Möbius Transformation and the Cassinian Metric

M.Sc. Thesis

by

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DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

JUNE 2023

Möbius Transformation and the Cassinian Metric

A THESIS

Submitted in partial fulfillment of the requirements for the award of the degree of

Master of Science

by

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JUNE 2023

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **Möbius Transformation and the Cassinian Metric** in the partial fulfilment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF MATHEMATICS**, **Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2022 to June 2023 under the supervision of **Prof. Swadesh Kumar Sahoo**, Professor, Department of Mathematics, IIT Indore. The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Date: 7 June 2023

Dedicated to my Family

We are what our thoughts have made us;

So take care about what you think.

Thoughts live; They travel far.

–***Swami Vivekananda***

Acknowledgements

I am incredibly grateful to my supervisor Prof. Swadesh Kumar Sahoo for his assistance and guidance throughout my whole M.Sc. course. He gave his most profound concern for my overall well-being in these challenging times. His friendly nature, enthusiasm, and supportive attitude encourage me to successfully complete this thesis. His confidence and trust in me always push me to do better in every direction. Also, he conveyed the spirit of adventure and pushed me forward toward my area of interest concerning research. I believe that it is an honor for me to work with him. I thank the Almighty for bestowing his blessings upon me. I am very thankful to my PSPC members Dr. Anand Parkash, Dr. Vijay Kumar Sohani as well as DPGC Convener and HOD for their continuous support. I am also thankful to all our teachers for their kind and valuable suggestion. I am too fortunate to have all these teachers that even my most profound gratitude is not enough. This acknowledgement note would be irrelevant without being thankful to my family members. Their support and faith gradually and firmly boost my confidence from inside. They gave me the strength to pursue my interests even in difficult times. I would like to extend my sincere thanks to my seniors Sheetal

Sanjay Wankhede and Ritwick Maity for their supportive and kind nature and last but not the least i also would to thanks my friends for always being there with me. My heart is full of favours received from every single person. I remain indebted and grateful to every one of you.

Abstract

In this thesis, we mainly consider Möbius transformations and the Cassinian metric in the complex plane. In view of the importance of the hyperbolic metric on the unit disk, we have focused on some results associated with Möbius transformation, the Cassinian metric and the hyperbolic metric.

The concept of inverse (or symmetric) points in circles plays a crucial role in Möbius transformation setting. Some of the basic results such as Ptolemy's identity, characterization of disk automorphism, finding Möbius transformations from a pair of circles onto concentric circles, are proved using the concept of inverse points in circles.

The construction of the Cassinian distance between two points in a subdomain of the complex plane is studied in two ways: (i) identifying the maximal Cassinian ovals inside the domain with the two points as its foci, and (ii) computing the Euclidean distance between the inversions of those two points in the unit circle circle centred at the boundary point at which the maximal Cassinian oval meets. Various classical properties of the Cassinian metric are presented. Some of them are listed below:

- The Cassinian metric defines a complete metric in bounded domains.
- The Cassinian metric is not Möbius invariant, however, it is Möbius quasi-invariant.
- The geodesics of the Cassinian metric, if exist, are nothing but part of some circular arcs.

Note that the hyperbolic metric is Möbius invariant whereas we already stated that the Cassinian metric does not satisfy the Möbius invariant property. Therefore, it is important to investigate how these two metrics are comparable. The thesis is ended with such comparisons followed by some problems for future directions.

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CHAPTER 1

Introduction

This thesis comprises of six chapters. The present chapter is of introductory type. The second chapter contains certain elementary properties and examples on Möbius transformations. The Cassinian metric, with it computations and several basic properties, is considered in Chapter 3. Most of the main results that we wish to highlight are covered in Chapter 4. The Poincare's hyperbolic metric of the unit disk, which got special attention in Chapter 5, is compared with the Cassinian metric. Finally, the last chapter concludes the thesis.

The Möbius transformation is one of the important mappings in complex analysis. This map is composed with certain elementary transformations such as translation, rotation, magnification, inversion. The concept of cross ratio and inverse (or symmetric) points play crucial roles in Möbius transformation setting. Both the concepts have interesting geometric behaviours. For instance, a Möbius transformation carries circles onto circles. Here, a circle with infinite radius is treated as a straight line. Also, the classical Ptolemy's identity which states that the product of two diagonals in a quadrilateral equals the sum of product of its two opposite sites provided the four vertices lie on a circle. This can also be proved by using the idea of symmetric points in a circle. One of the important applications of the classical symmetric principle is that it characterizes the most general Möbius transformation of the unit disk onto itself and to half-planes. Moreover, reflection of a circle in another circle can also be computed. This idea also helps us to find Möbius transformations from a pair of circles onto concentric circles.

The Cassinian metric [5] was introduced by Ibragimov in 2009 and it was further considered later in different prospectives (see [4, 9]). The classical Cassinian ovals play important roles in computing the Cassinian metric. The Cassinian ovals are very important in real life problems (see [7, 10]). Some of the applications are listed in Chapter 3. It defines a complete metric in bounded domains. This metric is not Möbius invariant, however, it is Möbius quasi-invariant. The geodesics of the Cassinian metric, if exist, are nothing but part of some circular arcs. One may ask a question about how the Cassinian metric is compared with the Poincare's hyperbolic metric [8] on the unit disk. Note that the hyperbolic metric is Möbius invariant whereas we already stated that the Cassinian metric does not satisfy the Möbius invariant property. Therefore, it is important to investigate how these two metrics are comparable. The thesis is ended with such comparisons followed by some problems for future directions.

We now list some of the common notations that are used in this thesis. The symbols $\mathbb{R}$ and $\mathbb{C}$ respectively stand for the real line and complex plane. The extended complex plane or the Riemann sphere is denoted by $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. The disk centered at a with radius r is denoted by $\mathbb{D}(a, r)$ and its boundary circle is denoted by $\mathbb{S}(a, r)$. We denote by $\mathbb{D} := \mathbb{D}(0, 1)$, the unit disk, and by $\mathbb{S} := \mathbb{S}(0, 1)$, the unit circle.

CHAPTER 2

Möbius Transformation

2.1 Definition

A Möbius transformation (linear fractional transformation or bilinear transformation) is a rational function of the form

$$T(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0,$$

where a, b, c, d are complex numbers. In general, it is a function $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined by

$$T(z) = \begin{cases} \dfrac{az + b}{cz + d}, & z \in \mathbb{C} \setminus \{-d/c\}; \\ \dfrac{a}{c}, & z = \infty; \\ \infty, & z = -\dfrac{d}{c}; \end{cases}$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. The Möbius transformation $T(z)$ is well-defined and bijective. Indeed, we have

$$\begin{aligned} T(z_1) = T(z_2) &\iff \frac{az_1 + b}{cz_1 + d} = \frac{az_2 + b}{cz_2 + d} \\ &\iff acz_1z_2 + adz_1 + bcz_2 + bd = acz_1z_2 + bcz_1 + adz_2 + bd \\ &\iff (ad - bd)z_1 = (ad - bc)z_2 \\ &\iff z_1 = z_2, \end{aligned}$$

since $ad - bc \neq 0$. This shows that T is well-defined and one-one.

Secondly, $w = T(z)$ has exactly one root and we have

$$z = T^{-1}(w) = \frac{dw - b}{-cw = a}, \quad ad - bc \neq 0.$$

Thus, T is also onto.

The following elementary transformations are nothing but special cases of Möbius transformation:

- **Translation:** $T(z) = z + b$;
- **Dilation/Magnification:** $T(z) = az$, $a \neq 0$;
- **Rotation:** $T(z) = \alpha z, |\alpha| = 1$ or $T(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$;
- **Inversion:** $T(z) = 1/z$.

Proposition 2.1.1. *[11, pp. 66-68] A Möbius transformation is the composition of elementary transformations.*

2.2 The Cross Ratio

Given three distinct points z_2, z_3, z_4 in $\mathbb{C}_\infty$, there exists a transformation T which carries them into $1, 0, \infty$ in this order. Such a transformation is represented in terms of cross ratio defined by

$$T(z) = (z, z_2, z_3, z_4) = \begin{cases} \dfrac{(z-z_3)(z_2-z_4)}{(z-z_4)(z_2-z_3)}, & z_2, z_3, z_4 \neq \infty; \\ \dfrac{z-z_3}{z-z_4}, & z_2 = \infty; \\ \dfrac{z_2-z_4}{z-z_4}, & z_3 = \infty; \\ \dfrac{z-z_3}{z_2-z_3}, & z_4 = \infty. \end{cases}$$

We now present some well-known results which are indeed used in the thesis. The cross ratio is invariant under Möbius transformations. More precisely, we have

Theorem 2.2.1. *[1, p. 79] If z_1, z_2, z_3, z_4 are distinct points in $\mathbb{C}_\infty$ and $T(z)$ is a Möbius transformation, then $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.*

The following theorem describes the positions of the four distinct points if their cross ratio is real.

Theorem 2.2.2. *[1, p. 79] The cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle.*

Proof. Let $S(z) = (z, z_2, z_3, z_4)$ be a Möbius transformation. It follows that $z = S^{-1}(z, z_2, z_3, z_4)$ and hence we have

$$S^{-1}(\mathbb{R}) = \{z \in \mathbb{C} : (z, z_2, z_3, z_4) \in \mathbb{R}\}.$$

We shall prove that this collection is a circle. In fact, we are proving that image of the real axis under any Möbius transformation is a circle. Let $w = S^{-1}(\mathbb{R})$. Since Möbius transformation are bijective, there exists a point $x \in \mathbb{R}$ such that $S^{-1}(x) = w$. Thus, $S(w) \in \mathbb{R}$. This is equivalent to $S(w) = \overline{S(w)}$. Since $S(w)$ is of the form $(aw+b)/(cw+d)$, the identity $S(w) = \overline{S(w)}$ is equivalent to

$$(a\overline{c} - \overline{a}c)|w|^2 + (a\overline{d} - \overline{b}c)w + (b\overline{c} - \overline{a}d)\overline{w} + (b\overline{d} - \overline{b}d) = 0,$$

which represent a circle. Indeed, after a simple computation we obtain

$$\left|z + \frac{\overline{a}d - \overline{c}b}{\overline{a}c - \overline{c}a}\right| = \left|\frac{ad - bc}{\overline{a}c - \overline{c}a}\right|,$$

which is the equation of a circle. Taking $z = z_1$, we conclude that z_1 lies on the circle. By considering $M(z) = (z_1, z, z_3, z_4)$, one can similarly verify that the collection

$$N^{-1}(\mathbb{R}) = \{z \in \mathbb{C} : (z_1, z, z_3, x_4) \in \mathbb{R}\}$$

describe a circle. Choosing $z = z_2$, we show that z_2 lies on the circle. Similarly, we can see that z_3 and z_4 lie on the circle. This completes the proof. □

We next provide a very important geometry of Möbius transformations. This can be proved by a general technique, however, here we provide its proof using the concept of cross ratio. Indeed, we use Theorems 2.2.1 and 2.2.2.

Theorem 2.2.3. *[1, p. 80] A Möbius transformation carries circles onto circles.*

Proof. Consider a circle containing the three distinct points z_2, z_3, z_4. Let z be an arbitrary point on that circle. Let $S(z)$ be a Möbius transformation. By Theorem 2.2.1, we know that the cross ratio is invariant under Möbius transformations. Then, we have

$$(z, z_2, z_3, z_4) = (S(z), S(z_2), S(z_3), S(z_4)).$$

Since z, z_2, z_3, z_4 lie on a circle, by Theorem 2.2.2, the cross ratio $(z, z_2, z_3, z_4) \in \mathbb{R}$ and hence $(S(z), S(z_2), S(z_3), S(z_4)) \in \mathbb{R}$. This implies that all the points $S(z), S(z_2), S(z_3), S(z_4)$ lie on a circle. Since $S(z)$ lies on a circle and z was an arbitrary point on the circle, conclusion follows. □

As an application of Theorems 2.2.1, 2.2.2 and 2.2.3, we now provide a proof of the classical Ptolemy's identity (see [1, p. 80, Exc. 3]).

Theorem 2.2.4 (Ptolemy's Theorem)**.** *If the consecutive vertices z_1, z_2, z_3, z_4 of a quadrilateral lie on a circle, then*

$$|z_1 - z_3||z_2 - z_4| = |z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_1 - z_4|.$$

If the vertices of the quadrilateral do not lie on a circle, then the inequality

$$|z_1 - z_3||z_2 - z_4| \leq |z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_1 - z_4|.$$

holds.

