Generalized Proofs for the Irrationality of $\zeta(2)$ and $\zeta(3)$

M.Sc. Thesis

by

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Generalized Proofs for the Irrationality of $\zeta(2)$ and $\zeta(3)$

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by

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DECLARATION BY CANDIDATE

I hereby affirm that the work which is being presented in this thesis entitled **"Generalized proofs for the irrationality of $\zeta(2)$ and $\zeta(3)$"** in the partial attainment of the requirements for the award of **MASTER OF SCIENCE** degree, submitted in the **DEPARTMENT OF MATHEMATICS**, **IIT Indore**, is an authentic record of my own work carried out from August 2022 to June 2023 under the supervision of **Dr. Bibekananda Maji**, an Assistant Professor in the Department of Mathematics, IIT Indore. The work illustrated by me in this thesis has not been submitted by anyone for the grant of any other degree of this or any other institute.

05/06/2023.

Student's signature with date

(**Muzamil Nazir Bhat**)

This is to certify that the statement made above by the candidate is accurate to the best of my knowledge.

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Date: 6 June 2023

Dedicated to my Family

No two things have been combined together better than knowledge and patience.

–***Prophet Muhammad (pbuh)***

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Abstract

Euler's famous formula for even zeta values immediately points out that they are irrational. Nevertheless, the arithmetic nature of odd zeta values remains a mystery. Roger Apéry [1], in 1978, made a breakthrough by proving that $\zeta(3)$ is irrational. Over the last four decades, many mathematicians have given different proofs of Apéry's theorem. The proof that Apéry presented was quite intricate however, Frits Beukers [4] gave an elementary proof for the same using the definite integrals. In this thesis, motivated by the elementary proof of irrationality of $\zeta(2)$ and $\zeta(3)$ due to Frits Beukers, we generalize some of the important lemmas, which played a crucial role in the Beukers' proof. We also investigate some expressions (multiple integrals) that seemingly look quite promising but lack the ability to prove the irrationality of some zeta-value. In this thesis, we generalize Beukers' proof to present a new proof of the irrationality of $\zeta(s)$ at $s = 2, 3$. We also mention a conjecture in which the integral expression is actually promising, proof of which may lead to the conclusion that all positive integer zeta-values are irrational.

Contents

CHAPTER 1

Introduction

Amongst the most popular special functions in mathematics is probably the Riemann zeta function due to the extremely challenging problem associated with it, namely, the Riemann hypothesis. The Riemann zeta function is denoted by Greek letter "ζ" and is defined as:

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}, \quad \text{for} \quad \Re(s) > 1,$$

and it has analytic continuation elsewhere, except at $s = 1$, where it has a simple pole. Before Riemann, Euler studied $\zeta(s)$ for even positive integers. In 1735, the following explicit formula was established by Euler for even zeta values:

$$\zeta(2m) = -\frac{1}{2}(2\pi i)^{2m}\frac{B_{2m}}{(2m)!}, \quad \forall \quad m \in \mathbb{N}, \tag{1.1}$$

where B_{2m} denotes the $2m$th Bernoulli number. The set of Bernoulli numbers is a subset of rational numbers, defined as for $|z| < 2\pi$,

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!}.$$

The Euler's formula (1.1) immediately concludes that even zeta values are all transcendental due to the presence of some power of π multiplied by some algebraic number. Although Ramanujan gave the following formula for odd zeta values as: For $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$ and $m \in \mathbb{Z}, m \neq 0$,

$$F(\alpha) = (-1)^m F(\beta) - 2^{2m} \sum_{j=0}^{m+1} \frac{(-1)^j B_{2j} B_{2m+2-2j}}{(2j)!(2m+2-2j)!} \alpha^{m+1-j} \beta^j,$$

where

$$F(x) = x^{-m} \left\{ \frac{1}{2}\zeta(2m+1) + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2xn}-1} \right\}.$$

But unlike Euler's formula, this formula does not say much about transcendence or irrationality of $\zeta(2m+1)$. Many mathematicians put a lot of effort in the direction of knowing whether the odd zeta values are algebraic or transcendental but none of them succeeded, although it is a widely believed conjecture that all odd ζ-values are transcendental. Even the question that whether the odd zeta values are rational or irrational is still open except for $\zeta(3)$, proved by Roger Apéry [1]. After that Rivoal [5] and in 2001, Rivoal and Ball [3] proved a ground breaking result in which they were able to prove the existence of infinitely many irrational zeta values at odd positive integers. Wadim Zudilin [7] by almost the same time showed that among $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ and $\zeta(11)$, at least one number should be irrational, which is the closest we have reached in this direction.

In 1978, Roger Apéry [1] gave an exceptional proof that $\zeta(3)$ is irrational after which, many proofs and expositions were given by different mathematicians like, F. Beukers [4], although the idea was motivated from the Apéry's proof. In this thesis we primarily study the Beukers' proof of irrationality of $\zeta(3)$, in which double and triple integrals are used, however, its idea is inspired by Apéry's formulas. Interestingly the technique is also good enough for proving the irrationality of $\zeta(2)$ like in Apéry's proof, although from Euler's formula it is clear that $\zeta(2)$ is irrational.

We would like to investigate many multiple-integral expression sequences which result into the expression of the form of

$$\frac{P_n + Q_n\zeta(K)}{R_n},$$

where $n, K \in \mathbb{N}$ and P_n, Q_n and R_n are sequence of integers. The reason to search for such integral expressions is that if the integrand tends to zero fast enough as $n \to \infty$, then we would be able to conclude the irrationality of that zeta-value. We also generalize the Beukers' proof of irrationality of $\zeta(2)$ and $\zeta(3)$, giving the family of integral expressions good enough to prove their irrationality. Finally we give two conjectures, one of which is very promising when it comes to proving the irrationality of all ζ-values on natural numbers greater than 1.

CHAPTER 2

Preliminaries

In this chapter, we define some important relevant functions and the theorems related to them. We also collect some essential results which will play crucial role in the Beukers' proof [4] of irrationality of $\zeta(2)$ and $\zeta(3)$. Let us first define the prime counting function followed by one of the important theorems related to it namely "The prime number theorem"(without proof).

Definition 2.1 (Prime Counting Function)**.** *Given $x \in \mathbb{R}$, we define*

$$\pi(x) := \#\{p \in \mathbb{N} : p \text{ is a prime and } 2 \leq p \leq [x]\}.$$

Theorem 2.1 (Prime Number Theorem)**.** *As $x \to \infty$, we have*

$$\pi(x) \sim \frac{x}{\log x}.$$

An important characterization of rational numbers and a criteria for irrationality is mentioned below.

A number β is rational if and only if for $l, m \in \mathbb{Z}$ $(m > 0)$ and $\beta \neq \frac{l}{m}$, there exists an integer $m_0 > 0$ such that

$$\left|\beta - \frac{l}{m}\right| \geq \frac{1}{mm_0}. \tag{2.1}$$

On the other hand, for an irrational number α there are always infinitely many $\frac{l}{m} \in \mathbb{Q}$ such that

$$\left|\alpha - \frac{l}{m}\right| < \frac{1}{m^2},$$

which yields the following criterion for irrationality. *Suppose for a sequence* $\{\frac{x_n}{y_n}\}$ *of rational numbers with* $\frac{x_n}{y_n} \neq \alpha$ *and* $\delta > 0$,

$$\left|\alpha - \frac{x_n}{y_n}\right| < \frac{1}{{y_n}^{1+\delta}}, \;\; n = 1, 2, \cdots,$$

then α *is irrational.*

Lemma 2.2. *Let* $K \in \mathbb{R}^+$, $\psi \in \mathbb{R}$, *and* (A_n, B_n) *be a sequence (with* $B_n \neq 0$*) in* $\mathbb{Z}^2$. *If* $0 < \delta < 1$, *such that*

$$0 < |A_n + B_n\psi| < K\delta^n,$$

for sufficiently large n, *then* ψ *is irrational.*

Proof. Given that

$$0 < |A_n + B_n\psi| < K\delta^n. \tag{2.2}$$

Therefore, the expression in the modulus sign can be made arbitrarily small for sufficiently large n. Now on contrary let ψ be a rational number say $\frac{p}{q}$ then by (2.1), for every $n \in \mathbb{N}$ we will have

$$|A_n + B_n\psi| = |B_n| \left|\psi - \left(-\frac{A_n}{B_n}\right)\right| \geq |B_n| \cdot \frac{1}{|B_n q|} = \frac{1}{|q|},$$

for some $q \in \mathbb{N}$. This is a contradiction to (2.2). □

Throughout this thesis l_n denotes the LCM$[\,1, 2, \cdots, n]$.

