Study of Dynamical Sampling and frame representations

M.Sc. Thesis

by

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"Study of Dynamical Sampling and frame representations"** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted to the **DEPARTMENT OF MATHEMATICS**, **Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2022 to June 2023 under the supervision of **Dr. Niraj Kumar Shukla**, Associate Professor, Department of Mathematics, IIT Indore. The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

07/06/2023

Signature of the student with date

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

7/6/2023

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Date: 7 June 2023

*Dedicated to my
Family*

"When you were born in the world,

you cried and everyone laughed.

Behave in such a way that you laugh

when you leave and they cry.".

–***Saint Kabir***

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Abstract

Let $Y = \{\phi(i), X\phi(i), \cdots, X^{m_i}\phi(i) : \phi \in \mathcal{H},\ X \in \mathcal{B}(\mathcal{H})$ and $i \in \Omega\}$, where $\mathcal{H}$ is a separable Hilbert space. Our main objective is to establish the essential requirements and criteria involving $X, \Omega, m_i : i \in \Omega$ that are necessary and sufficient to retrieve any function $\phi \in \mathcal{H}$ from the given sample set Y. This is known as the dynamical sampling problem. Our objective is to reconstruct ϕ by combining rough samples of ϕ with its future states $X^\ell \phi$. Dynamical sampling is widely applicable in various domains, including time-space sampling trade-off, super-resolution, on-chip sensing, satellite remote sensing, and more. In finite-dimensional spaces, we discuss this problem for a diverse class of bounded linear operators, including diagonalizable, convolution, Fourier multiplier, and translation invariant operators. Next, we analyze when a frame $\{\phi_n\}_{n=1}^N$ is dynamical frame for $\ell^2(\mathbb{Z}_d)$, i.e. there exist a bounded linear operator $X : \ell^2(\mathbb{Z}_d) \to \ell^2(\mathbb{Z}_d)$ and $\phi \in \ell^2(\mathbb{Z}_d)$ such that $\{\phi_n\}_{n=1}^N = \{X^n\phi\}_{n=1}^N$. We provide characterization results for dynamical frame and dynamical dual frame. Moreover, we demonstrate that each overcomplete frame possesses an infinite number of dynamical dual frames.

Contents

CHAPTER 1

Introduction

1.1 History of Dynamical Sampling

The origin of dynamical sampling is relatively recent in history. In 2009, Lu and Vetterli discussed the concept in their paper titled "Spatial super-resolution of a diffusion field by temporal oversampling in sensor networks" [24]. This is the first work on dynamical sampling. In 2011, Vetterli et al. further contributed to the field with their work on "Sampling and reconstructing diffusion fields with localized sources" [26]. These notable publications played a significant role in advancing our understanding and applications of dynamical sampling. In 2014, Aldroubi investigated into the field of dynamical sampling through a series of papers, including "Dynamical sampling" [3], "Dynamical sampling in shift-invariant spaces" [1] and "Dynamical Sampling: Time Space Trade-off" [4]. [1] Aldroubi's

[1]This chapter focuses on the dynamical sampling problem, which is elucidated using the research contributions of Aldroubi [3] and Martín, Medri, and Molter [25].

work shed light on the details of this specialized form of dynamical sampling, contributing valuable insights to the field. The most general form of dynamical sampling problem was studied by Aldroubi et al. in [3].

1.2 Difference between traditional & dynamical Sampling

Dynamical sampling and traditional sampling differ in the way that the sampling locations are selected. In traditional sampling, the sampling locations are fixed and predetermined, typically based on a uniform grid or some other regular pattern. Then signal is sampled at these fixed locations and the resulting samples are used for signal processing and analysis.

In contrast, dynamical sampling is a powerful signal processing technique that allows us to represent signals with a small number of samples accurately. The technique involves adapting the sampling locations to the signal's local features, such as frequency content and spatial structure. This approach can lead to more efficient sampling and better signal reconstruction compared to traditional, fixed sampling techniques.

Dynamical sampling can be beneficial when the signal of interest is highly variable and complex, with properties that vary over time and space. In such cases, traditional sampling techniques may miss important features of the signal or require a large number of samples to achieve high-fidelity representation. Dynamical sampling improves the efficiency of capturing essential signal information by adjusting the sampling locations based on the local properties of the signal.

1.3 Applications of Dynamical sampling

In environmental monitoring applications, sensors may be used to capture complex signals such as temperature, humidity, air quality, and water quality, which may exhibit spatial and temporal variations. By using dynamical sampling techniques, researchers can optimize the sampling locations to capture the most important features of these signals while minimizing the number of samples required. Here are a few examples of how it is used in different applications:

- **Time-space sampling trade-off:** The time-space sampling trade-off in dynamical sampling refers to finding the right balance between sampling frequency in time (temporal resolution) and data density in space (spatial resolution). Increasing temporal resolution captures finer changes but requires more computational resources, while higher spatial resolution captures more detail but increases storage requirements. Optimizing this trade-off involves choosing a sampling scheme that captures essential dynamics while minimizing resource usage [4].

- **Super-resolution:** The problem of super-resolution in dynamical sampling pertains to the challenge of reconstructing a high-resolution signal using a collection of low-resolution samples. In dynamical sampling, a signal is sampled in time and space, and the obtained samples are used to reconstruct the original signal. However, due to various factors such as noise, sensor limitations, and aliasing, the obtained samples may have a lower spatial resolution than the original signal [2].

- **On-chip sensing:** The application of dynamical sampling in on-chip sensing involves adapting the sampling rate of the sensing system to the changing characteristics of the measured signal. This technique allows for more efficient use of limited resources, such as power and memory, while maintaining the accuracy of the estimated data. For example, in a temperature sensor integrated into a microchip, the temperature may change slowly over time, requiring a lower sampling rate to capture the data accurately. However, if the temperature suddenly spikes, a higher sampling rate may be needed to capture the rapid change in temperature accurately [2].

- **Satellite remote sensing:** Satellite remote sensing involves the use of satellites equipped with sensors to acquire data about the Earth's surface and atmosphere. High spatial resolution refers to the ability to distinguish fine details in the imagery, while low temporal resolution refers to the frequency at which the data is acquired over time. By applying dynamical sampling in satellite remote sensing, the satellite can adjust its data acquisition rate based on the signal's characteristics, allowing for more efficient use of limited resources such as power, memory, and bandwidth. For example, in areas of the Earth's surface where no significant changes occur over a short time, the satellite can decrease the data acquisition rate to conserve resources while maintaining the desired level of spatial resolution [2].

- **Wireless Sensing Networks (WSN):** In biomedical sensing applications, such as electrocardiography (ECG) and electroencephalography (EEG), dynamical sampling can be used to accurately capture the signals while reducing the number of electrodes required. Dynamical sampling can be especially important in wearable sensing applications, where minimizing the number of sensors can improve patient comfort and compliance[3].

1.4 Problem formulation for dynamical sampling

Aldroubi [3] introduced the concept of the dynamical sampling problem within the context of a separable Hilbert space $\mathcal{H}$ of finite dimensions. Let $\phi \in \mathcal{H}$ and $X : \mathcal{H} \to \mathcal{H}$ be an evolution operator. $\phi^{(n)} = X^n \phi$ denotes the evolved function ϕ at time n. The Hilbert space $\mathcal{H}$ is associated with $\ell^2(I)$, where $I = 1, 2, \cdots, d$.

The expression $X^t\phi(i)$ denotes a sample in the time and space domain, specifically at time t within the set of natural numbers and location $i \in I$. We establish a connection between every combination of an element i from the set I and a natural number t, associating each pair with a specific sample value. The dynamical sampling problem can be formulated as follows: *Given an operator X and a set $P \subset I \times \mathbb{N}$, the question is to determine the conditions under which every $\phi \in \mathcal{H}$ can be recovered reliably using the samples in P.* We take samples of ϕ at specific locations $\Omega_n \subset I$ at time $t = n$, which gives us the samples $\{\phi^{(n)}(i) : i \in \Omega_n\}$. Where

$$\phi^{(n)}(i) = X^n\phi(i) = \langle X^n\phi, e_i\rangle,$$

where $\{e_i\}_{i\in I}$ is a standard basis for $\ell^2(I)$.

Generally, the measurements $\phi^{(0)}(i) : i \in \Omega_0$ acquired from the original signal $\phi = \phi^{(0)}$ do not possess enough information to reconstruct ϕ completely. In simpler terms, the signal ϕ is undersampled. The some extra information needed to recover ϕ is given by $\{\phi^{(n)}(i) = X^n\phi(i) : i \in \Omega_n\}$. Again, in general the samples $\{\phi^n(i) : i \in \Omega_n\}$ are insufficient to recover $X^n\phi$ for each n.

Problem: Consider the evolution operator $X : \ell^2(I) \to \ell^2(I)$, where I represents a set of indices. Let Ω be a fixed subset of I, and let $\{m_i : i \in \Omega\}$ be a collection of positive integers. The goal is to determine the conditions on X, Ω, and $\{m_i : i \in \Omega\}$ that enable the stable recovery of any vector $\phi \in \ell^2(I)$ from the given samples $S = \{\phi(i), X\phi(i), \ldots, X^{m_i}\phi(i) : i \in \Omega\}$.

Dynamical Sampling Problem in an Infinite-Dimensional Hilbert Space: In 2021, Martín, Medri, and Molter introduced the concept of the continuous dynamical sampling problem for infinite-dimensional Hilbert spaces [25]. This problem defines two distinct types of dynamical sampling problems within separable infinite-dimensional Hilbert spaces,

(i) Discrete dynamical sampling.

(ii) Continuous dynamical sampling.

Consider a separable Hilbert space $\mathcal{H}$, which could be $\ell^2(\mathbb{N})$ or $L^2([0, L])$ for a specific $L > 0$. Within this context, let ϕ represent a function in $\mathcal{H}$ that evolves over time through an evolution operator denoted as $X : \mathcal{H} \to \mathcal{H}$. We can describe the function ϕ at a particular time step n as $\phi^{(n)} = X^n\phi$. In this discussion, we can identify $\mathcal{H}$ with $\ell^2(I)$, where I corresponds to $\mathbb{N}$ for discrete dynamical sampling and $[0, L]$ or $[0, \infty)$ for continuous dynamical sampling. Consider a sequence of vectors $G = \{g_j\}_{j\in J}$ in the Hilbert space $\mathcal{H}$, where J represents a countable set of infinite indices. The time-space sample at time $t \in I$ and location $j \in J$ is defined as the value

$$\phi^{(t)}(j) = \langle X^t\phi, g_j\rangle.$$

In this approach, we assign a sample value to each pair $(j, t) \in J \times I$. The fundamental question of the dynamical sampling problem can be expressed as follows: *What are the conditions on the operator X, the vectors g_j, and the set $J \times I$ under which any vector $\phi \in \mathcal{H}$ can be reliably reconstructed from the given samples*

$$Y = \{\phi^{(t)}(j) : (j, t) \in J \times I\}.$$

Here, the term *stable way* refers to the property that the reconstruction is robust under small perturbations. We will make this notion precise in a moment.

Now assume that operator X is bounded operator and S_j is operator such that

$$S_j := \overline{span}\{g_j : j \in J\} \to L^2(I, \mu_I),$$

where $(S_j\phi)(t) = \phi^{(t)}(j)$ and μ_I is either the discrete measure or the Lebesgue measure depending on the set I . Define S to be the operator $S = \bigoplus_{j\in J} S_j$ for $j \in J$. We assert that the function ϕ can be recovered from a set of samples Y when the linear operator S, which is bounded and invertible on its range, is used. That is, there exist constants $A, B > 0$ such that for all $\phi \in \mathcal{H}$,

$$A\|\phi\|_2^2 \leq \|S\phi\|_2^2 = \sum_{j\in J} \|S_j\phi\|_{L^2(\mu_I)}^2 = \sum_{j\in J} \int_I |\langle X^{(t)}\phi, g_j\rangle|^2 d\mu_I(t) \leq B\|\phi\|_2^2. \quad (1.1)$$

The options for $X^{(t)} : t \in I$ are outlined below: The discrete dynamical sampling problem refers to a scenario where the iterations of the operator X are discrete. In this case, we define $X^{(t)}$ as X raised to the power of t, where $t \in I$ Using the countability of I for all $\phi \in \mathcal{H}$, (1.1) becomes

$$A\|\phi\|_2^2 \leq \sum_{j\in J}\sum_{t\in I} |\langle \phi, (X^*)^t g_j\rangle|^2 \leq B\|\phi\|_2^2,$$

where X^* is the adjoint operator of X. In the context of the continuous dynamical sampling problem, we take $X^{(t)} := e^{tX}$ for $t \in I$, where $I = [0, L]$ or $[0, \infty)$. Therefore for all $\phi \in \mathcal{H}$ condition (1.1) can be written as

$$A\|\phi\|_2^2 \leq \sum_{g\in G} \int_I |\langle \phi, e^{tX^*} g\rangle|^2 d\mu_I(t) \leq B\|\phi\|_2^2.$$

1.5 Organization

In Chapter 2, we establish a connection between dynamical sampling problems and frames in a separable Hilbert space. We specifically examine the dynamical sampling problem in cases where the evolution operator X is diagonalizable or has a Jordan form. Additionally, in Chapter 2, we provide solutions to the dynamical sampling problem for evolution operators that are translation invariant, Fourier multipliers, and convolution operators. In Chapter 3, we present a characterization of dynamical frames. Furthermore, in this chapter, we provide a characterization of dynamical dual frames and establish that every redundant frame has an infinite number of dual frames. Chapter 4 focuses on the practical applications of dynamical sampling. We explore its use in solving initial value problems and sampling signals in the presence of noise. Finally, in the last chapter, we provide a conclusion and outline future plans.

