Date:	Sign of Convener, DPGC Date:
Sign of Head of Discipline Date:		

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Dedicated to ...

the memory of My loving Father, Sh. Ashok Kumar

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and support made me what I am today ...

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prayers of day and night made me able
to get such success and honor ...

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ABSTRACT

KEYWORDS: Frame; orthogonal frames; locally compact abelian (LCA) group; unitary representation; translation invariant space; co-compact Gabor system; generalized translation invariant system; wave-packet system; wavelet system; dual Gramian; frame operator; dual framelet; group action; time-frequency localization; uncertainty principle.

The theory of frames has a very close connection with unitary group representations, since various structured systems, for example, wavelet and Gabor systems can be realized as a sequence obtained by applying a family of unitary operators to a particular window function. Among various frame properties, the "orthogonality or strongly disjointness" of a pair of frames for Hilbert spaces is a very useful concept introduced and studied by Balan, Han, and Larson. A pair of frames satisfying the above mentioned property are termed as pairwise orthogonal (simply, orthogonal). Orthogonal frames play a key role in frame theory, for example, in construction of new frames from existing ones, constructions related with duality, in multiple access communications, hiding data, and synthesizing super-frames, etc.

In this thesis we study the orthogonality of a pair of frames for function systems generated by regular representations of locally compact abelian (LCA) groups. To the best of our knowledge, we realize that the characterization results for pairwise orthogonal frames have not been studied earlier in the context of LCA groups, and are appearing first time in the literature via the work presented in this thesis.

Precisely, we examine pairwise orthogonality of frames generated by the action of a unitary representation ρ of a countable family of closed and co-compact subgroups $\{\Gamma_j\}_{j\in J} \subset G$ on a separable Hilbert space $L^2(G)$, where $J \subset \mathbb{Z}$ and G is a second countable LCA group. We consider frames of the form

$$E(\Psi) := \Big\{\rho(\gamma)\psi_{p,j} : \psi_{p,j} \in \Psi,\ \gamma \in \Gamma_j,\ p \in P_j,\ j \in J\Big\}$$

for a (not necessarily countable) family $\Psi := \{\psi_{p,j}\}_{p\in P_j, j\in J}$ in $L^2(G)$. We pay special attention to this problem by assuming ρ as the action of a countable family $\{\Gamma_j\}_{j\in J}$ on $L^2(G)$ by (left-)translation. The representation of Γ acting on $L^2(G)$ by (left-)translation is called the (left-)regular representation of Γ.

We obtain characterizations of orthogonal frame pairs of the form $E(\Psi)$ by considering different situations on a countable family of closed and co-compact subgroups $\{\Gamma_j\}_{j\in J}$ in G. Along with this, the resulting characterizations are used to construct new frames and super-frames by using various techniques including the unitary extension principle by Ron and Shen and its recent extension to LCA groups by Christensen and Goh.

Additionally, we investigate the orthogonality of structured function systems (e.g., wavelet, Gabor, and wave-packet systems) over LCA-group setting. Further, we relate the above-developed theory to the classical case as well as finite dimensional Hilbert spaces by letting LCA group G of the form $\mathbb{R}^d, \mathbb{Z}^d$, and $\mathbb{Z}_N^d$, etc.

Lastly, we provide a group-theoretic construction of a finite frame wavelet (simply, framelet) system associated with an induced group action. Along with this, we investigate the time-frequency localization properties of the framelet system. We also study the orthogonality and duality properties of a pair of framelet systems, and its applications in synthesizing super framelet systems over finite dimensional set-up.

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CHAPTER 1

INTRODUCTION

The thesis has been divided into seven chapters. The purpose of this chapter is to give some motivations and background knowledge about the research problems discussed in this thesis. Also, this chapter includes preliminaries for the upcoming chapters. We begin with a brief introduction to frame theory in Hilbert spaces, which is followed by motivation and objective of the work done in this thesis.

A *frame* is defined to be a set of vectors in a Hilbert space that provides robust, basis-like representations. Frames have found their significant role in signal and image processing, data compression, and sampling theory with ever-increasing applications to problems in both pure and applied mathematics, physics, engineering, and computer science, to mention a few (e.g. see [3, 4, 29, 30, 33, 34, 75, 87, 88] and references within).

In this scenario, a lot of mathematicians and researchers have contributed in analysing various interesting properties and results of frame theory in different set-ups (see [7, 8, 10, 12, 15, 21, 23, 34, 40–43, 48, 57, 58, 74, 76, 77, 85] and references therein).

In the last two decades, the study of frames in the context of locally compact abelian (LCA) groups has become the focus of an active research, both in theory as well as in applications. This is due to the following advantages of the LCA-group approach:

- It unifies the continuous theory (integral representations) and the discrete theory (series expansions), and enables us to consider key questions in frame analysis from an abstract angle.
- This abstract approach also provides a unified way to the analysis on the four elementary groups $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{T}$, $\mathbb{Z}_m$ and their higher dimensional variants.
- Since in signal processing one often considers products of the groups $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{T}$, $\mathbb{Z}_m$ which are also LCA groups, the approach is very useful in solving practical problems related to signal and image analysis. For example, multichannel video signal involves the group $\mathbb{Z}^d \times \mathbb{Z}_m$, where d is the number of channels and m the number of pixels of each image.

In view of the above advantages, several researchers have made remarkable contributions in establishing the theory required to analyse frame properties on LCA groups (see [10, 12, 15, 19, 28, 33, 40, 42, 59, 60, 63, 64, 77, 89] and various references therein).

Among these properties, the "orthogonality or strongly disjointness" of frame pairs in Hilbert spaces is very useful. It was initially introduced and investigated by Han and Larson [51], and Balan [4] in the context of multiplexing. *The aforementioned property of a pair of frames says that the ranges of the analysis operators for the two frames are orthogonal*. In this case, the corresponding frames are termed as *pairwise orthogonal* (simply, *orthogonal*). The orthogonality of frame and Bessel sequences plays a key role in frame theory, for example:

- In construction of new frames from existing ones, constructions related with duality, in multiple access communications, hiding data, and synthesizing super-frames and frames, etc. (see [3, 4, 35, 40, 51, 67, 89] and various references within).
- This property is used (by Han and Larson in [52]) as an essential ingredient in examining the density and connectedness results of the wavelet frames.

Inspired by the wide applications of pairwise orthogonal frames, a lot of mathematicians and engineering scientists have contributed in developing different aspects of this natural geometric concept in frame theory over various function spaces (see [35, 36, 40, 42, 50–52, 63, 67, 89] and various references therein).

In this scenario, the main concern of the present thesis is to study pairwise orthogonal frames for Hilbert spaces associated with LCA groups. Our focus is to investigate and explore the orthogonality of frame pairs for function systems generated by regular representations of LCA groups. In this direction, we obtain characterizations and constructions of orthogonal frame pairs for various function spaces on LCA groups. Additionally, we relate the above developed theory to the classical case as well as finite-dimensional Hilbert spaces by letting LCA groups of the form $\mathbb{R}^d, \mathbb{Z}^d$, and $\mathbb{Z}_N^d$, etc.

To the best of our knowledge, we realize that the characterization results for pairwise orthogonal frames have not been studied earlier in the context of LCA groups, and are appearing first time in the literature via the work presented in this thesis. The results obtained in the thesis are an interesting addition to the recent growing body of literature on frames generated by structured families of functions on LCA groups.

1.1. Motivation and objective

Literature says that the theory of frames has a very close connection with unitary group representations, since various structured systems, for example, wavelet and Gabor systems can be realized as a sequence obtained by applying a family of unitary operators to a particular window function. In this direction, several researchers have contributed in analysing various interesting properties and results of frame theory in different set-ups (e.g. see [7, 10, 12, 15, 59] and various references therein).

Motivated by this, our concern in the thesis is to examine the orthogonality of frames for subspaces of a separable Hilbert space $\mathcal{H}$ that are invariant under the action of unitary representations of a closed and (a not necessarily discrete) co-compact abelian group $\Gamma \subset G$ on $\mathcal{H}$, where G is a second countable LCA group. Note that a subgroup Γ in G is called *co-compact* if the quotient group G/Γ is compact, whereas Γ in G is said to be a *uniform lattice* if in addition, Γ is discrete. More specifically, in the present thesis we consider unitary representations of a countable family of co-compact subgroups $\{\Gamma_j\}_{j\in J}$ acting on $L^2(G)$ by left-translation, where J is a countable index set. Such unitary representations are called (*left-*)*regular representations* of $\{\Gamma_j\}_{j\in J}$.

In connection with this, we mention that our work on orthogonal frames over translation invariant (TI) spaces is actually inspired by the utility of recent notion on TI systems in LCA-group setting, which is given by Bownik and Ross in [10]. We also consider orthogonality in the context of generalized TI (simply, GTI) frame systems, which is a class of systems introduced recently by Jakobsen and Lemvig in [59]. The notion of GTI systems connects the well-established discrete frame theory of generalized shift-invariant (GSI) systems and its continuous version. Therefore, the study of pairwise orthogonality of GTI frame systems represents a unified way to deduce similar results for several other function systems including the case of GSI systems studied by Kutyniok and Labate in [64], and TI systems considered by Bownik and Ross in [10].

In view of the above discussion, note that the work of Kutyniok and Labate [64] presented a combined theory for many of the known function systems (e.g., Gabor systems and GSI systems on $\mathbb{R}^d$) by introducing the notion of GSI systems in the LCA-group setting. This approach is an extension of the theory of Hernández, Labate and Weiss [53], and Ron and Shen [81] on GSI systems in $L^2(\mathbb{R}^d)$.

Another significant fact which we wish to remark here is that among all function systems mentioned above, SI and GSI systems are based on translation along uniform lattices while TI and GTI systems, respectively, generalize the concept of SI and GSI systems for the continuous case by considering translation along co-compact subgroups of an LCA group. The motivation behind the consideration of co-compact subgroups for TI systems in [10] and GTI systems in [59] is related to the necessity of overcoming the limitation on the existence of uniform lattices for an LCA group, which says there exist LCA groups that do not contain any uniform lattices, for example, the p-adic numbers $\mathbb{Q}_p$, whose only discrete subgroup is the neutral element which is not a uniform lattice. Another example is the p-adic integers which have only trivial examples of uniform lattices but have a lot of non-trivial co-compact subgroups. Hence, the concept of co-compact subgroups in [10] and [59] generalizes the work on function systems with translation along uniform lattices considered in [19] and [64], respectively.

At this juncture, it is pertinent to note that in the Euclidean setting, Weber in [89] studied orthogonal frames of translates with various applications, and later, Kim et al. in [63] discussed such frames in a general shift-invariant subspace of $L^2(\mathbb{R}^d)$, while Lopez and Han in [67] described the orthogonality of discrete Gabor frame pairs in $\ell^2(\mathbb{Z}^d)$. Therefore, the motive of the proposed thesis is to extend the characterization results from the Euclidean case [89], discrete setting [67], and from the case of uniform lattices [63], to the set-up of (not necessarily discrete) co-compact subgroups over LCA groups. Along with this, the investigation of frame properties for structured function systems in different settings have got special attention (see [2,4,12,28,30,33,60,67]). Hence, we got motivated to study the orthogonality of such systems over LCA-group setting.

Since practical life applications mainly require frames in finite setting, this motivates us to study finite time-frequency localized constructions of frame and wavelet systems. It is known that a time-frequency localized basis plays an important role in extracting both time as well as frequency information of a given signal, and the uncertainty principle helps us to understand how much local information in time and frequency we can extract by using the above-mentioned basis. The theory of uncertainty principle related to the pair of bases and its applications has been developed by various authors (see [21,24,37,62,65,82]).

1.1.1. Objective of the thesis

The present thesis investigates the orthogonality of frame pairs generated by the action of a unitary representation ρ of Γ_j on a separable Hilbert space $L^2(G)$, where for each $j \in J \subset \mathbb{Z}$, we let Γ_j to be a closed and (a not necessarily discrete) co-compact subgroup in a second countable LCA group G with Haar measure denoted by μ_G, and the unitary representation $\rho : \Gamma_j \to \mathbb{U}(L^2(G))$; $\gamma \mapsto \rho(\gamma)$ that acts by a (left-)translation, i.e., $\rho(\gamma)f = f(\cdot - \gamma)$ for all $f \in L^2(G)$. Here, note that $\mathbb{U}(L^2(G))$ represents the group of linear unitary operators over $L^2(G)$. Further, for each $j \in J$, by assuming P_j as a (not necessarily countable) index set, consider $\Psi_j := \{\psi_{p,j}\}_{p \in P_j} \subset L^2(G)$, and define a family

$$E^{\Gamma_j}(\Psi_j) := \big\{\rho(\gamma)\psi_{p,j} : \gamma \in \Gamma_j,\, p \in P_j\big\} \subset L^2(G).$$

Precisely, we consider frames of the form $\langle \{\Psi_j\}_{j\in J}\rangle := \bigcup_{j\in J} E^{\Gamma_j}(\Psi_j)$ which coincides with the recent notion of GTI system introduced in [59]. Note that the GTI system reduces to TI system discussed in [10], if $\Gamma_j = \Gamma$ for each $j \in J$. However, if each P_j is countable and each Γ_j is a uniform lattice in a GTI system, we arrive at GSI system considered in [64]. The main focus of this thesis is to study the orthogonality of frame pairs in the context of above-mentioned systems over LCA groups. In view of this, the main objectives of the present thesis are as follows:

- To characterize the orthogonality of frame pairs of the form $\langle \{\Psi_j\}_{j\in J}\rangle$ by considering different situations on $J \subset \mathbb{Z}$, $\{P_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$. By using the resulting characterizations, we wish to provide constructions of new frames and super-frames.
- To investigate the orthogonality of structured function systems (for example, wavelet, Gabor, and wave-packet systems) over LCA-group setting.
- To establish a relation between the above-developed theory and the classical case as well as finite set-up by letting LCA group G of the form $\mathbb{R}^d, \mathbb{Z}^d, \mathbb{Z}_N^d$, etc.
- To provide a group-theoretic construction of a time as well as frequency localized finite orthonormal wavelet system and its generalization to frame wavelet (simply, framelet) system associated with an induced group action. Additionally, we are interested to examine the time-frequency localization properties of these systems via uncertainty principle.
- To study the orthogonality and duality properties of a pair of framelet systems, and its applications in synthesizing super framelet systems over finite set-up.

1.2. Preliminaries

In this section, we set-up some notation, assumptions, and pre-requisites used for the remaining chapters. We first review some basic results from Fourier analysis on locally compact abelian groups. Then, we recall some basic definitions and results from frame theory in Hilbert spaces. Throughout this thesis, we shall use the following notation: the symbols $\mathbb{N}, \mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, denote sets of positive integers, integers, real numbers and complex numbers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

1.2.1. Background on LCA groups

In this part of section, we refer to the classical books [12, 25, 55, 56, 83] along with research papers [10, 19, 59, 60, 64] for the terminology used and for various results and properties related to harmonic analysis over locally compact abelian groups.

Here and throughout, let G denote a second countable locally compact abelian (LCA) group, with the additive group composition, denoted by the symbol "+", and the neutral element 0. Note that the second countable property of G is equivalent to saying that G is metrizable and σ-compact.

It is well-known that on every LCA group G, there exists a unique *Haar measure*, that is, a non-negative, regular Borel measure, denoted as μ_G (not identically zero) which is translation invariant, *i.e.*, $\mu_G(E+x) = \mu_G(E)$ for every element $x \in G$ and every Borel set $E \subseteq G$. It should be noted that the Haar measure of a locally compact group is unique only up to a positive multiplicative constant.

Denote by $\widehat{G}$, the set of all continuous characters, that is, all continuous homomorphisms from G into the torus $\mathbb{T} \cong \{z \in \mathbb{C} : |z| = 1\}$. Then, under the pointwise multiplication $\widehat{G}$ forms an LCA group with unit element 1, that is called the *dual group* associated to G, when equipped with the compact convergence topology and the composition

$$(\gamma + \gamma')(x) := \gamma(x)\gamma'(x), \ \gamma, \gamma' \in \widehat{G}, \ x \in G,$$

and thus possesses a Haar measure that we denote by $\mu_{\widehat{G}}$. It turns out that there exists a topological group isomorphism mapping the group $\widehat{\widehat{G}}$, that is, the dual group of $\widehat{G}$, onto G. More precisely, $\widehat{\widehat{G}} \cong G$ [25, Pontryagin duality theorem]. Note that if an LCA group G is discrete then $\widehat{G}$ is compact, and vice versa.

Given an LCA group G with Haar measure μ_G, the integral over G is translation invariant in the sense that,

$$\int_G f(x+y)d\mu_G(x) = \int_G f(x)d\mu_G(x)$$

for each element $y \in G$ and for each Borel-measurable function f on G. For $1 \leq p < \infty$, we define the space $L^p(G, \mu_G)$ (simply, $L^p(G)$) as follows:

$$L^p(G) := \Big\{ f : G \to \mathbb{C} \text{ is a measurable function and } \int_G |f(x)|^p d\mu_G(x) < \infty \Big\}.$$

Since G is a second countable LCA group, $L^p(G)$ is separable, for all $1 \leq p < \infty$. In this thesis, we will focus only on $p = 2$ case. Here, note that $L^2(G)$ is a Hilbert space with inner product given by

$$\langle f, g \rangle = \int_G f(x)\overline{g(x)}d\mu_G(x), \text{ for all } f, g \in L^2(G).$$

Let the Fourier transform $\widehat{\ } : L^1(G) \to C_0(\widehat{G})$, $f \mapsto \widehat{f}$, be defined by the operator

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_G f(x)\overline{\xi(x)}d\mu_G(x), \ \xi \in \widehat{G},$$

where $C_0(\widehat{G})$ denotes the functions on $\widehat{G}$ vanishing at infinity. If $f \in L^1(G)$, $\widehat{f} \in L^1(\widehat{G})$, and the measures on G and $\widehat{G}$ are normalized appropriately so that the Plancherel theorem holds, then the inverse Fourier transform can be defined by

$$f(x) = \mathcal{F}^{-1}\widehat{f}(x) = \int_{\widehat{G}} \widehat{f}(\xi)\xi(x)d\mu_{\widehat{G}}(\xi), \ x \in G.$$

Note that the Fourier transform $\mathcal{F}$ can be extended from $L^1(G) \cap L^2(G)$ to a surjective isometry between $L^2(G)$ and $L^2(\widehat{G})$ [25, Plancherel theorem]. Thus, the Parseval formula holds and is given by

$$\langle f, g \rangle = \int_G f(x)\overline{g(x)}d\mu_G(x) = \int_{\widehat{G}} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\mu_{\widehat{G}}(\xi) = \langle \widehat{f}, \widehat{g} \rangle, \text{ for all } f, g \in L^2(G).$$

The following definitions will be used in the sequel: Given G an LCA group, we call a subgroup Γ in G as *co-compact* if the quotient group G/Γ is compact, whereas Γ in G is said to be a *uniform lattice* if in addition, Γ is discrete.

Let $\Gamma \subseteq G$ be a closed subgroup of an LCA group G. Then, the quotient G/Γ is a regular topological group. Further, we note that it is a second countable LCA group under the quotient topology by using the fact that G is second countable.

Note that for a subgroup Γ of an LCA group G, the symbol $\Gamma^\perp$ denotes the *annihilator* of Γ, which is a subgroup of $\widehat{G}$ defined by

$$\Gamma^\perp := \{\xi \in \widehat{G} : \xi(x) = 1, \text{ for all } x \in \Gamma\}.$$

It follows from the definition of the topology on $\widehat{G}$ that the annihilator $\Gamma^\perp$ is a closed subgroup in $\widehat{G}$. Moreover, if Γ is closed, then $(\Gamma^\perp)^\perp = \Gamma$ and the following hold:

(i) there exists a topological group isomorphism mapping $\widehat{G/\Gamma}$ onto $\Gamma^\perp$, that is, we have $\widehat{G/\Gamma} \cong \Gamma^\perp$;

(ii) there exists a topological group isomorphism mapping $\widehat{\widehat{G}/\Gamma^\perp}$ onto Γ, that is, we have $\widehat{\widehat{G}/\Gamma^\perp} \cong \Gamma$.

In the rest of this thesis, unless mentioned otherwise we assume Γ to be a closed and co-compact (not necessarily discrete) subgroup in G. Note that the symbol μ_Γ represents a Haar measure on the subgroup Γ. Since, $\Gamma^\perp$ is topologically isomorphic to the dual of quotient group G/Γ, that is, $\Gamma^\perp \cong \widehat{(G/\Gamma)}$, therefore, Γ is co-compact in G if, and only if, $\Gamma^\perp$ is a discrete subgroup of $\widehat{G}$ (for more details, see [10]). Thus, $\Gamma^\perp$ will always be discrete in our case, and hence preserves a counting measure.

Let $\widehat{\Gamma}$ (the dual group of Γ) be an LCA group with measure denoted by $\mu_{\widehat{\Gamma}}$. Observe that there exists a topological group isomorphism mapping $\widehat{G}/\Gamma^\perp$ onto $\widehat{\Gamma}$. Hence, by choosing a measure $\mu_{\widehat{G}/\Gamma^\perp}$ on $\widehat{G}/\Gamma^\perp$ appropriately, Weil's formula [25] can be stated as

$$\int_{\widehat{G}} \widehat{f}(\xi) d\mu_{\widehat{G}}(\xi) = \int_{\widehat{G}/\Gamma^\perp} \sum_{\alpha \in \Gamma^\perp} \widehat{f}(\xi+\alpha) d\mu_{\widehat{G}/\Gamma^\perp}(\xi+\Gamma^\perp) = \int_{\widehat{\Gamma}} \sum_{\alpha \in \Gamma^\perp} \widehat{f}(\xi+\alpha) d\mu_{\widehat{\Gamma}}(\xi), \tag{1.1}$$

for all $f \in L^1(G)$.

Note that the dual group $\widehat{G} = \Omega \oplus \Gamma^\perp$, therefore, every $\xi \in \widehat{G}$ has a unique representation $w + \alpha$ for some $w \in \Omega$ and $\alpha \in \Gamma^\perp$. Here Ω is a $\mu_{\widehat{G}}$-measurable subset of $\widehat{G}$ and represents a Borel section of $\Gamma^\perp$ in $\widehat{G}$, also known as a *fundamental domain* of $\widehat{G}/\Gamma^\perp$, whose existence is guaranteed by [27]. Moreover, it is relevant to note that every element v in $\widehat{\Gamma} \cong \widehat{G}/\Gamma^\perp$ can be thought of as an element in Ω as all cosets in $\widehat{G}/\Gamma^\perp \cong \widehat{\Gamma}$ are of the form $w + \Gamma^\perp$ for some (unique) $w \in \Omega$. For more details, we refer [10, Section 3].

1.2.2. Theory of frames

In this section, we recall some definitions and basic properties about continuous frames for Hilbert spaces. Such frames were introduced independently by Ali et al. [1] and Kaiser [61]. For a brief and self-sufficient introduction to continuous frames, we refer [38,78]. For more details on general theory and applications of frames and Bessel sequences, we refer to [1, 2, 10, 12, 19, 28, 33, 34, 51, 60, 61, 79, 84]. For theory of frames generated by unitary actions of LCA groups, we refer to the research articles [7,8,58,74] and various references therein. Note that for basic definitions and theory on finite frames, we use the books by Christensen [12], Casazza et al. [18], and by Han et al. [50].

Definition 1.1. Let $\mathcal{H}$ be a complex Hilbert space, and let $(M, \sum_M, \mu_M)$ be a measure space, where $\sum_M$ denotes the σ-algebra and μ_M the non-negative measure. Then, a family of functions $\{f_m\}_{m\in M}$ in $\mathcal{H}$, is called a *continuous frame* for $\mathcal{H}$ with respect to $(M, \sum_M, \mu_M)$ if

(1) $m \mapsto f_m$ is weakly measurable; that is, for all $h \in \mathcal{H}$, the mapping $M \to \mathbb{C}$, $m \mapsto \langle h, f_m\rangle$ is measurable, and

(2) there exist constants $0 < \alpha_1 \leq \alpha_2$ (called continuous frame bounds) such that

$$\alpha_1||h||^2 \leq \int_M |\langle h, f_m\rangle|^2 d\mu_M(m) \leq \alpha_2||h||^2, \text{ for all } h \in \mathcal{H}. \tag{1.2}$$

A continuous frame $\{f_m\}_{m\in M}$ is called *tight* if we can choose $\alpha_1 = \alpha_2$, and *tight frame with frame bound 1* (*or, Parseval*) if $\alpha_1 = \alpha_2 = 1$. The family $\{f_m\}_{m\in M}$ is called *Bessel* with constant α_2 as its *Bessel constant* if the right side of inequality in (1.2) holds. In this case, we say that the family $\{f_m\}_{m\in M}$ satisfies the *Bessel condition*.

Since this thesis deals with only separable Hilbert spaces, we can use Petti's theorem to replace weak measurability of $m \mapsto f_m$ with (strong) measurability with respect to the Borel algebra in $\mathcal{H}$.

If μ_M is a counting measure and $M = \mathbb{N}$, then $\{f_m\}_{m\in M}$ reduces to a discrete frame. In this sense, continuous frames can be realized as the generalization of discrete frames. Recall that for a countable index set $\mathfrak{J}$, a sequence $\{f_n\}_{n\in\mathfrak{J}}$ in a separable complex Hilbert space $\mathcal{H}$ is called a *discrete frame* for $\mathcal{H}$ if there exist frame constants $0 < \alpha \leq \beta < \infty$

such that for every $f \in \mathcal{H}$, we have

$$\alpha||f||^2_{\mathcal{H}} \leq \sum_{n \in \mathfrak{J}} |\langle f, f_n \rangle_{\mathcal{H}}|^2 \leq \beta||f||^2_{\mathcal{H}}.$$

Here onwards, for the sake of simplicity, we will call continuous/discrete frames as just frames by suppressing the term continuous/discrete.

Given the family of functions $\mathbb{F} := \{f_m\}_{m \in M}$, which is Bessel with respect to a measure space $(M, \sum_M, \mu_M)$, define the *synthesis operator* $\Theta_{\mathbb{F}} : L^2(M, \mu_M) \to \mathcal{H}$ defined by

$$\langle \Theta_{\mathbb{F}} \varphi, h \rangle = \int\limits_M \langle f_m, h \rangle \varphi_m d\mu_M(m), \ \ \varphi = \{\varphi_m\}_{m \in M} \in L^2(M, \mu_M), \ \ h \in \mathcal{H},$$

which is a well-defined, linear and bounded operator [78, Theorem 2.6].

Further, the adjoint of the synthesis operator, known as the *analysis operator* of $\mathbb{F}$, is defined by $\Theta^*_{\mathbb{F}} : \mathcal{H} \to L^2(M, \mu_M)$ with

$$(\Theta^*_{\mathbb{F}} h)(m) = \langle h, f_m \rangle, \ \ m \in M.$$

Given two Bessel families $\mathbb{F}$ and $\mathbb{G} := \{g_m\}_{m \in M}$ with respect to the measure space $(M, \sum_M, \mu_M)$ for $\mathcal{H}$, define the *mixed dual Gramian operator* corresponding to $\mathbb{F}$ and $\mathbb{G}$ as

$$\Theta_{\mathbb{G}} \Theta^*_{\mathbb{F}} : \mathcal{H} \to \mathcal{H}; \ \ h \mapsto \int\limits_M \langle h, f_m \rangle g_m d\mu_M(m). \tag{1.3}$$

Gabardo and Han in [38] defined a dual frame for a continuous frame as follows:

Definition 1.2. Let $\mathbb{F}$ and $\mathbb{G}$ be two Bessel families with respect to the measure space $(M, \sum_M, \mu_M)$ for $\mathcal{H}$. We call $\mathbb{G}$ a *dual frame* for $\mathbb{F}$ if the following holds true:

$$\langle h_1, h_2 \rangle = \int\limits_M \langle h_1, f_m \rangle \langle g_m, h_2 \rangle d\mu_M(m), \ \ \text{for all} \ \ h_1, h_2 \in \mathcal{H}. \tag{1.4}$$

In this case, $\mathbb{F}$ and $\mathbb{G}$ are actually (continuous) frames, and hence $(\mathbb{F}, \mathbb{G})$ is called a *dual frame pair*. If $\Theta_{\mathbb{F}}$ and $\Theta_{\mathbb{G}}$ denote the synthesis operators of $\mathbb{F}$ and $\mathbb{G}$, respectively, then (1.4) is equivalent to $\Theta_{\mathbb{G}} \Theta^*_{\mathbb{F}} = I_{\mathcal{H}}$, that is, an identity operator on $\mathcal{H}$. In this case, we say that the relation

$$h = \int\limits_M \langle h, f_m \rangle g_m d\mu_M(m), \ \ \text{for all} \ f \in \mathcal{H},$$

holds in the weak sense. This relation is generally known as a *reproducing formula* for $f \in \mathcal{H}$.

Next, we define the orthogonality of a pair of Bessel families (frames) as follows:

Definition 1.3. Let $\mathbb{F}$ and $\mathbb{G}$ be Bessel families (frames) with respect to $(M, \sum_M, \mu_M)$ for $\mathcal{H}$. Then, if the mixed dual Gramian operator of $\mathbb{F}$ and $\mathbb{G}$ (as defined in (1.3)) is zero, that is, $\Theta_{\mathbb{G}}\Theta_{\mathbb{F}}^* = 0$, the Bessel families (frames) are said to be *pairwise orthogonal* (simply, *orthogonal*). In other words, we say that $\mathbb{F}$ and $\mathbb{G}$ satisfy the *orthogonality* property.

1.3. Structure of the thesis

In **Chapter 2**, by investigating the dual Gramian analysis tools of Ron and Shen through a pre-Gramian operator over the set-up of LCA groups, we study and characterize a pair of orthogonal frames generated by the action of a unitary representation ρ of a co-compact subgroup $\Gamma \subset G$ on a separable Hilbert space $L^2(G)$. We pay special attention to this problem in the context of translation invariant space by assuming ρ as the action of Γ on $L^2(G)$ by (left-)translation. As an application, we illustrate our results for the case of co-compact Gabor systems over LCA groups.

The results of **Chapter 2** are from the accepted research article:

Gumber, A., Shukla, N. K. (2017), *Pairwise orthogonal frames generated by regular representations of LCA groups*, Bulletin des Sciences Mathématiques (Elsevier).

In **Chapter 3**, we give necessary and sufficient conditions for the orthogonality of two Bessel families when such families have the form of GTI systems over a second countable LCA group G. Consequently, we deduce similar results for several function systems including the case of TI systems, and GTI systems on compact abelian groups. We apply our results to the Bessel families having wave-packet structure (combination of wavelet as well as Gabor structure), and hence a characterization for pairwise orthogonal wave-packet frame systems over LCA groups is obtained.

The results of **Chapter 3** are from the published research article:

Gumber A., Shukla N. K., *Orthogonality of a pair of frames over locally compact abelian groups*, J. Math. Anal. Appl., 458(2) (2018), 1344-1360.

In **Chapter 4**, we have obtained a general construction method for arbitrarily many pairwise orthogonal GSI frames by utilizing the unitary extension principle (UEP) of

Christensen and Goh in [15]. The method is then applied to study constructions of frame pairs over super-spaces.

The results of **Chapter 4** are from the following manuscript:

Gumber A., Shukla N. K., *Constructions of pairs of generalized shift invariant orthogonal frames*, under preparation.

In **Chapter 5**, we first investigate a finite collection of functions in $\ell^2(\mathbb{Z}_N^d)$ that satisfies some localization properties in a region of the time-frequency plane. For this, a group-theoretic approach based on the complete digit set associated to an invertible matrix is used. It leads to the construction of an orthonormal wavelet system (ONWS) which is concentrated in time as well as frequency. We study and characterize the ONWS. Further, some results on the uncertainty principle corresponding to the ONWS are obtained.

The results of **Chapter 5** have been appeared in:

Gumber A., Shukla N. K., *Uncertainty Principle corresponding to an Orthonormal Wavelet System*, Appl. Anal., 97(3) (2018), 486-498.

In **Chapter 6**, we first induce an action of a topological group $\mathbb{G}$ on $\ell^2(\mathbb{Z}_N^d)$ from a given action of $\mathbb{G}$ on the space $\mathbb{C}$ of complex numbers. Then, for each $\mathrm{g} \in \mathbb{G}$, we introduce a framelet system (**g**-framelet system or **g**-FS) associated with an induced action of $\mathbb{G}$ on $\ell^2(\mathbb{Z}_N^d)$, and a super **g**-FS for the super-space in the same set-up. By applying the group-theoretic approach based on the complete digit set, we characterize the generators of two **g**-framelet systems (super **g**-framelet systems) such that they form a **g**-dual pair (super **g**-dual pair). As a consequence, characterizations for the Parseval **g**-FS and the Parseval super **g**-FS are obtained. Further, some properties of the frame operator corresponding to the **g**-FS are observed, which results in concluding that its canonical dual preserves the same structure.

The results of **Chapter 6** are from the published article:

Gumber A., Shukla N. K. (2017), *Finite dual* g*-framelet systems associated with an induced group action*, Complex Anal. Oper. Theory, `DOI:10.1007/s11785-017-0729-6`.

Finally, **Chapter 7** deals with concluding remarks and provides some directions for future study.

CHAPTER 2

PAIRWISE ORTHOGONAL FRAMES GENERATED BY REGULAR REPRESENTATIONS OF LCA GROUPS

The purpose of this chapter is to study and characterize pairwise orthogonal frames generated by unitary representations of LCA groups. For this, we first investigate the dual Gramian analysis tools of Ron and Shen [79] by introducing the notion of a pre-Gramian operator (in Section 2.2) associated with Bessel families generated by unitary actions of co-compact subgroups of LCA groups. Using this approach, our main focus is to examine the orthogonality of frames (in the sense of Definition 2.3) for subspaces of a separable Hilbert space $\mathcal{H}$ that are invariant under the action of unitary representations of a closed and co-compact abelian group $\Gamma \subset G$ on $\mathcal{H}$, where G is a second countable LCA group. Note that the above mentioned theory is developed by using a (left-)regular representation, which is a unitary representation of Γ acting on $L^2(G)$ by (left-)translation.

2.1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space. Then, by a *unitary representation* of $\Gamma \subset G$, we mean a pair $(\rho, \mathcal{H})$ with ρ as a group homomorphism of Γ into $\mathbb{U}(\mathcal{H})$, that is, the group of linear unitary operators over $\mathcal{H}$. We assume that the map $\Gamma \times \mathcal{H} \to \mathcal{H};\ (\gamma, h) \mapsto \rho(\gamma)h$ is continuous. Note that a subspace of $\mathcal{H}$ that is invariant under the action of unitary representation ρ of Γ on the Hilbert space $\mathcal{H}$ is called (ρ, Γ)-*invariant*. To be precise about our concern in this chapter, we let $\Gamma \ni \gamma \mapsto \rho(\gamma) \in \mathbb{U}(\mathcal{H})$ as a unitary representation of Γ on $\mathcal{H}$, and wish to study a very interesting research problem regarding the orthogonality of frame pairs for (ρ, Γ)-invariant subspaces in $\mathcal{H}$. The problem can be stated as follows:

(Q1) *For which families Ψ and Φ in $\mathcal{H}$ do the collections*

$$\langle \Psi \rangle := \big\{\rho(\gamma)\psi : \gamma \in \Gamma, \psi \in \Psi\big\} \quad \textit{and} \quad \langle \Phi \rangle := \big\{\rho(\gamma)\varphi : \gamma \in \Gamma, \varphi \in \Phi\big\}$$

form a pair of orthogonal frames for $\overline{span\langle \Psi \rangle}^{\mathcal{H}}$, that is, for the closed linear span of $\langle \Psi \rangle$, which is indeed the smallest (ρ, Γ)-invariant subspace in $\mathcal{H}$?

In connection with this, we mention that by considering ρ as the action of a closed, co-compact subgroup Γ on $\mathcal{H}$ by left-translation, the main focus of this chapter is to explore the question (Q1) for the (ρ, Γ)-invariant subspaces in the context of LCA groups. Note that the representation of Γ acting on $\mathcal{H}$ by (left-)translation is called the *(left-)regular representation*, and the invariant subspaces of this representation are called *translation-invariant* (*TI*). At this juncture, it is pertinent to note that our work extends the results from the discrete setting $G = \mathbb{Z}^n$ and from the case of uniform lattices in $L^2(\mathbb{R}^n)$, studied respectively by Lopez and Han in [67] and Kim et al. in [63], to the set-up of (not necessarily discrete) co-compact subgroups over LCA groups.

For the rest of this chapter, we consider $\mathcal{H}$ as $L^2(G)$. Unless mentioned otherwise we assume Γ to be a closed and co-compact (not necessarily discrete) subgroup in G. Now, we let the unitary representation as a map

$$\rho : \Gamma \to \mathbb{U}(L^2(G)); \ \gamma \mapsto \rho(\gamma)$$

that acts by a left-translation, that is,

$$\rho(\gamma)f = f(\cdot - \gamma), \text{ for all } f \in L^2(G),$$

and define $E^\Gamma(\Psi)$ to be the family

$$\big\{\rho(\gamma)\psi : \gamma \in \Gamma, \psi \in \Psi\big\} =: E^\Gamma(\Psi), \tag{2.1}$$

which is generated by a countable subset Ψ in $L^2(G)$.

Now, we proceed to formulate the statement of the question (Q1) in terms of TI subspaces which were introduced and studied by Bownik and Ross in [10]. For this, we observe the following definition:

Definition 2.1. Suppose that $\Gamma \subset G$ is a closed, co-compact subgroup of G. Let $V \subset L^2(G)$ be a closed subspace. Then

(i) we say that V is (ρ, Γ)*-invariant* or more appropriately *translation-invariant* (*TI*) under Γ, in short Γ-TI, if $f \in V$ implies $\rho(\gamma)f \in V$ for all $\gamma \in \Gamma$.

(ii) we call $E^\Gamma(\Psi)$, that is, the system generated by Ψ (defined in (2.1)), as a Γ-*TI system* with its closed linear span in $L^2(G)$ denoted by

$$\overline{\operatorname{span}(E^\Gamma(\Psi))}^{L^2(G)} =: S^\Gamma(\Psi)$$

as a Γ-*TI space* which is the smallest closed subspace in $L^2(G)$ containing $E^\Gamma(\Psi)$.

(iii) for $\Psi = \{\psi\}$ as a singleton subset in $L^2(G)$, let the *principle translation-invariant* (Γ-*PTI*) *system* be given by $E^\Gamma(\psi) := \{\rho(\gamma)\psi : \gamma \in \Gamma\}$. In this case, the family $\overline{\operatorname{span}(E^\Gamma(\psi))}^{L^2(G)} =: S^\Gamma(\psi)$ is called a *principle translation-invariant* (Γ-*PTI*) *space* generated by ψ.

(iv) in case Γ is a uniform lattice, the term translation-invariant is replaced by *shift-invariant* (*SI*).

Now, we state the main problem associated with (Q1) in the context of Γ-TI systems over LCA groups. For this, here and throughout, let $\Psi := \{\psi_p\}_{p\in\mathscr{P}}$ and $\Phi := \{\varphi_p\}_{p\in\mathscr{P}}$ be subsets in $L^2(G)$, where $\mathscr{P}$ is a countable index set. Then, by assuming $S^\Gamma(\Psi) = S^\Gamma(\Phi)$, the main problem that we investigate in the remainder of this chapter is:

(Q2) *To find necessary and sufficient conditions on the above discussed generators* Ψ *and* Φ *in the Fourier domain such that the* Γ-*TI systems* $E^\Gamma(\Psi)$ *and* $E^\Gamma(\Phi)$ *as defined in (2.1), form a pair of orthogonal frames* (*in the sense of Definition 2.3*) *in the* Γ-*TI space* $S^\Gamma(\Psi)$.

For answering the above question, we introduce the notion of pre-Gramian operator associated with a Γ-TI space $S^\Gamma(\Psi)$ and fiberize the analysis, synthesis and mixed dual-Gramian operators corresponding to the Γ-TI systems $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$ in the next section. Note that the fiberization is helpful in analysing various frame properties of the Γ-TI systems.

In this scenario, this chapter is devoted to the study of the orthogonality property of a pair of frames for Γ-TI spaces over LCA groups by using the approach of a pre-Gramian operator. The major tool used here is the dual Gramian analysis of Ron and Shen [79] over the above set-up. Note that the novelty of our results in this chapter lies in understanding that we do not require Γ to be discrete. As a consequence, our results for two Γ-TI systems to be pairwise orthogonal frames are new even in the classical setting of $G = \mathbb{R}^n$, where the results of [63] are not applicable. Moreover, it also applies to LCA groups G that do not have uniform lattices such as the p-adic field, $\mathbb{Q}_p$.

Along with this, we apply our results on systems with co-compact Gabor structure to obtain a characterization so that the above-structured systems become pairwise orthogonal Γ-TI Bessel (frame) systems. For more details on the definition of such systems, refer Section 2.2 of this chapter.

2.2. Fiberization of operators associated with Γ-TI frame systems

The current section deals with the development of the necessary machinery required to answer the question (Q2) imposed on the orthogonality of two Γ-TI frame systems in the introduction part. For this, let

$$L^2(\Gamma, \ell^2(\mathscr{P})) := \Big\{\text{measurable } h : \Gamma \to \ell^2(\mathscr{P}) \text{ with } ||h||^2 := \int_\Gamma ||h(\gamma)||^2_{\ell^2(\mathscr{P})} d\mu_\Gamma(\gamma) < \infty\Big\},$$

be the Hilbert space of $\ell^2(\mathscr{P})$-valued square integrable functions over Γ with inner product

$$\langle h^1, h^2\rangle := \int_\Gamma \langle h^1(\gamma), h^2(\gamma)\rangle_{\ell^2(\mathscr{P})} d\mu_\Gamma(\gamma),$$

for all $h^i := (h^i(\gamma))_{\gamma\in\Gamma}$ in $L^2(\Gamma, \ell^2(\mathscr{P}))$ with $i = 1, 2$, and $h^i(\gamma) := (h^i_p(\gamma))_{p\in\mathscr{P}}$ in $\ell^2(\mathscr{P})$ for each γ. Note that $\Gamma \subset G$ is a second countable locally compact abelian (LCA) group. Thus, Γ is σ-compact, and hence σ-finite. Further, $\mathscr{P}$ is a countable index set, therefore, $L^2(\Gamma, \ell^2(\mathscr{P})) = L^2(\mathscr{P} \times \Gamma)$ since $\ell^2(\mathscr{P})$ is separable. For more details on this, we refer to [6, Proposition Appendix A.3.]).

2.2.1. Γ-TI frame systems and associated operators

In this subsection, we give the following definition of a Γ-TI system (see [10] for more details) to be a frame, and study various operators associated with such systems. We refer to [8, 12, 60, 78] for the general definitions of frames and associated operators along with other basic concepts of frame theory.

Definition 2.2. The Γ-TI system $E^\Gamma(\Psi)$ is called a Γ-*TI frame system* for $S^\Gamma(\Psi)$, if there exist frame bounds $0 < \mathcal{A} \le \mathcal{B} < \infty$ such that the following relation is satisfied:

$$\mathcal{A}||f||^2 \le \sum_{p\in\mathscr{P}} \int_\Gamma |\langle f, \rho(\gamma)\psi_p\rangle|^2 d\mu_\Gamma(\gamma) \le \mathcal{B}||f||^2, \quad \text{for all } f \in S^\Gamma(\Psi). \tag{2.2}$$

A Γ-TI frame system $E^\Gamma(\Psi)$ is called *tight* Γ-*TI frame system* for $S^\Gamma(\Psi)$ with frame bound $\mathcal{A}$ if we can choose $\mathcal{A} = \mathcal{B}$. Note that $E^\Gamma(\Psi)$ is a Γ-*TI Bessel system* for $S^\Gamma(\Psi)$ with $\mathcal{B}$ as its *Bessel constant* if the right side of inequality in (2.2) holds. Similarly, we can define all the above terms for the case of $S^\Gamma(\Psi) = L^2(G)$.

Further, let $E^\Gamma(\Psi)$ be a Γ-TI Bessel system for $L^2(G)$. Then, for $h := (h(\gamma))_{\gamma\in\Gamma}$ in $L^2(\Gamma, \ell^2(\mathscr{P}))$ with $h(\gamma) := (h_p(\gamma))_{p\in\mathscr{P}}$ in $\ell^2(\mathscr{P})$ for each $\gamma \in \Gamma$, the synthesis operator

Θ_Ψ associated to $E^\Gamma(\Psi)$ is defined by

$$\Theta_\Psi : L^2(\Gamma, \ell^2(\mathscr{P})) \to L^2(G); \quad h \mapsto \sum_{p\in\mathscr{P}} \int_\Gamma h_p(\gamma)\rho(\gamma)\psi_p d\mu_\Gamma(\gamma). \tag{2.3}$$

Note that Θ_Ψ is well-defined, linear and bounded [78, Theorem 2.6], and hence its adjoint

$$\Theta_\Psi^* : L^2(G) \to L^2(\Gamma, \ell^2(\mathscr{P})); \quad f \mapsto \Big((\langle f, \rho(\gamma)\psi_p\rangle)_{p\in\mathscr{P}} \Big)_{\gamma\in\Gamma}, \tag{2.4}$$

is also linear and bounded, called the *analysis operator* corresponding to $E^\Gamma(\Psi)$. Now, by composing the analysis and synthesis operators of two Γ-TI Bessel systems $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$, define another bounded operator called *mixed dual-Gramian operator* as follows:

$$\Theta_\Phi\Theta_\Psi^* : L^2(G) \to L^2(G); \quad f \mapsto \sum_{p\in\mathscr{P}} \int_\Gamma \langle f, \rho(\gamma)\psi_p\rangle \rho(\gamma)\varphi_p d\mu_\Gamma(\gamma). \tag{2.5}$$

Definition 2.3. Let $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$ be Γ-TI Bessel (frame) systems in the Γ-TI space $S^\Gamma(\Psi) = S^\Gamma(\Phi)$. Then, if the operator $\Theta_\Phi\Theta_\Psi^*$ as defined in (2.5) is the zero operator, we call $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$ as *pairwise orthogonal* (simply, *orthogonal*) Γ-*TI Bessel* (*frame*) *systems*. In this case, we say that the Γ-TI Bessel (frame) systems satisfy the *orthogonality* property.

In the above definition, it should be noted that the desired equality $S^\Gamma(\Psi) = S^\Gamma(\Phi)$ also poses some conditions on the families $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$.

