BULK RECONSTRUCTION OF GAUGE FIELDS IN RINDLER COORDINATES

M.Sc. THESIS

by

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BULK RECONSTRUCTION OF GAUGE FIELDS IN RINDLER COORDINATES

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **Bulk Reconstruction of Gauge Fields in Rindler Coordinates** in the partial fulfilment of the requirements for the award of the degree of **Master of Science** and submitted in the **Discipline of Physics, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2022 to June 2023 under the supervision of **Dr. Debajyoti Sarkar, Assistant Professor, Indian Institute of Technology Indore.**

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any institute.

Signature of the student with date

23/05/2023 **(Laksha Pradip Das)**

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Signature of the Supervisor of M.Sc. thesis (with date)

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LAKSHA PRADIP DAS has successfully given his M.Sc. Oral Examination held on 19$^{\text{th}}$ **May, 2023**.

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ABSTRACT

In the AdS/CFT correspondence, we try to answer questions about bulk physics by reconstructing bulk fields in terms of boundary CFT operators. We develop the representation of spin one bulk gauge fields in anti de-Sitter space in Rindler lightcone coordinates, as non-local observables in the dual CFT. After studying the bulk reconstruction of scalar fields in Rindler coordinates, we employ a similar method to obtain the bulk reconstruction of spin one gauge fields in Rindler coordinates. Finally, we use these results to obtain the representation of bulk Wilson lines extending from one boundary of the Rindler patch to the other in terms of the CFT spin one currents on both the boundaries. We also find the bulk reconstruction for scalar fields in Rindler coordinates whose boundary counterpart (scalar conformal primary) has conformal dimension $\Delta = d - 1$.

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1 Introduction

Our understanding of one of the four fundamental forces of nature gravity is based on Albert Einstein's general theory of relativity, which is a classical theory. However, although general relativity has been the most successful in describing gravity, this description is incomplete. For instance, the space-time curvature at the center of a black hole diverges, which indicates the breakdown of the general theory of relativity and the need for a theory that goes beyond general relativity into the quantum realm, a quantum theory of gravity. A theory of quantum gravity seeks to describe gravity according to the principles of quantum mechanics. By far, most promising candidate for a theory of quantum gravity is "String Theory".

One of the most remarkable developments in the field of quantum gravity in the past few decades is the relation between gravity theories in asymptotically Anti de-Sitter space-times and conformal field theories living in the boundary of the AdS space-time. This is the AdS/CFT correspondence which was conjectured by Juan Maldacena in his paper [1] in 1997. He proposed that large N limits of certain conformal field theories in d dimensions can be described in terms of super-gravity (and string theory) on the product of $d+1$-dimensional AdS space-time with a compact manifold. One of the most prominent examples of the AdS/CFT correspondence is the AdS_5/CFT_4 correspondence (which was studied in [1]) which relates $\mathcal{N} = 4$ Super Yang-Mills theory in 3+1 dimensions and IIB superstring theory in $AdS_5 \times S^5$. Within this duality, the string theory is defined on the product of $AdS_5 \times S^5$ involving five-dimensional Anti-de Sitter space and a five-dimensional sphere. The type IIB superstring theory is the AdS side of the AdS/CFT correspondence while $\mathcal{N} = 4$ Super Yang-Mills theory, which is a conformally invariant theory, is the CFT side of this correspondence. A review of Maldacena's original work can be found in [2].

The AdS/CFT correspondence helps us to answer questions about bulk physics in AdS space-time by studying the boundary conformal field theory. In principle all bulk observables are encoded in correlation functions of local operators in the CFT. In practice, however, many of the quantum gravity questions we would like to address are not simply related to local boundary correlators. Thus, in [3], the authors Hamilton, Kabat, Lifschytz and Lowe developed a set of tools for recovering bulk physics from the boundary CFT. This formalism is known as the HKLL prescription of bulk reconstruction from boundary CFT named after the authors. The results obtained in this paper by the authors were generalized for higher dimensions in Poincaré , global and Rindler coordinates in [4] and in [5] the authors obtained the smearing function with compact support by working in complexified spatial boundary coordinates. In [5], it was also shown that scalar smearing functions in Rindler coordinates can only be constructed by analytically continuing the boundary coordinates to complex values, since the naive expression derived from mode sums (given in [4]) was divergent. Some interesting reviews on the HKLL prescription of bulk reconstruction are [6], [7] and [8].

The main purpose of this thesis is to find expressions for smearing functions for gauge fields and Wilson lines in Rindler AdS_3 and AdS_{d+1}. A general outline of this thesis is a follows. In chapter 2, we introduce some basic concepts like AdS space-time, different coordinate systems of AdS space used in this thesis and review the AdS/CFT duality. Chapter 3 is dedicated as review of scalar field bulk reconstruction in Poincaré coordinates as done in [3], [4]. We then study how scalar conformal primary operators transform under conformal transformations and use this relation and the transformation relations between Poincaré coordinates (t, x, z)

and Rindler coordinates $(\tilde{t},\, r,\, \tilde{\phi})$ to obtain a relation for scalar smearing function in Rindler coordinates. Similar result was obtained by the authors in [5] but while working in $(\hat{t}, r, \hat{\phi})$ Rindler coordinates. In [5], however, the bulk reconstruction for scalar fields with boundary counterpart having conformal dimension $\Delta = d - 1$ was not done, so in section 3.2.1 we obtain this trivial yet new result. Chapter 4 starts with the review of gauge field reconstruction in Poincaré coordinates as done in [9]. However, we find that it is easier find the relation between Poincaré boundary current and Rindler boundary current in lightcone coordinates, so we work in lightcone coordinates from this point onwards and in section 4.2 we write expressions bulk reconstruction of gauge fields in Poincaré lightcone coordinates. Then, in section 4.3 we use the transformation relation between Poincaré boundary current and Rindler boundary current in lightcone coordinates and relations between Poincaré lightcone coordinates and Rindler lightcone coordinates defined in (4.13) to obtain expressions for gauge fields $A^{Rindler}_{w^{\pm}}$. In section 4.4, we use the expressions for the representation of all the components of bulk gauge fields at all wedges in Rindler lightcone coordinates in terms of boundary spin one primary operators, to find the expression for a Wilson line which connects the left boundary of the Rindler patch with the right boundary. In appendix A, we show the relations between Poincaré coordinates and Rindler coordinates. Appendix B presents an alternate derivation of bulk reconstruction for scalar fields in Rindler coordinates as done in [5]. Finally, in appendix C, we derive expressions for bulk reconstruction for gauge fields in Rindler coordinates but not in lightcone coordinates $(w^+,\, w^-,\, r)$, but in Rindler coordinates $(\tilde{t},\, r,\, \tilde{\phi})$.

2 Basic concepts

2.1 AdS (Anti de-Sitter) space-time

AdS space-time is a maximally symmetric solution of Einstein's equations of general relativity with a negative cosmological constant. AdS space-time has a constant negative curvature, which means it is a hyperbolic space and has many of the same properties of two dimensionless hyperbolic plane.

$(d+1)$-dimensional AdS space-time is represented by hyperboloid

$$X_0^2 + X_{d+1}^2 - \sum_{i=1}^{d} X_i^2 \;=\; R^2 \tag{2.1}$$

(with R being the AdS radius) embedded in a $(d+2)$-dimensional flat space ($\mathbb{R}^{2,d}$) with the metric

$$ds^2 \;=\; -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2 \tag{2.2}$$

An AdS invariant distance function is provided by

$$\sigma(x|x') \;=\; \frac{1}{R^2} X_\mu X'^\mu \tag{2.3}$$

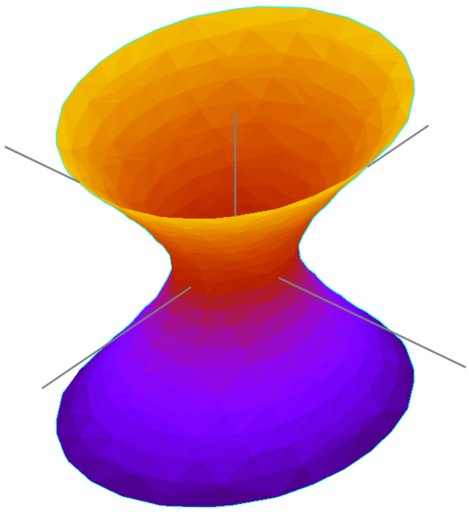

Figure 2.1: Image of (1 + 1)-dimensional anti-de Sitter space embedded in flat (1 + 2)-dimensional space (Source [10]).

2.1.1 Coordinate systems

Since, all calculations throughout this thesis would be carried out either in Poincaré or Rindler coordinates or both, so, the following subsections are dedicated as an introduction to Poincaré and Rindler coordinates of AdS space-time.

Poincaré coordinates

The Poincaré coordinates $(z, t, \vec{x})$ with $z > 0$, $\vec{x} \in \mathbb{R}^{d-1}$ are parameterized by:

$$X_0 = \frac{1}{2z}\left[z^2 + R^2 + \vec{x}^2 - t^2\right] \tag{2.4a}$$

$$X_i = \frac{R\,x_i}{z} \tag{2.4b}$$

$$X_d = \frac{1}{2z}\left[z^2 - R^2 = \vec{x}^2 - t^2\right] \tag{2.4c}$$

$$X_{d+1} = \frac{R\,t}{z} \tag{2.4d}$$

where, i = 1,2,....,$d - 1$.

Using these coordinate relations between the embedding coordinates and the Poincaré coordinates, we get the $(d + 1)$-dimensional AdS metric in

Poincaré coordinates as the following:

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right) \tag{2.5}$$

which can also be written as

$$ds^2 = \frac{R^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right) \tag{2.6}$$

where, $\eta_{\mu\nu}$ is the Minkowski metric and $\mu, \nu = 0, 1, ..., d-1$. For $d = 2$, the AdS_3 metric in Poincaré coordinates is

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + dx^2 + dz^2\right) \tag{2.7}$$

Using (2.3), we can calculate the AdS invariant distance in Poincaré coordinates. The AdS invariant distance between $(z, \vec{x}, t)$ and $(z', \vec{x}', t')$ in Poincaré coordinates is

$$\sigma(z, \vec{x}, t|z', \vec{x'}, t') = \frac{z^2 + z'^2 + |x - x'|^2 - (t - t')^2}{2zz'} \tag{2.8}$$

The invariant distance diverges as $z' \to 0$. So, we can define a regulated bulk-boundary distance as

$$(\sigma z')_{z' \to 0} = \frac{z^2 + z'^2 + |x - x'|^2 - (t - t')^2}{2z} \tag{2.9}$$

Rindler coordinates

Rindler coordinates of AdS_3 (t, r, ϕ) are parameterized as the following:

$$X_0 = \frac{Rr}{r_+} \cosh \frac{r_+ \phi}{R} \tag{2.10a}$$

$$X_1 = R\sqrt{\frac{r^2}{r_+^2} - 1} \sinh \frac{r_+ t}{R^2} \tag{2.10b}$$

$$X_2 = \frac{Rr}{r_+} \sinh \frac{r_+ \phi}{R} \tag{2.10c}$$

$$X_3 = R\sqrt{\frac{r^2}{r_+^2} - 1} \cosh \frac{r_+ t}{R^2} \tag{2.10d}$$

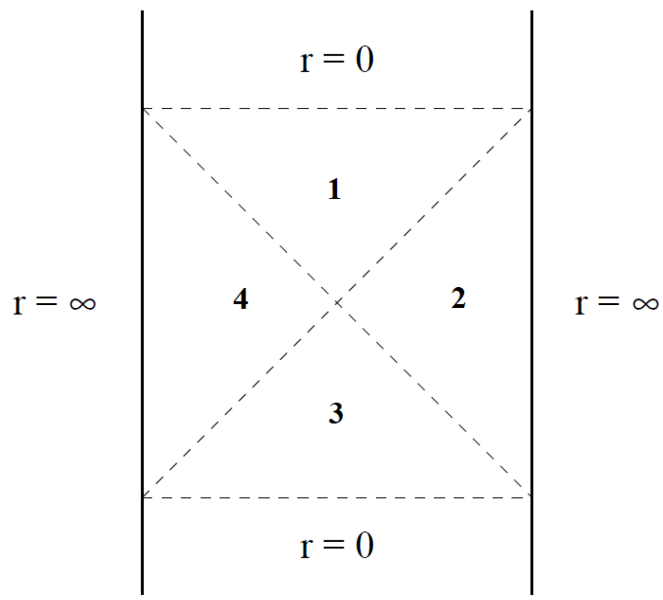

Figure 2.2: A slice of constant ϕ in AdS_3, drawn as an AdS_2 Penrose diagram. The four Rindler wedges are separated by horizons at $r = r_+$. Here, 1 is the future Rindler wedge, 2 is the Right Rindler wedge, 3 is the past Rindler wedge and 4 is the left Rindler wedge.