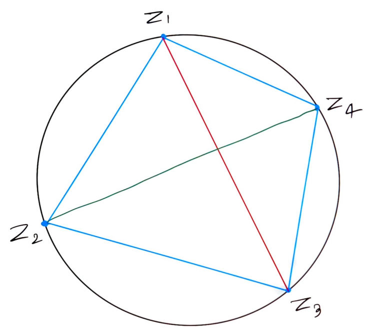

Figure 2.1: A cyclic quadrilateral

Remark. In the above inequality, the equality may also hold for some choices of points z_1, z_2, z_3, z_4 not lying on a circle. Indeed, one can find four such complex numbers. However, at this moment, we do not have a specific choices on this.

Proof of Theorem 2.2.4 Let C be the circle passing through z_1, z_2, z_3, z_4. Let T be the Möbius transformation such that

$$T(z_2) = 1,\ T(z_3) = 0,\ T(z_4) = \infty.$$

Since the Möbius transformation T preserves C (see Theorem 2.2.3) and also all the four points lie on C, by Theorem 2.2.2 it follows that $T(z_1) = a\,(> 1) \in \mathbb{R}$. Then the invariance property of the cross ratio under a Möbius transformation (Theorem 2.2.1) yields

$$(z_2, z_3, z_1, z_4) = (T(z_2), T(z_3), T(z_1), T(z_4)) = (1, 0, a, \infty) = \frac{a-1}{a} > 0.$$

Taking modulus on both sides, we have

$$|(z_2, z_3, z_1, z_4)| = \left|\frac{(z_2 - z_1)(z_3 - z_4)}{(z_2 - z_4)(z_3 - z_1)}\right| = \frac{a-1}{a}. \tag{2.1}$$

Further, we have

$$(z_2, z_1, z_3, z_4) = (T(z_2), T(z_1), T(z_3), T(z_4)) = (1, a, 0, \infty) = \frac{1}{a},$$

and thus

$$|(z_2, z_1, z_3, z_4)| = \left|\frac{(z_2 - z_3)(z_1 - z_4)}{(z_2 - z_4)(z_1 - z_3)}\right| = \frac{1}{a}. \tag{2.2}$$

Adding the expressions in equations (2.1) and (2.2), we get

$$\left|\frac{(z_2 - z_1)(z_3 - z_4)}{(z_2 - z_4)(z_3 - z_1)}\right| + \left|\frac{(z_2 - z_3)(z_1 - z_4)}{(z_2 - z_4)(z_1 - z_3)}\right| = \frac{a-1}{a} + \frac{1}{a} = 1.$$

This implies that

$$\left|\frac{(z_2 - z_1)(z_3 - z_4)}{(z_2 - z_4)(z_3 - z_1)}\right| = 1 - \left|\frac{(z_2 - z_3)(z_1 - z_4)}{(z_2 - z_4)(z_1 - z_3)}\right|$$

and hence

$$\left|\frac{(z_2 - z_1)(z_3 - z_4)}{(z_2 - z_4)(z_3 - z_1)}\right| = \frac{|(z_2 - z_4)(z_1 - z_3)| - |(z_2 - z_3)(z_1 - z_4)|}{|(z_2 - z_4)(z_1 - z_3)|}.$$

Equivalently, we finally obtain

$$|z_1 - z_3||z_2 - z_4| = |z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_1 - z_4|.$$

For the inequality, we consider the vertices z_1, z_2, z_3, z_4 as complex numbers. Then applying the triangle inequality to the simple identity

$$(z_1 - z_3)(z_2 - z_4) = (z_1 - z_2)(z_3 - z_4) + (z_2 - z_3)(z_1 - z_4),$$

we get the required inequality. □

Ptolemy's Theorem can also be proved using the inverse point technique, however, we skip that method of proof.

2.3 Inverse (or Symmetric) points

In order to generate the idea of inverse points with respect to a circle (which is not a line), we need to initiate the concept of inverse points with respect to a line.

Definition 2.3.1. *(Symmetric/Inverse points with respect to a line) Let L be a line in $\mathbb{C}$. Two points z and z^* are called the* inverse points (or symmetric points) *with respect to L if L is the perpendicular bisector of $[z, z^*]$, the line segment joining z and z^*.*

Definition 2.3.2. *(Reflection/Inversion with respect to a line) The mapping which carries z onto its inverse point z^* with respect to the line L is a one-one correspondence. This correspondence is called a* reflection/inversion *with respect to L.*

Example 2.3.3. *The points z and $z^* = \overline{z}$ are the inverse points with respect to the real axis. So, the correspondence $f(z) = \overline{z}$ is the reflection/inversion with respect to the real axis.*

Using a simple high school geometry, one can easily show that

Lemma 2.3.4. *Every circle passing through the inverse points z and z^* with respect to L intersects L at right angles.*

Definition 2.3.5. *(Inverse points with respect to a circle) Two points z and z^* in $\mathbb{C}$ are said to be inverse points with respect to a circle $\Gamma \subset \mathbb{C}$ if every circle passing through z and z^* intersects Γ at right angles.*

The following lemma says that a and $1/\overline{a}$ are inverse points with respect to the unit circle.

Lemma 2.3.6. *[1, Exc. 5, p. 17] All circles that pass through a and $1/\overline{a}$ intersect the unit circle $|z| = 1$ at right angles.*

Proof. Let $|z - \alpha| = r$ be an arbitrary circle that pass through a and $1/\overline{a}$. We need to show that $|z - \alpha| = r$ and $|z| = 1$ are orthogonal, i.e. $|\alpha|^2 = 1 + r^2$. By the assumption, we compute

$$|a - \alpha| = r \iff |a|^2 + |\alpha|^2 - 2\mathrm{Re}\,[\alpha\overline{a}] = r^2$$

and

$$\left|\frac{1}{\overline{a}} - \alpha\right| = r \iff 1 + |\alpha|^2|a|^2 - 2\mathrm{Re}\,[\alpha\overline{a}] = r^2|a|^2.$$

Solving these two equations, we obtain the required orthogonality conditions: $|\alpha|^2 = 1 + r^2$. □

Geometric view of the inverse points in a circle goes like this. Let Γ be a given circle centred at z_0. If z is a point inside Γ, then the point z^* can be located outside Γ lying on the line L passing through z_0 and z. The construction follows like this: take a line perpendicular to L at z. This meets Γ at a point, say, P. Then z^* is obtained by the point of intersection of the tangent line at P and L. See the figure below.

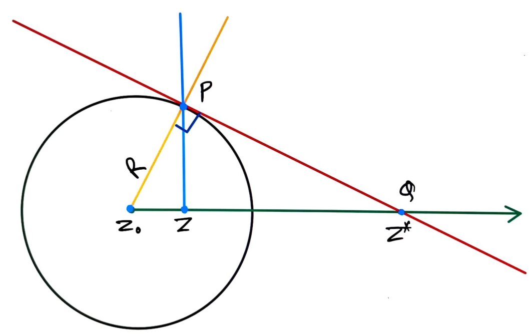

Figure 2.2: Construction of inverse points

For the detailed discussion about the geometric view of the inverse points z and z^* in a circle, first we provide a characterization which helps us to compute z^* when z is given.

Theorem 2.3.7. *[11, p. 78] Let* $\Gamma := \{\zeta \in \mathbb{C} : |\zeta - z_0| = R\}$ *be a circle. Two points* z *and* z^* *are inverse points with respect to* Γ *if and only if the following two geometric properties hold*

(i) z *and* z^* *are collinear with center* z_0*;*

(ii) $|z - z_0||z^* - z_0| = R^2$.

Proof. By Definition 2.3.5, every circle passing through z and z^* is orthogonal to Γ. In particular, the line passing through z_0, z and z^* is also orthogonal to Γ. Thus, z and z^* are collinear with the center z_0, proving (i).

For the proof of (ii), cosider the circle

$$\Gamma' := \{\zeta \in \mathbb{C} : |\zeta - z_0'| = R'\}$$

passing through the inverse points z and z^* which is orthogonal to the circle Γ such that z_0, z, z_0', z^* are collinear (see the figure in the next page).

Thus, it has the orthogonality condition:

$$|z_0 - z_0'|^2 = R^2 + R'^2.$$

This, along with a simple geometry, yields

$$\begin{aligned} |z - z_0||z^* - z_0| &= (|z_0 - z_0'| - R')(|z_0 - z_0'| + R') \\ &= |z_0 - z_0'|^2 - R'^2 = R^2, \end{aligned}$$

the relation (ii).

For the converse part, we provide two techniques.

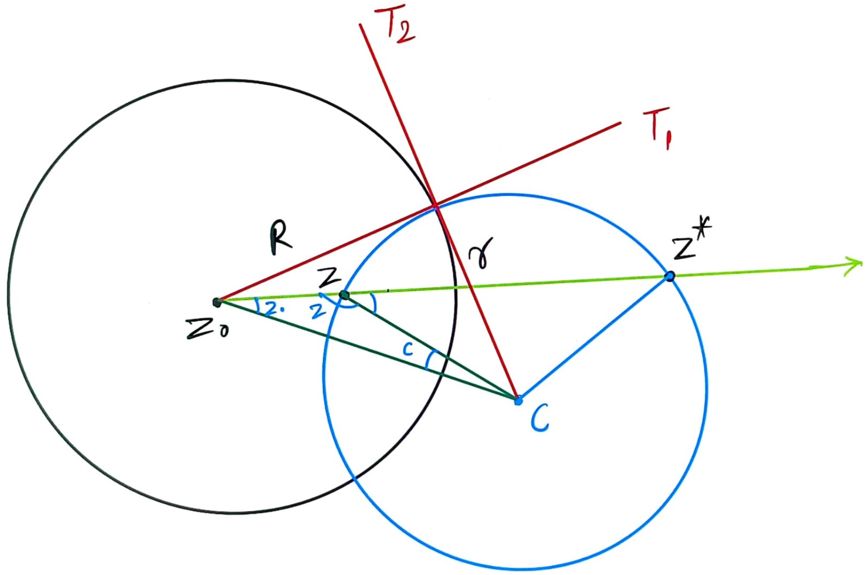

Figure 2.3: Construction of inverse points

Method-1: We require to show that any circle passing through z and z^* is orthogonal to Γ, i.e. to show the orthogonality condition $|z_0 - c|^2 = r^2 + R^2$.

Here, two cases arise.

Case-1: $z = z^*$.

By the given condition, the points z and z^* lie on the circle $|\zeta - z_0| = R$, and the circles passing through them merge into the line passing through z_0 and z. Trivially, this is orthogonal to Γ. So, there is nothing to analyse in this case.

Case-2: $z \neq z^*$.

We refer to the figure below. On the one hand, the cosine rule for the triangle $\triangle z_0 z c$ produces

$$r^2 = |z - z_0|^2 + |z_0 - c|^2 - 2|z - z_0||z_0 - c| \cos \angle z z_0 c. \tag{2.3}$$

On the other hand, the cosine rule for $\triangle z_0 z^* c$ gives

$$r^2 = |z^* - z_0|^2 + |z_0 - c|^2 - 2|z^* - z_0||z_0 - c| \cos \angle z^* z_0 c. \tag{2.4}$$

Solving the expressions obtained in (2.4) and (2.3), we obtain

$$0 = |z - z_0|^2 - |z^* - z_0|^2 + 2 \cos \angle z^* z_0 c \, |z_0 - c|(|z^* - z_0| - |z - z_0|),$$

because $\angle z z_0 c = \angle z^* z_0 c$. This implies

$$|z^* - z_0|^2 - |z - z_0|^2 = 2 \cos \angle z^* z_0 c \, |z_0 - c|(|z^* - z_0| - |z - z_0|),$$

and hence

$$|z^* - z_0| + |z - z_0| = 2 \cos \angle z^* z_0 c \, |z_0 - c|.$$

Squaring both sides and using Theorem 2.3.7(ii), we have

$$|z^* - z_0|^2 + |z - z_0|^2 + 2R^2 = 4 \cos^2 \angle z^* z_0 c \, |z_0 - c|^2. \tag{2.5}$$

Also, Theorem 2.3.7(i) says that the points z_0, z, z^* are collinear. It follows that

$$|z_0 - z^*| = |z_0 - z| + |z - z^*|.$$

Squaring both sides, Theorem 2.3.7(ii) produces

$$|z - z^*|^2 = |z_0 - z^*|^2 + |z_0 - z|^2 - 2R^2. \tag{2.6}$$

Solving the equations (2.5) and (2.6), we have

$$|z_0 - c|^2 = \frac{|z - z^*|^2 + 4R^2}{4 \cos^2 \angle z^* z_0 c}. \tag{2.7}$$

Applying the sine rule for the triangle $\triangle z_0 z c$, we have

$$\frac{\sin \angle z_0 z c}{|z_0 - c|} = \frac{\sin \angle z z_0 c}{r}.$$

A simplification after squaring it leads to

$$|z_0 - c|^2 = \frac{r^2 \sin^2 \angle z_0 z c}{\sin^2 \angle z z_0 c} \tag{2.8}$$

Again the cosine rule for the triangle $\triangle z z^* c$ gives

$$r^2 = |z - z^*|^2 + r^2 + 2r|z - z^*| \cos(\pi - \angle z_0 z c).$$

This simplifies to

$$0 = |z - z^*|(|z - z^*| + 2r \cos \angle z_0 z c) \implies |z - z^*| + 2r \cos \angle z_0 z c = 0,$$

since $z \neq z^*$. An easy simplification leads to

$$r^2 sin^2 \angle z_0 z c = r^2 - \frac{|z - z^*|^2}{4}.$$

Substituting this in (2.8) and simplifying, we obtain

$$|z - z^*|^2 = 4r^2 - 4|z_0 - c|^2 sin^2 \angle z z_0 c. \tag{2.9}$$

Eliminating the term $|z - z^*|^2$ from (2.7) and (2.9), we have

$$4|z_0 - c|^2 \cos^2 \angle z z_0 c = 4r^2 - 4|z_0 - c|^2 \sin^2 \angle z z_0 c + 4R^2.$$

By a simple calculation, the required orthogonality condition

$$|z_0 - c|^2 = r^2 + R^2$$

follows.

