Zeros of Ramanujan-type Polynomials

M.Sc. Thesis

by

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DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

JUNE 2023

Zeros of Ramanujan-type Polynomials

A THESIS

Submitted in partial fulfillment of the requirements for the award of the degree of

Master of Science

by

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JUNE 2023

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"Zeros of Ramanujan-type polynomials"** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF MATHEMATICS**, **Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2022 to June 2023 under the supervision of **Dr. Bibekananda Maji**, Assistant Professor, Department of Mathematics, IIT Indore. The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

Sarkar. 05/06/2023

Signature of the student with date

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Dr. Gourab Sil

Date: 06 June 2023

Dedicated to my
Maa and Baba

dream is not that you see in sleep,

dream is something that does not let you sleep.

–***A. P. J. Abdul Klam***

Acknowledgements

I am incredibly grateful to my supervisor Dr. Bibekananda Maji for his assistance and guidance throughout my whole M.Sc. course. He gave his most profound concern for my overall well-being in these challenging times. His friendly nature, enthusiasm, and supportive attitude encourage me to successfully complete this thesis. His confidence and trust in me always push me to do better in every direction. Also, he conveyed the spirit of adventure and pushed me forward toward my area of interest concerning research. I believe that it is an honor for me to work with him. I thank the Almighty for bestowing his blessings upon me. I am very thankful to my PSPC members Dr. Charitha Cherugondi, Dr. Gourab Sil as well as DPGC convener Dr. Vijay Kumar Sohani and HOD Dr. Niraj Kumar Shukla for their continous support. I am also thankful to all our teachers for their kind and valuable suggestion. I am too fortunate to have all these teachers that even my most profound gratitude is not enough. This acknowledgment note would be irrelevant without being thankful to my family members. Their support and faith gradually and firmly boost my confidence from inside. They gave me the strength to pursue my interests even in difficult times.

And lastly, I thank my friends for always being there with me. My heart is full of favours received from every single person. I remain indebted and grateful to every one of you.

Abstract

Ramanujan's notebooks contain many elegant identities and one of the well-known identities is a formula for $\zeta(2k+1)$. Grosswald [8] gave an extension of the aforementioned formula for $\zeta(2k+1)$ which contains a polynomial of degree $2k+2$. This polynomial is now known as the *Ramanujan polynomial* $R_{2k+1}(z)$. Murty, Smith and Wang [10] proved that all the complex zeros of $R_{2k+1}(z)$ lie on the unit circle. Recently, Chourasiya, Jamal, and Maji [5] found a new polynomial while obtaining a Ramanujan-type formula for Dirichlet L-function and named it as *Ramanujan-type polynomial* $R_{2k+1,p}(z)$. In the same paper, they conjectured that all the complex zeros of $R_{2k+1,p}(z)$ will lie on the unit circle. One of the main goals of this thesis is to present a proof of their conjecture.

Contents

CHAPTER 1

Prerequisites

1.1 Bernoulli numbers and its properties

The history of Bernoulli numbers goes back to the problem of finding a formula for the sum of k^{th} powers of first '$n-1$' natural numbers. The coefficient of 'n' in the formula is defined as the k^{th} Bernoulli number B_k. So it is clear that all the Bernoulli numbers are rational, as we know that the sum of k^{th} power of first '$n-1$' natural numbers is a polynomial of degree '$k+1$' with rational coefficients. The sequence of Bernoulli numbers can also be obtained from the following generating function:

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} c_k z^k, \quad |z| < 2\pi, \tag{1.1}$$

where

$$c_k = \frac{B_k}{k!}.$$

A few Bernoulli numbers are listed below.

$$\{B_k\}_{k=0}^{\infty} = \left\{1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, \cdots\right\}.$$

Definition 1.1. *For $k \in \mathbb{N}$ and $x \in \mathbb{R}$, Bernoulli polynomials $B_n(x)$ are defined as*

$$\frac{ze^{zx}}{e^z - 1} := \sum_{n=0}^{\infty} \frac{B_n(x)z^n}{n!} \quad |z| < 2\pi. \tag{1.2}$$

Now we mention some properties of Bernoulli polynomials which are going to be handy in proving our main results.

$$B_k(1) = B_k(0) = B_k,\ \forall k \neq 1, \tag{1.3}$$

$$B_k\left(\frac{1}{2}\right) = (2^{1-k} - 1)B_k, \tag{1.4}$$

$$B_{2k-1}\left(\frac{1}{2}\right) = 0,\ \forall k, \tag{1.5}$$

$$B_{2k} = (-1)^{k+1}|B_{2k}|. \tag{1.6}$$

$$\tag{1.7}$$

The following bounds [1, p. 805] hold for Bernoulli numbers.

Lemma 1.1. *For $n \in \mathbb{N}$, we have*

$$\frac{2(2n)!}{(2\pi)^{2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}(1 - 2^{1-2n})}. \tag{1.8}$$

We are now going to mention one useful identity about the convolution of Bernoulli polynomials [7, p. 31, eq. (3.2)].