So, the induced metric is

$$ds^2 = -\frac{r^2 - r_+^2}{R^2} dt^2 + \frac{R^2}{r^2 - r_+^2} dr^2 + r^2 d\phi^2 \tag{2.11}$$

Here, $-\infty < t, \phi < \infty$ and $r_+ < r < \infty$. $r = r_+$ is the position of the Rindler horizon. Re-scaling the coordinates as

$$\hat{t} = \tfrac{r_+ t}{R^2} \quad \text{and} \quad \hat{\phi} = \tfrac{r_+ \phi}{R}$$

we get the AdS_3 metric as the following:

$$ds^2 = \frac{R^2}{r_+^2}\left\{ -(r^2 - r_+^2)d\hat{t}^2 + \frac{r_+^2}{r^2 - r_+^2}dr^2 + r^2 d\hat{\phi}^2 \right\} \tag{2.12}$$

However, we find that for most of our cases it is easier to work with $(\tilde{t}, r, \tilde{\phi})$[1] coordinates where $\tilde{t} = R\hat{t}$ and $\tilde{\phi} = R\hat{\phi}$, which gives our AdS_3 metric to be:

$$ds^2 = \frac{1}{r_+^2}\left\{ -(r^2 - r_+^2)d\tilde{t}^2 + \frac{R^2 r_+^2}{r^2 - r_+^2}dr^2 + r^2 d\tilde{\phi}^2 \right\} \tag{2.13}$$

(2.13) is the AdS_3 metric in Rindler coordinates, the AdS_{d+1} metric is thus:

$$ds^2 = \frac{1}{r_+^2}\left\{ -(r^2 - r_+^2)d\tilde{t}^2 + \frac{R^2 r_+^2}{r^2 - r_+^2}dr^2 + r^2 d\Omega_{d-1}^2 \right\} \tag{2.14}$$

Using (2.3) the AdS invariant distance between two points in the right Rindler wedge for the metric (2.13) is found out to be

$$\sigma = \frac{rr'}{r_+^2}\cosh\left(\frac{\tilde{\phi}}{R} - \frac{\tilde{\phi}'}{R}\right) - \left(\frac{r^2}{r_+^2} - 1\right)^{\frac{1}{2}}\left(\frac{r'^2}{r_+^2} - 1\right)^{\frac{1}{2}}\cosh\left(\frac{\tilde{t}}{R} - \frac{\tilde{t}'}{R}\right) \tag{2.15}$$

while for a point $(\tilde{t}, r, \tilde{\phi})$ inside the future Rindler wedge 1 in figure 2.2 and a point $(\tilde{t}', r, \tilde{\phi}')$ in the right Rindler wedge 2, the AdS invariant distance is

$$\sigma_{IR} = \frac{rr'}{r_+^2}\cosh\left(\frac{\tilde{\phi}}{R} - \frac{\tilde{\phi}'}{R}\right) - \left(1 - \frac{r^2}{r_+^2}\right)^{\frac{1}{2}}\left(\frac{r'^2}{r_+^2} - 1\right)^{\frac{1}{2}}\sinh\left(\frac{\tilde{t}}{R} - \frac{\tilde{t}'}{R}\right) \tag{2.16}$$

similarly, for a point $(\tilde{t}, r, \tilde{\phi})$ inside the future Rindler wedge 1 in figure 2.2 and a point $(\tilde{t}', r, \tilde{\phi}')$ in the left Rindler wedge 4, the AdS invariant distance

[1] This is because the coordinates $(\hat{t}, \hat{\phi})$ are dimensionless, whereas the new coordinates $(\tilde{t}, \tilde{\phi})$ are of one length dimension.

is

$$\sigma_{IL} = \frac{rr'}{r_+^2} \cosh\left(\frac{\tilde{\phi}}{R} - \frac{\tilde{\phi}'}{R}\right) + \left(1 - \frac{r^2}{r_+^2}\right)^{\frac{1}{2}} \left(\frac{r'^2}{r_+^2} - 1\right)^{\frac{1}{2}} \sinh\left(\frac{\tilde{t}}{R} - \frac{\tilde{t}'}{R}\right) \quad (2.17)$$

The AdS invariant distance for the metric (2.12) is the following:

$$\sigma = \frac{rr'}{r_+^2} \cosh(\hat{\phi} - \hat{\phi}') - \left(\frac{r^2}{r_+^2} - 1\right)^{\frac{1}{2}} \left(\frac{r'^2}{r_+^2} - 1\right)^{\frac{1}{2}} \cosh(\hat{t} - \hat{t}') \quad (2.18)$$

2.2 Conformal field theory

Most quantum field theories have Poincaré symmetry, i.e they are invariant Lorentz transformations ($x^\mu \rightarrow x^{\mu\prime} = \Lambda^\mu_\nu x^\nu$) and translations ($x^\mu \rightarrow x^{\mu\prime} = x^\mu + a^\mu$). Any QFT which in addition to Poincaré symmetry is also invariant under dilations ($x^\mu \rightarrow x^{\mu\prime} = \lambda x^\mu$) and under special conformal transformation $\left(x^\mu \rightarrow x^{\mu\prime} = \frac{x^\mu + a^\mu x^2}{1 + 2a_\nu x^\nu + a^2 x^2}\right)$ is known as a conformal field theory. We know under the transformation of $x \rightarrow x'$, the metric $g_{\mu\nu}(x)$ transforms as

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \quad (2.19)$$

For conformal transformations, the metric remains invariant up to a scale change,

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \Omega\, g_{\alpha\beta}(x) \quad (2.20)$$

We employ the jacobian,

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{\sqrt{\det(g'_{\mu\nu})}} = \Omega^{-d/2} \quad (2.21)$$

to describe conformal transformations.

2.3 AdS/CFT duality

In 1997, Juan Maldacena in his paper [1] proposed that large N limits of certain conformal field theories in d dimensions can be described in terms of super-gravity (and string theory) on the product of $d+1$-dimensional AdS space with a compact manifold. In other words any theory of quantum gravity in the bulk ($\mathcal{M}$ or AdS space) can be corresponded to a quantum field theory with conformal symmetry in the boundary ($\partial\mathcal{M}$) having one less number of dimensions. This is the well known "AdS/CFT correspondence". AdS/CFT duality is also known as holographic duality. The basic idea of this duality is that a gravity theory in AdS space is equivalent to a quantum field theory with conformal symmetry on the boundary of that space. There are several advantages to embracing the AdS/CFT approach.

2.3.1 AdS/CFT dictionary

The map between observables on both sides of the AdS/CFT duality is known as AdS/CFT dictionary. The most basic way to view this duality is an isomorphism between the Hilbert spaces: $\phi : \mathcal{H}_{AdS} \to \mathcal{H}_{CFT}$. Another way to view the AdS/CFT duality is the extrapolate dictionary [11], [12]. According to the extrapolate dictionary, for scalar field in $d+1$ dimensions in pure AdS we have:

$$\lim_{r\to\infty} r^{n\Delta} \langle \phi(r_1, t_1, \Omega_1)\phi(r_2, t_2, \Omega_2).......\phi(r_n, t_n, \Omega_n)\rangle = \langle 0 \,|\, \mathcal{O}(t_1, \Omega_1) \mathcal{O}(t_2, \Omega_2)........\mathcal{O}(t_n, \Omega_n) \,|\, 0\rangle \quad (2.22)$$

Here, $\mathcal{O}$ is the scalar primary with conformal dimensions Δ which is dual to the bulk scalar field ϕ with mass m, with $\Delta = \frac{d}{2} + \frac{1}{2}\sqrt{d^2 + 4m^2}$.

3

Bulk reconstruction of scalar fields

Any local bulk field can be expressed in terms of non-local operators in the boundary. Bulk scalar field $\phi(t, \vec{x}, z)$ in Poincaré coordinates is related to the boundary scalar field $\phi_0(t, \vec{x})$ as:

$$\phi_0(t, \vec{x}) = \lim_{z \to 0} \frac{1}{z^\Delta} \phi(t, \vec{x}, z) \tag{3.1}$$

Here, z is the radial coordinate that vanishes in the boundary.
AdS/CFT correspondence imply that the boundary behaviour of the field corresponds to an operator of conformal dimensions Δ in the CFT ($\phi_0(x)$ $\leftrightarrow \mathcal{O}(x)$). This implies a correspondence between local fields in the bulk and non-local operators in the boundary CFT with conformal dimensions Δ, which is given as follows:

$$\phi(t, \vec{x}, z) = \int dt' d^{d-1}\vec{x}' K\left(t'\vec{x}' \mid t, \vec{x}, z\right) \mathcal{O}(t'\vec{x}') \tag{3.2}$$

This is the HKLL prescription of bulk reconstruction given in literature [3], [4].

3.1 Scalar field reconstruction in Poincaré coordinates

A scalar field of mass m satisfies the Klein-Gordon equation, which is $(\Box - m^2)\phi = 0$. In Poincaré coordinates, the box/d'Alembertian operator is:

$$\Box = \frac{z^2}{R^2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{z^2}{R^2}\partial_z^2 - (d-1)\frac{z}{R^2}\partial_z \tag{3.3}$$

This gives the scalar field equation in an AdS_{d+1} background in Poincaré coordinates as:

$$\left(\frac{z^2}{R^2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{z^2}{R^2}\partial_z^2 - (d-1)\frac{z}{R^2}\partial_z - m^2\right)\phi(t,\vec{x},z) = 0 \tag{3.4}$$

Doing Fourier transform on $\phi(t,\vec{x},z)$, we can write

$$\phi(t,\vec{x},z) = \int d\omega\, d^{d-1}\vec{k}\, a_{\omega\vec{k}}\, e^{-i\omega t}\, e^{i\vec{k}\vec{x}}\, f_{\omega\vec{k}}(z) \tag{3.5}$$

Using (3.4) in (3.5) we obtain,

$$z^2 f''(z) + (1-d)zf'(z) + \left[(\omega^2 - |\vec{k}|^2)z^2 - m^2\right] f(z) = 0 \tag{3.6}$$

$$z^2 f''(z) + \left(1 - 2.\frac{d}{2}\right) zf'(z) + \left\{(\omega^2 - |\vec{k}|^2)z^2 - \left(m^2 + \frac{d^2}{4}\right) + \frac{d^2}{4}\right\} f(z) = 0 \tag{3.7}$$

We know the conformal dimension Δ of the CFT operator is related to dimension d and mass m of the bulk scalar field $\phi(t,\vec{x},z)$ as $\Delta = \frac{d}{2} + \frac{\sqrt{d^2+4m^2}}{2}$. Thus we can write $\frac{d^2}{4} + m^2 = (\Delta - \frac{d}{2})^2$. The equation (3.7) then becomes,

$$z^2 f''(z) + \left(1 - 2.\frac{d}{2}\right) zf'(z) + \left\{(\omega^2 - |\vec{k}|^2)z^2 - \left(\Delta - \frac{d}{2}\right)^2 + \frac{d^2}{4}\right\} f(z) = 0 \tag{3.8}$$

A Bessel's differential equation of the form:

$$z^2 w''(z) + (1-2s)zw'(z) + [a^2 r^2 z^{2r} - \nu^2 r^2 + s^2]w(z) = 0 \tag{3.9}$$

has the solution $w(z) = C_1 z^s J_\nu(az^r) + C_2 z^s Y_\nu(az^r)$. Comparing equations (3.8) and (3.9), we get $r = 1$, $s = \frac{d}{2}$, $a^2 = (\omega^2 - |\vec{k}|^2)$, $\nu = (\Delta - \frac{d}{2})$. This gives the solution of the radial part of the scalar field equation $(\Box -$

$m^2)\,\phi(t,\vec{x},z) = 0$ in $(d{+}1)$-dimensional AdS in Poincaré coordinates to be:

$$f(z) = C_1 z^{\frac{d}{2}} J_\nu\left(\sqrt{\omega^2 - |\vec{k}|^2} z\right) + C_2 z^{\frac{d}{2}} Y_\nu\left(\sqrt{\omega^2 - |\vec{k}|^2} z\right) \tag{3.10}$$

with $\nu = \Delta - \frac{d}{2}$.

For $\nu > 1$, only Bessel's function of the first kind $J_\nu(x)$ is normalizable i.e. any arbitrary well behaved function $f(x)$ can be expanded in a Bessel series. Therefore,

$$\phi(t,\vec{x},z) = \int_{|\omega|>|\vec{k}|} d\omega\, d^{d-1}\vec{k}\, a_{\omega\vec{k}}\, e^{-i\omega t}\, e^{i\vec{k}\vec{x}} z^{\frac{d}{2}}\, J_\nu\left(\sqrt{\omega^2 - |\vec{k}|^2} z\right) \tag{3.11}$$

Using the expression of the bulk field obtained in (3.11) in the bulk-boundary correspondence (3.1), we get:

$$\phi_0(t,\vec{x}) = \lim_{z\to 0} \frac{1}{z^\Delta} \int_{|\omega|>|\vec{k}|} d\omega\, d^{d-1}\vec{k}\, a_{\omega\vec{k}}\, e^{-i\omega t}\, e^{i\vec{k}\vec{x}} z^{\frac{d}{2}}\, J_\nu\left(\sqrt{\omega^2 - |\vec{k}|^2} z\right) \tag{3.12}$$

We use a few identities of Bessel's functions and hyper-geometric functions:

1.
$$J_\nu(x) = \frac{\left(\frac{1}{2}\right)^\nu}{\Gamma(\nu+1)}\, {}_0F_1\left(;\nu-1;-\frac{1}{4}z^2\right) \tag{3.13}$$

   where, ${}_0F_1(;a;z) = \sum_{k=0}^{\infty} \frac{z^k}{a^k\, k!}$ is the hyper-geometric function of the first kind.

2.
$$\lim_{z\to 0} {}_0F_1\left(;\nu-1;-\frac{1}{4}(\omega^2 - |\vec{k}|^2)^{\frac{\nu}{2}} = 1\right) \tag{3.14}$$

Using (3.13) and (3.14) in (3.12), we obtain:

$$\phi_0(t,\vec{x}) = \frac{1}{2^\nu \Gamma(\nu+1)} \int_{|\omega|>|\vec{k}|} d\omega d^{d-1}\vec{k} a_{\omega\vec{k}} e^{-i\omega t} e^{i\vec{k}\vec{x}} (\omega^2 - |\vec{k}|^2)^{\frac{\nu}{2}} \tag{3.15}$$

Performing inverse Fourier transform on (3.15) gives the mode coefficients to be,

$$a_{\omega\vec{k}} = \frac{2^\nu\, \Gamma(\nu+1)}{(\omega^2 - |\vec{k}|^2)^{\frac{\nu}{2}}} \int dt\, d^{d-1}\vec{x}\, e^{i\omega t}\, e^{-i\vec{k}\vec{x}}\, \phi_0(t,\vec{x}) \tag{3.16}$$

Using (3.16) in (3.11), we obtain the representation of bulk scalar field $\phi(t,\vec{x},z)$ in terms of boundary field $\phi_0(t,\vec{x})$ as

$$\phi(t,\vec{x},z) = \int dt d^{d-1}\vec{x} K(t',\vec{x}'|t,\vec{x},z)\phi_0(t',\vec{x}') \tag{3.17}$$

where

$$K(t',\vec{x}'|t,\vec{x},z) = \frac{2^\nu\Gamma(\nu+1)}{(2\pi)^d}\int_{|\omega|>|\vec{k}|} d\omega\, d^{d-1}\vec{k}\, e^{-i\omega(t-t')}\, e^{i\vec{k}(\vec{x}-\vec{x}')} z^{\frac{d}{2}} \frac{J_\nu(\sqrt{\omega^2-|\vec{k}|^2}z)}{(\omega^2-|\vec{k}|^2)^{\frac{\nu}{2}}}$$

is the scalar smearing function in Poincaré coordinates. Following calculations along the lines of [3], [4], one obtains a smearing function with support on the entire boundary of the Poincaré patch, however by complexifying the boundary spatial coordinates one can obtain a smearing function with compact support. By representing bulk operators as operators on the boundary with compact support in fact with support that is as small as possible – we can have bulk operators whose dual boundary operators are spacelike separated. Such bulk operators will manifestly commute with each other just by locality of the boundary theory. In the following subsection, we discuss how to find scalar smearing function by complexifying the boundary spatial coordinates as done in [5].