Method-2: This method is based on the geometric view of the inverse points in a circle explained above (see Figure 2.2). In Figure 2.2, the line PQ is the tangent to the circle and it passes through z^*. Denote line which is joining the points P and z by Pz. Now to prove that the points z and z^* are inverse points with respect to the circle, it is sufficient to prove that the line Pz is perpendicular to the line $z_0 z^*$.

From the hypothesis (ii) we have

$$|z - z_0||z^* - z_0| = R^2.$$

Since $|z_0 - P| = R$, it simplifies to

$$\frac{|z - z_0|}{|z_0 - P|} = \frac{|z_0 - P|}{|z^* - z_0|}. \tag{2.10}$$

Now we consider the two triangles $\triangle P z_0 z^*$ and $\triangle P z_0 z$. Since the angles $\angle P z_0 z^*$ and $\angle P z_0 z$ are common to both the triangle, considering (2.10), the SAS property suggests that the triangles $\triangle P z_0 z^*$ and $\triangle P z_0 z$ are similar. It follows that $\angle z^* P z_0 = \angle z_0 z P$. Since PQ is the tangent at P and $z_0 P$ is a radius, the $\angle z^* P z_0 = \pi/2 = \angle z_0 z P$. That is, the line Pz is perpendicular to the line $z_0 z^*$. This completes the proof. □

One of the importances of Theorem 2.3.7 is to compute z^* if z is given to us. Since z and z^* lie on the same ray through z_0, it follows that

$$\operatorname{Arg}(z^* - z_0) = \operatorname{Arg}(z - z_0) = \theta \quad \text{(say)}.$$

Then the relation $|z - z_0||z^* - z_0| = R^2$ is equivalent to

$$z^* - z_0 = \frac{R^2}{|z - z_0|} e^{i\theta} = \frac{R^2}{re^{-i\theta}} = \frac{R^2}{\overline{z - z_0}},$$

if $z = z_0 + re^{i\theta}$, $r < R$. This brings the concept of inversion in a circle.

Definition 2.3.8. *[1, p. 81] The inversion in the circle $|\zeta - z_0| = R$ is a transformation $i : \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined by*

$$i(z) = \begin{cases} z_0 + \dfrac{R^2}{\overline{z - z_0}}, & \text{if } z \neq z_0, \\ \infty, & \text{if } z = z_0 \\ z_0 & \text{if } z = \infty. \end{cases}$$

Note that $i(\zeta) = \zeta$. We next discuss a result describing the inverse points in a circle in terms of the equation of the circle.

Theorem 2.3.9. *[11, Theorem 3.25, p. 80] Two points z and z^* are inverse points with respect to the circle*

$$\alpha w \overline{w} + \overline{\beta} w + \beta \overline{w} + \gamma = 0$$

in $\mathbb{C}_\infty$ *if and only if*

$$\alpha z\overline{z^*} + \overline{\beta} z + \beta\overline{z^*} + \gamma = 0$$

Now we see that Theorem 2.3.9 plays an important role in the proof of the next important result, namely, the symmetric principle. This says that a pair of inverse points in a circle are sent to a pair of inverse points under a Möbius transformation.

Theorem 2.3.10 (Symmetry Principle). *[1, Theorem 15, p. 82] If a Möbius transformation carries a circle* C_1 *into a circle* C_2*, then it transforms any pair of symmetric points with respect to* C_1 *into a pair of symmetric points with respect to* C_2 *.*

Proof. Let the equation of the circle C_1 be

$$\alpha_1 w_1\overline{w_1} + \overline{\beta_1} w_1 + \beta_1\overline{w_1} + \gamma_1 = 0, \tag{2.11}$$

and z_1 and z_1^* be a pair of inverse points with respect to C_1. Then by Theorem 2.3.9, we have

$$\alpha_1 z_1\overline{z_1^*} + \overline{\beta_1} z_1 + \beta_1\overline{z_1^*} + \gamma_1. \tag{2.12}$$

Let S be a Möbius transformation which caries C_1 onto C_2. Then we prove that $S(z_1)$ and $S(z_1^*)$ are inverse points with respect to C_2. Let

$$\zeta = S(w) = \frac{aw+b}{cw+d}, \quad ad - bc \neq 0$$

Then, we have

$$w = S^{-1}(\zeta) = \frac{d\zeta - b}{-c\zeta + a}, \quad ad - bc \neq 0.$$

If $\zeta_1 = S(w_1)$, then by (2.11), we obtain the equation of the circle C_2 as

$$\alpha_1 S^{-1}(\zeta_1)\overline{(S^{-1})(\zeta_1)} + \bar{\beta}_1 S^{-1}(\zeta_1) + \beta_1\overline{S^{-1}(\zeta_1)} + \gamma = 0.$$

This is equivalent to

$$\alpha_1\left(\frac{d\zeta - b}{-c\zeta + a}\right)\overline{\left(\frac{d\zeta - b}{-c\zeta + a}\right)} + \beta_1\left(\frac{d\zeta - b}{-c\zeta + a}\right) + \beta_1\overline{\left(\frac{d\zeta - b}{-c\zeta + a}\right)} + \gamma_1 = 0.$$

A simple computation takes the equation of the circle C_2 to the form

$$\alpha_2\zeta_1\overline{\zeta_1} + \overline{\beta_2}\zeta_1 + \beta_2\overline{\zeta_1} + \gamma_2 = 0,$$

equivalently,

$$\alpha_2 S(w_1)\overline{S(w_1)} + \overline{\beta_2}S(w_1) + \beta_2\overline{S(w)} + \gamma_2 = 0.$$

That is

$$\alpha_2\left(\frac{aw_1+b}{zw_1+d}\right)\left(\overline{\frac{aw_1+b}{cw_1+a}}\right) + \beta_2\left(\frac{aw_1+b}{cw_1+d}\right) + \beta_2\left(\overline{\frac{aw_1+b}{cw_1+d}}\right) + \gamma_2 = 0. \tag{2.13}$$

This is of the form

$$Aw_1\overline{w_1} + \overline{B}w_1 + B\overline{w_1} + D = 0.$$

Then by (2.12), a similar calculation can yield

$$Az_1\overline{z_1^*} + \overline{B}z_1 + B\overline{z_1^*} + D = 0.$$

Simplifying back to the form (2.13), we obtain

$$\alpha_2\left(\frac{az_1+b}{cz_1+d}\right)\left(\overline{\frac{az_1^*+b}{cz_1^*+d}}\right) + \beta_2\left(\frac{az_1+b}{cz_1+d}\right) + \beta_2\left(\overline{\frac{az_1^*+b}{cz_1^*+d}}\right) + \gamma_2 = 0.$$

This is equivalent to

$$\alpha_2 S(z_1)\overline{S(z_1)} + \overline{\beta_2}S(z_1) + \beta_2\overline{S(z)} + \gamma_2 = 0,$$

leading to the conclusion that $S(z_1)$ and $S^*(z_1)$ are inverse points with respect to C_2. This completes the proof. □

With the help of Theorem 2.3.10, we now very easily characterize the most general Möbius transformation that takes the unit disk onto itself. This is also used in the later chapters.

Theorem 2.3.11. *[11, Theorem 3.21, p. 77] The most general Möbius transformation that takes the unit disk $\mathbb{D}$ onto itself is of the form*

$$S(z) = e^{i\alpha}\frac{z-z_0}{1-\overline{z_0}z}, \quad \text{for } \alpha \in \mathbb{R} \text{ and } z_0 \in \mathbb{D}.$$

Proof. One can easily verify that the function

$$T(z) = \frac{z-z_0}{1-\overline{z_0}z}, \quad z \in \mathbb{D},$$

is a Möbius disk automorphism of the unit disk $\mathbb{D}$. By Lemma 2.3.6, we know that z_0 and $1/\overline{z_0}$ are inverse points with respect to the unit circle $|z| = 1$. By the symmetric principle (Theorem 2.3.10), $0 = T(z_0)$ and $\infty = T(1/\overline{z_0})$ are also inverse points with respect to the image circle $|w| = 1$. It follows that the most general equation of the required Möbius transformation is of the form

$$S(z) = K\left(\frac{z - z_0}{z - \frac{1}{\overline{z_0}}}\right) = A\left(\frac{z - z_0}{1 - \overline{z_0}z}\right) = AT(z), A = -K\overline{z_0},$$

for some complex constant K. Since $|S(z)| = 1$ for $|z| = 1$, in particular for $z = 1$, we too have $|S(1)| = 1$. A simple computation yields $|A| = 1$, i.e. $A = e^{i\alpha}$ for some $\alpha \in \mathbb{R}$. Thus, $S(z)$ has the required form. This completes the proof. □

We next provide two examples that are nothing but applications of the symmetric principle.

Example 2.3.12. *[1, Exc. 2, p. 83] Reflect the line $y = x$ and the circle $|z| = 1$ in the circle $|z - 2| = 1$.*

Proof. We first reflect the line of $x = y$ in the circle $|z - 2| = 1$.

Let $z = x + iy$. Then we obtain $z = x + ix$ and hence $\overline{z} = -iz$. Now by Definition 2.3.8, the inversion in the circle $|z - 2| = 1$ produces

$$z^* := i(z) = \frac{1}{\overline{z} - 2} + 2. \tag{2.14}$$

After simplification we get

$$-iz = \frac{2z^* - 3}{z^* - 2} \iff \overline{z} = \left(\frac{-2\bar{z^*} + 3}{-\overline{z^*} + 2}\right)i$$

Since $-iz = \overline{z}$, it follows that

$$\frac{2z^* - 3}{z^* - 2} = \left(\frac{-2\overline{z}^* + 3}{-\overline{z}^* + 2}\right)i$$

This simplifies to

$$z^*\overline{z}^* + z^*\left(\frac{4 - 3i}{-2 + 2i}\right) + \overline{z}^*\left(\frac{3 - 4i}{-2 + 2i}\right) + \left(\frac{6i - 6}{-2 + 2i}\right) = 0,$$

which represents a circle.

Secondly, we deal with the reflection of $|z| = 1$ in the given circle.

By the above simplification of (2.14), we get

$$1 = |z| = \left|\frac{2z^* - 3}{z^* - 2}\right|.$$

After simplification we get the circle

$$\left|z^* - \frac{4}{3}\right| = \frac{1}{3},$$

and the solution is complete. □

Example 2.3.13. *[1, Exc. 7, p. 83] Find a Möbius transformation which carries the circles $|z| = 1$ and $|z-(1/4)| = 1/4$ onto concentric circles. What is the ratio of radii?.*

Proof. Let

$$S(z) = \frac{z - a}{1 - \bar{a}z}, \quad 0 < a < 1, \tag{2.15}$$

be the Möbius transformation which carries the circles $C_1 : \ |z| = 1$ onto itself and $C_2 : \ |z - (1/4)| = 1/4$ onto $|w| = r$.