Lemma 2.3. *For any positive integer* $n \in \mathbb{N}$, *we have*

$$l_n = \prod_{\substack{p:\ prime \\ p \leq n}} p^{[\frac{\log n}{\log p}]} < \prod_{\substack{p:\ prime \\ p \leq n}} p^{\frac{\log n}{\log p}} = n^{\pi(n)}. \tag{2.3}$$

Furthermore, $n^{\pi(n)} < 3^n$ *for sufficiently large* n.

Proof. Let n be a fixed natural number. If p is a prime such that

$$p^m | \, l_n, \quad \text{and} \quad p^{m+1} \nmid l_n, \quad \text{for some} \quad m \in \mathbb{N},$$

then

$$p^m \leq n \Rightarrow m \leq \log n/ \log p \Rightarrow m = [\log n/ \log p].$$

Therefore,

$$l_n = \prod_{\substack{p:\ \text{prime} \\ p \leq n}} p^{[\log n/ \log p]} < \prod_{\substack{p:\ \text{prime} \\ p \leq n}} p^{\log n/ \log p}. \tag{2.4}$$

Now consider the right side expression,

$$\prod_{\substack{p:\ \text{prime} \\ p \leq n}} p^{\log n/ \log p} = \prod_{\substack{p:\ \text{prime} \\ p \leq n}} \exp(\log n) = \prod_{\substack{p:\ \text{prime} \\ p \leq n}} n \ = n^{\pi(n)}. \tag{2.5}$$

This completes the proof of (2.3). Now, from the prime number theorem, we know

$$\lim_{n \to \infty} \frac{\pi(n) \log(n)}{n} = 1.$$

Thus for any $\epsilon > 0$, one can find $S(\epsilon) \in \mathbb{N}$ such that

$$(1 - \epsilon) \ n/ \log n < \pi(n) < (1 + \epsilon \ )n/ \log n,$$

for all $n > S(\epsilon)$. Choose 'ϵ' such that $1+\epsilon = \log 3$, so $\exists \ K \in \mathbb{N}$ such that $\forall \ n > K$

$$\pi(n) \log n < n \log 3 \Rightarrow n^{\pi(n)} < 3^n, \quad \text{for all } n > K. \tag{2.6}$$

This finishes proof of the lemma. □

Remark 1. *Instead of '3' in "$1+\epsilon = \log 3$" any number $a \in \mathbb{R}$ such that $a > e$ can be chosen. Then we will have $n^{\pi(n)} < a^n$ for sufficiently large n.*

Now we mention an important lemma, due to Beukers, which played an important role in proving the irrationality of $\zeta(2)$ and $\zeta(3)$.

Lemma 2.4 (Beukers)**.** *For $r, s \in \mathbb{N} \cup \{0\}$, we define two definite integrals*

$$B_0[r, s] := \int_{[0,1]}^{(2)} \frac{x^r y^s}{1 - xy} \, dx \, dy, \quad B_1[r, s] := \int_{[0,1]}^{(2)} \frac{- \log xy}{1 - xy} x^r y^s \, dx \, dy. \tag{2.7}$$

1. *Then for $r > s$, we have*

$$B_0[r, s] = \frac{p_1}{q_1}, \tag{2.8}$$

where $p_1, q_1 \in \mathbb{Z}, q_1 \neq 0$. *Moreover,* $q_1 \big| l_r^2$.

2. *For* $r > s$, *we have*
$$B_1[r,s] = \frac{p_2}{q_2}, \tag{2.9}$$
with $q_2 \big| l_r^3$, *where* $p_2, q_2 \in \mathbb{Z}$ *and* $q_2 \neq 0$.

3. *For* $r = s$, *we have*
$$B_0[r,r] = \zeta(2) - \sum_{i=1}^{r} \frac{1}{i^2}, \tag{2.10}$$

4. *For* $r = s$, *we have*
$$B_1[r,r] = 2\left(\zeta(3) - \sum_{i=1}^{r} \frac{1}{i^3}\right). \tag{2.11}$$

Next, we establish the following simple generalization of Lemma 2.4.

Lemma 2.5. *For* $r, s, m \in \mathbb{N} \cup \{0\}$, *we define*
$$B_m[r,s] := \int_{[0,1]}^{(2)} \frac{(-1)^m (\log xy)^m}{1-xy} x^r y^s \, dx\, dy. \tag{2.12}$$
For $r > s$, *we have*
$$B_m[r,s] = \frac{P}{Q}, \tag{2.13}$$
with $Q \big| l_r^{m+2}$, *where* $P, Q \in \mathbb{Z}$ *and* $Q \neq 0$. *Again, when* $r = s$, *we have*
$$B_m[r,r] = (m+1)!\left(\zeta(m+2) - \sum_{i=1}^{r} \frac{1}{i^{m+2}}\right). \tag{2.14}$$

Remark 2. *In case* $r = 0$, *the finite sum* $\sum_{i=1}^{r} i^{-m-2}$ *is assumed to be zero.*

Remark 3. *One can clearly observe that the particular cases, corresponding to* $m = 0, 1$, *of the integral* (2.12) *have been studied by Beukers in* (2.7).

Proof. For $\sigma \geq 0$, we define
$$S[r,s;\sigma] := \int_{[0,1]}^{(2)} \frac{x^{r+\sigma} y^{s+\sigma}}{1-xy} \, dx\, dy. \tag{2.15}$$
Expanding $(1-xy)^{-1}$ as a geometric series and performing integration, we get
$$S[r,s;\sigma] = \sum_{i=0}^{\infty} \frac{1}{(\sigma+i+r+1)(\sigma+i+s+1)}. \tag{2.16}$$

For $r > s$, the above sum can be written as

$$\sum_{k=0}^{\infty}\frac{1}{r-s}\left(\frac{1}{\sigma+i+s+1}-\frac{1}{\sigma+i+r+1}\right)=\frac{1}{r-s}\left(\frac{1}{\sigma+s+1}+\cdots+\frac{1}{\sigma+r}\right). \tag{2.17}$$

Therefore, we get

$$S[r,s;\sigma]=\frac{1}{r-s}\sum_{i=1}^{r-s}\frac{1}{\sigma+s+i}.$$

Now, differentiating both sides of the above equation with respect to 'σ' m-times and substituting $\sigma = 0$ and utilizing the definition (2.12), we obtain

$$B_m[r,s]=\frac{m!}{r-s}\sum_{i=1}^{r-s}\frac{1}{(s+i)^{m+1}}.$$

As $(r-s) \mid l_r$ and $\mathrm{LCM}[(s+1)^{m+1},\cdots,r^{m+1}] \mid l_r^{m+1}$, so claim (2.13) becomes obvious.