3.1.1 Scalar smearing function

The bulk scalar field $\phi(t,\vec{x},z)$ can also be written as

$$\phi(t,\vec{x},z) = 2^\nu\Gamma(\nu+1)\int_{|\omega|>|\vec{k}|} d\omega d^{d-1}\vec{k} e^{-i\omega t} e^{i\vec{k}\vec{x}} z^{\frac{d}{2}} \frac{J_\nu\left(\sqrt{\omega^2-|\vec{k}|^2}z\right)}{\left(\omega^2-|\vec{k}|^2\right)^{\frac{\nu}{2}}}\phi_0(\omega,\vec{k}) \tag{3.18}$$

We use two identities of Bessel's polynomials:

1. $$(2\pi)^{\frac{d}{2}} J_0\left(r\sqrt{\omega^2-|\vec{k}|^2}\right) = \int_0^{2\pi} d\theta e^{-i|\vec{r}|\omega sin\theta-\vec{k}.\vec{r}cos\theta}$$

2.
$$2^{\nu-\frac{d}{2}}\Gamma(\nu-\frac{d}{2}+1)b^{-\nu}J_\nu(b)=\int_0^1 rd^{d-1}\vec{r}(1-r^2)^{(\nu-\frac{d}{2})}J_0(br)$$

Using these two identities we get,

$$\frac{J_\nu\left(\sqrt{\omega^2-|\vec{k}|^2}z\right)}{(\omega^2-|\vec{k}|^2)^{\frac{\nu}{2}}}=\frac{1}{(2z)^\nu\pi^{\frac{d}{2}}\Gamma(\nu-\frac{d}{2}+1)}\int_{t'^2+|\vec{y}|'^2<z^2}dt'd^{d-1}\vec{y}' \\ (z^2-t'^2+|\vec{y}|'^2)^{\nu-\frac{d}{2}}e^{-i\omega t'-\vec{k}.\vec{y}'} \quad (3.19)$$

Since $\nu=\Delta-\frac{d}{2}$, therefore $\Gamma(\nu+1)=\Gamma(\Delta-\frac{d}{2}+1)$ and $\Gamma(\nu-\frac{d}{2}+1)=\Gamma(\Delta-d+1)$.

Using (3.19) in (3.18), we obtain the following expression for bulk field

$$\phi(t,\vec{x},z)=\frac{\Gamma(\Delta-\frac{d}{2}+1}{\pi^{\frac{d}{2}}\Gamma(\Delta-d+1)}\int_{t'^2+|\vec{y}|'^2<z^2}dt'd^{d-1}\vec{y}'\left(\frac{z^2-t'^2+|\vec{y}|'^2}{z}\right)^{\Delta-d} \\ \phi_0(t+t',\vec{x}+i\vec{y}') \quad (3.20)$$

The Poincaré AdS invariant distance (2.8) with boundary coordinates $\vec{x}'=\vec{x}+i\vec{y}'$ and $t'=t+t'$ and $z'=0$ becomes $\lim_{z'\to 0}\left(2\sigma(z,\vec{x},t|z',\vec{x'},t')z'\right)=\frac{z^2-t'^2+|\vec{y}|'^2}{z}$. Thus,

$$\phi(t,\vec{x},z)=\frac{\Gamma(\Delta-\frac{d}{2}+1}{\pi^{\frac{d}{2}}\Gamma(\Delta-d+1)}\int_{t'^2+|\vec{y}|'^2<z^2}dt'd^{d-1}\vec{y}'\lim_{z'\to 0}(2\sigma z')^{\Delta-d}\phi_0(t+t',\vec{x}+i\vec{y}') \quad (3.21)$$

However, for the case of conformal dimension $\Delta=d-1$, $\Gamma(\Delta-d+1)$ is undefined. Using an integral representation for Bessel function,

$$J_\nu(a)=\frac{\left(\frac{a}{2}\right)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\frac{1}{\mathrm{Vol}(S^{d-2})}\int_{|\vec{n}|=1}d\vec{n}\,e^{-i\vec{a}.\vec{n}}$$

we obtain:

$$\frac{J_\nu\left(\sqrt{\omega^2-|\vec{k}|^2}z\right)}{(\omega^2-|\vec{k}|^2)^{\frac{\nu}{2}}}=\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\frac{d}{2}\right)\mathrm{Vol}(S^{d-1})}\int_{t^2+|\vec{y}|^2=1}dtd^{d-1}\vec{y}e^{-i\omega zt-z\vec{k}.\vec{y}} \quad (3.22)$$

Thus, the bulk reconstruction for a scalar field when $\Delta = d-1$ is

$$\phi(t, \vec{x}, z) = \frac{1}{\mathrm{Vol}(S^{d-1})} \int_{t'^2+|\vec{y}|'^2=z^2} dt' d^{d-1}\vec{y}' \phi_0(t+t', \vec{x}+i\vec{y}') \tag{3.23}$$

The integral is over a sphere of radius z on the complexified boundary, with the center of the sphere located at $(t, \vec{x})$. This can be written in the covariant form using the regulated bulk-boundary distance (2.9) as

$$\phi(t, \vec{x}, z) = \frac{1}{\mathrm{Vol}(S^{d-1})} \int_{t'^2+|\vec{y}|'^2=z^2} dt' d^{d-1}\vec{y}' \delta(\sigma z') \phi_0(t+t', \vec{x}+i\vec{y}') \tag{3.24}$$

Thus for general $\Delta \neq d-1$, the reconstructed bulk field in terms of boundary field is given by (3.20) with scalar smearing function

$$K(t+t', \vec{x}+i\vec{y}'|t, \vec{x}, z) = \frac{\Gamma(\Delta - \frac{d}{2}+1}{\pi^{\frac{d}{2}}\Gamma(\Delta-d+1)} \left(\frac{z^2 - t'^2 + |\vec{y}|'^2}{z}\right)^{\Delta-d} \tag{3.25}$$

While for $\Delta = d-1$, the reconstructed bulk field in terms of boundary field is given by (3.24) with scalar smearing function

$$K(t+t', \vec{x}+i\vec{y}'|t, \vec{x}, z) = \frac{1}{\mathrm{Vol}(S^{d-1})} \delta(\sigma z') \tag{3.26}$$

3.2 Scalar field reconstruction in Rindler coordinates

Under global conformal transformations, scalar quasi-primary operators of conformal dimensions Δ_j transform as the following:

$$\phi_j(x) \to \phi'_j(x) = \left|\frac{\partial x'}{\partial x}\right|^{\Delta_j/d} \phi_j(x') = \Omega^{-\Delta_j/2} \phi_j(x') \tag{3.27}$$

Using the value of conformal factor Ω from (A.14) in going from Poincaré boundary coordinates (t, x) to Rindler boundary coordinates $(\tilde{t}, \tilde{\phi})$, the boundary scalar operator transforms as

$$\phi_0^{Poincaré}(t, x) = \lim_{r\to\infty} \left(\frac{r_+ R}{r\, z}\right)^{\Delta} \phi_0^{Rindler}(\tilde{t}, \tilde{\phi}) \tag{3.28}$$

We know the transformation between bulk scalar fields $\phi^{Poincare}(x)$ and $\phi^{Rindler}(x')$ and boundary behaviour of bulk scalar field in Poincaré coordinates to be the following:

$$\begin{aligned} \phi_0^{Poincaré}(t,x) &= \lim_{z\to 0}\frac{1}{z^{\Delta}}\phi^{Poincaré}(t,z,x) \\ \phi^{Poincaré}(x) &= \phi^{Rindler}(x') \end{aligned} \tag{3.29}$$

Using the relations in (3.29) and the relation between boundary scalar operators $\phi_0^{Poincaré}$ and $\phi_0^{Rindler}$ given in (3.28), we have the boundary behaviour of bulk scalar field in Rindler coordinates to be the following:

$$\phi^{Rindler}(\tilde{t},r,\tilde{\phi}) = \lim_{r\to\infty}\left(\frac{Rr_+}{r}\right)^{\Delta}\phi_0^{Rindler}(\tilde{t},\tilde{\phi}) \tag{3.30}$$

From here onwards, we will be working in AdS_3/CFT_2. Setting $d = 2$ in (3.21), we obtain the bulk reconstruction of scalar fields in AdS_3 in Poincaré coordinates

$$\begin{aligned} \phi(t,x,z) = \frac{(\Delta-1)\,2^{\Delta-2}}{\pi}\int_{t'^2+y'^2<z^2} dt'dy' \lim_{z'\to 0}(\sigma z')^{\Delta-2} \\ \phi_0^{Poincaré}(t+t',x+iy') \end{aligned} \tag{3.31}$$

From (A.13) we have the boundary change of coordinates to be:

$$dt\,dx = \lim_{r\to\infty}\left(\frac{r\,z}{r_+R}\right)^2 d\tilde{t}\,d\tilde{\phi} \tag{3.32}$$

Using the relations (3.28) and (3.32) (and using primed coordinates for boundary) in (3.31), we obtain

$$\begin{aligned} \phi(\tilde{t},r,\tilde{\phi}) &= \frac{(\Delta-1)\,2^{\Delta-2}}{\pi}\int_{\sigma>0}\lim_{r'\to\infty}\left(\frac{r'z'}{r_+R}\right)d\tilde{t}'d\tilde{\phi}'\lim_{z'\to 0}(\sigma z')^{\Delta-2} \\ &\qquad \lim_{r'\to\infty}\left(\frac{r'z'}{r_+R}\right)^{\Delta}\phi_0^{Rindler}(\tilde{t}+\tilde{t}',\tilde{\phi}+i\tilde{\phi}') \\ \Rightarrow \phi(\tilde{t},r,\tilde{\phi}) &= \frac{(\Delta-1)\,2^{\Delta-2}}{\pi}\int_{\sigma>0}d\tilde{t}'d\tilde{\phi}'\lim_{r'\to\infty}\left(\frac{r_+R}{r'}\sigma\right)^{\Delta-2} \\ &\qquad \phi_0^{Rindler}(\tilde{t}+\tilde{t}',\tilde{\phi}+i\tilde{\phi}') \end{aligned} \tag{3.33}$$

where as $r' \to \infty$ the AdS-invariant distance (2.15) becomes

$$\sigma(\tilde{t}, r, \tilde{\phi} | \tilde{t} + \tilde{t}', r, \tilde{\phi} + i\tilde{\phi}') = \frac{rr'}{r_+^2} \left[\cos \frac{\tilde{\phi}'}{R} - \left(1 - \frac{r_+^2}{r^2}\right)^{\frac{1}{2}} \cosh \frac{\tilde{t}'}{R} \right] \tag{3.34}$$

and the integration is over space-like separated points on the boundary and $\sigma > 0$. A similar result but while working in Rindler coordinates $(\hat{t}, r, \hat{\phi})$ was also obtained in [5] and we show this derivation in appendix B. These results can also be obtained in a similar way as in [4] alternatively starting from Rindler mode sum and defining it via an analytic continuation, or alternatively from a de Sitter Green's function.

Scalar fields in the future Rindler wedge

But if we consider a scalar field at a point inside the Rindler horizon i.e. in the wedge 1 of figure 2.2, the smearing function extends outside the Rindler wedge, and covers points on the boundary which are to the future of the right Rindler patch[1], so [5] uses an antipodal map, $A : \tilde{t} \to \tilde{t} + i\pi$, $\tilde{\phi} \to \tilde{\phi} + i\pi$ under which the AdS invariant distance transform as $\sigma_{IR}(x|Ax') = -\sigma_{IL}(x|x')$ and the boundary fields transform as $\phi_0^R(Ax) = (-1)^{\Delta} \phi_0^L(x)$ to find the bulk reconstruction of a scalar field inside the Rindler horizon,

$$\begin{aligned}\phi(\tilde{t}, r, \hat{\phi}) = &\frac{(\Delta-1)2^{\Delta-2}}{\pi} \left[\int_{\sigma_{IR}>0} d\tilde{t}' d\tilde{\phi}' \lim_{r' \to \infty} \left(\frac{\sigma_{IR}}{r'}\right)^{\Delta-2} \phi_0^{Rindler,R}(\tilde{t} + \tilde{t}', \tilde{\phi} + i\tilde{\phi}') \right. \\ &\left. + \int_{\sigma_{IL}<0} d\tilde{t}' d\tilde{\phi}' \lim_{r' \to \infty} \left(\frac{-\sigma_{IL}}{r'}\right)^{\Delta-2} (-1)^{\Delta} \phi_0^{Rindler,L}(\tilde{t} + \tilde{t}', \tilde{\phi} + i\tilde{\phi}') \right]\end{aligned} \tag{3.35}$$

Here as $r \to \infty$, the AdS invariant distance (2.16) and (2.17) both become,

$$\begin{aligned}\sigma_{IR}(\tilde{t}, r, \tilde{\phi} | \tilde{t} + \tilde{t}', r, \tilde{\phi} + i\tilde{\phi}') &= \frac{rr'}{r_+^2} \left[\cos\left(\frac{\tilde{\phi}'}{R}\right) + \sqrt{\frac{r_+^2}{r^2} - 1}\, \sinh\left(\frac{\tilde{t}'}{R}\right) \right] \\ \sigma_{IL}(\tilde{t}, r, \tilde{\phi} | \tilde{t} + \tilde{t}', r, \tilde{\phi} + i\tilde{\phi}') &= \frac{rr'}{r_+^2} \left[\cos\left(\frac{\tilde{\phi}'}{R}\right) - \sqrt{\frac{r_+^2}{r^2} - 1}\, \sinh\left(\frac{\tilde{t}'}{R}\right) \right]\end{aligned} \tag{3.36}$$

[1] To visualize this we can choose a bulk point in the future Rindler wedge of figure (2.2) and follow the light rays to the right Rindler boundary.