Lemma 1.2. *For $m \in \mathbb{N}$, we have*

$$\sum_{j=0}^{m} \binom{m}{j} B_j(a)B_{m-j}(b) = m(a+b+1)B_{m-1}(a+b) - (m-1)B_m(a+b).$$

1.2 Riemann zeta function

The German mathematician Bernhard Riemann [14], in 1859, instigated Riemann zeta function while he was trying to understand the behavior of prime numbers.

The theory of the zeta function continuously motivated mathematicians to produce many elegant results. This zeta function is considered as one of the most important special functions in number theory due to its enormous applications in different branches of mathematics. This zeta function is defined as

$$\zeta(s) := \sum_{n=0}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1. \tag{1.9}$$

The analytic continuation of $\zeta(s)$ to the whole complex plane except at $s = 1$ has been shown by Riemann. Moreover, he showed that $\zeta(s)$ satisfies the following beautiful functional equation:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{1.10}$$

In the same paper, he conjectured that all the non trivial zeros of $\zeta(s)$ will lie on the line $\Re(s) = \frac{1}{2}$. This conjecture turned into the most celebrated hypothesis known as the Riemann hypothesis, which is one of the long standing Millennium Problems.

An exact formula for $\zeta(2k)$ was established by Euler. He represented the following formula for all even zeta values, for $k \in \mathbb{N}$,

$$\zeta(2k) = c_{2k}\pi^{2k}, \tag{1.11}$$

where $c_{2k} = (-1)^{k+1}\frac{2^{2k}B_{2k}}{2(2k)!}$. The above identity instantly gives transcendentality of $\zeta(2k)$. But we dot know a similar looking an exact formula for $\zeta(2k+1)$.

1.3 Reciprocal polynomials and their properties

In this thesis, our main results mostly concern with a special type of polynomials, namely, reciprocal polynomials. So we begin with the following definition.

Definition 1.2. *Given an nth degree polynomial*

$$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n,$$

where a_0, a_1,... a_n are from any arbitrary field, we say that $p(z)$ is reciprocal or self-inverse polynomial, if there exists a fixed complex number ω such that $|\omega| = 1$

so that,

$$p(z) = \omega p^*(z),$$

where

$$p^*(z) = z^n \bar{p}\left(\frac{1}{\bar{z}}\right) = \bar{a}_n + \bar{a}_{n-1}z + \ldots + \bar{a}_0 z^n.$$

Looking at the definition of reciprocal polynomial, we can make the following observations. The coefficients of a self-inverse or reciprocal polynomial satisfy the relation

$$a_j = \omega \bar{a}_{n-j}.$$

For a reciprocal polynomial with real coefficients the following observation can be easily made. Let $p(x) \in \mathbb{R}[x]$ be a non-zero polynomial of degree n with

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n.$$

We say that $p(x)$ is reciprocal polynomial if and only if

$$x^n p\left(\frac{1}{x}\right) = a_n + a_{n-1}x + a_{n-2}x^2 + \ldots + a_0 x^n = p(x).$$

That implies that the coefficients of real reciprocal polynomials satisfy

$$a_k = a_{n-k}, \ \forall \ k = 0, 1, \cdots, n.$$

Let us illustrate the reciprocal polynomials with some examples.

Example 1.1 (Examples of some reciprocal polynomials)**.** *For $x \in \mathbb{R}$ and $z \in \mathbb{C}$*
$p_1(x) = 1 + 7x^2 + 3x^3 + 7x^4 + x^5$,
$p_2(z) = z^4 + \left(\frac{1}{\sqrt{2}} + i\frac{1}{2}\right) z^3 + 7z^2 + z + \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$.

CHAPTER 2

Ramanujan type polynomials

2.1 Ramanujan's formula for $\zeta(2n+1)$

Srinivasa Ramanujan (1887 - 1920) is considered as one of the most genius mathematicians of all time. He has enormous contribution in the field of number theory. Ramanujan has gifted many elegant formulas to the world of mathematics throughout his life, especially in his famous notebooks and lost notebook. In the second notebook [33, p. 173, Ch. 14, Entry 21(i)] as well as in the lost notebook [34, p. 319, Entry (28)], Ramanujan mentioned the following remarkable identity about $\zeta(2k+1)$: For $a > 0$ and $b = \frac{\pi^2}{a}$ and non-zero integer k, we have

$$G_k(a) = (-1)^k G_k(b) - 2^{2k} \sum_{j=0}^{k+1} (-1)^{j-1} \frac{B_{2j}}{(2j)!} \frac{B_{2k+2-2j}}{(2k+2-2j)!} a^{k+1-j} b^j, \qquad (2.1)$$

where

$$G_k(x) = x^{-k} \left\{ \frac{1}{2}\zeta(2k+1) + \sum_{n=0}^{\infty} \frac{n^{-2k-1}}{e^{2xn} - 1} \right\}.$$