2.2.2. Pre-Gramian operator associated with Γ-TI systems

In this part, we extend the notion of pre-Gramian of Ron and Shen [79] to the case of LCA groups. The following is an important observation needed in the sequel:

Remark 2.4. We remark that for a countable family $\Psi = \{\psi_p\}_{p\in\mathscr{P}}$ in $L^2(G)$, $E^\Gamma(\Psi)$ is a Γ-TI Bessel system in $S^\Gamma(\Psi)$ with bound $\mathcal{B}$ if, and only if, we have

$$\sum_{\alpha\in\Gamma^\perp} \sum_{p\in\mathscr{P}} |\widehat{\psi}_p(w+\alpha)|^2 \le \mathcal{B}, \quad \text{for a.e. } w \in \Omega.$$

It is pertinent to note that the above fact can be proved easily by applying the technique used for the characterization result on frames obtained in [10].

Pre-Gramian operator: Now, we are ready to associate $E^\Gamma(\Psi)$ with a collection of 'fiber operators'. The fibers are indexed by Ω. We have the following definition:

Definition 2.5. For a.e. $w \in \Omega$, the fiber $\mathcal{J}_{\mathcal{G}}^{\Psi}(w)$, called the *pre-Gramian* operator (simply, *pre-Gramian*) associated with a Γ-TI Bessel system $E^{\Gamma}(\Psi)$, is defined by

$$\mathcal{J}_{\mathcal{G}}^{\Psi}(w) : \ell^2(\mathscr{P}) \to \ell^2(\Gamma^{\perp});$$
$$\eta \mapsto (\mathcal{J}_{\mathcal{G}}^{\Psi}(w))\eta = \left\{ \sum_{p\in\mathscr{P}} \eta(p)\widehat{\psi}_p(w+\alpha) \right\}_{\alpha\in\Gamma^{\perp}}.$$

Note that Remark 2.4 and the upcoming calculations show that the Bessel property of $E^{\Gamma}(\Psi)$ plays a very important role in the well-definedness of the pre-Gramian operator. Therefore, for *the rest of this chapter, we assume that $E^{\Gamma}(\Psi)$ and $E^{\Gamma}(\Phi)$ are Γ-TI Bessel systems in $S^{\Gamma}(\Psi)$ with $S^{\Gamma}(\Psi) = S^{\Gamma}(\Phi)$.*

We mention that a family of the form $\{\mathcal{J}_{\mathcal{G}}^{\Psi}(w)\}_{w\in\Omega}$ is called a collection of fiber operators, also known as *fiberization of the synthesis operator* Θ_{Ψ} (as defined in (2.3)) corresponding to the Γ-TI Bessel system $E^{\Gamma}(\Psi)$. Hence, $\{\mathcal{J}_{\mathcal{G}}^{\Psi}(w)\}_{w\in\Omega}$ can be regarded as the synthesis operator of $E^{\Gamma}(\Psi)$ represented in Fourier domain. This fact has been broadly studied for the case of $L^2(\mathbb{R}^d)$ by Ron and Shen in [79].

For a.e. $w \in \Omega$, clearly $\mathcal{J}_{\mathcal{G}}^{\Psi}(w)$ is a linear operator. Further, it is a well-defined and bounded operator since in view of Cauchy-Schwarz inequality and Remark 2.4, the following estimates hold:

$$\begin{aligned}
||(\mathcal{J}_{\mathcal{G}}^{\Psi}(w))\eta||^2 &= \sum_{\alpha\in\Gamma^{\perp}} \Big| \sum_{p\in\mathscr{P}} \eta(p)\widehat{\psi}_p(w+\alpha)\Big|^2 \\
&\le \sum_{\alpha\in\Gamma^{\perp}} ||\eta||^2 \Big(\sum_{p\in\mathscr{P}} |\widehat{\psi}_p(w+\alpha)|^2 \Big) \\
&< \infty, \quad \text{for all } \eta \in \ell^2(\mathscr{P}) \text{ and a.e. } w \in \Omega.
\end{aligned}$$

Furthermore, $\mathcal{J}_{\mathcal{G}}^{\Psi}(w)$ can be associated with a matrix whose rows are indexed by $\Gamma^{\perp}$, and whose columns are indexed by $\mathscr{P}$. For each $w \in \Omega$, let the symbol $\mathcal{M}_{\mathcal{G}}^{\Psi}(w)$ denote the matrix associated to $\mathcal{J}_{\mathcal{G}}^{\Psi}(w)$ with $(\alpha, p) \in \Gamma^{\perp} \times \mathscr{P}$ entry defined by $\widehat{\psi}_p(w+\alpha)$, and hence $\mathcal{M}_{\mathcal{G}}^{\Psi}(w) = \Big(\widehat{\psi}_p(w+\alpha)\Big)_{\alpha\in\Gamma^{\perp}, p\in\mathscr{P}}$. In this case, $(\mathcal{M}_{\mathcal{G}}^{\Psi}(w))^*$ represents the adjoint of $\mathcal{M}_{\mathcal{G}}^{\Psi}(w)$ with $(\mathcal{M}_{\mathcal{G}}^{\Psi}(w))^* = \Big(\overline{\widehat{\psi}_p(w+\alpha)}\Big)_{p\in\mathscr{P}, \alpha\in\Gamma^{\perp}}$.

Since $\mathcal{J}_{\mathcal{G}}^{\Psi}(w)$ is linear, well-defined, and bounded, we define the adjoint of $\mathcal{J}_{\mathcal{G}}^{\Psi}(w)$ as

$$(\mathcal{J}_{\mathcal{G}}^{\Psi}(w))^* : \ell^2(\Gamma^{\perp}) \to \ell^2(\mathscr{P});$$
$$\vartheta \mapsto ((\mathcal{J}_{\mathcal{G}}^{\Psi}(w))^*)\vartheta = \Big(\big\langle \vartheta, \{\widehat{\psi}_p(w+\alpha)\}_{\alpha\in\Gamma^{\perp}} \big\rangle \Big)_{p\in\mathscr{P}},$$

which is well-defined because $\left\{\widehat{\psi}_p(w+\alpha)\right\}_{\alpha\in\Gamma^\perp}$ belongs to $\ell^2(\Gamma^\perp)$ for each $p \in \mathscr{P}$, and

$$\begin{aligned}\left\langle \left((\mathcal{J}_{\mathcal{G}}^{\Psi}(w))^* \right)\vartheta, \eta \right\rangle &= \left\langle \vartheta, (\mathcal{J}_{\mathcal{G}}^{\Psi}(w))\eta \right\rangle = \left\langle \vartheta, \left\{ \sum_{p\in\mathscr{P}} \eta(p)\widehat{\psi}_p(w+\alpha) \right\}_{\alpha\in\Gamma^\perp} \right\rangle \\ &= \sum_{p\in\mathscr{P}} \overline{\eta(p)} \langle \vartheta, \{\widehat{\psi}_p(w+\alpha)\}_{\alpha\in\Gamma^\perp} \rangle \\ &= \left\langle \left(\langle \vartheta, \{\widehat{\psi}_p(w+\alpha)\}_{\alpha\in\Gamma^\perp} \rangle \right)_{p\in\mathscr{P}},\ \eta \right\rangle,\end{aligned}$$

is satisfied for all $\eta \in \ell^2(\mathscr{P})$. It follows that $(\mathcal{J}_{\mathcal{G}}^{\Psi}(w))^*$ is a bounded operator in view of Cauchy-Schwarz inequality, Remark 2.4 and the fact that $\left\{\widehat{\psi}_p(w+\alpha)\right\}_{\alpha\in\Gamma^\perp}$ is an element in $\ell^2(\Gamma^\perp)$ for each $p \in \mathscr{P}$.

Further, we term the collection $\left\{(\mathcal{J}_{\mathcal{G}}^{\Psi}(w))^*\right\}_{w\in\Omega}$ as the *fiberization of the analysis operator* Θ_Ψ^* (as defined in (2.4)) corresponding to the Γ-TI system $E^\Gamma(\Psi)$.

Now, for each $w \in \Omega$, by the notation $\widetilde{\mathbb{G}}^{\Psi}(w) := (\mathcal{J}_{\mathcal{G}}^{\Psi}(w))^* \mathcal{J}_{\mathcal{G}}^{\Psi}(w)$, we denote the *Gramian operator* corresponding to $E^\Gamma(\Psi)$, and, by the symbol $\mathbb{G}^{\Psi,\Phi}(w)$, we define the *mixed dual-Gramian operator* associated to the Γ-TI systems $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$ in terms of the pre-Gramain, where

$$\mathbb{G}^{\Psi,\Phi}(w) := \mathcal{J}_{\mathcal{G}}^{\Psi}(w)(\mathcal{J}_{\mathcal{G}}^{\Phi}(w))^* : \ell^2(\Gamma^\perp) \to \ell^2(\Gamma^\perp)$$

is a bounded operator by observing that the pre-Gramian $\mathcal{J}_{\mathcal{G}}^{\Psi}(w)$ is bounded, and the computation

$$||(\mathbb{G}^{\Psi,\Phi}(w))\vartheta||^2 = ||\mathcal{J}_{\mathcal{G}}^{\Psi}(w)\big(((\mathcal{J}_{\mathcal{G}}^{\Phi}(w))^*)\vartheta\big)||^2 < \infty, \quad \text{for all } \vartheta \in \ell^2(\Gamma^\perp).$$

Further, note that for all $\vartheta_1, \vartheta_2 \in \ell^2(\Gamma^\perp)$, we can write

$$\begin{aligned}\left\langle \big(\mathbb{G}^{\Psi,\Phi}(w)\big)\vartheta_1, \vartheta_2 \right\rangle &= \left\langle \left((\mathcal{J}_{\mathcal{G}}^{\Phi}(w))^* \right)\vartheta_1, \left((\mathcal{J}_{\mathcal{G}}^{\Psi}(w))^* \right)\vartheta_2 \right\rangle \\ &= \sum_{p\in\mathscr{P}} \left\langle \vartheta_1, \{\widehat{\varphi}_p(w+\alpha)\}_{\alpha\in\Gamma^\perp} \right\rangle \overline{\left\langle \vartheta_2, \{\widehat{\psi}_p(w+\beta)\}_{\beta\in\Gamma^\perp} \right\rangle}.\end{aligned}$$

Therefore, we get

$$\left\langle \big(\mathbb{G}^{\Psi,\Phi}(w)\big)\vartheta_1, \vartheta_2 \right\rangle = \sum_{p\in\mathscr{P}} \sum_{\alpha\in\Gamma^\perp} \sum_{\beta\in\Gamma^\perp} \vartheta_1(\alpha)\overline{\vartheta_2(\beta)}\widehat{\psi}_p(w+\beta)\overline{\widehat{\varphi}_p(w+\alpha)}. \tag{2.6}$$

We say that the collection $\left\{\mathbb{G}^{\Psi,\Phi}(w)\right\}_{w\in\Omega}$ is the *mixed dual-Gramian fiberization* of the operator $\Theta_\Phi\Theta_\Psi^*$ (as defined in (2.5)) corresponding to the Γ-TI systems $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$.

2.2.3. Fiberization in terms of the pre-Gramian operator

In this subsection, we first focus in proving some results which are required for the fiberization of analysis and synthesis operators associated with $E^{\Gamma}(\Psi)$. For this, next we define the space $L^2(\widehat{\Gamma}, \ell^2(\mathscr{P}))$ which appears as the image under the Fourier transform (FT) of the space $L^2(\Gamma, \ell^2(\mathscr{P}))$:

$$L^2(\widehat{\Gamma}, \ell^2(\mathscr{P})) := \big\{\text{measurable } \zeta : \widehat{\Gamma} \to \ell^2(\mathscr{P}) \text{ with } ||\zeta||^2 := \int_{\widehat{\Gamma}} ||\zeta(v)||^2_{\ell^2(\mathscr{P})} d\mu_{\widehat{\Gamma}}(v) < \infty\big\}.$$

Observe that $L^2(\widehat{\Gamma}, \ell^2(\mathscr{P}))$ is a Hilbert space of $\ell^2(\mathscr{P})$-valued square integrable functions over $\widehat{\Gamma}$ with inner product

$$\langle \zeta^1, \zeta^2 \rangle := \int_{\widehat{\Gamma}} \langle \zeta^1(v), \zeta^2(v) \rangle_{\ell^2(\mathscr{P})} d\mu_{\widehat{\Gamma}}(v),$$

for all $\zeta^i := (\zeta^i(v))_{v \in \widehat{\Gamma}}$ in $L^2(\widehat{\Gamma}, \ell^2(\mathscr{P}))$ with $i = 1, 2$, and $\zeta^i(v) := (\zeta^i_p(v))_{p \in \mathscr{P}}$ in $\ell^2(\mathscr{P})$ for each v. Since every element v in $\widehat{\Gamma} \cong \widehat{G}/\Gamma^{\perp}$ can be considered as an element in a fundamental domain $\Omega \subset \widehat{G}$ of the discrete subgroup $\Gamma^{\perp}$, we identify the space $L^2(\widehat{\Gamma}, \ell^2(\mathscr{P}))$ with $L^2(\Omega, \ell^2(\mathscr{P}))$ for the remainder of this chapter.

Proposition 2.6. *Let $E^{\Gamma}(\Psi)$ be a Γ-TI Bessel system in $S^{\Gamma}(\Psi)$ with Θ^*_{Ψ} as its analysis operator (defined in (2.4)). For $p \in \mathscr{P}$ and f in $L^2(G)$, let $(\Theta^*_{\Psi} f)_p := (\langle f, \rho(\gamma)\psi_p \rangle)_{\gamma \in \Gamma}$. Then, the following assertions are true:*

(i) *For each $p \in \mathscr{P}$, the FT of $(\Theta^*_{\Psi} f)_p$, that is, $\widehat{(\Theta^*_{\Psi} f)}_p : \Omega \to \mathbb{C}$ is given by*

$$\widehat{(\Theta^*_{\Psi} f)}_p(v) = \sum_{\alpha \in \Gamma^{\perp}} \widehat{f}(v+\alpha)\overline{\widehat{\psi}_p(v+\alpha)}.$$

Further, it is well-defined, belongs to $L^1(\Omega)$, and satisfies that

$$\widehat{(\Theta^*_{\Psi} f)}_p(v+\beta) = \widehat{(\Theta^*_{\Psi} f)}_p(v), \quad \text{for all } v \in \Omega, \text{ and } \beta \in \Gamma^{\perp}. \tag{2.7}$$

(ii) *The FT of $\Theta^*_{\Psi} f$ is given by $\widehat{\Theta^*_{\Psi} f} := \big(\widehat{\Theta^*_{\Psi} f}(v)\big)_{v \in \Omega}$ which is an element in $L^2(\Omega, \ell^2(\mathscr{P}))$ with $\widehat{\Theta^*_{\Psi} f}(v) := \big(\widehat{(\Theta^*_{\Psi} f)}_p(v)\big)_{p \in \mathscr{P}}$ for each $v \in \Omega$.*

Proof. Let $f \in L^2(G)$ and $p \in \mathscr{P}$. For each $v \in \Omega$, let $\mathrm{C}_p(v) := \sum_{\alpha \in \Gamma^{\perp}} \widehat{f}(v+\alpha)\overline{\widehat{\psi}_p(v+\alpha)}$. Then, from Weil's formula (1.1) and the relation between Ω and the dual group of Γ, it follows that

$$\int_{\Omega} \sum_{\alpha \in \Gamma^{\perp}} |\widehat{f}(v+\alpha)\overline{\widehat{\psi}_p(v+\alpha)}| d\mu_{\widehat{G}}(v) = \int_{\widehat{G}} |\widehat{f}(v)\overline{\widehat{\psi}_p(v)}| d\mu_{\widehat{G}}(v),$$

which is finite by the Cauchy-Schwarz inequality. This implies that $(\mathrm{C}_p(v))_{v\in\Omega} \in L^1(\Omega)$. Note that

$$\begin{aligned}
\mathrm{C}_p(v+\beta) &= \sum_{\alpha\in\Gamma^\perp} \widehat{f}(v+\alpha+\beta)\overline{\widehat{\psi}_p(v+\alpha+\beta)} \\
&= \sum_{\alpha\in\Gamma^\perp} \widehat{f}(v+\alpha)\overline{\widehat{\psi}_p(v+\alpha)} \\
&= \mathrm{C}_p(v), \quad \text{for all } v\in\Omega \text{ and } \beta\in\Gamma^\perp,
\end{aligned}$$

by changing the summation variable $\alpha \to \alpha-\beta$. Now, the proof for part (i) follows from the fact that

$$\begin{aligned}
(\widehat{\Theta_\Psi^* f})_p(v) &= \int_\Gamma \langle f, \rho(\gamma)\psi_p\rangle \overline{v(\gamma)} d\mu_\Gamma(\gamma) \\
&= \int_\Gamma \langle \widehat{f}, \widehat{\rho(\gamma)\psi_p}\rangle \overline{v(\gamma)} d\mu_\Gamma(\gamma) \\
&= \int_\Gamma \Big(\int_{\widehat{G}} \widehat{f}(\xi)\overline{(\widehat{\rho(\gamma)\psi_p})(\xi)} d\mu_{\widehat{G}}(\xi)\Big) \overline{v(\gamma)} d\mu_\Gamma(\gamma), \quad \text{for a.e. } v\in\Omega,
\end{aligned}$$

in view of the Parseval's formula. The above expression equivalently provides the following form by using Weil's formula (1.1) and the identity $(\widehat{\rho(\gamma)\psi_p})(\xi) = \overline{\xi(\gamma)}\widehat{\psi}_p(\xi)$, for all $\xi\in\widehat{G}$:

$$\begin{aligned}
(\widehat{\Theta_\Psi^* f})_p(v) &= \int_\Gamma \Big(\int_{\widehat{G}} \widehat{f}(\xi)\xi(\gamma)\overline{\widehat{\psi}_p(\xi)} d\mu_{\widehat{G}}(\xi)\Big) \overline{v(\gamma)} d\mu_\Gamma(\gamma) \\
&= \int_\Gamma \Big(\int_\Omega \sum_{\alpha\in\Gamma^\perp} \widehat{f}(\widetilde{v}+\alpha)\widetilde{v}(\gamma)\overline{\widehat{\psi}_p(\widetilde{v}+\alpha)} d\mu_{\widehat{G}}(\widetilde{v})\Big) \overline{v(\gamma)} d\mu_\Gamma(\gamma) \\
&= \int_\Gamma \overline{v(\gamma)} \Big(\int_\Omega \widetilde{v}(\gamma)\mathrm{C}_p(\widetilde{v}) d\mu_{\widehat{G}}(\widetilde{v})\Big) d\mu_\Gamma(\gamma) \\
&= \int_\Gamma (\mathcal{F}^{-1}(\mathrm{C}_p))(\gamma)\overline{v(\gamma)} d\mu_\Gamma(\gamma) \\
&= \mathcal{F}(\mathcal{F}^{-1}(\mathrm{C}_p))(v) \\
&= \mathrm{C}_p(v), \quad \text{for a.e. } v\in\Omega,
\end{aligned}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the FT on $L^2(\Gamma)$ and the inverse FT on $L^2(\Omega)$, respectively.

For proving part (ii), note that $\widehat{\Theta_\Psi^* f}$ is an element of $L^2(\Omega, \ell^2(\mathscr{P}))$ since for every $f\in L^2(G)$, the expression given by

$$\begin{aligned}\int_{\Omega}\|((\widehat{\Theta_{\Psi}^{*}f})_p(v))_{p\in\mathscr{P}}\|^2 d\mu_{\widehat{G}}(v) &= \int_{\Omega}\sum_{p\in\mathscr{P}}|(\widehat{\Theta_{\Psi}^{*}f})_p(v)|^2 d\mu_{\widehat{G}}(v)\\ &= \int_{\Omega}\sum_{p\in\mathscr{P}}|\sum_{\alpha\in\Gamma^{\perp}}\widehat{f}(v+\alpha)\overline{\widehat{\psi}_p(v+\alpha)}|^2 d\mu_{\widehat{G}}(v)\\ &\le \int_{\Omega}\big(\sum_{\alpha\in\Gamma^{\perp}}|\widehat{f}(v+\alpha)|^2\big)\sum_{p\in\mathscr{P}}\big(\sum_{\alpha\in\Gamma^{\perp}}|\widehat{\psi}_p(v+\alpha)|^2\big)d\mu_{\widehat{G}}(v),\end{aligned}$$

is finite by using Remark 2.4, Cauchy-Schwarz inequality and Weil's formula. Hence, the result follows.

□

Proposition 2.7. *Let the operator Θ_{Ψ} be as defined in (2.3). Then for $h\in L^2(\Gamma,\ell^2(\mathscr{P}))$, the Fourier transform of $\Theta_{\Psi}h$ is given by*

$$\widehat{(\Theta_{\Psi}h)}(\xi) = \sum_{p\in\mathscr{P}}\widehat{h}_p(\xi)\widehat{\psi}_p(\xi), \quad \textit{for a.e. } \xi\in\Omega.$$

Proof. Let $h\in L^2(\Gamma,\ell^2(\mathscr{P}))$. Then, using the definition of the operator Θ_{Ψ}, we can write

$$\begin{aligned}\widehat{(\Theta_{\Psi}h)}(\xi) &= \int\limits_{G}\sum_{p\in\mathscr{P}}\int\limits_{\Gamma}h_p(\gamma)\psi_p(x-\gamma)d\mu_{\Gamma}(\gamma)\overline{\xi(x)}d\mu_G(x)\\ &= \sum_{p\in\mathscr{P}}\int\limits_{G}\int\limits_{\Gamma}h_p(\gamma)\psi_p(y)\overline{\xi(y+\gamma)}d\mu_{\Gamma}(\gamma)d\mu_G(y),\end{aligned}$$

which further equals to the expression given by

$$\begin{aligned}&\sum_{p\in\mathscr{P}}\int\limits_{G}\bigg(\int\limits_{\Gamma}h_p(\gamma)\overline{\xi(\gamma)}d\mu_{\Gamma}(\gamma)\bigg)\psi_p(y)\overline{\xi(y)}d\mu_G(y)\\ &= \sum_{p\in\mathscr{P}}\widehat{h}_p(\xi)\bigg(\int\limits_{G}\psi_p(y)\overline{\xi(y)}d\mu_G(y)\bigg)\\ &= \sum_{p\in\mathscr{P}}\widehat{h}_p(\xi)\widehat{\psi}_p(\xi), \text{ for a.e. } \xi\in\Omega.\end{aligned}$$

□

The following result establishes a relation which represents the fiberization of operators associated with the Γ-TI Bessel systems $E^{\Gamma}(\Psi)$ and $E^{\Gamma}(\Phi)$ via the pre-Gramian operator defined in Definition 2.5.

Theorem 2.8. *For each f in $L^2(G)$, and for a.e. $\kappa \in \Omega$, the following expressions hold:*

$$\big(\widehat{(\Theta_\Psi^* f)}_p(\kappa)\big)_{p\in\mathscr{P}} = (\mathcal{J}_{\mathcal{G}}^{\Psi}(\kappa))^* \big(\widehat{f}(\kappa+\alpha)\big)_{\alpha\in\Gamma^\perp}, \tag{2.8}$$

$$\big(\widehat{\Theta_\Psi h}(\kappa+\alpha)\big)_{\alpha\in\Gamma^\perp} = \mathcal{J}_{\mathcal{G}}^{\Psi}(\kappa)\big(\ \widehat{h}_p(\kappa+\alpha)\big)_{p\in\mathscr{P}}, \ \textit{for all } h \in L^2(\Gamma, \ell^2(\mathscr{P})), \textit{and} \tag{2.9}$$

$$\big(\big(\widehat{\Theta_\Psi\Theta_\Phi^* f}\big)(\kappa+\alpha)\big)_{\alpha\in\Gamma^\perp} = \mathbb{G}^{\Psi,\Phi}(\kappa)\big(\widehat{f}(\kappa+\alpha)\big)_{\alpha\in\Gamma^\perp}, \tag{2.10}$$

where for a.e. $\kappa \in \Omega$, the symbol $\mathbb{G}^{\Psi,\Phi}(\kappa) = \mathcal{J}_{\mathcal{G}}^{\Psi}(\kappa)(\mathcal{J}_{\mathcal{G}}^{\Phi})^*(\kappa)$ denotes the mixed dual-Gramian operator corresponding to the Γ-TI Bessel systems $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$.

Proof. Let a.e. $\kappa \in \Omega$. Then, the proof for expression (2.8) follows by using the definition of operator $(\mathcal{J}_{\mathcal{G}}^{\Psi}(\kappa))^*$ along with Proposition 2.6.

Further, (2.9) holds by observing $\mathcal{J}_{\mathcal{G}}^{\Psi}(\kappa)$ from Definition 2.5, Proposition 2.7, and

$$\big(\widehat{\Theta_\Psi h}(\kappa+\alpha)\big)_{\alpha\in\Gamma^\perp} = \Big(\sum_{p\in\mathscr{P}} \widehat{h}_p(\kappa+\alpha)\widehat{\psi}_p(\kappa+\alpha)\Big)_{\alpha\in\Gamma^\perp}, \text{ for all } h \in L^2(\Gamma, \ell^2(\mathscr{P})).$$

Now, (2.10) follows by using the equalities (2.7), (2.8) and (2.9) in the computation:

$$\begin{aligned}\big(\big(\widehat{\Theta_\Psi(\Theta_\Phi^* f)}\big)(\kappa+\alpha)\big)_{\alpha\in\Gamma^\perp} &= \mathcal{J}_{\mathcal{G}}^{\Psi}(\kappa)\big(\widehat{(\Theta_\Phi^* f)}_p(\kappa+\alpha)\big)_{p\in\mathscr{P}} \\ &= \mathcal{J}_{\mathcal{G}}^{\Psi}(\kappa)\big(\widehat{(\Theta_\Phi^* f)}_p(\kappa)\big)_{p\in\mathscr{P}} \\ &= \mathcal{J}_{\mathcal{G}}^{\Psi}(\kappa)(\mathcal{J}_{\mathcal{G}}^{\Phi}(\kappa))^*\big(\widehat{f}(\kappa+\alpha)\big)_{\alpha\in\Gamma^\perp}, \text{ for all } f \in L^2(G).\end{aligned}$$

□

2.3. Orthogonality of Γ-TI frame systems over LCA groups

In this section, we characterize a pair of orthogonal Γ-TI Bessel (frame) systems over locally compact abelian (LCA) groups. For this, we use the notion of pre-Gramian operator in LCA-group setting along with the fiberization of operators associated with the Γ-TI systems which we have studied in Theorem 2.8. The next result characterizes the frame/Bessel property of a Γ-TI system in terms of the pre-Gramian operator. We refer [10, 12] for the proof of the following result:

Proposition 2.9. *$E^\Gamma(\Psi)$ is a Γ-TI frame system for $S^\Gamma(\Psi)$ if, and only if, $\mathcal{J}_{\mathcal{G}}^{\Psi}(w)(\mathcal{J}_{\mathcal{G}}^{\Psi}(w))^*$ is uniformly bounded with uniformly bounded inverse on the range of $\mathcal{J}_{\mathcal{G}}^{\Psi}(w)$ for a.e.*

$w \in \Omega$ such that ran $\mathcal{J}_{\mathcal{G}}^{\Psi}(w) \neq \{0\}$. In particular, if $S^{\Gamma}(\Psi) = L^2(G)$, then the following are equivalent:

(i) *The system $E^{\Gamma}(\Psi)$ is a Γ-TI frame system for $L^2(G)$,*

(ii) *there exist constants $0 < \mathcal{A} \leq \mathcal{B} < \infty$, such that*

$$\mathcal{A}I_{\ell^2(\Gamma^{\perp})} \leq \mathcal{J}_{\mathcal{G}}^{\Psi}(w)(\mathcal{J}_{\mathcal{G}}^{\Psi}(w))^* \leq \mathcal{B}I_{\ell^2(\Gamma^{\perp})}, \quad \text{for a.e. } w \in \Omega.$$

In addition, $E^{\Gamma}(\Psi)$ is a tight Γ-TI frame system with frame bound 1 for $L^2(G)$ if, and only if, $\mathcal{J}_{\mathcal{G}}^{\Psi}(w)(\mathcal{J}_{\mathcal{G}}^{\Psi}(w))^* = I_{\ell^2(\Gamma^{\perp})}$, for a.e. $w \in \Omega$.

Proof. The result follows from [10,12]. The key for the proof lies in using the computation done in (2.6) for the case of $\vartheta_1 = \vartheta_2 \in \ell^2(\Gamma^{\perp})$ along with the characterization result on frames obtained in [10]. $\square$

2.3.1. Characterizations of pairwise orthogonal Γ-TI Bessel (frame) systems

The following results provide necessary and sufficient conditions on a pair of Γ-TI frame systems to be orthogonal in the sense of Definition 2.3:

Theorem 2.10. *Let Ψ and Φ be countable subsets of $L^2(G)$. Suppose that $S^{\Gamma}(\Psi) = S^{\Gamma}(\Phi)$, and that both $E^{\Gamma}(\Psi)$ and $E^{\Gamma}(\Phi)$ are Γ-TI frame systems for $S^{\Gamma}(\Psi)$. Then, the following are equivalent:*

(i) *$E^{\Gamma}(\Psi)$ and $E^{\Gamma}(\Phi)$ form orthogonal Γ-TI frame systems in $S^{\Gamma}(\Psi)$.*

(ii) *$\mathcal{J}_{\mathcal{G}}^{\Psi}(\kappa)(\mathcal{J}_{\mathcal{G}}^{\Phi}(\kappa))^* \mathcal{J}_{\mathcal{G}}^{\Phi}(\kappa) = 0$ for a.e. $\kappa \in \Omega$.*

(iii) *$\mathcal{M}_{\mathcal{G}}^{\Psi}(\kappa)(\mathcal{M}_{\mathcal{G}}^{\Phi}(\kappa))^* \mathcal{M}_{\mathcal{G}}^{\Phi}(\kappa) = 0$ for a.e. $\kappa \in \Omega$.*

(iv) *$\widetilde{\mathbb{G}}^{\Psi}(\kappa)\widetilde{\mathbb{G}}^{\Phi}(\kappa) = 0$ for a.e. $\kappa \in \Omega$.*

In particular, when $S^{\Gamma}(\Psi) = L^2(G)$, $E^{\Gamma}(\Psi)$ and $E^{\Gamma}(\Phi)$ are pairwise orthogonal Γ-TI Bessel (frame) systems in $L^2(G)$ if, and only if, $\mathcal{J}_{\mathcal{G}}^{\Psi}(\kappa)(\mathcal{J}_{\mathcal{G}}^{\Phi}(\kappa))^* = 0$ for a.e. $\kappa \in \Omega$.

For proving Theorem 2.10, first we need to describe the decomposition theorem for a Γ-TI space of $L^2(G)$. Before stating this result, we have the following notation. Let $\psi \in L^2(G)$. We denote by $L^2(\Omega, \omega_{\psi})$, the space of all functions $r : \Omega \to \mathbb{C}$, which satisfy $\int_{\Omega} |r(\xi)|^2 \omega_{\psi}(\xi) d\mu_{\widehat{G}}(\xi) < \infty$, where $\omega_{\psi}(\xi) = \sum_{\alpha \in \Gamma^{\perp}} |\widehat{\psi}(\xi + \alpha)|^2$ for each $\xi \in \Omega$. Observe that by using the relation between the dual group of Γ with Ω, Weil's formula and the

Plancherel theorem, we can write

$$\int_{\Omega} \sum_{\alpha\in\Gamma^{\perp}} |\widehat{\psi}(\xi+\alpha)|^2 d\mu_{\widehat{G}}(\xi) = \int_{\widehat{G}} |\widehat{\psi}(\xi)|^2 d\mu_{\widehat{G}}(\xi) = ||\psi||^2.$$

Thus, $\omega := \{\omega_{\psi}(\xi)\}_{\xi\in\Omega}$ is a function in $L^1(\Omega)$. Note that in this case

$$||r||^2_{L^2(\Omega,\omega_{\psi})} = \int_{\Omega} |r(\xi)|^2 \omega_{\psi}(\xi) d\mu_{\widehat{G}}(\xi)$$

is a norm in $L^2(\Omega, \omega_{\psi})$. Further, we denote the support of ω_{ψ} by the set

$$\{\xi \in \Omega : \omega_{\psi}(\xi) \neq 0\} =: \mathbb{S}_{\psi}.$$

Here, note that the set $\mathbb{S}_{\psi}$ is called the *spectrum* of $S^{\Gamma}(\psi)$.

The next result shows the existence of a decomposition of a Γ-TI space of $L^2(G)$ into an orthogonal sum of spaces each of which is generated by a single function whose translates form a tight Γ-PTI frame system with frame bound 1.

Theorem 2.11. *Let V be a Γ-TI space of $L^2(G)$. Then, there exists a family of functions $\{\psi_n\}_{n\in\mathbb{N}}$ in V such that V can be decomposed as an orthogonal sum $V = \bigoplus_{n\in\mathbb{N}} S^{\Gamma}(\psi_n)$, and $E^{\Gamma}(\psi_n)$ is a tight Γ-PTI frame system in the Γ-PTI space $S^{\Gamma}(\psi_n)$ with frame bound 1. Moreover, $f \in V$ if, and only if,*

$$\widehat{f}(\xi) = \sum_{n\in\mathbb{N}} r_n(\xi)\widehat{\psi}_n(\xi), \text{ and hence } ||f||^2 = \sum_{n\in\mathbb{N}} ||r_n||^2_{L^2(\Omega\cap\mathbb{S}_{\psi_n},\omega_{\psi_n})}, \tag{2.11}$$

where $r_n \in L^2(\Omega \cap \mathbb{S}_{\psi_n}, w_{\psi_n})$ and $\mathbb{S}_{\psi_n}$ is the spectrum of $S^{\Gamma}(\psi_n)$, for every $n \in \mathbb{N}$.

Proof. The first part of the proof follows from [10, Theorem 5.3]. For the moreover part, let $n \in \mathbb{N}$ and $\psi_n \in L^2(G)$. Then, for each n by following the steps of [46, Proposition 2.2], we get $f_n \in S^{\Gamma}(\psi_n)$ if, and only if, $\widehat{f_n}(\xi) = r_n(\xi)\widehat{\psi}_n(\xi)$, for some $r_n \in L^2(\Omega, w_{\psi_n})$. Now, if (2.11) holds, then clearly in view of the above discussion $f \in V$.

Conversely, let P_n be the orthogonal projection onto the space $S^{\Gamma}(\psi_n)$. Note that for each $f \in V$, we have $f = \sum_{n\in\mathbb{N}} P_n f$. Thus,

$$\widehat{f} = \sum_{n\in\mathbb{N}} (\widehat{P_n f}) = \sum_{n\in\mathbb{N}} r_n \widehat{\psi}_n,$$

where $r_n \in L^2(\Omega \cap \mathbb{S}_{\psi_n}, w_{\psi_n})$ for each n. Therefore, we have

$$||f||^2 = ||\widehat{f}||^2 = \sum_{n\in\mathbb{N}} ||r_n||^2_{L^2(\Omega\cap\mathbb{S}_{\psi_n}, w_{\psi_n})} ||\widehat{\psi}_n||^2$$
$$= \sum_{n\in\mathbb{N}} ||r_n||^2_{L^2(\Omega\cap\mathbb{S}_{\psi_n}, w_{\psi_n})},$$

in view of Plancherel's formula and the fact that $E^\Gamma(\psi_n)$ is a tight Γ-PTI frame system in $S^\Gamma(\psi_n)$ with frame bound 1. Hence, the result follows. $\square$

Proof of Theorem 2.10. By following Definition 2.3, $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$ form pairwise orthogonal Γ-TI Bessel (frame) systems for $S^\Gamma(\Psi)$ if, and only if, for all $f \in S^\Gamma(\Psi)$, we have $\Theta_\Psi \Theta_\Phi^* f = 0$, which is equivalent to saying that $||\Theta_\Psi \Theta_\Phi^* f|| = 0$. Let $f \in L^2(G)$. Then, by Plancherel's formula and Weil's formula, the following expression holds:

$$(2.12)\qquad ||\Theta_\Psi \Theta_\Phi^* f||^2 = \int_{\widehat{G}} |(\widehat{\Theta_\Psi \Theta_\Phi^* f})(\xi)|^2 d\mu_{\widehat{G}}(\xi) = \sum_{\alpha\in\Gamma^\perp} \int_{\Omega} |(\widehat{\Theta_\Psi \Theta_\Phi^* f})(v+\alpha)|^2 d\mu_{\widehat{G}}(v).$$

Therefore, $\Theta_\Psi \Theta_\Phi^* f = 0$ if, and only if, in view of (2.12), $(\widehat{\Theta_\Psi \Theta_\Phi^* f})(v+\alpha) = 0$ for each $v \in \Omega$ and $\alpha \in \Gamma^\perp$. This means, $\Theta_\Psi \Theta_\Phi^* f = 0$ if, and only if, we have $((\widehat{\Theta_\Psi \Theta_\Phi^* f})(\kappa+\delta))_{\delta\in\Gamma^\perp} = 0$ for a.e. $\kappa \in \Omega$.

Further, by using Theorem 2.8 observe that for every $f \in L^2(G)$ and for a.e. $\kappa \in \Omega$, the following relation is satisfied:

$$\Big((\widehat{\Theta_\Psi \Theta_\Phi^* f})(\kappa+\alpha) \Big)_{\alpha\in\Gamma^\perp} = \mathbb{G}^{\Psi,\Phi}(\kappa) \big(\widehat{f}(\kappa+\alpha) \big)_{\alpha\in\Gamma^\perp},$$

where $\mathbb{G}^{\Psi,\Phi}(\kappa) = \mathcal{J}_{\mathcal{G}}^{\Psi}(\kappa) (\mathcal{J}_{\mathcal{G}}^{\Phi}(\kappa))^*$ is the mixed dual-Gramian operator corresponding to the Γ-TI Bessel (frame) systems $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$. Hence, it follows that for $S^\Gamma(\Psi) = L^2(G)$, we get $\Theta_\Psi \Theta_\Phi^* f = 0$ for all $f \in L^2(G)$ if, and only if, $\mathbb{G}^{\Psi,\Phi}(\kappa) = 0$ for a.e. $\kappa \in \Omega$.

Now, we start for proving the equivalence of (i) and (ii). From Theorem 2.11, we observe that $f \in S^\Gamma(\Phi)$ if, and only if, the Fourier transform of f can be written as $\widehat{f} = \sum_{p\in\mathscr{P}} r_p \widehat{\varphi}_p$ for some r_p in $L^2(\Omega \cap \mathbb{S}_{\varphi_p}, w_{\varphi_p})$, where p belongs to a countable index set $\mathscr{P}$. Moreover, in view of the above expression of $\widehat{f}$ and Definition 2.5 for the pre-Gramian operator $\mathcal{J}_{\mathcal{G}}^{\Phi}(\kappa)$, we get the following estimate:

$$\big(\widehat{f}(\kappa+\alpha) \big)_{\alpha\in\Gamma^\perp} = \mathcal{J}_{\mathcal{G}}^{\Phi}(\kappa) \big(r_p(\kappa+\alpha) \big)_{p\in\mathscr{P}} \quad \text{for a.e. } \kappa \in \Omega.$$

Thus, in view of the above discussion and the equality (2.10), part (i) is equivalent to the statement that for any $f \in S^{\Gamma}(\Phi)$, we have

$$\mathbb{G}^{\Psi,\Phi}(\kappa)\big(\widehat{f}(\kappa+\alpha)\big)_{\alpha\in\Gamma^{\perp}} = \mathcal{J}_{\mathcal{G}}^{\Psi}(\kappa)(\mathcal{J}_{\mathcal{G}}^{\Phi}(\kappa))^{*}\mathcal{J}_{\mathcal{G}}^{\Phi}(\kappa)\big(r_p(\kappa+\alpha)\big)_{p\in\mathscr{P}}, \tag{2.13}$$

equals to zero since $\Theta_{\Psi}\Theta_{\Phi}^{*}f = 0$, and hence

$$\mathcal{J}_{\mathcal{G}}^{\Psi}(\kappa)(\mathcal{J}_{\mathcal{G}}^{\Phi}(\kappa))^{*}\mathcal{J}_{\mathcal{G}}^{\Phi}(\kappa) = 0,$$

for a.e. $\kappa \in \Omega$.

Further, by using $\mathcal{M}_{\mathcal{G}}^{\Psi}(\kappa)$, that is, the matrix associated with $\mathcal{J}_{\mathcal{G}}^{\Psi}(\kappa)$, clearly (ii) holds if, and only if, the relation (iii) is satisfied.

Now, from (2.13), the equivalence of (ii) and (iv) follows since $E^{\Gamma}(\Phi)$ is a Γ-TI frame system in $S^{\Gamma}(\Phi)$, and hence for a.e. $\kappa \in \Omega$, $(\mathcal{J}_{\mathcal{G}}^{\Psi}(\kappa))^{*}$ has bounded inverse on the range of $\mathcal{J}_{\mathcal{G}}^{\Psi}(\kappa)$ by Proposition 2.9. □

The next result reflects a very useful property of pairwise orthogonal frames. By using a Γ-periodic function on G and a given pair of orthogonal Γ-TI Bessel (frame) generators, we construct another pair of generators providing orthogonal frames of the same structure. Moreover, one system in the newly constructed pair of Γ-TI Bessel (frame) systems remains the same while the second system acquires some extra properties due to the effect of a Γ-periodic function. Here, note that for a co-compact subgroup Γ of an LCA group G, a bounded function on G is called Γ-*periodic* in the sense that for every $\gamma \in \Gamma$, we have $f(x+\gamma) = f(x)$ for all $x \in G$.

Proposition 2.12. *Suppose that $E^{\Gamma}(\Psi)$ and $E^{\Gamma}(\Phi)$ are orthogonal Γ-TI Bessel (frame) systems in $L^2(G)$. Let h be a complex-valued measurable function on G which is Γ-periodic, such that the collection $h\Psi$ is a subset of $L^2(G)$, where $h\Psi$ is defined by*

$$h\Psi := \{h\psi_p \in L^2(G) : (h\psi_p)(x) = h(x)\psi_p(x);\ \ \psi_p \in \Psi, p \in \mathscr{P}, x \in G\}.$$

Then, the families $E^{\Gamma}(h\Psi)$ and $E^{\Gamma}(\Phi)$ also form orthogonal Γ-TI Bessel (frame) systems in $L^2(G)$.

Proof. Given that $E^{\Gamma}(\Psi)$ and $E^{\Gamma}(\Phi)$ form pairwise orthogonal Γ-TI Bessel (frame) systems in $L^2(G)$. Then, for all $f \in L^2(G)$, we have $\Theta_{\Psi}\Theta_{\Phi}^{*}f = 0$. That means, the following holds:

$$\sum_{p\in\mathscr{P}} \int_{\Gamma} \langle f(x), \varphi_p(x-\gamma)\rangle \psi_p(x-\gamma)\, d\mu_{\Gamma}(\gamma) = 0, \quad \text{for all } x \in G. \tag{2.14}$$

Now, the result follows by using h as a Γ-periodic function along with (2.14) in the following relation:

$$\begin{aligned}
&\sum_{p\in\mathscr{P}}\int_{\Gamma}\langle f(x),\varphi_p(x-\gamma)\rangle(h\psi_p)(x-\gamma)d\mu_\Gamma(\gamma)\\
&=\sum_{p\in\mathscr{P}}\int_{\Gamma}\langle f(x),\varphi_p(x-\gamma)\rangle h(x-\gamma)\psi_p(x-\gamma)d\mu_\Gamma(\gamma)\\
&=h(x)\sum_{p\in\mathscr{P}}\int_{\Gamma}\langle f(x),\varphi_p(x-\gamma)\rangle\psi_p(x-\gamma)d\mu_\Gamma(\gamma),
\end{aligned}$$

for all x belongs to G. □

Our next proposition provides a construction of pairwise orthogonal Bessel (frame) systems by using the similarity (equivalence) property. Here, note that for a measure space $(\mathbb{J},\mu_{\mathbb{J}})$ with $\mu_{\mathbb{J}}$ being a Haar measure on $\mathbb{J}$, we say that $X:=\{x_j\}_{j\in\mathbb{J}}$ and $Y:=\{y_j\}_{j\in\mathbb{J}}$ in the Hilbert space $\mathcal{H}$ are *similar* (*equivalent*) if there exists a bounded invertible operator $\mathbb{U}$ on $\mathcal{H}$ such that $x_j=\mathbb{U}y_j$, for all $j\in\mathbb{J}$.