CHAPTER 2

Dynamical sampling

In this chapter, we will give a brief introduction to frames in finite dimensional Hilbert spaces [22]. In Section 2.1, we will explore the relationship between the dynamical sampling problem and frames. To establish the essential conditions that must be met for a set Ω and its elements $\{m_i : i \in \Omega\}$ to be both necessary and sufficient, such that any vector ϕ in $\ell^2(I)$ can be recovered stably from a set of samples $S = \{\phi(i), X\phi(i), \cdots, X^{m_i}\phi(i) : i \in \Omega\}$ when evolution operator X to be diagonal matrix and Jordan matrix is main objective of Section 2.2. The main results of Sections 2.1 and 2.2 are taken from the work of Aldroubi [3]. [1] [2]

[1]This chapter provides a concise overview of frames in finite dimensional Hilbert spaces. The key findings are derived from the research conducted by Aldroubi [3] and the book authored by D. Han [22]

[2]In last sections, we will also investigate the results of dynamical sampling using the polar value decomposition and for some special operators.

In Sections 2.3, 2.4, 2.5, and 2.6, we will present our findings. These findings establish the conditions under which the set $\{X^t e_i : 0 \leq t \leq m, , i \in \Omega_t\}$ becomes a frame for $\ell^2(\mathbb{Z}_d)$. To achieve this, we use different types of evolution operators: polar value decomposition, translation invariant, convolution, and Fourier multiplier operators, as discussed in each respective section. In the last Section we discuss dynamical sampling in infinite dimensional Hilbert space.

2.1 Frames and its relation with dynamical sampling

The set $\{v_i\}_{i=1}^k$ in a vector space $\mathcal{V}$ is said to be *linearly independent set* if $\sum_{i=1}^{k} c_i v_i = 0$ for some $c_i \in \mathbb{F}$, implies $c_i = 0, \ \forall\ 1 \leq i \leq k$. A subset $\mathcal{S} = \{v_1, v_2, \ldots, v_n\}$ of a vector space $\mathcal{V}$ is called a *spanning set* if it has the property that any vector in $\mathcal{V}$ can be expressed as a linear combination of the vectors in $\mathcal{S}$. In such cases, we say that $\mathcal{S}$ spans $\mathcal{V}$, denoted by $L(\mathcal{S}) = \mathcal{V}$.

Definition 2.1. If every element of a vector space $\mathcal{V}$ is a linear combination of elements of $\mathcal{S}$ is linearly independent, then $\mathcal{S}$ is called a basis for $\mathcal{V}$.

Example 2.1. Consider the vector space $\ell^2(\mathbb{Z}_d) = \{(z(n))_{n=0}^{d-1} \ : z(n) \in \mathbb{C}\}$. We have two special orthonormal bases for $\ell^2(\mathbb{Z}_d)$.

(i) Set $\{e_0, e_1, ..., e_{d-1}\}$ is known as standard orthonormal basis, where

$$e_n[k] = \begin{cases} 0, & \text{if } n \neq k, \\ 1, & \text{if } n = k. \end{cases}$$

(ii) Set $\{f_0, f_1, ..., f_{d-1}\}$ is called Fourier basis for $\ell^2(\mathbb{Z}_d)$, if

$$f_n[k] = \frac{e^{2\pi ikn/d}}{\sqrt{d}}.$$

Example 2.2. Vector space $\ell^2(\mathbb{N})$ is defined as

$$\mathcal{V} = \left\{ (z(n))_{n\in\mathbb{N}} : \sum_{n\in\mathbb{N}} |z(n)|^2 < \infty \right\}.$$

The Set $\{e_n\}_{n\in\mathbb{N}}$ is known as standard orthonormal basis, where

$$e_n[k] = \begin{cases} 0, & \text{if } n \neq k, \\ 1, & \text{if } n = k. \end{cases}$$

Definition 2.2. For $k \geq n$, a finite sequence of vectors $\{v_1, v_2, v_3, \cdots, v_k\}$ in $\mathbb{R}^n$ is said to be *extension to a basis* for $\mathbb{R}^k$, if there exist vectors $\{w_1, w_2, w_3, \cdots, w_k\}$ in $\mathbb{R}^{k-n}$ such that

$$\left\{ \begin{pmatrix} v_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ w_2 \end{pmatrix}, \cdots, \begin{pmatrix} v_k \\ w_k \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^k$.

Example 2.3. Consider the collection of vectors,

$$\{v_1, v_2, v_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \subset \mathbb{R}^2.$$

If we take the vectors $\{w_1, w_2, w_3\} = \{[0], [0], [3]\} \subset \mathbb{R}$, then

$$\left\{ \begin{pmatrix} v_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ w_2 \end{pmatrix}, \begin{pmatrix} v_3 \\ w_3 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^3$. Hence, $\{v_1, v_2, v_3\}$ is an extension to a basis for $\mathbb{R}^3$.

Definition 2.3. Let $\mathbb{F} = \{\phi_n\}_{n=1}^k$, be a finite sequence of vectors in $\mathbb{R}^n$ is an extension to a basis for $\mathbb{R}^k$, where $k \geq n$. Then $\mathbb{F}$ is called an $\mathbb{R}^n$-*frame*.

Next, we will present a general definition of frames within the context of Hilbert spaces.

Definition 2.4. Let $\mathcal{H}$ be a separable Hilbert space. A sequence of vectors $\{\phi_i\}_{i\in I} \subset \mathcal{H}$ is said to be a *frame for* $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that for every $x \in \mathcal{H}$,

$$A\|x\|^2 \leq \sum_{i\in I} |\langle x, \phi_i \rangle|^2 \leq B\|x\|^2.$$

The constants A and B are known as lower and upper frame bounds, respectively. If $A = B$, then $\{\phi_i\}_{i\in I}$ is called *tight frame* and it is called Parseval frame if $A = B = 1$.

Proposition 2.1. *[16] Let $\mathcal{H}$ be a separable Hilbert space and $\{\phi_i\}_{i=1}^k$ be a sequence in a Hilbert space $\mathcal{H}$. If $\mathcal{W} := span\{\phi_i\}_{i=1}^k$, then there exist constants $0 < A \leq B < \infty$ such that*

$$A\|x\|^2 \leq \sum_{i=1}^{k} |\langle x, \phi_i\rangle|^2 \leq B\|x\|^2 \text{ for all } x \in \mathcal{W}.$$

Proof. Assume that not all $\{\phi_i\}$ are zero. Now consider finite families $\{\phi_i\}_{i=1}^k \subset \mathcal{V}$, $k \in \mathbb{N}$. Then by Cauchy-Schwarz' inequality,

$$\sum_{i=1}^{k} |\langle x, \phi_i\rangle|^2 \leq \sum_{i=1}^{k} \|\phi_i\|^2\|x\|^2 \text{ for all } x \in \mathcal{V}.$$

Now consider upper frame bound $B = \sum_{i=1}^k \|\phi_i\|^2$ and clearly $B > 0$.

For lower bound consider the continuous mapping $f : \mathcal{W} \to \mathbb{R}$,

$$f(x) = \sum_{i=1}^{k} |\langle x, \phi_i\rangle|^2$$

Since the unit sphere within $\mathcal{W}$ possesses compactness, so we can get $y \in \mathcal{W}$ with $\|y\| = 1$ such that

$$\begin{aligned} A &:= \sum_{i=1}^{k} |\langle y, \phi_i\rangle|^2 \\ &= inf\{\sum_{i=1}^{k} |\langle x, \phi_i\rangle|^2 \; x \in \mathcal{W}, \|x\| = 1\}. \end{aligned}$$

It is clear that $A > 0$, Now given $x \in \mathcal{W}, x \neq 0$, we have

$$\sum_{i=1}^{k} |\langle x, \phi_i\rangle|^2 = \sum_{i=1}^{k} \|x\|^2 |\langle \frac{x}{\|x\|}, \phi_i\rangle|^2 \geq A\|x\|^2.$$

□

In this section, we find a set $\mathbb{F}$ such that stable recovery of a signal from time-space samples set Y is possible if and only if $\mathbb{F}$ is a frame.

Theorem 2.2. *[3] Every $\phi \in \ell^2(\mathbb{Z}_d)$ can be recovered from the set*

$$Y = \{\phi(i), X\phi(i), \cdots, X^{m_i}\phi(i) : i \in \Omega\}$$

if and only if the set of vectors

$$\mathbb{F} = \{X^{*t}e_i : i \in \Omega, t = 0, ..., m_i\}$$

becomes a frame for $\ell^2(\mathbb{Z}_d)$.

Proof. Now Y recovers ϕ if and only if there exists $A, B > 0$ such that

$$A\|\phi\|_2^2 \leq \sum_{i\in\Omega}\sum_{t=0}^{m_i} |X^t\phi(i)|^2 \leq B\|\phi\|_2^2. \tag{2.1}$$

If $\{e_i\}$ are standard orthonormal basis for $\ell^2(\mathbb{Z}_d)$ then (2.1) is same as the following

$$\begin{aligned}
&A\|\phi\|_2^2 \leq \sum_{i\in\Omega}\sum_{t=0}^{m_i} |\sum_{j=0}^{N-1} X^t\phi(j)e_i(j)|^2 \leq B\|\phi\|_2^2 \\
\Longleftrightarrow\ &A\|\phi\|_2^2 \leq \sum_{i\in\Omega}\sum_{t=0}^{m_i} |\langle X^t\phi, e_i\rangle|^2 \leq B\|\phi\|_2^2 \\
\Longleftrightarrow\ &A\|\phi\|_2^2 \leq \sum_{i\in\Omega}\sum_{t=0}^{m_i} |\langle \phi, X^{*t}e_i\rangle\rangle|^2 \leq B\|\phi\|_2^2 \\
\Longleftrightarrow\ &\{X^{*t}e_i : i \in \Omega, t = 0, \cdots, m_i\} \text{ will be a frame for } \ell^2(\mathbb{Z}_d).
\end{aligned}$$

Which finishes the proof. □

Consider a full rank square matrix B with complex coefficients, and suppose Q is defined as $Q = BA^*B^{-1}$, where A^* denotes the conjugate transpose of matrix A. We can express A^* in terms of B and Q as $A^* = B^{-1}QB$. Let b_i represents the ith column of matrix B. Then $\{Q^tb_i : i \in \Omega, t = 0, \cdots, m_i\}$ will be a frame for $\ell^2(\mathbb{Z}_d)$ iff $\{X^tb_i : i \in \Omega, t = 0, \cdots, m_i\}$ will be a frame for $\ell^2(\mathbb{Z}_d)$.

Lemma 2.3. *Every vector* $\phi \in \ell^2(\mathbb{Z}_d)$ *can be reconstructed from the sampling set* $S = \{X^tb_i : i \in \Omega, t = 0, \cdots, m_i\}$ *iff the sequence of vectors of the set* $S = \{Q^jb_i : i \in \Omega, t = 0, \cdots, m_i\}$ *forms a frame for the space* $\ell^2(\mathbb{Z}_d)$.

2.2 Dynamical sampling with diagonalizable and Jordan matrix

In this section, let us start by considering a simpler case when evolution operator X^* is a daigonalizable matrix. Let $X^* = B^{-1}DB$, where D denotes a diagonal matrix as the following form

$$D = \begin{pmatrix} \lambda_1 I_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 I_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n I_n \end{pmatrix},$$

Where I_k denote a $t_k \times t_k$ identity matrix, and consider $B \in \ell^2(\mathbb{Z}_{d\times d})$. The *D-annihilator* of a vector $b \in B$ refers to the monic polynomial with the smallest degree that annihilates b and r_i denotes degree of *D-annihilator*. Let $P_j : \ell^2(\mathbb{Z}_d) \to \mathrm{Ker}(T - \lambda_j I)$ is an orthogonal projection associated to the eigenvalue λ_j . Then we have,

Theorem 2.4. *[3] Consider a diagonal matrix denoted by D. In this context, r_i represents the degree of the D-annihilator of vector b_i. We can define $m_i \in \mathbb{N}$ as $m_i = r_i - 1$. Consequently, the set $\{D^j b_i : i \in \Omega, 0 \leq j \leq m_i\}$ forms a frame of $\ell^2(\mathbb{Z}_d)$ if and only if the set $\{P_j(b_i) : i \in \Omega\}$ forms a frame of $P_j(\ell^2(\mathbb{Z}_d))$, where $j = 1 \leq j \leq n$.*

In this section, we express evolution operator X^* in Jordan form. Let J be Jordan form of a matrix X^*. if Jordan form of a matrix is J such that

$$J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_n \end{pmatrix}$$

for $x = 1, \cdots, n$, $J_x = \lambda_x I_x + N_x$ where I_x is an $h_x \times h_x$ identity matrix, and N_x denotes the block-nilpotent matrix of order h_x of the form

$$N_x = \begin{pmatrix} N_{x1} & 0 & \cdots & 0 \\ 0 & N_{x2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{x\gamma_x} \end{pmatrix}$$

where each N_{xi} is a $t_i^x \times t_i^x$ cyclic nilpotent matrix,

$$N_{xi} \in \ell^2(\mathbb{Z}_{t_i^x \times t_i^x}), N_{xi} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

with $t_1^x \geq t_2^x \geq \cdots$ and $t_1^x + t_2^x + \cdots + t_{\gamma_x}^x = h_x$. Also $h_1 + \cdots + h_n = d$. The matrix J is a square matrix with d rows and eigenvalues $\lambda_j, j = 1, \cdots, n$ such that $\lambda_i \neq \lambda_k$. And $\{b_i : i \in \Omega\} \subset \ell^2(\mathbb{Z}_d)$. Define $W_x := span\{e_{k_j^x} : j = 1, 2, \cdots, \gamma_x\}$ and γ_x is total number of nilpotent matrix.