In 1972, Emil Grosswald [8] gave a notable extension of Ramanujan's formula (2.1). Let us recall the divisor function $\sigma_z(m) = \sum_{d|m} d^z$ for any $z \in \mathbb{C}$. For $\xi \in \mathbb{H}$, $k \in \mathbb{Z} - \{0\}$, we define

$$\mathfrak{F}_k(\xi) := \sum_{n=1}^{\infty} \sigma_{-k}(n) e^{2\pi i n \xi}.$$

Then we have

$$\mathfrak{F}_{2k+1}(\xi) - \xi^{2k}\mathfrak{F}_{2k+1}\left(-\frac{1}{\xi}\right) = \frac{1}{2}\zeta(2k+1)(\xi^{2k} - 1) + \frac{(2\pi i)^{2k+1}}{2\xi} \sum_{j=0}^{k+1} \xi^{2k+2-2j} \frac{B_{2j}}{(2j)!} \frac{B_{2k+2-2j}}{(2k+2-2j)!}, \quad (2.2)$$

Substituting $\xi = ib/\pi$, $ab = \pi^2$, with $a, b > 0$, the identity (2.2) is same as Ramanujan's identity (2.1) for $\zeta(2k+1)$. Later Gun, Murty and Rath [9] studied the finite sum involving Bernoulli numbers present on the right hand side of (2.2) and named it as Ramanujan polynomial. In the following section, we will discuss the properties of Ramanujan polynomials in detail.

2.2 Ramanujan polynomials

Definition 2.1. *For $k \in \mathbb{N}$ and $z \in \mathbb{C}$, the Ramanujan polynomials are defined as follows:*

$$R_{2k+1}(z) := \sum_{j=0}^{k+1} z^{2k+2-2j} \frac{B_{2j}}{(2j)!} \frac{B_{2k+2-2j}}{(2k+2-2j)!}. \quad (2.3)$$

Let us look at some examples of the Ramanujan polynomials followed by their properties studied by Murty, Symth and Wang [10], in order to get a good idea about them.

Example 2.1 (Ramanujan polynomials)**.** *A few examples of Ramanujan polynomials are listed below.*

$R_1(z) = \frac{1}{2\cdot 3!}(z^2 + 1)$,

zeros of $R_1(z)$ *are* : $\pm i$.

$R_3(z) = \frac{1}{6!}(-z^4 + 5z^2 - 1)$,

$zeros\ of\ R_3(z) are: \pm\frac{\sqrt{5\pm\sqrt{21}}}{2}.$

$R_5(z) = \frac{1}{12\cdot 7!}(-2z^6 + 7z^4 + 7z^2 - 2),$

$zeros\ of\ R_5(z) are: \pm i, \pm\frac{\sqrt{9\pm\sqrt{65}}}{4}.$

$R_7(z) = \frac{1}{10!}(-3z^8 + 10z^6 + 7z^4 + 10z^2 - 3),$

$zeros\ of\ R_7(z) are: \pm\rho,\ \pm\bar{\rho}, \pm\frac{\sqrt{13\pm\sqrt{133}}}{6}.$

$R_9(z) = \frac{1}{10!}(10z^{10} - 33z^8 - 22z^6 - 22z^4 - 33z^2 + 10),$

$zeros\ of\ R_9(z) are: \pm i,\ \pm\sqrt{\frac{43}{40} + \frac{3\sqrt{210}}{40} \pm \frac{1}{2}\sqrt{\frac{1029}{200} + \frac{129\sqrt{210}}{200}}},\ \pm\sqrt{\frac{43}{40} - \frac{3\sqrt{210}}{40} \pm \frac{i}{2}\sqrt{\frac{-1029}{200} + \frac{129\sqrt{210}}{200}}},$

$where\ \rho = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$

Properties of Ramanujan polynomials

In the paper [10], the authors proved some properties of Ramanujan polynomials which motivated us to look for similar behaviour for Ramanujan-type polynomials. Upon looking at the characteristics of Ramanujan polynomials we observe that indeed they resemble with the characteristics of Ramanujan-type polynomials. Here we mention some characteristics of Ramanujan polynomials and their similarity with the Ramanujan-type polynomials.

Theorem 2.1. *Ramanujan polynomials $R_{2k+1}(z)$ are self-inverse.*

Later we will see that Ramanujan-type polynomials $R_{2k+1,p}(z)$ are also self-inverse.

Theorem 2.2. *The largest real root of Ramanujan polynomial $R_{2k+1}(z)$ approaches* 2 *and does not surpass* 2.2 *as $k \to \infty$.*

Subsequently, we will prove that that for any prime number p and $k \in \mathbb{N}$, $R_{2k+1,p}(z)$ has only one repeated real root which is 0.

Theorem 2.3. *All complex roots of Ramanujan polynomial $R_{2k+1}(z)$ lie on unit circle.*

In this thesis, we will prove that all the complex roots of $R_{2k+1,p}\left(\frac{z}{p}\right)$ are on the unit circle.