Proposition 2.13. *For a countable index set $\mathscr{P}$, let $\widetilde{\Psi}:=\{\widetilde{\psi}_p\}_{p\in\mathscr{P}}$ and $\widetilde{\Phi}:=\{\widetilde{\varphi}_p\}_{p\in\mathscr{P}}$ be subsets in $L^2(G)$. Further, let $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$ be orthogonal Γ-TI Bessel (frame) systems in $L^2(G)$, and that $E^\Gamma(\Psi)$ is similar to $E^\Gamma(\widetilde{\Psi})$, and $E^\Gamma(\Phi)$ is similar to $E^\Gamma(\widetilde{\Phi})$. Then, $E^\Gamma(\widetilde{\Psi})$ and $E^\Gamma(\widetilde{\Phi})$ also form orthogonal Γ-TI Bessel (frame) systems in $L^2(G)$.*

Proof. Clearly, $E^\Gamma(\widetilde{\Psi})$ and $E^\Gamma(\widetilde{\Phi})$ are Γ-TI Bessel (frame) systems in $L^2(G)$ since similarity among the systems preserves the frame property, including the frame bounds. Further, by the definition of similarity, there exist bounded invertible operators $\mathbb{U}_1$ and $\mathbb{U}_2$ on $L^2(G)$ such that we can write $\mathbb{U}_1\rho(\gamma)\psi_p=\rho(\gamma)\widetilde{\psi}_p$ and $\mathbb{U}_2\rho(\gamma)\varphi_p=\rho(\gamma)\widetilde{\varphi}_p$ for every $p\in\mathscr{P}$. We claim that $\Theta_{\widetilde{\Psi}}=\mathbb{U}_1\Theta_\Psi$ and $\Theta_{\widetilde{\Phi}}=\mathbb{U}_2\Theta_\Phi$. For this, it is enough to prove that for all $h\in L^2(\Gamma,\ell^2(\mathscr{P}))$, we have

$$\begin{aligned}
\Theta_{\widetilde{\Psi}}h=\sum_{p\in\mathscr{P}}\int_{\Gamma}h_p(\gamma)\rho(\gamma)\widetilde{\psi}_p d\mu_\Gamma(\gamma)&=\sum_{p\in\mathscr{P}}\int_{\Gamma}h_p(\gamma)\mathbb{U}_1\rho(\gamma)\psi_p d\mu_\Gamma(\gamma)\\
&=\mathbb{U}_1\sum_{p\in\mathscr{P}}\int_{\Gamma}h_p(\gamma)\rho(\gamma)\psi_p d\mu_\Gamma(\gamma)=\mathbb{U}_1(\Theta_\Psi h),
\end{aligned}$$

and hence the result follows in view of the fact that $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$ are orthogonal in $L^2(G)$, and $\Theta_{\widetilde{\Psi}}\Theta_{\widetilde{\Phi}}^*=\mathbb{U}_1\Theta_\Psi(\mathbb{U}_2\Theta_\Phi)^*=\mathbb{U}_1\Theta_\Psi\Theta_\Phi^*\mathbb{U}_2^*=\mathbb{U}_1 0\mathbb{U}_2^*=0$, where 0 denotes the zero operator on $L^2(G)$. □

2.3.2. Application of the characterization result on co-compact Gabor systems

In this subsection, we deduce a characterization result for the case of Gabor systems. For this, next we define these structured systems as a special case of Γ-TI systems given in Definition 2.1. Let a character χ in $\widehat{G}$, and define the *modulation operator* $\eta(\chi)$ on $L^2(G)$ as $\eta(\chi)(f)(x) = \chi(x)f(x)$, for all $f \in L^2(G)$ and $x \in G$, and observe that it is associated with the translation operator on $L^2(\widehat{G})$ by

$$\begin{aligned}(\widehat{\eta(\chi)f})(\xi) &= \int_G \chi(x)f(x)\overline{\xi(x)}d\mu_G(x) = \int_G f(x)\overline{(\xi-\chi)(x)}d\mu_G(x) \\ &= \widehat{f}(\xi-\chi) = \rho(\chi)\widehat{f}(\xi), \quad \text{for a.e. } \xi \in \widehat{G}.\end{aligned}$$

Let Γ and Λ be respectively, co-compact subgroups of G and $\widehat{G}$. For an index set $J \subset \mathbb{Z}$, let $\mathscr{A} := \{f_j\}_{j\in J}$ be a subset in $L^2(G)$. Then the collection $\mathscr{G}(\mathscr{A},\Gamma,\Lambda)$ defined by

$$\mathscr{G}(\mathscr{A},\Gamma,\Lambda) := \big\{\rho(\gamma)\eta(\chi)f_j : \gamma \in \Gamma,\ \chi \in \Lambda,\ j \in J\big\}, \tag{2.15}$$

is called the *Gabor system* generated by $\mathscr{A}$. Note that $\mathscr{G}(\mathscr{A},\Gamma,\Lambda)$ is a frame for $L^2(G)$ if, and only if, $\{\eta(\chi)\rho(\gamma)f_j : \gamma \in \Gamma, \chi \in \Lambda, j \in J\}$ is a frame for $L^2(G)$, where the later system is termed as a *co-compact Gabor system* in [60]. Further, observe that $\mathscr{G}(\mathscr{A},\Gamma,\Lambda)$ is a Γ-TI system of form $\big\{\rho(\gamma)\psi_j : \gamma \in \Gamma,\ \psi_j \in \Psi,\ j \in J\big\}$ defined in Definition 2.1, with $\psi_j = \eta(\chi)f_j$, where $(j,\chi) \in J \times \Lambda$.

Let $\mathscr{A} = \{f_j\}_{j\in J}$ and $\mathscr{B} := \{h_j\}_{j\in J}$ be countable subsets of $L^2(G)$. Then, our next result gives a characterization of $\mathscr{A}$ and $\mathscr{B}$ such that $\mathscr{G}(\mathscr{A},\Gamma,\Lambda)$ and $\mathscr{G}(\mathscr{B},\Gamma,\Lambda)$ form a pair of orthogonal Bessel families (frames) in $L^2(G)$, we call as a pair of *co-compact Gabor orthogonal Bessel (frame) systems* over LCA groups.

Proposition 2.14. *Suppose that $\mathscr{G}(\mathscr{A},\Gamma,\Lambda)$ and $\mathscr{G}(\mathscr{B},\Gamma,\Lambda)$ are co-compact Gabor Bessel (frame) systems in $L^2(G)$. Then, the following assertions are equivalent:*

(i) *$\mathscr{G}(\mathscr{A},\Gamma,\Lambda)$ and $\mathscr{G}(\mathscr{B},\Gamma,\Lambda)$ form a pair of co-compact Gabor orthogonal Bessel (frame) systems in $L^2(G)$ in the sense of Definition 2.3.*

(ii) *For each $\alpha,\beta \in \Gamma^\perp$ and $\chi \in \Lambda$, we have*

$$\sum_{j\in J} \widehat{f}_j(\kappa+\alpha-\chi)\overline{\widehat{h}_j(\kappa+\beta-\chi)} = 0,$$

for a.e. $\kappa \in \Omega$.

Proof. In view of computation (2.6) and Theorem 2.10, (i) holds if, and only if, for each $\chi \in \Lambda$, we have

$$\sum_{\alpha\in\Gamma^{\perp}} \vartheta_1(\alpha) \sum_{\beta\in\Gamma^{\perp}} \overline{\vartheta_2(\beta)} \Big(\sum_{j\in J} (\widehat{\eta(\chi) f_j})(\kappa+\alpha) \overline{(\widehat{\eta(\chi) h_j})(\kappa+\beta)} \Big) = 0, \text{ for a.e. } \kappa \in \Omega,$$

and for all $\vartheta_1, \vartheta_2 \in \ell^2(\Gamma^{\perp})$. Hence, the result follows since ϑ_1, ϑ_2 are arbitrary elements of $\ell^2(\Gamma^{\perp})$. $\square$

Note that Proposition 2.14 can be used to derive various results on a pair of co-compact Gabor orthogonal Bessel (frame) systems by considering different situations on Γ, Λ and G, etc.

Example 2.15. Let $G = \mathbb{Z}^d$, and let $\Gamma = A\mathbb{Z}^d$ and $\Lambda = B\mathbb{Z}^d$ be uniform lattices in $\mathbb{Z}^d$ for some invertible $d \times d$ matrices A and B with integer entries. Then, $\Gamma^{\perp} = \widetilde{A}\mathbb{Z}^d$, where $\widetilde{A} = (A^t)^{-1}$, that is, inverse of the transpose of matrix A. In this case, (2.15) reduces to the following collection:

$$\mathscr{G}(\mathscr{A}, A\mathbb{Z}^d, B\mathbb{Z}^d) := \big\{ \rho(\gamma)\eta(\chi) f_j : \gamma \in A\mathbb{Z}^d,\ \chi \in B\mathbb{Z}^d,\ j \in J \big\}.$$

Hence, $\mathscr{G}(\mathscr{A}, A\mathbb{Z}^d, B\mathbb{Z}^d)$ and $\mathscr{G}(\mathscr{B}, A\mathbb{Z}^d, B\mathbb{Z}^d)$ form a pair of orthogonal frames in $\ell^2(\mathbb{Z}^d)$ if, and only if, for $m, n, p \in \mathbb{Z}^d$, we have

$$\sum_{j\in J} \widehat{f}_j(\kappa + \widetilde{A}m - Bp) \overline{\widehat{h}_j(\kappa + \widetilde{A}n - Bp)} = 0,$$

for a.e. $\kappa \in \widetilde{A}([0,1]^d)$.

CHAPTER 3

ORTHOGONALITY OF GENERALIZED TRANSLATION INVARIANT FRAME PAIRS

In this chapter, we give a characterization of pairwise orthogonal frames which arise from translations of generating functions via a countable family of closed, co-compact subgroups of a second countable LCA group G. We call such frames as pairwise orthogonal GTI frame systems in the sense of Definition 1.3. As an application of our characterization result, we get necessary and sufficient conditions for the orthogonality of various structured frame systems such as wave-packet, wavelet, and Gabor frame systems over LCA groups.

3.1. Pairwise orthogonal GTI frame systems

In **Chapter 2**, we have characterized the orthogonality of frame pairs for translation invariant (TI) subspaces in $L^2(G)$ by generalizing the fiberization techniques of Ron and Shen [79] over locally compact abelian (LCA) groups. In the present chapter, we study and characterize pairwise orthogonal generalized TI (GTI) Bessel systems in $L^2(G)$. The characterization is obtained in the spirit of the characterizations of duality between GTI systems investigated in [59]. From this characterization, we deduce a series of similar results for the function systems related to GTI systems. We state our first main characterization result, that is, Theorem 3.5 (along with several deductions in Remark 3.6) in Subsection 3.2.1, while the proof for the result is discussed in Subsection 3.2.2.

Our second main result, that is, Theorem 3.9 provides a characterization for pairwise orthogonal wave-packet frame systems over LCA groups. We shall discuss the definition of wave-packet systems over LCA groups along with the characterization result in Section 3.3.

For this part of section, we begin by considering GTI systems introduced by Jakobsen and Lemvig in [59] along with the following definition and the standing hypotheses on it. Note that such systems model various discrete and continuous systems, e.g., the wavelet, shearlet and Gabor systems, etc.

Definition 3.1. Let $J \subset \mathbb{Z}$ be a countable index set. For each $j \in J$, let P_j be a countable or an uncountable index set, let $g_{j,p} \in L^2(G)$ for $p \in P_j$, and let Γ_j be a closed, co-compact subgroup in G. Then, the *generalized translation invariant* (*GTI*) *system* generated by $\{g_{j,p}\}_{p\in P_j, j\in J}$ with translation along closed, co-compact subgroups $\{\Gamma_j\}_{j\in J}$ is the family $\bigcup_{j\in J}\{T_\gamma g_{j,p}\}_{\gamma\in\Gamma_j, p\in P_j}$, where for $y \in G$, T_y is the *translation* by y, which is defined as

$$T_y : L^2(G) \to L^2(G),\ (T_y f)(x) = f(x-y),\ x \in G.$$

Note that the GTI system in Definition 3.1 reduces to the *translation invariant* (*TI*) system discussed in [10], if $\Gamma_j = \Gamma$ for each $j \in J$. However, if each P_j is countable and each Γ_j is a uniform lattice in Definition 3.1, we arrive at the *generalized shift-invariant* (*GSI*) system considered in [53, 64].

Standing Hypotheses: For GTI systems considered in Definition 3.1, we assume that these systems satisfy the following criterion for the remainder of this chapter. Before proceeding for this, we introduce some notation. Let $(P_j, \sum_{P_j}, \mu_{P_j})$ be a measure space for each $j \in J$, where $J \subset \mathbb{Z}$ is a countable index set. For a topological space X, by B_X, we denote the Borel algebra of X. By the symbol $P_j \times G$, we represent the product measure space formed by the Cartesian product of G with the measure space P_j, $\sum_{P_j} \otimes B_G$ denotes the tensor-product σ-algebra on $P_j \times G$ which is formed by the tensor-product of B_G with the σ-algebra $\sum_{P_j}$ on P_j, and the notation $\mu_{P_j} \otimes \mu_G$ specifies the product measure on $P_j \times G$. By assuming above notation, we state the conditions as follows. For each $j \in J$:

(1) $(P_j,\ \sum_{P_j},\ \mu_{P_j})$ is a σ-finite measure space,

(2) the mapping $p \mapsto g_{j,p}$, $(P_j,\ \sum_{P_j}) \to (L^2(G),\ B_{L^2(G)})$ is measurable,

(3) the mapping $(p, x) \mapsto g_{j,p}(x)$, that is, $(P_j \times G,\ \sum_{P_j} \otimes B_G) \to (\mathbb{C}, B_{\mathbb{C}})$ is measurable.

Further, it is relevant to note that in order to investigate frame properties for the GTI systems considered in Definition 3.1, we need to view the family of functions given by $\bigcup_{j\in J}\{T_\gamma g_{j,p}\}_{\gamma\in\Gamma_j, p\in P_j}$ in the set-up of continuous g-frames. Recall that these frames are a generalized version of continuous frames, more precisely, for a countable index set $J \subset \mathbb{Z}$, a family of functions $\bigcup_{j\in J}\{f_{j,m}\}_{m\in M_j}$ is a *continuous generalized frame* (*continuous* g-*frame*) for a complex Hilbert space $\mathcal{H}$ with respect to a collection of measure spaces $\{(M_j, \sum_{M_j}, \mu_j) : j \in J\}$ if

($\mathcal{C}_1$) $m \mapsto f_{j,m}$, $M_j \to \mathcal{H}$ is measurable for each $j \in J$, and

($\mathcal{C}_2$) there exist constants $0 < \alpha_1 \leq \alpha_2$ such that

$$\alpha_1 ||h||^2 \leq \sum_{j \in J} \int_{M_j} |\langle h, f_{j,m} \rangle|^2 d\mu_{M_j}(m) \leq \alpha_2 ||h||^2, \text{ for all } h \in \mathcal{H}.$$

Note that all the definitions and operators associated to Definition 1.1 can be easily visualized for the case of continuous g-frames. For more details, we refer [38, 84] and various references within.

GTI System as a Continuous g-*frame*: In order to study pairwise orthogonal GTI frame systems, first we need to define GTI frame systems. The GTI system as a continuous g-frame is well-explained in [59], but we include the details here for the sake of completion. Hence, our next motive is to compare the GTI system $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ with the family of functions $\bigcup_{j \in J} \{f_{j,m}\}_{m \in M_j}$ considered in the above definition of continuous g-frame. Then, it follows that we can view the GTI system as a family of functions in $L^2(G)$ with respect to the collection of measure spaces

$$\big\{(M_j, \sum_{M_j}, \mu_j) := (P_j \times \Gamma_j, \sum_{P_j} \otimes B_{\Gamma_j}, \mu_{P_j} \otimes \mu_{\Gamma_j}) : j \in J\big\}.$$

Next, to realize GTI system as a continuous g-frame, we first verify the condition ($\mathcal{C}_1$). Let $j \in J$. Consider a function $F : P_j \times \Gamma_j \to L^2(G)$; $(p, \gamma) \mapsto T_\gamma g_{j,p}$. The function F is continuous in γ and measurable in p, and hence represents a Carathéodory function $\widetilde{F}$ which is defined on P_j by $\widetilde{F}(p)(\gamma) = F(p, \gamma)$. Since $\Gamma_j \subset G$ is second countable and locally compact, and $L^2(G)$ is separable, it follows that $\widetilde{F}$, and hence the function F, is jointly measurable on $(M_j, \sum_{M_j}) = \big(P_j \times \Gamma_j, \sum_{P_j} \otimes B_{\Gamma_j}\big)$ (for more details, see [71]). Thus, the condition ($\mathcal{C}_1$) holds, and the GTI system $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ is automatically weakly measurable.

In addition, if the GTI system satisfies the condition ($\mathcal{C}_2$) with respect to the set of measure spaces $\{(P_j \times \Gamma_j, \sum_{P_j} \otimes B_{\Gamma_j}, \mu_{P_j} \otimes \mu_{\Gamma_j}) : j \in J\}$, we call $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ as the *generalized translation invariant frame system* (*GTI frame system*) for $L^2(G)$. But, in case only the right side of inequality in the condition ($\mathcal{C}_2$) holds, the system $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ is termed as a *GTI Bessel system* in $L^2(G)$. Similarly, we can define *TI and GSI Bessel* (*frame*) *systems* in $L^2(G)$ by replacing GTI system in the definition of GTI Bessel (frame) system with TI and GSI systems, respectively.

It is known that pairwise orthogonal frames for a Hilbert space play a key role in constructing superframes and frames [35,36,40,51,67,89], and also in developing the theory of frames and its applications (e.g., see [40, 51, 52, 63] and various references therein). In our setting, we define such frames as GTI systems satisfying a special case of Definition 1.3 as follows:

Definition 3.2. Orthogonal GTI Bessel (frame) systems: Suppose that the systems $\bigcup_{j\in J}\{T_\gamma g_{j,p}\}_{\gamma\in\Gamma_j,\, p\in P_j}$ and $\bigcup_{j\in J}\{T_\gamma h_{j,p}\}_{\gamma\in\Gamma_j,\, p\in P_j}$ are GTI Bessel (frame) systems in $L^2(G)$. Then, we term these systems as *pairwise orthogonal GTI Bessel (frame) systems*, or simply, *orthogonal GTI Bessel (frame) systems* in $L^2(G)$ if they satisfy the orthogonality property in the sense of Definition 1.3. In particular, by replacing GTI systems with TI systems and GSI systems, this definition corresponds to *orthogonal TI Bessel (frame) systems* and *orthogonal GSI Bessel (frame) systems*, respectively.

3.2. A characterization result on GTI orthogonal frame pairs

In this section, we give the statement of our first main result discussed in this chapter, that is, Theorem 3.5 which provides a characterization for orthogonal GTI Bessel (frame) systems defined in Definition 3.2 over LCA group set-up (proof shall be discussed in Subsection 3.2.2).

3.2.1. Statement of the characterization result

For stating our main characterization result, we require some technical assumption in the form of a local integrability condition as follows. For the case of GSI systems, such condition was originally introduced by Hernández, Labate, and Weiss in [53] for $L^2(\mathbb{R}^n)$, and later generalized by Kutyniok and Labate in [64] for $L^2(G)$. This condition was further proposed in a more generalized form by Jakobsen and Lemvig in [59] for GTI systems in $L^2(G)$. We state these conditions as follows:

Definition 3.3. Consider two GTI systems $\bigcup_{j\in J}\{T_\gamma g_{j,p}\}_{\gamma\in\Gamma_j,p\in P_j}$ and $\bigcup_{j\in J}\{T_\gamma h_{j,p}\}_{\gamma\in\Gamma_j,p\in P_j}$ in $L^2(G)$.

(i) We say that $\bigcup_{j\in J}\{T_\gamma g_{j,p}\}_{\gamma\in\Gamma_j,p\in P_j}$ satisfies the *local integrability condition* (*LIC*) if

$$\sum_{j\in J}\int_{P_j}\sum_{\alpha\in\Gamma_j^\perp}\int_{supp\,\widehat{f}}|\widehat{f}(\xi+\alpha)\widehat{g}_{j,p}(\xi)|^2 d\mu_{\widehat{G}}(\xi)d\mu_{P_j}(p)<\infty,\ \text{for all } f\in\mathfrak{D}, \tag{3.1}$$

where for a Borel set B in $\widehat{G}$ with $\mu_{\widehat{G}}(\overline{B}) = 0$, we define the subset $\mathfrak{D}$ in $L^2(G)$ as follows:

$$\mathfrak{D} := \{f \in L^2(G) : \widehat{f} \in L^\infty(\widehat{G}) \text{ and } \mathrm{supp}\widehat{f} \text{ is compact in } \widehat{G} \setminus B\}.$$

(ii) $\bigcup_{j\in J}\{T_\gamma g_{j,p}\}_{\gamma\in\Gamma_j, p\in P_j}$ and $\bigcup_{j\in J}\{T_\gamma h_{j,p}\}_{\gamma\in\Gamma_j, p\in P_j}$ satisfy the *dual α-local integrability condition* (*dual α-LIC*) if

$$\sum_{j\in J}\int_{P_j}\sum_{\alpha\in\Gamma_j^\perp}\int_{\widehat{G}} |\widehat{f}(\xi)\widehat{f}(\xi+\alpha)\widehat{g}_{j,p}(\xi)\widehat{h}_{j,p}(\xi+\alpha)| d\mu_{\widehat{G}}(\xi) d\mu_{P_j}(p) < \infty, \text{ for all } f \in \mathfrak{D}. \tag{3.2}$$

In case $g_{j,p} = h_{j,p}$ for each j and p, we refer to (3.2) as the *α-local integrability condition* (*α-LIC*) for the GTI system $\bigcup_{j\in J}\{T_\gamma g_{j,p}\}_{\gamma\in\Gamma_j, p\in P_j}$.

Note that the integrands in (3.1) and (3.2) are measurable on $P_j \times \widehat{G}$, therefore, we are allowed to reorder sums and integrals in the local integrability conditions.

Further, note that the subset $\mathfrak{D}$ used in Definition 3.3 is dense in $L^2(G)$, and since it is sufficient to prove the various frame properties on a dense subset of the Hilbert space, we may verify our results for $\mathfrak{D}$ and then extend on $L^2(G)$ by a density argument.

Remark 3.4. In view of [59, Lemma 3.9], it is clear that

(i) LIC implies the α-LIC while the converse need not be true (see [59, Example 1]).

(ii) if two GTI systems satisfy the LIC, then they satisfy the dual α-LIC.

Next, we provide the statement of our first main characterization result on orthogonal frames. Here, we would like to add that the characterization results for orthogonal frames with the form of GTI systems, TI systems, and GSI systems discussed here are new to the literature. The results obtained here are also helpful in studying the orthogonality of special structured systems which lead to our second main result on orthogonal wave-packet Bessel (frame) systems (that is, Theorem 3.9), and hence for the case of wavelet and Gabor systems over LCA groups (see Section 3.3 for more details).

Moreover, we can easily deduce the orthogonality conditions for a pair of frames in case of LCA group $G = \mathbb{R}^d, \mathbb{Z}^d$, etc. That means, our orthogonality results on GTI systems generalize the existing similar work done for the classical case (e.g., see [63,67,89]) as well.

Theorem 3.5. *Let the families* $\bigcup_{j\in J}\{T_\gamma g_{j,p}\}_{\gamma\in\Gamma_j, p\in P_j}$ *and* $\bigcup_{j\in J}\{T_\gamma h_{j,p}\}_{\gamma\in\Gamma_j, p\in P_j}$ *be GTI Bessel (frame) systems in* $L^2(G)$ *which satisfy the dual* α*-LIC. Then, the following assertions are equivalent:*

(i) $\bigcup_{j\in J}\{T_\gamma g_{j,p}\}_{\gamma\in\Gamma_j, p\in P_j}$ *and* $\bigcup_{j\in J}\{T_\gamma h_{j,p}\}_{\gamma\in\Gamma_j, p\in P_j}$ *are orthogonal GTI Bessel (frame) systems in* $L^2(G)$ *in the sense of Definition 1.3,*

(ii) *for each* $\alpha \in \bigcup_{j\in J} \Gamma_j^\perp \setminus \{0\}$, *we have*

$$\sum_{j\in J:\alpha\in\Gamma_j^\perp} \int_{P_j} \overline{\widehat{h}_{j,p}(\xi)}\widehat{g}_{j,p}(\xi+\alpha)d\mu_{P_j}(p) = 0, \text{ for a.e. } \xi\in\widehat{G}, \tag{3.3}$$

and

$$\sum_{j\in J} \int_{P_j} \overline{\widehat{h}_{j,p}(\xi)}\widehat{g}_{j,p}(\xi)d\mu_{P_j}(p) = 0, \text{ for a.e. } \xi\in\widehat{G}.$$

We point out that Theorem 3.5 can be used to deduce similar results for several function systems since in these cases some of the assumptions trivially hold. We have the following observation in this regard:

Remark 3.6. We remark that Theorem 3.5 leads to the characterization results on the orthogonality of TI Bessel (frame) systems, GSI Bessel (frame) systems, and GTI Bessel (frame) systems (over a compact abelian group). For this, observe that

(i) in the case for TI Bessel systems, and for GTI Bessel systems over compact abelian groups, the dual α-LIC is satisfied automatically in view of [59] and Remark 3.4;

(ii) for each j in J if we take P_j as countable and Γ_j as a uniform lattice in Theorem 3.5, then the orthogonality result for the case of GSI Bessel (frame) systems is obtained.

3.2.2. Proof of the characterization result

In the present part of section, we obtain a proof for Theorem 3.5, that gives necessary and sufficient conditions for two GTI systems to form orthogonal frames for $L^2(G)$ by following the Definition 1.3. For this, the next result plays an important role.

Proposition 3.7. *Suppose* $\bigcup_{j\in J}\{T_\gamma g_{j,p}\}_{\gamma\in\Gamma_j, p\in P_j}$ *and* $\bigcup_{j\in J}\{T_\gamma h_{j,p}\}_{\gamma\in\Gamma_j, p\in P_j}$ *are GTI Bessel systems in* $L^2(G)$ *satisfying the dual* α*-LIC. Then, the following statements are equivalent:*

(i) *the mixed dual Gramian operator corresponding to the systems* $\bigcup_{j\in J}\{T_\gamma g_{j,p}\}_{\gamma\in\Gamma_j, p\in P_j}$ *and* $\bigcup_{j\in J}\{T_\gamma h_{j,p}\}_{\gamma\in\Gamma_j, p\in P_j}$ *commutes with the family of translations* $\{T_x\}_{x\in G}$,

(ii) *for each* $\alpha \in \bigcup_{j\in J} \Gamma_j^\perp \setminus \{0\}$,

$$t_\alpha(\xi) := \sum_{j\in J:\alpha\in\Gamma_j^\perp} \int_{P_j} \overline{\widehat{h}_{j,p}(\xi)}\widehat{g}_{j,p}(\xi+\alpha)d\mu_{P_j}(p) = 0, \text{ for a.e. } \xi \in \widehat{G}. \tag{3.4}$$

Moreover, if (i) or (ii) holds, then the mixed dual Gramian operator is a Fourier multiplier whose symbol is

$$s(\xi) = \sum_{j\in J} \int_{P_j} \overline{\widehat{h}_{j,p}(\xi)}\widehat{g}_{j,p}(\xi)d\mu_{P_j}(p), \text{ for a.e. } \xi \in \widehat{G}. \tag{3.5}$$

We first remark that the equations (3.4) and (3.5) are well-defined which can be easily verified by using Cauchy-Schwartz inequality in the following computation:

$$\begin{aligned}
&\sum_{j\in J:\alpha\in\Gamma_j^\perp} \int_{P_j} |\overline{\widehat{h}_{j,p}(\xi)}\widehat{g}_{j,p}(\xi+\alpha)|d\mu_{P_j}(p)\\
&\leq \sum_{j\in J} \int_{P_j} |\widehat{h}_{j,p}(\xi)||\widehat{g}_{j,p}(\xi+\alpha)|d\mu_{P_j}(p)\\
&\leq \sum_{j\in J} \Big(\int_{P_j} |\widehat{h}_{j,p}(\xi)|^2 d\mu_{P_j}(p)\Big)^{1/2}\Big(\int_{P_j} |\widehat{g}_{j,p}(\xi+\alpha)|^2 d\mu_{P_j}(p)\Big)^{1/2}\\
&\leq \Big(\sum_{j\in J}\int_{P_j} |\widehat{h}_{j,p}(\xi)|^2 d\mu_{P_j}(p)\Big)^{1/2}\Big(\sum_{j\in J}\int_{P_j} |\widehat{g}_{j,p}(\xi+\alpha)|^2 d\mu_{P_j}(p)\Big)^{1/2},
\end{aligned}$$

and hence we can write

$$\sum_{j\in J:\alpha\in\Gamma_j^\perp} \int_{P_j} |\overline{\widehat{h}_{j,p}(\xi)}\widehat{g}_{j,p}(\xi+\alpha)|d\mu_{P_j}(p) \leq \beta, \text{ for a.e. } \xi \in \widehat{G}, \tag{3.6}$$

in view of [59, Proposition 3.3] and by letting β be a common Bessel constant for the two GTI systems. Now, in order to prove Proposition 3.7, we need the following result:

Lemma 3.8. *Suppose* $\bigcup_{j\in J}\{T_\gamma g_{j,p}\}_{\gamma\in\Gamma_j,p\in P_j}$ *and* $\bigcup_{j\in J}\{T_\gamma h_{j,p}\}_{\gamma\in\Gamma_j,p\in P_j}$ *satisfy the assumptions of Proposition 3.7, and assume that the symbol* Θ *is their corresponding mixed dual Gramian operator. For* $f \in \mathfrak{D}$*, define the function* $w_f : G \to \mathbb{C}$*,* $x \mapsto \langle \Theta T_x f, T_x f\rangle$*. Then, the following hold true:*

(i) *The operator* Θ *commutes with all translations* T_x *for* $x \in G$*, if, and only if,* w_f *is constant for all* $f \in \mathfrak{D}$*, that means,* $w_f(x) = w_f(0) = \langle \Theta f, f\rangle$ *for all* $x \in G$*, where* 0 *denotes the neutral element of the LCA group* G*.*

(ii) *Let* $f \in \mathfrak{D}$. *Then,* w_f *is a continuous function that coincides pointwise with its absolutely convergent* (*almost periodic*) *Fourier series*

$$w_f(x) = \sum_{\alpha \in \bigcup\limits_{j\in J} \Gamma_j^{\perp}} \alpha(x)\widehat{w}_f(\alpha), \tag{3.7}$$

where

$$\widehat{w}_f(\alpha) := \int\limits_{\widehat{G}} \widehat{f}(\xi)\overline{\widehat{f}(\xi+\alpha)}t_\alpha(\xi)d\mu_{\widehat{G}}(\xi), \tag{3.8}$$

and the last integral converges absolutely.

(iii) w_f *is constant for all* $f \in \mathfrak{D}$ *if, and only if, for all* $\alpha \in \bigcup\limits_{j\in J} \Gamma_j^{\perp} \setminus \{0\}$, $t_\alpha(\xi) = 0$ *a.e.* $\xi \in \widehat{G}$.

Proof. (i) Let $\Theta T_x = T_x\Theta$, for all $x \in G$. Then, the direct part of (i) can be concluded by observing that

$$\begin{aligned} w_f(x) &= \langle \Theta T_x f, T_x f\rangle = \langle T_x\Theta f, T_x f\rangle \\ &= \langle \Theta f, T_x^* T_x f\rangle = \langle \Theta f, f\rangle, \end{aligned}$$

for all $x \in G$ and $f \in \mathfrak{D}$, since for each x, T_x is an unitary operator.

Conversely, let w_f be constant for all $f \in \mathfrak{D}$. Then, for all $x \in G$,

$$w_f(x) = \langle \Theta T_x f, T_x f\rangle = \langle T_{-x}\Theta T_x f, f\rangle = \langle \Theta f, f\rangle,$$

which by using unitary nature of T_x for each x and polarization identity, leads to $T_{-x}\Theta T_x = \Theta$, and hence we get $\Theta T_x = T_x\Theta$.

(ii) For each $f \in \mathfrak{D}$ and $x \in G$, we can write the function

$$w_f(x) = \langle \Theta T_x f, T_x f\rangle = \Big\langle \sum_{j\in J} \int\limits_{p\in P_j} \int\limits_{\Gamma_j} \big\langle T_x f, T_\gamma h_{j,p}\big\rangle T_\gamma g_{j,p} d\mu_{\Gamma_j}(\gamma) d\mu_{P_j}(p), T_x f \Big\rangle,$$

which further gives

$$w_f(x) = \sum_{j\in J} \int\limits_{p\in P_j} \int\limits_{\Gamma_j} \big\langle T_x f, T_\gamma h_{j,p}\big\rangle \big\langle T_\gamma g_{j,p}, T_x f\big\rangle d\mu_{\Gamma_j}(\gamma) d\mu_{P_j}(p). \tag{3.9}$$

Now, by proceeding in the same way as in the proof of [59, Theorem 3.4], the result follows.

(iii) Given that dual α-LIC holds for all $f \in \mathfrak{D}$. From (3.7) and (3.9), it follows that

(3.10)
$$w_f(x) = \sum_{j\in J} \int_{p\in P_j} \int_{\Gamma_j} \langle T_x f, T_\gamma h_{j,p}\rangle \langle T_\gamma g_{j,p}, T_x f\rangle d\mu_{\Gamma_j}(\gamma) d\mu_{P_j}(p) = \sum_{\alpha \in \bigcup\limits_{j\in J} \Gamma_j^\perp} \alpha(x)\widehat{w}_f(\alpha).$$

Consider now the function $z_f(x) := w_f(x) - \langle \Theta f, f\rangle$ which is continuous in view of continuity of the function w_f.

Now, for the direct part, assume that the function w_f is constant for all $f \in \mathfrak{D}$. We claim that $t_\alpha(\xi) = 0$, for all $\alpha \in \bigcup\limits_{j\in J} \Gamma_j^\perp \setminus \{0\}$ and *a.e.* $\xi \in \widehat{G}$. Here, note that by the construction, z_f is identical to the zero function. Additionally, since w_f equals an absolute convergent generalized Fourier series, also z_f can be expressed as an absolute convergent generalized Fourier series $z_f(x) = \sum\limits_{\alpha \in \bigcup\limits_{j\in J} \Gamma_j^\perp} \alpha(x)\widehat{z}_f(\alpha)$, with

$$\widehat{z}_f(\alpha) = \begin{cases} \widehat{w}_f(0) - \langle \Theta f, f\rangle, & \text{if } \alpha = 0, \\ \widehat{w}_f(\alpha), & \text{if } \alpha \neq 0. \end{cases}$$

By the uniqueness theorem for generalized Fourier series [11, Theorem 7.12], the function $z_f(x)$ is identical to zero if, and only if, $\widehat{z}_f(\alpha) = 0$ for all $\alpha \in \bigcup\limits_{j\in J} \Gamma_j^\perp$.

Thus, for $\alpha \in \bigcup\limits_{j\in J} \Gamma_j^\perp$ and $f \in \mathfrak{D}$, we have

$$\widehat{w}_f(\alpha) = \delta_{\alpha,0}\langle \Theta f, f\rangle. \tag{3.11}$$

Let $\alpha \neq 0$. Then, for all $f \in \mathfrak{D}$, (3.11) reduces to $\widehat{w}_f(\alpha) = 0$, and hence we get

$$\int_{\widehat{G}} \widehat{f}(\xi)\overline{\widehat{f}(\xi+\alpha)} t_\alpha(\xi) d\mu_{\widehat{G}}(\xi) = 0, \text{ for a.e. } \xi \in \widehat{G}. \tag{3.12}$$

Now, define the multiplication operator $M_{\bar{t}_\alpha} : L^2(\widehat{G}) \to L^2(\widehat{G})$ by $M_{\bar{t}_\alpha}\widehat{f}(\xi) = \overline{t_\alpha(\xi)}\widehat{f}(\xi)$ which is a bounded linear operator in view of the fact that $t_\alpha(\xi) \in L^\infty(\widehat{G})$ (for details, see (3.6)). For all $f \in \mathfrak{D}$ and a.e. $\xi \in \widehat{G}$, we can now rewrite the term in left hand side of (3.12) as

$$\begin{aligned} \int_{\widehat{G}} \widehat{f}(\xi)\overline{\overline{t_\alpha(\xi)} T_\alpha \widehat{f}(\xi)} d\mu_{\widehat{G}}(\xi) &= \int_{\widehat{G}} \widehat{f}(\xi)\overline{M_{\bar{t}_\alpha}(T_\alpha \widehat{f})(\xi)} d\mu_{\widehat{G}}(\xi) \\ &= \int_{\widehat{G}} \widehat{f}(\xi)\overline{(M_{\bar{t}_\alpha} T_\alpha)\widehat{f}(\xi)} d\mu_{\widehat{G}}(\xi) = \langle \widehat{f}, M_{\bar{t}_\alpha} T_\alpha \widehat{f}\,\rangle_{L^2(\widehat{G})}, \end{aligned}$$

which is equal to zero in view of (3.12). From the above equality and the fact that $\mathfrak{D}$ is dense in the complex Hilbert space $L^2(G)$, it follows that $M_{\bar{t}_\alpha} T_\alpha \widehat{f} = 0$, which is if, and only if, $M_{\bar{t}_\alpha} T_\alpha = 0$, that means, $M_{\bar{t}_\alpha} T_\alpha(\widehat{g}) = 0$ for all $\widehat{g} \in L^2(\widehat{G})$, and hence $M_{\bar{t}_\alpha} T_\alpha \widehat{g}(\xi) = \overline{t_\alpha(\xi)} T_\alpha \widehat{g}(\xi) = 0$ for all $\widehat{g} \in L^2(\widehat{G})$ and a.e. $\xi \in \widehat{G}$. Thus, (3.12) holds if, and only if, for a.e. $\xi \in \widehat{G}$, we have $t_\alpha(\xi) = 0$, for all $\alpha \in \bigcup\limits_{j\in J} \Gamma_j^\perp \setminus \{0\}$.

Conversely, for each $\alpha \in \bigcup\limits_{j\in J} \Gamma_j^\perp \setminus \{0\}$, let $t_\alpha(\xi) = 0$ for a.e. $\xi \in \widehat{G}$, which implies that $\widehat{w}_f(\alpha) = 0$, and by using this in (3.10) along with the fact from (3.11) that for $\alpha = 0$, we have $\widehat{w}_f(\alpha) = \langle \Theta f, f\rangle$, and hence

$$w_f(x) = \sum_{\alpha \in \bigcup\limits_{j\in J} \Gamma_j^\perp \setminus\{0\}} \alpha(x)\widehat{w}_f(\alpha) + \sum_{\alpha\in\{0\}} \alpha(x)\widehat{w}_f(\alpha) = 0 + \widehat{w}_f(0) = \langle \Theta f, f\rangle,$$

for a.e. $x \in G$. Therefore, w_f is constant for all $f \in \mathfrak{D}$. $\square$

Proof of Proposition 3.7. Clearly, part (i) is true if, and only if, (3.4) holds in view of Lemma 3.8. Further, it is well-known that if the mixed dual Gramian operator, say Θ, commutes with T_x for all $x \in G$, then it is a Fourier multiplier (see [66, Theorem 4.1.1]), and hence there exists a unique $s \in L^\infty(\widehat{G})$ such that $\widehat{\Theta f}(\xi) = s(\xi)\widehat{f}(\xi)$, where $s(\xi)$ represents the symbol corresponding to Θ. Now, for a.e. $\xi \in \widehat{G}$, we are interested in finding the expression for $s(\xi)$. For this, observe that

$$\langle \Theta f, f\rangle = \langle \widehat{\Theta f}, \widehat{f}\rangle_{L^2(\widehat{G})} = \int\limits_{\widehat{G}} \widehat{\Theta f}(\xi)\overline{\widehat{f}(\xi)} d\mu_{\widehat{G}}(\xi) = \int\limits_{\widehat{G}} s(\xi)\widehat{f}(\xi)\overline{\widehat{f}(\xi)} d\mu_{\widehat{G}}(\xi). \tag{3.13}$$

Moreover, for $\alpha = 0$, it follows from (3.8) and (3.11) that for all $f \in \mathfrak{D}$,

$$\langle \Theta f, f\rangle = \widehat{w}_f(0) = \int\limits_{\widehat{G}} \widehat{f}(\xi)\overline{\widehat{f}(\xi)} \sum_{j\in J} \int\limits_{P_j} \overline{\widehat{h}_{j,p}(\xi)}\widehat{g}_{j,p}(\xi) d\mu_{P_j}(p) d\mu_{\widehat{G}}(\xi). \tag{3.14}$$

Since (3.13) and (3.14) are valid for all $f \in \mathfrak{D}$ and s is unique, it is clear that the symbol of Θ, that is, $s(\xi) = \sum\limits_{j\in J} \int\limits_{P_j} \overline{\widehat{h}_{j,p}(\xi)}\widehat{g}_{j,p}(\xi) d\mu_{P_j}(p)$. $\square$

Now, we are ready to prove our first main result, that is, Theorem 3.5, which is as follows:

Proof of Theorem 3.5. By Definition 1.3, the part (i) is equivalent to saying that the mixed dual Gramian operator corresponding to the GTI systems $\bigcup_{j\in J}\{T_\gamma g_{j,p}\}_{\gamma\in\Gamma_j, p\in P_j}$ and $\bigcup_{j\in J}\{T_\gamma h_{j,p}\}_{\gamma\in\Gamma_j, p\in P_j}$, say Θ, is equal to zero. Next, we claim that $\Theta = 0$ if, and only if, Θ commutes with the translations T_x for all $x \in G$, and, act as a Fourier multiplier with symbol

$$s(\xi) = \sum_{j \in J} \int_{P_j} \overline{\widehat{h}_{j,p}(\xi)} \widehat{g}_{j,p}(\xi) d\mu_{P_j}(p) = 0, \text{ for a.e. } \xi \in \widehat{G}.$$

For proving the above claim, let $\Theta = 0$. Then, $\Theta T_x(f) = 0$, for all $x \in G$ and $f \in L^2(G)$. Since for each x, the translation T_x is a linear operator, therefore $T_x(0) =$ zero function in $L^2(G) = 0$, and hence $T_x \Theta f = T_x(0) = 0$, which implies that $\Theta T_x = T_x \Theta$ for all $x \in G$. Thus by [66, Theorem 4.1.1], Θ is a Fourier multiplier. So for all $f \in L^2(G)$ we have $0 = \widehat{\Theta f}(\xi) = s(\xi)\widehat{f}(\xi)$, $\xi \in \widehat{G}$ a.e., where $s(\xi)$, the symbol of Θ as a Fourier multiplier, is given by (3.5).

Conversely, if Θ is a Fourier multiplier with symbol $s(\xi) = 0$, then $\widehat{\Theta f}(\xi) = 0$, which implies that $\Theta f = 0$ for all $f \in L^2(G)$, and hence $\Theta = 0$. Now, the result follows by considering the above claim along with Proposition 3.7. □

3.3. Applications of the characterization result

The purpose of this section is to discuss applications of our first main result (that is, Theorem 3.5 in Subsection 3.2.1) to the Bessel families having wave-packet structure, which are obtained by applying certain collections of dilations, modulations, and translations to a countable family of functions in $L^2(G)$. As a consequence, we obtain results for wavelet and Gabor systems in Subsection 3.3.2. Along with this, we connect the already existing results from the literature with the theory discussed in this chapter by providing various examples in case of $G = \mathbb{R}^d$, $\mathbb{Z}^d$, etc.

3.3.1. Wave-Packet Systems

For a given second countable LCA group G, let $\mathbf{Epi}(G)$, $\mathbf{Epick}(G)$, and $\mathbf{Aut}(G)$ denote respectively, the semigroup of continuous group homomorphisms α from G onto G, the semigroup of $\alpha \in \mathbf{Epi}(G)$ having compact kernel $\ker \alpha$, and the group of topological automorphisms α of G onto itself. Note that $\mathbf{Aut}(G) \subset \mathbf{Epick}(G) \subset \mathbf{Epi}(G)$. For $\alpha \in \mathbf{Epick}(G)$, we define the isometric *dilation operator* D_α by

$$D_\alpha : L^2(G) \to L^2(G);\ D_\alpha f(x) = (\Delta(\alpha))^{-1/2} f(\alpha(x)), \text{ for all } x \in G,$$

where the *modular function* $\Delta : \mathbf{Epick}(G) \to (0, \infty)$ is a semigroup homomorphism such that

$$\int_G (g \circ \alpha)(x) d\mu_G(x) = \Delta(\alpha) \int_G g(x) d\mu_G(x)$$

for all integrable functions g on G with respect to the Haar measure μ_G (see [10, Theorem 6.2]). For a character χ in $\widehat{G}$, we define the *modulation operator* M_χ on $L^2(G)$ as

$$M_\chi(f)(x) = \chi(x) f(x), \text{ for all } x \in G,$$

and observe that for each $\chi \in \widehat{G}$, it is associated with the translation operator on $L^2(\widehat{G})$ by the relation

$$(3.15) \qquad (\widehat{M_\chi f})(\xi) = \int_G \chi(x) f(x) \overline{\xi(x)} d\mu_G(x) = \int_G f(x) \overline{(\xi - \chi)(x)} d\mu_G(x) = T_\chi \widehat{f}(\xi),$$

for all $f \in L^2(G)$ and a.e. $\xi \in \widehat{G}$. Further, note that for each $\alpha \in \mathbf{Epick}(G)$, the dilation operator on $L^2(G)$ satisfies the following relation (see [10, Lemma 6.6]):

$$(3.16) \qquad \widehat{(D_\alpha f)}(\chi) = \begin{cases} (\Delta(\alpha))^{1/2} \widehat{f}(\beta^{-1}(\chi)) & \text{for } \chi \in \beta(\widehat{G}) = (\ker \alpha)^\perp, \\ 0 & \text{otherwise,} \end{cases}$$

for all $f \in L^2(G)$, where by $\beta := \alpha^*$, we denote the adjoint of $\alpha \in \mathbf{Epick}(G)$ which is a topological isomorphism $\beta : \widehat{G} \to (\ker \alpha)^\perp$; $\chi \mapsto \chi \circ \alpha$ in view of [10, Proposition 6.5].

Let $\mathcal{A}$ be a subset of $\mathbf{Epick}(G)$, let Γ and Λ be co-compact subgroups of G and $\widehat{G}$, respectively, and for some index set $J \subset \mathbb{Z}$, let $\Psi := \{\psi_j : j \in J\}$ be a subset of $L^2(G)$. Then, we define the *wave-packet system* generated by Ψ as:

$$(3.17) \qquad \mathcal{W}(\Psi, \mathcal{A}, \Gamma, \Lambda) := \{D_\alpha T_\gamma M_\chi \psi_j : \alpha \in \mathcal{A}, \gamma \in \Gamma, \chi \in \Lambda, j \in J\}.$$

In the case of $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^d)$, the systems of the above form have been studied by several authors, including [20, 54, 69], and various references within. The wave-packet systems were originally introduced by Córdoba and Fefferman [13], and the collection defined in (3.17) generalizes the notion of such systems in the context of LCA groups. In particular, the wavelet and Gabor systems can be seen as special cases of (3.17) which we shall discuss in Subsection 3.3.2.

The following commutator relation helps in representing the collection (3.17) in the form of a GTI system. This relation says that for each $\alpha \in \mathcal{A}$, $\gamma \in \Gamma$, $\chi \in \Lambda$, and $j \in J$, we have:

$$\begin{aligned}
D_\alpha T_\gamma M_\chi \psi_j(x) &= (\Delta(\alpha))^{-1/2} T_\gamma M_\chi \psi_j(\alpha(x)) = (\Delta(\alpha))^{-1/2} M_\chi \psi_j(\alpha(x) - \gamma) \\
&= (\Delta(\alpha))^{-1/2} M_\chi \psi_j(\alpha(x - \gamma_1)) = D_\alpha M_\chi \psi_j(x - \gamma_1) = T_{\gamma_1} D_\alpha M_\chi \psi_j(x),
\end{aligned}$$

for all $x \in G$, and for some $\gamma_1 \in \alpha^{-1}\Gamma$ such that $\alpha(\gamma_1) = \gamma$.