If $r = s$, then by (2.15) and (2.16),

$$S[r,r;\sigma]=\sum_{i=0}^{\infty}\frac{1}{(\sigma+i+r+1)^2}. \tag{2.18}$$

If we differentiate (2.18) m times with respect to σ. Then we obtain

$$\begin{aligned}\int_{[0,1]}^{(2)}\frac{(xy)^{\sigma+r}(\log xy)^m}{1-xy}\,dx\,dy&=(-1)^m\sum_{i=0}^{\infty}\frac{(m+1)!}{(\sigma+i+r+1)^{m+2}}\\&=(-1)^m(m+1)!\left(\zeta(m+2,\sigma)-\sum_{i=1}^{r}\frac{1}{i^{m+2}}\right).\end{aligned} \tag{2.19}$$

And putting $\sigma = 0$ in (2.19), we get

$$B_m[r,r]=(m+1)!\left(\zeta(m+2)-\sum_{i=1}^{r}\frac{1}{i^{m+2}}\right). \tag{2.20}$$

This proves (2.14) and consequently Lemma 2.5. $\square$

CHAPTER 3

Beukers' proof of irrationality of $\zeta(2)$ and $\zeta(3)$

Theorem 3.1 (Euler)**.** *The constant $\zeta(2)$ is an irrational number.*

Proof. The idea of this proof is due to Beukers. Here we explain all the details of the proof. For $n \in \mathbb{N}$ consider the integral

$$I[n] := \int_{[0,1]}^{(2)} \frac{(1-y)^n f_n(x)}{1-xy}\, dx\, dy, \tag{3.1}$$

where the symbol $\int_{[0,1]}^{(k)}$ means $\int_0^1 \cdots \int_0^1$ k-times, and $f_n(x)$ is an n-degree polynomial defined by

$$f_n(x) := \frac{1}{n!}\frac{d^n}{dx^n}\left\{x^n(1-x^n)\right\}. \tag{3.2}$$

Note that $f_n(x) \in \mathbb{Z}[x]$. If we write $(1-y)^n f_n(x)$ in the expanded form, the numerator of the integrand in (3.1) will have the terms of the form of $Kx^p y^q$ where $p, q \in \mathbb{N}$, $K \in \mathbb{Z}$, and $0 \le p, q \le n$. Now distributing the integral makes it clear form Lemma 2.4 that the integral (3.1) equals

$$\frac{P_n + Q_n\zeta(2)}{l_n^2} \tag{3.3}$$

for some $P_n, Q_n \in \mathbb{Z}$. Now we write the integral (3.1) in the following way

$$I[n] = \int_{[0,1]}^{(1)} \left(\int_{[0,1]}^{(1)} \frac{f_n(x)}{1-xy}\, dx \right) (1-y)^n\, dy. \tag{3.4}$$

Now to simply the middle integral, we perform integration by parts n-times with respect to the x. Finally, upon simplification, it becomes

$$I[n] = (-1)^n \int_{[0,1]}^{(2)} \frac{g(x,y)^n}{1-xy}\, dx\, dy, \tag{3.5}$$

where

$$g(x,y) = \frac{x(1-x)y(1-y)}{1-xy}. \tag{3.6}$$

Using multi-variable calculus, it can easily be verified that, for $0 < x, y < 1$,

$$g(x,y) \leq \left(\frac{\sqrt{5}-1}{2} \right)^5. \tag{3.7}$$

Therefore,

$$|I[n]| \leq \left(\frac{\sqrt{5}-1}{2} \right)^{5n} \int_{[0,1]}^{(2)} \frac{1}{1-xy}\, dx\, dy \leq \left(\frac{\sqrt{5}-1}{2} \right)^{5n} \zeta(2). \tag{3.8}$$

As (3.5) is non-zero, so using (3.3) and (3.8), we finally have

$$0 < \frac{|(P_n + Q_n\zeta(2)|}{l_n^2} < \left(\frac{\sqrt{5}-1}{2} \right)^{5n} \zeta(2).$$

This implies

$$0 < |P_n + Q_n\zeta(2)| < l_n^2 \left(\frac{\sqrt{5}-1}{2} \right)^{5n} \zeta(2) < 9^n \left(\frac{\sqrt{5}-1}{2} \right)^{5n} \zeta(2), \tag{3.9}$$

for sufficiently large n. In the last inequaltiy, we have used Lemma 2.3. Since $9 \cdot \left(\frac{\sqrt{5}-1}{2} \right)^5 < 1$, hence the irrationality of $\zeta(2)$ follows by Lemma 2.2. □

Theorem 3.2 (Beukers)**.** *The constant $\zeta(3)$ is an irrational number.*

Proof. For any $n \in \mathbb{N}$, let us define

$$J[n] := \int_{[0,1]}^{(2)} \frac{-\log xy}{1-xy} f_n(x) f_n(y)\, dx\, dy. \tag{3.10}$$

Here $f_n(x)$ is same as we defined in (3.2). Now writing the expansion of $f_n(x)f_n(y)$ and distributing the integral, and utilizing Lemma 2.4, we can write

$$J[n] = \frac{(P_n + Q_n\zeta(3))}{l_n^3} \tag{3.11}$$

for some $P_n, Q_n \in \mathbb{Z}$. Note that

$$\int_{[0,1]}^{(1)} \frac{1}{1-(1-xy)z}\, dz = -\frac{\log xy}{1-xy}. \tag{3.12}$$

Therefore, the above integral $J[n]$ can be written as

$$\int_{[0,1]}^{(3)} \frac{f_n(x) f_n(y)}{1-(1-xy)z}\, dx\, dy\, dz = \int_{[0,1]}^{(2)} \left(\int_{[0,1]}^{(1)} \frac{f_n(x)}{1-(1-xy)z}\, dx \right) f_n(y)\, dy\, dz. \tag{3.13}$$

Using integration by parts n-times with respect to x, with $\frac{1}{1-(1-xy)z}$ as the first function and $f_n(x)$ as the second function, one can verify that

$$\int_{[0,1]}^{(1)} \frac{f_n(x)}{1-(1-xy)z}\, dx = \int_{[0,1]}^{(1)} \frac{(xyz)^n (1-x)^n}{(1-(1-xy)z)^{n+1}} dx. \tag{3.14}$$

Substituting (3.14) in (3.13), we get

$$J[n] = \int_{[0,1]}^{(3)} \frac{(H(x,y,z))^n f_n(y)}{(1-(1-xy)z)}\, dx\, dy\, dz,$$

where

$$H(x,y,z) := \frac{xyz(1-x)}{1-(1-xy)z}.$$

Now we make a change of variable, namely,

$$z = \frac{1-w}{1-(1-xy)w} \Rightarrow \quad 1-w = \frac{xyz}{1-(1-xy)z}.$$

One can verify that

$$\frac{dz}{1-(1-xy)z} = -\frac{dw}{1-(1-xy)w}.$$

Under the above substitution, the integral (3.14) changes into

$$J[n] = -\int_{[0,1]}^{(3)} ((1-x)(1-w))^n \frac{f_n(y)}{1-(1-xy)w}\, dx\, dy\, dw.$$

Again invoke integration by parts n-times with respect to y, to see that

$$J[n] = -\int_{[0,1]}^{(3)} \frac{(G(x,y,w))^n}{(1-(1-xy)w)}\, dx\, dy\, dw. \tag{3.15}$$

where

$$G(x,y,w) := \frac{xyw(1-x)(1-y)(1-w)}{(1-(1-xy)w)}.$$

One can verified that the maximum of G occurs for $x = y$ and $w = \frac{1}{1+x}$. Therefore, $\frac{x^2(1-x)^2}{(1+x)^2}$ acts as a bound for G with maxima at $x = \sqrt{2}-1$ in $0 < x < 1$. Therefore,

$$|G(x,y,w)| \leq (\sqrt{2}-1)^4, \quad \forall \quad 0 < x, y, w < 1. \tag{3.16}$$

Thus, utilizing the above bound, we have

$$
\begin{aligned}
|J[n]| &\leq (\sqrt{2}-1)^{4n} \int_{[0,1]}^{(3)} \frac{1}{(1-(1-xy)w)}\, dx\, dy\, dw \\
&= (\sqrt{2}-1)^{4n} \int_{[0,1]}^{(2)} \frac{-\log(xy)}{1-xy}\, dx\, dy \\
&= 2(\sqrt{2}-1)^{4n}\zeta(3). \\
\Rightarrow \quad & 0 < |P_n + Q_n\zeta(3)| {l_n}^{-3} < 2\zeta(3)(\sqrt{2}-1)^{4n}.
\end{aligned}
$$

Employing the above bound and using the expression (3.11) for $J[n]$, we arrive at

$$
0 < |P_n + Q_n\zeta(3)| < 2\zeta(3){l_n}^3(\sqrt{2}-1)^{4n} < L(27(\sqrt{2}-1)^4)^n,
$$

for sufficiently large n, where $L = 2\zeta(3)$. As $27(\sqrt{2}-1)^4 < 1$, therefore, by Lemma 2.2 irrationality of $\zeta(3)$ follows. $\square$

CHAPTER 4

Some relevant integral expressions

In this chapter, we give some expressions which result in the expression of the form
$$\frac{P_n + Q_n\zeta(k)}{R_n},$$
where $P_n, Q_n, R_n \in \mathbb{Z}$ and $k > 1$ is some fixed natural number. Although concluding the irrationality of the corresponding zeta value from such expressions is difficult or not at all conclusive. Let us first prove some important lemmas in this direction. In the proof of irrationality of $\zeta(3)$, Beukers used the following identity:
$$\int_{[0,1]}^{(1)} \frac{1}{1-(1-xy)z}\,dz = -\frac{\log xy}{1-xy}.$$
Here we mention a simple one variable generalization of the above identity.