3.2.1 Rindler smearing function when $\Delta = d - 1 = 1$

Scalar field reconstruction in Poincaré coordinates when boundary scalar primary has conformal dimensions $\Delta = d-1$ is given by (3.25), putting $d = 2$ i.e. $\Delta = 1$ there we obtain:

$$\phi(t,z,x) = \frac{1}{2\pi}\int_{t'^2+y'^2=z^2} dt'dy'\,\delta(\sigma z')\phi_0(t,x) \tag{3.37}$$

Putting $\Delta = 1$ in (3.28) we have,

$$\phi_0^{Poincare}(t,x) \;=\; \lim_{r\to\infty}\left(\frac{r_+R}{r\,z}\right)\phi_0^{Rindler}(\tilde{t},\tilde{\phi}) \tag{3.38}$$

Using the relations (3.32) and (3.38) in (3.37) (and using primed for boundary coordinates) we have the following:

$$\phi(\tilde{t},r,\tilde{\phi}) \;=\; \frac{1}{2\pi}\int_S d\tilde{t}'d\tilde{\phi}'\,\lim_{r'\to\infty}\left(\frac{r'z'}{r_+R}\right)^2\left(\frac{r_+R}{r'z'}\right)\delta(\sigma z')\phi_0^{Rindler}(\tilde{t}+\tilde{t}',\tilde{\phi}+i\tilde{\phi}')$$

$$\Rightarrow \phi(\tilde{t},r,\tilde{\phi}) \;=\; \frac{1}{2\pi}\int_S d\tilde{t}'d\tilde{\phi}'\,\lim_{r'\to\infty}\left(\frac{r'z'}{r_+R}\right)\delta\left(\frac{r'\sigma z'}{r'}\right)\phi_0^{Rindler}(\tilde{t}+\tilde{t}',\tilde{\phi}+i\tilde{\phi}')$$

Using a property of δ-function which is $\delta(ax) = \frac{1}{|a|}\delta(x)$ in the above expression, we get

$$\phi(\tilde{t},r,\tilde{\phi}) \;=\; \frac{1}{2\pi}\int_S d\tilde{t}'d\tilde{\phi}'\,\delta\left(\frac{r_+R}{r'}\sigma\right)\phi_0^{Rindler}(\tilde{t}+\tilde{t}',\tilde{\phi}+i\tilde{\phi}') \tag{3.39}$$

here, the region of integration S is $\cos\frac{\tilde{\phi}'}{R} = \sqrt{1-\frac{r^2}{r_+^2}}\cosh\frac{\tilde{t}'}{R}$.
For a bulk scalar field at the point inside the Rindler horizon, we use the same antipodal mapping used in the above section to obtain the result (3.35) and obtain,

$$\begin{aligned}\phi(\tilde{t},r,\tilde{\phi}) =& \frac{1}{2\pi}\Bigg[\int_{S_1} d\tilde{t}'d\tilde{\phi}'\,\delta\left(\frac{r_+R}{r'}\sigma_{IR}\right)\phi_0^{Rindler,R}(\tilde{t}+\tilde{t}',\tilde{\phi}+i\tilde{\phi}') - \\ &\int_{S_2} d\tilde{t}'d\tilde{\phi}'\,\delta\left(\frac{r_+R}{r'}\sigma_{IL}\right)\phi_0^{Rindler,L}(\tilde{t}+\tilde{t}',\tilde{\phi}+i\tilde{\phi}')\Bigg]\end{aligned} \tag{3.40}$$

here, the regions of integration S_1 and S_2 are respectively

$$S_1 \,:\, \cos\left(\frac{\tilde{\phi}'}{R}\right) = -\sqrt{\frac{r_+^2}{r^2} - 1}\,\sinh\left(\frac{\tilde{t}'}{R}\right)$$

$$S_2 \,:\, \cos\left(\frac{\tilde{\phi}'}{R}\right) = \sqrt{\frac{r_+^2}{r^2} - 1}\,\sinh\left(\frac{\tilde{t}'}{R}\right)$$

4

Bulk reconstruction of gauge fields

4.1 Gauge field reconstruction in Poincaré coordinates

In [9], the authors reconstructed the bulk Maxwell field $A_M(x,z)$ in terms of boundary current $j_\mu(x)$, via a kernel of the form

$$A_M(x,z) = \int d^d x' K^\mu_M(x,z|x') j_\mu(x') \tag{4.1}$$

Thus, bulk gauge field can be represented in terms of a non-local operator in the dual CFT.

4.1.1 Gauge smearing function in Poincaré coordinates

The source free Maxwell equation in the bulk is

$$\nabla_M F^{MN} = 0 \tag{4.2}$$

In the holographic gauge we set $A_z = 0$ and use a residual gauge transformation to set $\partial_\mu A^\mu = 0$.
The remaining Maxwell equations then simplify to

$$\partial_\mu(\eta^{\mu\rho}\partial_\rho)A_\nu + z^{d-3}\left(\frac{1}{z^{d-3}}\partial_z A_\nu\right) = 0 \tag{4.3}$$

Defining $\phi_\nu(x,z) = zA_\nu(x,z)$, one finds that

$$z^2(\partial_\mu\eta^{\mu\rho}\partial_\rho)\phi_\nu + z^2\partial_z^2\phi_\nu - (d-1)z\partial_z\phi_\nu + (d-1)\phi_\nu = 0 \tag{4.4}$$

Equation (4.4) is just the scalar field equation $(\Box - m^2)\phi = 0$ with the AdS radius $R = 1$ in the box operator (3.3) and mass term $m^2 = 1-d$. Using $\Delta = \frac{d}{2} + \frac{\sqrt{d^2+4m^2}}{2}$ we obtain, $\Delta = d-1$. Thus, bulk field $\phi_\mu(x,z)$ is related to boundary current $j_\mu(x)$ (according to (3.1)) as

$$\phi_\mu^{Poincaré}(x,z) \sim z^{d-1} j_\mu^{Poincaré}(x) \text{ as } z \to 0$$

Since the field $\phi_\mu(x,z)$ satisfies the scalar field equation and $\Delta = d-1$, we obtain the reconstruction of bulk field in terms of boundary current by (3.23) and thus $A_\mu^{Poincaré}(t,\vec{x},z)$ can be written in terms of boundary current $j_\mu^{Poincaré}(t+t',\vec{x}+i\vec{y}')$ as

$$zA_\mu^{Poincaré}(t,\vec{x},z) = \frac{1}{\text{Vol}(S^{d-1})}\int_{t'^2+|\vec{y}|'^2=z^2} dt' d^{d-1}\vec{y}' j_\mu^{Poincaré}(t+t',\vec{x}+i\vec{y}') \tag{4.5}$$

In the covariant form (3.24), we have

$$zA_\mu^{Poincaré}(t,\vec{x},z) = \frac{1}{\text{Vol}(S^{d-1})}\int_{t'^2+|\vec{y}|'^2=z^2} dt'\, d^{d-1}\vec{y}'\, \delta(\sigma z')\; j_\mu^{Poincaré}(t+t',\vec{x}+i\vec{y}') \tag{4.6}$$

Thus, the gauge smearing function is $\frac{\delta(\sigma z')}{z\,\text{Vol}(S^{d-1})}$.

4.2 Gauge smearing function in AdS$_3$ Poincaré lightcone coordinates

Since, we are working in AdS$_3$/CFT$_2$, we find that it is easier for us to work with Poincaré and Rindler lightcone coordinates to obtain a simpler boundary current transformation between Poincaré and Rindler coordinates. We define Poincaré lightcone coordinates as $x^\pm = x \pm t$. Thus the bulk AdS_3 metric in Poincaré lightcone coordinates is:

$$ds^2 = \frac{R^2}{z^2}\left(dx^+dx^- + dz^2\right) \tag{4.7}$$

And the boundary CFT_2 metric in lightcone coordinates is

$$ds^2 = dx^{+\prime}dx^{-\prime} \tag{4.8}$$

Putting $d = 2$ in (4.5), we get the bulk reconstruction of gauge fields in AdS_3 in Poincaré coordinates as the following

$$zA_\mu^{Poincar\acute{e}}(t,x,z) = \frac{1}{2\pi}\int_{t'^2+y'^2=z^2} dt'dy' j_\mu^{Poincar\acute{e}}(t+t', x+iy') \tag{4.9}$$

Now, on going from (t', y') coordinates to $(x^{+\prime}, x^{-\prime})$ in the boundary, we have the following jacobian of transformation

$$dt'dy' = \frac{1}{2}dx^{+\prime}dx^{-\prime} \tag{4.10}$$

We can also write the boundary spin-one primary $j_\mu^{Poincar\acute{e}}$ alternatively as $j^{Poincar\acute{e}}(x^+, x^-)$ (there may be some overall, convention dependent constant factors between $j_\mu^{Poincar\acute{e}}$ and $j^{Poincar\acute{e}}(x^+, x^-)$, but in our case that is not that important). Thus in Poincaré lightcone coordinates the expressions of bulk reconstruction of gauge fields are the following:

$$zA_{x^+}^{Poincar\acute{e}}(x_+, x_-, z) = \frac{1}{4\pi}\int_{x^{+\prime 2}+x^{-\prime 2}=2z^2} dx^{+\prime}dx^{-\prime}\, j_{x^+}^{Poincar\acute{e}}(x^{+\prime}, x^{-\prime}) \tag{4.11}$$

$$zA_{x^-}^{Poincar\acute{e}}(x_+, x_-, z) = \frac{1}{4\pi}\int_{x^{+\prime 2}+x^{-\prime 2}=2z^2} dx^{+\prime}dx^{-\prime}\, j_{x^-}^{Poincar\acute{e}}(x^{+\prime}, x^{-\prime}) \tag{4.12}$$

4.3 Bulk reconstruction of gauge fields in AdS_3 Rindler lightcone coordinates

In Rindler coordinates, the boundary conformal metric is $ds^2 = -d\tilde{t}^2 + d\tilde{\phi}^2$. We can forget about the conformal factor, as the theory is invariant under an overall conformal factor (as the theory is conformal itself). We define lightcone Rindler coordinates as:

$$w^{\pm} = \tilde{\phi} \pm \tilde{t} \tag{4.13}$$

thus, the Rindler boundary conformal metric in the lightcone coordinates is then $ds^2 = dw^{+\prime} dw^{-\prime}$.[1]

4.3.1 Poincaré to Rindler transformation of boundary current

The transformation relations between the Poincaré lightcone coordinates (x^+, x^-, z) and Rindler lightcone coordinates (w^+, w^-, r) are:

$$x^- = R\left\{\frac{\tilde{r}\sinh\left(\frac{w^+ + w^-}{2R}\right) - \sqrt{\tilde{r}^2 - 1}\sinh\left(\frac{w^+ - w^-}{2R}\right)}{\tilde{r}\cosh\left(\frac{w^+ + w^-}{2R}\right) + \sqrt{\tilde{r}^2 - 1}\cosh\left(\frac{w^+ - w^-}{2R}\right)}\right\} \tag{4.14}$$

$$x^+ = R\left\{\frac{\tilde{r}\sinh\left(\frac{w^+ + w^-}{2R}\right) + \sqrt{\tilde{r}^2 - 1}\sinh\left(\frac{w^+ - w^-}{2R}\right)}{\tilde{r}\cosh\left(\frac{w^+ + w^-}{2R}\right) + \sqrt{\tilde{r}^2 - 1}\cosh\left(\frac{w^+ - w^-}{2R}\right)}\right\} \tag{4.15}$$

$$z = \frac{R}{\tilde{r}\cosh\left(\frac{w^+ + w^-}{2R}\right) + \sqrt{\tilde{r}^2 - 1}\cosh\left(\frac{w^+ - w^-}{2R}\right)} \tag{4.16}$$

where, $\tilde{r} = \frac{r}{r_+}$. In the limit $r \to \infty$ in the above relations we obtain the transformation relation between the boundary lightcone coordinates $(x^{+\prime}, x^{-\prime})$ and $(w^{+\prime}, w^{-\prime})$ to be the following:

$$w^{+\prime} = R\log\left(\frac{x^{+\prime} - R}{x^{+\prime} + R}\right) \tag{4.17}$$

[1] Primed quantities are used to represent the boundary coordinates to be integrated over.

$$w^{-\prime} = R\log\left(\frac{x^{-\prime}-R}{x^{-\prime}+R}\right) \tag{4.18}$$

Thus, the jacobian of transformation between the boundary coordinates is

$$dw^{+\prime}dw^{-\prime} \;=\; \left(\frac{dw^{+\prime}}{dx^{+\prime}}\right)\left(\frac{dw^{-\prime}}{dx^{-\prime}}\right)dx^{+\prime}dx^{-\prime} \tag{4.19}$$

From [13], we know that the transformation relation of spin-s primary operator $j(z,\bar{z})$ with conformal dimension Δ is

$$j(z,\bar{z}) \longrightarrow \left(\frac{\partial f}{\partial z}\right)^{h}\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} j(f(z),\bar{f}(\bar{z})) \tag{4.20}$$

where, $h = \frac{1}{2}(\Delta + s)$ and $\bar{h} = \frac{1}{2}(\Delta - s)$. In our case, $z = x^{+\prime}$, $\bar{z} = x^{-\prime}$ and $f(z) = w^{+\prime}$, $\bar{f}(\bar{z}) = w^{-\prime}$. For $j^{Poincaré}_{x^+}(x^{+\prime},x^{-\prime})$, we have $\Delta = 1$, $s = 1$ thus, $h = 1$, $\bar{h} = 0$ and $j^{Poincaré}_{x^-}(x^{+\prime},x^{-\prime})$, we have $\Delta = 1$, $s = -1$ thus, $h = 0$, $\bar{h} = 1$, hence the transformation of $j^{Poincaré}_{x^\pm}$ to $j^{Rindler}_{w^\pm}$ are as follows:

$$j^{Poincaré}_{x^+}(x^{+\prime},x^{-\prime}) \;=\; \left(\frac{dw^{+\prime}}{dx^{+\prime}}\right) j^{Rindler}_{w^+}(w^{+\prime},w^{-\prime}) \tag{4.21}$$

$$j^{Poincaré}_{x^-}(x^{+\prime},x^{-\prime}) \;=\; \left(\frac{dw^{-\prime}}{dx^{-\prime}}\right) j^{Rindler}_{w^-}(w^{+\prime},w^{-\prime}) \tag{4.22}$$

We know A_μ's are bulk spin-one gauge fields and should transform as vector under Lorentz transformation from Rindler to Poincaré coordinates i.e. as $A^{Rindler}_{\mu}(x') = \frac{\partial x^\alpha}{\partial x^{\mu\prime}} A^{Poincaré}_{\alpha}$. Thus, we can write $A^{Rindler}_{w^\pm}$ in terms of $A^{Poincaré}_{x^\pm}$ as

$$\begin{aligned}
A^{Rindler}_{w^+} &= \frac{\partial x^+}{\partial w^+} A^{Poincaré}_{x^+} + \frac{\partial x^-}{\partial w^+} A^{Poincaré}_{x^-} \\
A^{Rindler}_{w^-} &= \frac{\partial x^+}{\partial w^-} A^{Poincaré}_{x^+} + \frac{\partial x^-}{\partial w^-} A^{Poincaré}_{x^-} \\
A^{Rindler}_{r} &= \frac{\partial x^+}{\partial r} A^{Poincaré}_{x^+} + \frac{\partial x^-}{\partial r} A^{Poincaré}_{x^-}
\end{aligned} \tag{4.23}$$

Using the transformation relations (4.14) and (4.15) in (4.23), we obtain

$$\begin{aligned} A_{w^+}^{Rindler} &= \frac{1}{2\left\{\tilde{r}\cosh\left(\frac{w^+ + w^-}{2R}\right) + \sqrt{\tilde{r}^2 - 1}\cosh\left(\frac{w^+ - w^-}{2R}\right)\right\}^2} \\ &\left[A_{x^-}^{Poincaré} + \left(2\tilde{r}^2 - 1 + 2\tilde{r}\sqrt{\tilde{r}^2-1}\cosh\left(w^-/R\right)\right) A_{x^+}^{Poincaré}\right] \\ \Rightarrow A_{w^+}^{Rindler} &= \frac{1}{2\left\{\tilde{r}\cosh\left(\frac{w^+ + w^-}{2R}\right) + \sqrt{\tilde{r}^2 - 1}\cosh\left(\frac{w^+ - w^-}{2R}\right)\right\}} \\ &\left[zA_{x^-}^{Poincaré} + \left(2\tilde{r}^2 - 1 + 2\tilde{r}\sqrt{\tilde{r}^2-1}\cosh\left(w^-/R\right)\right) zA_{x^+}^{Poincaré}\right] \end{aligned} \quad (4.24)$$