In the rest of this section, let $\mathcal{A}$ be a countable subset of $\mathbf{Epick}(G)$. Then, by using the above commutator relation, the wave-packet system $\mathcal{W}(\Psi, \mathcal{A}, \Gamma, \Lambda)$ will represent a GTI system of the form $\bigcup_{\alpha \in \mathcal{A}} \{T_\gamma g_{\alpha,p}\}_{\gamma \in \Gamma_\alpha,\, p \in P_\alpha}$ for $\Gamma_\alpha := \alpha^{-1}\Gamma$ with $\alpha \in \mathcal{A}$, $g_{\alpha,p} = g_{\alpha,(j,\chi)} = D_\alpha M_\chi \psi_j$ for $(\alpha, p) = (\alpha, (j, \chi))$ in $\mathcal{A} \times (J \times \Lambda)$. In this case, for each $\alpha \in \mathcal{A}$, the measure space $P_\alpha := \{(j, \chi) : j \in J, \chi \in \Lambda\}$ is equipped with the measure $\mu_{P_\alpha} := \mu_{J \times \Lambda} = (\Delta(\alpha))^{-1}(\mu_J \otimes \mu_\Lambda)$, where the quantity $(\Delta(\alpha))^{-1}$ helps in avoiding the scaling factor in the calculations and μ_J represents the counting measure on J. Clearly, the measure μ_{P_α} is σ-finite. Here, note that $\Gamma_\alpha = \alpha^{-1}\Gamma$ is a closed co-compact subgroup of G for each $\alpha \in \mathcal{A}$, in view of [10, Proposition 6.4] and the fact that α is a continuous group homomorphism from G onto G along with Γ as a closed subgroup of G.

Next, we apply Theorem 3.5 to wave-packet systems $\mathcal{W}(\Psi, \mathcal{A}, \Gamma, \Lambda)$ and $\mathcal{W}(\Phi, \mathcal{A}, \Gamma, \Lambda)$, where for any index set $J \subset \mathbb{Z}$, $\Psi := \{\psi_j\}_{j \in J}$ and $\Phi := \{\varphi_j\}_{j \in J}$ are subsets in $L^2(G)$. Further, we simplify (3.3) by considering $\mathcal{W}(\Psi, \mathcal{A}, \Gamma, \Lambda)$ and $\mathcal{W}(\Phi, \mathcal{A}, \Gamma, \Lambda)$ as GTI systems $\bigcup_{\alpha \in \mathcal{A}} \{T_\gamma g_{\alpha,p}\}_{\gamma \in \Gamma_\alpha,\, p \in P_\alpha}$ and $\bigcup_{\alpha \in \mathcal{A}} \{T_\gamma h_{\alpha,p}\}_{\gamma \in \Gamma_\alpha,\, p \in P_\alpha}$, respectively, where

$$g_{\alpha,p} = g_{\alpha,(j,\chi)} = D_\alpha M_\chi \psi_j \text{ and } h_{\alpha,p} = h_{\alpha,(j,\chi)} = D_\alpha M_\chi \varphi_j$$

for $(\alpha, p) = (\alpha, (j, \chi)) \in \mathcal{A} \times P_\alpha = \mathcal{A} \times (J \times \Lambda)$. Hence, for each $\widetilde{\alpha} \in \bigcup_{\alpha \in \mathcal{A}} \Gamma_\alpha^\perp$ and for a.e. $\xi \in \bigcup_{\alpha \in \mathcal{A}} (\ker \alpha)^\perp$, the expression (3.3) takes the following form in view of (3.15) and (3.16) along with $\beta = \alpha^*$:

$$\begin{aligned}
\mathcal{T}_{\widetilde{\alpha}}(\xi) &:= \sum_{\alpha \in \mathcal{A}:\, \widetilde{\alpha} \in \Gamma_\alpha^\perp} \int_{P_\alpha} \overline{\widehat{h}_{\alpha,p}(\xi)} \widehat{g}_{\alpha,p}(\xi + \widetilde{\alpha}) d\mu_{P_\alpha}(p) \\
&= \sum_{\alpha \in \mathcal{A}:\, \widetilde{\alpha} \in \Gamma_\alpha^\perp} \int_{J \times \Lambda} \overline{\widehat{h}_{\alpha,(j,\chi)}(\xi)} \widehat{g}_{\alpha,(j,\chi)}(\xi + \widetilde{\alpha}) d\mu_{J \times \Lambda}((j, \chi)) \\
&= \sum_{\alpha \in \mathcal{A}:\, \widetilde{\alpha} \in (\alpha^{-1}\Gamma)^\perp} \sum_{j \in J} \int_{\Lambda} \overline{(\widehat{D_\alpha M_\chi \varphi_j})(\xi)} (\widehat{D_\alpha M_\chi \psi_j})(\xi + \widetilde{\alpha}) \frac{1}{\Delta(\alpha)} d\mu_\Lambda(\chi),
\end{aligned}$$

which further gives

$$\begin{aligned}
\mathcal{T}_{\widetilde{\alpha}}(\xi) &= \sum_{\alpha\in\mathcal{A}:\,\widetilde{\alpha}\in\beta\Gamma^{\perp}}\sum_{j\in J}\int_{\Lambda}\overline{\widehat{M_{\chi}\varphi_j}(\beta^{-1}\xi)}\widehat{M_{\chi}\psi_j}(\beta^{-1}(\xi+\widetilde{\alpha}))d\mu_{\Lambda}(\chi)\\
&= \sum_{\alpha\in\mathcal{A}:\,\widetilde{\alpha}\in\beta\Gamma^{\perp}}\sum_{j\in J}\int_{\Lambda}\overline{T_{\chi}\widehat{\varphi}_j(\beta^{-1}\xi)}T_{\chi}\widehat{\psi}_j(\beta^{-1}(\xi+\widetilde{\alpha}))d\mu_{\Lambda}(\chi)\\
&= \sum_{\alpha\in\mathcal{A}:\,\widetilde{\alpha}\in\beta\Gamma^{\perp}}\sum_{j\in J}\int_{\Lambda}\overline{\widehat{\varphi}_j(\beta^{-1}\xi-\chi)}\widehat{\psi}_j(\beta^{-1}(\xi+\widetilde{\alpha})-\chi)d\mu_{\Lambda}(\chi) =: \widetilde{\mathcal{T}}_{\widetilde{\alpha}}(\xi) \text{ (say)},
\end{aligned}$$

whereas for the case of $\xi\in\widehat{G}\setminus\bigcup_{\alpha\in\mathcal{A}}(\ker\alpha)^{\perp}$ a.e., we get $\mathcal{T}_{\widetilde{\alpha}}(\xi)=0$ by proceeding in the similar way as above. Hence, we can write

$$\mathcal{T}_{\widetilde{\alpha}}(\xi) = \begin{cases}\widetilde{\mathcal{T}}_{\widetilde{\alpha}}(\xi) & \text{for a.e. } \ \xi\in\bigcup_{\alpha\in\mathcal{A}}(\ker\alpha)^{\perp},\\ 0 & \text{otherwise.}\end{cases} \tag{3.18}$$

Now, to apply Theorem 3.5 on the wave-packet systems, we require that for a.e. $\xi\in\widehat{G}$, $\mathcal{T}_{\widetilde{\alpha}}(\xi)$ in (3.18) should be equal to 0 for all $\widetilde{\alpha}\in\bigcup\limits_{\alpha\in\mathcal{A}}\Gamma_{\alpha}^{\perp}$.

The above discussion leads to our second main result, that is, Theorem 3.9 which provides the conditions on Ψ and Φ such that the wave-packet systems generated by Ψ and Φ form pairwise orthogonal Bessel families (frames) which we call as *pairwise orthogonal* (simply, *orthogonal*) *wave-packet Bessel* (*frame*) *systems* in $L^2(G)$. Note that the general LCA group approach applies to all groups of the form $G=\mathbb{R}^s\times\mathbb{Z}^p\times\mathbb{T}^q\times\mathbb{Z}_m$. Therefore, the following characterization result on wave-packet systems can be easily used to verify the concrete conditions for any choice of G specified as above, while the direct derivation would be rather complex.

Theorem 3.9. *Let the wave-packet systems $\mathcal{W}(\Psi,\mathcal{A},\Gamma,\Lambda)$ and $\mathcal{W}(\Phi,\mathcal{A},\Gamma,\Lambda)$ be Bessel families* (*frames*) *in $L^2(G)$ satisfying the corresponding dual α-LIC, where $\mathcal{A}$ is a countable subset of* $\mathbf{Epick}(G)$. *Then, the following assertions are equivalent:*

(i) *$\mathcal{W}(\Psi,\mathcal{A},\Gamma,\Lambda)$ and $\mathcal{W}(\Phi,\mathcal{A},\Gamma,\Lambda)$ form orthogonal wave-packet Bessel* (*frame*) *systems in $L^2(G)$,*

(ii) *for a.e. $\xi\in\widehat{G}$ and $\widetilde{\alpha}\in\bigcup\limits_{\alpha\in\mathcal{A}}\Gamma_{\alpha}^{\perp}$, the following holds:*

$$\sum_{\alpha\in\mathcal{A}:\,\widetilde{\alpha}\in\Gamma_{\alpha}^{\perp}}\sum_{j\in J}\int_{\Lambda}\overline{\widehat{\varphi}_j(\beta^{-1}\xi-\chi)}\widehat{\psi}_j(\beta^{-1}(\xi+\widetilde{\alpha})-\chi)d\mu_{\Lambda}(\chi)=0,$$

where for $\beta=\alpha^*$, $\Gamma_{\alpha}^{\perp}$ is given by $\beta\Gamma^{\perp}$.

Proof. The proof can be concluded by observing that if we consider $\mathcal{W}(\Psi, \mathcal{A}, \Gamma, \Lambda)$ and $\mathcal{W}(\Phi, \mathcal{A}, \Gamma, \Lambda)$ as Bessel families (frames) satisfying the corresponding dual α-LIC, then, the wave-packet systems generated by Ψ and Φ form pairwise orthogonal Bessel families (frames) in $L^2(G)$ if, and only if, in view of (3.18), $\mathcal{T}_{\widetilde{\alpha}}(\xi)$ is equal to 0 for all $\widetilde{\alpha} \in \bigcup_{\alpha\in\mathcal{A}} \Gamma_\alpha^\perp$ and a.e. $\xi \in \widehat{G}$. □

In the following example, by applying Theorem 3.9 to the case $G = \mathbb{R}^d$, we get a characterization result for the orthogonality of a pair of wave-packet systems in $L^2(\mathbb{R}^d)$. Hence, the wave-packet systems within $L^2(\mathbb{R}^d)$ are easily covered within our framework.

Example 3.10. Let $G = \mathbb{R}^d$ (equipped with Lebesgue measure), $\Gamma = \mathbb{Z}^d$ and $\Lambda = \mathbb{R}^d$. Then, $\widehat{G} = \mathbb{R}^d$, with Euclidean metric, we have $\Gamma^\perp = \mathbb{Z}^d$ and $\Lambda^\perp = \{0\}$. Further, by assuming the matrix A in $\mathrm{GL}(d, \mathbb{R})$, set $\mathcal{A} = \{x \mapsto A^k x : k \in \mathbb{Z}\}$. Under these assumptions, from (3.17), the wave-packet system generated by $\Psi = \{\psi_l\}_{l=1}^L \subset L^2(\mathbb{R}^d)$ can be written as

$$\begin{aligned}\mathcal{W}(\Psi, \mathcal{A}, \mathbb{Z}^d, \mathbb{R}^d) &:= \big\{D_{A^k} T_\gamma M_\chi \psi_l(\cdot) : l = 1, \ldots, L,\, k \in \mathbb{Z}, \gamma \in \mathbb{Z}^d, \chi \in \mathbb{R}^d\big\} \\ &= \big\{|\det A|^{-k/2} \chi(A^k \cdot\, - \gamma)\psi_l(A^k \cdot\, - \gamma) : l = 1, \ldots, L,\, k \in \mathbb{Z}, \gamma \in \mathbb{Z}^d, \chi \in \mathbb{R}^d\big\}.\end{aligned}$$

By letting $\mathcal{W}(\Psi, \mathcal{A}, \mathbb{Z}^d, \mathbb{R}^d)$ as a wave-packet system in $L^2(\mathbb{R}^d)$ which satisfies the Bessel condition (frame inequality), we conclude from Theorem 3.9 that the wave-packet systems generated by Ψ and Φ form pairwise orthogonal Bessel families (frames) in $L^2(\mathbb{R}^d)$ if, and only if, the following holds: $\displaystyle\sum_{k\in\mathbb{Z}:\, \widetilde{\alpha}\in B^k\mathbb{Z}^d} \sum_{l=1}^{L} \int_{\mathbb{R}^d} \overline{\widehat{\varphi}_l(B^{-k}\xi - \chi)}\widehat{\psi}_l(B^{-k}(\xi + \widetilde{\alpha}) - \chi) d(\chi) = 0,$ for a.e. $\xi \in \mathbb{R}^d$ and for each $\widetilde{\alpha} \in \bigcup_{k\in\mathbb{Z}} B^k\mathbb{Z}^d$ along with $B = A^*$.

In the next subsection, by applying Theorem 3.9 we deduce the orthogonality conditions for the case of Gabor and wavelet systems over LCA groups.

3.3.2. Special cases of Wave-Packet Systems

3.3.2.1. *Gabor Systems.* In (3.17), by assuming $\mathcal{A} = \{I_G\}$, where I_G denotes the identity group homomorphism on G, we consider the following system as a special case of wave-packet system defined in (3.17) which we call the *Gabor system* generated by Ψ:

$$\mathcal{G}(\Psi, \Gamma, \Lambda) := \{T_\gamma M_\chi \psi_j : \gamma \in \Gamma, \chi \in \Lambda, j \in J\}.$$

At this juncture, it is relevant to note that the system $\mathcal{G}(\Psi,\Gamma,\Lambda)$ is a frame for $L^2(G)$ if, and only if, $\{M_\chi T_\gamma \psi_j : \gamma \in \Gamma, \chi \in \Lambda, j \in J\}$ is a frame for $L^2(G)$ (see [59, Lemma 2.4]), where the later system is termed as a *co-compact Gabor system* in [60]. Further, observe that $\mathcal{G}(\Psi,\Gamma,\Lambda)$ is a TI system of the form $\bigcup_{j\in J}\{T_\gamma g_{j,p}\}_{\gamma\in\Gamma_j,\, p\in P_j}$ with $\Gamma_j = \Gamma$ for $j \in J \subset \mathbb{Z}$ and $g_{j,p} = g_{j,\chi} = M_\chi \psi_j$, where $(j,p) = (j,\chi) \in J \times \Lambda$. In this case, for each $j \in J$, $P_j = \{\chi : \chi \in \Lambda\}$ is equipped with the measure $\mu_{P_j} := (\Delta(\alpha))^{-1}\mu_\Lambda$ that satisfies the standing hypothesis. Since for TI systems the dual α-LIC is automatically satisfied, Theorem 3.9 leads to the following result on the orthogonality of Gabor systems. Here, note that the Bessel families (frames) with co-compact Gabor structure which satisfy the orthogonality property are termed as *pairwise orthogonal* (simply, *orthogonal*) *co-compact Gabor Bessel* (*frame*) *systems* in $L^2(G)$:

Proposition 3.11. *Let the Gabor systems $\mathcal{G}(\Psi,\Gamma,\Lambda)$ and $\mathcal{G}(\Phi,\Gamma,\Lambda)$ be Bessel families (frames) in $L^2(G)$. Then, they form orthogonal co-compact Gabor Bessel (frame) systems in $L^2(G)$ if, and only if, for each $\widetilde{\alpha} \in \Gamma^\perp$ and for a.e. $\xi \in \widehat{G}$, the following assertion holds:*

$$\sum_{j\in J}\int_\Lambda \overline{\widehat{\varphi}_j(\xi-\chi)}\,\widehat{\psi}_j((\xi+\widetilde{\alpha})-\chi)\,d\mu_\Lambda(\chi) = 0.$$

Using the above result, we can deduce a characterization for the orthogonality of Gabor Bessel families (frames) in $\ell^2(\mathbb{Z}^d)$ given by Lopez and Han in [67]. This property of Gabor frames has found its significance in developing frame theory and its applications including the construction of Gabor superframes in various set-ups. For more details, see [2, 35, 36, 39, 51, 67, 68] and references within.

In the next part, we provide a characterization for pairwise orthogonal Bessel families (frames) with wavelet structure which we call as *pairwise orthogonal* (simply, *orthogonal*) *wavelet Bessel* (*frame*) *systems* in $L^2(G)$.

3.3.2.2. *Wavelet Systems.* By letting $\Lambda = \{\chi_0\} \subset \widehat{G}$ in (3.17), where χ_0 being the neutral element of $\widehat{G}$, we define the collection $\mathcal{U}(\Psi,\mathcal{A},\Gamma)$ as the *wavelet system* generated by Ψ:

$$\mathcal{U}(\Psi,\mathcal{A},\Gamma) := \{D_\alpha T_\gamma \psi_j : \alpha \in \mathcal{A}, \gamma \in \Gamma, j \in J\}, \tag{3.19}$$

as a special case of wave-packet system defined in (3.17). For a countable subset $\mathcal{A}$ in **Epick**(G), the system (3.19) is a GTI system of the form $\bigcup_{\alpha\in\mathcal{A}}\{T_\gamma g_{\alpha,p}\}_{\gamma\in\Gamma_\alpha,\, p\in P_\alpha}$ for

$\Gamma_\alpha = \alpha^{-1}\Gamma$ with $\alpha \in \mathcal{A}$, $g_{\alpha,p} = g_{\alpha,j} = D_\alpha \psi_j$ for $(\alpha, p) = (\alpha, j)$ in $\mathcal{A} \times J$. In this case, for each $\alpha \in \mathcal{A}$, the measure space $P_\alpha := \{j : j \in J\}$ is equipped with a counting measure $\mu_{P_\alpha} := (\Delta(\alpha))^{-1}(\mu_J)$ which is clearly σ-finite. Thus, Theorem 3.9 for the case of wave-packet systems now reduces to the following result on wavelet systems which generalizes similar results discussed in [63, 89] along with various applications:

Proposition 3.12. *Let $\mathcal{U}(\Psi, \mathcal{A}, \Gamma)$ and $\mathcal{U}(\Phi, \mathcal{A}, \Gamma)$ be Bessel families (frames) in $L^2(G)$ which satisfy the corresponding dual α-LIC, where $\mathcal{A}$ is a countable subset of* $\mathbf{Epick}(G)$. *Then, they form orthogonal wavelet Bessel (frame) systems in $L^2(G)$ if, and only if, for a.e. $\xi \in \widehat{G}$ and $\widetilde{\alpha} \in \bigcup_{\alpha \in \mathcal{A}} \Gamma_\alpha^\perp$, we have*

$$\sum_{\alpha \in \mathcal{A} \colon \widetilde{\alpha} \in \Gamma_\alpha^\perp} \sum_{j \in J} \overline{\widehat{\varphi}_j(\beta^{-1}\xi)} \widehat{\psi}_j(\beta^{-1}(\xi + \widetilde{\alpha})) = 0,$$

where for $\beta = \alpha^*$, $\Gamma_\alpha^\perp$ is given by $\beta\Gamma^\perp$.

Example 3.13. By assuming $\Lambda = \{\chi_0\} \subset \widehat{G}$ in Example 3.10, where χ_0 being the neutral element of $\widehat{G}$, we obtain a wavelet system generated by Ψ:

$$\begin{aligned}\mathcal{U}(\Psi, \mathcal{A}, \mathbb{Z}^d) &= \left\{D_{A^k} T_\gamma \psi_l(\cdot) : l = 1, \dots, L,\, k \in \mathbb{Z}, \gamma \in \mathbb{Z}^d\right\} \\ &= \left\{|\det A|^{-k/2} \psi_l(A^k \cdot - \gamma) : l = 1, \dots, L,\, k \in \mathbb{Z}, \gamma \in \mathbb{Z}^d\right\},\end{aligned}$$

which is a special case of wave-packet system $\mathcal{W}(\Psi, \mathcal{A}, \mathbb{Z}^d, \mathbb{R}^d)$. It follows that two Bessel families (frames) $\mathcal{U}(\Psi, \mathcal{A}, \mathbb{Z}^d)$ and $\mathcal{U}(\Phi, \mathcal{A}, \mathbb{Z}^d)$ are pairwise orthogonal Bessel families (frames) if, and only if, we have

$$\sum_{k \in \mathbb{Z} \colon \widetilde{\alpha} \in B^k \mathbb{Z}^d} \sum_{l=1}^{L} \overline{\widehat{\varphi}_l(B^{-k}\xi)} \widehat{\psi}_l(B^{-k}(\xi + \widetilde{\alpha})) = 0, \text{ for a.e. } \xi \in \mathbb{R}^d,$$

and for all $\widetilde{\alpha} \in \bigcup_{k \in \mathbb{Z}} B^k \mathbb{Z}^d$. Note that the above result coincides with the characterization of two wavelet systems to be pairwise orthogonal frames in $L^2(\mathbb{R}^d)$ which is given by Weber in [89].

CHAPTER 4

CONSTRUCTIONS OF GENERALIZED SHIFT INVARIANT ORTHOGONAL FRAME PAIRS

In this chapter, we obtain a general construction for arbitrarily many orthogonal GSI frame pairs in the context of LCA groups by utilizing the main characterization result obtained in **Chapter 3** and the unitary extension principle (UEP) of Christensen and Goh [15]. The construction is then applied to synthesize GSI frame pairs over super-spaces. Note that the above developed constructions are useful in the derivation of explicit constructing procedures for pairwise orthogonal GSI frames generated by B-splines on the group itself as well as on characteristic functions on the dual group.

4.1. Introduction

In **Chapter 3**, we have obtained a characterization for orthogonal frame pairs with GTI structure. The characterization represents a unified way to deduce similar results for several function systems including the case of GSI systems which is studied by Kutyniok and Labate in [64]. Using the above result along with a unitary extension principle on LCA groups [15] as basic ingredients, this chapter provides a general construction technique for pairwise orthogonal GSI frames over LCA groups.

Recently, the construction techniques for some special types of frames, such as tight wavelet frames, dual wavelet frames, Gabor frames, and pairwise orthogonal wavelet frames have attracted many researchers (see [5, 9, 16, 23, 47, 48, 57, 63, 89] and references within). In [89], Weber has investigated orthogonal wavelet frames, which are useful in multiplexing techniques and in characterizing frames for super-spaces. Moreover, an explicit construction of symmetric orthogonal wavelet frames is given by Li and Yang in [70]. In this chapter, motivated by the work of Weber et al. in [9], we obtain a general construction for pairwise orthogonal frames with GSI structure. Note that generalized shift-invariant (GSI) systems in $L^2(\mathbb{R}^d)$ were introduced by Hernández, Labate and Weiss [53], and by Ron and Shen [81] as a common framework to present a unified theory for many of the familiar discrete systems, most notably the Gabor system and the wavelet system.

The GSI systems in $L^2(\mathbb{R}^d)$ are collections of functions of the form $\bigcup_{j\in J}\{T_\gamma g_j\}_{\gamma\in\Gamma_j}$, where J is a countable index set, T_γ denotes translation by γ, the generators $\{g_j\}_{j\in J}\subset L^2(\mathbb{R}^d)$, and for each $j\in J$, Γ_j is a *uniform lattice*, that is, a discrete subgroup in $\mathbb{R}^d$ such that $\mathbb{R}^d/\Gamma_j$ is compact. If $\Gamma_j=\Gamma$ for each j, we recover the case of shift-invariant (SI) system. To give examples of such systems in case of $L^2(\mathbb{R})$, by letting $a,b\in\mathbb{R}$ and $f\in L^2(\mathbb{R})$, we define the *modulation* operator $(E_bf)(x)=e^{2\pi ibx}f(x)$, and (for $a>0$) the *dilation* operator $(D_af)(x)=a^{1/2}f(ax)$. Then, the discrete wavelet system $\{D_{2^j}T_k\psi\}_{j,k\in\mathbb{Z}}$ is a GSI system of the form $\{T_{2^{-j}k}D_{2^j}\psi\}_{j,k\in\mathbb{Z}}$ which is generated by $\psi\in L^2(\mathbb{R})$ with $J=\mathbb{Z},\Gamma_j=2^{-j}\mathbb{Z}$, and $g_j=D_{2^j}\psi$. For $c,b\in\mathbb{R}$, the discrete Gabor system $\{T_{ck}E_{bt}\psi\}_{t,k\in\mathbb{Z}}$ generated by $\psi\in L^2(\mathbb{R})$ describes an SI system for the choice of $J=\{j_0\},\Gamma=c\mathbb{Z}$, and $g_{j_0}=E_{bt}\psi$.

The theory of GSI frame systems given in [53] is further generalized by Kutyniok and Labate [64] to the LCA-group setting by letting GSI systems of the form $\bigcup_{j\in J}\{T_\gamma g_j\}_{\gamma\in\Gamma_j}$ in $L^2(G)$, where G is an LCA group and for a countable index set J, the collection $\{\Gamma_j\}_{j\in J}$ denotes a family of uniform lattices in G. Since GSI systems provide a unified way for the analysis of a large class of function systems, various researchers have contributed in the investigation of frame properties for such systems (see [5, 14, 16, 53, 59, 64, 81] and references within). In this direction, our main focus is to provide a construction method for orthogonal GSI frames pairs over LCA groups. The major tool used in the construction procedure is the recent unitary extension principle technique of Christensen and Goh [15].

4.2. Unitary extension principle (UEP) on LCA groups

The unitary extension principle (UEP) investigated by Ron and Shen [80] for the Euclidean case and its many variants (e.g. [17, 23], to mention a few) play a key role in constructing tight wavelet frames with compact support, desired smoothness, and good approximation theoretic properties. Recently, the principle is further generalized to the set-up of LCA groups by Christensen and Goh in [15]. The generalization covers several variants of the unitary extension principle including the Euclidian case and the periodic case corresponding to the torus group at the same time. In [15], the authors have derived explicit conditions for the UEP construction of tight frames in the context of general LCA groups. For this, they have proved the general version of the UEP over LCA groups. In this chapter, we have utilized the UEP by Christensen and Goh [15] to give a general construction of pairwise orthogonal frames for $L^2(G)$ with GSI structure.

In order to obtain the general construction procedure of pairwise orthogonal GSI frames, we first provide a "General set-up" consisting of notation and standing assumptions needed for the remainder of this chapter. For this, we follow the general set-up of [15] for describing the UEP with some additional assumptions which we need in the sequel.

Next, we state the standing assumptions for the rest of this chapter:

General set-up: Let $\mathcal{I} := \{k\}_{k=k_0}^{\infty}$ be a sequence of consecutive numbers in $\mathbb{Z}$, $\{\Lambda_k\}_{k\in\mathcal{I}}$ be a nested sequence of lattices in G and $\{\Phi_k\}_{k\in\mathcal{I}} \subset L^2(\widehat{G})$. For each $k \in \mathcal{I}$, consider V_k as a fundamental domain associated with the lattice $\Lambda_k^{\perp}$. Then, for each $k \in \mathcal{I}$, we have

$$\widehat{G} = \bigcup_{w\in\Lambda_k^{\perp}} (w + V_k),\ (w + V_k) \cap (w' + V_k) = \phi \text{ for } w \neq w', w, w' \in \Lambda_k^{\perp}. \tag{4.1}$$

Further, we need to restate the following conditions which are used by Christensen and Goh in [15] for proving the UEP over LCA groups:

($\mathcal{O}_1$) For every compact set S in $\widehat{G}$, there exists $K_1 \in \mathcal{I}$ such that

$$\mu_{\widehat{G}}((w + S) \cap (w' + S)) = 0 \text{ for } w \neq w',\ w, w' \in \Lambda_{K_1}^{\perp}.$$

($\mathcal{O}_2$) For every compact set S in $\widehat{G}$ and any $\epsilon > 0$, there exists $K_2 \in \mathcal{I}$ such that for all $k \geq K_2, k \in \mathcal{I}$,

$$\left|\mu_{\widehat{G}}(V_k)|\Phi_k(\gamma)|^2 - 1\right| \leq \epsilon,\ \forall\gamma \in S.$$

($\mathcal{O}_3$) For all $k \in \mathcal{I}$ and some periodic functions $H_{k+1} \in L^{\infty}(V_{k+1})$, we define

$$\Phi_k(\gamma) = H_{k+1}(\gamma)\Phi_{k+1}(\gamma) \text{ for a.e. } \gamma \in \widehat{G}. \tag{4.2}$$

($\mathcal{O}_4$) For $k \in \mathcal{I}$, choose a sequence $\{v_{k,l}\}_{l=1,\ldots,d_k} \subset \widehat{G}$ such that $v_{k,1} = 0$ and

$$\Lambda_k^{\perp} = \bigcup_{l=1}^{d_k}(v_{k,l} + \Lambda_{k+1}^{\perp}),\ \ (v_{k,l} + \Lambda_{k+1}^{\perp}) \cap (v_{k,l'} + \Lambda_{k+1}^{\perp}) = \phi \ \text{ for } l \neq l',$$

where $d_k = \mu_{\widehat{G}}(V_{k+1})/\mu_{\widehat{G}}(V_k)$.

Additionally, for all $k \in \mathcal{I}$ and for $i = 1, 2$, consider the periodic functions given by

$$\Psi_k^{(i)(m)}(\gamma) = G_{k+1}^{(i)(m)}(\gamma)\Phi_{k+1}(\gamma); \, \gamma \in \widehat{G} \;\text{ and } m = 1, 2, \ldots, s_k, \tag{4.3}$$

where $\{G_{k+1}^{(i)(m)} \in L^\infty(V_{k+1}) : m = 0, 1, \ldots, s_k\}$ is a collection of given periodic functions with $G_{k+1}^{(1)(0)} = G_{k+1}^{(2)(0)} = H_{k+1}$. Further, for all $k \in \mathcal{I}$ and for $i = 1, 2$, let the matrix-valued functions $M_k^{(i)}$ and $N_k^{(i)}$ be defined for a.e. $\gamma \in V_k$ by

$$M_k^{(i)}(\gamma) := \left(G_{k+1}^{(i)(m)}(\gamma + v_{k,n})\right)_{\substack{0 \leq m \leq s_k \\ 1 \leq n \leq d_k}} \text{ and } N_k^{(i)}(\gamma) := \left(G_{k+1}^{(i)(m)}(\gamma + v_{k,n})\right)_{\substack{1 \leq m \leq s_k \\ 1 \leq n \leq d_k}}, \tag{4.4}$$

respectively, with $(M_k^{(i)})^*$ as the conjugate transpose of the matrix $M_k^{(i)}$.

For each $k \in \mathcal{I}$ such that $k \geq k_0 + 1$, let the index set

$$P_k := \{(m, k) : m = 1, 2, \ldots, s_k\} \tag{4.5}$$

with $P_{k_0} := \{(m, k_0) : m = 0, 1, 2, \ldots, s_{k_0}\}$.

Now, for each $i = 1, 2$ and $p \in P_k$, we assume the functions $g_p^{(i)}$ in $L^2(G)$ given by

$$g_p^{(i)} = \begin{cases} g_{(m,k_0)}^{(i)} = \mathcal{F}^{-1}\Phi_{k_0} & \text{if } p = (m, k_0) \in P_{k_0}; \\ g_{(m,k)}^{(i)} = \mathcal{F}^{-1}\Psi_k^{(i)(m)} & \text{if } \; p = (m, k) \in P_k, \; k \geq k_0 + 1, \end{cases} \tag{4.6}$$

with $\mathcal{F}^{-1}$ as a symbol for the inverse Fourier transform on $L^2(\widehat{G})$.

Using the above assumptions, we state the modified form of the UEP [15] as follows:

Theorem 4.1. *In addition to the assumptions $(\mathcal{O}_1) - (\mathcal{O}_3)$ in the general set-up, assume that for each $k \in \mathcal{I}$, the matrix-valued function $M_k^{(1)}$ defined in (4.4) satisfies the following:*

$$(M_k^{(1)})^*(\gamma) M_k^{(1)}(\gamma) = d_k I_{d_k}, \;\; \textit{for a.e. } \gamma \in V_k.$$

Then, for each $p \in P_k$, by assuming the functions $g_p^{(1)}$ defined as in (4.6), the GSI system $\bigcup_{k \in \mathcal{I}} \{T_\lambda g_p^{(1)}\}_{\lambda \in \Lambda_k, p \in P_k}$ forms a Parseval frame for $L^2(G)$.

4.3. A general construction for orthogonal frame pairs via UEP

In this section, by using the assumptions from the "General set-up"and the modified form of UEP given in Theorem 4.1, we obtain a general construction procedure for arbitrarily many pairwise orthogonal GSI frames. The next result provides a general method to construct a pair of orthogonal frames with GSI structure:

Theorem 4.2. *In addition to the assumptions* $(\mathcal{O}_1)-(\mathcal{O}_4)$ *and* (4.4)–(4.6) *in the general set-up, assume that for each* $k \in \mathcal{I}$*, the periodic function* $H_{k+1} \in L^\infty(V_{k+1})$ *(as defined in (4.2)) satisfies the conditions of Theorem 4.1. Additionally, for each* $k \in \mathcal{I}$*, let the matrix-valued functions* $M_k^{(1)}$*,* $M_k^{(2)}$*,* $N_k^{(1)}$ *and* $N_k^{(2)}$ *satisfy the following conditions:*

(i) $(M_k^{(i)})^*(\gamma)M_k^{(i)}(\gamma) = d_k I_{d_k}$ *for each* $i = 1, 2$ *and for a.e.* $\gamma \in V_k$*;*

(ii) $(N_k^{(1)})^*(\gamma)N_k^{(2)}(\gamma) = 0$ *for a.e.* $\gamma \in V_k$*;*

(iii) $(M_{k_0}^{(1)})^*(\gamma)M_{k_0}^{(2)}(\gamma) = 0$ *for a.e.* $\gamma \in V_{k_0}$*.*

Then, the collection of functions given by $\bigcup_{k\in\mathcal{I}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Lambda_k,p\in P_k}$ *and* $\bigcup_{k\in\mathcal{I}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Lambda_k,p\in P_k}$ *generate orthogonal Parseval GSI frame pairs in* $L^2(G)$*.*

For proving Theorem 4.2, the following result is needed in the sequel:

Theorem 4.3. *For each* $k \in \mathcal{I}$ *and* $p \in P_k$*, let* $\mu_{\widehat{G}}\Big(supp(\widehat{g_p^{(1)}}) \bigcap supp(\widehat{g_p^{(2)}})(\cdot + \alpha)\Big) = 0$ *for all* $\alpha \in \Gamma_k^\perp \setminus \{0\}$*. Suppose that the GSI systems given by* $\bigcup_{k\in\mathcal{I}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Lambda_k,p\in P_k}$ *and* $\bigcup_{k\in\mathcal{I}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Lambda_k,p\in P_k}$ *are Bessel families in* $L^2(G)$*. Then, the systems form:*

(i) *dual frames for* $L^2(G)$ *if, and only if, we have*

$$\sum_{k\in\mathcal{I}}\sum_{p\in P_k} \overline{(\widehat{g_p^{(1)}})(w)}(\widehat{g_p^{(2)}})(w) = 1, \text{ for a.e. } w \in \widehat{G},$$

(ii) *orthogonal frames for* $L^2(G)$ *if, and only if, we have*

$$\sum_{k\in\mathcal{I}}\sum_{p\in P_k} \overline{(\widehat{g_p^{(1)}})(w)}(\widehat{g_p^{(2)}})(w) = 0, \text{ for a.e. } w \in \widehat{G}.$$

Proof. Given that $\text{supp}(\widehat{g_p^{(1)}}) \cap \text{supp}(\widehat{g_p^{(2)}})(\cdot + \alpha) = \phi$ for all $p \in P_k$ and $k \in \mathcal{I}$ with $\alpha \in \Gamma_k^\perp \setminus \{0\}$. Therefore, the proof can be easily deduced by using $\alpha = 0$ in the duality characterization result given by Jakobsen and Lemvig in [59, Theorem 3.4] and in the characterization of pairwise orthogonal GTI systems, that is, Theorem 3.5 which we have obtained in Chapter 3. □

Proof of Theorem 4.2. By using the given condition in part (i) and the modified UEP stated in Theorem 4.1, it follows that $\bigcup_{k\in\mathcal{I}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Lambda_k,p\in P_k}$ and $\bigcup_{k\in\mathcal{I}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Lambda_k,p\in P_k}$ generate Parseval GSI frames for $L^2(G)$.

Before proceeding further, we first simplify the conditions given in part (ii) and part (iii). In part (ii), it is given that for each $k \in \mathcal{I}$, the matrix-valued functions $N_k^{(1)}$ and

$N_k^{(2)}$ (as defined in (4.4)) satisfy the following condition:

$$(N_k^{(1)})^*(\gamma)N_k^{(2)}(\gamma) = \left(\sum_{m=1}^{s_k} \overline{G_{k+1}^{(1)(m)}(\gamma+v_{k,i})}G_{k+1}^{(2)(m)}(\gamma+v_{k,j})\right)_{\substack{1\le i\le d_k\\ 1\le j\le d_k}} = 0,$$

for a.e. $\gamma \in V_k$, which further holds if, and only if, we have

$$\sum_{m=1}^{s_k} \overline{G_{k+1}^{(1)(m)}(\gamma+v_{k,i})}G_{k+1}^{(2)(m)}(\gamma+v_{k,j}) = 0, \text{ for all } 1\le i,j\le d_k, \tag{4.7}$$

which by observing the condition $(\mathcal{O}_4)$ from the general set-up along with the equality (4.1), is further equivalent to the following equation:

$$\sum_{m=1}^{s_k} \overline{G_{k+1}^{(1)(m)}(w)}G_{k+1}^{(2)(m)}(w) = 0 \text{ for a.e. } w\in\widehat{G}. \tag{4.8}$$

Similarly, we can prove that the condition given in part (iii), $(M_{k_0}^{(1)})^*(\gamma)M_{k_0}^{(2)}(\gamma) = 0$ for a.e. $\gamma\in V_{k_0}$ holds if, and only if, we have

$$\sum_{m=0}^{s_{k_0}} \overline{G_{k_0+1}^{(1)(m)}(w)}G_{k_0+1}^{(2)(m)}(w) = 0 \text{ for a.e. } w\in\widehat{G}, \tag{4.9}$$

Next, we proceed for proving the orthogonality of GSI frame systems. For this, we use the characterization equations of Theorem 4.3(ii) to prove the orthogonality conditions. For a.e. $w\in\widehat{G}$, consider the following:

$$\begin{aligned}
&\sum_{k\in\mathcal{I}}\sum_{p\in P_k}\overline{\widehat{(g_p^{(1)})}(w)}\widehat{(g_p^{(2)})}(w)\\
&\quad=\sum_{p\in P_{k_0}}\overline{\widehat{(g_p^{(1)})}(w)}\widehat{(g_p^{(2)})}(w)+\sum_{k\in\mathcal{I}\setminus\{k_0\}}\sum_{p\in P_k}\overline{\widehat{(g_p^{(1)})}(w)}\widehat{(g_p^{(2)})}(w)\\
&\quad=\sum_{m=0}^{s_{k_0}}\overline{\widehat{(g_{(m,k_0)}^{(1)})}(w)}\widehat{(g_{(m,k_0)}^{(2)})}(w)+\sum_{k=k_0+1}^{\infty}\sum_{m=1}^{s_k}\overline{\widehat{(g_{(m,k)}^{(1)})}(w)}\widehat{(g_{(m,k)}^{(2)})}(w)\\
&\quad=\overline{\widehat{(g_{(0,k_0)}^{(1)})}(w)}\widehat{(g_{(0,k_0)}^{(2)})}(w)+\sum_{m=1}^{s_{k_0}}\overline{\widehat{(g_{(m,k_0)}^{(1)})}(w)}\widehat{(g_{(m,k_0)}^{(2)})}(w)\\
&\qquad+\sum_{k=k_0+1}^{\infty}\sum_{m=1}^{s_k}\overline{\widehat{(g_{(m,k)}^{(1)})}(w)}\widehat{(g_{(m,k)}^{(2)})}(w)\\
&\quad=\overline{\widehat{(g_{(0,k_0)}^{(1)})}(w)}\widehat{(g_{(0,k_0)}^{(2)})}(w)+\sum_{k\in\mathcal{I}}\sum_{m=1}^{s_k}\overline{\widehat{(g_{(m,k)}^{(1)})}(w)}\widehat{(g_{(m,k)}^{(2)})}(w).
\end{aligned}$$

By using (4.2), (4.3) and (4.6), the above expression is further equivalent to the following:

$$\overline{(\widehat{\mathcal{F}^{-1}\Phi_{k_0}})(w)}(\widehat{\mathcal{F}^{-1}\Phi_{k_0}})(w) + \sum_{k\in\mathcal{I}}\sum_{m=1}^{s_k} \overline{(\widehat{\mathcal{F}^{-1}\Psi_k^{(1)(m)}})(w)}(\widehat{\mathcal{F}^{-1}\Psi_k^{(2)(m)}})(w)$$

$$= \overline{\Phi_{k_0}(w)}\Phi_{k_0}(w) + \sum_{k\in\mathcal{I}}\sum_{m=1}^{s_k} \overline{\Psi_k^{(1)(m)}(w)}\Psi_k^{(2)(m)}(w)$$

$$= \overline{H_{k_0+1}(w)\Phi_{k_0+1}(w)}H_{k_0+1}(w)\Phi_{k_0+1}(w) + \sum_{k\in\mathcal{I}}\sum_{m=1}^{s_k} \overline{G_{k+1}^{(1)(m)}(w)\Phi_{k+1}(w)}G_{k+1}^{(2)(m)}(w)\Phi_{k+1}(w)$$

$$= \overline{G_{k_0+1}^{(1)(0)}(w)}G_{k_0+1}^{(2)(0)}(w)|\Phi_{k_0+1}(w)|^2 + \sum_{k\in\mathcal{I}}|\Phi_{k+1}(w)|^2\big(\sum_{m=1}^{s_k}\overline{G_{k+1}^{(1)(m)}(w)}G_{k+1}^{(2)(m)}(w)\big)$$

$$= \overline{G_{k_0+1}^{(1)(0)}(w)}G_{k_0+1}^{(2)(0)}(w)|\Phi_{k_0+1}(w)|^2 + |\Phi_{k_0+1}(w)|^2\sum_{m=1}^{s_{k_0}}\overline{G_{k_0+1}^{(1)(m)}(w)}G_{k_0+1}^{(2)(m)}(w)$$

$$+ \sum_{k\in\mathcal{I}\setminus\{k_0\}}|\Phi_{k+1}(w)|^2\big(\sum_{m=1}^{s_k}\overline{G_{k+1}^{(1)(m)}(w)}G_{k+1}^{(2)(m)}(w)\big)$$

$$= |\Phi_{k_0+1}(w)|^2\big(\sum_{m=0}^{s_{k_0}}\overline{G_{k_0+1}^{(1)(m)}(w)}G_{k_0+1}^{(2)(m)}(w)\big) + \sum_{k\in\mathcal{I}\setminus\{k_0\}}|\Phi_{k+1}(w)|^2\big(\sum_{m=1}^{s_k}\overline{G_{k+1}^{(1)(m)}(w)}G_{k+1}^{(2)(m)}(w)\big).$$

In view of the equations (4.8)-(4.9) along with the above computation, we get

$$\sum_{k\in\mathcal{I}}\sum_{p\in P_k}\overline{(\widehat{g_p^{(1)}})(w)}(\widehat{g_p^{(2)}})(w) = 0$$

for a.e. $w \in \widehat{G}$. Hence, the result follows.

□

For the better understanding of the construction method obtained in Theorem 4.2, we provide the following example in the set-up of $L^2(\mathbb{R})$:

Example 4.4. By following Section 4.1 for the modulation, translation and dilation operators on $L^2(\mathbb{R})$, consider a function $\varphi \in L^2(\mathbb{R})$ satisfying that $\widehat{\varphi}(\gamma) \to 1$ as $\gamma \to 0$. In the "General set-up", let $G = \mathbb{R}$, $\mathcal{I} := \mathbb{Z}$ and for each $k \in \mathbb{Z}$, consider $\Phi_k := \widehat{D_{a^k}E_{bk}\varphi}$ and $\Lambda_k := a^{-k}\mathbb{Z}$. Then, $\Lambda^\perp := a^k\mathbb{Z}$, with the fundamental domain $V_k := [0, a^k)$. By following the initial steps of [15, Example 3.7], we get $d_k := a$ and $\{v_{k,l}\}_{l=1,\ldots,d_k} := \{(l-1)a^k\}_{l=1,\ldots,a}$ for each $k \in \mathbb{Z}$. Now, the scaling equation described in [15, equation (3.25)] in our case means that

$$(\widehat{D_{a^{-1}}E_{-b}\varphi})(\gamma) = H_0(\gamma)(\widehat{E_{-b}\varphi})(\gamma), \quad \text{for } \gamma \in \mathbb{R},$$

with $H_0 \in L^\infty([0,1)$. The above expression is further equivalent to the following:

$$a^{1/2}\widehat{\varphi}(a\gamma + b) = H_0(\gamma)\widehat{\varphi}(\gamma + b), \text{ for } \gamma \in \mathbb{R},$$

which is the classical scaling equation in wave-packet analysis. By using the above discussion and assumptions of "General set-up"along with the procedure of [15, Example 3.7], note that the matrix-valued functions $M_k^{(i)}$ and $N_k^{(i)}$ becomes

$$M_k^{(i)}(\gamma) = M_{-1}^{(i)}(a^{-k-1}\gamma), \text{ and } N_k^{(i)}(\gamma) = N_{-1}^{(i)}(a^{-k-1}\gamma)$$

for each $\gamma \in V_k = [0, a^k)$, where $k \in \mathbb{Z}$ and $i = 1, 2$. Since $a^{-k-1}\gamma$ runs through $V_{-1} := [0, a^{-1})$ when γ runs through V_k, the assumptions given in Theorem 4.2 now reduces to the following for all $k \in \mathbb{Z}$:

(a_1) $(M_{-1}^{(i)})^*(\gamma)M_{-1}^{(i)}(\gamma) = aI_a$ for each $i = 1, 2$ and for a.e. $\gamma \in V_{-1} := [0, a^{-1})$;
(a_2) $(N_{-1}^{(1)})^*(\gamma)N_{-1}^{(2)}(\gamma) = 0$ for a.e. $\gamma \in V_{-1}$;
(a_3) $(M_{-1}^{(1)})^*(\gamma)M_{-1}^{(2)}(\gamma) = 0$ for a.e. $\gamma \in V_{-1}$.

Under the assumptions $(a_1) - (a_3)$ along with the conditions used in Theorem 4.2, we can generate orthogonal Parseval wave-packet frame pairs in $L^2(\mathbb{R})$ by choosing the suitable generators according to the choice of $\Phi_k := \widehat{D_{a^k}E_{bk}\varphi}$ for each $k \in \mathbb{Z}$.