Theorem 2.5. *[3] Consider a matrix J in Jordan form. Suppose Ω is a subset of index set $\{1, \cdots, d\}$, and let $\{b_i : i \in \Omega\}$ represent a set of vectors in the Hilbert space $\ell^2(\mathbb{Z}_d)$. Here, r_i denotes the degree of the J-annihilator of vector b_i. and define $m_i = r_i - 1$ and $W_x = span\{e_{k_{jx}} : j = 1, 2, 3, \cdots, \gamma_x\}$ and γ_x is total number of nilpotent matrix. Then the following assertions are equivalent:*

(i) The collection of vectors $\{J^j b_i : i \in \Omega, j = 0, \cdots, m_i\}$ forms a frame for $\ell^2(\mathbb{Z}_d)$.

(ii) $\{P_x(b_i), i \in \Omega\}$ forms a frame for W_x, for all $x = 1, \cdots, n$.

Example 2.4. Let J be a Jordan matrix defined by

$$J = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

The characteristic polynomial of J is $ch(t) = (t-2)^4(t-5)^3$ and minimal polynomial is $m(t) = (t-2)^2(t-5)^3$. Let $J_x = \lambda I_x + N_x$, where N_x is nilpotent matrix.

Then we have

$$J_1 = 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = N_1.$$

$$J_2 = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = N_2.$$

$$J_3 = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = N_3.$$

Now $\Omega \subset \{1, ..., 7\}$ and $b_i \in \ell^2(\mathbb{Z}_7)$. If we take $\Omega = \{1, 2, 3\}$, $j = 0, 1, 2, 3$ and

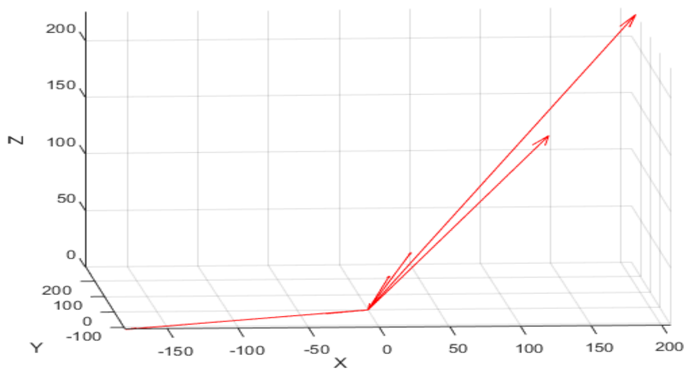
$$\{b_i\}_{i=1}^{3} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 2 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Then the sequence of elements of the set $\{J^j b_i : i \in \{1, 2, 3\}, j = 0, \cdots, 3\}$ forms a frame for $\ell^2(\mathbb{Z}_7)$. Now for each $i \in \Omega$ the index of N_{s_i} is k_{i^s}. Here $k_{1^s} = 3, k_{2^s} = 2$ and $k_{3^s} = 2$. Then $W_x = span\{e_3, e_2\} \cong\ \ell^2(\mathbb{Z}_2)$.

Now P_x is orthogonal projection onto W_x and

$$P_x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $\{P_x(b_i), i \in \Omega\}$ form a frame of W_x.



Graph of the frame $\{J^j b_i : i \in \Omega = \{1, 2, 3\}, j = 0, 1, 2, 3\}$

2.3 Dynamical sampling with PVD

Now we are familiar with the concept of dynamical sampling in finite dimensional Hilbert spaces. In this section, we will investigate the results of dynamical sampling using the polar value decomposition(PVD) of a square matrix.

Definition 2.5. The polar value decomposition (PVD) is a matrix factorization that decomposes any complex or real square matrix A into two parts: a unitary matrix W and a positive semidefinite matrix P, i.e. $A = WP$. Furthermore, if A is invertible matrix then representation is unique.

Theorem 2.6. *Let A be a square matrix of order d and polar value decomposition of matrix A is $A = WP$, where W is unitary matrix and P is positive semi-definite matrix. Let $W^*A = Q$ and it can be written as, $Q = VSV^*$, where $VV^* = V^*V = I$. Then the following are equivalent,*

(i) The collection of vectors $\{Q^j f(i) : i \in \Omega, j = 0, \cdots, m_i\}$ forms a frame for $\ell^2(\mathbb{Z}_d)$.

(ii) The collection of vectors $\{S^j f(i) : i \in \Omega, j = 0, \cdots, m_i\}$ forms a frame for $\ell^2(\mathbb{Z}_d)$.

Let $A \in \mathbb{C}^{n\times n}$ be a matrix that can be written as $Q = VSV^*$, where S is a diagonal matrix which has the form,

$$S = \begin{pmatrix} \sigma_1 I_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 I_2 & & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & & \sigma_k I_k \end{pmatrix},$$

and $I_1, I_2, \cdots, I_k$ are identity matrix with distinct singular values $\sigma_1, \sigma_2, \cdots, \sigma_k$ of A and $1 + 2 + \cdots + k = d$.

Theorem 2.7. *Consider a square matrix A of order d and Q is a matrix defined in Theorem 2.6. Let B be an invertible matrix of order d with column vectors $\{b_i : i \in \Omega\}$, where Ω represents the set of indices. Each column vector b_i has a D-annihilator of degree r_i. Define $m_i \in \mathbb{N}$, by $m_i = r_i - 1$. Then the following statements are equivalent.*

(i) The collection of vectors $\{S^j b_i : i \in \Omega, j = 0, \cdots, m_i\}$ form a frame of $\ell^2(\mathbb{Z}_d)$.

(ii) For each $j = 1, ..., k$, the set $\{P_j(b_i), i \in \Omega\}$ forms a frame for $P_j(\ell^2(\mathbb{Z}_d))$.

Proof. Suppose $P_j(b_i), i \in \Omega$ form a frame for $P_j(\ell^2(\mathbb{Z}_d))$ for each $j = 1, \cdots, k$. Because we are dealing with d-dimensional Hilbert spaces $\mathcal{H}$, therefore our goal is to demonstrate that the set $\{S^j b_i : i \in \Omega, j = 0, \cdots, m_i\}$ forms a frame for $\ell^2(\mathbb{Z}_d)$. For this it is sufficient to show that $\text{span}\{S^j b_i : i \in \Omega, j = 0, \cdots, m_i\} = \ell^2(\mathbb{Z}_d)$. Since $V = W_1 \oplus \cdots \oplus W_k$, it follows that there exist k linear projection operators $P_1, P_2, \cdots, P_k$ such that

$$P_1 + P_2 + \cdots + P_k = I.$$

Let $x \in \ell^2(\mathbb{Z}_d)$ then

$$x = Ix = \sum_{j=1}^{k} P_j x.$$

Suppose that $\langle S^l b_i, x\rangle = 0$ holds for all $i \in \Omega$ and $l = 0, 1, \ldots, m_i$, where $m_i = r_i - 1$ and r_i represents the degree of the D-annihilator of b_i. Consequently, it can be inferred that $\langle S^l b_i, x\rangle = 0$ holds for all $i \in \Omega$ and $l = 0, 1, \ldots, d$, since $k \leq d$. Moreover, this condition holds for $l = 0, 1, \ldots, k$. Then

$$\langle S^l b_i, x\rangle = \sum_{j=1}^{k} \langle S^l b_i, P_j x\rangle = \sum_{j=1}^{k} \sigma_j^l \langle P_j b_i, P_j x\rangle = 0, \tag{2.2}$$

for all $i \in \Omega$ and $l = 0, 1, \cdots, k$. Consider that z_i is the vectors $(\langle P_j b_i, P_j x\rangle) \in \ell^2(\mathbb{Z}_k)$. The matrix form of the equation 2.2 for each value of i can be written as $V z_i = 0$, where V represents an $n \times n$ Vandermonde matrix.

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \sigma_1 & \sigma_2 & \cdots & \sigma_k \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1^{k-1} & \sigma_2^{k-1} & \cdots & \sigma_k^{k-1} \end{pmatrix},$$

since $\sigma_r \neq \sigma_s$ for all $r, s \in \{1, 2. \cdots, k\}$. Hence V is invertible matrix.

$$\implies z_i = 0 \text{ for all } i \in \Omega$$

$$\langle P_j b_i, P_j x\rangle = 0 \text{ for all } i \in \Omega \text{ and for each } j = 1, 2, \cdots, k.$$

Because sequence of vectors $\{P_j(b_i), i \in \Omega\}$ forms a frame of $P_j(\ell^2(\mathbb{Z}_d))$.

$$\implies \langle P_j b_i, P_j x\rangle = 0 \text{ for all } i \in \Omega \text{ and for each } j = 1, 2, \cdots, k.$$

$$\implies \|P_j x\|^2 = 0,$$

$$\implies P_j x = 0,$$

$$\implies \sum_{j=1}^{k} P_j x = 0.$$

Hence $x = 0$ for all $i \in \Omega$ and for each $j = 1, 2, \cdots, k$. □

Theorem 2.8. *Let A be a square matrix of order d and Q, S be matrices same as defined in theorem 2.6 with $\Omega \subset \{1, 2, \cdots, d\}$ and $\phi \in \ell^2(\mathbb{Z}_d)$, given a collection of vectors $\{b_i : i \in \Omega\}$ and using the operator P_j, where $1 \leq j \leq k$, we have that $\{P_j(b_i), i \in \Omega\}$ forms a frame for $P_j(\ell^2(\mathbb{Z}_d))$. Now, consider a fixed integer L. We construct the set $E = \bigcup_{i \in \Omega : b_i \neq 0} \{b_i, S b_i, ..., S^L b_i\}$. Then E forms a frame for $\ell^2(\mathbb{Z}_d)$ if and only if all the vectors $\{S^{L+1} b_i : i \in \Omega\}$ are contained within the span$E$.*

Proof. ($\Longrightarrow$) It is given that $E = \bigcup_{i\in\Omega : b_i\neq 0} \{b_i, Sb_i, ..., S^L b_i\}$ is a frame of $\ell^2(\mathbb{Z}_d)$ then by definition of frame in finite dimension case, each vector in $\ell^2(\mathbb{Z}_d)$ will belong to $span(E)$. Hence

$$\{S^{L+1}b_i : i \in \Omega\} \subset span(E).$$

($\Longleftarrow$) If the set $S^{L+1}b_i : i \in \Omega$ is contained within the span$E$, it implies that $S(span(E)) \subset span(E)$. Consequently, according to Theorem 2.7, E forms a frame for $\ell^2(\mathbb{Z}_d)$.

□

Theorem 2.9. *Let A be a normal matrix of order d. Let polar value decomposition of matrix A is $A = WP$, and matrix Q is same as defined in Theorem 2.6. Then $Q = W^*A = AW^*$ and the following are equivalent,*

(i) The collection of vectors $\{A^j b_i : i \in \Omega, j = 0, \cdots, m_i\}$ forms a frame for $\ell^2(\mathbb{Z}_d)$.

(ii) The collection of vectors $\{S^j b_i : i \in \Omega, j = 0, \cdots, m_i\}$ forms a frame for $\ell^2(\mathbb{Z}_d)$.

Proof. Note that A is normal matrix of order n, and its polar value decomposition is $A = WP$, where W is unitary matrix and P is positive semi-definite matrix. Since A is normal then

$$A = WP = PW \implies P = W^*A = AW^*.$$

Since matrix W^* and A are commutative and $Q = W^*A \implies Q^n = (W^*)^n A^n$. Since W^* is invertible matrix then by Theorem 2.6 statement (i) and (ii) are equivalent. □

Example 2.5. Let A be discrete Fourier transform matrix(DFT) of order N. Then polar value decomposition of matrix A is $A = WP$, where $W = UV^*$ and

$P = VSV^*$. For $w_N = e^{-2\pi i/N}$ matrix A is defined as,

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w_N & w_N^2 & \cdots & w_N^{(N-1)} \\ 1 & w_N^2 & w_N^4 & \cdots & w_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w_N^{(N-1)} & w_N^{2(N-1)} & \cdots & w_N^{(N-1)(N-1)} \end{pmatrix}_{N\times N}.$$

Then note that $U = \frac{1}{\sqrt{N}}A$ and

$$S = \begin{pmatrix} \sqrt{N} & 0 & \cdots & 0 \\ 0 & \sqrt{N} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{N} \end{pmatrix}_{N\times N},$$

and matrix $V = I_{N\times N} \implies V^* = I_{N\times N}$.

Then semi definite matrix $P = VSV^* = S$ and $W = UV^* = \frac{1}{\sqrt{N}}A$.

Then matrix Q of Theorem 2.9 is $Q = W^*A \implies Q = (\frac{1}{\sqrt{N}}A)(\sqrt{N}A) = A^2$.

In this example we assume that $m_i = n$ and for each j, Ω is a singleton and

$$\bigcup_{j=1}^{n} \Omega_j = \{1, 2, \cdots, n\}.$$

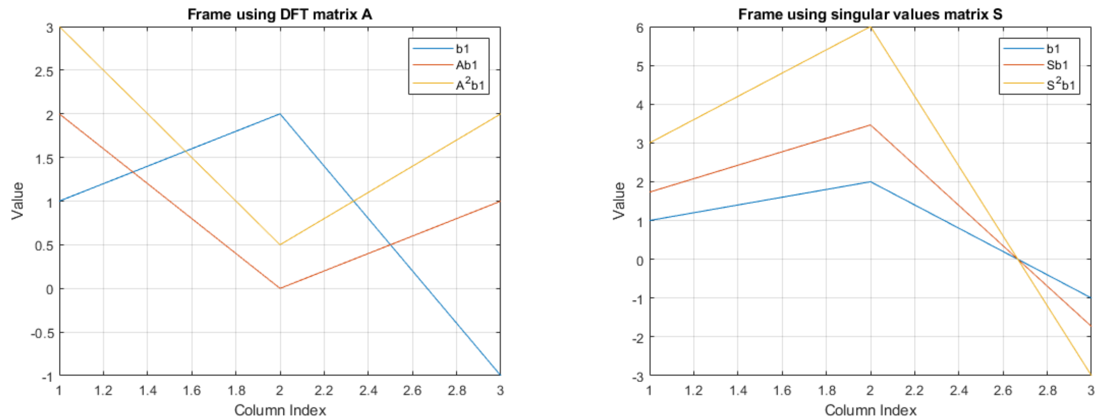
Let $\{b_i\}_{i\in I}$ be column vectors of some invertible matrix of order N, then Theorem 2.9 holds true.