By Lemma 2.3.6, we know that a and $1/\bar{a} = 1/a$ are inverse points with respect to C_1. We choose $1/4 < a < 1$ in such a way that a and $1/a$ are also inverse points with respect to C_2. By Theorem 2.3.7(ii), we have

$$\left(a - \frac{1}{4}\right)\left(\frac{1}{a} - \frac{1}{4}\right) = \frac{1}{16}.$$

Solving this quadratic equation in a, we get $a = 2 - \sqrt{3}$, since we have chosen $1/4 < a < 1$. Therefore, from (2.15), we obtain

$$S(z) = \frac{z - (2 - \sqrt{3})}{1 - (2 - \sqrt{3})z}.$$

Now we go for finding the ratio of radii: $1 : r$. Since 0 and ∞ are inverse points with respect to the circle C_1, so $S(0)$ and $S(\infty)$ are inverse points with respect to the image circle C_1. This implies

$$|S(0)||S(\infty)| = 1. \tag{2.16}$$

Again since $1/4$ and ∞ are inverse points with respect to the circle C_2, so $S(1/4)$ and $S(\infty)$ are inverse points with respect to the image circle $|w| = r$. This implies

$$|S(1/4)||S(\infty)| = r^2. \tag{2.17}$$

Solving the equations (2.16) and (2.17), we have

$$\frac{|S(0)||S(\infty)|}{|S(1/4)||S(\infty)|} = \frac{1}{r^2}.$$

Thus, the ratio of radii becomes

$$\frac{1}{r} = \sqrt{\frac{|S(0)|}{|S(1/4)|}} = \sqrt{\frac{1}{7 - 4\sqrt{3}}},$$

completing the solution. $\square$

CHAPTER 3

The Cassinian metric

3.1 Cassinian ovals

A Cassinian oval is an algebraic curve of degree four in the plane. Geometrically, it is a locus of points $P(x, y)$ with foci $F_1(a, 0)$ and $F_2(-a, 0)$ such that the product of the distances $|P - F_1|$ and $|P - F_2|$ is a positive constant b^2 (see Figure 3.1). Thus, its mathematical representation becomes

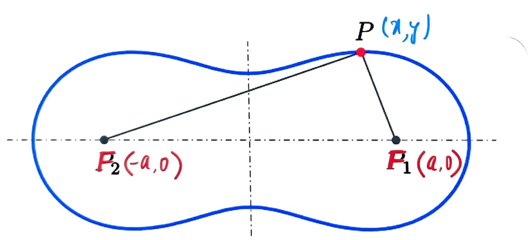

Figure 3.1: Cassinian oval: $|PF_1| \times |PF_2| = b^2$ for any location of P on the curve.

$$[(x-a)^2 + y^2][(x+a)^2 + y^2] = b^4 \iff (x^2 + y^2 + a^2)^2 - 4a^2x^2 = b^4.$$

In polar form, after substituting $x = r\cos\theta$ and $y = r\sin\theta$, it reduces to

$$[(r\cos\theta - a)^2 + (r\sin\theta^2][(r\cos\theta + a)^2 + (r\sin\theta)^2] = b^4,$$

equivalently,

$$r^4 + a^4 - 2a^2r^2\cos(2\theta) = b^4.$$

The Cassinian oval's shapes depend on the parameters a and b, and they are generally categorized by evaluating the ratio a/b. In Figure 3.2, a family of Cassinian ovals is drawn for a fixed a on the left and a fixed b on the right.

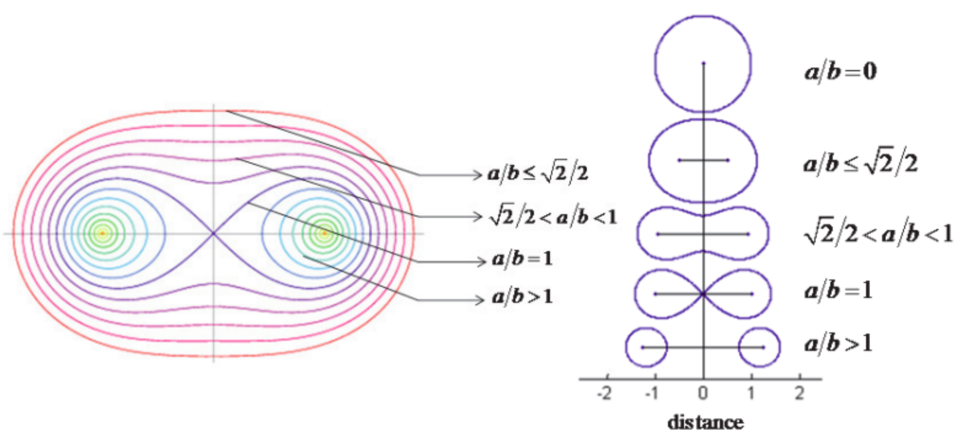

Figure 3.2: A family of Cassinian ovals

The form of the ovals can be characterized as follows:

- For $a/b \leq \sqrt{2}/2$, the curve is a single loop that looks like an ellipse and intersects x-axis at $x = \pm\sqrt{a^2+b^2}$.

- For $\sqrt{2}/2 < a/b < 1$, the oval attains a dent on top and bottom or peanut shape or dog bone shape.

- When $a/b = 1$, the curve is a Lemniscate of Bernoulli which has a point of self intersection. This shapes pass through the origin and has shape similar to the ∞ symbol.

- For $a/b > 1$, the curve splits into two mirror-imaged disjoint ovals and there are two additional real x-intercepts at $x = \pm\sqrt{a^2 - b^2}$.

The Cassinian ovals are extensively used in real life problems (see [7, 10]). Some of the applications are listed below.

- Model to determine Earth's orbit.
  - Cassinian ovals were first studied in 1680 by Giovanni Domenico Cassinian as a model to determine Earth's orbit. After discovering few satellites, he rejected the theory of Kepler ellipses and discovered the Cassinian ovals suggesting that stellar bodies followed paths traced out by one of these curves.
- Modelling population growths.
  - Growth of population between the two metropolitan cities, Beijing and Tianjin of China, is modelled by the a/b characteristic index of Cassinian ovals. In the study when $a/b > 1$, the population density is more than 3000 $persons/km^2$. When $a/b = 1$, the population density is about 3000 $persons/km^2$. For $\sqrt{2}/2 < a/b < 1$, the density is between $500 - 3000$ $persons/km^2$.
- Human red blood cell is treated using the Cassinian ovals for modelling its profile.
- Simulating light scattering by small particles.

3.2 The Cassinian distance

Let $z_1, z_2 \in D \subset\neq \mathbb{C}$ be a pair of distinct points. The Cassinian distance between them can be defined through two concepts: (i) the maximal Cassinian ovals, and (ii) inversion formula.

(i) The maximal Cassinian oval approach. For a finite number $k > 0$, let $C(z_1, z_2; k)$ be the Cassinian ovals

$$\{z \in \mathbb{R}^n : |z_1 - z||z_2 - z| = k^2\}$$

with foci z_1 and z_2. Since D is open, the Cassinian ovals $C(z_1, z_2; k)$ is contained in D for a small enough k. As the boundary ∂D of D contains at least two points, there exists a largest number k with the property that $C(z_1, z_2; k) \cap \partial D \neq 0$ and that the interior of $C(z_1, z_2; k)$ is contained in D. Let $p \in C(z_1, z_2; k) \cap \partial D$. Then

$$k^2 = |z_1 - p||z_2 - p| = \min\{|z_1 - q||z_2 - q| : q \in \partial D\}.$$

This generates a maximal Cassinian oval in D with foci $z_1, z_2 \in D$ and it is denoted by $C(z_1, z_2; k(z_1, z_2))$. Using this notion, the Cassinian distance is first defined as follows (see [5]):

$$c_D(z_1, z_2) := \frac{|z_1 - z_2|}{k^2(z_1, z_2)} = \frac{|z_1 - z_2|}{|z_1 - p||z_2 - p|}.$$

Alternatively,

$$c_D(z_1, z_2) = \frac{|z_1 - z_2|}{\min\{|z_1 - q||z_2 - q| : q \in \partial D\}} = \max_{q \in \partial D} \frac{|z_1 - z_2|}{|z_1 - q||z_2 - q|}.$$

In particular, for each $z_1, z_2 \in D$, there exists a point $p \in \partial D$ such that

$$c_D(z_1, z_2) = \frac{|z_1 - z_2|}{|z_1 - p||z_2 - p|}.$$

(ii) The inversion formula approach. Suppose that $p \in \partial D$ such that

$$c_D(z_1, z_2) = \frac{|z_1 - z_2|}{|z_1 - p||z_2 - p|}$$

holds. Then geometrically the Cassinian distance between z_1 and z_2 is nothing but the Euclidean distance between the inversions of z_1 and z_2 in the unit circle $\mathbb{S}(p, 1)$ centred at p. Indeed, if the inversion in the circle $\mathbb{S}(p, 1)$ is denoted by i_p then we have

$$\begin{aligned} |i_p(z_1) - i_p(z_2)| &= \left| \left(p + \frac{1}{\overline{\overline{z_1 - p}}} \right) - \left(p + \frac{1}{\overline{\overline{z_2 - p}}} \right) \right| \\ &= \frac{|z_1 - z_2|}{|z_1 - p||z_2 - p|} = c_D(z_1, z_2), \end{aligned}$$

for all $z_1, z_2 \in D$.

Note that the Cassinian distance is easy to compute, which can be seen from the following observations and examples.

Observations. If $z_2 \to \infty$, then we see that

$$\begin{aligned} c_D(z_1, \infty) &= \lim_{z_2 \to \infty} c_D(z_1, z_2) \\ &= \lim_{z_2 \to \infty} \max_{p \in \partial D} \frac{|z_1 - z_2|}{|z_1 - p||z_2 - p|} \\ &= \lim_{z_2 \to \infty} \max_{p \in \partial D} \frac{|z_2||\frac{z_1}{z_2} - 1|}{|z_2||z_1 - p||1 - \frac{p}{z_2}|} \\ &= \lim_{z_2 \to \infty} \max_{p \in \partial D} \frac{1}{|z_1 - p|} \\ &= 1/\delta(z_1), \end{aligned}$$

where $\delta(z_1) := \text{dist}\,(z_1, \partial D) = \min\{|z_1 - p| : p \in \partial D\}$.

If $D = \mathbb{C}_p = \mathbb{C} \setminus \{p\}, p \neq \infty$, then

$$c_D(z_1, z_2) = \frac{|z_1 - z_2|}{|z_1 - p||z_2 - p|}.$$

If $z_1, z_2 \in D_p := \{z \in D : \delta(z) = |z - p|\}$ for some $p \in \partial D$, then

$$c_D(z_1, z_2) = \frac{|z_1 - z_2|}{|z_1 - p||z_2 - p|}.$$

The next two examples compute the Cassinian distance between points in a punctured disk and a half-plane, respectively.