4.4. Applications: Construction of frame pairs over super-spaces

Let $\mathbb{H}$ be a complex-valued Hilbert space. In order to study the applications of our construction technique which we have obtained in the last section, we need to introduce an $\mathbb{H}$-*valued generalized translation invariant system*. To define such systems, let $n \in \mathbb{N} \cup \{\infty\}$ and consider the following notation:

$$\mathcal{Z}_n := \begin{cases} \mathbb{Z}/n\mathbb{Z} \text{ (under addition modulo } n) & \text{if } n < \infty; \\ \mathbb{Z} & \text{if } n = \infty, \end{cases} \tag{4.10}$$

and $\ell^2(\mathcal{Z}_n)$ as a collection of all square summable sequences α with norm

$$||\alpha||^2_{\ell^2(\mathcal{Z}_n)} = \sum_{k \in \mathcal{Z}_n} |\alpha(k)|^2 < \infty.$$

4.4.1. $\mathbb{H}$-valued generalized translation invariant systems

We begin by defining a Hilbert space valued system (also known as vector-valued (super) space) in the context of locally compact abelian (LCA) groups. For this, let the notation $\mathbb{H}$ represents a Hilbert space $\mathbb{H} := \ell^2(\mathcal{Z}_d)$, where $\mathcal{Z}_d$ is as described in (4.10) for $d \in \mathbb{N} \cup \{\infty\}$. Then, the collection of functions

$$L^2(G, \mathbb{H}) := \Big\{\text{measurable } \mathbf{f} : G \to \mathbb{H} \text{ such that } ||\mathbf{f}||^2 := \int_G ||\mathbf{f}(x)||^2_{\mathbb{H}} d\mu_G(x) < \infty\Big\}, \tag{4.11}$$

is a Hilbert space, we call as a *Hilbert space valued* ($\mathbb{H}$-*valued*) *system, or, vector-valued* (*super*) *space* endowed with the inner product given by

$$\langle \mathbf{f}^1, \mathbf{f}^2 \rangle := \int_G \langle \mathbf{f}^1(x), \mathbf{f}^2(x) \rangle_{\mathbb{H}} d\mu_G(x),$$

for all $\mathbf{f}^i := (\mathbf{f}^i(x))_{x \in G}$ in $L^2(G, \mathbb{H})$ with $i = 1, 2$, and $\mathbf{f}^i(x) := (\mathbf{f}^i(x, m))_{m \in \mathcal{Z}_d}$ in $\mathbb{H}$ for each x in G. Note that G is a second countable LCA group which implies G is σ-compact, and hence, σ-finite. Thus, the $\mathbb{H}$-valued system satisfies the following equality:

$$L^2(G, \mathbb{H}) = L^2(G, \ell^2(\mathcal{Z}_d)) = L^2(G \times \mathcal{Z}_d), \tag{4.12}$$

in view of Proposition Appendix A.3 from [6] and the fact that $\ell^2(\mathcal{Z}_d)$ is separable.

By using the above mentioned set-up, we first modify the definition of GTI system given in Definition 3.1. Then, we introduce the notion of the $\mathbb{H}$-*valued generalized translation invariant system*. Note that the families of functions in the definitions given below satisfy the assumptions of standing hypothesis which are stated in **Chapter 3**.

Remark 4.5. For each $n \in \mathbb{N} \cup \{\infty\}$ and for $\mathrm{j} \in \mathcal{Z}_n$, let P_j be a countable or an uncountable index set, let $g_p \in L^2(G \times \mathcal{Z}_n)$ for $p \in P_\mathrm{j}$, and let Γ_j be a closed, co-compact subgroup in G. Then, the GTI system generated by $\{g_p\}_{p \in P_\mathrm{j}, \mathrm{j} \in \mathcal{Z}_n} \subset L^2(G \times \mathcal{Z}_n)$ with translation along closed, co-compact subgroups $\{\Gamma_j\}_{j \in \mathcal{Z}_n}$ is the family of functions in $L^2(G)$ which is given by $\bigcup_{j \in \mathcal{Z}_n} \{T_\gamma g_p\}_{\gamma \in \Gamma_\mathrm{j}, p \in P_\mathrm{j}}$.

Definition 4.6. For each $d, n \in \mathbb{N} \cup \{\infty\}$ and for $\mathrm{j} \in \mathcal{Z}_n$, let P_j be a countable or an uncountable index set, let $g_p \in L^2(G \times \mathcal{Z}_d \times \mathcal{Z}_n)$ for $p \in P_\mathrm{j}$, and let Γ_j be a closed, co-compact subgroup in G. Then, the $\mathbb{H}$-*valued generalized translation invariant* ($\mathbb{H}$-*valued GTI*) *system* generated by $\{g_p\}_{p \in P_\mathrm{j}, \mathrm{j} \in \mathcal{Z}_n} \subset L^2(G \times \mathcal{Z}_d \times \mathcal{Z}_n)$ with translation along closed, co-compact subgroups $\{\Gamma_{\mathrm{j}} \times \{0\}\}_{\mathrm{j} \in \mathcal{Z}_n}$ is the family of functions in $L^2(G \times \mathcal{Z}_d)$ which is given by $\bigcup\limits_{j \in \mathcal{Z}_n} \{T_\gamma g_p\}_{\gamma \in \Gamma_{\mathrm{j}} \times \{0\},\, p \in P_{\mathrm{j}}}$.

Note that the above definition can be used to deduce the case of GSI system over super-spaces. Further, in view of Remark 4.5 and Definition 4.6, the following is another important observation related to the concept of frames over super-spaces in context of LCA groups:

Remark 4.7. We mention that while the concept of vector-valued frames is an addition to the frame theory of functions on $L^2(\mathbb{R})$ it is not for frames on $L^2(G)$, where G is a second countable LCA group. This is because the "$\mathbb{H}$-valued system" $L^2(G, \mathbb{H})$ defined in (4.11) coincides with the space $L^2(G \times \mathcal{Z}_d)$, where $\mathcal{Z}_d$ is the locally compact abelian, in fact, countable and discrete group as described in (4.10). Hence the "$\mathbb{H}$-valued system" $L^2(G, \mathbb{H})$ is just L^2 for the group $G \times \mathcal{Z}_d$.

By using the important observation of Remark 4.7, we conclude that Theorem 4.2 also provides the construction technique for orthogonal Parseval frame pairs over super-spaces in the context of LCA groups by choosing generators for GSI systems for super-spaces in accordance with the Definition 4.6, and by using corresponding modifications in the assumptions of "General set-up".

CHAPTER 5

GROUP THEORETIC CONSTRUCTION OF FINITE TIME-FREQUENCY LOCALIZED ONWS

In this chapter, we give a group-theoretic construction of a finite orthonormal wavelet system (ONWS) which is concentrated in time as well as frequency. Further, we study and characterize the ONWS. Finally, some results on the uncertainty principle corresponding to the ONWS are obtained.

5.1. Introduction

In **Chapters 2-4**, we have characterized and constructed orthogonal frame pairs in the context of a general (second countable) locally compact abelian (LCA) group G. Since practical life applications mainly require frames in finite setting, this motivates us to relate our work in the thesis with the finite set-up by letting LCA group G of the form $\mathbb{Z}_N^d$. Therefore, the main focus of **Chapters 5-6** is to give a construction of finite time-frequency localized frame systems with wavelet structure.

In this direction, we construct a finite time-frequency localized orthonormal wavelet system in the current chapter. The construction is obtained by using a new group-theoretic approach based on the complete digit set associated to an invertible matrix. By continuing this work in **Chapter 6**, we generalize the above construction to frame set-up and study orthogonality and duality properties of such finite frame pairs, with applications to frames over super-spaces.

In the present chapter, we provide a group-theoretic construction of a finite time-frequency localized basis with wavelet structure, and study time-frequency localization properties of this basis via uncertainty principle. Note that a time-frequency localized basis plays an important role in extracting both time as well as frequency information of a given signal, and the uncertainty principle helps us to understand how much local information in time and frequency we can extract by using the above-mentioned basis.

A function (signal) f with domain D is said to be *time localized/ frequency localized* near a point $n_0 \in D$, if for $n \in D$, most of the (components $f(n)$ of f)/(components of Fourier transform of f) are 0 or relatively small except for a few values of n close to n_0. By a *time-frequency localized basis* (TFL-basis), we mean that every vector in the basis is time localized as well as frequency localized. In the last two decades, a lot of research on time-frequency analysis has been done by several authors for the various spaces, namely, finite and infinite abelian groups, Euclidean spaces, etc. (e.g. [26, 37, 44, 45, 49, 65, 82, 88, 89]).

A TFL-basis has enormous applications in the problems related to real life as well as in different areas of mathematics. Due to this, many researchers are attracted towards the problem of finding good bases having both time as well as frequency localization properties (e.g. [26, 72, 82, 88]). Furthermore, in this day and age of computers, processing can be done only when the signal can be stored in memory. Therefore, the importance of discrete and finite signals cannot be ignored.

At this juncture, it is pertinent to note that the standard orthonormal basis for $\mathbb{C}^n$ is time localized but not frequency localized, while its Fourier basis is frequency localized but not time localized. Therefore, we need to search another basis for $\mathbb{C}^n$, which is localized in time as well as frequency.

Wavelets are the latest and most successful tools to extract information from many different kinds of data including but certainly not limited to audio signals and images. Motivated from the construction of wavelets on $\mathbb{Z}_N$ by Frazier (see [26, Definition 3.4]), the main goal of this chapter is to study an orthonormal wavelet system for $\ell^2(\mathbb{Z}_N^d)$ along with its time-frequency localization properties via uncertainty principle, which is obtained by generalizing the set-up mentioned above in [26] to multidimensional case. We proceed by providing motivation for this particular form of system, which we introduce in (5.2). It is actually inspired from the following system of translates in $\ell^2(\mathbb{Z}_N)$ having time-frequency localization properties, which is defined and studied by Frazier [26].

Definition 5.1. Let $N = 2M$ for some $M \in \mathbb{N}$ and let $u_1, u_2 \in \ell^2(\mathbb{Z}_N)$. An orthonormal basis for $\ell^2(\mathbb{Z}_N)$ of the form

$$\{u_i(\cdot - 2k) : 1 \leq i \leq 2,\ 0 \leq k \leq M-1\}, \tag{5.1}$$

is called an orthonormal wavelet system for $\ell^2(\mathbb{Z}_N)$.

Inspired from the collection of the form (5.1), we wish to develop the theory related to the collection in a multidimensional set-up for the more general case. In particular, we want to find conditions on the sets I_1, I_2 and $A \in GL(d, \mathbb{R})$, and to characterize $\Phi = \{\varphi_p\}_{p \in I_2} \subset \ell^2(\mathbb{Z}_N^d)$ such that the collection $\mathfrak{B}(\Phi, A, I_1)$ (with distinct elements) defined by

$$\mathfrak{B}(\Phi, A, I_1) := \{T_{Ak}\varphi_p : k \in I_1 \subseteq \mathbb{Z}_N^d,\ p \in I_2 \subset \mathbb{N}\} \tag{5.2}$$

forms a TFL-basis for $\ell^2(\mathbb{Z}_N^d)$, where for each $k \in \mathbb{Z}_N^d$ and $f \in \ell^2(\mathbb{Z}_N^d)$, the *translation operator* T_k on $\ell^2(\mathbb{Z}_N^d)$ is defined by $T_k(f)(m) = f(m-k)$, for all $m \in \mathbb{Z}_N^d$. For $N \in \mathbb{N}$, the space $\ell^2(\mathbb{Z}_N^d)$ denotes the collection of all $N^d \times 1$-complex vectors. Here, note that it is an N^d-dimensional Hilbert space with usual inner product $\langle f, g\rangle = \sum\limits_{n \in \mathbb{Z}_N^d} f(n)\overline{g(n)}$, for all $f, g \in \ell^2(\mathbb{Z}_N^d)$, and associated norm $||\cdot|| := \langle \cdot, \cdot \rangle^{1/2}$, where $\mathbb{Z}_N := \{0, 1, 2, \ldots, N-1\}$ denotes a group of integers under addition modulo N.

In order to compare $\mathfrak{B}(\Phi, A, I_1)$ with the classical notion of an orthonormal multiwavelet [90], let us consider $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\} \subset L^2(\mathbb{R}^d)$ with the dilation matrix A_0 and translation $\mathbb{Z}^n$, which provides an orthonormal basis $\mathcal{A}(\Psi, A_0)$ for $L^2(\mathbb{R}^d)$, where

$$\mathcal{A}(\Psi, A_0) := \{\psi_{j,k}^l := |\det(A_0)|^{j/2}\psi_l(A_0^j \cdot -k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d \text{ and } \psi_l \in \Psi\}.$$

In view of the system $\mathcal{A}(\Psi, A_0)$, note that the elements of $\mathfrak{B}(\Phi, A, I_1)$ can be written as $\varphi_{0,Ak}^p = T_{Ak}\varphi_p$, where $p \in I_2, k \in I_1$. Further, we remark that if $\mathfrak{B}(\Phi, A, I_1)$ is an orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$, then one cannot construct an orthogonal set by adding elements into $\mathfrak{B}(\Phi, A, I_1)$. This, in turn, implies that we are able to get an orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$ in terms of translations of Φ, without using any dilation operator. Thus, it leads us to conclude that we cannot use the natural definition of orthonormal multiwavelet for $L^2(\mathbb{R}^n)$ in our case. Throughout the chapter, we call such system $\mathfrak{B}(\Phi, A, I_1)$ as an orthonormal wavelet system (ONWS) in $\ell^2(\mathbb{Z}_N^d)$.

Further, in this chapter, we study the uncertainty principle in terms of the pair of representations of a given signal by coupling ONWS with the standard basis and Fourier basis. This study can be useful to provide uniqueness properties of the sparse representation of the signal. The theory of uncertainty principle related to the pair of bases and its applications has been developed by many authors (e.g. [22, 24, 37, 62, 65, 82]).

Throughout the chapter, for $d, N \in \mathbb{N}$, the notations $\mathcal{M}(d,\mathbb{K})/\ \widetilde{\mathcal{M}}(d,\mathbb{K})$ denote the collection of all $d \times d$ (matrices)/(invertible matrices) over $\mathbb{K}$, respectively, where $\mathbb{K}$ is used as $\mathbb{Z}$ (the set of integers) or $\mathbb{R}$ (the set of real numbers). The notation $|\cdot|$ denotes the cardinality of a set, or, the absolute value of a complex number. For $m, n \in \mathbb{N}$, the symbol $\overline{M}^t$ represents the conjugate transpose of any $m \times n$ matrix M. For $A \in \mathcal{M}(d,\mathbb{Z})$ and $\widetilde{I} \subseteq \mathbb{Z}_N^d$, we get $A\widetilde{I} \subseteq \mathbb{Z}_N^d$, where its entries are obtained by applying addition modulo N (denoted by $mod\, N$) to each coordinate of $A\widetilde{I}$. Note that we represent any arbitrary element of $\mathbb{Z}_N^d$ as a $d \times 1$ column vector.

For $f, g \in \ell^2(\mathbb{Z}_N^d)$, the *discrete Fourier transform* (DFT) and *inverse discrete Fourier transform* (IDFT) on $\ell^2(\mathbb{Z}_N^d)$ are defined for each $m, n \in \mathbb{Z}_N^d$ by

$$\widehat{f}(m) = \sum_{s \in \mathbb{Z}_N^d} f(s) e^{-2\pi i \langle m,s\rangle/N} \text{ and } g^{\vee}(n) = \frac{1}{N^d} \sum_{l \in \mathbb{Z}_N^d} g(l) e^{2\pi i \langle n,l\rangle/N},$$

respectively. Here, note that the formulas for DFT and IDFT can be defined for all $m, n \in \mathbb{Z}^d$ by observing the relations $\widehat{f}(m + jN) = \widehat{f}(m)$, and $g^{\vee}(n + jN) = g^{\vee}(n)$, for all $f, g \in \ell^2(\mathbb{Z}_N^d)$, $j \in \mathbb{Z}^d$. For proving various results in upcoming sections, we need to state the following properties related to DFT and the translation operator on $\ell^2(\mathbb{Z}_N^d)$:

(P1) For $f, g \in \ell^2(\mathbb{Z}_N^d)$, the *Plancherel's formula* is given by

$$\langle f, g\rangle = \frac{1}{N^d} \sum_{m \in \mathbb{Z}_N^d} \widehat{f}(m)\overline{\widehat{g}(m)} = \frac{1}{N^d}\langle \widehat{f}, \widehat{g}\rangle,$$

which leads to *Parseval's formula* for $f = g$.

(P2) Let $k, k_1 \in \mathbb{Z}_N^d$. Then for $f, g \in \ell^2(\mathbb{Z}_N^d)$, we can easily observe the following:

   (i) $\widehat{(T_{Ak} f)}(m) = e^{-2\pi i \langle m, Ak\rangle/N}\widehat{f}(m)$, for all $m \in \mathbb{Z}_N^d$.

   (ii) $\langle T_{Ak_1} f, T_{Ak} g\rangle = \langle f, T_{(Ak - Ak_1)} g\rangle$.

(P3) $|\widehat{\delta}(k)| = 1$, where $\delta(k) = 1$ for $k = 0$, and zero for $0 \neq k \in \mathbb{Z}_N^d$.

In the next section, we study some group-theoretic results, which help us to construct a TFL-basis for $\ell^2(\mathbb{Z}_N^d)$:

5.2. Group-theoretic construction of an ONWS in $\ell^2(\mathbb{Z}_N^d)$

Our main motive in this section is to answer the question imposed for the system $\mathfrak{B}(\Phi, A, I_1)$ defined in (5.2). For this, we need to recall and establish some group-theoretic results. From (5.2), it is clear that $A\mathbb{Z}_N^d \subseteq \mathbb{Z}_N^d$, and $|\mathfrak{B}(\Phi, A, I_1)| = |AI_1||I_2|$. Further, we note that the system $\mathfrak{B}(\Phi, A, I_1)$ should have N^d distinct elements to become an orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$. Hence, we get $N^d = |\mathfrak{B}(\Phi, A, I_1)| = |AI_1||I_2|$, which implies that $|I_2| = N^d/|AI_1|$. To find $|I_2|$, firstly we are interested in computing $|AI_1|$, where $I_1 \subseteq \mathbb{Z}_N^d$. Next, we observe that in order to get the distinctness property in $\mathfrak{B}(\Phi, A, I_1)$, the form of I_1 should be such that AI_1 behaves like a group.

Now, by using the fact that $\mathbb{Z}_N^d$ is an abelian group, we conclude that $A\mathbb{Z}_N^d$ is a normal subgroup of $\mathbb{Z}_N^d$. The motivation for considering subgroups of $\mathbb{Z}_N^d$ of this type comes from the fact that every subgroup of $\mathbb{Z}^d$ is of the form $\widetilde{A}\mathbb{Z}^d$ for some $\widetilde{A} \in \mathcal{M}(d, \mathbb{Z})$.

Note that if $\widetilde{A}$ is a $d \times d$ expansive matrix with integer entries, then the order of group $\dfrac{\mathbb{Z}^d}{\widetilde{A}\mathbb{Z}^d}$ is $|\det(A)|$ (see [90, Proposition 5.5]), while the result is also true for $\widetilde{A} \in \widetilde{\mathcal{M}}(d, \mathbb{Z})$. Here, by an *expansive matrix*, we mean that all of its eigenvalues λ satisfy $|\lambda| > 1$.

Next, we proceed for the computation of $|AI_1|$, where $I_1 \subseteq \mathbb{Z}_N^d$. In general, here we are interested in finding the order of group $A\mathbb{Z}_N^d$. Therefore, we have the following result:

Proposition 5.2. *For $N \in \mathbb{N}$, let the matrix $A \in \widetilde{\mathcal{M}}(d, \mathbb{Z})$ be such that $B := NA^{-1} \in \widetilde{\mathcal{M}}(d, \mathbb{Z})$. Then, the determinant $\det(A)$ of A divides N^d, and hence the order of the groups $A\mathbb{Z}_N^d$ and $\dfrac{\mathbb{Z}_N^d}{B\mathbb{Z}_N^d}$ are respectively given by $|A\mathbb{Z}_N^d| = \dfrac{N^d}{|\det(A)|}$ and $\left|\dfrac{\mathbb{Z}_N^d}{B\mathbb{Z}_N^d}\right| = |\det(B)|$.*

Proof. The key for this proof lies in understanding the significance of assuming the matrix $B = NA^{-1} \in \widetilde{\mathcal{M}}(d, \mathbb{Z})$, which implies that $\frac{\mathbb{Z}^d}{B\mathbb{Z}^d}$ is defined, and hence $\left|\frac{\mathbb{Z}^d}{B\mathbb{Z}^d}\right| = |\det(B)|$ (see [90, Proposition 5.5]). Now, by using this particular form of B, we claim to establish an isomorphism between the groups $A\mathbb{Z}_N^d$ and $\frac{\mathbb{Z}^d}{B\mathbb{Z}^d}$, that means, $A\mathbb{Z}_N^d \cong \frac{\mathbb{Z}^d}{B\mathbb{Z}^d}$, which will imply that $|A\mathbb{Z}_N^d| = \left|\frac{\mathbb{Z}^d}{B\mathbb{Z}^d}\right| = |\det(B)|$, and hence $\left|\frac{\mathbb{Z}_N^d}{B\mathbb{Z}_N^d}\right| = |\det(B)|$. Firstly, observe that N^d divides $\det(A)$ since entries of B are integers. Next, for proving our claim, we consider a map $f : \mathbb{Z}^d \to A\mathbb{Z}_N^d;\ (n_1, \ldots, n_d)^t \mapsto A(r_1, \ldots, r_d)^t$, where for $1 \leq i \leq d,\ r_i \equiv n_i (\,mod\, N)$, that means, N divides $(n_i - r_i)$. It can be seen easily that the map f is well defined, onto and homomorphism with kernel $B\mathbb{Z}^d$. Hence, we conclude our claim from the fundamental theorem of group homomorphism. $\square$

Remark 5.3. (i) Observe that if $A \in \widetilde{\mathcal{M}}(d, \mathbb{Z})$ is a matrix such that $\det(A)$ divides N^d, then the matrix NA^{-1} need not be a member of $\widetilde{\mathcal{M}}(d, \mathbb{Z})$. For example, let

$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and $N = 4$. Then, $\det(A)$ divides N^2, but $NA^{-1} = \frac{4}{8}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix} \notin \widetilde{\mathcal{M}}(2, \mathbb{Z})$.

(ii) Note that this particular form of B defined in Proposition 5.2 (which will be used for getting A in collection (5.2)) may play a significant role when one has to look at reconstruction formula or the windowed Fourier transform.

Now, assume that the matrix A in the collection (5.2) satisfies the assumptions of Proposition 5.2. Then, from the same result, it is clear that $|I_2| = N^d/|AI_1| = |\det(A)|$. Next, we wish to find the structure of the set I_1 used in the collection (5.2). We need following definition in this regard:

Definition 5.4. Let H be a subgroup of $\mathbb{Z}_N^d$. We say that a set $\widetilde{\mathfrak{D}}$ *tiles* $\mathbb{Z}_N^d$ by H if $\{H + m : m \in \widetilde{\mathfrak{D}}\}$ is a disjoint partition of $\mathbb{Z}_N^d$. In this situation, $\widetilde{\mathfrak{D}}$ is also called a *complete digit set* for the quotient group $\mathbb{Z}_N^d/H$ and we say that $\widetilde{\mathfrak{D}}$ is a H*-tile* of $\mathbb{Z}_N^d$.

Now, by letting $\mathfrak{D}$ as a $B\mathbb{Z}_N^d$-tile of $\mathbb{Z}_N^d$ for the rest of this chapter, we list the following properties, which suggest us to choose I_1 as $\mathfrak{D}$, and then AI_1 will behave like a group:

Proposition 5.5. *For $N \in \mathbb{N}$, let the matrix $A \in \widetilde{\mathcal{M}}(d, \mathbb{Z})$ be such that $B = NA^{-1} \in \widetilde{\mathcal{M}}(d, \mathbb{Z})$. Let $\mathfrak{D}$ be a $B\mathbb{Z}_N^d$-tile of $\mathbb{Z}_N^d$ containing 0 as zero element and satisfying $Ad_1 \neq Ad_2$ whenever $d_1 \neq d_2 \in \mathfrak{D}$. Then, $\mathfrak{D}$ possesses the following properties:*

(i) $|\mathfrak{D}| = |A\mathbb{Z}_N^d| = |A(\mathfrak{D})| = \frac{N^d}{|\det(A)|}$.

(ii) $A\mathbb{Z}_N^d = A(\mathfrak{D})$.

(iii) *For some $\beta \in \mathfrak{D}, A\beta = 0$ if, and only if, $\beta = 0$.*

(iv) *For $\overline{\beta}_1, \overline{\beta}_2 \in \left\{\overline{\beta} = \beta + B\mathbb{Z}_N^d : \beta \in \mathfrak{D}\right\}$, $\overline{\beta}_1 \cap \overline{\beta}_2 = \phi$, and we can write $\mathbb{Z}_N^d = \bigcup_{\beta \in \mathfrak{D}}(\beta + B\mathbb{Z}_N^d)$.*

(v) *For $\beta_1, \beta_2 \in \mathfrak{D}$, we have $(\beta_1 + \beta_2) + B\mathbb{Z}_N^d = \beta + B\mathbb{Z}_N^d$ for some $\beta \in \mathfrak{D}$.*

(vi) *For $\beta_1, \beta_2 \in \mathfrak{D}$, there exists $\beta \in \mathfrak{D}$ such that $A\beta_1 - A\beta_2 = A\beta$. Moreover, $\beta_1 = \beta_2$ if, and only if, $\beta = 0$.*

Proof. The proof for (i) can be seen by observing Proposition 5.2 along with the fact that $|\mathfrak{D}| = \left|\frac{\mathbb{Z}_N^d}{B\mathbb{Z}_N^d}\right| = |\det(B)| = |A\mathbb{Z}_N^d|$, and hence, $|\mathfrak{D}| = |A(\mathfrak{D})|$ by noting that $Ad_1 \neq Ad_2$

whenever $d_1 \neq d_2 \in \mathfrak{D}$. Now, the result $A\mathbb{Z}_N^d = A(\mathfrak{D})$ in (ii) follows from $A(\mathfrak{D}) \subseteq A\mathbb{Z}_N^d$ and $|A(\mathfrak{D})| = |A\mathbb{Z}_N^d|$. The rest of properties can be proved easily. $\square$

Now, by considering the matrix A with assumptions of Proposition 5.5, we conclude that in the collection (5.2), $|I_2| = N^d/|AI_1| = |\det(A)|$, and $I_1 = \mathfrak{D}$ yields a group structure for AI_1. Further, note that we should take $|I_2| > 1$. Because for $|I_2| = 1 = |\det(A)|$ case, we get $|AI_1| = N^d$, and since $AI_1 \subset \mathbb{Z}_N^d$, this results in $AI_1 = \mathbb{Z}_N^d$. It converts (5.2) into a collection obtained through translates of a single function, say φ_0. Hence, for $\Phi = \{\varphi_0\}$, the collection (5.2) takes the following form $\mathfrak{B}(\varphi_0, A, \mathfrak{D})$, defined by

$$\mathfrak{B}(\varphi_0, A, \mathfrak{D}) := \{T_{Ak}\varphi_0 : Ak \in A(\mathfrak{D}) = \mathbb{Z}_N^d\}, \tag{5.3}$$

which is just a reordering of the collection $\{T_k\varphi_0\}_{k\in\mathbb{Z}_N^d}$. Now, the next result shows that we will not get frequency localization properties in the collection of the form defined in the equation (5.3):

Lemma 5.6. *The collection $\mathfrak{B}(\varphi_0, A, \mathfrak{D})$ defined in (5.3) forms an orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$ if, and only if, $|\widehat{\varphi}_0(n)|^2 = 1$ for all $n \in \mathbb{Z}_N^d$. Moreover, an orthonormal basis of this form is not frequency localized.*

Proof. The system $\mathfrak{B}(\varphi_0, A, \mathfrak{D})$ forms an orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$ if, and only if, $\delta(k) = \langle \varphi_0, T_k\varphi_0\rangle$, for $k \in \mathbb{Z}_N^d$. Now, by using the Plancherel's formula, we can write

$$\delta(k) = \frac{1}{N^d}\langle \widehat{\varphi}_0, \widehat{T_k\varphi_0}\rangle = \frac{1}{N^d}\sum_{n\in\mathbb{Z}_N^d} |\widehat{\varphi}_0(n)|^2 e^{2\pi i\langle n,k\rangle/N} = \mathcal{G}^\vee(k),$$

where $\mathcal{G}(n) = |\widehat{\varphi}_0(n)|^2$, for all $n \in \mathbb{Z}_N^d$. Hence, the system $\mathfrak{B}(\varphi_0, A, \mathfrak{D})$ forms an orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$ if, and only if, for all $k \in \mathbb{Z}_N^d$, we have $\delta(k) = \mathcal{G}^\vee(k)$, which is if, and only if, $|\widehat{\varphi}_0(n)|^2 = 1$ for all $n \in \mathbb{Z}_N^d$, in view of property (P3). This implies that the vector φ_0 is not frequency localized, and hence the orthonormal basis of the form $\mathfrak{B}(\varphi_0, A, \mathfrak{D})$ is not frequency localized. $\square$

In order to get a TFL-basis for $\ell^2(\mathbb{Z}_N^d)$, we modify our approach by considering a collection whose elements are translations of two or more vectors. This means, we consider the case when $|I_2| = |\det(A)| > 1$. From Proposition 5.5, we have $|AI_1| = |A(\mathfrak{D})| = \frac{N^2}{|\det(A)|}$, and hence in this case, we must have $I_1 = \mathfrak{D} \subsetneq \mathbb{Z}_N^d$. Thus, we have the following assumptions for the rest of the chapter, which are needed to redefine the collection (5.3):

($\mathcal{A}_1$): For some $N \in \mathbb{N}$, we consider the matrix $A \in \widetilde{\mathcal{M}}(d,\mathbb{Z})$ such that $B = NA^{-1} \in \widetilde{\mathcal{M}}(d,\mathbb{Z})$.

($\mathcal{A}_2$): For $I_2 = |\det(A)| =: q \geq 2$, we have $\Phi = \{\varphi_p\}_{p=0}^{q-1} \subset \ell^2(\mathbb{Z}_N^d)$.

($\mathcal{A}_3$): Under the assumptions of $\mathcal{A}_1$ and $\mathcal{A}_2$, the system in (5.3) can be redefined as:

$$\mathfrak{B}(\Phi, A, \mathfrak{D}) := \{T_{Ak}\varphi_p : k \in \mathfrak{D} \subsetneq \mathbb{Z}_N^d,\ 0 \leq p \leq q-1\} \subset \ell^2(\mathbb{Z}_N^d). \tag{5.4}$$

Remark 5.7. An important point that one should keep in mind while choosing Φ in (5.4) is to select Φ in such a way that there should not exist any element $\psi \in \ell^2(\mathbb{Z}_N^d)$ such that $\{T_k\psi : k \in \mathbb{Z}_N^d\} = \mathfrak{B}(\Phi, A, \mathfrak{D})$, otherwise $\mathfrak{B}(\Phi, A, \mathfrak{D})$ will become similar with the system (5.3), and Lemma 5.6 says that the orthonormal basis of this form is not frequency localized. For a better explanation of this fact, we need the following result in the sequel:

Proposition 5.8. *Let $\{e_m\}_{m\in\mathbb{Z}_N^d}$ be a standard orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$, where for each $m \in \mathbb{Z}_N^d$, we define $e_m(m') = 1$ if $m = m'$, and 0 otherwise. Then, for some $N \in \mathbb{N}$ and $A \in \widetilde{\mathcal{M}}(d,\mathbb{Z})$ (which satisfies the condition ($\mathcal{A}_1$)) with $|\det(A)| = q \geq 2$, there exists a set $\{\gamma_0, \gamma_1, \cdots, \gamma_{q-1}\} \subsetneq \mathbb{Z}_N^d$ for which*

$$\{T_{Ak}e_{\gamma_j} : k \in \mathfrak{D},\ 0 \leq j \leq q-1\} = \{T_k e_n\}_{k\in\mathbb{Z}_N^d},\ \textit{for some } n \in \mathbb{Z}_N^d.$$

Proof. Let $n \in \mathbb{Z}_N^d$. Then, both the collections $\{T_k e_n\}_{k\in\mathbb{Z}_N^d}$ and $\{e_m\}_{m\in\mathbb{Z}_N^d}$ are the same, and hence we claim that $\{T_{Ak}e_{\gamma_j} : k \in \mathfrak{D},\ 0 \leq j \leq q-1\} = \{e_m\}_{m\in\mathbb{Z}_N^d}$. For this, it is enough to show that there exists $\{\gamma_j\}_{j=0}^{q-1} \subset \mathbb{Z}_N^d$ such that the collection

$$\{\gamma_j + Ak : k \in \mathfrak{D},\ 0 \leq j \leq q-1\} = \mathbb{Z}_N^d,$$

since for $k \in \mathfrak{D}$ and $0 \leq j \leq q-1$, we have $T_{Ak}e_{\gamma_j} = e_{(\gamma_j + Ak)}$. Now, from Proposition 5.2, we have $\left|\dfrac{\mathbb{Z}_N^d}{A\mathbb{Z}_N^d}\right| = q$. Further, by $\{\overline{\alpha}_j\}_{j=0}^{q-1}$, we denote the q distinct coset representatives for $\frac{\mathbb{Z}_N^d}{A\mathbb{Z}_N^d}$, where for $0 \leq j \leq q-1$, we define $\overline{\alpha}_j = \alpha_j + A\mathbb{Z}_N^d$. Let $\mathfrak{D}_0 := \{\alpha_j\}_{j=0}^{q-1}$, which satisfies properties of Proposition 5.5. Then, we can write $\mathbb{Z}_N^d = \{\alpha + Ak : \alpha \in \mathfrak{D}_0,\ k \in \mathfrak{D}\}$. This implies that there exists $\{\alpha_j\}_{j=0}^{q-1} = \mathfrak{D}_0 \subset \mathbb{Z}_N^d$ such that $\{T_{Ak}e_{\alpha_j} : k \in \mathfrak{D},\ 0 \leq j \leq q-1\} = \{e_m\}_{m\in\mathbb{Z}_N^d}$. □

Now, we are defining an orthonormal wavelet system for $\ell^2(\mathbb{Z}_N^d)$ that is motivated from Definition 5.1, with terminology adapted from Frazier in [26, Definition 3.4].

Definition 5.9. Orthonormal Wavelet System: We call the system $\mathfrak{B}(\Phi, A, \mathfrak{D})$, which is defined in (5.4), an *orthonormal wavelet system (ONWS)* for $\ell^2(\mathbb{Z}_N^d)$, if it forms an orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$.

The following is our main result of this section which characterizes $\Phi \subset \ell^2(\mathbb{Z}_N^d)$ such that the system $\mathfrak{B}(\Phi, A, \mathfrak{D}))$ defined in (5.4) provides an ONWS for $\ell^2(\mathbb{Z}_N^d)$:

Theorem 5.10. *Let* $A \in \widetilde{\mathcal{M}}(d, \mathbb{Z})$ *be a matrix such that for some* $N \in \mathbb{N}$*, we have another matrix* $C := (NA^{-1})^t \in \widetilde{\mathcal{M}}(d, \mathbb{Z})$*. Consider* $\Phi = \{\varphi_p\}_{p=0}^{q-1} \subset \ell^2(\mathbb{Z}_N^d)$*, where* $|\det(A)| = q \geq 2$*. Then, the following statements are equivalent:*

(i) $\mathfrak{B}(\Phi, A, \mathfrak{D}) \subset \ell^2(\mathbb{Z}_N^d)$ *forms an ONWS in* $\ell^2(\mathbb{Z}_N^d)$*.*

(ii) *For all* $m \in \mathfrak{D}^*$ *and* $(p_1, p_2) \in \{(n_1, n_2) : 0 \leq n_1 \leq n_2 \leq q-1\}$*, we have*

$$\sum_{\gamma \in C\mathbb{Z}_N^d} \widehat{\varphi}_{p_1}(m+\gamma)\overline{\widehat{\varphi}_{p_2}(m+\gamma)} = q\delta_{p_1,p_2},$$

where $\mathfrak{D}^*$ *is a* $C\mathbb{Z}_N^d$*-tile of* $\mathbb{Z}_N^d$*.*

(iii) *For each* $m \in \mathfrak{D}^*$*, the system matrix* $\mathcal{S}_\Phi(m)$ *of* Φ *is unitary, where the* $q \times q$ *matrix* $\mathcal{S}_\Phi(m)$ *is defined as*

$$\mathcal{S}_\Phi(m) := \frac{1}{\sqrt{q}}\Big(\widehat{\varphi}_p(m+\gamma)\Big)_{\substack{\gamma \in C\mathbb{Z}_N^d \\ 0 \leq p \leq q-1}}.$$

For proving the above result, we need to set up the background in the form of following results:

Lemma 5.11. *For* $\alpha \in A\mathbb{Z}_N^d$ *and* $\beta \in C\mathbb{Z}_N^d$*, the inner product* $\langle \alpha, \beta \rangle \in N\mathbb{Z}$*, where* A *and* C *are as defined in Theorem 5.10.*

Proof. Let $\alpha \in A\mathbb{Z}_N^d$ and $\beta \in C\mathbb{Z}_N^d$. Then, there exist n, $m \in \mathbb{Z}_N^d$ such that $\alpha = An$ and $\beta = Cm$. Now, the result follows by observing that $\langle \alpha, \beta \rangle = \langle An, (NA^{-1})^t m \rangle = N\langle n, m \rangle = Np$, for some $p \in \mathbb{Z}$. $\square$

Our main motive is to prove the Theorem 5.10, and the upcoming Proposition 5.12 plays an important role in this, which is true in view of the following identity: for $k_1, k_2 \in \mathfrak{D}$, we have

$$\sum_{m \in \mathfrak{D}^*} e^{2\pi i \langle m, (Ak_1 - Ak_2) \rangle / N} = \begin{cases} |\mathfrak{D}^*| & \text{for } k_1 = k_2, \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

Note that the above key identity is well known (see for example [32]), but we include its proof for completing the details for our next result, which is as follows:

Proposition 5.12. *Let $\{E_k\}_{k\in\mathfrak{D}} \subseteq \ell^2(\mathfrak{D}^*)$, where for each $k \in \mathfrak{D}$, we define $E_k(m) = \frac{1}{\sqrt{|\mathfrak{D}^*|}} e^{2\pi i\langle m, Ak\rangle/N}$, for all $m \in \mathfrak{D}^*$ (here, A and $\mathfrak{D}^*$ are as defined in Theorem 5.10). Then, the system $\{E_k\}_{k\in\mathfrak{D}}$ forms an orthonormal basis for $\ell^2(\mathfrak{D}^*)$.*

Proof. Observe that for $k_1, k_2 \in \mathfrak{D}$, we have

$$\langle E_{k_1}, E_{k_2}\rangle_{\ell^2(\mathfrak{D}^*)} = \sum_{m\in\mathfrak{D}^*} E_{k_1}(m)\overline{E_{k_2}(m)} = \frac{1}{|\mathfrak{D}^*|} \sum_{m\in\mathfrak{D}^*} e^{2\pi i\langle m,(Ak_1-Ak_2)\rangle/N}. \tag{5.6}$$

Now, we claim that the identity (5.5) holds true. For this, let $m_1 \in \mathfrak{D}^*$ be an arbitrary element. Then, we can write

$$e^{2\pi i\langle m_1,(Ak_1-Ak_2)\rangle/N} \sum_{m\in\mathfrak{D}^*} e^{2\pi i\langle m,(Ak_1-Ak_2)\rangle/N} = \sum_{m\in\mathfrak{D}^*} e^{2\pi i\langle (m+m_1),(Ak_1-Ak_2)\rangle/N}.$$

Now, in view of Proposition 5.5, we have $Ak_1 - Ak_2 = Ak$, for some $k \in \mathfrak{D}$. Therefore, by substituting $m + m_1 = m_2$, we get

$$e^{2\pi i\langle m_1,Ak\rangle/N} \sum_{m\in\mathfrak{D}^*} e^{2\pi i\langle m,Ak\rangle/N} = \sum_{m\in\mathfrak{D}^*} e^{2\pi i\langle m+m_1,Ak\rangle/N} = \sum_{m_2\in\mathfrak{D}^*} e^{2\pi i\langle m_2,Ak\rangle/N},$$

which implies that either $e^{2\pi i\langle m_1,Ak\rangle/N} = 1$, or, $\sum_{m\in\mathfrak{D}^*} e^{2\pi i\langle m,Ak\rangle/N} = 0$. But, $e^{2\pi i\langle m_1,Ak\rangle/N} = 1$ if, and only if, $\langle m_1, Ak\rangle \in N\mathbb{Z}$, which is if, and only if, we have $Ak \in N\mathbb{Z}^d$ as $m_1 \in \mathfrak{D}^*$ is arbitrary. Equivalently, we can say that $e^{2\pi i\langle m_1,Ak\rangle/N} = 1$ if, and only if, $k \in NA^{-1}\mathbb{Z}^d \cap \mathfrak{D} = \{0\}$, which in view of Proposition 5.5 implies that $k_1 = k_2$.

Now, by using (5.5) and (5.6), we get $\langle E_{k_1}, E_{k_2}\rangle_{\ell^2(\mathfrak{D}^*)} = \delta_{k_1,k_2}$. Hence, the result. □

Proof of Theorem 5.10. The system $\mathfrak{B}(\Phi, A, \mathfrak{D})$ will form an orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$ if, and only if, for $k \in \mathfrak{D}$ and $\varphi_{p_1}, \varphi_{p_2} \in \Phi$, where $0 \le p_1, p_2 \le q-1$, we have

$$\delta_{p_1,p_2}\delta(k) = \langle \varphi_{p_1}, T_{Ak}\varphi_{p_2}\rangle = \frac{1}{N^d}\langle \widehat{\varphi}_{p_1}, \widehat{T_{Ak}\varphi_{p_2}}\rangle = \frac{1}{N^d}\sum_{n\in\mathbb{Z}_N^d} \widehat{\varphi}_{p_1}(n)\overline{\widehat{\varphi}_{p_2}(n)} e^{2\pi i\langle n,Ak\rangle/N},$$

in view of the Plancherel's formula. Next, by applying Lemma 5.11 in the above equation, we get

$$\begin{aligned}
\delta_{p_1,p_2}\delta(k) &= \frac{1}{N^d}\sum_{m\in\mathfrak{D}^*}\sum_{\gamma\in C\mathbb{Z}_N^d}\widehat{\varphi}_{p_1}(m+\gamma)\overline{\widehat{\varphi}_{p_2}(m+\gamma)}e^{2\pi i\langle (m+\gamma),Ak\rangle/N}\\
&= \frac{1}{N^d}\sum_{m\in\mathfrak{D}^*}\sum_{\gamma\in C\mathbb{Z}_N^d}\widehat{\varphi}_{p_1}(m+\gamma)\overline{\widehat{\varphi}_{p_2}(m+\gamma)}e^{2\pi i\langle m,Ak\rangle/N}\\
&= \frac{1}{q|\mathfrak{D}^*|}\sum_{m\in\mathfrak{D}^*}\Psi(m)e^{2\pi i\langle m,Ak\rangle/N},
\end{aligned}$$

where $\Psi(m) = \sum_{\gamma\in C\mathbb{Z}_N^d}\widehat{\varphi}_{p_1}(m+\gamma)\overline{\widehat{\varphi}_{p_2}(m+\gamma)}$, for all $m\in\mathfrak{D}^*$. Next, by noting that $(\Psi(m))_{m\in\mathfrak{D}^*}\in\ell^2(\mathfrak{D}^*)$ and Proposition 5.12, we get $\delta_{p_1,p_2}\delta(k) = \dfrac{1}{q}\mathcal{F}^{-1}(\Psi(k))$, for all $k\in\mathfrak{D}$, where $\mathcal{F}^{-1}$ denotes the inverse discrete Fourier transform on $\ell^2(\mathfrak{D}^*)$, which further yields $\Psi(m) = q\delta_{p_1,p_2}$ for all $m\in\mathfrak{D}^*$. Hence, the system $\mathfrak{B}(\Phi,A,\mathfrak{D})$ forms an ONWS in $\ell^2(\mathbb{Z}_N^d)$ if, and only if, for $\varphi_{p_1},\varphi_{p_2}\in\Phi$, where $0\le p_1,p_2\le q-1$, we have

$$\mathfrak{R}_{p_1,p_2} := \sum_{\gamma\in C\mathbb{Z}_N^d}\widehat{\varphi}_{p_1}(m+\gamma)\overline{\widehat{\varphi}_{p_2}(m+\gamma)} = q\delta_{p_1,p_2} \text{ for all } m\in\mathfrak{D}^*.$$

Further, we note that $\mathfrak{R}_{p_2,p_1} = \overline{\mathfrak{R}}_{p_1,p_2}$, which proves (i) $\Leftrightarrow$ (ii) part. Observe that the above equation is equivalent to the fact that for $m\in\mathfrak{D}^*$, columns of $\mathcal{S}_\Phi(m)$, the system matrix of Φ having order $q\times q$, defined by

$$\mathcal{S}_\Phi(m) = \frac{1}{\sqrt{q}}\Big(\widehat{\varphi}_p(m+\gamma)\Big)_{\substack{\gamma\in C\mathbb{Z}_N^d\\ 0\le p\le q-1}},$$

forms an orthonormal basis for $\mathbb{C}^q$. Equivalently, the collection $\mathfrak{B}(\Phi,A,\mathfrak{D})\subset\ell^2(\mathbb{Z}_N^d)$ is an ONWS in $\ell^2(\mathbb{Z}_N^d)$ if, and only if, the system matrix of Φ is unitary for each $m\in\mathfrak{D}^*$. Hence (i) $\Leftrightarrow$ (iii) follows. $\square$

The following are some examples of ONWS in $\ell^2(\mathbb{Z}_N^d)$ with respect to expansive as well as non-expansive matrices. In the case of $L^2(\mathbb{R}^n)$, the expansive and non-expansive nature of a dilation matrix plays an important role in the existence of an orthonormal wavelet.