We can write (4.24) as the following expression

$$A_{w^+}^{Rindler} = \frac{1}{C}\left[zA_{x^-}^{Poincaré} + \left(2\tilde{r}^2 - 1 + 2\tilde{r}\sqrt{\tilde{r}^2-1}\cosh\left(w^-/R\right)\right) zA_{x^+}^{Poincaré}\right] \quad (4.25)$$

where, $C = 2R\left\{\tilde{r}\cosh\left(\frac{w^+ + w^-}{2R}\right) + \sqrt{\tilde{r}^2 - 1}\cosh\left(\frac{w^+ - w^-}{2R}\right)\right\}$. Similarly, we can write $A_{w^-}^{Rindler}$ in terms of $A_{x^+}^{Poincaré}$ and $A_{x^-}^{Poincaré}$ as:

$$A_{w^-}^{Rindler} = \frac{1}{C}\left[zA_{x^+}^{Poincaré} + \left(2\tilde{r}^2 - 1 + 2\tilde{r}\sqrt{\tilde{r}^2-1}\cosh\left(w^+/R\right)\right) zA_{x^-}^{Poincaré}\right] \quad (4.26)$$

Using (4.19) and (4.21) in (4.11), we have

$$\begin{aligned} zA_{x^+}^{Poincaré} &= \frac{1}{4\pi}\int_S \frac{1}{\left(\frac{dw^{+\prime}}{dx^{+\prime}}\right)\left(\frac{dw^{-\prime}}{dx^{-\prime}}\right)} dw^{+\prime}dw^{-\prime}\left(\frac{dw^{+\prime}}{dx^{+\prime}}\right) j_{w^+}^{Rindler}(w^{+\prime}, w^{-\prime}) \\ zA_{x^+}^{Poincaré} &= \frac{1}{4\pi}\int_S \frac{1}{\left(\frac{dw^{-\prime}}{dx^{-\prime}}\right)} dw^{+\prime}dw^{-\prime} j_{w^+}^{Rindler}(w^{+\prime}, w^{-\prime}) \\ zA_{x^+}^{Poincaré} &= \frac{1}{4\pi}\int_S dw^{+\prime}dw^{-\prime}\left[\frac{(e^{w^{-\prime}/R} - 1)^2}{2e^{w^{-\prime}/R}}\right] j_{w^+}^{Rindler}(w^{+\prime}, w^{-\prime}) \end{aligned} \quad (4.27)$$

Similarly, we find

$$zA_{x^-}^{Poincaré} = \frac{1}{4\pi}\int_S dw^{+\prime}dw^{-\prime}\left[\frac{(e^{w^{+\prime}/R}-1)^2}{2e^{w^{+\prime}/R}}\right] j_{w^-}^{Rindler}(w^{+\prime}, w^{-\prime}) \tag{4.28}$$

Thus, the representation of bulk field $A_{w^+}^{Rindler}(w^+, w^-, r)$ in terms of boundary spin-one conformal operators $j_{w^+}^{Rindler}(w^{+\prime}, w^{-\prime})$ and $j_{w^-}^{Rindler}(w^{+\prime}, w^{-\prime})$ is

$$\begin{aligned} A_{w^+}^{Rindler}(w^+, w^-, r) &= \frac{1}{4\pi C}\Bigg[\int_S dw^{+\prime}dw^{-\prime}\left[\frac{(e^{w^{+\prime}/R}-1)^2}{2e^{w^{+\prime}/R}}\right] j_{w^-}^{Rindler}(w^{+\prime}, w^{-\prime}) \\ &\quad + \left(2\tilde{r}^2 - 1 + 2\tilde{r}\sqrt{\tilde{r}^2-1}\cosh(w^-/R)\right) \\ &\quad \int_S dw^{+\prime}dw^{-\prime}\left[\frac{(e^{w^{-\prime}/R}-1)^2}{2e^{w^{-\prime}/R}}\right] j_{w^+}^{Rindler}(w^{+\prime}, w^{-\prime})\Bigg] \end{aligned} \tag{4.29}$$

Similarly, the representation of bulk field $A_{w^-}^{Rindler}(w^+, w^-, r)$ in terms of boundary spin-one conformal operators $j_{w^+}^{Rindler}(w^{+\prime}, w^{-\prime})$ and $j_{w^-}^{Rindler}(w^{+\prime}, w^{-\prime})$ is

$$\begin{aligned} A_{w^-}^{Rindler}(w_+, w_-, r) &= \frac{1}{4\pi C}\Bigg[\int_S dw^{+\prime}dw^{-\prime}\left[\frac{(e^{w^{-\prime}/R}-1)^2}{2e^{w^{-\prime}/R}}\right] j_{w^+}^{Rindler}(w^{+\prime}, w^{-\prime}) \\ &\quad + \left(2\tilde{r}^2 - 1 + 2\tilde{r}\sqrt{\tilde{r}^2-1}\cosh(w_+/R)\right) \\ &\quad \int_S dw^{+\prime}dw^{-\prime}\left[\frac{(e^{w^{+\prime}/R}-1)^2}{2e^{w^{+\prime}/R}}\right] j_{w^-}^{Rindler}(w^{+\prime}, w^{-\prime})\Bigg] \end{aligned} \tag{4.30}$$

where, the region of integration S is $\cos\left(\frac{w^+ + w^-}{2R}\right) = \sqrt{1-\tilde{r}^2}\cosh\left(\frac{w^+ - w^-}{2R}\right)$. The expressions (4.29) and (4.30) give the bulk reconstruction of bulk gauge fields $A_{w^\pm}^{Rindler}(w^+, w^-, r)$ which are inside the right Rindler wedge in Rindler lightcone coordinates. As shown in figure (4.1), these bulk fields are reconstructed in terms of only right Rindler boundary currents. Even though we have $A_r^{Rindler}$, the radial component of bulk gauge field in Rindler coordinates, but a deeper look in the holographic gauge condition $A_z^{Poincaré} = 0$ in Poincaré coordinates reveals that $A_r^{Rindler}$ is dependent on $A_{w^+}^{Rindler}$ and

$A_{w^-}^{Rindler}$ as:

$$A_r^{Rindler} = -\left\{ \left(\frac{\frac{\partial w^+}{\partial z}}{\frac{\partial r}{\partial z}} \right) A_{w^+}^{Rindler} + \left(\frac{\frac{\partial w^-}{\partial z}}{\frac{\partial w^+}{\partial z}} \right) A_{w^-}^{Rindler} \right\}$$

Thus, by finding the bulk reconstruction of $A_{w^+}^{Rindler}$ and $A_{w^-}^{Rindler}$, we can find the bulk reconstruction of $A_r^{Rindler}$.

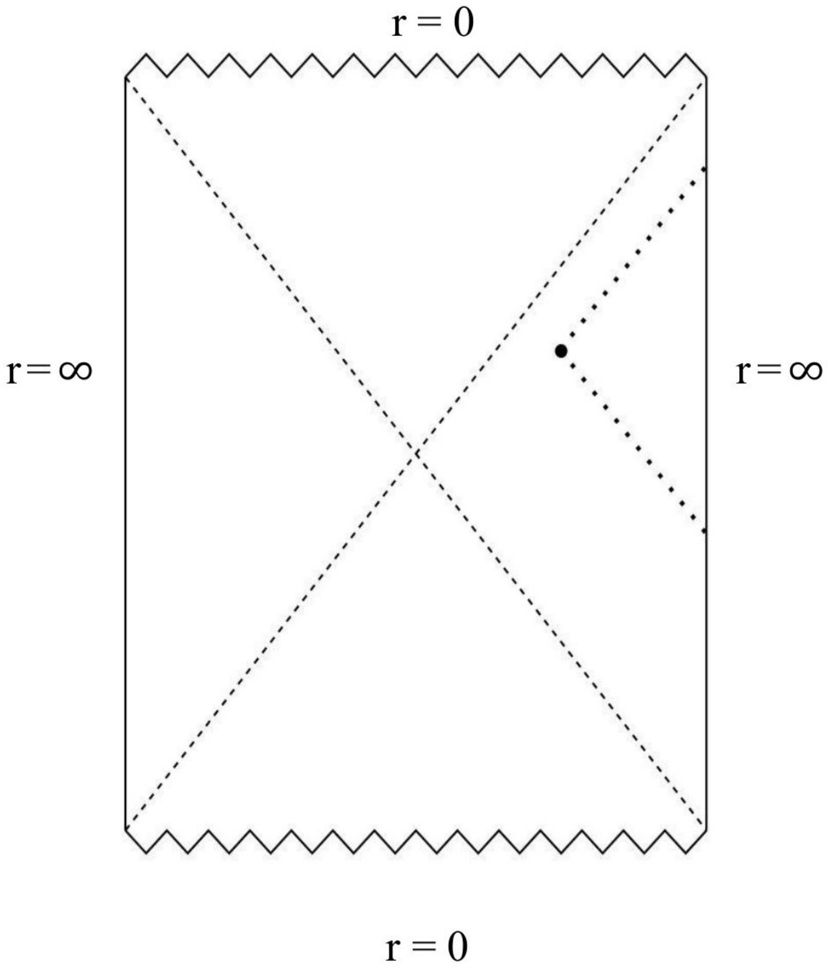

Figure 4.1: Diagram of a gauge field inside the right Rindler wedge represented by the black dot and its support on the boundary.

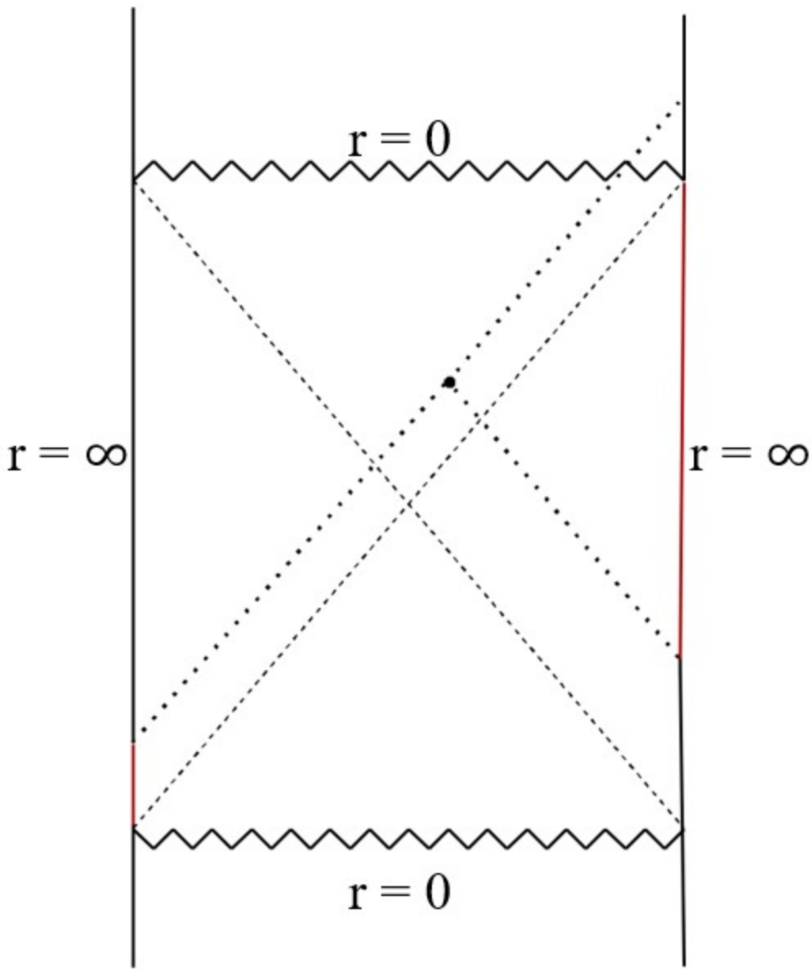

Figure 4.2: Diagram that shows how the smearing function support on the boundary changes for gauge fields inside the Rindler horizon when we impose an antipodal mapping.

Gauge fields inside the Rindler horizon

For bulk gauge fields inside the Rindler horizon ($r = r_+$), we impose an antipodal mapping A which acts on w^+ and w^- as $A\ :\ w^+ = w^+ + 2i\pi \quad \text{and} \quad w^- = w^-$. Fields with integer conformal dimension Δ transform simply under the antipodal map as,

$$j^{Rindler,R}(Ax) = (-1)^{\Delta} j^{Rindler,L}(x) \tag{4.31}$$

Thus, in our case we have,

$$j^{Rindler,R}_{w^{\pm}}(Ax) = -j^{Rindler,L}_{w^{\pm}}(x) \tag{4.32}$$

This gives the bulk reconstruction of gauge field $A^{(I)\,Rindler}_{w^+}(w^+, w^-, r)$ inside the Rindler horizon to be

$$
\begin{aligned}
A^{(I)\,Rindler}_{w^+}(w^+, w^-, r) \;=\; \\
\frac{1}{4\pi C}\Bigg[\int_{S_1} dw^{+\prime}dw^{-\prime}\left[\frac{(e^{w^{+\prime}/R}-1)^2}{2e^{w^{+\prime}/R}}\right] j^{Rindler,R}_{w^-}(w^{+\prime}, w^{-\prime}) \\
-\int_{S_2} dw^{+\prime}dw^{-\prime}\left[\frac{(e^{w^{+\prime}/R}-1)^2}{2e^{w^{+\prime}/R}}\right] j^{Rindler,L}_{w^-}(w^{+\prime}, w^{-\prime}) \\
+\Big(2\tilde{r}^2-1+2\tilde{r}\sqrt{\tilde{r}^2-1}\cosh\left(w^-/R\right)\Big) \\
\int_{S_1} dw^{+\prime}dw^{-\prime}\left[\frac{(e^{w^{-\prime}/R}-1)^2}{2e^{w^{-\prime}/R}}\right] j^{Rindler,R}_{w^+}(w^{+\prime}, w^{-\prime}) \\
-\Big(2\tilde{r}^2-1+2\tilde{r}\sqrt{\tilde{r}^2-1}\cosh\left(w^-/R\right)\Big) \\
\int_{S_2} dw^{+\prime}dw^{-\prime}\left[\frac{(e^{w^{-\prime}/R}-1)^2}{2e^{w^{-\prime}/R}}\right] j^{Rindler,L}_{w^+}(w^{+\prime}, w^{-\prime})\Bigg] \qquad (4.33)
\end{aligned}
$$