Example 3.2.1. *Find $c_D(z_1, z_2)$ on the domain $D = \{z \in \mathbb{C} : 0 < |z| < k\}$.*

Solution. Let a point $\zeta \in \partial D$, where boundary $\partial D = \{0, |z| = k\}$. By the definition of the Cassinian distance, we have

$$c_D(z_1, z_2) = \max_{\zeta \in \partial D} \frac{|z_1 - z_2|}{|z_1 - \zeta||z_2 - \zeta|} = \frac{|z_1 - z_2|}{\min\limits_{\zeta \in \partial D} |z_1 - \zeta||z_2 - \zeta|}.$$

Since $|z_1 - \zeta| \geq ||z_1| - |\zeta||$ and $|z_2 - \zeta| \geq ||z_2| - |\zeta||$, for all $z_1, z_2, \zeta \in \mathbb{C}$, it follows that

$$|z_1 - \zeta||z_2 - \zeta| \geq ||z_1| - |\zeta||\,||z_2| - |\zeta||.$$

It implies that

$$\min_{\zeta\in\partial D}|z_1-\zeta||z_2-\zeta| \geq \min_{\zeta\in\partial D}||z_1|-|\zeta||\,||z_2|-|\zeta||. \tag{3.1}$$

In right side of the inequality (3.1) if we take boundary point as $\zeta = 0$ then it becomes $|z_1||z_2|$ and if we take boundary point as $|\zeta| = k$ then it becomes $||z_1|-k|\,||z_2|-k| = ||z_1||z_2|-k(|z_1|+|z_2|)+k^2|$. Now when $|z_1|+|z_2|>k$, then minimum occurs for $|\zeta| = k$ and when $|z_1|+|z_2|\leq k$, then minimum occurs for $\zeta = 0$. i.e., right side of inequality (3.1) becomes

$$\min_{\zeta\in\partial D}||z_1|-|\zeta||\,||z_2|-|\zeta|| = \begin{cases} |z_1||z_2|, & |z_1|+|z_2|\leq k; \\ ||z_1|-k|\,||z_2|-k|, & |z_1|+|z_2|>k. \end{cases}$$

So equation (3.1) becomes

$$\min_{\zeta\in\partial D}|z_1-\zeta||z_2-\zeta| \geq \begin{cases} |z_1||z_2|, & |z_1|+|z_2|\leq k; \\ ||z_1|-k|\,||z_2|-k|, & |z_1|+|z_2|>k. \end{cases}$$

Thus

$$c_D(z_1,z_2) = \frac{|z_1-z_2|}{\min_{\zeta\in\partial D}|z_1-\zeta||z_2-\zeta|} \leq \begin{cases} \dfrac{|z_1-z_2|}{|z_1||z_2|}, & |z_1|+|z_2|\leq k; \\ \dfrac{|z_1-z_2|}{||z_1|-k|\,||z_2|-k|}, & |z_1|+|z_2|>k. \end{cases} \tag{3.2}$$

This is the upper bound of c_D in a punctured disk with radius k. Now we are interested to find a lower bound of c_D. Since $|z_1-\zeta|\leq|z_1|+|\zeta|$ and $|z_2-\zeta|\leq|z_2|+|\zeta|$, it implies that

$$|z_1-\zeta||z_2-\zeta| \leq (|z_1|+|\zeta|)(|z_2|+|\zeta|).$$

It implies that

$$\min_{\zeta\in\partial D}|z_1-\zeta||z_2-\zeta| \leq \min_{\zeta\in\partial D}(|z_1|+|\zeta|)(|z_2|+|\zeta|). \tag{3.3}$$

In right side of equation (3.3) if we take boundary point as $\zeta = 0$ then it becomes $|z_1||z_2|$ and if we take boundary point as $|\zeta| = k$ then it becomes

$$(|z_1|+k)(|z_2|+k) = |z_1||z_2| + k((|z_1|+|z_2|)+k),$$

where $k>0$, then one can easily see that minimum occurs for $\zeta = 0$ in right side of (3.3). So (3.3) becomes

$$\min_{\zeta\in\partial D}|z_1-\zeta||z_2-\zeta|\leq|z_1||z_2|.$$

Thus

$$c_D(z_1,z_2)=\frac{|z_1-z_2|}{\min\limits_{\zeta\in\partial D}|z_1-\zeta||z_2-\zeta|}\geq\frac{|z_1-z_2|}{|z_1||z_2|}. \tag{3.4}$$

From (3.2) and (3.4), we have

$$\frac{|z_1-z_2|}{|z_1||z_2|}\leq c_D(z_1,z_2)\leq\begin{cases}\dfrac{|z_1-z_2|}{|z_1||z_2|}, & |z_1|+|z_2|\leq k;\\[2ex] \dfrac{|z_1-z_2|}{||z_1|-k|\,||z_2|-k|}, & |z_1|+|z_2|>k.\end{cases} \tag{3.5}$$

This establishes

$$c_D(z_1,z_2)=\frac{|z_1-z_2|}{|z_1|\,|z_2|}\quad \text{if } |z_1|+|z_2|\leq k.$$

However, for $|z_1|+|z_2|>k$, we fail to compute the Cassinian distance exactly. Moreover, we see that when $k\to\infty$, the punctured disk is converted to the punctured plane. In this limiting situation, the Cassinian distance is coinciding with the case of the punctured plane. □

Example 3.2.2. *Find $c_D(z_1,z_2)$ on the upper half-plane $D=\{z:\operatorname{Im} z>0\}$.*

Solution. Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2\in D$. We discuss the precise formula for $c_D(z_1,z_2)$ in two situations: (i) when the two points lie on a vertical line, (ii) when the two points lie on a horizontal line.

Case (i): $x_1=x_2$.

We have $z_1=x_1+iy_1$ and $z_2=x_1+iy_2$. Now Let a point $\zeta\in\partial D$, where boundary $\partial D=\{z:\Im(z)=0\}$. By the definition of Cassinian distance, we have

$$c_D(z_1,z_2)=\max_{\zeta\in\partial D}\frac{|z_1-z_2|}{|z_1-\zeta||z_2-\zeta|}=\frac{|z_1-z_2|}{\min\limits_{\zeta\in\partial D}|z_1-\zeta||z_2-\zeta|}.$$

Now

$$|z_1-z_2|=|x_1+iy_1-x_1-iy_2|=|iy_1-iy_2|=|y_1-y_2|$$

Let the maximal Cassinian oval touch at the point $\zeta=t$ on the x-axis with

$|t| \leq x_1$, then

$$\min_{-x_1 \leq t \leq x_1} |z_1 - t||z_2 - t| = \min_{-x_1 \leq t \leq x_1} \sqrt{((x_1 - t)^2 + y_1^2)((x_1 - t)^2 + y_2^2)}.$$

Consider the function

$$f(t) = ((x_1 - t)^2 + y_1^2)((x_1 - t)^2 + y_2^2).$$

Differentiation of it gives

$$\begin{aligned} f'(t) &= ((x_1 - t)^2 + y_2^2)(-2(x_1 - t)) + ((x_1 - t)^2 + y_1^2)(-2(x_1 - t)) \\ &= -2(x_1 - t)(2(x_1 - t)^2 + y_1^2 + y_2^2). \end{aligned}$$

For finding the minimum value of $f(t)$, we make $f'(t) = 0$ i.e., $-2(x_1 - t) = 0$ or $(2(x_1 - t)^2 + y_1^2 + y_2^2) = 0$. Since $(2(x_1 - t)^2 + y_1^2 + y_2^2) = 0$ is not possible so we have $-2(x_1 - t) = 0$ i.e., at $x_1 = t, f(t)$ gives its minimum value. Then minimum of $f(t)$ at $x_1 = t$ is $y_1^2 y_2^2$. i.e.,

$$\min_{-x_1 \leq t \leq x_1} |z_1 - t||z_2 - t| = \min_{-x_1 \leq t \leq x_1} \sqrt{((x_1 - t)^2 + y_1^2)((x_1 - t)^2 + y_2^2)} = \sqrt{y_1^2 y_2^2} = y_1 y_2.$$

Therefore, finally we have

$$c_D(z_1, z_2) == \frac{|z_1 - z_2|}{\min_{\zeta \in \partial D} |z_1 - \zeta||z_2 - \zeta|} = \frac{|y_1 - y_2|}{y_1 y_2}.$$

Case (ii): $y_1 = y_2$.

Without loss of generality one can assume that $x_1 > 0$ and $x_2 = -x_1$. Then $z_1 = x_1 + iy_1$ and $z_2 = -x_1 + iy_1$. By the definition of Cassinian distance, we have

$$c_D(z_1, z_2) = \max_{\zeta \in \partial D} \frac{|z_1 - z_2|}{|z_1 - \zeta||z_2 - \zeta|} = \frac{|z_1 - z_2|}{\min_{\zeta \in \partial D} |z_1 - \zeta||z_2 - \zeta|}.$$

Now

$$|z_1 - z_2| = |x_1 + iy_1 + x_1 - iy_1| = 2x_1.$$

Let the maximal Cassinian oval touch at the point $\zeta = t$ on the x-axis with $|t| \leq x_1$, then

$$\min_{-x_1 \leq t \leq x_1} |z_1 - t||z_2 - t| = \min_{-x_1 \leq t \leq x_1} \sqrt{((x_1 + t)^2 + y_1^2)((x_1 - t)^2 + y_2^2)}.$$

Consider the function

$$g(t) = ((x_1 + t)^2 + y_1^2)((x_1 - t)^2 + y_1^2).$$

We need to find minimum value of above function for t so find its derivative i.e.,

$$g'(t) = 4t(t^2 + y_1^2 - x_1^2).$$

$$g^{'}(t) = 0 \quad i.e., \quad t = 0 \quad \text{or} \quad t^2 = x_1^2 - y_1^2.$$

Now we have to find its double derivative i.e.,

$$g^{''}(t) = 4(t^2 + y_1^2 - x_1^2) + 8t^2.$$

The function $g(t)$ achieves its minimum value when $g^{''}(t) > 0$ i.e.,

$$\min_{-x_1 \le t \le x_1} g(t) = \begin{cases} (x_1^2 + y_1^2)^2, & y_1 \ge x_1; \\ 4x_1^2 y_1^2, & y_1 \le x_1. \end{cases}$$

Therefore, finally we have

$$c_D(z_1, z_2) = \frac{|z_1 - z_2|}{\min\limits_{\zeta \in \partial D} |z_1 - \zeta||z_2 - \zeta|} = \begin{cases} \dfrac{2x_1}{x_1^2 + y_1^2}, & y_1 \ge x_1; \\ \dfrac{1}{y_1}, & y_1 \le x_1. \end{cases}$$

This obtains the desired formula. □

3.3 Basic Properties

This section is devoted to some elementary properties of the Cassinian distance. The first property confirms that the Cassinian distance defines a metric on a domain $D \subsetneq \mathbb{C}$.

Theorem 3.3.1. *[5, Lemma 3.1] The distance function c_D defines a metric on D.*

Proof. Clearly, $c_D(z_1, z_2)$ is symmetric, $c_D(z_1, z_2) \ge 0$ and $c_D(z_1, z_2) = 0$ if and only if $z_1 = z_2$. It remains to prove the triangle inequality. For this, consider the three arbitrary points $z_1, z, z_2 \in D$. Let $p \in \partial D$ be such that

$$c_D(z_1, z_2) = \frac{|z_1 - z_2|}{|z_1 - p||z_2 - p|}.$$

Then

$$\begin{aligned} c_D(z_1, z) + c_D(z, z_2) &= \max_{p_1 \in \partial D} \frac{|z_1 - z|}{|z_1 - p_1||z - p_1|} + \max_{p_2 \in \partial D} \frac{|z - z_2|}{|z - p_2||z_2 - p_2|} \\ &\geq \frac{|z_1 - z|}{|z_1 - p||z - p|} + \frac{|z - z_2|}{|z - p||z_2 - p|} \\ &= \frac{1}{|z_1 - p||z_2 - p|} \frac{|z_1 - z||p - z_2| + |z - z_2||p - z_1|}{|z - p|} \\ &\geq \frac{|z_1 - z_2||z - p|}{|z_1 - p||z_2 - p||z - p|} = c_D(z_1, z_2), \end{aligned}$$

where the last inequality is nothing but Ptolemy's inequality (see Theorem 2.2.4). This completes the proof. □

Corollary 3.3.2 (Monotone property)**.** *[5, Corollary 3.2] The Cassinian metric is monotonic with respect to domains. That is, $D \subseteq D'$ implies $c_{D'}(z_1, z_2) \leq c_D(z_1, z_2)$ for all $z_1, z_2 \in D$.*

Proof. For $z_1, z_2 \in D$ the Cassinian distance is defined by

$$c_D(z_1, z_2) = \max_{p \in \partial D} \frac{|z_1 - z_2|}{|z_1 - p||z_2 - p|}.$$

By geometry (see Figure 3.3), for all $z_1, z_2 \in D \subset D'$ we have

$$\min_{p \in D} |z_1 - p||z_2 - p| \leq \min_{p' \in D'} |z_1 - p'||z_2 - p'|.$$

This implies that

$$\max_{p \in \partial D} \frac{|z_1 - z_2|}{|z_1 - p||z_2 - p|} \geq \max_{p' \in \partial D'} \frac{|z_1 - z_2|}{|z_1 - p'||z_2 - p'|},$$

equivalently, we have

$$c_D(z_1, z_2) \geq c_{D'}(z_1, z_2).$$

This completes the proof. □

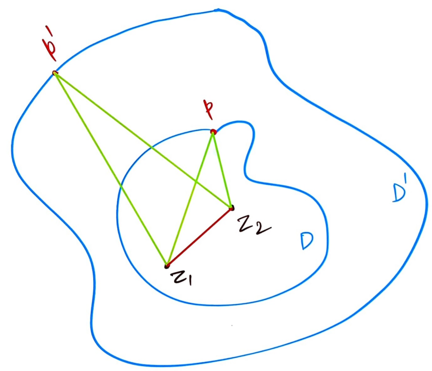

Figure 3.3: Comparison of $|z_1 - p||z_2 - p|$ and $|z_1 - p'||z_2 - p'|$ for $z_1, z_2 \in D \subset D'$

Next we discuss the completeness property of the Cassinian metric.