Lemma 4.1. *For $n \in \mathbb{N} \cup \{0\}$, we have*
$$\int_{[0,1]}^{(1)} \frac{(\log(1-(1-xy)z))^n}{1-(1-xy)z}\,dz = \frac{-1}{n+1}\frac{(\log xy)^{n+1}}{1-xy}. \tag{4.1}$$

Proof. Substitute

$$\log(1-(1-xy)z)=t,$$

the left hand side of (4.1) becomes,

$$\begin{aligned}&\frac{-1}{1-xy}\int_0^{\log xy} t^n dt\\ &=\frac{-1}{n+1}\frac{(\log xy)^{n+1}}{1-xy}.\end{aligned}$$

□

The next lemma gives a one variable generalization of Lemma 2.4.

Lemma 4.2. *Let $r,s,t\in\mathbb{N}\cup\{0\}$. We define*

$$D[r,s,t]:=\int_{[0,1]}^{(3)}\frac{x^r y^s z^t}{1-xyz}\,dx\,dy\,dz. \tag{4.2}$$

For $r>s>t$, we have

$$D[r,s,t]=\frac{L}{M} \tag{4.3}$$

and $M\big|l_r^3$, where $L,M\in\mathbb{Z}$. If $r=s=t$, then

$$D[r,r,r]=\zeta(3)-\sum_{i=1}^{r}\frac{1}{i^3}. \tag{4.4}$$

If $r=s>t$, then

$$D[r,r,t]=\frac{P+Q\zeta(2)}{l_r^3}, \tag{4.5}$$

where $P,Q\in\mathbb{Z}$. If $r=s<t$, then

$$R[r,r,t]=\frac{R+S\zeta(2)}{l_t^3}, \tag{4.6}$$

where $R,S\in\mathbb{Z}$.

Proof. Write $\frac{1}{1-xyz}$ into geometric series, and interchange the integral and summation to get

$$\begin{aligned}D[r,s,t]&=\sum_{i=0}^{\infty}\int_{[0,1]}^{(3)}x^{r+i}y^{s+i}z^{t+i}\,dx\,dy\,dz\\ &=\sum_{i=0}^{\infty}\frac{1}{(r+i+1)(s+i+1)(t+i+1)}.\end{aligned} \tag{4.7}$$

Assume $r > s > t$, then we can write (4.7) as

$$\begin{aligned}
&\sum_{i=0}^{\infty} \frac{1}{r+i+1}\left(\frac{1}{s-t}\left(\frac{1}{t+i+1} - \frac{1}{s+i+1}\right)\right) \\
&= \frac{1}{s-t}\left(\sum_{i=0}^{\infty} \frac{1}{(r+i+1)(t+i+1)} - \sum_{i=0}^{\infty} \frac{1}{(r+i+1)(s+i+1)}\right) \\
&= \frac{1}{(s-t)(r-t)}\left(\frac{1}{t+1} + \cdots + \frac{1}{r}\right) - \frac{1}{(s-t)(r-s)}\left(\frac{1}{s+1} + \cdots + \frac{1}{r}\right).
\end{aligned}$$

Since $(s-t), (r-t), (r-s)$, LCM$[(t+1), \cdots, r]$ and LCM$[(s+1), \cdots, r]$ all divide l_r, so this proves (4.2). Now we assume $r = s = t$, then from (4.7) we have

$$\begin{aligned}
D[r, r, r] &= \sum_{i=0}^{\infty} \frac{1}{(r+i+1)^3} \\
&= \zeta(3) - \frac{1}{1^3} - \cdots - \frac{1}{r^3}.
\end{aligned}$$

This proves (4.4). Now if $r = s > t$, then from (4.7) we obtain

$$\begin{aligned}
D[r, r, t] &= \sum_{i=0}^{\infty} \frac{1}{(r+i+1)(r+i+1)(t+i+1)} \\
&= \sum_{i=0}^{\infty} \frac{1}{r+i+1}\left(\frac{1}{r-t}\left(\frac{1}{t+i+1} - \frac{1}{r+i+1}\right)\right) \\
&= \frac{1}{r-t}\left(\sum_{i=0}^{\infty} \frac{1}{(r+i+1)(t+i+1)} - \sum_{i=0}^{\infty} \frac{1}{(r+i+1)^2}\right) \\
&= \frac{1}{r-t}\left(\sum_{i=0}^{\infty} \frac{1}{r-t}\left(\frac{1}{t+i+1} - \frac{1}{r+i+1}\right) - \left(\zeta(2) - \frac{1}{1^2} - \cdots - \frac{1}{r^2}\right)\right) \\
&= \frac{1}{r-t}\left(\frac{1}{r-t}\left(\frac{1}{t+1} + \cdots + \frac{1}{r}\right) + \frac{1}{1^2} + \cdots + \frac{1}{r^2}\right) - \frac{\zeta(2)}{r-t}.
\end{aligned}$$

From the above calculation (4.5) follows immediately as the denominator of the above expression is divisor of l_r^3. Similarly one can do for (4.6). This completes proof of the lemma. □

Now following assertions, mentioned as remarks, can be inductively concluded from the above lemma. Let $r_1, r_2, \cdots, r_k \in \mathbb{N} \cup \{0\}$.

Remark 4. *If $r_1 > r_2 > \cdots > r_k$, then*

$$\int_{[0,1]}^{(k)} \frac{{x_1}^{r_1} {x_2}^{r_2} \cdots {x_k}^{r_k}}{1 - x_1 x_2 \cdots x_k} \, dx_1 \, dx_2 \cdots dx_k = \frac{P}{Q} \tag{4.8}$$

and $Q \big| l_{r_1}^k$, *where* $P, Q \in \mathbb{Z}$.