Example 5.13. (i) **For non-expansive matrix:** Let $N=2$ and $A = \begin{pmatrix}2 & 2\\ 1 & 2\end{pmatrix}$. Then, $A\mathbb{Z}_2^2 = B\mathbb{Z}_2^2 = \Big\{(0,0)^t,(0,1)^t\Big\}$, where $B = NA^{-1} = \begin{pmatrix}2 & -2\\ -1 & 2\end{pmatrix}$ and hence, we can choose $\mathfrak{D} = \Big\{(0,0)^t, (1,0)^t\Big\} \subset \mathbb{Z}_2^2$. Note that the matrix A is non-expansive since one of its eigenvalues $(2 \pm \sqrt{2})$ is less than 1. Further, consider $\Phi_1 = \{\varphi_0, \varphi_1\} \subset \ell^2(\mathbb{Z}_2^2)$ defined by

$$\begin{aligned}\varphi_0 &= \Big(\varphi_0((0,0)^t), \varphi_0((0,1)^t), \varphi_0((1,0)^t), \varphi_0((1,1)^t)\Big)^t \\ &= \Big(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\Big)^t, \\ \text{and } \varphi_1 &= \Big(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0\Big)^t.\end{aligned}$$

Then,

$$\begin{aligned}\mathfrak{B}(\Phi_1, A, \mathfrak{D}) &= \{T_{Ak}\varphi_p : k \in \mathfrak{D},\ 0 \le p \le 1\} \\ &= \{\varphi_0, \varphi_1, T_{(0,1)^t}\varphi_0, T_{(0,1)^t}\varphi_1\},\end{aligned}$$

is an ONWS for $\ell^2(\mathbb{Z}_2^2)$, where

$$T_{(0,1)^t}\varphi_0 = (0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})^t$$

and

$$T_{(0,1)^t}\varphi_1 = (0, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})^t.$$

Further, the discrete Fourier transforms of φ_0 and φ_1 are given by $\widehat{\varphi}_0 = \Big(\sqrt{2}, \sqrt{2}, 0, 0\Big)^t$ and $\widehat{\varphi}_1 = \Big(0, 0, \sqrt{2}, \sqrt{2}\Big)^t$, respectively. We can easily verify that for each $m \in \mathfrak{D}^* = \{(0,0)^t, (0,1)^t\}$ and $0 \le p_1, p_2 \le 1$,

$$\sum_{\gamma \in C\mathbb{Z}_2^2} \widehat{\varphi}_{p_1}(m+\gamma)\overline{\widehat{\varphi}_{p_2}(m+\gamma)} = 2\delta_{p_1 p_2},$$

where $C\mathbb{Z}_2^2 = B^t\mathbb{Z}_2^2 = \{(0,0)^t, (1,0)^t\}$. Hence,

$$\mathfrak{B}(\Phi_1, A, \mathfrak{D}) = \{T_{Ak}\varphi_p : k \in \mathfrak{D},\ 0 \le p \le 1\} \subset \ell^2(\mathbb{Z}_2^2)$$

forms an ONWS in $\ell^2(\mathbb{Z}_2^2)$, by using Theorem 5.10.

(ii) **For expansive matrix:** Let $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$ be an expansive matrix with eigenvalues 2, 2. Then, $A\mathbb{Z}_4^2 = B\mathbb{Z}_4^2 = \Big\{(0,0)^t, (1,3)^t, (2,2)^t, (3,1)^t\Big\}$, where $B = NA^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$, and hence, we can choose

$$\mathfrak{D} = \Big\{(0,0)^t, (0,1)^t, (0,2)^t, (0,3)^t\Big\} \subset \mathbb{Z}_4^2,$$

which satisfies all properties from Proposition 5.5. Next, we consider $\Phi_2 = \{\varphi_0, \varphi_1, \varphi_2, \varphi_3\} \subset \ell^2(\mathbb{Z}_4^2)$ whose discrete Fourier transforms are given by

$$\begin{aligned}
\widehat{\varphi}_0 &= \Big(\widehat{\varphi}_0((0,0)^t), \widehat{\varphi}_0((0,1)^t), \ldots, \widehat{\varphi}_0((0,3)^t), \widehat{\varphi}_0((1,0)^t), \widehat{\varphi}_0((1,1)^t), \ldots, \widehat{\varphi}_0((3,3)^t)\Big)^t \\
&= \Big(\sqrt{2}, 0, \sqrt{2}i, 0, -\sqrt{2}i, 0, -\sqrt{2}, 0, 0, 1-i, 0, -\sqrt{2}, 0, \sqrt{2}, 0, -1-i\Big)^t, \\
\widehat{\varphi}_1 &= \Big(0, \sqrt{2}, 0, \sqrt{2}, 0, \sqrt{2}i, 0, -\sqrt{2}i, -\sqrt{2}, 0, \sqrt{2}i, 0, \sqrt{2}i, 0, \sqrt{2}, 0\Big)^t, \\
\widehat{\varphi}_2 &= \Big(0, \sqrt{2}i, 0, \sqrt{2}i, 0, -1+i, 0, 1-i, -\sqrt{2}i, 0, 1-i, 0, \sqrt{2}, 0, -\sqrt{2}i, 0\Big)^t, \\
\text{and } \widehat{\varphi}_3 &= \Big(-1-i, 0, 1-i, 0, 1+i, 0, 1-i, 0, 0, -\sqrt{2}i, 0, -1+i, 0, 1+i, 0, -\sqrt{2}i\Big)^t.
\end{aligned}$$

Then, for $0 \le p_1, p_2 \le 3$, we can check that

$$\sum_{\gamma \in C\mathbb{Z}_4^2} \widehat{\varphi}_{p_1}(m+\gamma)\overline{\widehat{\varphi}_{p_2}(m+\gamma)} = 4\delta_{p_1p_2} \text{ for all } m \in \mathfrak{D}^*,$$

where $C\mathbb{Z}_4^2 = B^t\mathbb{Z}_4^2 = \{(0,0)^t, (1,1)^t, (2,2)^t, (3,3)^t\}$. Hence, from Theorem 5.10, we conclude that the collection

$$\mathfrak{B}(\Phi_2, A, \mathfrak{D}) = \{T_{Ak}\varphi_p : k \in \mathfrak{D},\ 0 \le p \le 3\} \subset \ell^2(\mathbb{Z}_4^2)$$

forms an ONWS in $\ell^2(\mathbb{Z}_4^2)$.

In the next section, we discuss some results on the uncertainty principle by coupling the Fourier basis and the standard orthonormal basis with the orthonormal wavelet system $\mathfrak{B}(\Phi, A, \mathfrak{D})$ defined in (5.4). For this, let us denote the standard orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$ by $\mathfrak{B}_S := \{T_{Ak}e_{\alpha_j} : k \in \mathfrak{D},\ 0 \le j \le q-1\}$, where $\{\alpha_j\}_{j=0}^{q-1} = \mathfrak{D}_0 \subset \mathbb{Z}_N^d$ (see proof of Proposition 5.8). By $\mathfrak{F} := \{F_v\}_{v \in \mathbb{Z}_N^d}$, we denote a d-dimensional Fourier basis for $\ell^2(\mathbb{Z}_N^d)$ defined for each $v \in \mathbb{Z}_N^d$ by $F_v(n) = \frac{1}{N^{d/2}} e^{2\pi i\langle n,v\rangle/N}$, for all $n \in \mathbb{Z}_N^d$. For a non-zero function $f \in \ell^2(\mathbb{Z}_N^d)$, we can write

$$f = \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} t_{j,k} T_{Ak} e_{\alpha_j} = \sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} s_{p,m} T_{Am}\varphi_p = \sum_{v \in \mathbb{Z}_N^d} w_v F_v, \tag{5.7}$$

and hence, we have

$$\|f\|^2 = \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} |t_{j,k}|^2 = \sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} |s_{p,m}|^2 = \sum_{v \in \mathbb{Z}_N^d} |w_v|^2. \tag{5.8}$$

Further, we denote the number of non-zero coefficients from among $\{t_{j,k} : k \in \mathfrak{D}, 0 \le j \le q-1\}$, $\{w_v : v \in \mathbb{Z}_N^d\}$ and $\{s_{p,m} : m \in \mathfrak{D}, 0 \le p \le q-1\}$ by S_f, C_f and W_f, respectively.

5.3. Uncertainty principle corresponding to an ONWS in $\ell^2(\mathbb{Z}_N^d)$

Uncertainty principles put restrictions on how well frequency localized a good time localized signal can be and vice versa. In the case of a signal defined on a finite abelian group, localization is generally expressed through the cardinality of the support of the signal. Uniqueness of sparse representation of a signal depends upon the bound provided by uncertainty relations in terms of pair of bases.

In this direction, for the setup of $\ell^2(\mathbb{Z}_N^d)$, we prove the following results on the uncertainty principle with respect to the collection $\mathfrak{B}(\Phi, A, \mathfrak{D})$ defined in (5.4):

Theorem 5.14. *Let $\Phi = \{\varphi_p\}_{p=0}^{q-1} \subset \ell^2(\mathbb{Z}_N^d)$ be such that $\mathfrak{B}(\Phi, A, \mathfrak{D})$ is an ONWS in $\ell^2(\mathbb{Z}_N^d)$. Consider two positive real numbers R_0 and E_0 defined by*

$$R_0 = max\left\{|\varphi_p(\alpha + A\beta)| : \alpha \in \mathfrak{D}_0, \beta \in \mathfrak{D}, 0 \leq p \leq q-1\right\}, \text{ and}$$
$$E_0 = max\left\{\frac{1}{N^{d/2}} \sum_{n \in \mathbb{Z}_N^d} |\varphi_p(n)| : 0 \leq p \leq q-1\right\},$$

where the max Z represents the maximum of all elements of the set $Z \subset \mathbb{R}$. Then, the following inequalities hold true:

(i) *The bounds for R_0 and E_0 are given by $\frac{1}{N^{d/2}} \leq R_0, E_0 \leq 1$. In the case of $R_0 = 1$, the system $\mathfrak{B}(\Phi, A, \mathfrak{D})$ is not frequency localized. Moreover, a similar situation arises for $E_0 = \frac{1}{N^{d/2}}$.*

(ii) *For $\frac{1}{N^{d/2}} \leq R_0 < 1$, the representations of $f \in \ell^2(\mathbb{Z}_N^d)$ in terms of $\mathfrak{B}(\Phi, A, \mathfrak{D})$ and $\mathfrak{B}_S$ provide the following relations:*

$$S_f W_f \geq max\left\{2, \frac{1}{R_0^2}\right\}, \quad \text{and} \quad S_f + W_f \geq max\left\{3, \frac{2}{R_0}\right\}.$$

(iii) *For $\frac{1}{N^{d/2}} < E_0 \leq 1$, the representations of $f \in \ell^2(\mathbb{Z}_N^d)$ in terms of $\mathfrak{B}(\Phi, A, \mathfrak{D})$ and $\mathfrak{F}$ provide the following relations:*

$$C_f W_f \geq \frac{1}{E_0^2}, \quad \text{and} \quad C_f + W_f \geq \frac{2}{E_0}.$$

Proof. By considering the representations of $f \in \ell^2(\mathbb{Z}_N^d)$ in terms of $\mathfrak{B}(\Phi, A, \mathfrak{D})$ and $\mathfrak{B}_S$ from (5.7), we have the following:

$$\begin{aligned}
||f||^2 &= \left| \left\langle \sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} s_{p,m} T_{Am} \varphi_p, \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} t_{j,k} T_{Ak} e_{\alpha_j} \right\rangle \right| \\
&= \left| \sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} s_{p,m} \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} \overline{t_{j,k}} \left\langle T_{Am} \varphi_p, T_{Ak} e_{\alpha_j} \right\rangle \right| \\
&\leq \sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} |s_{p,m}| |\overline{t_{j,k}}| |\langle T_{Am} \varphi_p, T_{Ak} e_{\alpha_j} \rangle|,
\end{aligned}$$

and hence, we have

$$||f||^2 \leq \sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} |s_{p,m}| \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} |\overline{t_{j,k}}| |\langle T_{Am} \varphi_p, T_{Ak} e_{\alpha_j} \rangle|. \tag{5.9}$$

Similarly, if we proceed by using the representations of $f \in \ell^2(\mathbb{Z}_N^d)$ in terms of $\mathfrak{B}(\Phi, A, \mathfrak{D})$ and $\mathfrak{F}$, we get

$$||f||^2 \leq \sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} |s_{p,m}| \sum_{v \in \mathbb{Z}_N^d} |\overline{w_v}| |\langle T_{Am} \varphi_p, F_v \rangle|. \tag{5.10}$$

In view of Proposition 5.5, we observe that for $k, m \in \mathfrak{D}$ and $0 \leq j, p \leq q-1$,

$$\begin{aligned}
|\langle T_{Am} \varphi_p, T_{Ak} e_{\alpha_j} \rangle| &= |\langle \varphi_p, T_{(Ak-Am)} e_{\alpha_j} \rangle| = |\langle \varphi_p, T_{A\beta} e_{\alpha_j} \rangle|, \text{ for some } \beta \in \mathfrak{D} \\
&= \left| \sum_{n \in \mathbb{Z}_N^d} \varphi_p(n) \overline{e_{\alpha_j + A\beta}(n)} \right| = |\varphi_p(\alpha_j + A\beta)|,
\end{aligned}$$

which implies that $R_1 = R_2$, for $R_1 = \{|\langle T_{Am} \varphi_p, T_{Ak} e_{\alpha_j} \rangle| : k, m \in \mathfrak{D}, 0 \leq j, p \leq q-1\}$, and $R_2 = \{|\varphi_p(\alpha_j + A\beta)| : \alpha_j \in \mathfrak{D}_0, \beta \in \mathfrak{D}, 0 \leq p, j \leq q-1\}$, and hence for any $h \in R_1$, we have $h \leq R_0$, where $R_0 = max\, R_2$. Using this fact in (5.9), along with (5.8) and the Cauchy-Schwarz inequality, it is clear that

$$\begin{aligned}
||f||^2 &\leq \sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} |s_{p,m}| \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} |\overline{t_{j,k}}| R_0 \\
&\leq \sqrt{\sum_{m,p:|s_{p,m}| \neq 0} |s_{p,m}|^2} \sqrt{\sum_{m,p:|s_{p,m}| \neq 0} \Big(\sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} |\overline{t_{j,k}}| R_0 \Big)^2},
\end{aligned}$$

which further gives the following estimates:

$$
\begin{aligned}
||f||^2 &\leq \sqrt{\sum_{m\in\mathfrak{D}}\sum_{p=0}^{q-1}|s_{p,m}|^2}\sqrt{W_f\Big(\sum_{k\in\mathfrak{D}}\sum_{j=0}^{q-1}|\overline{t_{j,k}}|R_0\Big)^2} \\
&= ||f||\sqrt{W_f}\sum_{k\in\mathfrak{D}}\sum_{j=0}^{q-1}|\overline{t_{j,k}}|R_0 \\
&\leq ||f||\sqrt{W_f}\sqrt{\sum_{j,k:|t_{j,k}|\neq 0}|\overline{t_{j,k}}|^2}\sqrt{\sum_{j,k:|t_{j,k}|\neq 0}\big(R_0\big)^2} \\
&= ||f||\sqrt{W_f}\sqrt{\sum_{k\in\mathfrak{D}}\sum_{j=0}^{q-1}|t_{j,k}|^2}\sqrt{S_f\big(R_0\big)^2} \\
&= ||f||^2\sqrt{W_f}\sqrt{S_f}R_0.
\end{aligned}
$$

Therefore, we have the inequality $\sqrt{S_fW_f} \geq \frac{1}{R_0}$. Now, by using the inequality of arithmetic and geometric means, we have $S_f + W_f \geq 2\sqrt{S_fW_f} \geq \frac{2}{R_0}$. Further, by assuming part (i) [which we will prove later], we cannot consider R_0 equal to 1, otherwise the system $\mathfrak{B}(\Phi, A, \mathfrak{D})$ will not remain frequency localized. Thus, for $\frac{1}{N^{d/2}} \leq R_0 < 1$, we get

$$
S_f + W_f \geq \frac{2}{R_0} > 2 \text{ and } S_fW_f \geq \frac{1}{R_0^2} > 1,
$$

which results in part (ii), in view of the fact that S_f and W_f are natural numbers.

Next, we prove (iii) part. For this, first we observe the following:

$$
\begin{aligned}
|\langle T_{Am}\varphi_p, F_v\rangle| &= |\sum_{n\in\mathbb{Z}_N^d} T_{Am}\varphi_p(n)\overline{F_v(n)}| \\
&= |\sum_{n\in\mathbb{Z}_N^d}\varphi_p(n-Am)\frac{1}{N^{d/2}}e^{-2\pi i\langle n,v\rangle/N}| \leq \frac{1}{N^{d/2}}\sum_{t\in\mathbb{Z}_N^d}|\varphi_p(t)|,
\end{aligned}
$$

for $m \in \mathfrak{D}, 0 \leq p \leq q-1$ and $v \in \mathbb{Z}_N^d$, which implies that $max\, X \leq max\, Y$, where the sets X and Y are given by

$$
\{|\langle T_{Am}\varphi_p, F_v\rangle| : m \in \mathfrak{D}, 0 \leq p \leq q-1, v \in \mathbb{Z}_N^d\}
$$

and

$$
\Big\{\frac{1}{N^{d/2}}\sum_{t\in\mathbb{Z}_N^d}|\varphi_p(t)| : 0 \leq p \leq q-1\Big\},
$$

respectively, and hence for any $x \in X$, we have $x \leq E_0$, where $E_0 = max\, Y$. Therefore, by using the above discussion in (5.10) along with the way we proceeded in case of part

(ii) provide the following inequalities:

$$C_f W_f \geq \frac{1}{E_0^2}, \text{ and } C_f + W_f \geq \frac{2}{E_0}, \text{ for } \frac{1}{N^{d/2}} \leq E_0 \leq 1.$$

Again, by considering part (i) [which we will prove next], note that we cannot consider E_0 equal to $\frac{1}{N^{d/2}}$, otherwise the system $\mathfrak{B}(\Phi, A, \mathfrak{D})$ will not remain frequency localized. Therefore, for $\frac{1}{N^{d/2}} < E_0 \leq 1$, we can write

$$C_f + W_f \geq \frac{2}{E_0} \geq 2 \text{ and } C_f W_f \geq \frac{1}{E_0^2} \geq 1,$$

which leads to part (iii).

For the part (i), let us consider elements of $\mathfrak{B}(\Phi, A, \mathfrak{D})$ and $\mathfrak{B}_S$. Then, from Cauchy-Schwarz inequality, we have

$$|\langle T_{Am}\varphi_p, T_{Ak}e_{\alpha_j}\rangle| \leq ||T_{Am}\varphi_p|| ||T_{Ak}e_{\alpha_j}|| = 1,$$

for all $k, m \in \mathfrak{D}$ and $0 \leq j, p \leq q-1$. Therefore, 1 is an upper bound for the set R_1 and hence its maximum element $R_0 \leq 1$. Next, we note that the real number R_0 cannot take the value 1. For this, let by contradiction we assume $R_0 = 1$. Then, we have $max\{|\varphi_p(n)| : n \in \mathbb{Z}_N^d, 0 \leq p \leq q-1\} = 1$, since the collection $\{\alpha + Ak : \alpha \in \mathfrak{D}_0,\ k \in \mathfrak{D}\}$ is a partition of $\mathbb{Z}_N^d$. This assures the existence of $p_1 \in \{0, 1, \ldots, q-1\}$ and $n_1 \in \mathbb{Z}_N^d$ such that $|\varphi_{p_1}(n_1)| = 1$. Therefore, we have

$$||\varphi_{p_1}||^2 = \sum_{m \in \mathbb{Z}_N^d} |\varphi_{p_1}(m)|^2 = |\varphi_{p_1}(n_1)|^2 + \sum_{m \neq n_1 \in \mathbb{Z}_N^d} |\varphi_{p_1}(m)|^2 = 1, \tag{5.11}$$

in view of the fact that $\varphi_{p_1} \in \Phi$ and $\mathfrak{B}(\Phi, A, \mathfrak{D})$ forms an ONWS in $\ell^2(\mathbb{Z}_N^d)$. Since $|\varphi_{p_1}(n_1)| = 1$, therefore from (5.11), it is clear that

$$\sum_{m \neq n_1 \in \mathbb{Z}_N^d} |\varphi_{p_1}(m)|^2 = 0,$$

which allows us to write

$$|\varphi_{p_1}(m)| = \begin{cases} 1, & \text{if } m = n_1, \\ 0, & \text{if } m \neq n_1 \in \mathbb{Z}_N^d. \end{cases} \tag{5.12}$$

It follows from (5.12) that $\varphi_{p_1}(m) = 0$ for all $m \neq n_1 \in \mathbb{Z}_N^d$, and hence we can define $\varphi_{p_1} \in \Phi \subset \ell^2(\mathbb{Z}_N^d)$ by $\varphi_{p_1}(n) = e^{i\theta(n)}$ for $n = n_1 \in \mathbb{Z}_N^d$, and zero otherwise, where the real number $\theta(n)$ depends on n. Therefore, we have

$$\widehat{\varphi}_{p_1}(m) = \sum_{n \in \mathbb{Z}_N^d} \varphi_{p_1}(n) e^{-2\pi i \langle m,n \rangle/N} = e^{i\theta(n_1)} e^{-2\pi i \langle m,n_1 \rangle/N},$$

and hence $|\widehat{\varphi}_{p_1}(m)| = 1$ for all $m \in \mathbb{Z}_N^d$. Thus, we conclude that φ_{p_1} is not frequency localized which implies that the system $\mathfrak{B}(\Phi, A, \mathfrak{D})$ is not frequency localized.

Now, for computing a lower bound of R_0 (greater than zero), we consider an $N^d \times N^d$ matrix $M = \left(\langle T_{Am}\varphi_p, T_{Ak}e_{\alpha_j}\rangle\right)$, where rows of the matrix are varying over $m \in \mathfrak{D}$ and $0 \leq p \leq q-1$, and columns over $k \in \mathfrak{D}$ and $0 \leq j \leq q-1$. Observe that, for each $k, m \in \mathfrak{D}$ and $0 \leq p, j \leq q-1$, we have

$$\sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} |\langle T_{Am}\varphi_p, T_{Ak}e_{\alpha_j}\rangle|^2 = ||T_{Ak}e_{\alpha_j}||^2 = 1,$$

and

$$\sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} |\langle T_{Ak}e_{\alpha_j}, T_{Am}\varphi_p\rangle|^2 = ||T_{Am}\varphi_p||^2 = 1.$$

Therefore, rows and columns of M have unit norm. Further, for $0 \leq j_1 \neq j_2 \leq q-1$ and $k_1 \neq k_2 \in \mathfrak{D}$, the inner product of any two columns of M given by

$$\sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} \langle T_{Am}\varphi_p, T_{Ak_1}e_{j_1}\rangle \overline{\langle T_{Am}\varphi_p, T_{Ak_2}e_{j_2}\rangle} = \langle T_{Ak_2}e_{j_2}, T_{Ak_1}e_{j_1}\rangle = 0,$$

implies that columns of M are orthogonal. Similarly, we can check that rows of M are orthogonal, and hence M is an orthonormal matrix having the sum of squares of its entries equal to N^d. Therefore, all of its entries cannot be less than $1/N^{d/2}$. This follows by noting that if we assume that all entries of M are less than $1/N^{d/2}$, then, sum of squares of all N^{2d} entries of M will be less than $N^{2d} \times \frac{1}{N^d}$, that is, N^d, which is not true. Further, observe that the absolute values of all entries of the matrix M are exactly the elements of the set R_1. Hence, we conclude that there exists $h \in R_1$ such that $h \geq 1/N^{d/2}$. Thus, $R_0 \geq 1/N^{d/2}$. Hence, we have $1/N^{d/2} \leq R_0 < 1$.

Similarly, we can show that $1/N^{d/2} \leq E_0 \leq 1$ by considering elements of $\mathfrak{B}(\Phi, A, \mathfrak{D})$ and $\mathfrak{F}$. Note that we cannot consider $E_0 = \frac{1}{N^{d/2}}$, otherwise the system $\mathfrak{B}(\Phi, A, \mathfrak{D})$ will not remain frequency localized. For verifying this fact, let us consider

$$E_0 = max\{\frac{1}{N^{d/2}} \sum_{n \in \mathbb{Z}_N^d} |\varphi_p(n)| : 0 \leq p \leq q-1\} = \frac{1}{N^{d/2}},$$

which assures the existence of some $p_2 \in \{0, 1, \ldots, q-1\}$ such that

$$\sum_{n \in \mathbb{Z}_N^d} |\varphi_{p_2}(n)| = 1,$$

and hence

$$\Big(\sum_{n \in \mathbb{Z}_N^d} |\varphi_{p_2}(n)|\Big)^2 = \sum_{n \in \mathbb{Z}_N^d} |\varphi_{p_2}(n)|^2 + 2\Big(\sum_{n_1 \neq n_2 \in \mathbb{Z}_N^d} |\varphi_{p_2}(n_1)||\varphi_{p_2}(n_2)|\Big) = 1,$$

which further implies that

$$\sum_{n_1 \neq n_2 \in \mathbb{Z}_N^d} |\varphi_{p_2}(n_1)||\varphi_{p_2}(n_2)| = 0, \tag{5.13}$$

in view of the fact that $\|\varphi_{p_2}\|^2 = 1$ as $\varphi_{p_2} \in \Phi$ and $\mathfrak{B}(\Phi, A, \mathfrak{D})$ forms an ONWS in $\ell^2(\mathbb{Z}_N^d)$. It follows from (5.13) that $|\varphi_{p_2}(n_1)||\varphi_{p_2}(n_2)| = 0$, for all $n_1 \neq n_2 \in \mathbb{Z}_N^d$, which in turn says that either $|\varphi_{p_2}(n_1)| = 0$ for all $n_1 \neq n_2 \in \mathbb{Z}_N^d$, or $|\varphi_{p_2}(n_2)| = 0$ for all $n_2 \neq n_1 \in \mathbb{Z}_N^d$. Now, without loss of generality, let us assume that $|\varphi_{p_2}(n_1)| = 0$ for all $n_1 \neq n_2 \in \mathbb{Z}_N^d$, which in view of $\sum_{n \in \mathbb{Z}_N^d} |\varphi_{p_2}(n)| = 1$ implies that $|\varphi_{p_2}(n_2)| = 1$. Therefore, we conclude that φ_{p_2} satisfies (5.12), and hence the required result can be concluded by following the approach used after (5.12). $\square$

CHAPTER 6

FINITE DUAL g-FRAMELET SYSTEMS ASSOCIATED WITH AN INDUCED GROUP ACTION

This chapter is the continuation of our work discussed in **Chapter 5** which is actually inspired by the study of Frazier in [26, Chapter 3], and Frazier and Kumar in [31]. The purpose of this chapter is to study the frame properties of a finite collection consisting of time-frequency localized functions associated with an induced action of a topological group $\mathbb{G}$ on $\ell^2(\mathbb{Z}_N^d)$. We refer **Chapter 5** for preliminaries on finite-dimensional Hilbert spaces and notation used throughout this chapter.

6.1. Introduction

In **Chapter 3**, we have investigated the orthogonality of GTI frame pairs over LCA groups, which can be used to deduce the orthogonality results for finite frames of translates by considering an LCA group of the form $\mathbb{Z}_N^d$. However, this chapter is concerned with the study of duality and orthogonality of frame pairs with a special representation which is associated with an induced group action.

By using the above-mentioned representation, in the present chapter, we first generalize the construction techniques of an ONWS (obtained in **Chapter 4**) to a framelet system in finite set-up. Then, we investigate the orthogonality and duality of such finite framelet systems by adapting techniques which are different from that are used in [59] and **Chapter 3**. Note that the framelet systems obtained in this chapter have a special representation with additional properties due to the role of an induced group action.

It is well known that one of the important factor behind the stable decomposition of a signal for analysis or transmission is related to the type of representation used for its spanning set (representation system). A careful choice of the spanning set enables us to solve a variety of analysis tasks. In the last two decades, many researchers have contributed in the designing and time-frequency analysis of these representation systems for the various spaces, namely, finite and infinite abelian groups, Euclidean spaces, locally compact abelian groups, etc. (see [10, 31, 33, 44, 49, 51, 59, 67, 68] and references therein).

In this connection, we first introduce a framelet system for each $\mathrm{g} \in \mathbb{G}$ (we call as **g**-framelet system) in $\ell^2(\mathbb{Z}_N^d)$, and then, establish the characterizations for the generators of two **g**-framelet systems (super **g**-framelet systems) associated with an induced action of the topological group $\mathbb{G}$ on $\ell^2(\mathbb{Z}_N^d)$ such that they form a **g**-dual pair (super **g**-dual pair) for $\ell^2(\mathbb{Z}_N^d)$. Among different types of representation systems, frame wavelet (simply, framelet) systems are useful tools in various research areas including but certainly not limited to signal detection, image representation, object recognition, noise reduction, sampling theory, harmonic analysis, non linear sparse approximation, wireless communications and filter banks, etc. (e.g. [18, 28, 33]).

6.2. g-Framelet systems through an induced group action

In this section, by taking motivation from the collection of the form (5.1) which was studied by Frazier in [26, Chapter 3], we wish to generalize the time-frequency localized collection (5.4) along with the theory related to its construction to the frame set-up. For this, we consider a system of translates in terms of a general invertible matrix, in which elements are represented through an induced action of a topological group $\mathbb{G}$ on $\ell^2(\mathbb{Z}_N^d)$. We refer [55] for more details about topological groups.

In order to introduce a generalized version of (5.4), let θ be an action of $\mathbb{G}$ on the space $\mathbb{C}$ of complex numbers, that means, a continuous map

$$\theta : \mathbb{G} \times \mathbb{C} \to \mathbb{C};\ (\mathrm{g}, x) \mapsto \mathrm{g}x$$

so that $\theta(\mathbf{e}, x) = x$ and $\theta(\mathrm{g}, \theta(h, x)) = \theta(\mathrm{g}h, x)$, for all $\mathrm{g}, h \in \mathbb{G}$ and $x \in \mathbb{C}$, where $\mathbf{e} \in \mathbb{G}$ is the identity element. Now, we consider a map

$$\widetilde{\theta} : \mathbb{G} \times \ell^2(\mathbb{Z}_N^d) \to \ell^2(\mathbb{Z}_N^d);\ (\mathrm{g}, f) \mapsto \widetilde{\theta}(\mathrm{g}, f)$$

defined for all $n \in \mathbb{Z}_N^d$ by $\widetilde{\theta}(\mathrm{g}, f)(n) = \theta(\mathrm{g}, f(n))$. Note that the map $\widetilde{\theta}$ induces an action of $\mathbb{G}$ on $\ell^2(\mathbb{Z}_N^d)$. Throughout this chapter, *the map $\widetilde{\theta}$ will be termed as the induced action on $\ell^2(\mathbb{Z}_N^d)$ by the action θ of $\mathbb{G}$ on $\mathbb{C}$.*

Further, by fixing an arbitrary element g in $\mathbb{G}$ and the matrix A in $\widetilde{\mathcal{M}}(d,\mathbb{Z})$, we consider the following collection in $\ell^2(\mathbb{Z}_N^d)$:

$$\mathfrak{B}_F(\mathrm{g},A,\Psi) := \left\{\widetilde{\theta}(\mathrm{g},T_{Ak}\psi_j) : k \in I_1 \subseteq \mathbb{Z}_N^d,\ \psi_j \in \Psi,\ j \in I_2 \subset \mathbb{N}_0\right\}, \tag{6.1}$$

where $\Psi := \{\psi_j\}_{j\in I_2} \subset \ell^2(\mathbb{Z}_N^d)$. Now, our motive is to find conditions on Ψ, and the finite sets I_1 and I_2 such that the collection $\mathfrak{B}_F(\mathrm{g},A,\Psi)$ defines a framelet system for $\ell^2(\mathbb{Z}_N^d)$ having time-frequency localization properties.

Here, note that by letting the group element g in $\mathbb{G}$ as $\mathbf{e}$, that is, the identity element of the group $\mathbb{G}$, and by using the definition of the induced group action, the collection (6.1) reduces to the following form:

$$\mathfrak{B}_F(\mathbf{e},A,\Psi) := \left\{T_{Ak}\psi_j : k \in I_1 \subseteq \mathbb{Z}_N^d,\ \psi_j \in \Psi,\ j \in I_2 \subset \mathbb{N}_0\right\}, \tag{6.2}$$

which yields a generalization for the collection (5.1) in the multidimensional set-up. Further, in order to see the time-frequency localization properties in the system (6.1), we consider the possible situations of the sets I_1 and I_2, and the matrix A as follows.

In the collection (6.1), if we consider I_1 such that $AI_1 = \mathbb{Z}_N^d$, then the cardinality of I_2 should be equal to one, in order to make the system $\mathfrak{B}_F(\mathrm{g},A,\Psi)$ an orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$. That means, if we let $\Psi = \{\psi\}$, then (6.1) can be redefined as follows:

$$\mathfrak{B}_F(\mathrm{g},A,\{\psi\}) = \left\{\widetilde{\theta}(\mathrm{g},T_{Ak}\psi) : Ak \in AI_1 = \mathbb{Z}_N^d\right\}. \tag{6.3}$$

But the above defined system is not frequency localized, which follows by noting the following result:

Proposition 6.1. *Let the map $\widetilde{\theta}$ be the induced action on $\ell^2(\mathbb{Z}_N^d)$ by the additive as well as multiplicative action θ of $\mathbb{G}$ on $\mathbb{C}$. Then, for each $\mathrm{g} \in \mathbb{G}$, the collection defined by (6.3) forms an orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$ if, and only if, $|\widehat{\widetilde{\theta}(\mathrm{g},\psi)}(n)|^2 = 1$, for all $n \in \mathbb{Z}_N^d$. Moreover, an orthonormal basis of this form is not frequency localized.*

In order to prove Proposition 6.1, we need the following definitions and result that provide the notion of an additive action and a multiplicative action of $\mathbb{G}$ on $\mathbb{C}$, which will be useful for obtaining various results in this chapter. For this, we define the map

$$\theta^2 : \mathbb{G}\times\mathbb{C}^2 \to \mathbb{C}^2;\ (\mathrm{g},(z_1,z_2)) \mapsto (\theta(\mathrm{g},z_1),\theta(\mathrm{g},z_2)).$$

Inductively, we can define θ^n for a natural number n. Note that in the following definitions, the notation $I_{\mathbb{G}}$ denotes the identity map on $\mathbb{G}$:

Definition 6.2. An action θ of the topological group $\mathbb{G}$ on $\mathbb{C}$ is called

(i) *additive* if the following diagram commutes, where $'+'$ is the addition of complex numbers:

$$\begin{tikzcd} \mathbb{G} \times \mathbb{C}^2 \arrow[r, "\theta^2"] \arrow[d, "(I_{\mathbb{G}}, +)"'] & \mathbb{C}^2 \arrow[d, "+"] \\ \mathbb{G} \times \mathbb{C} \arrow[r, "\theta"] & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} (\mathrm{g}, (z_1, z_2)) \arrow[r, "\theta^2"] \arrow[d, "(I_{\mathbb{G}}, +)"'] & \theta^2(\mathrm{g}, (z_1, z_2)) \arrow[d, "+"] \\ (\mathrm{g}, z_1 + z_2) \arrow[r, "\theta"] & \theta(\mathrm{g}, z_1 + z_2) \end{tikzcd}$$

This means, θ is additive if for all $\mathrm{g} \in \mathbb{G}$ and $(z_1, z_2) \in \mathbb{C}^2$, we have

$$\theta(\mathrm{g}, z_1 + z_2) = \theta(\mathrm{g}, z_1) + \theta(\mathrm{g}, z_2).$$

(ii) *multiplicative* if the following diagram commutes, where $' \cdot '$ is the multiplication of complex numbers and p_2 is the second projection of $\mathbb{C}^2$ onto $\mathbb{C}$:

$$\begin{tikzcd} \mathbb{G} \times \mathbb{C}^2 \arrow[r, "(\theta, p_2)"] \arrow[d, "(I_{\mathbb{G}}, \cdot)"'] & \mathbb{C}^2 \arrow[d, "\cdot"] \\ \mathbb{G} \times \mathbb{C} \arrow[r, "\theta"] & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} (\mathrm{g}, (z_1, z_2)) \arrow[r, "(\theta, p_2)"] \arrow[d, "(I_{\mathbb{G}}, \cdot)"'] & (\theta(\mathrm{g}, z_1), z_2)) \arrow[d, "\cdot"] \\ (\mathrm{g}, z_1.z_2) \arrow[r, "\theta"] & \theta(\mathrm{g}, z_1.z_2) \end{tikzcd}$$

This means, θ is multiplicative if for all $\mathrm{g} \in \mathbb{G}$ and $(z_1, z_2) \in \mathbb{C}^2$, we have

$$\theta(\mathrm{g}, z_1.z_2) = \theta(\mathrm{g}, z_1).z_2.$$

Now, we provide some examples on additive and multiplicative group actions, which are as follows:

Example 6.3. (i) For $\mathbb{G} = \mathbb{Z}_2 = \{[0], [1]\}$, that is, the group of integers under addition modulo 2, the group action $\theta : \mathbb{Z}_2 \times \mathbb{C} \to \mathbb{C}$, defined for all $z \in \mathbb{C}$ by $[0].z = z$ and $[1].z = -z$, is additive as well as multiplicative.

(ii) Let the Klein's four group as $\mathbb{G} = \{I, f, g, h\}$, where I, f, g and h are mapped on $\mathbb{C}$, defined respectively by $I(z) = z, f(z) = \overline{z}, g(z) = -z$ and $h(z) = -\overline{z}$, for all $z \in \mathbb{C}$. Here, $\mathbb{G}$ forms a discrete topological group under the operation of the composition of maps, and the group action $\theta : \mathbb{G} \times \mathbb{C} \to \mathbb{C}$; $(\varphi, z) \mapsto \varphi(z)$, is additive but not multiplicative.

(iii) Let $\mathbb{G} = \mathrm{Iso}(\mathbb{C})$ be a set of all isometries on $\mathbb{C}$, where every element $f \in \mathbb{G}$, is a bijective map $f : \mathbb{C} \to \mathbb{C}$ satisfying the condition $|f(z_1) - f(z_2)| = |z_1 - z_2|$, for all $z_1, z_2 \in \mathbb{C}$. Then, $\mathbb{G}$ forms a group under the operation of composition of maps, and the group action θ on $\mathbb{C}$ defined by

$$\theta : \mathbb{G} \times \mathbb{C} \to \mathbb{C};\ (\varphi, z) \mapsto \varphi(z),$$

where

$$\varphi(z) = \begin{cases} az + b, & \text{if } \varphi \text{ is a direct isometry,} \\ a\overline{z} + b, & \text{if } \varphi \text{ is an indirect isometry,} \end{cases}$$

for some $a, b \in \mathbb{C}$ with $|a| = 1$, is neither additive nor multiplicative.

Next, we define an equivarient map which is required in the sequel:

Definition 6.4. An *equivarient map* is a function between two sets that commutes with the action of a group. Specifically, let θ be a group action of the topological group $\mathbb{G}$ on the associated $\mathbb{G}$-sets X and Y. Then, a function $w : X \to Y$ is said to be equivarient if the following diagram commutes:

$$\begin{tikzcd} \mathbb{G} \times X \arrow[r, "\theta"] \arrow[d, "(I_{\mathbb{G}}, w)"'] & X \arrow[d, "w"] \\ \mathbb{G} \times Y \arrow[r, "\theta"] & Y \end{tikzcd} \qquad \begin{tikzcd} (\mathrm{g}, x) \arrow[r, "\theta"] \arrow[d, "(I_{\mathbb{G}}, w)"'] & \theta(\mathrm{g}, x) \arrow[d, "w"] \\ (\mathrm{g}, w(x)) \arrow[r, "\theta"] & \theta(\mathrm{g}, w(x)) \end{tikzcd}$$

In other words, we say w is an equivarient map if for all $\mathrm{g} \in \mathbb{G}$ and $x \in X$, we have

$$w(\theta(\mathrm{g}, x)) = \theta(\mathrm{g}, w(x)).$$

Lemma 6.5. *Let θ be an action of a group $\mathbb{G}$ on $\mathbb{C}$, and let $\widetilde{\theta}$ be an induced action on $\ell^2(\mathbb{Z}_N^d)$ by the action θ of $\mathbb{G}$ on $\mathbb{C}$. Then, we have following:*

(i) *For each $k \in \mathbb{Z}^d$, the translation map $T_k : \ell^2(\mathbb{Z}_N^d) \to \ell^2(\mathbb{Z}_N^d)$ is an equivarient map under the action of $\widetilde{\theta}$.*

(ii) *By assuming additive and multiplicative properties of θ, the DFT map given by $\widehat{\ } : \ell^2(\mathbb{Z}_N^d) \to \ell^2(\mathbb{Z}_N^d)$ is an equivarient map under the action of $\widetilde{\theta}$.*

Proof. The result can be easily seen by observing that for each $\mathrm{g} \in \mathbb{G}$ and

(i) for $k, n \in \mathbb{Z}_N^d$, we have

$$\begin{aligned} T_k(\widetilde{\theta}(\mathrm{g}, f))(n) &= \widetilde{\theta}(\mathrm{g}, f)(n-k) = \theta(\mathrm{g}, f(n-k)) \\ &= \theta(\mathrm{g}, T_k f(n)) = \widetilde{\theta}(\mathrm{g}, T_k f)(n), \quad \text{for all } f \in \ell^2(\mathbb{Z}_N^d). \end{aligned}$$

(ii) for $f \in \ell^2(\mathbb{Z}_N^d)$, we have

$$\begin{aligned} \widehat{\widetilde{\theta}\big(\mathrm{g}, f\big)}(n) &= \sum_{m \in \mathbb{Z}_N^d} \widetilde{\theta}\big(\mathrm{g}, f\big)(m) e^{-2\pi i \langle m,n\rangle/N} = \sum_{m \in \mathbb{Z}_N^d} \theta\big(\mathrm{g}, f(m)\big) e^{-2\pi i \langle m,n\rangle/N} \\ &= \sum_{m \in \mathbb{Z}_N^d} \theta\left(\mathrm{g}, f(m).e^{-2\pi i \langle m,n\rangle/N}\right) = \theta\left(\mathrm{g}, \sum_{m \in \mathbb{Z}_N^d} f(m).e^{-2\pi i \langle m,n\rangle/N}\right) \\ &= \theta\big(\mathrm{g}, \widehat{f}(n)\big) = \widetilde{\theta}\big(\mathrm{g}, \widehat{f}\,\big)(n), \quad \text{for all } n \in \mathbb{Z}_N^d, \end{aligned}$$

in view of the additive and multiplicative nature of θ.

□

Proof of Proposition 6.1. The system $\mathfrak{B}_F(\mathrm{g}, A, \{\psi\})$ forms an orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$ if, and only if, for $k_1, k_2 \in \mathbb{Z}_N^d$, we have $\langle \widetilde{\theta}(\mathrm{g}, T_{k_1}\psi), \widetilde{\theta}(\mathrm{g}, T_{k_2}\psi)\rangle = \delta_{k_1,k_2}$, which by using (P2) and Lemma 6.5 becomes equivalent to $\delta_{k,0} = \langle \widetilde{\theta}(\mathrm{g}, \psi), T_k(\widetilde{\theta}(\mathrm{g}, \psi))\rangle$, for $k \in \mathbb{Z}_N^d$. Now, by using the Plancherel's formula and (P2), we can write

$$\delta_{k,0} = \frac{1}{N^d} \langle \widehat{\widetilde{\theta}(\mathrm{g}, \psi)}, \widehat{T_k(\widetilde{\theta}(\mathrm{g}, \psi))}\rangle = \frac{1}{N^d} \sum_{n \in \mathbb{Z}_N^d} |\widehat{\widetilde{\theta}(\mathrm{g}, \psi)}(n)|^2 e^{2\pi i \langle n,k\rangle/N} = \mathcal{G}^{\vee}(k),$$

where for each $n \in \mathbb{Z}_N^d$, we have $\mathcal{G}(n) := |\widehat{\widetilde{\theta}(\mathrm{g}, \psi)}(n)|^2$, and hence, the result follows by following the steps of Lemma 5.6 of **Chapter 5**. □

By continuing in the same way as in case of **Chapter 5**, we give the final form of the collection (6.1) under the following list of assumptions for the remainder of this chapter:

($\mathcal{B}_1$): *For some $N \in \mathbb{N}$, we consider the matrix $A \in \widetilde{\mathcal{M}}(d, \mathbb{Z})$ such that $\widetilde{A} = NA^{-1} \in \widetilde{\mathcal{M}}(d, \mathbb{Z})$.*

($\mathcal{B}_2$): *For $L := |I_2| \geq |\det(A)| \geq 2$, we assume $\Psi = \{\psi_j\}_{j=0}^{L-1} \subset \ell^2(\mathbb{Z}_N^d)$.*

($\mathcal{B}_3$): *Let $I_1 = \mathfrak{D}_{\widetilde{A}}$ (which is an $\widetilde{A}\mathbb{Z}_N^d$-tile of $\mathbb{Z}_N^d$) to get a group representation for AI_1.*

($\mathcal{B}_4$): *Under the assumptions of $(\mathcal{B}_1) - (\mathcal{B}_3)$, for each g in $\mathbb{G}$, the system in (6.1) can be redefined as follows:*

$$(6.4)\qquad \widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi) := \left\{\widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j) : k \in \mathfrak{D}_{\widetilde{A}},\ \psi_j \in \Psi,\ 0 \le j \le L-1\right\}.$$

Consequently, the collection defined in (6.2) takes the following form:

$$\widetilde{\mathfrak{B}}_F(\mathbf{e}, A, \Psi) := \left\{\widetilde{\theta}(\mathbf{e}, T_{Ak}\psi_j) = T_{Ak}\psi_j : k \in \mathfrak{D}_{\widetilde{A}},\ \psi_j \in \Psi,\ 0 \le j \le L-1\right\}.$$

Next, we provide the definitions for the framelet systems associated with an induced action of the group $\mathbb{G}$ on $\ell^2(\mathbb{Z}_N^d)$:

Definition 6.6. Let $\Psi = \{\psi_j\}_{j=0}^{L-1} \subset \ell^2(\mathbb{Z}_N^d)$, where $L \ge |\det(A)| \ge 2$. Then, for each $\mathrm{g} \in \mathbb{G}$, the collection $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi)$ (defined in (6.4)), is termed as a

(i) $\mathbf{g}$-*framelet system* ($\mathbf{g}$-FS) for $\ell^2(\mathbb{Z}_N^d)$ if it is a frame for $\ell^2(\mathbb{Z}_N^d)$.