Lemma 3.3.3. *[5, Lemma 3.11] If D is a domain with $\infty \notin \partial D$, then the Cassinian metric space (D, c_D) is complete.*

Proof. Generally, the Cassinian metric is not complete. Here we have to prove if $\infty \notin \partial D$, then (D, c_D) is complete. First we see what happen if $\infty \in \partial D$. For this we let $D = \mathbb{C}\backslash\{0\}$ i.e., $\partial D = \{0, \infty\}$ and consider the sequence $\{z_n\} = n^2$. For any z_n we have

$$c_D(z_n, \infty) = \max_{p \in \partial D} \frac{|z_n - \infty|}{|z_n - p||\infty - p|} = \max_{p \in \partial D} \frac{1}{|z_n - p|}.$$

That is,

$$c_D(z_n, \infty) = \max \begin{cases} \dfrac{1}{|z_n|}, & p = 0; \\ 0, & p = \infty. \end{cases}$$

i.e., $c_D(n^2, \infty) \to 0$ as n increases. This shows that z_n converges to ∞ in the Cassinian metric. This implies that z_n is a Cauchy sequence. Also by explicitly calculation we can show the sequence $\{z_n\}$ is Cauchy. i.e.,

$$c_D(z_n, z_m) = \sup_{p\in\partial D} \frac{|z_n - z_m|}{|z_n - p||z_m - p|}, \quad p \in \partial D.$$

i.e.,

$$c_D(z_n, z_m) = \max \begin{cases} \left|\frac{1}{n^2} - \frac{1}{m^2}\right|, & p = 0; \\ 0, & p = \infty. \end{cases}$$

Therefore, $c_D(z_n, z_m) = \left|\frac{1}{n^2} - \frac{1}{m^2}\right| \to 0$ as $m, n \to \infty$. This implies that the sequence $\{z_n\} = n^2$ is Cauchy. But it is converging to ∞ which is outside the domain $D = \mathbb{C}\backslash\{0\}$. i.e., $(D = \mathbb{C}\backslash\{0\}, c_D)$ is not complete. Now we comeback to our main proof. Since $\infty \notin \partial D$. This implies that D is a bounded domain. To prove (D, c_D) is complete, by definition, it is required to show that every Cauchy sequence in D is converges in D. Now two scenarios arise:

(i) Sequence of points approaching to a boundary point:these kind of sequence are not Cauchy sequences (so, they are not convergent) as they are not even bounded. Let z_n be the sequence approaches to $q \in \partial D$ then

$$\min_{p\in\partial D} |z_0 - p||z_n - p| \leq |z_0 - q||z_n - q|.$$

This implies that

$$c_D(z_0, z_n) = \frac{|z_0 - z_n|}{\min_{p\in\partial D} |z_0 - p||z_n - p|} \geq \frac{|z_0 - z_n|}{|z_0 - q||z_n - q|}$$

as D is bounded domain i.e., $|z_0 - z_n| < k$, for some k and if $n \to \infty$ then the points of z_n will be approaches to q i.e., $|z_n - q| \to 0$. This implies that $\frac{|z_0 - z_n|}{|z_0 - q||z_n - q|} \to \infty$ i.e., $c_D(z_0, z_n)$ is unbounded. So the sequence $\{z_n\}$ is not a bounded sequence and hence it is not a Cauchy sequence.

(ii) Sequence of points not approaching to a boundary point:Let $\{z_n\}$ be a Cauchy sequence. Now we wish to show it is convergent in D. Since $\{z_n\}$ be a Cauchy sequence then $c_D(z_n, z_m) = \dfrac{|z_n - z_m|}{\min\limits_{p \in \partial D} |z_n - p||z_m - p|} \to 0$ if $n, m \to \infty$. Since $\{z_n\}$ is not approaching to any boundary points so, $\min\limits_{p \in \partial D} |z_n - p||z_m - p| \neq 0$ as $n \to \infty$. This implies $|z_n - z_m| \to 0$ as $n, m \to \infty$. i.e., the sequence $\{z_n\}$ approaches to some point, say z in D (as it is not approaches to some boundary point). Thus

$$c_D(z_n, z_m) = \frac{|z_n - z_m|}{\min\limits_{p \in \partial D} |z_n - p||z - p|} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence the sequence $\{z_n\}$ converges to $z \in D$. This completes the proof. □

Next we investigate about the Möbius invariance property of the Cassinian metric.

Lemma 3.3.4. *In general, the Cassinian metric c_D is not Möbius invariant.*

Proof. We prove our assertion in the unit disk $\mathbb{D}$. Let $z_1, z_2 \in \mathbb{D}$. We show that $c_{f(\mathbb{D})}(f(z_1), f(z_2)) \neq c_{\mathbb{D}}(z_1, z_2)$ under a Möbius transformation $f : \mathbb{D} \to f(\mathbb{D})$. Take $f(z) = az, a > 0$. This implies $f(\mathbb{D}) = \{z : |z| < a\}$; $\quad \partial f(\mathbb{D}) = \{z : |z| = a\}$ also $f(z_1) = az_1$ and $f(z_2) = az_2$ also there exists $p \in \{z : |z| = 1\}$ such that $p^{'} = ap = \{z : |z| = a\}$. Now

$$\begin{aligned}
c_{f(\mathbb{D})}(f(z_1), f(z_2)) &= \max_{p^{'} \in \partial f(\mathbb{D})} \frac{|f(z_1) - f(z_2)|}{|f(z_1) - p^{'}||f(z_2) - p^{'}|} \\
&= \max_{p^{'} \in \partial f(\mathbb{D})} \frac{|az_1 - az_2|}{|az_1 - p^{'}||az_2 - p^{'}|} \\
&= \max_{ap \in \partial f(\mathbb{D})} \frac{a|z_1 - z_2|}{a^2|z_1 - p||z_2 - p|} \\
&= \frac{1}{a} \max_{ap \in \partial f(\mathbb{D})} \frac{|z_1 - z_2|}{|z_1 - p||z_2 - p|} \\
&= \frac{1}{a} \max_{p \in \partial \mathbb{D}} \frac{|z_1 - z_2|}{|z_1 - p||z_2 - p|} \\
&= \frac{1}{a} c_{\mathbb{D}}(z_1, z_2).
\end{aligned}$$

Thus $c_{\mathbb{D}}$ is not a Möbius invariant. This completes the proof. □

CHAPTER 4

Main Results

This chapter is highlighted on some of the important results concerning the Cassinian metric. Mainly, the Möbius quasi-invariance property in the unit disk and the geodesic properties of the Cassinian metric are presented.

4.1 Möbius quasi-invariance property

In Lemma 3.3.4, we have shown that the Cassinian metric is not necessarily a Möbius transformation. However, one can show that the Cassinian metric is Möbius quasi-invariant. This property is already proved by Ibragimov in the higher dimension [6, Theorem 4.1], however, looking into different nature of the proof, we state and prove the following in the plane.

Theorem 4.1.1. *Let ϕ be a Möbius transformation with $\phi(\mathbb{D}) = \mathbb{D}$. Then*

$$\frac{1-|\phi(0)|}{1+|\phi(0)|}\, c_{\mathbb{D}}(z_1, z_2) \le c_{\mathbb{D}}(\phi(z_1), \phi(z_2)) \le \frac{1+|\phi(0)|}{1-|\phi(0)|}\, c_{\mathbb{D}}(z_1, z_2)$$

for all $z_1, z_2 \in \mathbb{D}$.

Proof. Consider any Möbius transformation of the form

$$\phi(z) = \frac{z+a}{1+\overline{a}z}, \quad \text{with} \quad \phi(0) = a.$$

Also consider another Möbius transformation of the form

$$\sigma(z) = \frac{z-a}{1-\overline{a}z}, \quad \text{with} \quad \sigma(a) = 0.$$

Now we define the composition of these two Möbius transformation which is also a Möbius transformation like

$$\sigma \circ \phi(z) : \mathbb{D} \to \mathbb{D} \;\text{ such that }\; \sigma \circ \phi(z) = e^{i\theta} z \;\text{ with }\; \sigma \circ \phi(0) = 0.$$

Now

$$\sigma \circ \phi(z) = \frac{\phi(z) - a}{1 - \overline{a}\phi(z)} = \frac{\dfrac{z+a}{1+\overline{a}z} - a}{1 - \overline{a}\dfrac{z+a}{1+\overline{a}z}} = \frac{z + a - a - |a|^2 z}{1 + \overline{a} - \overline{a}z + |a|^2}$$

$$= \frac{z(1-|a|^2)}{1-|a|^2} = z.$$

Next

$$\begin{aligned}
|\sigma(z_1) - \sigma(z_2)| &= \left| \frac{z_1 - a}{1 - \overline{a}z_1} - \frac{z_2 - a}{1 - \overline{a}z_2} \right| \\
&= \left| \frac{z_1 - a - \overline{a}z_1 z_2 + |a|^2 z_2 - z_2 + a + \overline{a}z_1 z_2 - |a|^2 z_1}{(1 - \overline{a}z_1)(1 - \overline{a}z_2)} \right| \\
&= \left| \frac{(z_1 - z_2)(1 - |a|^2)}{(1 - \overline{a}z_1)(1 - \overline{a}z_2)} \right|. \qquad (4.1)
\end{aligned}$$

Similarly

$$|\phi(z_1) - \phi(z_2)| = \left| \frac{z_1 + a}{1 + \overline{a}z_1} - \frac{z_2 + a}{1 + \overline{a}z_2} \right| = \left| \frac{(z_1 - z_2)(1 - |a|^2)}{(1 + \overline{a}z_1)(1 + \overline{a}z_2)} \right|.$$

Let $\eta \in \mathbb{S}$, the unit circle. Then we have

$$|\phi(z_1) - \phi(\eta)| = \left| \frac{(z_1 - \eta)(1 - |a|^2)}{(1 + \overline{a}z_1)(1 + \overline{a}\eta)} \right| \;\text{ and }\; |\phi(z_2) - \phi(\eta)| = \left| \frac{(z_2 - \eta)(1 - |a|^2)}{(1 + \overline{a}z_2)(1 + \overline{a}\eta)} \right|$$

where $\phi(\eta) = \lim_{z \to \eta} \phi(z)$. Now we have

$$|\phi(z_1) - \phi(\eta)||\phi(z_2) - \phi(\eta)| = \left| \frac{(z_1 - \eta)(1 - |a|^2)}{(1 + \overline{a}z_1)(1 + \overline{a}\eta)} \right| \left| \frac{(z_2 - \eta)(1 - |a|^2)}{(1 + \overline{a}z_2)(1 + \overline{a}\eta)} \right|.$$

This implies

$$\frac{|(z_1-\eta)(z_2-\eta)|}{|\phi(z_1)-\phi(\eta)||\phi(z_2)-\phi(\eta)|}=\frac{|(1+\overline{a}z_1)(1+\overline{a}z_2)(1+\overline{a}\eta)^2|}{(1-|a|^2)^2}. \tag{4.2}$$

Now we compute

$$|z_1-z_2|=|\sigma(\phi(z_1))-\sigma(\phi(z_2))|=\frac{|1-|a|^2|\,|\phi(z_1)-\phi(z_2)|}{|1-\overline{a}z_1|\,|1-\overline{a}z_2|}.$$

This implies

$$|\phi(z_1)-\phi(z_2)|=\frac{|(1-\overline{a}z_1)||(1-\overline{a}z_2)|}{|1-|a|^2|}|z_1-z_2|.$$

Divide both sides by $|\phi(z_1)-\phi(\eta)||\phi(z_2)-\phi(\eta)|$, we have

$$\begin{aligned}\frac{|\phi(z_1)-\phi(z_2)|}{|\phi(z_1)-\phi(\eta)||\phi(z_2)-\phi(\eta)|}&=\frac{|(1-\overline{a}z_1)||(1-\overline{a}z_2)|}{|\phi(z_1)-\phi(\eta)||\phi(z_2)-\phi(\eta)|}|z_1-z_2|\\&=\frac{|z_1-z_2|}{|z_1-\eta||z_2-\eta|}\frac{|z_1-\eta||z_2-\eta|}{|\phi(z_1)-\phi(\eta)||\phi(z_2)-\phi(\eta)|}\\&\quad\frac{|(1-\overline{a}\phi(z_1))||(1-\overline{a}\phi(z_2))|}{|1-|a|^2|}.\end{aligned}$$