2.3 Ramanujan-type polynomials

After discussing about Ramanujan polynomials and their properties, now we are ready to define Ramanujan-type polynomials. Recently Chourasiya, Jamal and Maji [5] found a new polynomial which is an analogue of Ramanujan polynomial $R_{2k+1}(z)$, they named it as Ramanujan-type polynomial. Let us define Ramanujan-type polynomials formally.

Definition 2.2. *Let $k \in \mathbb{N}$, $z \in \mathbb{C}$ and p be any prime number,*

$$R_{2k+1,p}(z) := \sum_{j=1}^{k} (p^{2j} - 1)(p^{2k+2-2j} - 1) \frac{B_{2j} B_{2k+2-2j}}{(2j)!(2k+2-2j)!} (pz)^{2k+2-2j}. \quad (2.4)$$

In the same paper [5, p. 7] authors have given one conjecture about the $R_{2k+1,p}(z)$.

Conjecture 1. *For $k \in \mathbb{N}$ and p be any prime number, the only real root of Ramanujan-type polynomial $R_{2k+1,p}(z)$ is $z = 0$ of multiplicity 2. Also the non-real roots are simple with all of them lying on the circle $|z| = \frac{1}{p}$.*

In this thesis, we prove this conjecture. Next we illustrate some examples of $R_{2k+1,p}(z)$.

Example 2.2 (Ramanujan-type polynomials)**.** *For $k \in \{1, 2, 3, 4\}$ and fixing $p = 2$ and $p = 3$.*

$R_{3,2}(z) = \frac{z^2}{4}$,

$zeros: 0, 0.$

$R_{5,2}(z) = -\frac{z^4}{12} - \frac{z^2}{48}$,

$zeros: 0, 0, \pm\frac{i}{2}.$

$R_{7,2}(z) = \frac{z^6}{30} + \frac{z^4}{144} + \frac{z^2}{480}$,
$zeros: 0, 0, \pm\sqrt{-\frac{5}{48} - \frac{i\sqrt{119}}{48}}, \pm\sqrt{-\frac{5}{48} + \frac{i\sqrt{119}}{48}}$.
$R_{9,2}(z) = -\frac{17z^8}{1260} - \frac{z^6}{360} - \frac{z^4}{1440} - \frac{17z^2}{80640}$,
$zeros: 0, 0, \pm\frac{i}{2}, \pm\sqrt{\frac{3}{136} - \frac{i\sqrt{1147}}{136}}, \pm\sqrt{\frac{3}{136} + \frac{i\sqrt{1147}}{136}}$.
$R_{3,3}(z) = 4z^2$,

$zeros: 0, 0$.

$R_{5,3}(z) = -6z^4 - \frac{2z^2}{3}$,

$zeros: 0, 0, \pm\frac{i}{3}$.

$R_{7,3}(z) = \frac{117z^6}{10} + z^4 + \frac{13z^2}{90}$,
$zeros: 0, 0, \pm\frac{\frac{2}{3}+i}{\sqrt{13}}, \pm\frac{\frac{2}{3}-i}{\sqrt{13}}$.
$R_{9,3}(z) = -\frac{3321z^8}{140} - \frac{29z^6}{20} - \frac{13z^4}{60} - \frac{41z^2}{1260}$,
$zeros: 0, 0 \pm \frac{i}{3}, \pm\sqrt{\frac{16}{1107} - \frac{i\sqrt{14873}}{110}}, \pm\sqrt{\frac{16}{1107} + \frac{i\sqrt{14873}}{110}}$.

CHAPTER 3

Main Results

In this chapter, we state a few results that we found while observing the behavior of Ramanujan-type polynomials. First we mention the conjecture given by Chourasiya, Jamal and Maji [5, p. no. 7], which is our main theorem.

Theorem 3.1. *For $k \in \mathbb{N}$ and any prime number p, the only real root of Ramanujan-type polynomial*

$$R_{2k+1,p}(z) = \sum_{j=1}^{k} (p^{2j} - 1)(p^{2k+2-2j} - 1) \frac{B_{2j} B_{2k+2-2j}}{(2j)!(2k+2-2j)!} (pz)^{2k+2-2j}, \tag{3.1}$$

is at $z = 0$ of multiplicity 2. Moreover, all the non-real roots are simple and lie on the circle $|z| = \frac{1}{p}$.

Let us mention some important lemmas in this direction.

Lemma 3.2. *Let $f_p : [0,\ 2k-2] \to \mathbb{R}$ be a function defined as*

$$f_p(x) := (p^{x+2} - 1)(p^{2k-x} - 1), \tag{3.2}$$

where $k \geq 1$ and p be any prime number. Then $f_p(x)$ has its minima at $x = 0, 2k-2$, with the same minimum value $(p^2 - 1)(p^{2k} - 1)$ at both of the points.