Example 2.6. Let A be discrete Fourier transform matrix(DFT) of order 3. Then polar value decomposition of matrix A is $A = WP$, where $W = UV^*$ and $P = VSV^*$. Now

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{-1+\sqrt{3}i}{2} & \frac{-1-\sqrt{3}i}{2} \\ 1 & \frac{-1-\sqrt{3}i}{2} & \frac{-1+\sqrt{3}i}{2} \end{pmatrix}, \text{ and singular values matrix is } S = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix}.$$

For $\Omega = \{1\} \subset I$ and $b_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, degree of the D-annihilator is, $r_1 = 3 \implies$ $m_1 = 2$. Since A is normal matrix, then by Theorem 2.9, the set of vectors

$\{A^j b_i : i \in \Omega, j = 0,1,2\}$ forms a frame for $\ell^2(\mathbb{Z}_3)$ iff the collection of vectors $\{S^j b_i : i \in \Omega, j = 0,1,2\}$ forms a frame for $\ell^2(\mathbb{Z}_3)$.



(a) Frame $\{A^j b_i : i \in \Omega, j = 0,1,2\}$ (b) Frame $\{S^j b_i : i \in \Omega, j = 0,1,2\}$

Figure 2.1: Dynamical sampling with PVD of a DFT matrix.

2.4 Dynamical sampling with Translation invariant operator

In Section 2.4 we will present our result, which gives the sufficient conditions such that $\{X^t e_i : 0 \leq t \leq m,\ i \in \Omega_t\}$ becomes a frame for $\ell^2(\mathbb{Z}_d)$ by fixing evolution operator X to be translation invariant operator.

Theorem 2.10. *Suppose that $\{\Lambda_k\}_{k\in I}$ is a countable family of lattices in G, and let $\{V_k\}_{k\in I}$ be associated fundamental domains with lattice Λ_k in $\widehat{G}$. Consider the two collections of elements $\{\phi_k\}_{k\in I}$, $\{\widetilde{\phi_k}\}_{k\in I}$ in $L^2(G)$. Then the following hold:*

(i) $\{R_\lambda \phi_k\}_{\lambda\in\Lambda_k, k\in I}$ *is a Bessel sequence in $L^2(G)$ if*

$$B := \sup_{\gamma\in\widehat{G}} \sum_{k\in I} \mu_{\widehat{G}}(V_k) \sum_{\omega\in\Lambda_k^\perp} |\widehat{\phi_k(\gamma)}\widehat{\phi_k(\gamma+\omega)}| < \infty.$$

(ii) If (i) holds, then $\{R_\lambda \phi_k\}_{\lambda\in\Lambda_k, k\in I}$ *is a frame for* $L^2(G)$ *if*

$$A =: \inf_{\gamma\in\widehat{G}} \left(\sum_{k\in I} \mu_{\widehat{G}}(V_k)|\widehat{\phi_k(\gamma)}|^2 - \sum_{k\in I} \mu_{\widehat{G}}(V_k) \sum_{\omega\in\Lambda_k^\perp/\{0\}} |\widehat{\phi_k(\gamma)}\widehat{\phi_k(\gamma+\omega)}| \right) > 0.$$

Definition 2.6. Let $X : \ell^2(\mathbb{Z}_d) \to \ell^2(\mathbb{Z}_d)$ be a linear transformation. It is called translation invariant if

$$X(R_k z) = R_k X(z),$$

for all $z \in \ell^2(\mathbb{Z}_d)$ and all $k \in \mathbb{Z}$. Where R_k is translation operator defined as $(R_k z)(n) = z(n-k)$.

Theorem 2.11. *Suppose the following:*

(i) $X : \ell^2(\mathbb{Z}_d) \to \ell^2(\mathbb{Z}_d)$ *is a translation invariant,*

(ii) $\Big(a_{i,j}\Big)_{0\le i,j\le d-1}$ *is a matrix representation of* X *with respect to standard basis,*

(iii) $m = 1$, *and*

(iv) $\Omega_0 = \{0, 2, 4, \cdots, d-2\}$ *and* $\Omega_1 = \{0, 1, 2, \cdots, d-1\}$.

If $|\widehat{a_{i,0}}| \neq 0$ *for each* $i \in \{0, \cdots, d-1\}$, *then* $\{X^t e_i : 0 \le t \le m,\ i \in \Omega_t\}$ *is a frame for* $\ell^2(\mathbb{Z}_d)$.

Proof. Given $\Omega_0, \Omega_1 \subset \ell^2(\mathbb{Z}_d)$. Define $\Omega_t^\perp = \{\gamma \in \widehat{\mathbb{Z}_d} : \gamma(x) = 1, \forall x \in \Omega_t\}$. Since $\Omega_0 = \{0, 2, 4, \cdots, d-2\}$ and $\Omega_1 = \{0, 1, 2, \cdots, d-1\}$, therefore $\Omega_0^\perp = \{0, \frac{d}{2}\}, \Omega_1^\perp = \{0\}$. And we have $X^t e_\lambda = X^t R_\lambda e_0 = R_\lambda X^t e_0$, define $\phi_t = X^t e_0$. Here $V_0 = \{0, 1, \cdots, d/2-1\}$ and $V_1 = \{0, 1, \cdots, d-1\}$ applying Theorem 2.10 (i) is always true in finite dimensional case. Therefore only remains to prove (ii) that is

$$A = \inf_{n\in\widehat{G}} \Big(\mu_{\widehat{G}}(V_0)|\widehat{\phi}_0(n)|^2 + \mu_{\widehat{G}}(V_1)|\widehat{\phi}_1(n)|^2 - \mu_{\widehat{G}}(V_0)|\widehat{\phi}_1(n)\widehat{\phi}_1(n+\omega)| \Big) > 0.$$

For this it is sufficient to show that for each $n \in \mathbb{Z}_d$, we have

$$\mu_{\widehat{G}}(V_0)|\widehat{\phi}_0(n)|^2 + \mu_{\widehat{G}}(V_1)|\widehat{\phi}_1(n)|^2 - \mu_{\widehat{G}}(V_0)|\widehat{\phi}_1(n)\widehat{\phi}_1(n+\omega)| > 0. \tag{2.3}$$

Now take $n \in \mathbb{Z}_d$ then

$$\begin{aligned}\mu_{\widehat{G}}(V_0)|\widehat{\phi}_0(n)|^2 + \mu_{\widehat{G}}(V_1)|\widehat{\phi}_1(n)|^2 - \mu_{\widehat{G}}(V_0)|\widehat{\phi}_1(n)\widehat{\phi}_1(n+\omega)|\\ = \frac{d}{2} + d|\widehat{Xe_0(m)}|^2 - \frac{d}{2}\\ = d|\widehat{Xe_0}(m)|^2\end{aligned}$$

Now equation (2.3) is true if and only if $|\widehat{Xe_0}(n)| \neq 0$ for all $n \in \mathbb{Z}_d$, that is $\widehat{Xe_0}(n) \neq 0$, equivalently $\widehat{a_{(n,1)}} \neq 0$. Which finishes the proof.

□

2.5 Dynamical sampling with convolution operator

In Section 2.5 we will present our result, which gives the sufficient conditions such that $\{X^t e_i : 0 \leq t \leq m,\ i \in \Omega_t\}$ becomes a frame for $\ell^2(\mathbb{Z}_d)$ by fixing evolution operator X to be convolution operator.

Definition 2.7. Suppose $b \in \ell^2(\mathbb{Z}_d)$, define $X_b : \ell^2(\mathbb{Z}_d) \to \ell^2(\mathbb{Z}_d)$ by $X_b(z) = b * z$, for all $z \in \ell^2(\mathbb{Z}_d)$. Then it is called a convolution operator.

Theorem 2.12. *Suppose the following:*

(i) X_b *is a convolution operator for some* $b \in \ell^2(\mathbb{Z}_d)$*,*

(ii) $m = 1$*, and*

(iii) $\Omega_0 = \{0, 2, 4, \cdots, d-2\}$ *and* $\Omega_1 = \{0, 1, 2, \cdots, d-1\}$.

If $|\widehat{b(m)}| \neq 0$ *for all* $m \in \{0, 1, \cdots, d-1\}$*, then* $\{X_b^t e_i : 0 \leq t \leq m,\ i \in \Omega_t\}$ *is a frame for* $\ell^2(\mathbb{Z}_d)$.

Proof. Given $\Omega_0, \Omega_1 \subset \ell^2(\mathbb{Z}_d)$. Define $\Omega_t^\perp = \{\gamma \in \widehat{\mathbb{Z}_d} : \gamma(x) = 1, \forall x \in \Omega_t\}$. Since $\Omega_0 = \{0, 2, 4, \cdots, d-2\}$ and $\Omega_1 = \{0, 1, 2, \cdots, d-1\}$, therefore $\Omega_0^\perp = \{0, \frac{d}{2}\}, \Omega_1^\perp =$

$\{0\}$. And we have $X_b^n e_\lambda = X_b^n T_\lambda e_0$, define $\phi_n = X_b^n e_0$ and $X_b^n e_0 = b * b * \cdots * e_0$. Here $V_0 = \{0, 1, \cdots, d/2 - 1\}$ and $V_1 = \{0, 1, \cdots, d-1\}$ applying Theorem 2.10 (i) is always true in finite dimensional case. Therefore only remains to prove (ii) that is

$$A = \inf_{n \in \widehat{G}} \left(\mu_{\widehat{G}}(V_0)|\widehat{\phi}_0(n)|^2 + \mu_{\widehat{G}}(V_1)|\widehat{\phi}_1(n)|^2 - \mu_{\widehat{G}}(V_0)|\widehat{\phi}_1(n)\widehat{\phi}_1(n+\omega)| \right) > 0.$$

For this it is sufficient to show that for each $n \in \mathbb{Z}_d$, we have

$$\mu_{\widehat{G}}(V_0)|\widehat{\phi}_0(n)|^2 + \mu_{\widehat{G}}(V_1)|\widehat{\phi}_1(n)|^2 - \mu_{\widehat{G}}(V_0)|\widehat{\phi}_1(n)\widehat{\phi}_1(n+\omega)| > 0. \tag{2.4}$$

Now take $n \in \mathbb{Z}_d$ then

$$\begin{aligned} \mu_{\widehat{G}}(V_0)|\widehat{\phi}_0(n)|^2 + \mu_{\widehat{G}}(V_1)|\widehat{\phi}_1(n)|^2 - \mu_{\widehat{G}}(V_0)|\widehat{\phi}_1(n)\widehat{\phi}_1(n+\omega)| \\ = \frac{d}{2} + d|\widehat{b(n)}|^2 - \frac{d}{2} \\ = d|\widehat{b(n)}|^2 \end{aligned}$$

Now equation (2.4) is true if and only if $|\widehat{b(n)}| \neq 0$ for all $n \in \mathbb{Z}_d$. Which finishes the proof. □

2.6 Dynamical sampling with Fourier multiplier operator

In Section 2.6 we will present our result, which gives the sufficient conditions such that $\{X^t e_i : 0 \leq t \leq m,\ i \in \Omega_t\}$ becomes a frame for $\ell^2(\mathbb{Z}_d)$ by fixing evolution operator X to be Fourier multiplier operator.

Definition 2.8. Let $m \in \ell^2(\mathbb{Z}_d)$. Define $X_{(m)} : \ell^2(\mathbb{Z}_d) \to \ell^2(\mathbb{Z}_d)$ by $X_{(m)}(z) = (m\widehat{z})\check{}$, where $(m\widehat{z})$ is the vector obtained from multiplying m and $\widehat{z}$ component wise, i.e. $m\widehat{z}(n) = m(n)\widehat{z}(n)$ for each n. Any transformation of this form is called a Fourier multiplier operator.

Proposition 2.13. *Let $X_{(m)} : \ell^2(\mathbb{Z}_d) \to \ell^2(\mathbb{Z}_d)$ be Fourier multiplier operator* *then it is translation invariant, i.e.*

$$X_{(m)}(R_k z) = R_k(X_{(m)} z).$$

Proof. Since X is a Fourier multiplier operator if and only if X is a convolution operator,

$$X_{(m)}(z) = X_b(z) \text{ where } \widehat{b} = m \implies b = m^{\vee},$$

$$\begin{aligned}
&\implies X_{(m)}(z) = X_{m^{\vee}}(z),\\
&\implies X_{(m)}(R_k(z)) = X_{m^{\vee}}(R_k(z)),\\
&\implies X_{(m)}(R_k(z)) = R_k(X_{m^{\vee}}(z)),\\
&\implies X_{(m)}(R_k(z)) = R_k(X_{(m)}(z)).
\end{aligned}$$

□

Theorem 2.14. *Suppose the following:*

(i) $X_{(m)}$ *is a Fourier multiplier operator for some* $m \in \ell^2(\mathbb{Z}_d)$*,*

(ii) $m = 1$*, and*

(iii) $\Omega_0 = \{0, 2, 4, \cdots, d-2\}$ *and* $\Omega_1 = \{0, 1, 2, \cdots, d-1\}$.

If $|m(n)| \neq 0$ *for all* $n \in \{0, 1, \cdots, d-1\}$*, then* $\{X^t_{(m)} e_i : 0 \leq t \leq m,\ i \in \Omega_t\}$ *is a frame for* $\ell^2(\mathbb{Z}_d)$.