Remark 5. *If* $r_1 = r_2 = \cdots = r_k = r$, *then*
$$\int_{[0,1]}^{(k)} \frac{{x_1}^r {x_2}^r \cdots {x_k}^r}{1 - x_1 x_2 \cdots x_k}\, dx_1\, dx_2 \cdots dx_k = \zeta(k) - \frac{1}{1^k} \cdots - \frac{1}{r^k}. \tag{4.9}$$
In particular, if we put $r = 0$, *we obtain*
$$\int_{[0,1]}^{(k)} \frac{1}{1 - x_1 x_2 \cdots x_k}\, dx_1\, dx_2 \cdots dx_k = \zeta(k). \tag{4.10}$$
If we define
$$I := \int_0^1 \int_0^{x_k} \cdots \int_0^{x_3} \int_0^{x_2} \frac{1}{(1 - x_1) x_2 \cdots x_k}\, dx_1\, dx_2 \cdots dx_k. \tag{4.11}$$
Substitute $x_1 = x_2 y_1$, *the above integral changes into*
$$\int_0^1 \int_0^{x_k} \cdots \int_0^{x_3} \int_0^1 \frac{1}{(1 - y_1 x_2) x_3 \cdots x_k}\, dy_1\, dx_2 \cdots dx_k.$$
Similarly substituting $x_2 = x_3 y_2$ *and so on upto* $x_{k-1} = x_k y_{k-1}$, *then we obtain*
$$I = \int_{[0,1]}^{(k)} \frac{1}{1 - y_1 y_2 \cdots y_{k-1} x_k}\, dy_1\, dy_2 \cdots dy_{k-1} dx_k. \tag{4.12}$$
Finally, from (4.10), (4.11) *and* (4.12), *we conclude*
$$\begin{aligned} \zeta(k) &= \int_0^1 \int_0^{x_k} \cdots \int_0^{x_3} \int_0^{x_2} \frac{1}{(1 - x_1) x_2 \cdots x_k}\, dx_1\, dx_2 \cdots dx_k \\ &= \underset{0<x_1<x_2<\cdots<x_k<1}{\underbrace{\int \cdots \int}_{k\text{-times}}} \frac{1}{(1 - x_1) x_2 x_3 \cdots x_k}\, dx_1\, dx_2 \cdots dx_k. \end{aligned} \tag{4.13}$$

Remark 6. *If* $r_1 = r_2 = \cdots = r_m = r > r_{m+1} > \cdots > r_k$, *where* $m < k$ *then*
$$\begin{aligned} &\int_{[0,1]}^{(k)} \frac{{x_1}^r {x_2}^r \cdots {x_m}^r {x_{m+1}}^{r_{m+1}} \cdots {x_k}^{r_k}}{1 - x_1 x_2 \cdots x_k}\, dx_1\, dx_2 \cdots dx_k \\ &\qquad = \frac{P_1 + P_2 \zeta(2) + \cdots + P_m \zeta(m)}{l_r^k}, \end{aligned} \tag{4.14}$$
where $P_1, P_2, \cdots, P_m \in \mathbb{Z}$.

Proposition 4.3. *For* $n \in \mathbb{N}$, *we have*
$$\int_{[0,1]}^{(2)} \frac{(xy(1-x)(1-y))^n}{1 - xy}\, dx\, dy = \frac{P_n + Q_n \zeta(2)}{l_{2n}^2},$$
for some $P_n, Q_n \in \mathbb{Z}$.

Proof. Consider the integral
$$S[n] := \int_{[0,1]}^{(2)} \frac{(xy(1-x)(1-y))^n}{1 - xy}\, dx\, dy. \tag{4.15}$$

Note that the numerator of the integrand is a polynomial of two variables with degree $2n$. Therefore, Lemma 2.4 immediately tells that (4.15) is of the form

$$\frac{P_n + Q_n\zeta(2)}{{l_{2n}}^2}.$$

Now we will show that this does not suffice to show the irrationality of $\zeta(2)$. Using some basic calculus one can see that

$$xy(1-x)(1-y) \leq \frac{1}{2^4}, \quad \forall \quad 0 < x, y < 1. \tag{4.16}$$

Therefore,

$$\left| \int_{[0,1]}^{(2)} \frac{(x(1-x)y(1-y))^n}{1-xy}\, dx\, dy. \right| \leq \left(\frac{1}{2^4}\right)^n \int_{[0,1]}^{(2)} \frac{1}{1-xy}\, dx\, dy = \left(\frac{1}{2^4}\right)^n \zeta(2).$$

Since (4.15) is non-zero, so using the remark of Lemma 2.3, we have

$$0 < |P_n + Q_n\zeta(2)| \leq l_{2n}^2 \left(\frac{1}{2^4}\right)^n \zeta(2) < \left(\frac{(e+\epsilon)^4}{(16)}\right)^n \zeta(2),$$

for sufficiently large n and some $\epsilon > 0$. Clearly, $\left(\frac{(e+\epsilon)^4}{16}\right) > 1$ so we are inadequate to say anything about the irrationality of $\zeta(2)$. □

Proposition 4.4. *For $n \in \mathbb{N}$, we have*

$$\int_{[0,1]}^{(3)} \frac{(x(1-x)y(1-y)z(1-z))^{2n+1}}{1-xyz}\, dx\, dy\, dz = \frac{P_n + Q_n\zeta(2)}{l_{4n+2}^3},$$

for some $P_n, Q_n \in \mathbb{Z}$.

Proof. Define the integral as follows:

$$J[n] := \int_{[0,1]}^{(3)} \frac{(x(1-x)y(1-y)z(1-z))^{2n+1}}{1-xyz}\, dx\, dy\, dz. \tag{4.17}$$

Employing Lemma 4.2 in (4.17), it follows that

$$J[n] = \frac{P_n + Q_n\zeta(2) + R_n\zeta(3)}{l_{4n+2}^3},$$

where P_n, Q_n, and $R_n \in \mathbb{Z}$. Now we prove that $R_n = 0$ for all $n \in \mathbb{N}$. We write $\frac{1}{1-xyz}$ as the geometric series, then $J[n]$ can be written as

$$\sum_{i=0}^{\infty} \int_{[0,1]}^{(3)} x^{2n+i+1}(1-x)^{2n+1} y^{2n+i+1}(1-y)^{2n+1} z^{2n+i+1}(1-z)^{2n+i+1}\, dx\, dy\, dz$$

$$= \sum_{i=0}^{\infty} \left(\int_{[0,1]}^{(1)} x^{2n+i+1}(1-x)^{2n+1} dx \right)^3.$$

Performing integration by parts $2n+1$ times, with $(1-x)^{2n+1}$ as the first function and x^{2n+i+1} as the second function, we get

$$J[n] = ((2n+1)!)^3 \sum_{i=0}^{\infty} \left(\frac{1}{(2n+i+2)(2n+i+3)\cdots(4n+i+3)} \right)^3 . \tag{4.18}$$

Using partial fraction decomposition, we can write

$$\frac{1}{(2n+i+2)(2n+i+3)\cdots(4n+i+3)} = \frac{A_1}{2n+i+2} + \cdots + \frac{A_{2n+2}}{4n+i+3},$$

where

$$A_1 = \frac{1}{(2n+1)!},\ A_2 = -\frac{1}{(2n)!}, \cdots ,\ A_{2n+1} = \frac{1}{(2n)!},\ A_{2n+2} = -\frac{1}{(2n+1)!}.$$

One can see that $A_i = -A_{2n+3-i}$ also the terms are even in number, so the terms in (4.18) which produces the coefficients of $\zeta(3)$, will get cancelled, that is, $R_n = 0$. Hence

$$J[n] = \frac{P_n + Q_n\zeta(2)}{l_{4n+2}^3}. \tag{4.19}$$

Unfortunately here also we show that we can not conclude the irrationality of $\zeta(2)$. Since

$$xyz(1-x)(1-y)(1-z) \leq \frac{1}{2^6} \quad \text{for all} \quad 0 < x,\ y,\ z < 1.$$

Therefore,

$$\left| \int_{[0,1]}^{(3)} \frac{(x(1-x)y(1-y)z(1-z))^{2n+1}}{1-xyz}\, dx\, dy\, dz \right| = \left| \frac{P_n + Q_n\zeta(2)}{l_{4n+2}^3} \right| \leq \left(\frac{1}{2^6} \right)^{2n+1} \zeta(3).$$

Using remark of the Lemma 2.3, we have

$$|P_n + Q_n\zeta(2)| \leq l_{4n+2}^3 \left(\frac{1}{2^6} \right)^{2n+1} \zeta(3) < \left(\frac{(e+\epsilon)^6}{2^6} \right)^{2n+1} \zeta(3)$$

for sufficiently large n and for some $\epsilon > 0$. Clearly, $\left(\frac{(e+\epsilon)^6}{2^6} \right) > 1$ and thus we can not conclude that $\zeta(2)$ is irrational. □

Proposition 4.5. *For $n \in \mathbb{N}$, we have*

$$\int_{[0,1]}^{(3)} \frac{(xyz(1-x)(1-y)(1-z))^{2n}}{1-xyz}\, dx\, dy\, dz = \frac{P_n + Q_n\zeta(3)}{l_{4n}^3}.$$