Similarly, the bulk reconstruction of gauge field $A^{(I)\,Rindler}_{w_-}(w_+, w_-, r)$ inside the Rindler horizon is

$$
\begin{aligned}
A^{(I)\,Rindler}_{w^-}(w^+, w^-, r) \;=\; \\
\frac{1}{4\pi C}\Bigg[\int_{S_1} dw^{+\prime}dw^{-\prime}\left[\frac{(e^{w^{-\prime}/R}-1)^2}{2e^{w^{-\prime}/R}}\right] j^{Rindler,R}_{w^+}(w^{+\prime}, w^{-\prime}) \\
-\int_{S_2} dw^{+\prime}dw^{-\prime}\left[\frac{(e^{w^{-\prime}/R}-1)^2}{2e^{w^{-\prime}/R}}\right] j^{Rindler,L}_{w^+}(w^{+\prime}, w^{-\prime}) \\
+\Big(2\tilde{r}^2-1+2\tilde{r}\sqrt{\tilde{r}^2-1}\cosh\left(w^+/R\right)\Big) \\
\int_{S_1} dw^{+\prime}dw^{-\prime}\left[\frac{(e^{w^{+\prime}/R}-1)^2}{2e^{w^{+\prime}/R}}\right] j^{Rindler,R}_{w^-}(w^{+\prime}, w^{-\prime}) \\
-\Big(2\tilde{r}^2-1+2\tilde{r}\sqrt{\tilde{r}^2-1}\cosh\left(w_-/R\right)\Big) \\
\int_{S_2} dw^{+\prime}dw^{-\prime}\left[\frac{(e^{w^{+\prime}/R}-1)^2}{2e^{w^{+\prime}/R}}\right] j^{Rindler,L}_{w^-}(w^{+\prime}, w^{-\prime})\Bigg] \qquad (4.34)
\end{aligned}
$$

The regions of integration S_1 is $\sigma_{IR} = 0$ or $\cos\left(\frac{w^+ + w^-}{2R}\right) = -\sqrt{\tilde{r}^2-1}$ $\sinh\left(\frac{w^+ - w^-}{2R}\right)$ and S_2 is $\sigma_{IL} = 0$ or $\cos\left(\frac{w^+ + w^-}{2R}\right) = \sqrt{\tilde{r}^2-1}\sinh\left(\frac{w^+ - w^-}{2R}\right)$. Thus, we obtain the representation of bulk field present at a point in the future Rindler wedge inside the Rindler horizon in terms of spin one conformal operators $j^{Rindler,R}_{w^+}(w^{+\prime}, w^{-\prime})$, $j^{Rindler,R}_{w^-}(w^{+\prime}, w^{-\prime})$ on the right boundary and spin one conformal operators $j^{Rindler,L}_{w^+}(w^{+\prime}, w^{-\prime})$, $j^{Rindler,L}_{w^-}(w^{+\prime}, w^{-\prime})$ on the left boundary. In (4.11) and (4.12), we see that $A^{Poincaré}_{x^+}$ and $A^{Poincaré}_{x^-}$ are represented in terms of $j^{Poincaré}_{x^+}$ and $j^{Poincaré}_{x^-}$ respectively, but in Rindler light-cone coordinates both $A^{Rindler}_{w^+}$ and $A^{Rindler}_{w^-}$ are represented as a function of both $j^{Rindler}_{w^+}$ and $j^{Rindler}_{w^-}$ as seen in (4.29), (4.30), (4.33) and (4.34). This is because the authors in [9] worked in the holographic gauge in Poincaré coordinates i.e. $A^{Poincaré}_{z} = 0$, but in our case in Rindler coordinates we have not chosen any gauge to work with. Also, while working in Rindler lightcone coordinates we can obtain the bulk reconstruction for light-like components of the bulk gauge fields only. We will be working in Rindler coordinates $(\tilde{t}, r, \tilde{\phi})$ in appendix C to resolve this limitation and to represent the space-like and time-like components of bulk gauge fields in terms of space-like and time-like boundary spin one conformal operators ($j_{\tilde{\phi}}$ and $j_{\tilde{t}}$).

4.4 Bulk reconstruction of Wilson line

In [14], [15], the authors suggested that in the presence of two boundaries, the expression for a charged operator inside the Rindler horizon in the future Rindler wedge must contain a contribution from a new gauge invariant operator which is a boundary-to-boundary Wilson line W_{LR}, in addition to well known smeared boundary operator contributions. In this section, we want to find the representation of this new operator W_{LR} in terms of spin one operators in both the boundary CFTs. Alternately, we can also study Wilson lines which connect charged scalar operators in the right/left Rindler wedges of the bulk to points in the right/left boundary to make the entire thing a gauge invariant bulk operator (but we will not focus on these Wilson lines in this section). Wilson loops are gauge invariant operators which are defined as:

$$W[\gamma] \;=\; \exp\left(i\int_{\gamma} A_\mu dx^\mu\right) \tag{4.35}$$

where, γ is a closed loop. Wilson lines can be defined in a similar fashion where the start and end points of the loop γ are not the same, thus Wilson lines are defined as

$$W[x_i, x_f] \;=\; \exp\left(i\int_{x_i}^{x_f} A_\mu dx^\mu\right) \tag{4.36}$$

Let us consider a Wilson line $W_{LR} = W[A, E]$ going from the left CFT_2 boundary of the AdS_3 Rindler patch to the right CFT_2 boundary. Since, we know representation of all the components of bulk gauge fields of all wedges in Rindler lightcone coordinates in terms of boundary spin one primary operators, we can easily obtain the representation of Wilson lines in terms of these boundary operators.

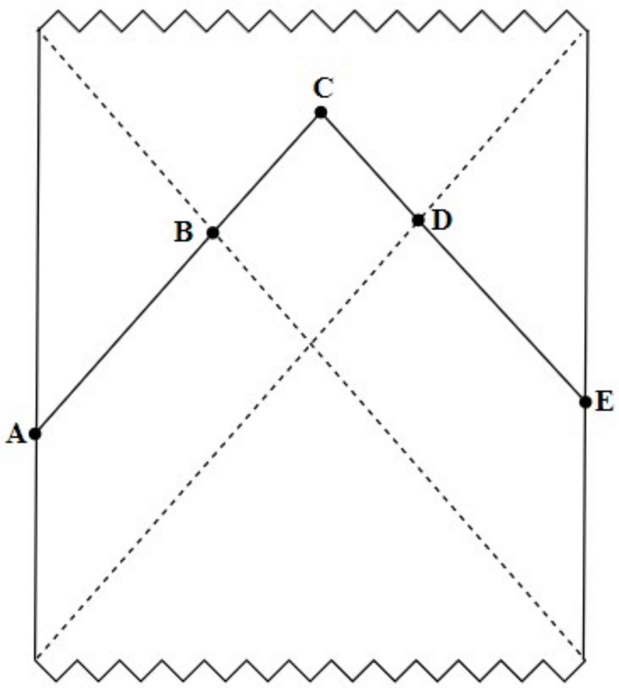

Figure 4.3: Wilson line going from a point A on the left CFT_2 boundary of Rindler patch to a point E on the right boundary. The dotted lines represent the Rindler horizon at $r = r_+$.

From the point A on the left CFT_2 boundary of the AdS_3 Rindler patch to the point B on the left Rindler horizon, we have

$$W[A,B] \;=\; \lim_{r\to r_+}\left\{\exp\left(i\int_A^B f_1\left(A_{w^+}^{(L)Rindler},\, A_{w^-}^{(L)Rindler}\right)dr\right)\right\} \tag{4.37}$$

here, f_1 is a function of the light-like components of a bulk gauge field in the left Rindler wedge. Using (4.29) and (4.30), we can express $W[A,B]$ as a function of spin one conformal operators $j_{w^+}^{Rindler,L}$ and $j_{w^-}^{Rindler,L}$ on the left boundary.

Again from B to C and from C to D, we have,

$$\begin{aligned} W[B,C] \;&=\; \lim_{r\to r_+}\left\{\exp\left(i\int_B^C f_2\left(A_{w^+}^{(I)Rindler},\, A_{w^-}^{(I)Rindler}\right)dr\right)\right\} \\ W[C,D] \;&=\; \lim_{r\to r_+}\left\{\exp\left(i\int_C^D f_3\left(A_{w^+}^{(I)Rindler},\, A_{w^-}^{(I)Rindler}\right)dr\right)\right\} \end{aligned} \tag{4.38}$$

here, f_2 and f_3 are two different functions of the light-like components of a bulk gauge field in the future Rindler wedge. Using (4.33) and (4.34), we can express $W[B,C]$ and $W[C,D]$ as functions of both spin one conformal operators $j_{w^+}^{Rindler,L}$ and $j_{w^-}^{Rindler,L}$ on the left boundary and spin one conformal operators $j_{w^+}^{Rindler,R}$ and $j_{w^-}^{Rindler,R}$ on the right boundary. Similarly, from a point D on the right Rindler horizon to a point E on the right CFT_2 boundary of the AdS_3 Rindler patch, we have,

$$W[D,E] \;=\; \lim_{r\to r_+}\left\{\exp\left(i\int_D^E f_4\left(A_{w^+}^{(R)Rindler},\, A_{w^-}^{(R)Rindler}\right)dr\right)\right\} \tag{4.39}$$

here, f_4 is a function of the light-like components of a bulk gauge field in the left Rindler wedge. Using (4.29) and (4.30), we can express $W[D,E]$ as a function of spin one conformal operators $j_{w^+}^{Rindler,R}$ and $j_{w^-}^{Rindler,R}$ on the right boundary.

Thus, using the expressions for $W[A,B]$, $W[B,C]$, $W[C,D]$, and $W[D,E]$ from (4.37), (4.38) and (4.39) in (4.36), we can find the expression for the entire Wilson line $W[A,E]$ which connects the left boundary of the Rindler patch with the right boundary. Thus, we have an expression for the representation of a Wilson line that stretches between the two boundaries of the Rindler patch in terms of some functions of spin one CFT operators of the right boundary of the Rindler patch ($j_{w^+}^{Rindler,R}$ and $j_{w^-}^{Rindler,R}$) and spin one CFT operators of the left boundary of the Rindler patch ($j_{w^+}^{Rindler,L}$ and $j_{w^-}^{Rindler,L}$).

5 Summary and outlook

In this thesis, we found out expressions for the representation of bulk gauge fields in Rindler lightcone coordinates (w^+, w^-, r) in terms of boundary spin one conformal operators $j^{Rindler}_{w^\pm}$. In chapter 4, we started with the revision of bulk reconstruction of gauge fields in Poincaré coordinates as done in [9] in section 4.1. Then, we define Poincaré lightcone coordinates as $x^\pm = x \pm t$ and obtained the bulk reconstruction result for gauge fields in Poincaré lightcone coordinates as given in (4.11) and (4.12) of the section 4.2. Since in [9], the authors used the holographic gauge conditions in Poincaré coordinates i.e. $A^{Poincaré}_z = 0$, we can see in (4.11) that $A^{Poincaré}_{x^+}$ is represented only in terms of $j^{Poincaré}_{x^+}$ and similarly in (4.12), $A^{Poincaré}_{x^-}$ is represented only in terms of $j^{Poincaré}_{x^-}$. In section 4.3, we define the Rindler lightcone coordinates $w^\pm = \tilde{\phi} \pm \tilde{t}$, where $\tilde{\phi}$ is the Rindler spatial coordinate and $\tilde{t}$ is the Rindler temporal coordinate. Equations (4.14), (4.15) and (4.16) are the relations between the Poincaré lightcone coordinates (x^+, x^-, z) and Rindler lightcone coordinates (w^+, w^-, r). Using these coordinate transformation relations and the relation between Poincaré boundary spin one conformal operators $j^{Poincaré}_{x^\pm}$ and Rindler boundary spin one conformal operators $j^{Rindler}_{w^\pm}$ given in (4.21) and (4.22), we obtain the representation of both the light-like components of bulk gauge field in Rindler coordinates in terms of Rindler boundary spin one conformal operators. (4.29) and (4.30) are the expressions which represent bulk fields in the right/left Rindler wedge $A^{Rindler}_{w^+}$ and $A^{Rindler}_{w^-}$ in terms of combinations of $j^{Rindler}_{w^+}$ and $j^{Rindler}_{w^-}$. Similarly, (4.33) and (4.34) are expressions which represent bulk fields in the future Rindler wedge $A^{(I)Rindler}_{w^+}$ and $A^{(I)Rindler}_{w^-}$ in terms of $j^{Rindler,L}_{w^+}$, $j^{Rindler,R}_{w^+}$, $j^{Rindler,L}_{w^-}$ and $j^{Rindler,R}_{w^-}$. Unlike in Poincaré coordinates, we find that in Rindler light-cone coordinates both $A^{Rindler}_{w^+}$ and $A^{Rindler}_{w^-}$ are represented as a function of both $j^{Rindler}_{w^+}$ and $j^{Rindler}_{w^-}$, this is because unlike in Poincaré coordinates we have not worked with any particular gauge in Rindler coordinates. Finally in section 4.4, we try to find the representation of a Wilson line (figure 4.3) that stretches from the left CFT_2 boundary to right CFT_2 boundary of the AdS_3 Rindler patch in terms of boundary spin one operators using (4.29), (4.30), (4.33) and (4.34).

In [14], Harlow suggests that cutting of Wilson lines requires the existence of bulk fields of fundamental charge. These oppositely charged

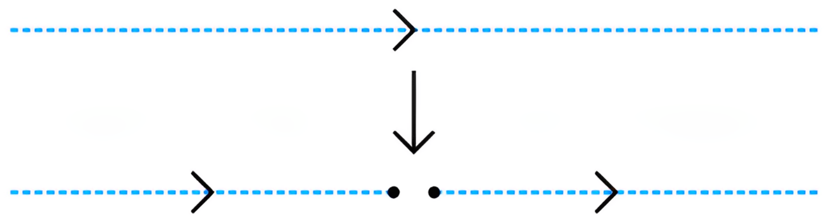

Figure 5.1: Cutting a Wilson line by a pair of oppositely charged fields.

fields are related to "edge modes" [16]. As possible future work, we want to check for the existence of these edge modes. We want to develop a full mathematical formulation of this cutting of Wilson lines. We want to check while cutting the Wilson line $W[A, E]$ which connects the left boundary of the Rindler patch with the right boundary, whether the bulk reconstruction result obtained in the previous section reduces to bulk reconstruction of gauge fields coupled with charged scalar fields as done in [17].

A

Poincaré to Rindler transformation

$(t,\, x,\, z)$ are the Poincaré coordinates and $(\hat{t},\, r,\, \hat{\phi})$ are the Rindler coordinates of AdS_3. The coordinate transformation between these two sets of coordinates is given the following relations:

$$t = \frac{R\sqrt{\tilde{r}^2 - 1}\, \sinh(\hat{t})}{\sqrt{\tilde{r}^2 - 1}\cosh(\hat{t}) + \tilde{r}\, \cosh(\hat{\phi})} \tag{A.1}$$

$$x = \frac{R\,\tilde{r}\, \sinh(\hat{\phi})}{\sqrt{\tilde{r}^2 - 1}\cosh(\hat{t}) + \tilde{r}\, \cosh(\hat{\phi})} \tag{A.2}$$

$$z = \frac{R}{\sqrt{\tilde{r}^2 - 1}\cosh(\hat{t}) + \tilde{r}\, \cosh(\hat{\phi})} \tag{A.3}$$

where, $\tilde{r} = \frac{r}{r_+}$.