Proof. Given $\Omega_0, \Omega_1 \subset \ell^2(\mathbb{Z}_d)$. Define $\Omega_t^{\perp} = \{\gamma \in \widehat{\mathbb{Z}_d} : \gamma(x) = 1, \forall x \in \Omega_t\}$. Since $\Omega_0 = \{0, 2, 4, \cdots, d-2\}$ and $\Omega_1 = \{0, 1, 2, \cdots, d-1\}$, therefore $\Omega_0^{\perp} = \{0, \frac{d}{2}\}, \Omega_1^{\perp} = \{0\}$. And we have $X^n_{(m)} e_\lambda = X^n_{(m)} R_\lambda e_0 = R^\lambda X_{(m)} e_0$, define $\phi_n = X^n e_0$ then $X^n_{(m)} e_\lambda = R^\lambda \phi_n$.

Here $V_0 = \{0, 1, \cdots, d/2 - 1\}$ and $V_1 = \{0, 1, \cdots, d-1\}$ applying Theorem 2.10 (i) is always true in finite dimensional case. Therefore only remains to prove (ii) that is

$$A = \inf_{n \in \widehat{G}} \Big(\mu_{\widehat{G}}(V_0) |\widehat{\phi}_0(n)|^2 + \mu_{\widehat{G}}(V_1) |\widehat{\phi}_1(n)|^2 - \mu_{\widehat{G}}(V_0) |\widehat{\phi}_1(n) \widehat{\phi}_1(n + \omega)| \Big) > 0.$$

For this it is sufficient to show that for each $n \in \mathbb{Z}_d$, we have

$$\mu_{\widehat{G}}(V_0)|\widehat{\phi}_0(n)|^2 + \mu_{\widehat{G}}(V_1)|\widehat{\phi}_1(n)|^2 - \mu_{\widehat{G}}(V_0)|\widehat{\phi}_1(n)\widehat{\phi}_1(n+\omega)| > 0. \tag{2.5}$$

Now take $n \in \mathbb{Z}_d$ then

$$\begin{aligned}
&\mu_{\widehat{G}}(V_0)|\widehat{\phi}_0(n)|^2 + \mu_{\widehat{G}}(V_1)|\widehat{\phi}_1(n)|^2 - \mu_{\widehat{G}}(V_0)|\widehat{\phi}_1(n)\widehat{\phi}_1(n+\omega)| \\
&= \frac{d}{2} + d|\widehat{X_{(m)}e_0}|^2 - \frac{d}{2} \\
&= d|\widehat{X_{(m)}e_0}|^2
\end{aligned}$$

Now equation (2.5) is true if and only if $|\widehat{X_{(m)}e_0}| \neq 0$, that is $|m(n)\widehat{e_0(n)}| \neq 0$, equivalently $|m(n)| \neq 0$ for all $n \in \mathbb{Z}_d$. Which finishes the proof. □

Example 2.7. Let $X : \ell^2(\mathbb{Z}_4) \to \ell^2(\mathbb{Z}_4)$ be translation invariant linear transformation defined as

$$X(z)(n) = z(n) + 2z(n+1) + z(n+3),$$

then matrix representation of X is

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 \\ 2 & 0 & 1 & 1 \end{pmatrix}.$$

If we take fix $b(n) = a_{n,0}$ then X will be convolution operator and $b = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$.

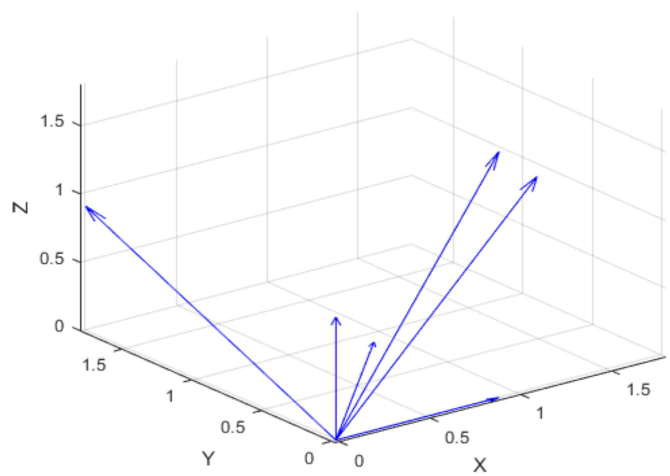
Now by definition of discrete Fourier transform, $\widehat{b} = \begin{pmatrix} 4 \\ 1+i \\ -2 \\ 1-i \end{pmatrix}$.

Now one can observe that $|\widehat{b}(m)| \neq 0$ for all $m = 0, 1, 2, 3$. Here $N = 4$ and $\Omega_0 = \{0, 2\}$, $\Omega_1 = \{0, 1, 2, 3\}$, then $\{X^t e_i : 0 \leq t \leq 1,\, i \in \Omega_t\} = Y(Say)$.

Therefore

$$Y = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

And note that $span\{Y\} = \ell^2(\mathbb{Z}_4)$. Hence $\{X^t e_i : 0 \le t \le 1,\, i \in \Omega_t\}$ is frame for $\ell^2(\mathbb{Z}_4)$.



Graph of frame $\{X^t e_i : 0 \le t \le 1,\, i \in \Omega_t\}$

Example 2.8. Let $X : \ell^2(\mathbb{Z}_4) \to \ell^2(\mathbb{Z}_4)$ be translation invariant linear transformation defined as

$$X(z)(n) = z(n) + 2z(n+1) + z(n+3).$$

Now take $b = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$ and $m = \widehat{b} = \begin{pmatrix} 4 \\ 1+i \\ -2 \\ 1-i \end{pmatrix}$.

Then $X = X_{(m)}$ and we can proceed same as example 2.7. Hence $\{X^t e_i : 0 \le t \le m,\, i \in \Omega_t\}$ is frame for $\ell^2(\mathbb{Z}_4)$.

2.7 For infinite dimensional case

In this section we are going to introduced dynamical sampling problem in case of infinite dimensional separable Hilbert space from the work of Aldroubi [3]. Let $\mathcal{H}$ be a infinite dimensional separable Hilbert space. Therefore, without loss of generality assume that $\mathcal{H} = \ell^2(\mathbb{N})$. Let the evolution operator $X \in \mathcal{X}$, where $\mathcal{X}$ be defined as:

$$\mathcal{X} = \{X \in \mathcal{B}(\ell^2(\mathbb{N})) : X = X^*, \text{ and eigenvectors of X forms a basis of } \ell^2(\mathbb{N})\}.$$

The notation $\mathcal{B}(\mathcal{H})$ denotes the collection of bounded linear maps that operate on the separable Hilbert space $\mathcal{H}$. If an operator $X \in \mathcal{X}$, it implies the existence of a unitary operator B such that $X = B^*DB$. Here, D can be expressed as $D = \sum_j \lambda_j P_j$, where λ_j are the eigenvalues of X forming a pure spectrum $\sigma_p(X) = \lambda_j : j \in \mathbb{N}$, and these eigenvalues are real numbers. It is also required that the supremum of the absolute values of the eigenvalues, denoted by $\sup_j |\lambda_j|$, is finite. The set of orthogonal projections, denoted as P_j, satisfies two key conditions: the sum of all projections is equal to the identity operator, i.e. $\sum_j P_j = I$, and any two distinct projections are orthogonal ($P_jP_k = 0$ for $j \neq k$). It is important to note that the class $\mathcal{X}$ includes all bounded self-adjoint compact operators.

Definition 2.9. A set $\{v_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ is said to be complete if for $\phi \in \mathcal{H}$ satisfying $\langle \phi, v_k \rangle = 0$ for all $1 \leq k < \infty \implies \phi = 0$.

Definition 2.10. A sequence of vectors $\{v_k\}$ in a Hilbert space $\mathcal{H}$ is considered minimal if, for every index j, the vector v_j does not belong to $\overline{span}\{v_k\}_{k \neq j}$.

Remark 1. *By taking into consideration the definition of $\mathcal{X}$, it follows that for any $\phi \in \ell^2(\mathbb{N})$ and $m \in \mathbb{N}_0$*

$$\langle \phi, X^m e_i \rangle = \langle \phi, B^* D^m B e_j \rangle = \langle Bf, D^m b_j \rangle \text{ and } \|X^m\| = \|D^m\|.$$

The completeness (minimality, frame property) of the set $F_\Omega = \{X^m e_i : i \in \Omega, m \in \mathbb{N}_0\}$ is equivalent to the completeness (minimality, frame property) of the set $\{D^m b_i : i \in \Omega, m \in \mathbb{N}_0\}$.

By utilizing Remark 1, the following theorem presents a characterization outcome in the case where F_Ω forms a frame for a set with $|\Omega| = 1$.

Theorem 2.15. *[3] If we consider a linear operator D defined as $D = \sum_j \lambda_j P_j$, where P_j represents rank 1 operators for all $j \in \mathbb{N}$, and $v = \{v(k)\}_{k\in\mathbb{N}}$ is an element of $\ell^2(\mathbb{N})$, then the set $\{D^m v : m = 0, 1, \cdots\}$ forms a frame if and only if,*

(i) $|\lambda_k| < 1$ for all k.

(ii) $|\lambda_k| \to 1$.

(iii) $\{\lambda_k\}$ satisfies Carleson's condition

$$\inf_n \prod_{k\neq n} \frac{|\lambda_n - \lambda_k|}{|1 - \bar{\lambda}_n \lambda_k|} \geq \delta \text{ for some } \delta > 0.$$

(iv) $b(k) = m_k\sqrt{1 - |\lambda_k|^2}$ for some sequence $\{m_k\}$ satisfying $0 < A \leq |m_k| \leq B < \infty$, where A and B are constants such that $A, B > 0$.

This theorem implies the following Corollary.

Corollary 2.16. *Consider a matrix X defined as $X = B^*DB \in \mathcal{X}$, where $D = \sum_j \lambda_j P_j$, and P_j represents rank-1 matrices for all $j \in \mathbb{N}$. Our claim is that there exists an index $i_0 \in \mathbb{N}$ such that the set $F_\Omega = \{X^m e_{i_0} : m = 0, 1, ...\}$ forms a frame for $\ell^2(\mathbb{N})$ if and only if the sequence of eigenvalues $\{\lambda_j\}$, satisfies the conditions described in Theorem 2.15, and there exists an index $i_0 \in \mathbb{N}$ such that $b = Be_{i_0}$ satisfies condition (iv) of Theorem 2.15.*

CHAPTER 3

Dynamical Frames in $\ell^2(\mathbb{Z}_d)$

In previous Chapter 2, we find the conditions such that $\{X^n\phi\}_{n\in I}$ is a frame for $\ell^2(\mathbb{Z}_d)$, where I is an index set, $X : \mathcal{H} \to \mathcal{H}$ is a linear operator and $\phi \in \mathcal{H}$. In this chapter, we approach the problems in an another way. Let $\{\phi_n\}_{n=1}^N$ be a frame for $\ell^2(\mathbb{Z}_d)$ when this frame has a representation of the form $\{X^n\phi\}_{n=1}^N$, for some operator X, i.e., $\{\phi_n\}_{n=1}^N = \{X^n\phi\}_{n=1}^N$. In Section 3.1, we provide basic results and definitions. In Section 3.2, we characterize a frame $\{\phi_n\}_{n=1}^N$ for which representation of the form $\{X^n\phi\}_{n=1}^N$, with a bounded operator X [11]. In Section 3.3 we give the basic definition and properties of dual frame. In Section 3.4 we provide a characterization result for the existence of dynamical dual frame. [1] [2]

[1] This chapter will encompass the definition of a dynamical frame along with the presentation of several examples utilizing the research conducted by Jonathan A. and M. Powell[11].

[2] In the last sections of this chapter, we will establish the concept of a dynamical dual frame and illustrate its application through the research conducted by Jonathan A. and M. Powell, as referenced in their work[11].

In the final section, we provide a proof that every redundant frame possesses an infinite number of dynamical dual frames. Furthermore, we offer a constructive result that outlines a method for finding dynamical dual frames. The main result of this chapter is taken from a recent publication by Ashbrock and Powell in 2023.

3.1 Frame representation

Let $\mathcal{H}$ be a finite dimensional Hilbert space. A sequence of vectors $\{\phi_n\}_{n=1}^N \subset \mathcal{H}$ is said to be a *frame for* $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that for every $x \in \mathcal{H}$,

$$A\|x\|^2 \leq \sum_{n=1}^{N} |\langle x, \phi_n \rangle|^2 \leq B\|x\|^2.$$

The *analysis operator* for the frame is denoted by Θ and defined as

$$\Theta : \mathcal{H} \to \mathbb{C}^N \text{ and } \Theta x = \begin{pmatrix} \langle x, \phi_1 \rangle \\ \langle x, \phi_2 \rangle \\ \vdots \\ \langle x, \phi_N \rangle \end{pmatrix}.$$

Also $\Theta x = \sum_{n=1}^{N} \langle x, \phi_i \rangle e_n$, where $\{e_n\}_{n=1}^N$ is the standard orthonormal basis for $\mathbb{C}^N$. The adjoint of analysis operator is called *synthesis operator* and denoted by Θ^*, i.e.,

$$\Theta^* : \mathbb{C}^N \to \mathcal{H} \text{ and } \Theta^* \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} = \sum_{n=1}^{N} c_n \phi_n.$$

The operator $S := \Theta^*\Theta : \mathcal{H} \to \mathcal{H}$ is called a *frame operator* of frame $\{\phi_n\}_{n=1}^N$ and defined by

$$S(x) = \Theta^*\Theta(x) = \sum_{n=1}^{N} \langle x, \phi_n \rangle \phi_n.$$

The frame operator S is an invertible operator.