Proof of Lemma 3.2. For a fixed prime p and $k \geq 1$, we have defined

$$f_p(x) := (p^{x+2} - 1)(p^{2k-x} - 1).$$

Computing derivatives with respect to x, we have

$$f_p'(x) = -p^{x+2}\log(p) + p^{2k-x}\log(p),$$
$$\Rightarrow f_p''(x) = -\left(p^{x+2} + p^{2k-x}\right)\log^2(p).$$

We can check easily that $f_p''(x)$ is always negative in the given domain, so all the extrema points obtained by equating $f_p'(x)$ to zero will correspond to the local maximum values. Now we find extrema points of the function $f_p(x)$ by equating $f_p'(x)$ to zero, that is,

$$f_p'(x) = 0 \Rightarrow p^{x+2} = p^{2k-x}.$$

Therefore,

$$x = k - 1.$$

Thus, $x = k-1$ is the only local maxima point in the interval $[0, 2k-2]$. Since the function is continuously differentiable, so we conclude that $x = 2k - 2$ and $x = 0$ are the only points of minima. Moreover,

$$f_p(0) = f_p(2k-2) = (p^2 - 1)(p^{2k} - 1).$$

Hence the result follows. $\square$

Lemma 3.3. *Let p be a fixed prime number. The function $g : [0, 2k-2] \to \mathbb{R}$ defined as*

$$g(x) := \frac{(p^{x+2} - 1)}{(2^{x+2} - 1)}\frac{(p^{2k-x} - 1)}{(2^{2k-x} - 1)}, \quad \forall k \geq 1, \tag{3.3}$$

has an upper bound $\frac{(p^{k+1}-1)^2}{3(2^{2k}-1)}$.

Proof of Lemma 3.3. We can write $g(x)$ in terms of $f_p(x)$ as follows:

$$g(x) = \frac{f_p(x)}{f_2(x)}.$$

Therefore, the ratio of maximum value of $f_p(x)$ and the minimum value of $f_2(x)$ will provide us the upper bound for $g(x)$. From Lemma 3.2, we know that the maxima of $f_p(x)$ is attained at $x = k-1$ and the corresponding maximum value is,

$$f_p(k-1) = (p^{k+1}-1)^2. \tag{3.4}$$

Also the minimum value of $f_p(x)$ is attained at $x = 0$ and in particular for $p = 2$ the minimum value will be,

$$f_2(0) = (2^2-1)(2^{2k}-1) = 3(2^{2k}-1). \tag{3.5}$$

Hence, from (3.4) and (3.5), we have

$$g(x) \leq \frac{(p^{k+1}-1)^2}{3(2^{2k}-1)}.$$

$\square$

CHAPTER 4

Some well known results

In this chapter, we mention some familiar results which are going to be useful in proving the main theorem.

Theorem 4.1 (Lakatos [11]). *If a reciprocal polynomial*

$$p(x) = \sum_{i=0}^{k} a_i x^i$$

of degree k with real coefficients satisfies the condition

$$|a_k| \geq \sum_{i=0}^{k} |a_i - a_k|, \tag{4.1}$$

then the polynomial $p(x)$ certainly bears all its zeros on the unit circle. If the inequality (4.1) *is strict then the multiplicity of all the zeros of $p(x)$ is one.*

Later, this theorem has been modified by many mathematicians, thus the condition (4.1) for reciprocal polynomials has many improvements. Schinzel [16] gave one generalization of Theorem 4.1 for any reciprocal polynomial over $\mathbb{C}$.

Theorem 4.2 (Schinzel [16]). *Let $p(z)$ be a reciprocal polynomial, say,*

$$p(z) = \sum_{j=0}^{k} a_j z^j$$

of degree k with $a_j \in \mathbb{C}, \forall j$. If

$$|a_k| \geq \inf_{\substack{c,d \in \mathbb{C} \\ |d|=1}} \sum_{j=0}^{k} |ca_j - d^{k-j} a_k|, \tag{4.2}$$

then all the zeros of $p(z)$ will be located on $|z| = 1$.

In 2011, Lalin and Roger [13] used these above mentioned theorems to prove that various reciprocal polynomials have all their roots on $|z| = 1$.

Theorem 4.3 (Lalin and Roger [13]). *If B_n denotes the nth Bernoulli number, then for $k \geq 2$ the polynomial*

$$P_k(z) = \pi^{2k} \sum_{j=0}^{k} (2^{2j} - 1)(2^{2k-2j} - 1) \frac{B_{2j} B_{2k-2j}}{(2j)!(2k-2j)!} z^j, \tag{4.3}$$

is a reciprocal polynomial with all its complex zeros lying on $|z| = 1$.

For $p = 2$, our Theorem 3.1 resembles with Theorem 4.3, so we can easily observe that Theorem 4.3 is a particular case of our main Theorem 3.1. This encouraged us to prove Theorem 3.1 for any prime number p.

CHAPTER 5

Proof of main theorem

Before proving our main theorem, we need the following lemma.

Lemma 5.1. *For $k \in \mathbb{N}$, we have*

$$|A_{k-1}| \geq \inf_{\substack{c,d \in \mathbb{C} \\ |d|=1}} \sum_{j=0}^{k-1} |cA_j - dA_{k-1}|,$$

where A_j is the $(2k-2-2j)^{th}$ coefficient of $z^{-2} R_{2k+1,p}\left(\frac{z}{p}\right)$.