These coordinate transformations in the boundary where $r \to \infty$ reduces to the following relations between the $(t,\, x)$ and $(\hat{t},\, \hat{\phi})$ coordinates.

$$t = \frac{R\, \sinh(\hat{t})}{\cosh(\hat{t}) + \cosh(\hat{\phi})} \tag{A.4}$$

$$x = \frac{R \sinh(\hat{\phi})}{\cosh(\hat{t}) + \cosh(\hat{\phi})} \tag{A.5}$$

The jacobian of transformation between Poincaré boundary coordinates to Rindler boundary coordinates is the following:

$$dt\, dx \;=\; \begin{vmatrix} \frac{dt}{d\hat{t}} & \frac{dt}{d\hat{\phi}} \\ \frac{dx}{d\hat{t}} & \frac{dx}{d\hat{\phi}} \end{vmatrix} d\hat{t}\, d\hat{\phi}$$

which gives,

$$dt\,dx \;=\; \lim_{r\to\infty}\left(\frac{r\,z}{r_+}\right)^2\, d\hat{t}\,d\hat{\phi} \tag{A.6}$$

However, while doing the bulk reconstruction for scalar and gauge fields in Rindler coordinates, we find that it is easier to work with dimension-full coordinates defined as $\tilde{t} = R\,\hat{t}$ and $\tilde{\phi} = R\,\hat{\phi}$, as this makes the conformal factor Ω in the transformation from Poincaré to Rindler coordinates dimensionless. The the coordinate transformation relations (A.1), (A.2) and (A.3) then becomes,

$$t = \frac{R\sqrt{\tilde{r}^2-1}\,\sinh(\tilde{t}/R)}{\sqrt{\tilde{r}^2-1}\cosh(\tilde{t}/R)+\tilde{r}\,\cosh(\tilde{\phi}/R)} \tag{A.7}$$

$$x = \frac{R\,\tilde{r}\,\sinh(\tilde{\phi}/R)}{\sqrt{\tilde{r}^2-1}\cosh(\tilde{t}/R)+\tilde{r}\,\cosh(\tilde{\phi}/R)} \tag{A.8}$$

$$z = \frac{R}{\sqrt{\tilde{r}^2-1}\cosh(\tilde{t}/R)+\tilde{r}\,\cosh(\tilde{\phi}/R)} \tag{A.9}$$

and, the boundary coordinate transformation relations (A.4) and (A.5) then becomes,

$$t = \frac{R\,\sinh(\frac{\tilde{t}}{R})}{\cosh(\frac{\tilde{t}}{R})+\cosh(\frac{\tilde{\phi}}{R})} \tag{A.10}$$

$$x = \frac{R\,\sinh(\frac{\tilde{\phi}}{R})}{\cosh(\frac{\tilde{t}}{R})+\cosh(\frac{\tilde{\phi}}{R})} \tag{A.11}$$

The Poincaré boundary metric is $ds^2 = -dt^2 + dx^2$.
Using the transformation relation $g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}(x)$ and (A.10) and (A.11), we get,

$$g^{\text{Rindler}}_{\mu\nu}(x) \;=\; \frac{1}{\left[\cosh(\frac{\tilde{t}}{R})+\cosh(\frac{\tilde{\phi}}{R})\right]^2}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} \tag{A.12}$$

Thus, the conformal factor is $\Omega = \frac{1}{\left[\cosh(\frac{\tilde{t}}{R})+\cosh(\frac{\tilde{\phi}}{R})\right]^2}$. Using the coordinate relations (A.10) and (A.11) the Jacobian of transformation between Poincaré boundary coordinates to Rindler boundary coordinates is the following:

$$dt\,dx \;=\; \begin{vmatrix} \frac{dt}{d\tilde{t}} & \frac{dt}{d\tilde{\phi}} \\ \frac{dx}{d\tilde{t}} & \frac{dx}{d\tilde{\phi}} \end{vmatrix} d\tilde{t}\,d\tilde{\phi}$$

which gives,

$$dt\,dx \;=\; \lim_{r\to\infty}\left(\frac{r\,z}{r_+R}\right)^2 d\tilde{t}\,d\tilde{\phi} \tag{A.13}$$

And, conformal factor Ω can be written in terms of r, z, r_+, R as the following:

$$\Omega \;=\; \frac{1}{\left[\cosh(\frac{\tilde{t}}{R}) + \cosh(\frac{\tilde{\phi}}{R})\right]^2} \;=\; \lim_{r\to\infty}\left(\frac{r\,z}{r_+R}\right)^2 \tag{A.14}$$

$\mathcal{B}$

Alternate derivation for bulk reconstruction of scalar fields in Rindler coordinates

In this appendix, we obtain the representation of bulk scalar fields in Rindler coordinates $(\hat{t},\, r,\, \hat{\phi})$ in terms of scalar primary operators in the CFT_2 boundary of the AdS_3 Rindler patch by using the coordinate transformation relations between the Poincaré coordinates $(t,\, z,\, x)$ and Rindler coordinates $(\hat{t},\, r,\, \hat{\phi})$ as done by the authors in [5]. According to the HKLL prescription [5], we can express the value of bulk field anywhere in the right Rindler wedge in terms of data on the right Rindler boundary. We define the boundary field in Rindler coordinates as

$$\phi_0^{Rindler,R}(\hat{t},\hat{\phi}) = \lim_{r\to\infty} r^{\Delta}\ \phi(\hat{t},r,\hat{\phi})\Big|_{right\,boundary} \tag{B.1}$$

B.1 Rindler smearing function

We start with the bulk scalar field reconstruction in Poincaré coordinates (3.21) and set $d = 2$ for AdS_3 and obtain

$$\phi(t,x,z) = \frac{\Delta-1}{\pi}\int_{t'^2+y'^2<z^2} dt'dy' \lim_{z'\to 0}\,(2\sigma z')^{\Delta-2}\phi_0(t+t',x+iy') \tag{B.2}$$

The right boundary field in Rindler coordinates given by (B.1) is related to the Poincare boundary field (3.1) as

$$\phi_0^{Rindler,\,R}(\hat{t},\hat{\phi}) = \lim_{r\to\infty}(rz)^{\Delta}\phi_0^{Poincaré}(t,z) \tag{B.3}$$

We also have the boundary change of coordinates from (A.6)

$$dt\,dx \;=\; \lim_{r\to\infty}\left(\frac{r\,z}{r_+}\right)^2 d\hat{t}\,d\hat{\phi} \tag{B.4}$$

Using the relations (B.3) and (B.4) in equation for bulk field (B.2)

$$\phi(\hat{t},r,\hat{\phi}) = \frac{(\Delta-1)2^{\Delta-2}}{\pi r_+^2}\lim_{z'\to 0}\int r'^2 z'^2 d\hat{t}' d\hat{\phi}' (\sigma z')^{\Delta-2}\lim_{r'\to\infty}\left(\frac{1}{z' r'}\right)^{\Delta} \phi_0^{Rindler,R}(\hat{t}+\hat{t}',\hat{\phi}+i\hat{\phi}')$$

$$\Rightarrow \phi(\hat{t},r,\hat{\phi}) = \frac{(\Delta-1)2^{\Delta-2}}{\pi r_+^2}\int_{Spacelike} d\hat{t}' d\hat{\phi}' \lim_{r'\to\infty}\left(\frac{\sigma}{r'}\right)^{\Delta-2} \phi_0^{Rindler,R}(\hat{t}+\hat{t}',\hat{\phi}+i\hat{\phi}') \tag{B.5}$$

where as $r'\to\infty$ the AdS-invariant distance (2.18) becomes

$$\sigma(\hat{t},r,\hat{\phi}|\hat{t}+\hat{t}',r,\hat{\phi}+i\hat{\phi}') = \frac{rr'}{r_+^2}\left[\cos\hat{\phi}' - \sqrt{1-\frac{r_+^2}{r^2}}\cosh\hat{t}'\right] \tag{B.6}$$

and the integration is over space-like separated points on the boundary and $\sigma > 0$. This result in these coordinates was obtained by the authors in [5].

B.1.1 Scalar fields inside the Rindler horizon

For scalar fields inside the Rindler horizon we impose an antipodal mapping: $A\;:\;\hat{t}\to\hat{t}+i\pi$, and $\hat{\phi}\to\hat{\phi}+i\pi$, under which the AdS invariant distance transform as $\sigma_{IR}(x|Ax') = -\sigma_{IL}(x|x')$ and the boundary scalar primary transform as $\phi_0^{Rindler,R}(A\,x) = (-1)^{\Delta}\phi_0^{Rindler,L}(x)$ to find the bulk reconstruction of a scalar field inside the Rindler horizon,

$$\begin{aligned}\phi(\hat{t},r,\hat{\phi}) = \frac{(\Delta-1)2^{\Delta-2}}{\pi r_+^2}\Bigg[&\int_{\sigma_{IR}>0} d\hat{t}' d\hat{\phi}' \lim_{r'\to\infty}\left(\frac{\sigma_{IR}}{r'}\right)^{\Delta-2} \phi_0^{Rindler,R}(\hat{t}+\hat{t}',\hat{\phi}+i\hat{\phi}')\\ &+\int_{\sigma_{IL}<0} d\hat{t}' d\hat{\phi}' \lim_{r'\to\infty}\left(\frac{-\sigma_{IL}}{r'}\right)^{\Delta-2} (-1)^{\Delta}\,\phi_0^{Rindler,L}(\hat{t}+\hat{t}',\hat{\phi}+i\hat{\phi}')\Bigg]\end{aligned} \tag{B.7}$$

where, as $r \to \infty$, the AdS invariant distance σ_{IR} in $(\hat{t},\, r,\, \hat{\phi})$ coordinates is:

$$\sigma_{IR}(\hat{t}, r, \hat{\phi}|\hat{t}+\hat{t}', r, \hat{\phi}+i\hat{\phi}') = \frac{rr'}{r_+^2}\left[\cos\hat{\phi}' + \sqrt{\frac{r_+^2}{r^2}-1}\,\sinh\hat{t}'\right] \tag{B.8}$$

and the AdS invariant distance σ_{IL} in $(\hat{t},\, r,\, \hat{\phi})$ coordinates is:

$$\sigma_{IR}(\hat{t}, r, \hat{\phi}|\hat{t}+\hat{t}', r, \hat{\phi}+i\hat{\phi}') = \frac{rr'}{r_+^2}\left[\cos\hat{\phi}' - \sqrt{\frac{r_+^2}{r^2}-1}\,\sinh\hat{t}'\right] \tag{B.9}$$

B.2 Rindler smearing function when $\Delta = d - 1 = 1$

Scalar field reconstruction in Poincaré coordinates for bulk scalar fields whose boundary counterpart has conformal dimensions $\Delta = d - 1$ is given by (3.25), putting $d = 2$ i.e. $\Delta = 1$ there we obtain:

$$\phi(t, z, x) = \frac{1}{2\pi}\int_{t'^2+y'^2=z^2} dt'dy'\,\delta(\sigma z')\phi_0(t, x) \tag{B.10}$$

Using the relations (B.3) and (B.4) in equation for bulk field (B.10), we obtain,

$$\begin{aligned}
\phi(\hat{t}, r, \hat{\phi}) &= \frac{1}{2\pi r_+^2}\int_S d\hat{t}'d\hat{\phi}' \lim_{r'\to\infty} r'z'\delta(\sigma z')\,\phi_0^{Rindler}(\hat{t}+\hat{t}', \hat{\phi}+i\hat{\phi}') \\
\phi(\hat{t}, r, \hat{\phi}) &= \frac{1}{2\pi r_+^2}\int_S d\hat{t}'d\hat{\phi}' \lim_{r'\to\infty} r'z'\delta\left(\frac{r'\sigma z'}{r'}\right)\,\phi_0^{Rindler}(\hat{t}+\hat{t}', \hat{\phi}+i\hat{\phi}') \\
\phi(\hat{t}, r, \hat{\phi}) &= \frac{1}{2\pi r_+^2}\int_S d\hat{t}'d\hat{\phi}'\,\delta\left(\frac{\sigma}{r'}\right)\,\phi_0^{Rindler}(\hat{t}+\hat{t}', \hat{\phi}+i\hat{\phi}')
\end{aligned} \tag{B.11}$$

(B.11) above represents the bulk reconstruction of scalar fields $\phi(\hat{t}, r, \hat{\phi})$ in the right Rindler wedge with conformal dimension $\Delta = d - 1$, where, the region of integration S is $\sigma = 0$ or $\cos\hat{\phi}' = \sqrt{1-\frac{r_+^2}{r^2}}\cosh\hat{t}'$.
For a bulk scalar field at the point inside the Rindler horizon, we use the same antipodal mapping used in the above section to obtain the result

(B.7) and obtain,

$$
\begin{aligned}
\phi(\hat{t}, r, \hat{\phi}) =& \frac{1}{2\pi r_+^2} \left[\int_{S_1} d\hat{t}' d\hat{\phi}' \, \delta\left(\frac{\sigma}{r'}\right) \phi_0^{Rindler,R}(\hat{t}+\hat{t}', \hat{\phi}+i\hat{\phi}') - \right. \\
& \left. \int_{S_2} d\hat{t}' d\hat{\phi}' \, \delta\left(\frac{-\sigma}{r'}\right) \, \phi_0^{Rindler,L}(\hat{t}+\hat{t}', \hat{\phi}+i\hat{\phi}') \right]
\end{aligned} \tag{B.12}
$$

where, the regions of integration S_1 and S_2 are respectively,

$$
S_1: \quad \cos\hat{\phi}' \; = \; -\sqrt{\frac{r_+^2}{r^2} - 1} \sinh \hat{t}'
$$

$$
S_2: \quad \cos\hat{\phi}' \; = \; +\sqrt{\frac{r_+^2}{r^2} - 1} \sinh \hat{t}'
$$

C
Gauge field reconstruction in Rindler coordinates $(\tilde{t}, r, \tilde{\phi})$

In this appendix, we want to find the representation of the time-like and space-like components $A_{\tilde{t}}^{Rindler}$ and $A_{\tilde{\phi}}^{Rindler}$ of bulk gauge fields in the AdS_3 Rindler patch in terms of boundary spin one conformal operators $j_{\tilde{t}}^{Rindler}$ and $j_{\tilde{\phi}}^{Rindler}$. In section 4.3, while working in Rindler lightcone coordinates, we obtained only the expressions for bulk reconstruction of light-like components of bulk gauge fields in Rindler coordinates. This method is useful to find the representation of a Wilson line that stretches between the two boundaries of the Rindler patch in terms of functions of the time-like and space-like components of spin one CFT operators of the boundary.