(ii) *tight* $\mathbf{g}$-*framelet system* (*tight* $\mathbf{g}$-*FS*) for $\ell^2(\mathbb{Z}_N^d)$ if it is a tight frame for $\ell^2(\mathbb{Z}_N^d)$.

(iii) *Parseval* $\mathbf{g}$-*framelet system* (*Parseval* $\mathbf{g}$-*FS*) for $\ell^2(\mathbb{Z}_N^d)$ if it is a tight frame for $\ell^2(\mathbb{Z}_N^d)$ with frame bound equal to one.

(iv) $\mathbf{g}$-*orthonormal wavelet system* ($\mathbf{g}$-ONWS) if it is a Parseval frame for $\ell^2(\mathbb{Z}_N^d)$ along with $L = |\det(A)| \ge 2$.

Here, we mention that for $\mathrm{g} = \mathbf{e}$, the above defined $\mathbf{g}$-framelet systems will be called as $\mathbf{e}$-*framelet systems* (simply, *framelet systems*).

Note that the frames in Hilbert spaces allow stable representation of all the elements of the space via a given frame and its dual frame. Now, for defining dual of a $\mathbf{g}$-FS, we need to first recall $(\mathcal{B}_1)$ and then assume the following throughout this chapter:

$(\mathcal{B}_5)$: *For each positive integer* $1 \le a \le 2$, *let* $\Psi^{(a)} := \{\psi_j^{(a)}\}_{j=0}^{L-1} \subset \ell^2(\mathbb{Z}_N^d)$, *where* $L \ge |\det(A)| \ge 2$.

$(\mathcal{B}_6)$: *By assuming* $(\mathcal{B}_1)$, $(\mathcal{B}_3)$ *and* $(\mathcal{B}_5)$, *for each* $\mathrm{g} \in \mathbb{G}$ *and positive integer* $1 \le a \le 2$, *we define the collection in* $\ell^2(\mathbb{Z}_N^d)$ *by*

$$(6.5)\qquad \widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(a)}) := \left\{\widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(a)}) : k \in \mathfrak{D}_{\widetilde{A}},\ \psi_j^{(a)} \in \Psi^{(a)},\ 0 \le j \le L-1\right\}.$$

Definition 6.7. A system $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(1)})$ in $\ell^2(\mathbb{Z}_N^d)$ will form a *dual* $\mathbf{g}$-*framelet system* (*dual* $\mathbf{g}$-*FS*) for the $\mathbf{g}$-framelet system $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(2)})$ in $\ell^2(\mathbb{Z}_N^d)$, if for all $f \in \ell^2(\mathbb{Z}_N^d)$, the following reconstruction formula is satisfied:

$$f = \sum_{k \in \mathfrak{D}_{\widetilde{A}}} \sum_{j=0}^{L-1} \langle f, \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(1)}) \rangle \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(2)})$$
$$= \sum_{k \in \mathfrak{D}_{\widetilde{A}}} \sum_{j=0}^{L-1} \langle f, \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(2)}) \rangle \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(1)}).$$

In this case, we say that the systems $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(1)})$ and $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(2)})$ form a **g**-*dual pair* in $\ell^2(\mathbb{Z}_N^d)$ or $\Psi^{(1)}$ is a **g**-*dual* of $\Psi^{(2)}$. Similarly, for $\mathrm{g} = \mathbf{e}$, we say that the collections $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(1)})$ and $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(2)})$ form an **e**-*dual pair* in $\ell^2(\mathbb{Z}_N^d)$.

In the next section, we provide the characterization results for the above defined **g**-framelet systems. Several such systems have been introduced very successfully in mathematics and its applications by many researchers (see [10, 12, 51, 59]).

6.3. Characterization results for the g-framelet systems

Our first aim in this section is to characterize dual of a **g**-FS, from which the characterizations for the Parseval **g**-FS and **g**-ONWS can be easily deduced. For this, we use equivarient property of DFT and translation operator, and, the additive and multiplicative properties of the group action θ. Therefore in view of Lemma 6.5, *here and throughout, we assume the action θ of $\mathbb{G}$ on $\mathbb{C}$ is additive as well as multiplicative.*

6.3.1. Characterization for a dual g-framelet system

We begin this subsection by stating some notation to be used in the remainder of this chapter. By $\mathfrak{D}^*_{\widetilde{A}}$, we denote the complete digit set of $\mathcal{P}$ in $\mathbb{Z}_N^d$, where $\mathcal{P} := C\mathbb{Z}_N^d$ for $C = (\widetilde{A})^t$, that is, transpose of the matrix $\widetilde{A}$. Here, observe that $|\mathfrak{D}^*_{\widetilde{A}}| = |\mathfrak{D}_{\widetilde{A}}|$.

Motivated from a result of Frazier [26, Theorem 3.8], that is, the system (defined in (5.1)) generated by u_1 and u_2 forms an orthonormal basis for $\ell^2(\mathbb{Z}_N)$ if, and only if, for each $0 \leq n \leq M-1$, the system matrix given by

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \widehat{u}_1(n) & \widehat{u}_2(n) \\ \widehat{u}_1(n+M) & \widehat{u}_2(n+M) \end{pmatrix}$$

is unitary, we expect the required characterization for dual **g**-FS in terms of system matrices for the generators $\Psi^{(a)} = \{\psi_j^{(a)}\}_{j=0}^{L-1}; a = 1, 2$ defined in $(\mathcal{B}_5)$. Here, we provide the definition for this:

Definition 6.8. For each $\mathrm{g} \in \mathbb{G}$, positive integer $1 \leq a \leq 2$ and $m \in \mathfrak{D}^*_{\widetilde{A}}$, we define the *system matrix of* $\Psi^{(a)}$, that is, $\mathcal{S}^{(a)}_{\mathrm{g}}(m)$ as follows:

$$\mathcal{S}^{(a)}_{\mathrm{g}}(m) := \frac{1}{\sqrt{|\det(A)|}} \Big(\widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}^{(a)}_j\big)(m+p) \Big)_{\substack{p \in \mathcal{P} \\ 0 \leq j \leq L-1}}.$$

Further, in case of frame theory, the dual (cross dual) Gramian operators play a significant role in deducing the important characteristics of frame vectors. Thus, we want to relate the system matrices of $\Psi^{(a)}; a = 1, 2$ with the dual (cross dual) Gramian operators corresponding to some suitable collections. Therefore, our next step is to define these collections.

For this, for each $\mathrm{g} \in \mathbb{G}$, positive integer $1 \leq a \leq 2$ and $m \in \mathfrak{D}^*_{\widetilde{A}}$, we consider the collection in the space $\ell^2(\mathcal{P})$, given as

(6.6)

$$\widetilde{\mathcal{K}}(\mathrm{g}, m, \Psi^{(a)}) := \left\{ \mathcal{K}^{\mathrm{g}}(\psi^{(a)}_j)(m) := \frac{1}{\sqrt{|\det(A)|}} \mathcal{T}^{\mathrm{g}}\big(\widetilde{\theta}(\mathrm{g}, \psi^{(a)}_j)\big)(m);\ 0 \leq j \leq L-1 \right\},$$

where we define

$$\mathcal{T}^{\mathrm{g}}\big(\widetilde{\theta}(\mathrm{g}, \psi^{(a)}_j)\big)(m) = \Big(\widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}^{(a)}_j\big)(m+p) \Big)_{p \in \mathcal{P}}$$

in view of equivarient property of DFT. Here, the map $\mathcal{T}^{\mathrm{g}}$ described by

$$\mathcal{T}^{\mathrm{g}} : \ell^2(\mathbb{Z}^d_N) \to \ell^2(\mathfrak{D}^*_{\widetilde{A}}, \ell^2(\mathcal{P}));$$

$$\mathcal{T}^{\mathrm{g}} f(m)(p) = \widehat{f}\,(m+p), \quad \text{for all } m \in \mathfrak{D}^*_{\widetilde{A}} \text{ and } p \in \mathcal{P},$$

is a well defined isometric isomorphism, which follows by observing that the collection $\mathcal{H}_0 := \ell^2(\mathfrak{D}^*_{\widetilde{A}}, \ell^2(\mathcal{P}))$ is an $|\mathfrak{D}^*_{\widetilde{A}}||\mathcal{P}|$, that is, N^d-dimensional Hilbert space with the usual inner product, and

$$\begin{aligned} ||\mathcal{T}^{\mathrm{g}} f||^2_{\mathcal{H}_0} &= \sum_{l \in \mathfrak{D}^*_{\widetilde{A}}} ||\mathcal{T}^{\mathrm{g}} f(l)||^2_{\ell^2(\mathcal{P})} = \sum_{l \in \mathfrak{D}^*_{\widetilde{A}}} \sum_{p \in \mathcal{P}} |\ \mathcal{T}^{\mathrm{g}} f(l)(p)\ |^2 \\ &= \sum_{l \in \mathfrak{D}^*_{\widetilde{A}}} \sum_{p \in \mathcal{P}} |\ \widehat{f}(l+p)\ |^2 = \sum_{n \in \mathbb{Z}^d_N} |\ \widehat{f}(n)\ |^2 = N^d ||f||^2, \text{ for all } f \in \ell^2(\mathbb{Z}^d_N), \end{aligned}$$

since $\mathbb{Z}^d_N = \bigcup\limits_{l \in \mathfrak{D}^*_{\widetilde{A}}} (l + \mathcal{P})$. The above discussion leads to the following result:

Proposition 6.9. *For each* $\mathrm{g} \in \mathbb{G}$*, the map* $\mathcal{T}^{\mathrm{g}}$ *is well defined and establishes an isometric isomorphism between* $\ell^2(\mathbb{Z}^d_N)$ *and* $\mathcal{H}_0$*. Moreover, we have* $||\mathcal{T}^{\mathrm{g}} f||_{\mathcal{H}_0} = \sqrt{N^d}||f||$*, for all* $f \in \ell^2(\mathbb{Z}^d_N)$.

Next, we define the dual (cross dual) Gramian operators corresponding to the collections in (6.6).

Definition 6.10. For each $\mathrm{g} \in \mathbb{G}$, positive integers $1 \leq a, b \leq 2$, and $m \in \mathfrak{D}^*_{\widetilde{A}}$, the composition operator $G^{(a),(b)}_{\mathrm{g}}(m) := \big(\Theta^{(a)}_{\mathrm{g}}(m)\big)^* \Theta^{(b)}_{\mathrm{g}}(m)$ defined by

$$G^{(a),(b)}_{\mathrm{g}}(m) : \ell^2(\mathcal{P}) \to \ell^2(\mathcal{P});\ f \mapsto \sum_{j=0}^{L-1} \langle f, \mathcal{K}^{\mathrm{g}}(\psi^{(b)}_j)(m)\rangle \mathcal{K}^{\mathrm{g}}(\psi^{(a)}_j)(m),$$

will be termed as the *dual* (*cross dual*) *Gramian operator* corresponding to the collections $\widetilde{\mathcal{K}}(\mathrm{g}, m, \Psi^{(a)})$ and $\widetilde{\mathcal{K}}(\mathrm{g}, m, \Psi^{(b)})$, when $a = b$ ($a \neq b$). Here, $\Theta^{(a)}_{\mathrm{g}}(m)$ (respectively, adjoint of $\Theta^{(a)}_{\mathrm{g}}(m)$, that is, $\big(\Theta^{(a)}_{\mathrm{g}}(m)\big)^*$) is the analysis operator (respectively, synthesis operator) corresponding to the collection $\widetilde{\mathcal{K}}(\mathrm{g}, m, \Psi^{(a)})$, where the map $\Theta^{(a)}_{\mathrm{g}}(m) : \ell^2(\mathcal{P}) \to \mathbb{C}^L$.

The system matrices for $\Psi^{(a)}; a = 1, 2$ and the dual (cross dual) Gramian operators corresponding to the collections in (6.6) are related as follows:

Proposition 6.11. *For each* $\mathrm{g} \in \mathbb{G}$, *positive integers* $1 \leq a, b \leq 2$, *and* $m \in \mathfrak{D}^*_{\widetilde{A}}$, *the dual* (*cross dual*) *Gramian operator* $G^{(a),(b)}_{\mathrm{g}}(m)$ *defined as above, satisfy the following relation:*

$$\begin{aligned} G^{(a),(b)}_{\mathrm{g}}(m) &= \mathcal{S}^{(a)}_{\mathrm{g}}(m)\big(\mathcal{S}^{(b)}_{\mathrm{g}}(m)\big)^* \\ &= \frac{1}{|\det(A)|}\left(\sum_{j=0}^{L-1} \widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}^{(a)}_j\big)(m + p_1)\overline{\widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}^{(b)}_j\big)(m + p_2)}\right)_{p_1, p_2 \in \mathcal{P}}, \end{aligned}$$

where $\big(\mathcal{S}^{(b)}_{\mathrm{g}}(m)\big)^*$ *represents the conjugate transpose of* $\mathcal{S}^{(b)}_{\mathrm{g}}(m)$ (*that is, the system matrix for* $\Psi^{(b)}$).

Proof. Let $\mathrm{g} \in \mathbb{G}$, $m \in \mathfrak{D}^*_{\widetilde{A}}$ and positive integers $1 \leq a, b \leq 2$. Then,

$$\mathcal{S}^{(a)}_{\mathrm{g}}(m)\big(\mathcal{S}^{(b)}_{\mathrm{g}}(m)\big)^* = \frac{1}{|\det(A)|}\left(\sum_{j=0}^{L-1} \widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}^{(a)}_j\big)(m + p_1)\overline{\widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}^{(b)}_j\big)(m + p_2)}\right)_{p_1, p_2 \in \mathcal{P}}.$$

Now, we wish to compute the dual (cross dual) Gramian operators corresponding to the collections $\widetilde{\mathcal{K}}(\mathrm{g}, m, \Psi^{(a)}); a = 1, 2$. For this, let $\{e_p\}_{p \in \mathcal{P}}$ be the standard orthonormal basis for $\ell^2(\mathcal{P})$. Then, for $s, q \in \mathcal{P}$, the entry in the s^{th} row and q^{th} column of $G^{(a),(b)}_{\mathrm{g}}(m)$

is given by

$$\begin{aligned}\big\langle G_{\mathrm{g}}^{(a),(b)}(m)e_q, e_s\big\rangle &= \big\langle \big(\Theta_{\mathrm{g}}^{(a)}(m)\big)^* \Theta_{\mathrm{g}}^{(b)}(m)e_q, e_s\big\rangle \\ &= \Big\langle \sum_{j=0}^{L-1} \langle e_q, \mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(m)\rangle \mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(m), e_s\Big\rangle \\ &= \sum_{j=0}^{L-1} \langle e_q, \mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(m)\rangle \overline{\langle e_s, \mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(m)\rangle} \\ &= \frac{1}{|\det(A)|} \sum_{j=0}^{L-1} \widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}_j^{(a)}\big)(m+s)\overline{\widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}_j^{(b)}\big)(m+q)},\end{aligned}$$

since

$$\begin{aligned}\langle e_q, \mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(m)\rangle &= \frac{1}{\sqrt{|\det(A)|}} \sum_{q'\in\mathcal{P}} e_q(q')\overline{\widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}_j^{(b)}\big)(m+q')} \\ &= \frac{1}{\sqrt{|\det(A)|}} \overline{\widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}_j^{(b)}\big)(m+q)}\end{aligned}$$

and

$$\overline{\langle e_s, \mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(m)\rangle} = \frac{1}{\sqrt{|\det(A)|}} \widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}_j^{(a)}\big)(m+s).$$

Therefore, we conclude that

$$G_{\mathrm{g}}^{(a),(b)}(m) = \frac{1}{|\det(A)|} \left(\sum_{j=0}^{L-1} \widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}_j^{(a)}\big)(m+s)\overline{\widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}_j^{(b)}\big)(m+q)} \right)_{s,q\in\mathcal{P}}.$$

□

The following result relates the collections $\widetilde{\mathcal{K}}(\mathrm{g}, m, \Psi^{(a)})$ and $\widetilde{\mathcal{K}}(\mathrm{g}, m, \Psi^{(b)})$ with the operator $G_{\mathrm{g}}^{(a),(b)}(m)$:

Lemma 6.12. *For each $\mathrm{g} \in \mathbb{G}$, $m \in \mathfrak{D}^*_{\widetilde{A}}$, and positive integers $1 \le a, b \le 2$, the collections $\widetilde{\mathcal{K}}(\mathrm{g}, m, \Psi^{(a)})$ and $\widetilde{\mathcal{K}}(\mathrm{g}, m, \Psi^{(b)})$ are dual frames for $\ell^2(\mathcal{P})$ if, and only if, the dual (cross dual) Gramian operator $G_{\mathrm{g}}^{(a),(b)}(m) = I_Q$, where I_Q is an identity matrix of order $Q := |\det(A)|$.*

Proof. Let $\mathrm{g} \in \mathbb{G}$, $m \in \mathfrak{D}^*_{\widetilde{A}}$, and positive integers $1 \le a, b \le 2$. Then, the collections $\widetilde{\mathcal{K}}(\mathrm{g}, m, \Psi^{(a)})$ and $\widetilde{\mathcal{K}}(\mathrm{g}, m, \Psi^{(b)})$ are dual frames for $\ell^2(\mathcal{P})$ if, and only if, the following reconstruction formula holds:

$$h=\sum_{j=0}^{L-1}\langle h,\mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(m)\rangle\mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(m),\ \text{ for all } h\in\ell^2(\mathcal{P}),$$

which in view of Definition 6.10 is equivalent to the following equality:

$$\begin{aligned}\langle h,\widetilde{h}\rangle &=\sum_{j=0}^{L-1}\langle h,\mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(m)\rangle\langle\mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(m),\widetilde{h}\rangle\\ &=\langle G_{\mathrm{g}}^{(a),(b)}(m)h,\widetilde{h}\rangle,\ \text{ for all } h,\widetilde{h}\in\ell^2(\mathcal{P}),\end{aligned}$$

that means, $G_{\mathrm{g}}^{(a),(b)}(m)$ is an identity operator on $\ell^2(\mathcal{P})$, and hence, the result follows. □

Now, we state the main result of this subsection that represents the conditions under which the systems $\widetilde{\mathfrak{B}}_F(\mathrm{g},A,\Psi^{(a)})$ and $\widetilde{\mathfrak{B}}_F(\mathrm{g},A,\Psi^{(b)})$; $1\le a,b\le 2$ (defined in (6.5)) form a $\mathbf{g}$-dual pair in $\ell^2(\mathbb{Z}_N^d)$:

Theorem 6.13. *For each* $\mathrm{g}\in\mathbb{G}$ *and positive integers* $1\le a,b\le 2$, *let* $\Psi^{(a)}=\{\psi_j^{(a)}\}_{j=0}^{L-1}$ *and* $\Psi^{(b)}=\{\psi_j^{(b)}\}_{j=0}^{L-1}$ *be subsets of* $\ell^2(\mathbb{Z}_N^d)$, *where* $L\ge|\det(A)|\ge 2$. *Then, the following assertions are equivalent:*

(i) *The collections* $\widetilde{\mathfrak{B}}_F(\mathrm{g},A,\Psi^{(a)})$ *and* $\widetilde{\mathfrak{B}}_F(\mathrm{g},A,\Psi^{(b)})$ *form a* $\mathbf{g}$*-dual pair in* $\ell^2(\mathbb{Z}_N^d)$.

(ii) *For each* $m\in\mathfrak{D}^*_{\widetilde{A}}$, *the following relation between the system matrices for* $\Psi^{(a)}$ *and* $\Psi^{(b)}$ *holds*:

$$\mathcal{S}_{\mathrm{g}}^{(a)}(m)\big(\mathcal{S}_{\mathrm{g}}^{(b)}(m)\big)^*=I_Q,$$

where I_Q *is an identity matrix of order* $Q=|\det(A)|$, *equivalently, it means*

$$\sum_{j=0}^{L-1}\widetilde{\theta}\big(\mathrm{g},\widehat{\psi}_j^{(a)}\big)(m+p_1)\overline{\widetilde{\theta}\big(\mathrm{g},\widehat{\psi}_j^{(b)}\big)(m+p_2)}=|\det(A)|\delta_{p_1p_2},\ \text{ for all } p_1,p_2\in\mathcal{P}. \tag{6.7}$$

In particular, by using $a=b$ *in the above result, we get a characterization of* $\Psi^{(a)}=\Psi^{(b)}$ *such that for each* $\mathrm{g}\in\mathbb{G}$, *the system* $\widetilde{\mathfrak{B}}_F(\mathrm{g},A,\Psi^{(a)})$ *defined in (6.5) forms a Parseval* $\mathbf{g}$*-FS for* $\ell^2(\mathbb{Z}_N^d)$, *which in addition by considering* L *equals to the* $|\det(A)|$ *leads to a* $\mathbf{g}$*-ONWS in* $\ell^2(\mathbb{Z}_N^d)$, *and hence a generalization of the result [26, Theorem 3.8] is obtained for the multidimensional set-up.*

Observe that by using $\mathrm{g}=\mathbf{e}$ and the definition of an induced group action $\widetilde{\theta}$ in Theorem 6.13, we can easily deduce a characterization for an $\mathbf{e}$-dual pair. We state this result in the following form:

Corollary 6.14. *For positive integers $1 \le a, b \le 2$, let $\Psi^{(a)} = \{\psi_j^{(a)}\}_{j=0}^{L-1}$ and $\Psi^{(b)} = \{\psi_j^{(b)}\}_{j=0}^{L-1}$ be subsets of $\ell^2(\mathbb{Z}_N^d)$, where $L \ge |\det(A)| \ge 2$. Then, the following statements are equivalent:*

(i) *The collections $\widetilde{\mathfrak{B}}_F(\mathbf{e}, A, \Psi^{(a)})$ and $\widetilde{\mathfrak{B}}_F(\mathbf{e}, A, \Psi^{(b)})$ form an $\mathbf{e}$-dual pair in $\ell^2(\mathbb{Z}_N^d)$.*

(ii) *For each $m \in \mathfrak{D}^*_{\widetilde{A}}$, we have*

$$\sum_{j=0}^{L-1} \widehat{\psi}_j^{(a)}(m+p_1)\overline{\widehat{\psi}_j^{(b)}(m+p_2)} = |\det(A)|\delta_{p_1p_2}, \quad \text{for all } p_1, p_2 \in \mathcal{P}.$$

In particular, by using $a = b$ in the above result, we get a characterization of $\Psi^{(a)} = \Psi^{(b)}$ such that the system $\widetilde{\mathfrak{B}}_F(\mathbf{e}, A, \Psi^{(a)})$ (defined in (6.5) for $\mathrm{g} = \mathbf{e}$) forms a Parseval $\mathbf{e}$-FS for $\ell^2(\mathbb{Z}_N^d)$, which in addition by considering $L = |\det(A)|$ leads to an $\mathbf{e}$-ONWS in $\ell^2(\mathbb{Z}_N^d)$.

The following result is useful in finding a $\mathbf{g}$-ONWS with the help of one generator:

Proposition 6.15. *For $N \in \mathbb{N}$, let the matrix $A \in \widetilde{\mathcal{M}}(d, \mathbb{Z})$ with $|\det(A)| = 2$ be such that the matrix $\widetilde{A} = NA^{-1} \in \widetilde{\mathcal{M}}(d, \mathbb{Z})$. Let $\psi_0 \in \ell^2(\mathbb{Z}_N^d)$ be such that for each $\mathrm{g} \in \mathbb{G}$, the collection $\left\{\widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_0) : k \in \mathfrak{D}_{\widetilde{A}}\right\}$ forms an orthonormal set in $\ell^2(\mathbb{Z}_N^d)$ with $|\mathfrak{D}_{\widetilde{A}}|$ elements. Further, by assuming $\mathcal{P} = \{\gamma_1, \gamma_2\}$, we let $\psi_1 \in \ell^2(\mathbb{Z}_N^d)$ satisfying the following relation:*

$$\widetilde{\theta}(\mathrm{g}, \widehat{\psi}_1)(m + \gamma_{j_1}) = (-1)^{j_1-1}e^{-2\pi i m/N}\overline{\widetilde{\theta}(\mathrm{g}, \widehat{\psi}_0)(m + \gamma_{j_2})},$$

for all $m \in \mathfrak{D}^*_{\widetilde{A}}$ and $1 \le j_1 \ne j_2 \le 2$. Then, for each $\mathrm{g} \in \mathbb{G}$, the collection given by $\left\{\widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j) : k \in \mathfrak{D}_{\widetilde{A}}, 0 \le j \le 1\right\}$ forms a $\mathbf{g}$-ONWS for $\ell^2(\mathbb{Z}_N^d)$.

Proof. Following the technique used in Proposition 6.1, it can be easily verified that the orthonormality of the collection $\left\{\widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_0) : k \in \mathfrak{D}_{\widetilde{A}}\right\}$ is equivalent to saying that for each $m \in \mathfrak{D}^*_{\widetilde{A}}$, we can write:

$$\sum_{\gamma \in \mathcal{P}} |\widetilde{\theta}(\mathrm{g}, \widehat{\psi}_0)(m+\gamma)|^2 = |\det(A)| = 2. \tag{6.8}$$

Therefore, the result follows by using the given form of $\psi_1 \in \ell^2(\mathbb{Z}_N^d)$ along with (6.8), which says

$$\sum_{\gamma \in \mathcal{P}} |\widetilde{\theta}(\mathrm{g}, \widehat{\psi}_1)(m+\gamma)|^2 = \sum_{\gamma \in \mathcal{P}} |\overline{\widetilde{\theta}(\mathrm{g}, \widehat{\psi}_0)(m+\gamma)}|^2 = 2,$$

and for each $0 \le i \ne j \le 1$, $\sum_{\gamma \in \mathcal{P}} \widetilde{\theta}(\mathrm{g}, \widehat{\psi}_i)(m+\gamma)\overline{\widetilde{\theta}(\mathrm{g}, \widehat{\psi}_j)(m+\gamma)} = 0$, for all $m \in \mathfrak{D}^*_{\widetilde{A}}$. $\square$

Our next motive is to prove Theorem 6.13. For this, the following result plays an important role, which is true in view of Proposition 5.12:

Lemma 6.16. *Let $\{E_k\}_{k\in\mathfrak{D}_{\widetilde{A}}} \subseteq \ell^2(\mathfrak{D}^*_{\widetilde{A}})$, where for each $k \in \mathfrak{D}_{\widetilde{A}}$, we define $E_k(m) = \frac{1}{\sqrt{|\mathfrak{D}^*_{\widetilde{A}}|}} e^{2\pi i\langle m,Ak\rangle/N}$, for all $m \in \mathfrak{D}^*_{\widetilde{A}}$. Then, the system $\{E_k\}_{k\in\mathfrak{D}_{\widetilde{A}}}$ forms an orthonormal basis for $\ell^2(\mathfrak{D}^*_{\widetilde{A}})$.*

Proof of Theorem 6.13. Let $\mathrm{g} \in \mathbb{G}$ and positive integers $1 \leq a, b \leq 2$. Then, by using the approach of [10], it can be justified that the system $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(a)})$ is a $\mathbf{g}$-FS for $\ell^2(\mathbb{Z}_N^d)$ with frame bounds α_1 and α_2 if, and only if, for each $m \in \mathfrak{D}^*_{\widetilde{A}}$, the collection $\widetilde{\mathcal{K}}(\mathrm{g}, m, \Psi^{(a)})$ is a frame for $\ell^2(\mathcal{P})$ with the same bounds.

Also, if $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(a)})$ and $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(b)})$ are collections in $\ell^2(\mathbb{Z}_N^d)$, then, for $f, h \in \ell^2(\mathbb{Z}_N^d)$, we get the following estimates:

$$\begin{aligned}
&\sum_{j=0}^{L-1}\sum_{k\in\mathfrak{D}_{\widetilde{A}}} \big\langle \widetilde{\theta}\big(\mathrm{g}, T_{Ak}\psi_j^{(b)}\big), h\big\rangle \overline{\big\langle \widetilde{\theta}\big(\mathrm{g}, T_{Ak}\psi_j^{(a)}\big), f\big\rangle} \\
&= \sum_{j=0}^{L-1}\sum_{k\in\mathfrak{D}_{\widetilde{A}}} \big\langle T_{Ak}\big(\widetilde{\theta}(\mathrm{g}, \psi_j^{(b)})\big), h\big\rangle \overline{\big\langle T_{Ak}\big(\widetilde{\theta}(\mathrm{g}, \psi_j^{(a)})\big), f\big\rangle} \\
&= \frac{1}{N^{2d}}\sum_{j=0}^{L-1}\sum_{k\in\mathfrak{D}_{\widetilde{A}}} \big\langle T_{Ak}\big(\widehat{\widetilde{\theta}(\mathrm{g}, \psi_j^{(b)})}\big), \widehat{h}\big\rangle \overline{\big\langle T_{Ak}\big(\widehat{\widetilde{\theta}(\mathrm{g}, \psi_j^{(a)})}\big), \widehat{f}\,\big\rangle},
\end{aligned}$$

where the last equality follows from Lemma 6.5 and the Plancherel's formula. Further, by using the property (P2), the definition of map $\mathcal{T}^{\mathrm{g}}$, equivarient property of the DFT map, and the fact that $\langle \alpha, \beta\rangle \in N\mathbb{Z}$, for all $\alpha \in A\mathbb{Z}_N^d$ and $\beta \in C\mathbb{Z}_N^d$, the above quantity is equivalent to the following:

$$\begin{aligned}
&\frac{1}{N^{2d}}\sum_{j=0}^{L-1}\sum_{k\in\mathfrak{D}_{\widetilde{A}}}\sum_{n\in\mathbb{Z}_N^d} e^{-2\pi i\langle n,Ak\rangle/N}\widehat{\widetilde{\theta}\big(\mathrm{g}, \psi_j^{(b)}\big)}(n)\overline{\widehat{h}(n)} \sum_{m\in\mathbb{Z}_N^d} e^{2\pi i\langle m,Ak\rangle/N}\overline{\widehat{\widetilde{\theta}\big(\mathrm{g}, \psi_j^{(a)}\big)}(m)\widehat{f}(m)} \\
&= \frac{1}{N^{2d}}\sum_{j=0}^{L-1}\sum_{k\in\mathfrak{D}_{\widetilde{A}}}\sum_{m_1\in\mathfrak{D}^*_{\widetilde{A}}} e^{-2\pi i\langle m_1,Ak\rangle/N} \sum_{p_1\in\mathcal{P}} \widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}_j^{(b)}\big)(m_1+p_1)\overline{\widehat{h}(m_1+p_1)} \\
&\qquad \times \sum_{m_2\in\mathfrak{D}^*_{\widetilde{A}}} e^{2\pi i\langle m_2,Ak\rangle/N} \sum_{p_2\in\mathcal{P}} \overline{\widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}_j^{(a)}\big)(m_2+p_2)\widehat{f}(m_2+p_2)}
\end{aligned}$$

$$= \frac{|\det(A)|}{N^{2d}} \sum_{j=0}^{L-1} \sum_{k\in\mathfrak{D}_{\widetilde{A}}} \sum_{m_1\in\mathfrak{D}^*_{\widetilde{A}}} e^{-2\pi i\langle m_1,Ak\rangle/N} \big\langle \mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(m_1), \mathcal{T}^{\mathrm{g}}h(m_1)\big\rangle$$
$$\times \sum_{m_2\in\mathfrak{D}^*_{\widetilde{A}}} e^{2\pi i\langle m_2,Ak\rangle/N} \overline{\big\langle \mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(m_2), \mathcal{T}^{\mathrm{g}}f(m_2)\big\rangle}.$$

Next, we define

$$\mathcal{G}^{(a)}_{j,\mathrm{g}}(m) = \langle \mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(m), \mathcal{T}^{\mathrm{g}}f(m)\rangle$$

and

$$\mathcal{G}^{(b)}_{j,\mathrm{g}}(m) = \langle \mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(m), \mathcal{T}^{\mathrm{g}}h(m)\rangle,$$

for each $0 \le j \le L-1$ and $m \in \mathfrak{D}^*_{\widetilde{A}}$. Thus, by using $\{E_k\}_{k\in\mathfrak{D}_{\widetilde{A}}} \subseteq \ell^2(\mathfrak{D}^*_{\widetilde{A}})$ as an orthonormal basis for $\ell^2(\mathfrak{D}^*_{\widetilde{A}})$ from Lemma 6.16, we can write

$$\sum_{j=0}^{L-1} \sum_{k\in\mathfrak{D}_{\widetilde{A}}} \big\langle \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(b)}), h\big\rangle \overline{\big\langle \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(a)}), f\big\rangle}$$
$$= \frac{|\det(A)|}{N^{2d}} \sum_{j=0}^{L-1} \sum_{k\in\mathfrak{D}_{\widetilde{A}}} \sum_{m_1\in\mathfrak{D}^*_{\widetilde{A}}} e^{-2\pi i\langle m_1,Ak\rangle/N} \mathcal{G}^{(b)}_{j,\mathrm{g}}(m_1) \overline{\sum_{m_2\in\mathfrak{D}^*_{\widetilde{A}}} e^{-2\pi i\langle m_2,Ak\rangle/N} \mathcal{G}^{(a)}_{j,\mathrm{g}}(m_2)}$$
$$= \frac{|\det(A)||\mathfrak{D}^*_{\widetilde{A}}|}{N^{2d}} \sum_{j=0}^{L-1} \sum_{k\in\mathfrak{D}_{\widetilde{A}}} \langle \mathcal{G}^{(b)}_{j,\mathrm{g}}, E_k\rangle \overline{\langle \mathcal{G}^{(a)}_{j,\mathrm{g}}, E_k\rangle}$$
$$= \frac{1}{N^d} \sum_{j=0}^{L-1} \langle \mathcal{G}^{(b)}_{j,\mathrm{g}}, \mathcal{G}^{(a)}_{j,\mathrm{g}}\rangle$$
$$= \frac{1}{N^d} \sum_{j=0}^{L-1} \sum_{\xi\in\mathfrak{D}^*_{\widetilde{A}}} \mathcal{G}^{(b)}_{j,\mathrm{g}}(\xi) \overline{\mathcal{G}^{(a)}_{j,\mathrm{g}}(\xi)}$$
$$(*) \qquad = \frac{1}{N^d} \sum_{j=0}^{L-1} \sum_{\xi\in\mathfrak{D}^*_{\widetilde{A}}} \langle \mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(\xi), \mathcal{T}^{\mathrm{g}}h(\xi)\rangle \overline{\langle \mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(\xi), \mathcal{T}^{\mathrm{g}}f(\xi)\rangle}.$$

Now, in order to prove the sufficient conditions, assume that the relation (6.7) holds true, which in view of Proposition 6.11 and Lemma 6.12, is equivalent to the fact that for each $m \in \mathfrak{D}^*_{\widetilde{A}}$, the collections $\widetilde{\mathcal{K}}(\mathrm{g}, m, \Psi^{(a)})$ and $\widetilde{\mathcal{K}}(\mathrm{g}, m, \Psi^{(b)})$ are dual frames for $\ell^2(\mathcal{P})$, that is if, and only if, the following relation holds:

$$h_1 = \sum_{j=0}^{L-1} \langle h_1, \mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(m)\rangle \mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(m), \text{ for all } h_1 \in \ell^2(\mathcal{P}),$$

which can be equivalently written as

$$\langle h_1, h_2\rangle = \sum_{j=0}^{L-1} \langle h_1, \mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(m)\rangle \langle \mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(m), h_2\rangle, \text{ for all } h_1, h_2 \in \ell^2(\mathcal{P}).$$

If $f, h \in \ell^2(\mathbb{Z}_N^d)$, both $\mathcal{T}^{\mathrm{g}} f(m)$ and $\mathcal{T}^{\mathrm{g}} h(m)$ belong to $\ell^2(\mathcal{P})$, for all $m \in \mathfrak{D}^*_{\widetilde{A}}$, by using the definition and properties of the map $\mathcal{T}^{\mathrm{g}}$. Therefore,

$$\langle \mathcal{T}^{\mathrm{g}} f(m), \mathcal{T}^{\mathrm{g}} h(m)\rangle = \sum_{j=0}^{L-1} \langle \mathcal{T}^{\mathrm{g}} f(m), \mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(m)\rangle \langle \mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(m), \mathcal{T}^{\mathrm{g}} h(m)\rangle,$$

for all $f, h \in \ell^2(\mathbb{Z}_N^d)$. Further, by taking summation over $\mathfrak{D}^*_{\widetilde{A}}$ in the above equation, we get the following:

$$\begin{aligned}
&\sum_{m \in \mathfrak{D}^*_{\widetilde{A}}} \langle \mathcal{T}^{\mathrm{g}} f(m), \mathcal{T}^{\mathrm{g}} h(m)\rangle \\
&= \sum_{m \in \mathfrak{D}^*_{\widetilde{A}}} \sum_{j=0}^{L-1} \langle \mathcal{T}^{\mathrm{g}} f(m), \mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(m)\rangle \langle \mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(m), \mathcal{T}^{\mathrm{g}} h(m)\rangle,
\end{aligned}$$

for all $f, h \in \ell^2(\mathbb{Z}_N^d)$. Moreover, by using the isometric properties of the map $\mathcal{T}^{\mathrm{g}}$ and equation $(*)$ in the above equality, we get

$$\begin{aligned}
N^d \langle f, h\rangle = \langle \mathcal{T}^{\mathrm{g}} f, \mathcal{T}^{\mathrm{g}} h\rangle &= \sum_{m \in \mathfrak{D}^*_{\widetilde{A}}} \sum_{j=0}^{L-1} \langle \mathcal{T}^{\mathrm{g}} f(m), \mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(m)\rangle \langle \mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(m), \mathcal{T}^{\mathrm{g}} h(m)\rangle \\
&= N^d \sum_{j=0}^{L-1} \sum_{k \in \mathfrak{D}_{\widetilde{A}}} \big\langle f, \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(a)})\big\rangle \overline{\big\langle h, \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(b)})\big\rangle},
\end{aligned}$$

for all $f, h \in \ell^2(\mathbb{Z}_N^d)$. Thus, the collections $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(a)})$ and $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(b)})$ form a **g**-dual pair in $\ell^2(\mathbb{Z}_N^d)$.

Conversely, if the collection $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(a)})$ is a dual **g**-FS for $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(b)})$ in $\ell^2(\mathbb{Z}_N^d)$, or equivalently by the previous computations, if the identity

(6.9)

$$\sum_{m \in \mathfrak{D}^*_{\widetilde{A}}} \langle \mathcal{T}^{\mathrm{g}} f(m), \mathcal{T}^{\mathrm{g}} h(m)\rangle = \sum_{m \in \mathfrak{D}^*_{\widetilde{A}}} \sum_{j=0}^{L-1} \langle \mathcal{T}^{\mathrm{g}} f(m), \mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(m)\rangle \langle \mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(m), \mathcal{T}^{\mathrm{g}} h(m)\rangle,$$

holds for all $f, h \in \ell^2(\mathbb{Z}_N^d)$, we would like to deduce that, for each $m \in \mathfrak{D}^*_{\widetilde{A}}$ and all $f, h \in \ell^2(\mathbb{Z}_N^d)$,

$$\langle \mathcal{T}^{\mathrm{g}} f(m), \mathcal{T}^{\mathrm{g}} h(m)\rangle = \sum_{j=0}^{L-1} \langle \mathcal{T}^{\mathrm{g}} f(m), \mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(m)\rangle \langle \mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(m), \mathcal{T}^{\mathrm{g}} h(m)\rangle. \tag{6.10}$$

Let $\{e_i\}_{i\in\mathcal{P}}$ denote the standard basis in $\ell^2(\mathcal{P})$. Note that the validity of (6.10) for all $f,h\in\ell^2(\mathbb{Z}_N^d)$ is equivalent to

$$\langle e_s,e_t\rangle=\sum_{j=0}^{L-1}\langle e_s,\mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(m)\rangle\langle\mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(m),e_t\rangle,\quad s,t\in\mathcal{P}.$$

If (6.10) fails, there is $s_0,t_0\in\mathcal{P}$ such that,

$$\mathcal{Z}(m):=\left(\sum_{j=0}^{L-1}\langle e_{s_0},\mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(m)\rangle\langle\mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(m),e_{t_0}\rangle-\langle e_{s_0},e_{t_0}\rangle\right)\neq 0$$

on a finite set, say, $\mathfrak{D}\subset\mathfrak{D}^*_{\widetilde{A}}$ with $|\mathfrak{D}|>0$. Thus, one of the inequalities Real part of $\mathcal{Z}(m)$, that means, $\mathrm{Re}(\mathcal{Z}(m))>0$, $\mathrm{Re}(\mathcal{Z}(m))<0$, Imaginary part of $\mathcal{Z}(m)$, that is, $\mathrm{Im}(\mathcal{Z}(m))>0$, or $\mathrm{Im}(\mathcal{Z}(m))<0$, holds on the set $\mathfrak{D}$. Suppose, for example, that the inequality $\mathrm{Re}(\mathcal{Z}(m))>0$ is true on such a set $\mathfrak{D}$. Letting

$$p_1(m)=\begin{cases}e_{s_0}, & \text{for } m\in\mathfrak{D},\\ 0, & \text{for } m\in\mathfrak{D}^*_{\widetilde{A}}\setminus\mathfrak{D},\end{cases}\qquad\text{and}\quad p_2(m)=\begin{cases}e_{t_0}, & \text{for } m\in\mathfrak{D},\\ 0, & \text{for } m\in\mathfrak{D}^*_{\widetilde{A}}\setminus\mathfrak{D},\end{cases}$$

there exist $f,h\in\ell^2(\mathbb{Z}_N^d)$ such that $p_1(m)=\mathcal{T}^{\mathrm{g}}f(m)$, and $p_2(m)=\mathcal{T}^{\mathrm{g}}h(m)$ for each $m\in\mathfrak{D}^*_{\widetilde{A}}$. Therefore, using (6.9), we can write

$$\begin{aligned}0&=\mathrm{Re}\left(\sum_{m\in\mathfrak{D}^*_{\widetilde{A}}}\sum_{j=0}^{L-1}\langle\mathcal{T}^{\mathrm{g}}f(m),\mathcal{K}^{\mathrm{g}}(\psi_j^{(a)})(m)\rangle\langle\mathcal{K}^{\mathrm{g}}(\psi_j^{(b)})(m),\mathcal{T}^{\mathrm{g}}h(m)\rangle-\sum_{m\in\mathfrak{D}^*_{\widetilde{A}}}\langle\mathcal{T}^{\mathrm{g}}f(m),\mathcal{T}^{\mathrm{g}}h(m)\rangle\right)\\&=\mathrm{Re}\left(\sum_{m\in\mathfrak{D}^*_{\widetilde{A}}}\mathcal{Z}(m)\right)=\sum_{m\in\mathfrak{D}^*_{\widetilde{A}}}\big(\mathrm{Re}(\mathcal{Z}(m))\big),\end{aligned}$$

which being sum of non negative terms implies that each term, that is, $\mathrm{Re}(\mathcal{Z}(m))=0$, and hence contradicts our assumption that $\mathrm{Re}(\mathcal{Z}(m))>0$. Therefore, (6.10) holds true, or equivalently, for each $m\in\mathfrak{D}^*_{\widetilde{A}}$, the collections $\widetilde{\mathcal{K}}(\mathrm{g},m,\Psi^{(a)})$ and $\widetilde{\mathcal{K}}(\mathrm{g},m,\Psi^{(b)})$ are dual frames for $\ell^2(\mathcal{P})$, and hence by using Lemma 6.12, the equality (6.7) can be obtained easily, which completes the proof. □

6.3.2. Relation between a dual g-FS and a dual e-FS

In this subsection, for each $\mathrm{g}\in\mathbb{G}$ and positive integer $1\leq a\leq 2$, we want to relate the collections $\widetilde{\mathfrak{B}}_F(\mathrm{g},A,\Psi^{(a)})$ and $\widetilde{\mathfrak{B}}_F(\mathbf{e},A,\Psi^{(a)})$. For this, we need to state the notion of a circle orbit which is defined as follows:

Definition 6.17. Let $f_1, f_2 \in \ell^2(\mathbb{Z}_N^d)$. Then, we say that f_1 lies in the *circle orbit* of f_2, written as $f_1 \in S^1.f_2$, if f_1 is a product of f_2 by an element of the unit circle S^1 in the complex plane.