Proof. Define the integral

$$H[n] := \int_{[0,1]}^{(3)} \frac{(xyz(1-x)(1-y)(1-z))^{2n}}{1-xyz}\, dx\, dy\, dz. \tag{4.20}$$

As the numerator of the integrand is a polynomial of degree $4n$, so we have from Lemma 4.2 that

$$H[n] = \frac{P_n + Q_n\zeta(2) + R_n\zeta(3)}{l_{4n}^3},$$

where P_n, Q_n, and $R_n \in \mathbb{Z}$. Now we prove that $Q_n = 0$ for all $n \in \mathbb{N}$. Writing $\frac{1}{1-xyz}$ as the geometric series, then $H[n]$ can be written as,

$$\begin{aligned} H[n] &= \sum_{i=0}^{\infty} \int_{[0,1]}^{(3)} x^{2n+i}(1-x)^{2n} y^{2n+i}(1-y)^{2n} z^{2n+i}(1-z)^{2n}\, dx\, dy\, dz \\ &= \sum_{i=0}^{\infty} \left(\int_{[0,1]}^{(1)} x^{2n+i}(1-x)^{2n}\, dx \right)^3. \end{aligned}$$

Doing integration by parts $2n$ times, we have

$$H[n] = ((2n)!)^3 \sum_{i=0}^{\infty} \left(\frac{1}{(2n+1+i)(2n+2+i)\cdots(4n+1+i)} \right)^3. \tag{4.21}$$

Using partial fraction decomposition, one can write

$$\frac{1}{(2n+1+i)(2n+2+i)\cdots(4n+1+i)} = \frac{A_1}{2n+1+i} + \cdots + \frac{A_{2n+1}}{4n+1+i},$$

where

$$A_1 = \frac{1}{(2n)!},\ A_2 = -\frac{1}{(2n-1)!}, \cdots,\ A_{2n} = -\frac{1}{(2n-1)!},\ A_{2n+1} = \frac{1}{(2n)!}.$$

Clearly $A_j = A_{2n+2-j}$ where $j \in \{1, 2, \cdots, 2n+1\}$. Therefore, the right hand side of (4.21) becomes

$$((2n)!)^3 \sum_{i=0}^{\infty} \left(\frac{A_1}{(2n+1+i)} + \frac{A_2}{(2n+2+i)} + \cdots + \frac{A_{2n+1}}{(4n+1+i)} \right)^3. \tag{4.22}$$

The only terms in the expansion of (4.22) which contribute a value of $\zeta(2)$ are of the form

$$\sum_{i=0}^{\infty} \frac{A_j^2 A_k}{(2n+j+i)^2(2n+k+i)},$$

where $j,\ k \in \{1, 2, \cdots, 2n+1\}$ and $j \neq k$, also we have even number of such terms. Now without loss of generality, for a given j and k with $j > k$,

$$\begin{aligned}
&\sum_{i=0}^{\infty} \frac{A_j^2 A_k}{(2n+j+i)^2(2n+k+i)} \\
&= \sum_{i=0}^{\infty} A_j^2 A_k \left(\frac{1}{2n+j+i} \cdot \frac{1}{j-k} \left(\frac{1}{2n+k+i} - \frac{1}{2n+j+i} \right) \right) \\
&= \sum_{i=0}^{\infty} \frac{A_j^2 A_k}{(j-k)^2} \left(\frac{1}{2n+k+i} - \frac{1}{2n+j+i} \right) - \sum_{i=0}^{\infty} \frac{A_j^2 A_k}{j-k} \frac{1}{(2n+j+i)^2} \\
&= \frac{P_{j,k}}{Q_{j,k}} - \frac{A_j^2 A_k}{j-k} \zeta(2), \qquad (4.23)
\end{aligned}$$

where $P_{j,k}, Q_{j,k} \in \mathbb{Z}$ with $Q_{j,k} \neq 0$. Similarly using the fact that $A_j = A_{2n+2-j}$, we have

$$\sum_{i=0}^{\infty} \frac{A_{2n+2-j}^2 A_{2n+2-k}}{(2n+2n+2-j+i)^2(2n+2n+2-k+i)} = \frac{A_j^2 A_k}{j-k} \zeta(2) - \frac{P_{2n+2-j,2n+2-k}}{Q_{2n+2-j,2n+2-k}}, \qquad (4.24)$$

where $P_{2n+2-j,2n+2-k}, Q_{2n+2-j,2n+2-k} \in \mathbb{Z}$ with $Q_{2n+2-j,2n+2-k} \neq 0$. As coefficient of $\zeta(2)$ in (4.23) and (4.24) is same with opposite sign. Therefore, for all $j,\ k \in \{1, 2, \cdots, 2n+1\}$ and $j \neq k$, we have

$$\sum_{i=0}^{\infty} \left(\sum_{\substack{j,k \in \{1,2,\cdots,2n+1\} \\ j \neq k}} \frac{A_j^2 A_k}{(2n+i+j)^2(2n+i+k)} \right)$$

a rational number. Hence we have then

$$H[n] = \frac{P_n + R_n \zeta(3)}{l_{4n}^3}.$$

The effort here are also in vain as we show that it does not imply the irrationality of $\zeta(3)$. Since

$$x(1-x)y(1-y)z(1-z) \leq \frac{1}{2^6} \text{ for all } 0 < x < 1,\ 0 < y < 1,\ 0 < z < 1.$$

Therefore,

$$\begin{aligned}
\left| \int_{[0,1]}^{(3)} \frac{(x(1-x)y(1-y)z(1-z))^{2n}}{1-xyz} \, dx \, dy \, dz \right| &= \left| \frac{P_n + R_n \zeta(3)}{{l_{4n}}^3} \right| \\
&\leq \left(\frac{1}{2^6} \right)^{2n} \zeta(3).
\end{aligned}$$

Using the remark of Lemma 2.3, we have

$$|P_n + R_n\zeta(3)| \leq l_{4n}^3 \left(\frac{1}{2^6}\right)^{2n} \zeta(3) < \left(\frac{(e+\epsilon)^6}{2^6}\right)^{2n} \zeta(3),$$

for sufficiently large n and for some $\epsilon > 0$. As $\left(\frac{(e+\epsilon)^6}{2^6}\right) > 1$, hence we can not draw the conclusion that $\zeta(3)$ is irrational. □

Now we mention an identity which is a natural generalization of the expressions used by Frits Beukers [4]. However, concluding from it, the irrationality of zeta-values greater than 3 is difficult. If $f_n(x)$ is defined the same way as in Theorem 1.1, then using Lemma 2.5, we have for $m \in \mathbb{N} \cup \{0\}$,

$$\int_{[0,1]}^{(2)} \frac{(-1)^m (\log xy)^m}{1-xy} f_n(x) f_n(y)\, dx\, dy = \frac{P_n + Q_n\zeta(m+2)}{l_n^{m+2}}. \tag{4.25}$$

Employing Lemma 4.1 in (4.25), we obtain

$$\begin{aligned} m(-1)^{m+1} \int_{[0,1]}^{(3)} & \frac{(\log(1-(1-xy)z))^{m-1}}{1-(1-xy)z} f_n(x) f_n(y)\, dx\, dy\, dz \\ &= \frac{P_n + Q_n\zeta(m+2)}{l_n^{m+2}}. \end{aligned} \tag{4.26}$$

CHAPTER 5

A new generalized proof

5.1 A new idea for the irrationality of $\zeta(2)$ and $\zeta(3)$

In this chapter, we show that the expression used by Frits Beukers [4] for proving the irrationality of $\zeta(2)$ and $\zeta(3)$ is not unique. In fact a whole class of expressions (generalized expression) can be used to conclude that $\zeta(2)$ and $\zeta(3)$ are irrational. In the next chapter, we will also mention a conjecture regarding a general expression which has a potential of proving the irrationality of all ζ-values at positive integers greater than or equal to 4.