Proof. By appropriate shifting, from the definition (3.1) of $R_{2k+1,p}(z)$, we can write

$$R_{2k+1,p}(z) = \sum_{j=0}^{k-1} (p^{2j+2}-1)(p^{2k-2j}-1) \frac{B_{2j+2}B_{2k-2j}}{(2j+2)!(2k-2j)!} (pz)^{2k-2j}. \tag{5.1}$$

Replacing z by $\frac{z}{p}$, we have

$$\begin{aligned} R_{2k+1,p}\left(\frac{z}{p}\right) &= z^2 \sum_{j=0}^{k-1} (p^{2j+2}-1)(p^{2k-2j}-1) \frac{B_{2j+2}B_{2k-2j}}{(2j+2)!(2k-2j)!} (z)^{2k-2-2j} \\ &= z^2 H(z), \end{aligned} \tag{5.2}$$

where $H(z)$ is defined as

$$H(z) := \sum_{j=0}^{k-1} (p^{2j+2}-1)(p^{2k-2j}-1)\frac{B_{2j+2}B_{2k-2j}}{(2j+2)!(2k-2j)!}(z)^{2k-2-2j}. \tag{5.3}$$

We denote the coefficient of $z^{2k-2-2j}$ in (5.3) as

$$A_j = (p^{2j+2}-1)(p^{2k-2j}-1)\frac{B_{2j+2}B_{2k-2j}}{(2j+2)!(2k-2j)!}. \tag{5.4}$$

Now we show that the polynomial $H(z)$ satisfies Theorem 4.2, that is, the coefficients of $H(z)$ satisfy the following inequality:

$$|A_{k-1}| \geq \inf_{\substack{c,d\in\mathbb{C} \\ |d|=1}} \sum_{j=0}^{k-1} |cA_j - d^{k-1-j}A_{k-1}|. \tag{5.5}$$

In particular, for $d=1$, we only need to show

$$|A_{k-1}| \geq \sum_{j=0}^{k-1} |cA_j - A_{k-1}|. \tag{5.6}$$

Using the lower bound for Bernoulli numbers i.e., Lemma 1.1 in (5.3), we have

$$\begin{aligned} |A_j| &= (p^{2j+2}-1)(p^{2k-2j}-1)\frac{|B_{2j+2}||B_{2k-2j}|}{(2j+2)!(2k-2j)!} \\ &> \frac{4(p^{2j+2}-1)(p^{2k-2j}-1)}{(2\pi)^{2k+2}}. \end{aligned} \tag{5.7}$$

Similarly, using the upper bound for B_{2k}, we get

$$\begin{aligned} |A_{k-1}| &= (p^{2k}-1)(p^2-1)\frac{|B_{2k}||B_2|}{(2k)!2!} \\ &< (p^{2k}-1)(p^2-1)\frac{2.(2k)!}{(2\pi)^{2k}(1-2^{1-2k})(2k)!}\frac{2.2!}{(2\pi)^2(1-2^{1-2})2!} \\ &= \frac{8(p^{2k}-1)(p^2-1)}{(2\pi)^{2k+2}(1-2^{1-2k})}. \end{aligned} \tag{5.8}$$

Since the sign of B_{2n} is $(-1)^{n+1}$, thus the sign of A_j is $(-1)^{k+1}, \forall\ j$. Therefore,

$$|cA_j - A_{k-1}| = (-1)^{k+1}(cA_j - A_{k-1}).$$

Now we shall try to find a value of positive c for which

$$\begin{aligned} &|cA_j - A_{k-1}| = (-1)^{k+1}(cA_j - A_{k-1}) > 0 \\ \Leftrightarrow \quad & c|A_j| - |A_{k-1}| > 0. \end{aligned} \tag{5.9}$$

Now use properties (1.3), (1.4) of Bernoulli numbers to see that

$$B_{2j+2}\left(\frac{1}{2}\right) - B_{2j+2}(0) = B_{2j+2}(2^{1-2j-2}-1) - B_{2j+2} = B_{2j+2}(2^{1-2j-2}-2)$$

$$\Rightarrow B_{2j+2} = \frac{B_{2j+2}(\frac{1}{2}) - B_{2j+2}(0)}{2(2^{-2j-2}-1)}. \tag{5.10}$$

Similarly,

$$B_{2k-2j} = \frac{B_{2k-2j}(\frac{1}{2}) - B_{2k-2j}(0)}{2(2^{-2k+2j}-1)}. \tag{5.11}$$