Definition 3.1. Let $\{\phi_n\}_{n=1}^N$ be a frame for $\ell^2(\mathbb{Z}_d)$. Then $\{\phi_n\}_{n=1}^N$ is called a dynamical frame if there exists a linear operator $X : \ell^2(\mathbb{Z}_d) \to \ell^2(\mathbb{Z}_d)$ and $\phi \in \ell^2(\mathbb{Z}_d)$ such that $X^n\phi = \phi_n$.

Lemma 3.1. *[11] Suppose $X : \ell^2(\mathbb{Z}_d) \to \ell^2(\mathbb{Z}_d)$ is linear operator and $\{X^n\phi\}_{n=1}^N$ is a frame for $\ell^2(\mathbb{Z}_d)$. Subsequently, for any $d \in \mathbb{N}$, a consecutive sequence of d elements from the frame $\{X^n\phi\}_{n=1}^N$ form a basis for $\ell^2(\mathbb{Z}_d)$. Additionally, $X^n\phi \neq 0$ for each $1 \leq n \leq N$.*

Proof. Our aim is to prove for all $m \in \{1, \cdots, N-d+1\}$, the set $\{X^n\phi\}_{n=m}^{m+d-1}$ forms a basis for $\ell^2(\mathbb{Z}_d)$. Since X is invertible, therefore it is sufficent to show that $\{X^n\phi\}_{n=1}^d$ is basis for $\ell^2(\mathbb{Z}_d)$. Now, Suppose $k \geq 1$ be the maximal index such that $\{X^n\phi\}_{n=1}^k$ is linearly independent. Define $S := \text{span}\{X^n\phi\}_{n=1}^k$. We will show first that $X(S) = S$. Now the set $\{X^n\phi\}_{n=1}^k + 1$ is linealrly dependent, by choice of the index k. Therefore, if

$$\sum_{n=1}^{k+1} a^n X^n \phi = 0,$$

then all $a_1, \cdots, a_{k+1}$, can not be zero and one must have $a_{k+1} \neq 0$, so that $X^{k+1} \in S = span\{X^n\phi\}_{n=1}^k$.

For any vector $v \in S$, we can express it as a linear combination of the vectors $\{X^n\phi\}_{n=1}^k$ with scalars $\{b_n\}_{n=1}^k$, such that $v = \sum_{n=1}^k b_n X^n\phi$. Since $\{X^n\phi\}_{n=1}^{k+1}$ is a subset of S, we can conclude that $Xv \in S$. Therefore, $X(S) \subset S$. Furthermore, since X is an invertible operator, it preserves the dimension of subspaces. Hence, if $X(S) \subset S$, it implies that $X(S) = S$. From this, we can deduce that $X^n(S) = S$ for all $n \geq 1$. Consequently, every vector of the frame satisfies $X^n\phi \in S$. Based on the above observations, we can conclude that S is equal to $\ell^2(\mathbb{Z}_d)$ and the set $\{X^n\phi\}_{n=1}^k$ forms a basis for S, where $k = d$. $\square$

Definition 3.2. A vector $x \in \ell^2(\mathbb{Z}_N)$, is called d-suitable within $\ell^2(\mathbb{Z}_N)$ if the following conditions are satisfied:

(i) $x(1)$ and $x(d+1)$ are non zero.

(ii) $x(i) = 0$ for all $i = d+2, \cdots, N$.

Essentially, a d-suitable vector in $\ell^2(\mathbb{Z}_N)$ has non-zero values at indices 1 and $d+1$, while all other indices greater than $d+1$ are zero.

Consider a frame $\{\phi_n\}_{n=1}^N$ for the Hilbert space $\ell^2(\mathbb{Z}_d)$, where N is greater than d. Suppose T is the synthesis matrix of $\{\phi_n\}_{n=1}^N$. We aim to show that the null space of T can be represented as the linear combination of right-shifts of a d-suitable vector. Additionally, we will show that if $Null(T)$ possesses this property, it implies that the frame is dynamical. Let $R : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)$ and $L : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)$ be the right and left shift operators, respectively and defined as:

$$R(b) = \begin{pmatrix} 0 \\ b(1) \\ b(2) \\ \vdots \\ b(N-1) \end{pmatrix} \quad \text{and} \quad L(b) = \begin{pmatrix} b(2) \\ b(3) \\ \vdots \\ b(N-1) \\ b(N) \\ 0 \end{pmatrix}$$

Lemma 3.2. *[11] Let $\{\phi_n\}_{n=1}^N$ be a frame for $\ell^2(\mathbb{Z}_d)$, $N > d$ and T denotes its synthesis matrix. Suppose $Null(T) = span\{R^{i-1}b : i = 1, 2, \cdots, N-d\}$ for some d-suitable vector $b \in \ell^2(\mathbb{Z}_N)$. Then the following holds true:*

(i) If $v \in Null(T)$ satisfies $v(N) = 0$, then $Rv \in Null(T)$.

(ii) If $v \in Null(T)$ satisfies $v(1) = 0$, then $Lv \in Null(T)$.

(iii) Every consecutive set of d elements of $\{\phi_n\}_{n=1}^N$ forms a basis for the space $\ell^2(\mathbb{Z}_d)$.

(iv) If $1 \leq i < j < N$ and $\phi_i = \phi_j$ then $\phi_{i+1} = \phi_{j+1}$.

Proof. (i) Since $N > d$, it follows that either $N = d + 1$ or $N > d + 1$.

Case I: First suppose $N = d + 1$ and v is a non-zero vector such that $v \in Null(T)$. Then $v \in \text{span}\{b\} = \text{Null(T)}$. Additionally, since b is a d-suitable vector, $v(N)$ will not be zero. Therefore (i) holds trivially in this case.

Case II: Suppose $N > d + 1$ and $v \in Null(T)$ such that $v(N) = 0$. Now $span\{R^{i-1}b : 1 \le i \le N - 1\} = Null(T)$ and $\dim\{R^{i-1}b : 1 \le i \le N - 1\} = \dim(Null(T)) = N - d$, therefore $\{R^{i-1}b : 1 \le i \le N - 1\}$ is basis for $Null(T)$. Hence there exist scalars $\{a_i\}_{i=1}^{N-d-1}$ such that

$$v = \sum_{i=1}^{N-d-1} a_i R^{i-1} b,$$

the coefficient on $R^{N-d-1}b$ will be zero. Then from above equation

$$R(v) = R\left(\sum_{i=1}^{N-d-1} a_i R^{i-1} b\right) = \sum_{i=1}^{N-d-1} a_i R^i b = \sum_{i=2}^{N-d} a_i R^{i-1} b \in Null(T).$$

(ii) If $N = d + 1$ (ii) holds trivially same as (i). Next suppose $N > d + 1$. Given that b is a d-suitable vector, we have $R^0 b = b$, which implies that b is the only vector in the set $\{R^{i-1}b : i = 1, 2, \cdots, N - d\}$ where $b(1) \neq 0$. Now, suppose that v belongs to the null space of T and that $v(1) = 0$. Writing v in linear combination of $\{R^{i-1}b : i = 2, 3, \cdots, N - d\}$, i.e.,

$$v = \sum_{i=2}^{N-d} a_i R^{i-1} b \implies Lv = LR\left(\sum_{i=1}^{N-d-1} a_{i+1} R^{i-1} b\right)$$

Define Z:subspace of vectors whose N^{th} coordinate is zero. Now $LR(z) = z$ for all $z \in Z$. Since $\sum_{i=1}^{N-d-1} a_{i+1} R^{i-1} b \in Z$, it follows that

$$Lv = \sum_{i=1}^{N-d-1} a_{i+1} R^{i-1} b \in Null(T).$$

(iii) In view of Lemma 3.1 it is sufficent to show that $\phi_1, \phi_2, \cdots, \phi_d$ is basis for $\ell^2(\mathbb{Z}_d)$. Define $S = span\{\phi_1, \phi_2, \cdots, \phi_d\}$. Take $b \in Null(T)$ and $b(d + 1) \neq 0$, then

$$\sum_{i=1}^{d+1} b(i)\phi_i = 0 \implies \phi_{d+1} = \frac{-1}{b(d+1)} \sum_{i=1}^{d} b(i)\phi_i \in S.$$

(iv) For $1 \le i < j < N$ it is given that $\phi_i = \phi_j$ and we have to show $\phi_{i+1} = \phi_{j+1}$. Let $\psi_1, \psi_2, \cdots, \psi_N$ denote the canonical basis for $\ell^2(\mathbb{Z}_N)$. Then $v = \psi_i - \psi_j \in$

$Null(T)$ and notice that $v(N) = 0$. Therefore by part (1), $Rv = \psi_{i+1} - \psi_{j+1} \in Null(T)$ and hence $\phi_{i+1} = \phi_{j+1}$. $\square$

Example 3.1. Let $\{\phi_n\}_{n=1}^5 \subset \ell^2(\mathbb{Z}_3)$ be a frame for $\ell^2(\mathbb{Z}_3)$, where

$$\{\phi_n\}_{n=1}^5 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Note that synthesis matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

Now null space of matrix is

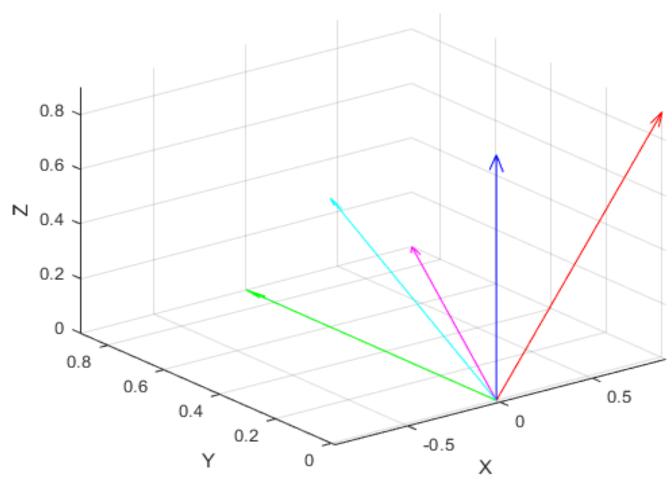
$$\mathcal{N}(T) = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\},$$

In this case d-suitable vector $b \in \ell^2(\mathbb{Z}_5)$ and $b = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}$ and $Rb = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$,

Therefore $Null(T) = span\{R^{i-1}b : 1 \leq i \leq 2\}$.

Now from (i) part of Lemma 3.2 take $v \in \mathcal{N}(\mathcal{T})$ such that $v = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}$ satisfying $v(5) = 0$ then $Rv \in \mathcal{N}(\mathcal{T})$ and it holds true.

Now from (ii) part, take $v = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$ satisfying $v(0) = 0$ then $Lv \in \mathcal{N}(\mathcal{T})$ and it holds true. Now from (iii) part, if we take 3 consecutive vectors of frame $\{\phi_n\}_{n=1}^5$ then each set of vectors $\{\phi_n\}_{n=1}^3$,$\{\phi_n\}_{n=2}^4$,$\{\phi_n\}_{n=3}^5$ will form the basis for $\ell^2(\mathbb{Z}_3)$. Now from (iv) part, in frame $\{\phi_n\}_{n=1}^5$ each vector is distinct, hence vacuously it holds true.



Graph of dynamical frame $\{\phi_n\}_{n=1}^5 \in \mathbb{R}^2$

3.2 Characterization of dynamical frames

In this section, we present a characterization result that determines whether a given frame is dynamical or not.

Theorem 3.3. *[11] Let $\{\phi_n\}_{n=1}^N$ be a frame for $\ell^2(\mathbb{Z}_d)$, $N > d$ and T denotes its synthesis matrix . Then the following assertions are equivalent:*

(i) $\{\phi_n\}_{n=1}^N$ is dynamical frame.

(ii) Null(T) is linear combination of right shifts of a d-suitable vector, i.e. $Null(T) = span\{R^{i-1}b : i = 1, 2, \cdots, N-d\}$, *b is a d-suitable vector.*

Proof. $(i) \implies (ii)$ Suppose $\{X_n\phi\}_{n=1}^N$ is a frame for $\ell^2(\mathbb{Z}_d)$. Now, it is not possible for the vectors $X^1\phi, X^2\phi, ..., X^{d+1}\phi$ to be linearly independent in the space $\ell^2(\mathbb{Z}_d)$. Therefore we can choose a $b \in Null(T)$ and $b \neq 0$. This vector b satisfy $b(n) = 0$ for $n > d$.

Let if possible $b(1) = 0$, then there exist a non-trivial linear combination of $\phi_2, \cdots, \phi_{d+1}$ that results in zero, which is contradiction. Similarly it is easy to see $b(d+1) \neq 0$. Therefore b is a d-suitable vector.

Firstly, note that the set $\{R^{i-1}b\}_{i=1}^{N-d}$ is linear independent. This can be proven by considering the fact that $b(d+1)$ is non-zero, while $b(i)$ equals zero for all $i > d+1$. Consequently, for $i \leq N-d$ we have $R^{i-1}b$ cannot be expressed as a linear combination of $\{R^{j-1}b : j < i\}$.

Next we show that $R^{i-1}b \in Null(T)$ for all $i \in \{1, \cdots, N-d\}$. Since $b \in Null(T)$, it follows that

$$TR^{i-1}b = \sum_{k=1}^{d+1} b_k X^{k+i-1}\phi = X^{i-1}\left(\sum_{k=1}^{d+1} b_k X^k\phi\right) = X^{i-1}(Tb) = 0.$$

Since $dim(Null(T)) = N-d$. and $\{R^{i-1}b : 1 \leq i \leq N-d\} \subset \mathrm{Null}(T)$ is linearly independent having cardinality $N-d$. Hence the result follows.