Also

$$\begin{aligned}1-\overline{a}\phi(z)&=1-\overline{a}\left(\frac{z+a}{1+\overline{a}z}\right)\\&=\frac{1+\overline{a}z-\overline{a}z-|a|^2}{1+\overline{a}z}\\&=\frac{1-|a|^2}{1+\overline{a}z}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{|\phi(z_1)-\phi(z_2)|}{|\phi(z_1)-\phi(\eta)||\phi(z_2)-\phi(\eta)|}&=\frac{|z_1-z_2|}{|z_1-\eta||z_2-\eta|}\frac{|1+\overline{a}z_1||1+\overline{a}z_2||1+\overline{a}\eta|}{(1-|a|^2)^2}\\&\quad\frac{(1-|a|^2)(1-|a|^2)}{|1+\overline{a}z_1||1+\overline{a}z_2|(1-|a|^2)}\\&=\frac{|z_1-z_2|}{|z_1-\eta||z_2-\eta|}\frac{|1+\overline{a}\eta|^2}{(1-|a|^2)}.\end{aligned}$$

Now

$$(1-|a|)^2\le|1+\overline{a}\eta|^2\le(1+|a|)^2.$$

This implies
$$\frac{(1-|a|)^2}{1-|a|^2}c_D(z_1,z_2) \le c_D(\phi(z_1),\phi(z_2)) \le \frac{(1+|a|)^2}{1-|a|^2}c_D(z_1,z_2).$$
This implies
$$\frac{(1-|a|)}{1+|a|}c_D(z_1,z_2) \le c_D(\phi(z_1),\phi(z_2)) \le \frac{(1+|a|)}{1-|a|}c_D(z_1,z_2)$$
and the proof is complete. □

4.2 Cassinian geodesic

This section characterizes Cassinian geodesics in some domains. For such characterizations, the concept of maximal Cassinian ovals are used associated with Euclidean disks. In this regard, the following lemma is recalled from [5] without it proof.

Lemma 4.2.1. *[5, Lemma 3.3] Let C be the interior of a Cassinian oval*
$$\{P \in \mathbb{C} : |P - F_1||P - F_2| = b^2\}$$
with foci $F_1 = (a, 0)$ and $F_2(-a, 0), a > 0$. Then
$$\mathbb{D}(F_1, \sqrt{a^2+b^2} - a) \cup \mathbb{D}(F_2, \sqrt{a^2+b^2} - a) \subset C$$
and
$$C \subset \mathbb{D}(F_1, b) \cup \mathbb{D}(F_2, b).$$
Moreover, if $b \ge a$, then
$$C \subset \mathbb{D}(Q_1, r) \cup \mathbb{D}(Q_2, r) \subset \mathbb{D}(F_1, b) \cup \mathbb{D}(F_2, b),$$
where $r = b^2/\sqrt{a^2+b^2}$, $\quad Q_1 = \dfrac{a^2}{\sqrt{a^2+b^2}}$ and $Q_2 = -\dfrac{a^2}{\sqrt{a^2+b^2}}$.

Definition 4.2.2. *An arc $\gamma \subset D$ is said to be a Cassinian geodesic if*
$$c_D(z_1, z_2) = c_D(z_1, z) + c_D(z, z_2)$$
for all ordered points $z_1, z, z_2 \in \gamma$.

Lemma 4.2.3. *[5, Lemma 3.7] For $z_1, z, z_2 \in D$, we have $c_D(z_1, z) + c_D(z, z_2) = c_D(z_1, z_2)$ if and only if there exists a point $p \in \partial D \cap C(z_1, z_2; k(z_1, z_2)) \cap C(z_1, z; k(z_1, z)) \cap C(z, z_2; k(z, z_2))$ such that the points p, z_1, z, z_2 lie on a circle in $\mathbb{C}^n$ in this order.*

Proof. We first deal with the proof of necessary part.

The necessary part. For $z_1, z, z_2 \in D$, it is given that $c_D(z_1, z) + c_D(z, z_2) = c_D(z_1, z_2)$.

We show that there exists a point p lying in the intersection

$$\partial D \cap C(z_1, z_2; k(z_1, z_2)) \cap C(z_1, z; k(z_1, z)) \cap C(z, z_2; k(z, z_2))$$

such that the points p, z_1, z, z_2 lie on a circle in $\mathbb{C}$.

Let $p \in \partial D$ be such that $c_D(z_1, z_2) = |z_1 - z_2|/(|z_1 - p|\,|z_2 - p|)$. For the same p, since

$$c_D(z_1, z_2) = c_D(z_1, z) + c_D(z, z_2) \geq \frac{|z_1 - z|}{|z_1 - p||z - p|} + \frac{|z - z_2|}{|z - p||z_2 - p|},$$

it follows that

$$\frac{|z_1 - z_2|}{|z_1 - p||z_2 - p|} \geq \frac{|z_1 - z|}{|z_1 - p||z - p|} + \frac{|z - z_2|}{|z - p||z_2 - p|}.$$

However, the reverse inequality also holds as described in the proof of Theorem 3.3.1. Therefore, for the same $p \in \partial D$, we have the equality

$$\begin{aligned} &c_D(z_1, z_2) = c_D(z_1, z) + c_D(z, z_2) \\ &\qquad \iff \frac{|z_1 - z_2|}{|z_1 - p||z_2 - p|} = \frac{|z_1 - z|}{|z_1 - p||z - p|} + \frac{|z - z_2|}{|z - p||z_2 - p|}. \end{aligned}$$

Hence, we obtain

$$|z_1 - z||p - z_2| + |z - z_2||p - z_1| = |z_1 - z_2||z - p|$$

for some $p \in \partial D$. By the definition of the Cassinian metric, it implies that $p \in \partial D \cap C(z_1, z_2; k(z_1, z_2)) \cap C(z_1, z; k(z_1, z)) \cap C(z, z_2; k(z, z_2))$, and by Ptolemy's Theorem (see Theorem 2.2.4), the last equality implies that the points p, z_1, z, z_2 lie on a circle in this order.

The sufficient part. For $z_1, z, z_2 \in D$ it is given that there exists a point $p \in \partial D \cap C(z_1, z_2; k(z_1, z_2)) \cap C(z_1, z; k(z_1, z)) \cap C(z, z_2; k(z, z_2))$ such that the points p, z_1, z, z_2 lie on a circle in $\mathbb{C}^n$. It implies that

$$|z_1 - z||p - z_2| + |z - z_2||p - z_1| = |z_1 - z_2||z - p|.$$

Dividing by $|z_1 - p||z_2 - p||z - p|$ on both sides, we get

$$\frac{|z_1 - z|}{|z_1 - p||z - p|} + \frac{|z - z_2|}{|z - p||z_2 - p|} = \frac{|z_1 - z_2|}{|z_1 - p||z_2 - p|}.$$

From the hypothesis we finally get

$$c_D(z_1, z) + c_D(z, z_2) = c_D(z_1, z_2),$$

This completes the proof. □

We next discuss another useful characterization of the Cassinian geodesic. For this, we use the following notation. $[z_1, z_2]_p$: the line segment in $\mathbb{C}_p$ joining the points $z_1, z_2 \in \mathbb{C}_p$ which is the arc of the unique circle in $\mathbb{C}$ passing through the points z_1, z_2, p and not containing p.

Theorem 4.2.4. *[5, Theorem 3.8] An arc $\gamma \subset D$ joining the points z_1 and z_2 is a Cassinian geodesic if and only if there exists $p \in \partial D$ such that*

$$\gamma = [z_1, z_2]_p \cap D_p.$$

Proof. **The necessary part.** For an arc $\gamma \subset D$ joining the point z_1 and z_2 is a Cassinian geodesic. We will show that there exists a point $p \in \partial D$ such that

$$\gamma = [z_1, z_2]_p \cap D_p.$$

For this, let a, b, c be any triple of ordered complex points on γ. By hypothesis we have

$$c_D(a, c) = c_D(a, b) + c_D(b, c).$$

By Lemma 4.2.3 there exists a point $p \in \partial D$ so that the points p, a, b, c lie on a circle in $\mathbb{C}^n$in this order. In particular, γ is a subarc of this circle, i.e.,$\gamma \subset [z_1, z_2]$. Also, by Lemma 4.2.3 we have

$$p \in C(a, b; k(a, b)), \quad \forall a, b \in \gamma.$$

In particular, if we let $b \to a$ we obtained that $\delta(a) = |a - p|$. Since $a \in \gamma$ is arbitrary, we obtain$\gamma \subset D_p$. Since γ is an arc, we have

$$\gamma = [z_1, z_2]_p \cap D_p.$$

The sufficient part. There exists a point $p \in \partial D$ such that

$$\gamma = [z_1, z_2]_p \cap D_p.$$

We require to show that the arc $\gamma \subset D$ joining the point z_1 and z_2 is a Cassinian geodesic, i.e. we have to show

$$c_D(a, c) = c_D(a, b) + c_D(b, c)$$

for all ordered points a, b, c. i.e., we have to show there exists $p \in \partial D$ such that the points p, a, b, c lie on a circle in $\mathbb{C}$ in this order. This is nothing but the direct consequence of Lemma 4.2.3. This completes the proof. $\square$

CHAPTER 5

Comparison with the hyperbolic metric

5.1 The hyperbolic metric

In this chapter we consider the hyperbolic metric of the unit disk and compare it with the Cassinian metric.

Definition 5.1.1. *The* **hyperbolic metric** $\rho_{\mathbb{D}}$ *of the unit disk* $\mathbb{D} = \{z : |z| < 1\}$ *is given by*

$$\rho_{\mathbb{D}}(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \frac{2|dz|}{1 - |z|^2},$$

where the infimum is taken over all rectifiable curves $\subset \mathbb{D}$ *joining* z_1 *and* z_2*.*

As noted in [8, Section 2.1.1], the hyperbolic geodesic from the origin to any point $z_2 \in \mathbb{D}$ is the line segment $[0, z_2]$ joining 0 and z_2. More precisely, we have

Lemma 5.1.2. *[8, Section 2.1.1] Let $z_2 \in \mathbb{D}$ be arbitrary. Then we have*

$$\int_{[0,z_2]} \frac{2|dz|}{1-|z|^2} \le \int_\gamma \frac{2|dz|}{1-|z|^2}$$

for all rectifiable curves $\gamma \subset \mathbb{D}$ connecting 0 and z_2.

Lemma 5.1.3. *[2, p. 124] For the unit disk $\mathbb{D}, 0 \le r < 1$, we have*

$$\rho_{\mathbb{D}}(0,r) = 2\tan^{-1}(r) = \log\frac{1+r}{1-r}.$$

Proof. By Lemma 5.1.2, it follows that

$$\rho_{\mathbb{D}}(0,r) = \int_{[0,r]} \frac{2|dz|}{1-|z|^2}.$$

Let $z(t) = t, \quad i.e., \quad d(z(t)) = dt, \quad 0 \le t \le r$. Then

$$\rho_{\mathbb{D}}(0,r) = \int_0^r \frac{2\,dt}{1-t^2} = 2\tan^{-1}(r) = \int_0^r \left(\frac{1}{1+t} + \frac{1}{1-t}\right)dt = \log\frac{1+r}{1-r},$$

completing the proof. □

Lemma 5.1.4. *[2, p. 125] Let $M : \mathbb{D} \to \mathbb{D}$ be a Möbius transformation, then for any $z_1, z_2 \in \mathbb{D}$, we have*

$$\frac{|z_1 - z_2|^2}{(1-|z_1|^2)(1-z_2|^2)} = \frac{|M(z_1) - M(z_2)|^2}{(1-|M(z_1)|^2)(1-|M(z_2)|^2)}.$$

Proof. Consider any Möbius transformation of the form

$$M(z) = e^{i\theta}\left(\frac{z-a}{1-\overline{a}z}\right), \quad \theta \in \mathbb{R}. \tag{5.1}$$

Now we compute

$$\begin{aligned}
(M(z_1) - M(z_2))^2 &= \left(e^{i\theta}\left(\frac{z_1-a}{1-\overline{a}z_1}\right) - \left(\frac{z_2-a}{1-\overline{a}z_2}\right)\right)^2 \\
&= e^{2i\theta}\left(\frac{z_1 - \overline{a}z_1z_2 - a + |a|^2z_2 - z_2 + \overline{a}z_1z_2 + a - |a|^2z_1}{(1-\overline{a}z_1)(1-\overline{a}z_2)}\right)^2 \\
&= e^{2i\theta}\frac{(1-|a|^2)^2(z_1-z_2)^2}{(1-\overline{a}z_1)^2(1-\overline{a}z_2)^2}.
\end{aligned} \tag{5.2}$$