Now employing the above identities (5.10) and (5.11) in (5.26), we obtain

$$\begin{aligned} A_j =&(p^{2j+2}-1)(p^{2k-2j}-1)\frac{B_{2j+2}B_{2k-2j}}{(2j+2)!(2k-2j)!} \\ =&\binom{2k+2}{2j+2}(p^{2j+2}-1)(p^{2k-2j}-1)\frac{B_{2j+2}B_{2k-2j}}{(2k+2)!} \\ =&\binom{2k+2}{2j+2}\frac{(p^{2j+2}-1)(p^{2k-2j}-1)}{(2k+2)!}\frac{B_{2j+2}(\frac{1}{2}) - B_{2j+2}(0)}{2(2^{-2j-2}-1)}\frac{B_{2k-2j}(\frac{1}{2}) - B_{2k-2j}(0)}{2(2^{-2k+2j}-1)}. \end{aligned} \tag{5.12}$$

Recall that the sum on the right hand side of (5.6) is

$$\sum_{j=0}^{k-1} |cA_j - A_{k-1}| = (-1)^{k+1}\sum_{j=0}^{k-1}(cA_j - A_{k-1}).$$

Using (5.12), we can write

$$\begin{aligned} &(-1)^{k+1}\sum_{j=0}^{k-1} cA_j = \sum_{j=0}^{k-1} c|A_j| \\ &= \sum_{j=0}^{k-1} c\binom{2k+2}{2j+2}\frac{(p^{2j+2}-1)(p^{2k-2j}-1)}{(2k+2)!}\left|\frac{B_{2j+2}(\frac{1}{2}) - B_{2j+2}(0)}{2(2^{-2j-2}-1)}\frac{B_{2k-2j}(\frac{1}{2}) - B_{2k-2j}(0)}{2(2^{-2k+2j}-1)}\right| \\ &= \sum_{j=0}^{k-1} D(j)\binom{2k+2}{2j+2}\left|\left(B_{2j+2}\left(\frac{1}{2}\right) - B_{2j+2}(0)\right)\left(B_{2k-2j}\left(\frac{1}{2}\right) - B_{2k-2j}(0)\right)\right|, \end{aligned} \tag{5.13}$$

where

$$D(j) = \frac{c}{4(2k+2)!}\frac{(p^{2j+2}-1)(p^{2k-2j}-1)}{(2^{-2j-2}-1)(2^{-2k+2j}-1)} = \frac{2^{2k}c}{(2k+2)!}\frac{(p^{2j+2}-1)}{(2^{2j+2}-1)}\frac{(p^{2k-2j}-1)}{(2^{2k-2j}-1)}. \tag{5.14}$$

Using the definition (3.3) of $g(x)$ and Lemma 3.3, one can find the following upper bound for $D(j)$:

$$D(j) = \frac{2^{2k}c}{(2k+2)!}g(2j) < \frac{2^{2k}c}{(2k+2)!}\frac{(p^{k+1}-1)^2}{3(2^{2k}-1)}. \tag{5.15}$$

Utilize (5.15) in (5.13) to obtain

$$\sum_{j=0}^{k-1} c|A_j| < \frac{2^{2k}c}{(2k+2)!}\frac{(p^{k+1}-1)^2}{3(2^{2k}-1)}\sum_{j=0}^{k-1}\binom{2k+2}{2j+2}$$
$$\left|\left(B_{2j+2}\left(\frac{1}{2}\right) - B_{2j+2}(0)\right)\left(B_{2k-2j}\left(\frac{1}{2}\right) - B_{2k-2j}(0)\right)\right|. \tag{5.16}$$

Now we will use Lemma 1.2 to calculate the value of the sum given in (5.16). That is,

$$\begin{aligned}
&\sum_{j=0}^{k-1}\binom{2k+2}{2j+2}\left|\left(B_{2j+2}\left(\frac{1}{2}\right) - B_{2j+2}(0)\right)\left(B_{2k-2j}\left(\frac{1}{2}\right) - B_{2k-2j}(0)\right)\right| \\
&= (-1)^{k-1}\sum_{j=0}^{k-1}\binom{2k+2}{2j+2}\left[B_{2j+2}\left(\frac{1}{2}\right)B_{2k-2j}\left(\frac{1}{2}\right) - B_{2j+2}\left(\frac{1}{2}\right)B_{2k-2j}(0)\right. \\
&\left. - B_{2j+2}(0)B_{2k-2j}\left(\frac{1}{2}\right) + B_{2j+2}(0)B_{2k-2j}(0)\right] \\
&= (-1)^k 4(2k+1)(1-2^{-2k-2})B_{2k+2}.
\end{aligned} \tag{5.17}$$

Employing (5.17) in (5.16), we get

$$\sum_{j=0}^{k-1} c|A_j| < \frac{2^{2k+2}c}{(2k+2)!}\frac{(p^{k+1}-1)^2}{3(2^{2k}-1)}(2k+1)(1-2^{-2k-2})|B_{2k+2}|. \tag{5.18}$$

Our main aim is to find a positive c such that (5.6) holds. From (5.6), we have

$$\begin{aligned}
&|A_{k-1}| > \sum_{j=0}^{k-1}|cA_j - A_{k-1}| \\
&\Rightarrow (-1)^{k+1}A_{k-1} > (-1)^{k+1}\sum_{j=0}^{k-1}(cA_j - A_{k-1}) \\
&\Rightarrow (1+k)(-1)^{k+1}A_{k-1} > (-1)^{k+1}\sum_{j=0}^{k-1}cA_j \\
&\Rightarrow (1+k)|A_{k-1}| > \sum_{j=0}^{k-1}c|A_j|.
\end{aligned} \tag{5.19}$$