$(ii) \implies (i)$ Suppose b is a d-suitable vector and the span of $\{R^{i-1}b : 1 \leq i \leq N-d\}$ is $Null(T)$. Since $N-1 \geq d$, it follows that the vectors $\phi_1, \cdots, \phi_{N-1}$ must span $\ell^2(\mathbb{Z}_d)$, in view of Lemma 3.2. Let us define a linear map $X : \ell^2(\mathbb{Z}_d) \to \ell^2(\mathbb{Z}_d)$ by $X : \sum_{n=1}^{N-1} a_n\phi_n \mapsto \sum_{n=1}^{N-1} a_n\phi_{n+1}$. First we will show that map X is well define. Suppose

$$\sum_{n=1}^{N-1} a_n\phi_n = \sum_{n=1}^{N-1} b_n\phi_n \implies \sum_{n=1}^{N-1} a_n\phi_{n+1} = \sum_{n=1}^{N-1} a_n\phi_{n+1}.$$

Now define $c_n = a_n - b_n$ for $1 \leq n \leq N-1$ and $c(N) = 0$. Then $c = (c(n))_{n=1}^N \in$

$Null(T)$ and $c(N) = 0 \implies Rc \in Null(T)$. Thus

$$\sum_{n=1}^{N-1}(a_n - b_n)\phi_{N+1} = \sum_{n=1}^{N-1}(c_n)\phi_{N+1} = TRc = 0.$$

Therefore $X : \ell^2(\mathbb{Z}_d) \to \ell^2(\mathbb{Z}_d)$ is a well-defined linear map. Since $\phi_2, \cdots, \phi_N$ are in the range of X, therefore by Lemma 3.2 we get $\text{span}\{\phi_2, \cdots, \phi_N\} = \ell^2(\mathbb{Z}_d)$. Define $\phi = X^{-1}\phi_1$ as X is invertible. Since $X\phi_j = X\phi_{j+1}$ for each $1 \leq j \leq N-1$, one has that ϕ_n, defined as the action of an operator X on a vector ϕ raised to the power of n for all $1 \leq n \leq N$. Consequently, the set of vectors $\{\phi_n\}_{n=1}^N$ forms a dynamical frame. $\square$

Example 3.2. Let $X \in \mathcal{B}(\ell^2(\mathbb{Z}_2))$ and defined by

$$X = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Here $d = 2$ and suppose $N = 4$ and $\phi = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Then dynamical frame generated by ϕ is $\{\phi_n\}_{n=1}^4 = \{X^n\phi\}_{n=1}^4$ and

$$\{\phi_n\}_{n=1}^4 = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \end{pmatrix} \right\}.$$

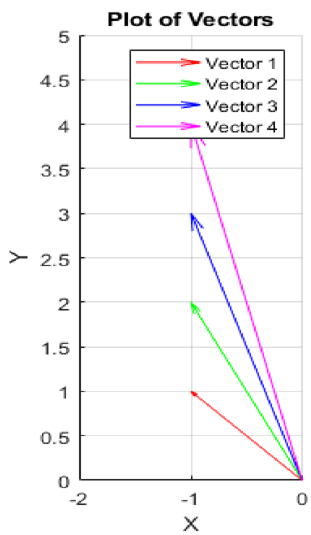
Now note that synthesis matrix of dynamical frame is

$$T = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 2 & 3 & 4 \end{pmatrix},$$

and null space of matrix T is

$$Null(T) = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

If we take d-suitable vector $b = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$ then (ii) part of Theorem 3.3 holds true, i.e. $Null(T) = span\{R^{i-1}b : 1 \leq i \leq 2\}$ for d-suitable b.



Graph of dynamical frame $\{\phi_n\}_{n=1}^{4} = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \end{pmatrix} \right\}$

3.3 Dual frame in $\ell^2(\mathbb{Z}_d)$

In this section, we provide a brief introduction to the concept of a dual frame and outline its fundamental properties.

Let $\{\phi_n\}_{n=1}^{N} \subset \mathcal{H}$ be a frame for $\mathcal{H}$. Define a operator $S : \mathcal{H} \to \mathcal{H}$ by

$$S(x) = \Theta^*\Theta(x) = \sum_{n=1}^{N} \langle x, \phi_n \rangle \phi_n.$$

The Operator S is called a *frame operator*. Also S is invertible operator.

Let $\{\phi_n\}_{n=1}^{N}$ and $\{\psi_n\}_{n=1}^{N}$ be frames for a Hilbert space $\mathcal{H}$ and T, U denote the synthesis matrices of $\{\phi_n\}_{n=1}^{N}$ and $\{\psi_n\}_{n=1}^{N}$, respectively. Then $UT^* : \mathcal{H} \to \mathcal{H}$ is called the *mixed dual Grammian matrix* of $\{\phi_n\}_{n=1}^{N}$ and $\{\psi_n\}_{n=1}^{N}$, defined by

$$UT^*(x) = \sum_{n=1}^{N} \langle x, \psi_n \rangle \phi_n = \sum_{n=1}^{N} \langle x, \phi_n \rangle \psi_n.$$

The frame $\{\psi_n\}_{n=1}^{N}$ is called *dual frame* of $\{\phi_n\}_{n=1}^{N}$ if mixed dual Grammian matrix is identity matrix, i.e., $UT^* = I$. In this case following Formula holds:

$$x = \sum_{n=1}^{N} \langle x, \psi_n \rangle \phi_n = \sum_{n=1}^{N} \langle x, \phi_n \rangle \psi_n \text{ for all } x \in \mathcal{H},$$

which is known as *reproducing formula.* The set $\{S^{-1}\phi_n : n = 1, 2, \cdots, N\}$ is dual frame for the frame $\{\phi_n\}_{n=1}^{N}$ is known as the *canonical dual frame* and it is unique.

Definition 3.3. Let $\mathcal{H}$ be a n dimensional Hilbert space and $\{\phi_i\}_{i=1}^{N} \subset \mathcal{H}$ be a frame for $\mathcal{H}$. If matrix $F = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_k \end{bmatrix}_{n\times k}$ has a rank n, then F is known as *frame matrix.* If $FF^* = \lambda I_{n\times n}$ for some $\lambda > 0$, then F is called a *tight frame matrix.* If $FF^* = I_{n\times n}$, then F is known as *Parseval frame matrix.* In addition, F is called an *equi-norm tight frame matrix* if each of the columns of tight frame matrix have the same norm.

Example 3.3. Let $\mathcal{H} = \mathbb{R}^2$ be a Hilbert space. Consider the frame,

$$\{x_1, x_2, x_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\} \subset \mathbb{R}^2,$$

and frame matrix is

$$F = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \end{pmatrix}.$$

Then adjoint matrix

$$F^* = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 3 \end{pmatrix}.$$

Then by definition frame Matrix is,

$$S = \Theta^*\Theta = F^*F = \begin{pmatrix} 6 & 1 \\ 1 & 10 \end{pmatrix}.$$

and $S^{-1} = \dfrac{1}{59} \begin{pmatrix} 10 & -1 \\ -1 & 6 \end{pmatrix}$.

then canonical dual frame of given frame is

$$\{y_i\}_{i=1}^{3} = \{S^{-1}x_i\}_{i=1}^{3} = \left\{ \frac{1}{59} \begin{pmatrix} 10 \\ -1 \end{pmatrix}, \frac{1}{59} \begin{pmatrix} 21 \\ -8 \end{pmatrix}, \frac{1}{59} \begin{pmatrix} 27 \\ 17 \end{pmatrix} \right\}.$$

3.4 Characterization of dynamical dual frames

In this section, we provide a characterization result for the existence of dynamical dual frame. Consider a frame $\{\phi_n\}$ along with its corresponding dual frame $\{\psi_n\}$. We define the dual frame $\{\psi_n\}$ as a *dynamical dual frame* if it possesses dynamical properties. In other words, there exists a linear operator $X : \mathcal{H} \to \mathcal{H}$ and a vector $\psi \in \mathcal{H}$ such that each element ψ_n of the dual frame can be expressed as $\psi_n = X^n\psi$

Proposition 3.4. *[11] If $\{\phi_n\}_{n=1}^d$ is a basis for $\ell^2(\mathbb{Z}_d)$ then $\{\phi_n\}_{n=1}^d$ is dynamical.*

Proof. It is given that $\{\phi_n\}_{n=1}^d$ is basis for $\ell^2(\mathbb{Z}_d)$, then every $x \in \ell^2(\mathbb{Z}_d)$ has unique representation i.e., there exist scalars $\{c_n\}_{n=1}^d$ such that

$$x = \sum_{n1}^{d} c_n\phi_n.$$

Now define a linear operator $X : \ell^2(\mathbb{Z}_d) \to \ell^2(\mathbb{Z}_d)$ by

$$T : \sum_{n=1}^{d} c_n\phi_n \to c_d\phi_1 + \sum_{n=1}^{d-1} c_n\phi_{n+1}.$$

Note that $X\phi_d = \phi_1$ and $X\phi_n = \phi_{n+1}$ for $n = 1, 2, \cdots, d-1$, we have $\phi_n = X^n\phi$ for all $1 \leq n \leq d$. □

The following theorem demonstrates that if a frame is dynamical, then its its canonical dual frame is also dynamical.

Theorem 3.5. *[11] If a sequence $\{\phi_n\}_{n=1}^N$ is a dynamical frame for $\ell^2(\mathbb{Z}_d)$, then its corresponding canonical dual frame $\{\widetilde{\phi_n}\}_{n=1}^N$ is also dynamical.*

Proof. Let $\{\phi_n\}_{n=1}^N$ be a dynamical frame for $\ell^2(\mathbb{Z}_d)$ and its synthesis matrix is T, then the corresponding canonical dual frame can be represented as $\{\widetilde{\phi_n}\}_{n=1}^N$, where $\widetilde{\phi_n} = S^{-1}\phi_n$ and has synthesis matrix $\widetilde{T} = (TT^*)^{-1}T$. In the case where N is equal to d, the proof can be derived from Proposition 3.4. However, If $N > d$, then $\widetilde{T} = (TT^*)^{-1}T$ implies $Null(T) = Null(\widetilde{T})$. Then by Theorem 3.3, canonical dual frame $\{\widetilde{\phi_n}\}_{n=1}^N$ is dynamical. □

In view of Theorem 3.5 any dynamical frame $\{\phi_n\}_{n=1}^N$ whose synthesis matrix is denoted by T, then its canonical dual frame is dynamical and can be expressed as $\{H^n h\}_{n=1}^N$, where $H : \ell^2(\mathbb{Z}_d) \to \ell^2(\mathbb{Z}_d)$ and $h \in \ell^2(\mathbb{Z}_d)$. When considering $S = TT^*$, we can rewrite this expression using $H = S^{-1}XS$ and $h = S^{-1}\phi$. Since $\widetilde{T} = (TT^*)^{-1}T$, then columns of $\widetilde{T}$ are $S^{-1}X\phi, S^{-1}X^2\phi, \cdots, S^{-1}X^N\phi$. Note that $S^{-1}X^nS = (S^{-1}XS)^n$ so that $S^{-1}X^nSh = (S^{-1}XS)^n h = S^{-1}X^n\phi$.

Example 3.4. Let $\{\phi_n\}_{n=1}^4 = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \end{pmatrix} \right\}$ be dynamical frame for $\ell^2(\mathbb{Z}_2)$. Then synthesis matrix T and analysis matrix T^* are given by

$$T^* = \begin{pmatrix} -1 & 1 \\ -1 & 2 \\ -1 & 3 \\ -1 & 4 \end{pmatrix}$$

and

$$T = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Then frame matrix is

$$S = TT^* = \begin{pmatrix} 4 & -1 \\ -1 & 30 \end{pmatrix}.$$

Since frame matrix is always invertible, then inverse of frame matrix is

$$S^{-1} = \frac{1}{10} \begin{pmatrix} 15 & 5 \\ 5 & 2 \end{pmatrix}$$

Now canonical dynamical dual frame of this dynamical frame is $\{\psi_n\}_{n=1}^4 = \{S^{-1}\phi_n\}_{n=1}^4$ and

$$\{\psi_n\}_{n=1}^4 = \left\{ \frac{1}{10} \begin{pmatrix} -10 \\ -3 \end{pmatrix}, \frac{1}{10} \begin{pmatrix} -5 \\ -1 \end{pmatrix}, \frac{1}{10} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \frac{1}{10} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \right\}.$$

Now using Theorem 3.3 we will show that this canonical dual frame is dynamical.

Synthesis matrix of canonical dual frame is

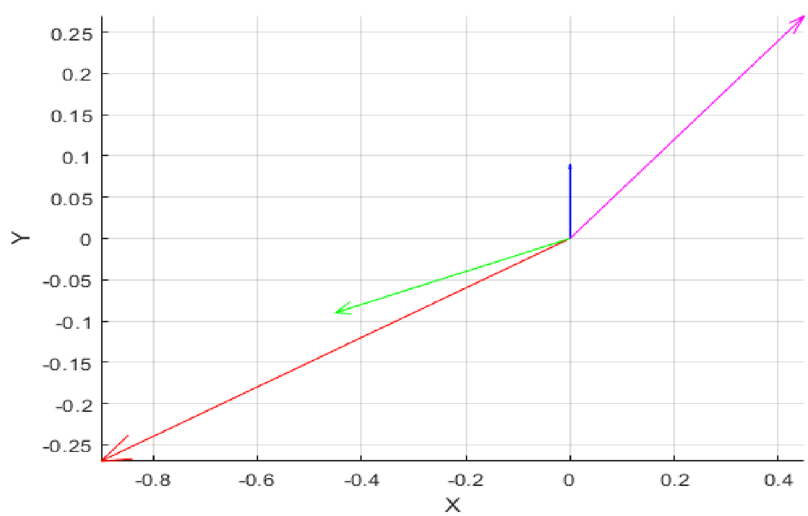
$$\widetilde{T} = \frac{1}{10}\begin{pmatrix} -10 & -5 & 0 & 5 \\ -3 & -1 & 1 & 3 \end{pmatrix}.$$

Now null space of $\widetilde{T}$ is

$$\mathrm{Null}(\widetilde{T}) = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

For vector $b = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$ by Theorem 3.3, $\mathrm{Null}(\widetilde{T}) = span\{R^{i-1}b : 1 \leq i \leq 2\}$.