Now we have to find

$$M^{'}(z) = e^{i\theta}\frac{(1-\overline{a}z) + \overline{a}(z-a)}{(1-\overline{a}z)^2} = e^{i\theta}\frac{1-|a|^2}{(1-\overline{a}z)^2}.$$

Therefore,

$$M'(z_1) = e^{i\theta}\frac{1-|a|^2}{(1-\overline{a}z_1)^2} \quad \text{and} \quad M'(z_2) = e^{i\theta}\frac{1-|a|^2}{(1-\overline{a}z_2)^2}.$$

Also, we have

$$\begin{aligned} M'(z_1)M'(z_2)(z_1-z_2)^2 &= e^{2i\theta}\left(\frac{1-|a|^2}{(1-\overline{a}z_1)^2}\right)\left(\frac{1-|a|^2}{(1-\overline{a}z_2)^2}\right)\left((z_1-z_2)^2\right) \\ &= (M(z_1)-M(z_2))^2. \end{aligned} \tag{5.3}$$

Also we find

$$\begin{aligned} 1-|M(z)|^2 &= 1-\frac{|z-a|^2}{|1-\overline{a}z|^2} \\ &= \frac{|1-\overline{a}z|^2-|z-a|^2}{|1-\overline{a}z|^2} \\ &= \frac{1+|a|^2|z|^2-2\mathrm{Re}\,(\overline{a}z)-|z|^2-|a|^2-2\mathrm{Re}\,(\overline{a}z))}{|1-\overline{a}z|^2} \\ &= \frac{(1-|z|^2)(1-|a|^2)}{|1-\overline{a}z|^2}. \end{aligned} \tag{5.4}$$

Now by the help of equation (5.4) we write

$$|M'(z_1)| = \frac{1-|a|^2}{|1-\overline{a}z_1|^2} = \frac{1-|M(z_1)|^2}{1-|z_1|^2},$$

which implies

$$\frac{|M'(z_1)|}{1-|M(z_1)|^2} = \frac{1}{1-|z_1|^2}. \tag{5.5}$$

Similarly one can obtain

$$\frac{|M'(z_2)|}{1-|M(z_2)|^2} = \frac{1}{1-|z_2|^2}. \tag{5.6}$$

Therefore, by the help of equations (5.5), (5.6) and (5.3), we write

$$\begin{aligned} \frac{|z_1-z_2|^2}{(|1-|z_1|^2)(1-|z_2|^2)} &= |z_1-z_2|\frac{|M'(z_1)|}{1-|M(z_1)|^2}\frac{|M'(z_2)|}{1-|M(z_2)|^2} \\ &= \frac{|M(z_1)-M(z_2)|^2}{(1-|M(z_1)|^2)(1-|M(z_2)|^2)}. \end{aligned}$$

This completes the proof. □

An useful formula for the hyperbolic metric can now be derived as follows (see [3, p. 131] and [8, Exc. 2.11]).

Lemma 5.1.5. *For all $z_1, z_2 \in \mathbb{D}$, we have*

$$\sinh\left(\frac{\rho_{\mathbb{D}}(z_1,z_2)}{2}\right) = \frac{|z_1 - z_2|}{\sqrt{(1-|z_1|^2)(1-|z_2|^2)}}. \tag{5.7}$$

Proof. Consider any Möbius transformation of the form

$$M(z) = e^{i\theta}\left(\frac{z-a}{1-\overline{a}z}\right), \quad \theta \in \mathbb{R}$$

with $z_1 \to 0$ and $z_2 \to r$. From Lemma 5.1.4, we have

$$\frac{|z_1-z_2|^2}{(1-|z_1|^2)(1-z_2|^2)} = \frac{|M(z_1)-M(z_2)|^2}{(1-|M(z_1)|^2)(1-|M(z_2)|^2)}.$$

By the above argument this can be written as

$$\frac{|z_1-z_2|^2}{(1-|z_1|^2)(1-z_2|^2)} = \frac{r}{1-r^2} = \sinh^2\left(\frac{1}{2}\rho_{\mathbb{D}}(0,r)\right) = \sinh^2\left(\frac{1}{2}\rho_{\mathbb{D}}(z_1,z_2)\right).$$

This implies that

$$\sinh\left(\frac{\rho_{\mathbb{D}}(z_1,z_2)}{2}\right) = \frac{|z_1 - z_2|}{\sqrt{(1-|z_1|^2)(1-|z_2|^2)}}.$$

Hence the result.

Now we have to verify the equality $\dfrac{r}{1-r^2} = \sinh^2(\frac{1}{2}\rho_{\mathbb{D}}(0,r))$. From Lemma 5.1.3, we have

$$\rho_{\mathbb{D}}(0,r) = \log\frac{1+r}{1-r}.$$

That is,

$$\begin{aligned}
\sinh^2\left(\frac{1}{2}\rho_{\mathbb{D}}(0,r)\right) &= \sinh^2\left(\log\sqrt{\frac{1+r}{1-r}}\right) \\
&= \left(\sqrt{\frac{1+r}{1-r}} - \sqrt{\frac{1-r}{1+r}}\right)^2 \\
&= \left(\frac{r}{\sqrt{1-r}}\right)^2 \\
&= \frac{r^2}{1-r^2}.
\end{aligned}$$

This completes the proof. □

Using Lemmas 5.1.4 and 5.1.5, we now conclude that the hyperbolic metric is Möbius invariant.

Corollary 5.1.6. *Let $M : \mathbb{D} \to \mathbb{D}$ be a Möbius transformation. Then for any $z_1, z_2 \in \mathbb{D}$, we have*

$$\rho_{\mathbb{D}}(z_1, z_2) = \rho_{\mathbb{D}}(M(z_1), M(z_2)).$$

Proof. By Lemmas 5.1.4 and 5.1.5, we see that

$$\sinh\left(\frac{\rho_{\mathbb{D}}(z_1, z_2)}{2}\right) = \frac{|z_1 - z_2|}{\sqrt{(1-|z_1|^2)(1-|z_2|^2)}}$$
$$= \frac{|M(z_1) - M(z_2)|}{\sqrt{(1-|M(z_1)|^2)(1-|M(z_2)|^2)}} = \sinh\left(\frac{\rho_{\mathbb{D}}(M(z_1), M(z_2))}{2}\right).$$

Thus, the conclusion follows. □

In the next section, we compare the hyperbolic metric with the Cassinian metric.

5.2 Comparison with the Cassinian metric

Theorem 5.2.1. *[6, Theorem 3.1] For $z_1, z_2 \in \mathbb{D}$, we have*

$$\sinh\left(\frac{\rho_{\mathbb{D}}(z_1, z_2)}{2}\right) \le c_{\mathbb{D}}(z_1, z_2).$$

Here, the equality holds for $z_2 = -z_1$.

Proof. Without loss of generality we consider $|z_2| > |z_1|$.

Case-1: $z_2 = 0$.

In this case, Theorem 5.2.1 trivially holds as $z_1 = 0$ as well.

Case-2: $z_2 \neq 0$.

For this, we compute

$$\inf_{z\in\partial D}|z_1-z||z_2-z| \le |z_1-\frac{z_2}{|z_2|}||z_2-\frac{z_2}{|z_2|}|$$
$$= |z_1-\frac{z_2}{|z_2|}(1-|z_2|)$$
$$\le (|z_1|+1)(1-|z_2|)$$
$$\le \sqrt{(1-|z_1|^2)(1-|z_2|^2)},$$

Verification of last inequality: let it be true, then

$$(|z_1|+1)(1-|z_2|) \le \sqrt{(1-|z_1|^2)(1-|z_2|^2)}.$$

Squaring both sides and expanding, we obtain

$$(1+|z_1|^2+2|z_1|)(1+|z_2|^2-2|z_2|) \le (1-|z_1|^2-|z_2|^2+|z_1|^2|z_2|^2).$$

After some simplification, we get

$$(|z_2|-|z_1|)^2+(|z_2|-|z_1|)(|z_1||z_2|-1) \le 0,$$

equivalently,

$$(|z_2|-|z_1|)(|z_2|-|z_1|+|z_1||z_2|-1) \le 0$$

equivalently,

$$(|z_2|-|z_1|)(|z_2|-1)(|z_1|+1) \le 0.$$

Since $|z_2| \ge |z_1|$ then, $|z_2|-|z_1| \ge 0$. it implies that $|z_1| < 1$, So the last inequality is true. Thus

$$\min_{z\in\partial\mathbb{D}}|z_1-z||z_2-z| \le \sqrt{(1-|z_1|^2)(1-|z_2|^2)}$$

That is

$$\frac{|z_1-z_2|}{\min_{z\in\partial\mathbb{D}}|z_1-z||z_2-z|} \ge \frac{|z_1-z_2|}{\sqrt{(1-|z_1|^2)(1-|z_2|^2)}}$$

$$c_{\mathbb{D}}(z_1,z_2) \ge \sinh\left(\frac{\rho_{\mathbb{D}}(z_1,z_2)}{2}\right).$$

This completes the proof.

For the equality, take $z_2=-z_1$. Then we see that

$$c_{\mathbb{D}}(z_1,-z_1) = \frac{2|z_1|}{1-|z_1|^2} = \sinh(\rho_{\mathbb{D}}(z_1,-z_1)/2),$$

using Lemma 5.1.5. □

Corollary 5.2.2. *[6, Corollary 3.3] For $z_1, z_2 \in \mathbb{D}$, we have the following sharp inequality*

$$\rho_{\mathbb{D}}(z_1, z_2) \leq 2c_{\mathbb{D}}(z_1, z_2).$$

Proof. We know that

$$\sinh(t) \geq t, \qquad \text{for all } t \geq 0.$$

This leads to

$$\sinh\left(\frac{\rho_{\mathbb{D}}(z_1, z_2)}{2}\right) \geq \frac{\rho_{\mathbb{D}}(z_1, z_2)}{2}.$$

By Theorem 5.2.1, we have

$$c_{\mathbb{D}}(z_1, z_2) \geq \frac{\rho_{\mathbb{D}}(z_1, z_2)}{2}$$

and hence

$$\rho_{\mathbb{D}}(z_1, z_2) \leq 2c_{\mathbb{D}}(z_1, z_2),$$

completing the proof. □

CHAPTER 6

Conclusion

6.1 Overview

This thesis comprises of six chapters. The first chapter is of introductory type. The second chapter contains certain elementary properties and examples on Möbius transformations. The Cassinian metric, with it computations and several basic properties, is considered in Chapter 3. Most of the main results that we wish to highlight are covered in Chapter 4. The Poincare's hyperbolic metric of the unit disk, which got special attention in Chapter 5, is compared with the Cassinian metric.

6.2 Future Direction

Recall from complex analysis that the density of the chordal metric on the Riemann sphere generates the spherical metric. This is computed by taking the limit of the ratio of the chordal metric and the Euclidean metric between two points, when they are nearer. In a similar manner, we here compute the density of the Cassinian metric.

Lemma 6.2.1. *[5, Corollary 3.6] For a proper subdomain D of $\mathbb{C}$ and for $z \in D$, the density $\tau_D(z)$ of the Cassinian metric c_D at z is given by*

$$\tau_D(z) = \frac{1}{(\delta(z))^2}.$$

Proof. For $z, w \in D$, we have

$$\tau_D(z) = \lim_{w \to z} \frac{c_D(z,w)}{|z-w|} = \lim_{w \to z} \frac{1}{\min_{p \in \partial D} |z-p|\,|w-p|} = \frac{1}{\min_{p \in \partial D} |z-p|^2}.$$

It follows that

$$\tau_D(z) = \frac{1}{(\delta(z))^2},$$

as required. □

Let $z_1, z_2 \in D$. As in [5], considering this density function, the inner Cassinian metric is defined by

$$\tilde{c}_D(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{\delta(z)^2},$$

where the infimum is taken over all the rectifiable curves $\gamma \subset D$ joining z_1 and z_2. This metric is further considered in [6]. However, only a few elementary results concerning this are studied therein. Therefore, as a future scope in this direction, many open problems connected to the inner Cassinian metric may be investigated. For instance, one can ask questions about characterizations of certain domains having nice geometric properties in terms of inequalities in the Cassinian and its inner metrics. Secondly, comparisons of it with the hyperbolic and other related metrics can also be studied.

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