Theorem 5.1. *Let $k \in \mathbb{N} \cup \{0\}$ be a fixed number and $n \in \mathbb{N}$, then we have*

$$\int_{[0,1]}^{(2)} \frac{(xy(1-x)(1-y))^{n+k}}{(1-xy)^{n+1}}\, dx\, dy = \frac{P_n + Q_n\zeta(2)}{l_{n+2k}^2},$$

where $P_n, Q_n \in \mathbb{Z}$. Furthermore, the upper bound of the integrand tends to zero (fast enough) as $n \to \infty$ consequently showing that $\zeta(2)$ is irrational.

Proof. For a given $k \in \mathbb{N}$, define

$$f_{n,k}(x) := \frac{1}{n!}\frac{d^n}{dx^n}x^{n+k}(1-x)^{n+k}, \tag{5.1}$$

and

$$I[n,k] := \int_{[0,1]}^{(2)} \frac{f_{n,k}(x)y^k(1-y)^{k+n}}{(1-xy)}\,dx\,dy. \tag{5.2}$$

Since the terms in the numerator of integrand in (5.2) are of the form of Kx^py^q, where $K \in \mathbb{Z}$ and $k \leq p,q \leq n+2k$. So employing Lemma 2.4 it follows that (5.2) is of the form of

$$I[n,k] = \frac{A_n + B_n\zeta(2)}{l_{n+2k}^2},$$

where $A_n, B_n \in \mathbb{Z}$. Performing integration by parts n-times, one can verify that

$$\int_{[0,1]}^{(1)} \frac{f_{n,k}(x)}{1-xy}dx = (-1)^n \int_{[0,1]}^{(1)} \frac{y^n x^{n+k}(1-x)^{n+k}}{(1-xy)^{n+1}}dx. \tag{5.3}$$

Utilizing (5.3) in (5.2), we have

$$I[n,k] = (-1)^n \int_{[0,1]}^{(2)} \frac{\left(x(1-x)y(1-y)\right)^{k+n}}{(1-xy)^{n+1}}\,dx\,dy = \frac{P_n + Q_n\zeta(2)}{l_{n+2k}^2},$$

where $P_n = (-1)^n A_n$ and $Q_n = (-1)^n B_n$. Now using bounds (3.7), (4.16), and Lemma 2.3, we have

$$\begin{aligned}
\left|\frac{P_n + Q_n\zeta(2)}{l_{n+2k}^2}\right| &= \left|\int_{[0,1]}^{(2)} \frac{\left(x(1-x)y(1-y)\right)^{k+n}}{(1-xy)^{n+1}}\,dx\,dy\right| \\
&\leq \left(\frac{\sqrt{5}-1}{2}\right)^{5(n+1)} \left(\frac{1}{16}\right)^{k-1} \\
\Rightarrow \quad |P_n + Q_n\zeta(2)| &\leq l_{n+2k}^2 \left(\frac{\sqrt{5}-1}{2}\right)^{5} \left(\frac{1}{16}\right)^{k-1} \left(\frac{\sqrt{5}-1}{2}\right)^{5n} \\
\Rightarrow \quad |P_n + Q_n\zeta(2)| &< 9^{n+2k} \left(\frac{\sqrt{5}-1}{2}\right)^{5} \left(\frac{1}{16}\right)^{k-1} \left(\frac{\sqrt{5}-1}{2}\right)^{5n},
\end{aligned}$$

for sufficiently large n. Here in the last line we have used the bound (2.3). Hence,

$$0 < |P_n + Q_n\zeta(2)| < L \cdot \left(9\left(\frac{\sqrt{5}-1}{2}\right)^5\right)^n, \tag{5.4}$$

for sufficiently large n, where

$$L = 9^{2k}\left(\frac{\sqrt{5}-1}{2}\right)^5 \left(\frac{1}{16}\right)^{k-1}.$$

As $9\left(\frac{\sqrt{5}-1}{2}\right)^5 < 1$, so employing Lemma 2.2 in (5.4), it becomes clear that $\zeta(2)$ is irrational. $\square$

Theorem 5.2. *Let $k \in \mathbb{N} \cup \{0\}$ be a fixed number and $n \in \mathbb{N}$, then we have*
$$\int_{[0,1]}^{(2)} \frac{(x(1-x)y(1-y))^{k+n}\left((w(1-w))^n\right.}{(1-(1-xy)w)^{n+1}}\, dx\, dy = \frac{P_n + Q_n\zeta(3)}{l_{n+2k}^3},$$
where $P_n, Q_n \in \mathbb{Z}$. Furthermore, the upper bound of the integrand tends to zero (fast enough) as $n \to \infty$ and hence proving the irrationality of $\zeta(3)$.

Proof. Let $f_{n,k}(x)$ be defined as in (3.2) and let us consider
$$J[n,k] := \int_{[0,1]}^{(2)} \frac{f_{n,k}(x) f_{n,k}(y) \log(xy)}{(1-xy)}\, dx\, dy. \tag{5.5}$$
Note that $f_{n,k}(x)f_{n,k}(y)$ is a polynomial in $\mathbb{Z}[x,y]$ of degree $n+2k$ and each monomial is of the form $Kx^p y^q$, for some $K \in \mathbb{Z}$ and $k \leq p, q \leq n+2k$, so using Lemma 2.4 it follows that
$$J[n,k] = \frac{P_n + Q_n\zeta(3)}{l_{n+2k}^3}.$$
Now applying Lemma 4.1 in (5.5) we get
$$J[n,k] = -\int_{[0,1]}^{(3)} \frac{f_{n,k}(x) f_{n,k}(y)}{1-(1-xy)z}\, dx\, dy\, dz. \tag{5.6}$$
Performing integration by parts n-times with respect to x, one can check that
$$\int_{[0,1]}^{(1)} \frac{f_{n,k}(x)}{1-(1-xy)z} dx = \int_{[0,1]}^{(1)} \frac{(yz)^n x^{n+k}(1-x)^{n+k}}{(1-(1-xy)z)^{n+1}} dx. \tag{5.7}$$
Applying (5.7) in the right hand side of (5.6) becomes
$$J[n,k] = -\int_{[0,1]}^{(3)} \frac{(yz)^n x^{n+k}(1-x)^{n+k} f_{n,k}(y)}{(1-(1-xy)z)^{n+1}}\, dx\, dy\, dz. \tag{5.8}$$
Now we make a change of variable, namely, replace
$$z = \frac{1-w}{1-(1-xy)w}$$
$$\Rightarrow 1-w = \frac{xyz}{1-(1-xy)z}.$$
Under the above substitution, the above integral changes into
$$J[n,k] = -\int_{[0,1]}^{(3)} \frac{x^k(1-x)^{n+k}(1-w)^n f_{n,k}(y)}{1-(1-xy)w}\, dx\, dy\, dw. \tag{5.9}$$
Again, use integration by parts n-times with respect to y to see that
$$J[n,k] = -\int_{[0,1]}^{(3)} \frac{(x(1-x)y(1-y))^{n+k}(w(1-w))^n}{(1-(1-xy)w)^{n+1}}\, dx\, dy\, dw. \tag{5.10}$$

This proves the first part of the theorem. We now use the bounds (3.16) and (4.16) to derive

$$|J[n,k]| \leq (\sqrt{2}-1)^{4n}\left(\frac{1}{16}\right)^{k}\zeta(3)$$

$$\Rightarrow |A_n + B_n\zeta(3)| \leq l_{n+2k}^{3}(\sqrt{2}-1)^{4n}\left(\frac{1}{16}\right)^{k}\zeta(3) < L\left(27(\sqrt{2}-1)^{4}\right)^{n} \tag{5.11}$$

for sufficiently large n, where $L = 27^{2k}\left(\frac{1}{16}\right)^{k}\zeta(3)$. Here in the final inequality we have used the bound (2.3). As $27(\sqrt{2}-1)^{4} < 1$, so applying Lemma 2.2 in (5.11), it immediately follows that $\zeta(3)$ is irrational. $\square$

CHAPTER 6

A few conjectures

In this chapter, we look at some multi-integrals which are conjectured to yield the expression of the form of

$$\frac{P_n + Q_n\zeta(k)}{C_n},$$

where $k = 2, 3, \cdots$.