To prove (5.19), we shall show that the left hand side of (5.19) is bigger than the upper bound of $\sum_{j=0}^{k-1} c|A_j|$, which we have already found in (5.18). So we need to find a positive c such that the following inequality holds:

$$(1+k)|A_{k-1}| > \frac{c}{(2k+2)!}\frac{(p^{k+1}-1)^2}{3(2^{2k}-1)}(2k+1)(2^{2k+2}-1)|B_{2k+2}|. \tag{5.20}$$

This implies that c should satisfy the below inequality:

$$\frac{3(1+k)(2^{2k}-1)(2k+2)!|A_{k-1}|}{(2k+1)(p^{k+1}-1)^2(2^{2k+2}-1)|B_{2k+2}|} > c. \tag{5.21}$$

Moreover, using the lower bound (5.7) of $|A_{k-1}|$ and upper bound (1.1) for $|B_{2k+2}|$, one can show that

$$\frac{3(1+k)(2^{2k}-1)(2k+2)!|A_{k-1}|}{(2k+1)(p^{k+1}-1)^2(2^{2k+2}-1)|B_{2k+2}|} > c_{p,k} \tag{5.22}$$

where

$$c_{p,k} := \frac{(1+k)(p^{2k}-1)(p^2-1)(2^{2k}-1)(2^2-1)(2^{2k+1}-1)}{2^{2k}(2k+1)(p^{k+1}-1)^2(2^{2k+2}-1)}. \tag{5.23}$$

Note the constant $c_{p,k}$ is a positive constant. Thus for any positive $c < c_{p,k}$ will satisfy the equation (5.19). This proves the inequality (5.6) for the polynomial $H(z)$ and consequently the result follows. □

Now we are ready for proving the main theorem.

Proof of Theorem 3.1. From (5.2), we know

$$\begin{aligned} R_{2k+1,p}\left(\frac{z}{p}\right) &= z^2\sum_{j=0}^{k-1}(p^{2j+2}-1)(p^{2k-2j}-1)\frac{B_{2j+2}B_{2k-2j}}{(2j+2)!(2k-2j)!}(z)^{2k-2-2j} \\ &= z^2H(z), \end{aligned} \tag{5.24}$$

where $H(z)$ is defined as

$$H(z) = \sum_{j=0}^{k-1} A_j z^{2k-2-2j}, \tag{5.25}$$

and

$$A_j = (p^{2j+2}-1)(p^{2k-2j}-1)\frac{B_{2j+2}B_{2k-2j}}{(2j+2)!(2k-2j)!}. \tag{5.26}$$

We can easily check that A_j and A_{k-1-j} are same. Hence the Ramanujan-type polynomial (5.24) is reciprocal. Now from equation (5.24), we can easily see that $R_{2k+1,p}\left(\frac{z}{p}\right)$ has degree $2k$ containing only even powers of z, with z^2 being the least power of z. Therefore, $R_{2k+1,p}\left(\frac{z}{p}\right)$ has a real zero at $z=0$ of multiplicity

2. Utilizing Lemma 5.1, we can see that $H(z)$ satisfies Theorem 4.2, so we can conclude that $H(z)$ has all its roots on the unit circle. Also, since the inequality (5.6) is strict so all the zeros of $H(z)$ are simple. Thus, the Ramanujan-type polynomial $R_{2k+1,p}\left(\frac{z}{p}\right)$ has all the zeros on the unit circle except $z = 0$.

Next we show that there is no real root except zero. Let us suppose $R_{2k+1,p}\left(\frac{z}{p}\right)$ has a real root except $z = 0$, then by Lemma 5.1 it will be on the unit circle. Consequently, the possibility of the non-trivial real root will be either $z = 1$ or $z = -1$ or both. So, putting $z = 1$ in (5.3), we get

$$\begin{aligned} & H(1) = 0, \\ \Rightarrow & \sum_{j=0}^{k-1} (p^{2j+2} - 1)(p^{2k-2j} - 1) \frac{B_{2j+2} B_{2k-2j}}{(2j+2)!(2k-2j)!} = 0, \\ \Rightarrow & \sum_{j=0}^{k-1} A_j = 0. \end{aligned} \tag{5.27}$$

We have already seen that the sign of A_j is $(-1)^{k+1}$ for all j. Note that all even Bernoulli numbers are never zero, therefore depending on the parity of k the sum $\sum_{j=0}^{k-1} A_j$ is either negative or positive. This is a contradiction to (5.27). This completes the proof that $z = 1$ is not a root of $H(z)$. Also, since $H(z)$ is an even function, so by the same argument we can conclude that $z = -1$ is not a root of $H(z)$ as well. Hence, all roots of $H(z)$ are complex and simple, and lie on the unit circle. This completes the proof. □

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