From [9], we know that bulk gauge field $A_{\mu}^{Poincaré}$ satisfies the following equation

$$(\Box - m^2) z A_{\mu}^{Poincaré} = 0 \qquad \text{with, } m^2 = 1 - d \tag{C.1}$$

and, the boundary behaviour of bulk gauge field $A_{\mu}^{Poincaré}$ is the following:

$$\begin{aligned} A_{\mu}^{Poincaré} &= \lim_{z\to 0} z^{\Delta-1} j_{\mu}^{Poincaré} \\ \text{or, } zA_{\mu}^{Poincaré} &= \lim_{z\to 0} z^{\Delta} j_{\mu}^{Poincaré} \end{aligned}$$

Thus, it can be interpreted that $zA_{\mu}^{Poincaré}$ behaves like a scalar field $\phi_{\mu}^{Poincaré}$ with a boundary counterpart $\mathcal{O}_{\mu}$, which behaves as a scalar primary with conformal dimension $\Delta = d-1$ and $\mathcal{O}_\mu \equiv j_\mu^{Poincaré}$. From (3.27), we know how scalar primary operators transform under conformal transformations and using the value of conformal factor Ω, we find the relation between $j_\mu^{Poincaré}$ and $j_\mu^{Rindler}$ to be

$$j_\mu^{Poincaré}(t,x) \;=\; \lim_{r'\to\infty}\left(\frac{Rr_+}{r'z'}\right)^{\Delta} j_\mu^{Rindler}(\tilde{t},\tilde{\phi}) \tag{C.2}$$

For AdS$_3$, we have $d=2$ and $\Delta=1$, thus

$$j_\mu^{Poincaré}(t,x) \;=\; \lim_{r'\to\infty}\left(\frac{Rr_+}{r'z'}\right) j_\mu^{Rindler}(\tilde{t},\tilde{\phi}) \tag{C.3}$$

which can be interpreted as $j_\mu(x) \;=\; \frac{\partial x'_\alpha}{\partial x_\mu} j_\alpha(x')$, thus

$$\begin{aligned} j_t^{Poincaré}(t,x) \;&=\; \lim_{r'\to\infty}\left(\frac{Rr_+}{r'z'}\right)\left\{ j_{\tilde{t}}^{Rindler}(\tilde{t},\tilde{\phi}) + j_{\tilde{\phi}}^{Rindler}(\tilde{t},\tilde{\phi})\right\} \\ j_x^{Poincaré}(t,x) \;&=\; \lim_{r'\to\infty}\left(\frac{Rr_+}{r'z'}\right)\left\{ j_{\tilde{t}}^{Rindler}(\tilde{t},\tilde{\phi}) + j_{\tilde{\phi}}^{Rindler}(\tilde{t},\tilde{\phi})\right\} \end{aligned} \tag{C.4}$$

Putting $d=2$ in (4.5), we get the bulk reconstruction of gauge fields in AdS$_3$ in Poincaré coordinates as the following

$$zA_\mu^{Poincaré}(t,x,z) = \frac{1}{2\pi}\int_{t'^2+y'^2=z^2} dt'dy'\delta(\sigma z') j_\mu^{Poincaré}(t+t',x+iy') \tag{C.5}$$

Using (C.4) and (A.13) in (B.4), we get:

$$\begin{aligned} zA_t^{Poincaré}(t,x,z) = \frac{1}{2\pi}\int_\Sigma d\tilde{t}'d\tilde{\phi}'\,\delta\left(\frac{Rr_+}{r'}\sigma\right)\Big[& j_{\tilde{t}}^{Rindler}(\tilde{t}+\tilde{t}',\tilde{\phi}+i\tilde{\phi}') + \\ & j_{\tilde{\phi}}^{Rindler}(\tilde{t}+\tilde{t}',\tilde{\phi}+i\tilde{\phi}')\Big] \\ zA_x^{Poincaré}(t,x,z) = \frac{1}{2\pi}\int_\Sigma d\tilde{t}'d\tilde{\phi}'\,\delta\left(\frac{Rr_+}{r'}\sigma\right)\Big[& j_{\tilde{t}}^{Rindler}(\tilde{t}+\tilde{t}',\tilde{\phi}+i\tilde{\phi}') + \\ & j_{\tilde{\phi}}^{Rindler}(\tilde{t}+\tilde{t}',\tilde{\phi}+i\tilde{\phi}')\Big] \end{aligned} \tag{C.6}$$

We know A_μ's are bulk spin-one gauge fields and should transform as vector under Lorentz transformation from Rindler to Poincaré coordinates i.e. as $A_\mu^{Rindler}(x') \;=\; \frac{\partial x^\alpha}{\partial x^{\mu\prime}} A_\alpha^{Poincaré}$. Thus, we can write $A_{\tilde{t},\tilde{\phi}}^{Rindler}$ in terms of

$A_{t,x}^{Poincaré}$ as

$$\begin{aligned} A_{\tilde{t}}^{Rindler} &= \frac{\partial t}{\partial \tilde{t}} A_t^{Poincaré} + \frac{\partial x}{\partial \tilde{t}} A_x^{Poincaré} \\ A_{\tilde{\phi}}^{Rindler} &= \frac{\partial t}{\partial \tilde{\phi}} A_t^{Poincaré} + \frac{\partial x}{\partial \tilde{\phi}} A_x^{Poincaré} \end{aligned} \tag{C.7}$$

Using the transformation relations (A.7) and (A.8) in (C.7), we obtain

$$\begin{aligned} A_{\tilde{t}}^{Rindler} = \frac{1}{D^2}\Bigg\{ &\left(\tilde{r}^2 - 1 + \tilde{r}\sqrt{\tilde{r}^2-1}\cosh(\tilde{t}/R)\cosh(\tilde{\phi}/R)\right) A_t^{Poincaré} \\ &- \left(\tilde{r}\sqrt{\tilde{r}^2-1}\sinh(\tilde{t}/R)\sinh(\tilde{\phi}/R)\right) A_x^{Poincaré}\Bigg\} \end{aligned} \tag{C.8}$$

and,

$$\begin{aligned} A_{\tilde{\phi}}^{Rindler} = \frac{1}{D^2}\Bigg\{ &\left(\tilde{r}^2 + \tilde{r}\sqrt{\tilde{r}^2-1}\cosh(\tilde{t}/R)\cosh(\tilde{\phi}/R)\right) A_x^{Poincaré} \\ &- \left(\tilde{r}\sqrt{\tilde{r}^2-1}\sinh(\tilde{t}/R)\sinh(\tilde{\phi}/R)\right) A_t^{Poincaré}\Bigg\} \end{aligned} \tag{C.9}$$

where, $D = \sqrt{\tilde{r}^2-1}\cosh(\tilde{t}/R) + \tilde{r}\cosh(\tilde{\phi}/R)$. Using the expressions of bulk gauge fields $A_t^{Poincaré}$ and $A_x^{Poincaré}$ from (C.6) in (C.8), we get the representation of bulk gauge field $A_{\tilde{t}}^{Rindler}$ in terms of boundary spin-one currents $j_{\tilde{t}}^{Rindler}$ and $j_{\tilde{\phi}}^{Rindler}$

$$\begin{aligned} A_{\tilde{t}}^{Rindler}(\tilde{t}, r, \tilde{\phi}) = \frac{1}{2\pi D}&\left\{\tilde{r}^2 - 1 + \tilde{r}\sqrt{\tilde{r}^2-1}\cosh\left(\frac{\tilde{t}-\tilde{\phi}}{R}\right)\right\} \\ &\int_\Sigma d\tilde{t}' d\tilde{\phi}'\, \delta\left(\frac{Rr_+}{r'}\sigma\right)\Big[j_{\tilde{t}}^{Rindler}(\tilde{t}+\tilde{t}', \tilde{\phi}+i\tilde{\phi}') + \\ &\qquad j_{\tilde{\phi}}^{Rindler}(\tilde{t}+\tilde{t}', \tilde{\phi}+i\tilde{\phi}')\Big] \end{aligned} \tag{C.10}$$

Similarly, we have,

$$
\begin{aligned}
A_{\tilde{\phi}}^{Rindler}(\tilde{t},r,\tilde{\phi}) \;=\; & \frac{1}{2\pi D}\left\{\tilde{r}^2+\tilde{r}\sqrt{\tilde{r}^2-1}\cosh\left(\frac{\tilde{t}-\tilde{\phi}}{R}\right)\right\} \\
& \int_{\Sigma} d\tilde{t}'d\tilde{\phi}'\,\delta\left(\frac{Rr_+}{r'}\sigma\right)\Big[j_{\tilde{t}}^{Rindler}(\tilde{t}+\tilde{t}',\tilde{\phi}+i\tilde{\phi}')+ \\
& \qquad j_{\tilde{\phi}}^{Rindler}(\tilde{t}+\tilde{t}',\tilde{\phi}+i\tilde{\phi}')\Big]
\end{aligned}
\tag{C.11}
$$

with $D = \sqrt{\tilde{r}^2-1}\cosh(\tilde{t}/R)+\tilde{r}\cosh(\tilde{\phi}/R)$ and the region of integration (Σ) is $\cos(\frac{\tilde{\phi}'}{R}) = \sqrt{1-\frac{r^2}{r_+^2}}\cosh(\frac{\tilde{t}'}{R})$. Thus, we find that the reconstructed expression for both the time-like and space-like components of bulk gauge field $A_{\tilde{t}}^{Rindler}(\tilde{t},r,\tilde{\phi})$ and $A_{\tilde{\phi}}^{Rindler}(\tilde{t},r,\tilde{\phi})$ respectively depends on combinations of both the time-like and space-like components of the spin one boundary conformal operator $j_{\tilde{t}}^{Rindler}(\tilde{t},\tilde{\phi})$ and $j_{\tilde{\phi}}^{Rindler}(\tilde{t},\tilde{\phi})$.

Gauge fields inside the Rindler horizon in the future Rindler wedge

For a bulk gauge field at the point inside the Rindler horizon, we use an antipodal mapping $A:\hat{t}\to\hat{t}+i\pi$, and $\hat{\phi}\to\hat{\phi}+i\pi$, under which the AdS invariant distance transform as $\sigma_{IR}(x|Ax') = -\sigma_{IL}(x|x')$ and the boundary spin one primary transform as $j_{\mu}^{Rindler,R}(A\,x) = (-1)^{\Delta} j_{\mu}^{Rindler,L}(x)$. For our case of AdS_3/CFT_2 we have:

$$
\begin{aligned}
j_{\tilde{t}}^{Rindler,R}(A\,x) &= -j_{\tilde{t}}^{Rindler,L}(x) \\
j_{\tilde{\phi}}^{Rindler,R}(A\,x) &= -j_{\tilde{\phi}}^{Rindler,L}(x)
\end{aligned}
$$

Thus, the expression for bulk field $A_{\tilde{t}}^{(I)Rindler}(\tilde{t},r,\tilde{\phi})$ in terms of boundary spin one CFT operators is:

$$
\begin{aligned}
A_{\tilde{t}}^{(I)Rindler}(\tilde{t},r,\tilde{\phi}) \;=\; & \frac{1}{2\pi D}\left\{\tilde{r}^2-1+\tilde{r}\sqrt{\tilde{r}^2-1}\cosh\left(\frac{\tilde{t}-\tilde{\phi}}{R}\right)\right\} \\
\int_{\Sigma_1} d\tilde{t}'d\tilde{\phi}'\,\delta\left(\frac{Rr_+}{r'}\sigma_{IR}\right) & \Big[j_{\tilde{t}}^{Rindler,R}(\tilde{t}+\tilde{t}',\tilde{\phi}+i\tilde{\phi}')+j_{\tilde{\phi}}^{Rindler,R}(\tilde{t}+\tilde{t}',\tilde{\phi}+i\tilde{\phi}')\Big] \\
-\int_{\Sigma_2} d\tilde{t}'d\tilde{\phi}'\,\delta\left(\frac{Rr_+}{r'}\sigma_{IL}\right) & \Big[j_{\tilde{t}}^{Rindler,L}(\tilde{t}+\tilde{t}',\tilde{\phi}+i\tilde{\phi}')+j_{\tilde{\phi}}^{Rindler,L}(\tilde{t}+\tilde{t}',\tilde{\phi}+i\tilde{\phi}')\Big]
\end{aligned}
\tag{C.12}
$$

and, the expression for bulk field $A^{(I)Rindler}_{\tilde{\phi}}(\tilde{t}, r, \tilde{\phi})$ in terms of boundary spin one CFT operators is:

$$\begin{aligned} A^{(I)Rindler}_{\tilde{\phi}}(\tilde{t}, r, \tilde{\phi}) &= \frac{1}{2\pi D}\left\{\tilde{r}^2 - 1 + \tilde{r}\sqrt{\tilde{r}^2-1}\cosh\left(\frac{\tilde{t}-\tilde{\phi}}{R}\right)\right\} \\ \int_{\Sigma_1} d\tilde{t}' d\tilde{\phi}'\, \delta\left(\frac{Rr_+}{r'}\sigma_{IR}\right) &\left[j^{Rindler,R}_{\tilde{t}}(\tilde{t}+\tilde{t}', \tilde{\phi}+i\tilde{\phi}') + j^{Rindler,R}_{\tilde{\phi}}(\tilde{t}+\tilde{t}', \tilde{\phi}+i\tilde{\phi}')\right] \\ -\int_{\Sigma_2} d\tilde{t}' d\tilde{\phi}'\, \delta\left(\frac{Rr_+}{r'}\sigma_{IL}\right) &\left[j^{Rindler,L}_{\tilde{t}}(\tilde{t}+\tilde{t}', \tilde{\phi}+i\tilde{\phi}') + j^{Rindler,L}_{\tilde{\phi}}(\tilde{t}+\tilde{t}', \tilde{\phi}+i\tilde{\phi}')\right] \end{aligned} \tag{C.13}$$

Here, the regions of integration Σ_1 and Σ_2 are respectively:

$$\Sigma_1: \quad \cos\hat{\phi}' = -\sqrt{\frac{r_+^2}{r^2}-1}\,\sinh\hat{t}'$$

$$\Sigma_2: \quad \cos\hat{\phi}' = +\sqrt{\frac{r_+^2}{r^2}-1}\,\sinh\hat{t}'$$

In the bulk reconstruction for gauge fields Poincaré coordinates as done in [9], the authors used holographic gauge i.e. $A^{Poincaré}_z = 0$ due to which they found the representation time-like and space-like components of the bulk gauge field in Poincaré coordinates in terms of the time-like and space-like components of the boundary spin one conformal operators $j^{Poincaré}_t$ and $j^{Poincaré}_x$ respectively. However, in our calculations in Rindler coordinates we haven't used any particular gauge choice, thus we can see from (C.10), (C.11), (C.12) and (C.13) that the representation of the time-like and space-like components of bulk gauge fields of any wedge of the Rindler patch depends on combinations of both the time-like and space-like components of the boundary spin one conformal operators $j^{Rindler}_{\tilde{t}}$ and $j^{Rindler}_{\tilde{\phi}}$.

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