Conjecture 1. *For $k = 2, 3, \cdots$ and $n \in \mathbb{N}$,*

$$\int_0^1 \int_0^{x_k} \cdots \int_0^{x_3} \int_0^{x_2} \frac{(x_1(1-x_2)\cdots(1-x_k))^n}{(1-x_1)x_2\cdots x_k}\, dx_1\, dx_2 \cdots dx_k = \zeta(k) - \frac{A_n}{B_n}, \tag{6.1}$$

where $A_n, B_n \in \mathbb{Z}$ and $B_n \neq 0$.

We prove the above conjecture for $k = 2$.

Proof. For $k = 2$ and $n \in \mathbb{N}$, replacing x_1 by x and x_2 by y we need to prove that

$$\int_0^1 \int_0^y \frac{(x(1-y))^n}{y(1-x)}\, dx\, dy = \zeta(2) - \frac{A_n}{B_n}, \tag{6.2}$$

where $A_n, B_n \in \mathbb{Z}$. Note that the numerator of integrand in (6.2) is either of the form of x^n or, $x^n y^r$, where $1 \le r \le n$. So it sufficient to prove that

$$\int_0^1 \int_0^y \frac{x^n}{(1-x)y}\,dx\,dy + \int_0^1 \int_0^y \frac{x^n y^r}{y(1-x)}\,dx\,dy = \zeta(2) - \frac{P_n}{Q_n}, \tag{6.3}$$

where $P_n, Q_n \in \mathbb{Z}$. Now consider

$$I[n] = \int_0^1 \int_0^y \frac{x^n}{y(1-x)}\,dx\,dy.$$

Writing $1/(1-x)$ in geometric series and interchanging integration and summation, we get

$$\begin{aligned} I[n] &= \sum_{i=0}^{\infty} \int_0^1 \int_0^y \frac{x^{n+i}}{y}\,dx\,dy \\ &= \sum_{i=0}^{\infty} \int_0^1 \frac{y^{n+i}}{p+i+1} dy, \\ &= \sum_{i=0}^{\infty} \frac{1}{(n+i+1)^2} = \zeta(2) - \frac{1}{1^2} - \cdots - \frac{1}{n^2}. \end{aligned} \tag{6.4}$$

Now let

$$J[n,r] = \int_0^1 \int_0^y \frac{x^n y^r}{(1-x)y}\,dx\,dy. \tag{6.5}$$

Again, in a similar way, writing $1/(1-x)$ in geometric series and performing the integration term by term, we get

$$\begin{aligned} J[n,r] &= \sum_{i=0}^{\infty} \int_0^1 \int_0^y x^{n+i} y^{r-1}\,dx\,dy \\ &= \sum_{i=0}^{\infty} \int_0^1 \frac{y^{n+r+i}}{n+i+1} dy, \\ &= \sum_{i=0}^{\infty} \frac{1}{(n+i+1)(n+r+i+1)} \\ &= \sum_{i=0}^{\infty} \frac{1}{r} \left(\frac{1}{n+i+1} - \frac{1}{n+r+i+1} \right). \end{aligned} \tag{6.6}$$

Since the series (6.6) is telescopic and hence a rational number. Therefore, the result together with (6.4) becomes obvious. □

Next we are going to establish a conjecture which is very promising for proving the irrationality of zeta-values. The following integral is considered by Buekers

[4] for the proving irrationality of $\zeta(2)$:

$$\int_{[0,1]}^{(2)} \frac{(1-y)^n f_n(x)}{1-xy}\,dx\,dy,$$

which upon performing integration by parts n times, becomes

$$(-1)^n \int_{[0,1]}^{(2)} \frac{(x(1-x)y(1-y))^n}{(1-xy)^{n+1}}\,dx\,dy.$$

In a similar manner for $\zeta(3)$, instead of considering the integral in (3.10), we can consider the following integral

$$\int_{[0,1]}^{(3)} \frac{(1-y)^n(1-z)^n f_n(x)}{1-(1-xy)z}\,dx\,dy\,dz, \tag{6.7}$$

which upon performing integration by parts n times, changes into

$$\int_{[0,1]}^{(3)} \frac{(x(1-x)y(1-y)z(1-z))^n}{(1-(1-xy)z)^{n+1}}\,dx\,dy\,dz.$$

The above triple integral is same as the integral in (3.15) obtained by Beukers. So we can conclude that

$$\int_{[0,1]}^{(3)} \frac{(1-y)^n(1-z)^n f_n(x)}{1-(1-xy)z}\,dx\,dy\,dz = \frac{L_n + M_n\zeta(3)}{l_n^3}.$$

Note: One can replace x by $1-x$ in (6.7) for the integral to look more symmetric as the value of the integral will remain unchanged. Based on the integral in (6.7) we are going to state the following conjecture.

Conjecture 2. *For $k, n \in \mathbb{N}$ and $k \geq 4$,*

$$\int_{[0,1]}^{(k)} \frac{f_n(x_1)(1-x_2)^n \cdots (1-x_k)^n}{1-(1-(1-\cdots(1-x_1x_2)x_3)\cdots)x_k}\,dx_1\cdots dx_k = \frac{A_n + B_n\zeta(k)}{l_n^k}. \tag{6.8}$$

Or equivalently

$$\begin{aligned}\int_{[0,1]}^{(k)} &\frac{(x_1(1-x_1)x_2(1-x_2)\cdots x_k(1-x_k))^n}{(1-(1-(1-\cdots(1-x_1x_2)x_3)\cdots)x_k)^{n+1}}\,dx_1\cdots dx_k \\ &= \frac{A_n + B_n\zeta(k)}{l_n^k}.\end{aligned} \tag{6.9}$$

Let us analyze this conjecture for $k = 4$. Assume that the conjecture is true for $k = 4$, then

$$\begin{aligned}R[n] &:= \int_{[0,1]}^{(4)} \frac{(x(1-x)y(1-y)z(1-z)w(1-w))^n}{(1-(1-(1-xy)z)w)^{n+1}}\,dx\,dy\,dz\,dw \\ &= \frac{A_n + B_n\zeta(4)}{l_n^4}.\end{aligned} \tag{6.10}$$

Using "Mathematica software", we see that

$$\left|\frac{x(1-x)y(1-y)z(1-z)w(1-w)}{1-(1-(1-xy)z)w}\right| \leq 0.0063, \quad \forall \quad 0 < x, y, z, w < 1$$

So, we have

$$|R[n]| \leq (0.0063)^n \left|\int_{[0,1]}^{(4)} \frac{1}{1-(1-(1-xy)z)w}\, dx\, dy\, dz\, dw\right| = (0.0063)^n \frac{7\zeta(4)}{4}.$$

Hence from Lemma 2.3 and (6.10), we get

$$0 < |A_n + B_n\zeta(4)| < \frac{7}{4}81^n(0.0063)^n\zeta(4),$$

for sufficiently large n. Thus, we have

$$|A_n + B_n\zeta(4)| \leq L(0.52)^n,$$

for sufficiently large n, where

$$L = \frac{7\zeta(4)}{4}.$$

Therefore, using Lemma 2.2 the irrationality of $\zeta(4)$ follows.

Remark 7. *Using "Mathematica software" we have verified for $5 \leq k \leq 20$ that*

$$\left|\frac{x_1(1-x_1)x_2(1-x_2)\cdots x_k(1-x_k)}{1-(1-(1-\cdots(1-x_1x_2)x_3)\cdots)x_k}\right| \cdot 3^k < 1, \quad \forall \quad 0 < x_1, x_2, \cdots, x_k < 1.$$

This indicates that if the conjecture is true, it will immediately follow that all zeta values at positive integers are irrational. So, indeed Conjecture 2 looks promising.

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