Proposition 6.18. *For each* $\mathrm{g} \in \mathbb{G}$ *and positive integers* $1 \leq a, b \leq 2$*, let* $\Psi^{(a)} = \{\psi_j^{(a)}\}_{j=0}^{L-1}$ *and* $\Psi^{(b)} = \{\psi_j^{(b)}\}_{j=0}^{L-1}$ *be subsets of* $\ell^2(\mathbb{Z}_N^d)$*, where* $L \geq |\det(A)| \geq 2$*. Then, for each* $0 \leq j \leq L-1$*, by assuming* $\big(\widetilde{\theta}(\mathrm{g}, \widehat{\psi}_j^{(a)}), \widetilde{\theta}(\mathrm{g}, \widehat{\psi}_j^{(b)})\big)$ *in the circle orbit of* $\big(\widehat{\psi}_j^{(a)}, \widehat{\psi}_j^{(b)}\big)$*, the following statements are equivalent:*

(i) *The collections* $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(a)})$ *and* $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(b)})$ *form a* $\mathbf{g}$*-dual pair in* $\ell^2(\mathbb{Z}_N^d)$*.*

(ii) *The collections* $\widetilde{\mathfrak{B}}_F(\mathbf{e}, A, \Psi^{(a)})$ *and* $\widetilde{\mathfrak{B}}_F(\mathbf{e}, A, \Psi^{(b)})$ *form an* $\mathbf{e}$*-dual pair in* $\ell^2(\mathbb{Z}_N^d)$*.*

Moreover, a similar type of relation can be established between the Parseval $\mathbf{g}$*-FS* ($\mathbf{g}$*-ONWS*) *and the Parseval* $\mathbf{e}$*-FS* ($\mathbf{e}$*-ONWS*)*.*

Proof. Let $\mathrm{g} \in \mathbb{G}$ and positive integers $1 \leq a, b \leq 2$. Then, from Theorem 6.13, we recall that the collections $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(a)})$ and $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(b)})$ form a $\mathbf{g}$-dual pair in $\ell^2(\mathbb{Z}_N^d)$ if, and only if, for all $m \in \mathfrak{D}^*_{\widetilde{A}}$ and $p_1, p_2 \in \mathcal{P}$, the following condition is satisfied:

$$\sum_{j=0}^{L-1} \widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}_j^{(a)}\big)(m+p_1)\overline{\widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}_j^{(b)}\big)(m+p_2)} = |\det(A)|\delta_{p_1p_2},$$

which in turn is equivalent to the following equality:

$$\sum_{j=0}^{L-1} e^{i\beta_{j,g}}\widehat{\psi}_j^{(a)}(m+p_1)\overline{e^{i\beta_{j,g}}\widehat{\psi}_j^{(b)}(m+p_2)} = |\det(A)|\delta_{p_1p_2},$$

in view of the fact that the tuple $\big(\widetilde{\theta}(\mathrm{g}, \widehat{\psi}_j^{(a)}), \widetilde{\theta}(\mathrm{g}, \widehat{\psi}_j^{(b)})\big)$ belongs to the circle orbit of $\big(\widehat{\psi}_j^{(a)}, \widehat{\psi}_j^{(b)}\big)$, and hence by using Definition 6.17, we get the existence of some $\beta_{j,g} \in [0, 2\pi)$ such that the tuple

$$\big(\widetilde{\theta}(\mathrm{g}, \widehat{\psi}_j^{(a)}), \widetilde{\theta}(\mathrm{g}, \widehat{\psi}_j^{(b)})\big) = e^{i\beta_{j,g}}\big(\widehat{\psi}_j^{(a)}, \widehat{\psi}_j^{(b)}\big).$$

Hence, we can conclude the required result from Corollary 6.14. □

Next, we want to see conditions needed to put on the elements of $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi)$ (defined in (6.4)), in order to ensure that Ψ generates a framelet system having a dual of the same structure:

6.3.3. Structure of a canonical dual g-framelet system

This subsection deals with the study of the canonical dual of a framelet system associated with each $\mathrm{g} \in \mathbb{G}$. We define this type of **g**-framelet system as follows:

Definition 6.19. Let $\Psi = \{\psi_j\}_{j=0}^{L-1} \subset \ell^2(\mathbb{Z}_N^d)$, where $L \geq |\det(A)| \geq 2$. For each $\mathrm{g} \in \mathbb{G}$, let the collection $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi)$ be a **g**-FS in $\ell^2(\mathbb{Z}_N^d)$ with frame bounds α and β, and let S_{g} be the corresponding frame operator (we call as **g**-*frame operator*), which is a bounded, positive and invertible operator given by

$$S_{\mathrm{g}} : \ell^2(\mathbb{Z}_N^d) \to \ell^2(\mathbb{Z}_N^d);$$
$$f \mapsto \sum_{k \in \mathfrak{D}_{\widetilde{A}}} \sum_{j=0}^{L-1} \langle f, \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j)\rangle \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j).$$

Then, we can define the collection $C_{\mathrm{g}}(\Psi) := S_{\mathrm{g}}^{-1}(\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi))$, which is a framelet system for $\ell^2(\mathbb{Z}_N^d)$ with frame bounds α^{-1} and β^{-1}. We call $C_{\mathrm{g}}(\Psi)$ as the *canonical dual* **g**-*framelet system* (*canonical dual* **g**-*FS*) for $\ell^2(\mathbb{Z}_N^d)$.

Now, we wish to prove that under certain conditions the collection $C_{\mathrm{g}}(\Psi)$ shares a similar structure with $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi)$. For this, we need the following result:

Lemma 6.20. *Suppose $\Psi = \{\psi_j\}_{j=0}^{L-1} \subset \ell^2(\mathbb{Z}_N^d)$, where $L \geq |\det(A)| \geq 2$. Then, for each $\mathrm{g} \in \mathbb{G}$, the* **g**-*frame operator corresponding to the* **g**-*framelet system $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi)$ in $\ell^2(\mathbb{Z}_N^d)$, that is, S_{g} and its inverse denoted by S_{g}^{-1}, are equivarient maps under the action of $\widetilde{\theta}$.*

Proof. Let $\mathrm{g} \in \mathbb{G}$, $m \in \mathbb{Z}_N^d$ and $f \in \ell^2(\mathbb{Z}_N^d)$. Then, Lemma 6.5 allows us to write

$$\begin{aligned}
S_{\mathrm{g}}(\widetilde{\theta}(\mathrm{g}, f))(m) &= \sum_{k \in \mathfrak{D}_{\widetilde{A}}} \sum_{j=0}^{L-1} \langle \widetilde{\theta}(\mathrm{g}, f), T_{Ak}(\widetilde{\theta}(\mathrm{g}, \psi_j))\rangle T_{Ak}(\widetilde{\theta}(\mathrm{g}, \psi_j))(m) \\
&= \sum_{k \in \mathfrak{D}_{\widetilde{A}}} \sum_{j=0}^{L-1} \left(\sum_{n \in \mathbb{Z}_N^d} \widetilde{\theta}(\mathrm{g}, f)(n) T_{Ak}(\widetilde{\theta}(\mathrm{g}, \psi_j))(n) \right) T_{Ak}(\widetilde{\theta}(\mathrm{g}, \psi_j))(m) \\
&= \sum_{k \in \mathfrak{D}_{\widetilde{A}}} \sum_{j=0}^{L-1} \left(\sum_{n \in \mathbb{Z}_N^d} \theta(\mathrm{g}, f(n)) T_{Ak}(\widetilde{\theta}(\mathrm{g}, \psi_j))(n) \right) T_{Ak}(\widetilde{\theta}(\mathrm{g}, \psi_j))(m)
\end{aligned}$$

$$= \sum_{k\in\mathfrak{D}_{\widetilde{A}}} \sum_{j=0}^{L-1} \left(\sum_{n\in\mathbb{Z}_N^d} \theta\left(\mathrm{g}, f(n) T_{Ak}(\widetilde{\theta}(\mathrm{g},\psi_j))(n)\right)\right) T_{Ak}(\widetilde{\theta}(\mathrm{g},\psi_j))(m)$$

$$= \sum_{k\in\mathfrak{D}_{\widetilde{A}}} \sum_{j=0}^{L-1} \left(\theta\left(\mathrm{g}, \sum_{n\in\mathbb{Z}_N^d} f(n) T_{Ak}(\widetilde{\theta}(\mathrm{g},\psi_j))(n)\right)\right) T_{Ak}(\widetilde{\theta}(\mathrm{g},\psi_j))(m)$$

$$= \theta\left(\mathrm{g}, \sum_{k\in\mathfrak{D}_{\widetilde{A}}} \sum_{j=0}^{L-1} \langle f, T_{Ak}(\widetilde{\theta}(\mathrm{g},\psi_j))\rangle T_{Ak}(\widetilde{\theta}(\mathrm{g},\psi_j))(m)\right),$$

in view of the additive and multiplicative nature of the group action θ of $\mathbb{G}$ on $\mathbb{C}$. Hence, we obtain

$$S_{\mathrm{g}}(\widetilde{\theta}(\mathrm{g},f))(m) = \widetilde{\theta}(\mathrm{g}, S_{\mathrm{g}}f)(m), \quad \text{for all } m\in\mathbb{Z}_N^d \text{ and } f\in\ell^2(\mathbb{Z}_N^d).$$

Therefore, for each $0 \le j \le L-1$, the above relation implies that

$$S_{\mathrm{g}}(\widetilde{\theta}(\mathrm{g}, S_{\mathrm{g}}^{-1}\psi_j)) = \widetilde{\theta}(\mathrm{g}, S_{\mathrm{g}}S_{\mathrm{g}}^{-1}\psi_j),$$

which is if, and only if,

$$S_{\mathrm{g}}^{-1}\widetilde{\theta}(\mathrm{g},\psi_j) = \widetilde{\theta}(\mathrm{g},\varphi_j),$$

where $\varphi_j = S_{\mathrm{g}}^{-1}\psi_j$. Hence, the result follows. □

Theorem 6.21. *Suppose $\Psi = \{\psi_j\}_{j=0}^{L-1} \subset \ell^2(\mathbb{Z}_N^d)$, where $L \ge |\det(A)| \ge 2$. Then, for each $\mathrm{g}\in\mathbb{G}$, $C_{\mathrm{g}}(\Psi)$, that is, the canonical dual of* **g***-framelet system $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi)$, preserves the same structure.*

Proof. Observe that the canonical dual of $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi)$ is given by

$$\begin{aligned} C_{\mathrm{g}}(\Psi) &= S_{\mathrm{g}}^{-1}(\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi)) \\ &= \left\{ S_{\mathrm{g}}^{-1}\left(\widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j)\right) : k\in\mathfrak{D}_{\widetilde{A}},\, 0\le j\le L-1 \right\} \subset \ell^2(\mathbb{Z}_N^d). \end{aligned}$$

Let $\mathrm{g}\in\mathbb{G}$ and $k\in\mathfrak{D}_{\widetilde{A}}$. Then, we claim that

$$S_{\mathrm{g}}T_{Ak}f = T_{Ak}S_{\mathrm{g}}f \quad \text{and} \quad S_{\mathrm{g}}^{-1}T_{Ak}f = T_{Ak}S_{\mathrm{g}}^{-1}f, \text{ for all } f\in\ell^2(\mathbb{Z}_N^d), \tag{6.11}$$

and hence by using the above equation in $C_{\mathrm{g}}(\Psi)$ along with Lemma 6.5 and Lemma 6.20, we obtain the required proof as follows:

$$
\begin{aligned}
C_{\mathrm{g}}(\Psi) &= \left\{S_{\mathrm{g}}^{-1}T_{Ak}\left(\widetilde{\theta}(\mathrm{g},\psi_j)\right) : k \in \mathfrak{D}_{\widetilde{A}},\, 0 \le j \le L-1\right\} \\
&= \left\{T_{Ak}S_{\mathrm{g}}^{-1}\left(\widetilde{\theta}(\mathrm{g},\psi_j)\right) : k \in \mathfrak{D}_{\widetilde{A}},\, 0 \le j \le L-1\right\} \\
&= \left\{T_{Ak}\left(\widetilde{\theta}(\mathrm{g},\eta_j)\right) : k \in \mathfrak{D}_{\widetilde{A}},\, 0 \le j \le L-1\right\} \\
&= \left\{\widetilde{\theta}(\mathrm{g},T_{Ak}(\eta_j)) : k \in \mathfrak{D}_{\widetilde{A}},\, 0 \le j \le L-1\right\},
\end{aligned}
$$

where $S_{\mathrm{g}}^{-1}\psi_j = \eta_j$ for each $0 \le j \le L-1$.

Now, in order to prove (6.11), let $m \in \mathfrak{D}_{\widetilde{A}}$. Then, by using Lemma 6.5 and (P3), we get the following:

$$
\begin{aligned}
S_{\mathrm{g}}T_{Am}f &= \sum_{k\in\mathfrak{D}_{\widetilde{A}}}\sum_{j=0}^{L-1}\langle T_{Am}f, T_{Ak}(\widetilde{\theta}(\mathrm{g},\psi_j))\rangle T_{Ak}(\widetilde{\theta}(\mathrm{g},\psi_j)) \\
&= \sum_{k\in\mathfrak{D}_{\widetilde{A}}}\sum_{j=0}^{L-1}\langle f, T_{A(k-m)}(\widetilde{\theta}(\mathrm{g},\psi_j))\rangle T_{Ak}(\widetilde{\theta}(\mathrm{g},\psi_j)),
\end{aligned}
$$

which on replacement of the summation index k with $k+m$ gives

$$
\begin{aligned}
S_{\mathrm{g}}T_{Am}f &= \sum_{k\in\mathfrak{D}_{\widetilde{A}}}\sum_{j=0}^{L-1}\langle f, T_{Ak}(\widetilde{\theta}(\mathrm{g},\psi_j))\rangle T_{A(k+m)}(\widetilde{\theta}(\mathrm{g},\psi_j)) \\
&= \sum_{k\in\mathfrak{D}_{\widetilde{A}}}\sum_{j=0}^{L-1}\langle f, T_{Ak}(\widetilde{\theta}(\mathrm{g},\psi_j))\rangle T_{Ak}T_{Am}(\widetilde{\theta}(\mathrm{g},\psi_j)) \\
&= T_{Am}\left(\sum_{k\in\mathfrak{D}_{\widetilde{A}}}\sum_{j=0}^{L-1}\langle f, T_{Ak}(\widetilde{\theta}(\mathrm{g},\psi_j))\rangle T_{Ak}(\widetilde{\theta}(\mathrm{g},\psi_j))\right) \\
&= T_{Am}S_{\mathrm{g}}f.
\end{aligned}
$$

Since we know that the operator S_{g} is invertible, therefore for all $f \in \ell^2(\mathbb{Z}_N^d)$, $S_{\mathrm{g}}^{-1}f \in \ell^2(\mathbb{Z}_N^d)$, and hence using this fact in the above relation, we get

$$S_{\mathrm{g}}T_{Am}S_{\mathrm{g}}^{-1}f = T_{Am}S_{\mathrm{g}}S_{\mathrm{g}}^{-1}f = T_{Am}f,$$

which is if, and only if,

$$S_{\mathrm{g}}^{-1}S_{\mathrm{g}}T_{Am}S_{\mathrm{g}}^{-1}f = S_{\mathrm{g}}^{-1}T_{Am}f.$$

Hence, the claim follows. $\square$

6.3.4. Examples

To understand the above theory in a better way, next we are providing some examples related to **g**-FS, tight **g**-FS, **g**-ONWS, and the canonical dual of a **g**-FS in $\ell^2(\mathbb{Z}_N^d)$:

Example 6.22. Let $\mathbb{G} = \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ be a multiplicative group of real numbers and let

$$\theta : \mathbb{R}^* \times \mathbb{C} \to \mathbb{C}; \quad (r, z) \mapsto rz$$

be a group action on $\mathbb{C}$. Clearly, θ is an additive as well as multiplicative group action. Further, consider $A = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$, which is a non-expansive matrix with eigenvalues $(2 \pm \sqrt{2})$, and $\mathfrak{D}_{\widetilde{A}} = \{(0,0)^t, (1,0)^t\}$. Now, assume that $\Psi_1 = \{\psi_0, \psi_1\} \subset \ell^2(\mathbb{Z}_2^2)$, where for any arbitrary element $\mathrm{g} \in \mathbb{R}^*$, we have

$$\psi_0 = \big(\psi_0((0,0)^t), \psi_0((0,1)^t), \psi_0((1,0)^t), \psi_0((1,1)^t)\big)^t = \Big(\frac{1}{\mathrm{g}\sqrt{2}}, 0, \frac{1}{\mathrm{g}\sqrt{2}}, 0\Big)^t, \text{ and}$$

$$\psi_1 = \Big(\frac{1}{\mathrm{g}\sqrt{2}}, 0, -\frac{1}{\mathrm{g}\sqrt{2}}, 0\Big)^t.$$

(i) **g-ONWS**: Then, for each $\mathrm{g} \in \mathbb{R}^*$, the collection $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi_1)$ defined by

$$\begin{aligned}\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi_1) &= \Big\{\widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j) : k \in \mathfrak{D}_{\widetilde{A}},\ 0 \le j \le 1\Big\} \\ &= \big\{\mathrm{g}T_{Ak}\psi_j : k \in \mathfrak{D}_{\widetilde{A}},\ 0 \le j \le 1\big\}\end{aligned}$$

forms a **g**-ONWS for $\ell^2(\mathbb{Z}_2^2)$.

(ii) **g-FS**: Let $\Psi_2 = \{\psi_0, \psi_1, \psi_2\} \subset \ell^2(\mathbb{Z}_2^2)$, where $\psi_2 = \psi_0$. Then, for each $\mathrm{g} \in \mathbb{R}^*$, the collection given by

$$\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi_2) = \{\widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j) : k \in \mathfrak{D}_{\widetilde{A}},\ 0 \le j \le 2\}$$

forms a **g**-FS for $\ell^2(\mathbb{Z}_2^2)$ with frame bounds 1 and 2.

(iii) **tight g-FS**: Let $\Psi_3 = \{\psi_0, \psi_1, \psi_2, \psi_3\} \subset \ell^2(\mathbb{Z}_2^2)$, where $\psi_2 = \psi_0$ and $\psi_3 = \psi_1$. Then, for each $\mathrm{g} \in \mathbb{R}^*$, the collection

$$\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi_3) = \{\widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j) : k \in \mathfrak{D}_{\widetilde{A}},\ 0 \le j \le 3\}$$

forms a tight **g**-FS for $\ell^2(\mathbb{Z}_2^2)$ with frame bound 2.

Example 6.23. (Canonical dual g-FS): Consider the **g**-FS defined as $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi_2)$ in the Example 6.22(ii). Then, for each $\mathrm{g} \in \mathbb{R}^*$, the canonical dual of this **g**-FS is given by

$$S_{\mathrm{g}}^{-1}(\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi_2)) = \{S_{\mathrm{g}}^{-1}(\widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j)) : k \in \mathfrak{D}_{\widetilde{A}},\, 0 \leq j \leq 2\},$$

where $S_{\mathrm{g}} : \ell^2(\mathbb{Z}_2^2) \to \ell^2(\mathbb{Z}_2^2);\ f \mapsto \sum_{k \in \mathfrak{D}_{\widetilde{A}}} \sum_{j=0}^{2} \langle f, \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j)\rangle \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j)$ is termed as the **g**-frame operator corresponding to the **g**-framelet system $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi_2)$.

From the above definition of S_{g}, it can be easily verified that

$$S_{\mathrm{g}}(\widetilde{\theta}(\mathrm{g}, \psi_0)) = 2\,\widetilde{\theta}(\mathrm{g}, \psi_0), \quad S_{\mathrm{g}}(\widetilde{\theta}(\mathrm{g}, \psi_1)) = \widetilde{\theta}(\mathrm{g}, \psi_1),$$

$$S_{\mathrm{g}}(\widetilde{\theta}(\mathrm{g}, T_{(0,1)}\psi_0)) = 2\,\widetilde{\theta}(\mathrm{g}, T_{(0,1)}\psi_0), \quad \text{and } S_{\mathrm{g}}(\widetilde{\theta}(\mathrm{g}, T_{(0,1)}\psi_1)) = \widetilde{\theta}(\mathrm{g}, T_{(0,1)}\psi_1),$$

which in turn implies that

$$S_{\mathrm{g}}^{-1}(\widetilde{\theta}(\mathrm{g}, \psi_0)) = \frac{1}{2}\,\widetilde{\theta}(\mathrm{g}, \psi_0), \quad S_{\mathrm{g}}^{-1}(\widetilde{\theta}(\mathrm{g}, \psi_1)) = \widetilde{\theta}(\mathrm{g}, \psi_1),$$

$$S_{\mathrm{g}}^{-1}(\widetilde{\theta}(\mathrm{g}, T_{(0,1)}\psi_0)) = \frac{1}{2}\,\widetilde{\theta}(\mathrm{g}, T_{(0,1)}\psi_0), \quad \text{and } S_{\mathrm{g}}^{-1}(\widetilde{\theta}(\mathrm{g}, T_{(0,1)}\psi_1)) = \widetilde{\theta}(\mathrm{g}, T_{(0,1)}\psi_1).$$

Hence, the canonical dual of $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi_2)$, that is, $S_{\mathrm{g}}^{-1}(\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi_2))$ is given by

$$\left\{\frac{1}{2}\,\widetilde{\theta}(\mathrm{g}, \psi_0), \widetilde{\theta}(\mathrm{g}, \psi_1), \frac{1}{2}\,\widetilde{\theta}(\mathrm{g}, T_{(0,1)}\psi_0), \widetilde{\theta}(\mathrm{g}, T_{(0,1)}\psi_1), \frac{1}{2}\,\widetilde{\theta}(\mathrm{g}, \psi_0), \frac{1}{2}\,\widetilde{\theta}(\mathrm{g}, T_{(0,1)}\psi_0)\right\}$$

$$= \{\widetilde{\theta}(\mathrm{g}, T_{Ak}\varphi_j) : k \in \mathfrak{D}_{\widetilde{A}},\, 0 \leq j \leq 2\} = \widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Phi),$$

where $\Phi = \{\varphi_0 = \frac{\psi_0}{2} = \varphi_2,\, \varphi_1 = \psi_1\}$, and $\varphi_j = S_{\mathrm{g}}^{-1}\psi_j, \quad$ for all $0 \leq j \leq 2$.

6.4. A characterization for super dual g-framelet system

Frames have found use in potential applications to data transmission by using a technique called multiplexing (an important method in telecommunications, computer networks and digital video, etc.) [3, 50, 67, 68], it motivates us to study super-framelet and super-wavelet systems associated with an induced action of the group $\mathbb{G}$ on $\ell^2(\mathbb{Z}_N^d)$. For the purpose of defining such systems, we consider $\Psi^{(p)} := \{\psi_j^{(p)}\}_{j=0}^{q-1} \subset \ell^2(\mathbb{Z}_N^d)$ and $\mathbb{P} := \{1, 2, \ldots, P\}$, for some $q, P \in \mathbb{N}$, and also, we define the following collection:

$$\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(p)}) := \left\{\widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(p)}) : k \in \mathfrak{D}_{\widetilde{A}}\,;\, 0 \leq j \leq q-1\right\} \subset \ell^2(\mathbb{Z}_N^d), \tag{6.12}$$

for each $\mathrm{g} \in \mathbb{G}$ and $p \in \mathbb{P}$. Thus, we have the following definition:

Definition 6.24. For each $\mathrm{g} \in \mathbb{G}$ and $p \in \mathbb{P} = \{1, 2, \ldots, P\}$, assume the collection $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(p)})$ as defined in (6.12). Then for a finite sequence $\{\Psi^{(p)}\}_{p\in\mathbb{P}}$ in $\bigoplus_{p\in\mathbb{P}} \ell^2(\mathbb{Z}_N^d) \equiv \ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$, we say that

$$(6.13) \qquad \mathfrak{B}(\mathrm{g}, A, \{\Psi^{(p)}\}_{p\in\mathbb{P}}) := \left(\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(1)}), \widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(2)}), \ldots, \widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(P)})\right),$$

is a *super* $\mathbf{g}$*-FS* (*respectively Parseval super* $\mathbf{g}$*-FS, super* $\mathbf{g}$*-ONWS*) for the super-space $\ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$ if the collection

$$(6.14) \qquad \widetilde{\mathfrak{B}}(\mathrm{g}, A, \{\Psi^{(p)}\}_{p\in\mathbb{P}}) = \left\{\widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(1)}) \oplus \ldots \oplus \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(P)}) : k \in \mathfrak{D}_{\widetilde{A}}\,;\, 0 \le j \le q-1\right\},$$

forms a frame (respectively Parseval frame, orthonormal basis) for the super-space $\ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$.

Remark 6.25. Note that in order to make the collection (6.13) an orthonormal basis for the space $\ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$, we need to assume $q = P|\det(A)|$ in the collection defined in (6.12). Therefore, we should consider $q \ge P|\det(A)|$ to study super $\mathbf{g}$-FS for $\ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$.

For each $p \in \mathbb{P} = \{1, 2, \ldots, P\}$, let $\Psi^{(p)} = \{\psi_j^{(p)}\}_{j=0}^{q-1}$ and $\Phi^{(p)} = \{\varphi_j^{(p)}\}_{j=0}^{q-1}$ be subsets of $\ell^2(\mathbb{Z}_N^d)$, where $q \ge P|\det(A)|$, and let $\{\Psi^{(p)}\}_{p\in\mathcal{P}}$ and $\{\Phi^{(p)}\}_{p\in\mathcal{P}}$ be sequences in $\ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$. Then, for each $\mathrm{g} \in \mathbb{G}$, the collection $\widetilde{\mathfrak{B}}(\mathrm{g}, A, \{\Phi^{(p)}\})$ (defined in (6.14)) will form a *super dual* $\mathbf{g}$*-framelet system* (*super dual* $\mathbf{g}$*-FS*) for the super $\mathbf{g}$-framelet system $\widetilde{\mathfrak{B}}(\mathrm{g}, A, \{\Psi^{(p)}\})$ in $\ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$, if for all $f \in \ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$, the following reconstruction formula holds:

$$\begin{aligned} f &= \sum_{k\in\mathfrak{D}_{\widetilde{A}}} \sum_{j=0}^{q-1} \langle f, \bigoplus_{p=1}^{P} \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(p)})\rangle \bigoplus_{p=1}^{P} \widetilde{\theta}(\mathrm{g}, T_{Ak}\varphi_j^{(p)}) \\ &= \sum_{k\in\mathfrak{D}_{\widetilde{A}}} \sum_{j=0}^{q-1} \langle f, \bigoplus_{p=1}^{P} \widetilde{\theta}(\mathrm{g}, T_{Ak}\varphi_j^{(p)})\rangle \bigoplus_{p=1}^{P} \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(p)}). \end{aligned}$$

In this case, we say that the systems $\widetilde{\mathfrak{B}}(\mathrm{g}, A, \{\Psi^{(p)}\})$ and $\widetilde{\mathfrak{B}}(\mathrm{g}, A, \{\Phi^{(p)}\})$ form a *super* $\mathbf{g}$*-dual pair* in $\ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$ or $\{\Psi^{(p)}\}$ is a *super* $\mathbf{g}$*-dual* of $\{\Phi^{(p)}\}$. Similarly, for $\mathrm{g} = \mathbf{e}$, we say that the collections $\widetilde{\mathfrak{B}}(\mathrm{g}, A, \{\Psi^{(p)}\})$ and $\widetilde{\mathfrak{B}}(\mathrm{g}, A, \{\Phi^{(p)}\})$ form a *super* $\mathbf{e}$*-dual pair* in $\ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$.

In order to find conditions on $\{\Psi^{(p)}\}$ and $\{\Phi^{(p)}\}$ such that $\widetilde{\mathfrak{B}}(\mathrm{g}, A, \{\Psi^{(p)}\})$ form a super dual $\mathbf{g}$-FS for $\widetilde{\mathfrak{B}}(\mathrm{g}, A, \{\Psi^{(p)}\})$ in $\ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$, we need to prove a result on orthogonal $\mathbf{g}$-framelet systems. We define these $\mathbf{g}$-framelet systems as follows:

Definition 6.26. For each $p \in \mathbb{P} = \{1, 2, \ldots, P\}$, we assume $\Psi^{(p)} = \{\psi_j^{(p)}\}_{j=0}^{q-1} \subset \ell^2(\mathbb{Z}_N^d)$ for some $q \in \mathbb{N}$. Then, for each $\mathrm{g} \in \mathbb{G}$ and $p_1 \ne p_2 \in \mathbb{P}$, the $\mathbf{g}$-framelet systems

$\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(p_1)})$ and $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(p_2)})$ in $\ell^2(\mathbb{Z}_N^d)$ will be termed as *orthogonal* **g**-*framelet systems* if, for all $f \in \ell^2(\mathbb{Z}_N^d)$, we have

$$\sum_{j=0}^{q-1} \sum_{k \in \mathfrak{D}_{\widetilde{A}}} \langle f, \widetilde{\theta}\big(\mathrm{g}, T_{Ak}\psi_j^{(p_1)}\rangle \widetilde{\theta}\big(\mathrm{g}, T_{Ak}\psi_j^{(p_2)}\big) = \sum_{j=0}^{L-1} \sum_{k \in \mathfrak{D}_{\widetilde{A}}} \langle f, \widetilde{\theta}\big(\mathrm{g}, T_{Ak}\psi_j^{(p_2)}\rangle \widetilde{\theta}\big(\mathrm{g}, T_{Ak}\psi_j^{(p_1)}\big) = 0.$$

Lemma 6.27. *For each $p \in \mathbb{P} = \{1, 2, \ldots, P\}$, we assume $\Psi^{(p)} = \{\psi_j^{(p)}\}_{j=0}^{q-1} \subset \ell^2(\mathbb{Z}_N^d)$ for some $q \in \mathbb{N}$. Then, for each $\mathrm{g} \in \mathbb{G}$ and $p_1 \neq p_2 \in \mathbb{P}$, the* **g**-*framelet systems $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(p_1)})$ and $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(p_2)})$ form orthogonal* **g**-*framelet systems in $\ell^2(\mathbb{Z}_N^d)$ if, and only if, the following equality holds:*

$$\sum_{j=0}^{q-1} \widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}_j^{(p_1)}\big)(\xi + \gamma_1) \overline{\widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}_j^{(p_2)}\big)(\xi + \gamma_2)} = 0, \tag{6.15}$$

for all $\gamma_1, \gamma_2 \in \mathcal{P}$ and $\xi \in \mathfrak{D}^*_{\widetilde{A}}$.

Proof. Let $\mathrm{g} \in \mathbb{G}$ and $p_1 \neq p_2 \in \mathbb{P}$. Then, the **g**-framelet systems $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(p_1)})$ and $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(p_2)})$ form orthogonal **g**-framelet systems in $\ell^2(\mathbb{Z}_N^d)$ if, and only if, by equivalent form of Definition 6.26, we have $\big(\Theta_{\mathrm{g}}^{(p_1)}\big)^* \Theta_{\mathrm{g}}^{(p_2)} = 0$, where $\big(\Theta_{\mathrm{g}}^{(p_1)}\big)^*$ and $\Theta_{\mathrm{g}}^{(p_2)}$ are respectively the synthesis and analysis operators corresponding to the collections $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(p_1)})$ and $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(p_2)})$, which in turn holds if, and only if, $\big(\Theta_{\mathrm{g}}^{(p_1)}\big)^* \Theta_{\mathrm{g}}^{(p_2)} f = 0$, for all $f \in \ell^2(\mathbb{Z}_N^d)$, equivalently, it represents the following identity:

$$\Big\langle \big(\Theta_{\mathrm{g}}^{(p_1)}\big)^* \Theta_{\mathrm{g}}^{(p_2)} f, h \Big\rangle = 0, \quad \text{for all } f, h \in \ell^2(\mathbb{Z}_N^d). \tag{6.16}$$

Moreover, by recalling the technique used in computing the equation $(*)$ in the proof of Theorem 6.13, we can write

$$\begin{aligned} \Big\langle \big(\Theta_{\mathrm{g}}^{(p_1)}\big)^* \Theta_{\mathrm{g}}^{(p_2)} f, h \Big\rangle &= \sum_{j=0}^{L-1} \sum_{k \in \mathfrak{D}_{\widetilde{A}}} \Big\langle f, \widetilde{\theta}\big(\mathrm{g}, T_{Ak}\psi^{(p_2)}\big) \Big\rangle \overline{\Big\langle h, \widetilde{\theta}\big(\mathrm{g}, T_{Ak}\psi^{(p_1)}\big) \Big\rangle} \\ &= \frac{1}{N^d} \sum_{j=0}^{L-1} \sum_{\xi \in \mathfrak{D}^*_{\widetilde{A}}} \langle \mathcal{T}^{\mathrm{g}} f(\xi), \mathcal{K}^{\mathrm{g}}(\psi_j^{(p_2)})(\xi) \rangle \overline{\langle \mathcal{T}^{\mathrm{g}} h(\xi), \mathcal{K}^{\mathrm{g}}(\psi_j^{(p_1)})(\xi) \rangle}, \end{aligned} \tag{$**$}$$

for all $f, h \in \ell^2(\mathbb{Z}_N^d)$. Hence, from (6.16) and $(**)$, we get

$$0 = \sum_{j=0}^{L-1} \sum_{\xi \in \mathfrak{D}^*_{\widetilde{A}}} \langle \mathcal{T}^{\mathrm{g}} f(\xi), \mathcal{K}^{\mathrm{g}}(\psi_j^{(p_2)})(\xi) \rangle \overline{\langle \mathcal{T}^{\mathrm{g}} h(\xi), \mathcal{K}^{\mathrm{g}}(\psi_j^{(p_1)})(\xi) \rangle}, \quad \text{for all } f, h \in \ell^2(\mathbb{Z}_N^d),$$

which in view of the fact proved in the converse of Theorem 6.13 is if, and only if, for each $\xi \in \mathfrak{D}^*_{\widetilde{A}}$,

$$0 = \sum_{j=0}^{L-1} \langle \mathcal{T}^{\mathrm{g}} f(\xi), \mathcal{K}^{\mathrm{g}}(\psi_j^{(p_2)})(\xi)\rangle \overline{\langle \mathcal{T}^{\mathrm{g}} h(\xi), \mathcal{K}^{\mathrm{g}}(\psi_j^{(p_1)})(\xi)\rangle}, \quad \text{for all } f, h \in \ell^2(\mathbb{Z}_N^d).$$

Since for $f, h \in \ell^2(\mathbb{Z}_N^d)$, $h_1 = \mathcal{T}^{\mathrm{g}} f(\xi)$ and $h_2 = \mathcal{T}^{\mathrm{g}} h(\xi)$ are elements of $\ell^2(\mathcal{P})$, for all $\xi \in \mathfrak{D}^*_{\widetilde{A}}$, the above equation can be rewritten as

(6.17)
$$0 = \sum_{j=0}^{L-1} \langle h_1, \mathcal{K}^{\mathrm{g}}(\psi_j^{(p_2)})(\xi)\rangle \overline{\langle h_2, \mathcal{K}^{\mathrm{g}}(\psi_j^{(p_1)})(\xi)\rangle} = \langle G_{\mathrm{g}}^{(p_1)(p_2)}(\xi) h_2, h_1\rangle, \text{ for all } h_1, h_2 \in \ell^2(\mathcal{P}),$$

where $G_{\mathrm{g}}^{(p_1)(p_2)}(\xi)$ represents the cross dual Gramian operator corresponding to the collections $\widetilde{\mathcal{K}}(\mathrm{g}, \xi, \Psi^{(p_1)})$ and $\widetilde{\mathcal{K}}(\mathrm{g}, \xi, \Psi^{(p_2)})$. Now, (6.17) holds if, and only if, for all $\xi \in \mathfrak{D}^*_{\widetilde{A}}$, we have $G_{\mathrm{g}}^{(p_1)(p_2)}(\xi) = 0$, which by using Proposition 6.11 holds if, and only if, the relation (6.15) is satisfied. $\square$

Our next result provides necessary and sufficient conditions on $\{\Psi^{(p)}\}$ and $\{\Phi^{(p)}\}$ such that they form a super **g**-dual pair.

Theorem 6.28. *For each $p \in \mathbb{P} = \{1, 2, \ldots, P\}$, let $\Psi^{(p)} = \{\psi_j^{(p)}\}_{j=0}^{q-1}$ and $\Phi^{(p)} = \{\varphi_j^{(p)}\}_{j=0}^{q-1}$ be subsets of $\ell^2(\mathbb{Z}_N^d)$, where $q \geq P|\det(A)|$, and let $\{\Psi^{(p)}\}_{p\in\mathcal{P}}$ and $\{\Phi^{(p)}\}_{p\in\mathcal{P}}$ be sequences in $\ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$. For each $\mathrm{g} \in \mathbb{G}$, consider the collections $\widetilde{\mathfrak{B}}(\mathrm{g}, A, \{\Phi^{(p)}\})$ and $\widetilde{\mathfrak{B}}(\mathrm{g}, A, \{\Psi^{(p)}\})$ in $\ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$. Then, $\{\Psi^{(p)}\}$ is a super **g**-dual of $\{\Phi^{(p)}\}$ if, and only if, both of the following hold:*

(i) *For each $1 \leq p \leq P$ and $m \in \mathfrak{D}^*_{\widetilde{A}}$, we have*

$$\sum_{j=0}^{q-1} \widetilde{\theta}(\mathrm{g}, \widehat{\psi}_j^{(p)})(m + \gamma_1) \overline{\widetilde{\theta}(\mathrm{g}, \widehat{\varphi}_j^{(p)})(m + \gamma_2)} = |\det(A)| \delta_{\gamma_1, \gamma_2}, \text{ for all } \gamma_1, \gamma_2 \in \mathcal{P}.$$

(ii) *For each $1 \leq p_1 \neq p_2 \leq P$ and $m \in \mathfrak{D}^*_{\widetilde{A}}$, the following equality is satisfied:*

$$\sum_{j=0}^{q-1} \widetilde{\theta}\big(\mathrm{g}, \widehat{\psi}_j^{(p_1)}\big)(m + \gamma_1) \overline{\widetilde{\theta}\big(\mathrm{g}, \widehat{\varphi}_j^{(p_2)}\big)(m + \gamma_2)} = 0, \text{ for all } \gamma_1, \gamma_2 \in \mathcal{P}.$$

*In particular, for each $p \in \mathbb{P}$, by using $\Psi^{(p)} = \Phi^{(p)}$ in the above result, we get a characterization of $\{\Psi^{(p)}\}$ such that for each $\mathrm{g} \in \mathbb{G}$, the system $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \{\Psi^{(p)}\})$ forms a Parseval super **g**-FS for $\ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$, which in addition by considering q equals to the $P|\det(A)|$ leads to a super **g**-ONWS in $\ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$.*

Proof. Let $\{\Psi^{(p)}\}$ be a super **g**-dual of $\{\Phi^{(p)}\}$, which is if, and only if, $\widetilde{G}_{(p_1)(p_2)}$, that is, the cross dual Gramian operator corresponding to the collections $\widetilde{\mathfrak{B}}(\mathrm{g}, A, \{\Psi^{(p)}\})$ and $\widetilde{\mathfrak{B}}(\mathrm{g}, A, \{\Phi^{(p)}\})$, is identity on $\ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$, where $\widetilde{G}_{(p_1)(p_2)} : \ell^2(\mathbb{Z}_N^d, \mathbb{C}^P) \to \ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$ is defined by

$$\widetilde{G}_{(p_1)(p_2)} f = \sum_{k \in \mathfrak{D}_{\widetilde{A}}} \sum_{j=0}^{q-1} \big\langle f, \bigoplus_{p=1}^{P} \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(p)}) \big\rangle \bigoplus_{p=1}^{P} \widetilde{\theta}(\mathrm{g}, T_{Ak}\varphi_j^{(p)}),$$

for all $f = (f^{(1)}, f^{(2)}, \cdots, f^{(P)}) \in \ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$, where $f^{(i)} \in \ell^2(\mathbb{Z}_N^d)$ for $1 \leq i \leq P$. Since $\{\Psi^{(p)}\}$ is a super **g**-dual of $\{\Phi^{(p)}\}$, this means,

$$\sum_{k \in \mathfrak{D}_{\widetilde{A}}} \sum_{j=0}^{q-1} \big\langle f, \bigoplus_{p=1}^{P} \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(p)}) \big\rangle \bigoplus_{p=1}^{P} \widetilde{\theta}(\mathrm{g}, T_{Ak}\varphi_j^{(p)}) = f, \quad \text{for all } f \in \ell^2(\mathbb{Z}_N^d, \mathbb{C}^P),$$

which is if, and only if, we can write

$$\begin{aligned} &(f^{(1)}, f^{(2)}, \cdots, f^{(P)}) \\ &= \sum_{k \in \mathfrak{D}_{\widetilde{A}}} \sum_{j=0}^{q-1} \big\langle (f^{(1)}, f^{(2)}, \cdots, f^{(P)}), \bigoplus_{p=1}^{P} \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(p)}) \big\rangle \bigoplus_{p=1}^{P} \widetilde{\theta}(\mathrm{g}, T_{Ak}\varphi_j^{(p)}) \end{aligned}$$

for all $f^{(i)} \in \ell^2(\mathbb{Z}_N^d); 1 \leq i \leq P$, which in turn is equivalent to

$$\left(\sum_{k \in \mathfrak{D}_{\widetilde{A}}} \sum_{j=0}^{q-1} \langle f^{(1)}, \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(1)}) \rangle \widetilde{\theta}(\mathrm{g}, T_{Ak}\varphi_j^{(1)}), \ldots, \sum_{k \in \mathfrak{D}_{\widetilde{A}}} \sum_{j=0}^{q-1} \langle f^{(P)}, \widetilde{\theta}(\mathrm{g}, T_{Ak}\psi_j^{(P)}) \rangle \widetilde{\theta}(\mathrm{g}, T_{Ak}\varphi_j^{(P)}) \right)$$

if, and only if, the collections $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(p_1)})$ and $\widetilde{\mathfrak{B}}_F(\mathrm{g}, A, \Psi^{(p_2)})$ form orthogonal **g**-framelet systems in $\ell^2(\mathbb{Z}_N^d)$. Comparing both sides of the above equation, it leads us to conclude that $\{\Psi^{(p)}\}$ will be a super **g**-dual of $\{\Phi^{(p)}\}$ if, and only if, each $\Psi^{(p)}$ is a **g**-dual of $\Phi^{(p)}$ for each $1 \leq p \leq P$, and $\widetilde{\mathfrak{B}}(\mathrm{g}, A, \Psi^{(p_1)})$ and $\widetilde{\mathfrak{B}}(\mathrm{g}, A, \Psi^{(p_2)})$ are orthogonal **g**-framelet systems for $1 \leq p_1 \neq p_2 \leq P$. Hence, by using Lemma 6.27 and Theorem 6.13, we obtain the required result. □

Further, we establish the following relation between a super **g**-dual pair and a super **e**-dual pair:

Theorem 6.29. *For each $p \in \mathbb{P} = \{1, 2, \ldots, P\}$, let $\Psi^{(p)} = \{\psi_j^{(p)}\}_{j=0}^{q-1}$ and $\Phi^{(p)} = \{\varphi_j^{(p)}\}_{j=0}^{q-1}$ be subsets of $\ell^2(\mathbb{Z}_N^d)$, where $q \geq P|\det(A)|$, and let $\{\Psi^{(p)}\}_{p \in \mathcal{P}}$ and $\{\Phi^{(p)}\}_{p \in \mathcal{P}}$ be sequences in $\ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$. For each $\mathrm{g} \in \mathbb{G}$ and $1 \leq j \leq q-1$, assume that the tuple $\big(\widetilde{\theta}(\mathrm{g}, \widehat{\psi}_j^{(p)}), \widetilde{\theta}(\mathrm{g}, \widehat{\varphi}_j^{(p)})\big)$ is in the circle orbit of $\big(\widehat{\psi}_j^{(p)}, \widehat{\varphi}_j^{(p)}\big)$. Then, the following statements are equivalent:*

(i) *The collections* $\widetilde{\mathfrak{B}}(\mathrm{g}, A, \{\Psi^{(p)}\})$ *and* $\widetilde{\mathfrak{B}}(\mathrm{g}, A, \{\Phi^{(p)}\})$ *form a super* $\mathbf{g}$*-dual pair in* $\ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$.

(ii) *The collections* $\widetilde{\mathfrak{B}}(\mathbf{e}, A, \{\Psi^{(p)}\})$ *and* $\widetilde{\mathfrak{B}}(\mathbf{e}, A, \{\Phi^{(p)}\})$ *form a super* $\mathbf{e}$*-dual pair in* $\ell^2(\mathbb{Z}_N^d, \mathbb{C}^P)$.

Proof. Let $\mathrm{g} \in \mathbb{G}$, $p \in \mathbb{P}$ and $1 \leq j \leq q-1$. Then, it is given that $\big(\widetilde{\theta}(\mathrm{g}, \widehat{\psi}_j^{(p)}), \widetilde{\theta}(\mathrm{g}, \widehat{\varphi}_j^{(p)})\big)$ is in the circle orbit of $\big(\widehat{\psi}_j^{(p)}, \widehat{\varphi}_j^{(p)}\big)$, therefore by using Definition 6.17, we get the existence of some $\beta_{j,\mathrm{g},p} \in [0, 2\pi)$ such that $\big(\widetilde{\theta}(\mathrm{g}, \widehat{\psi}_j^{(p)}), \widetilde{\theta}(\mathrm{g}, \widehat{\varphi}_j^{(p)})\big) = e^{i\beta_{j,\mathrm{g},p}}\big(\widehat{\psi}_j^{(p)}, \widehat{\varphi}_j^{(p)}\big)$. Hence, we conclude the required result from the Theorem 6.28. $\square$

CHAPTER 7

SUMMARY AND FUTURE DIRECTIONS

The first chapter of this thesis gives an introduction to the research area, the motivation used for this work, and available literatures along with a basic introduction to each of the chapters presented in the thesis. In the second chapter, we introduce the notion of pre-Gramian operator associated to Γ-TI systems over LCA groups, and discuss the fiberization of operators associated with these systems. By using the fiberization approach, we study orthogonal Γ-TI Bessel (frame) systems, and apply our results on co-compact Gabor frame systems over LCA groups.

In the third chapter, we characterize pairwise orthogonal GTI frame systems in $L^2(G)$ and deduce similar results for several function systems including the case of TI systems, GSI systems and GTI systems on compact abelian groups. We also discuss applications of our characterization result on the Bessel families with wave-packet, Gabor, and wavelet structure over LCA groups. By using recent unitary extension principle (techniques of Christensen and Goh on LCA groups), the resulting characterizations (from Chapter 3) are used to develop construction procedures for orthogonal frames, which are recorded in the fourth chapter. The fourth chapter also consists of the construction of dual frame pairs over super-spaces.

The fifth and sixth chapters relate the theory developed in Chapters 2-4 to the finite dimensional Hilbert spaces by letting LCA group G of the form $\mathbb{Z}_N^d$. The connection of work with finite-dimensional set-up is necessary since in this day and age of computers, processing can be done only when the signal can be stored in memory. Therefore, the importance of discrete and finite signals cannot be ignored. The chapters are actually devoted to investigate a finite collection of functions in $\ell^2(\mathbb{Z}_N^d)$ that satisfies some localization properties in a region of the time-frequency plane using the group theoretic approach based on the set of digits. Using these functions, we first introduce an orthonormal wavelet system (ONWS) in $\ell^2(\mathbb{Z}_N^d)$ and then provide a characterization for its generators in the fifth chapter. Further, we present some results on the uncertainty principle corresponding to this ONWS.

The sixth chapter is the continuation of our work on the construction of a time-frequency localized ONWS which is discussed in Chapter 5. We generalize the above-mentioned construction to the **g**-framelet systems associated with an induced group action in Chapter 6. Along with this, we focus on developing the theory required to study the duality properties of the **g**-framelet systems for $\ell^2(\mathbb{Z}_N^d)$. Further, we investigate the structure of canonical dual for a **g**-FS. At last we concentrate on characterizing the generators of two super **g**-framelet systems associated with an induced group action such that they form a super **g**-dual pair.

Note that we have characterized orthogonality of TI and GTI frame systems in Chapter 2 and Chapter 3, respectively. The characterizations are further useful in Chapter 4 to construct orthogonal GSI systems via the approach of unitary extension principle on LCA groups. It would be interesting to use results discussed in Chapter 4 to study general constructions of orthogonal frames, based on B-splines on the group itself as well as on characteristic functions on the dual group. Moreover, we do expect that using the results of Chapter 5 and by considering a number of concrete groups, one can derive explicit constructions of the resulting orthogonal frames. These constructions might be useful in multiplexing of signals via super-frames, in the synthesis of new frames from existing ones, in the construction of super-frames, etc. Along with this, we expect that some of the investigations would lead to new results in different areas of research in frame theory.

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