Hence $\{\psi_n\}_{n=1}^4$ is canonical dynamical dual of dynamical frame $\{\phi_n\}_{n=1}^4$.



Graph of dynamical frame $\{\psi_n\}_{n=1}^4 = \left\{ \frac{1}{10}\binom{-10}{-3}, \frac{1}{10}\binom{-5}{-1}, \frac{1}{10}\binom{0}{1}, \frac{1}{10}\binom{5}{3} \right\}$

Lemma 3.6. *[11] Consider there are two properties, P_1 and P_2, such that a frame $\{\phi_n\}_{n=1}^N$ for $\ell^2(\mathbb{Z}_d)$ having synthesis matrix T possesses P_1 if and only if the Null$T$ possesses P_2. Then, the existence of a dual frame for the frame $\{\phi_n\}_{n=1}^N$ in $\ell^2(\mathbb{Z}_d)$ with property P_1 on the assumption of existence of a subspace $V \subset \ell^2(\mathbb{Z}_N)$ that* *satisfies the following conditions:*

(i) $\ell^2(\mathbb{Z}_N) = V \oplus Range(T^*)$.

(ii) V *has property* P_2.

Proof. ($\Longrightarrow$) Suppose dual frame of frame $\{\phi_n\}_{n=1}^N$ $\{\psi_n\}_{n=1}^N$ is the frame $\{\phi_n\}_{n=1}^N$ has property P_1, and let $\widetilde{T}$ be synthesis matrix of $\{\psi_n\}_{n=1}^N$. Then $V = Null(\widetilde{T})$ satisfies the property P_2. Since $\{\phi_n\}_{n=1}^N$ is frame, it follows that $dim(Range(T^*) = d$. Also $\{\psi_n\}_{n=1}^N$ is frame therefore $dim(Null(\widetilde{T})) = N - d$. Now $\{\psi_n\}_{n=1}^N$ is dual of $\{\phi_n\}_{n=1}^N$ and T^* is analysis matrix of the frame $\{\phi_n\}_{n=1}^N$ then $\widetilde{T}T^* = I_{d\times d}$. This gives that 0 is the only common element between $Range(T^*)$ and $Null(\widetilde{T})$. Then $\ell^2(\mathbb{Z}_N) = Null(\widetilde{T}) + Range(T^*) \implies \ell^2(\mathbb{Z}_N) = V + Range(T^*)$.

($\Longleftarrow$) Consider that P is the orthogonal projection onto $V^\perp$, where $V^\perp$ is the orthogonal complement of V. This projection satisfies the property $Null(P) = V$. Considering our assumption that intersection of V and $Range(T^*)$ is $\{0\}$, and knowing that the null space of T^* is trivial $\{0\}$, we can conclude that the linear map PT^* is one-one when operating $\ell^2(\mathbb{Z}_d) \to \ell^2(\mathbb{Z}_N)$. Thus define a operator $A : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_d)$ such that $APT^* = I_{d\times d}$ that is A is a left inverse to PT^*. Let $\widetilde{T} = AP$ and noting that $\widetilde{T}T^* = I_{d\times d}$, let $\{\psi_n\}_{n=1}^N$ be a dual frame of the frame $\{\phi_n\}_{n=1}^N$ with synthesis matrix $\widetilde{T}$. Given that the dimension of the range of the adjoint operator T^* is d, it follows that the dimension of the space V is $N - d$. Consequently, the dimension of the null space of the projection operator P is equal to the dimension of V, and $dim(V) = N - d$. As the matrix $\widetilde{T}$ with dimensions $d \times N$ is of full rank, the dimension of $Null(\widetilde{T})$, can be determined. Consequently, $dim(Null(\widetilde{T}))$ is equal to $dim(Null(P))$. Since $Null(P)$ is a subset of $Null(\widetilde{T})$, it can be concluded that $Null(\widetilde{T}) = Null(P) = V$. Therefore $\widetilde{T}$ has property P_1. □

Corollary 3.7. *[11] Consider a frame $\{\phi_n\}_{n=1}^N$ for $\ell^2(\mathbb{Z}_d)$ with $N > d$ and synthesis matrix T. Then the following statements are equivalent:*

(i) $\{\phi_n\}_{n=1}^N$ has a dynamical dual frame.

(ii) There exists a subspace V of $\ell^2(\mathbb{Z}_N)$ and a d-suitable vector b such that $V = span\{R^{i-1}b : i = 1, 2, \cdots, N-d\}$ and $\ell^2(\mathbb{Z}_N) = V \oplus Range(T^*)$.

Proof. Let P_1 denote the dynamical property of the frame, while P_2 represents the property that a subspace can be expressed as $span\{R^{i-1}b : i = 1, \cdots, N-d\}$. Then using above Lemma 3.6 proof is done. □

3.5 Construction of dynamical dual frames

Next we show that when $\{\phi_n\}_{n=1}^N$ forms a frame for $\ell^2(\mathbb{Z}_d)$ with $N > d$, then it has always a dynamical frame. In fact there exists an infinite number of dynamical dual frames associated with $\{\phi_n\}_{n=1}^N$. We also provide a construction method for the construction of existing dynamical dual frame.

Proposition 3.8. *[11] Consider a frame $\{\phi_n\}_{n=1}^N$ for $\ell^2(\mathbb{Z}_d)$, where $N > d$, and let the synthesis matrix be denoted by T. Assuming that $\{\phi_n\}_{n=1}^N$ possesses a dynamical dual frame, we define V as the $V = span\{R^{i-1}b : i = 1, 2, \cdots, N - d\}$, as stated in Corollary 3.7. Now, let N_b be a matrix with columns given by $b, Rb, \cdots, R^{N-d-1}b$, and assume that*

$$G_b = (T(I - N_b(N_b^* N_b)^{-1} N_b^*)T^*)^{-1} T(I - N_b(N_b^* N_b)^{-1} N_b^*). \tag{3.1}$$

Then the dynamical dual of frame $\{\phi_n\}_{n=1}^N$ are the column vectors of matrix G_b. Moreover, $Null(G_b) = V$.

According to Corollary 3.7, the existence of dynamical dual depended on the existence of a vector b which is d- suitable vector b such that $V = span\{R^{i-1}b : i = 1, 2, \cdots, N-d\}$ satisfy $\ell^2(\mathbb{Z}_N) = V \oplus Range(T^*)$.

Definition 3.4. Define a matrix $M_{T,b}$ of size $N \times N$ such that its first d column are transpose of rows of a matrix T of order $d \times N$ and last $N-d$ columns are given by right shifts of a d-suitable vector b, i.e. $\{R^{i-1}b : i = 1, 2, \cdots, N-d\}$.

Corollary 3.9. *[11] Let $N > d$ and $\{\phi_n\}_{n=1}^N$ be a frame for $\ell^2(\mathbb{Z}_d)$. Now if T denotes the synthesis matrix of $\{\phi_n\}_{n=1}^N$, then there exists a vector $b \in \ell^2(\mathbb{Z}_N)$, which is d-suitable such that the matrix $M_{T,b}$ is invertible. Hence $\{\phi_n\}_{n=1}^N$ possesses a dynamical dual frame $\{\psi_n\}_{n=1}^N$. The synthesis matrix for $\{\psi_n\}_{n=1}^N$ can be represented by G_b, in (3.1).*

Theorem 3.10. *[11] If $N > d$ and $\{\phi_n\}_{n=1}^N$ is a frame for $\ell^2(\mathbb{Z}_d)$ having synthesis matrix T, then there exist infinite many d-suitable vector b such that the matrix $M_{T,b}$ is invertible and hence there exist infinitely many dynamical dual frames for $\{\phi_n\}_{n=1}^N$.*

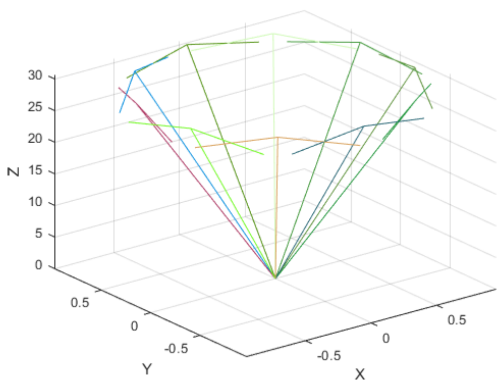
Example 3.5. Given $\theta \in \mathbb{R}$, for fix N define the matrix $\mathbb{R}_\theta$ by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

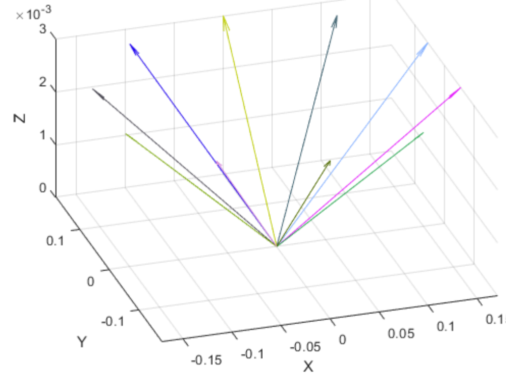
If $e = (0, 1, 30)'$ then for $\theta = \frac{2\pi n}{N}$, frame $\{\phi_n^N\}_{n=1}^N \subset \mathbb{R}^3$ is defined by

$$\phi_n^N = (R_{\frac{2\pi n}{N}})^n e = \begin{pmatrix} -\sin(\frac{2\pi n}{N}) \\ \cos(\frac{2\pi n}{N}) \\ 30 \end{pmatrix}.$$

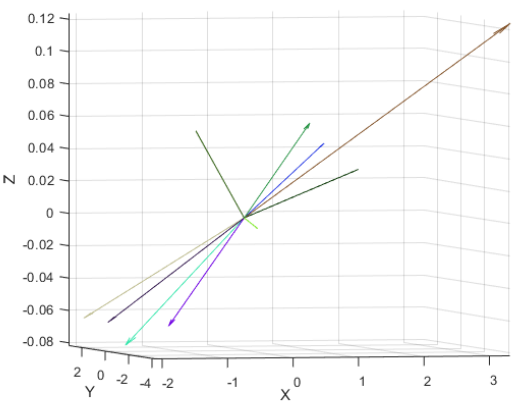
This is a dynamical frame of size N. Now take $N = 10$ then we are going to calculate canonical and alternate dynamical dual frames using Theorem 3.10 and Proposition 3.8.



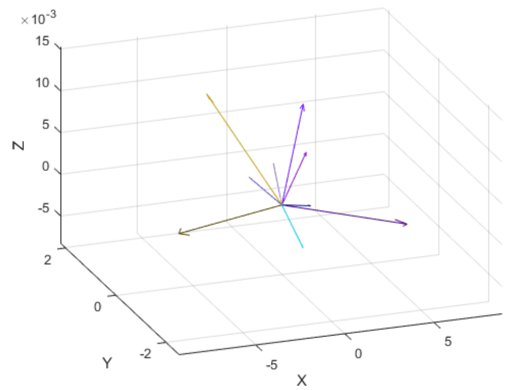
(a) Graph of frame $\{\phi_n\}_{n=1}^{10}$



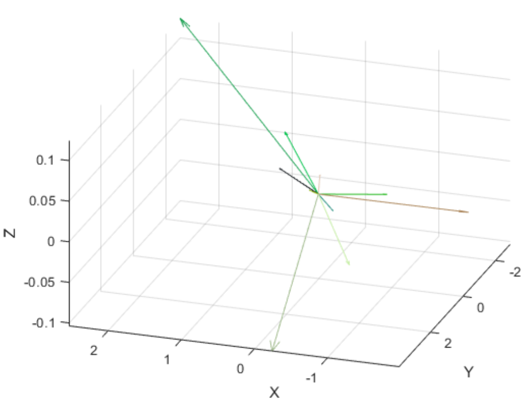
(b) Frame $\{S^{-1}\phi_n\}_{n=1}^{10}$ with frame operator S



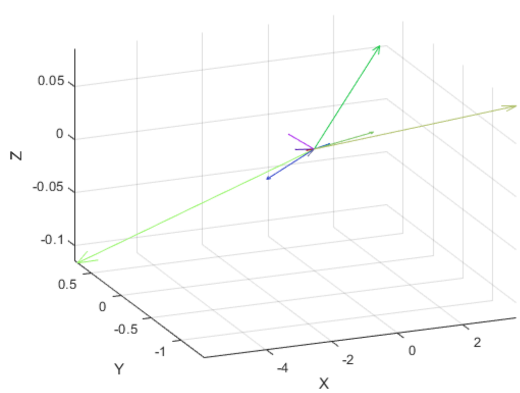
(c) For $b = (5, 3, -1, 6, 0, 0, 0, 0, 0, 0)'$



(d) For $b = (\pi, 3, -1, -6, 0, 0, 0, 0, 0, 0)'$



(e) For $b = (\pi, -30, -10, -4\pi, 0, 0, 0, 0, 0, 0)'$



(f) For $b = (5\pi, 3, -10, -60, 0, 0, 0, 0, 0, 0)'$

Figure 3.1: These are 5-different duals of frame $\{\phi_n^N\}_{n=1}^{10} \subset \ell^2(\mathbb{Z}_3)$ for given d-suitable vectors b.

Example 3.6. Given $x \in \mathbb{R}$, define the matrix $\mathbb{R}_x$ by

$$R_x = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

Then

$$R_x^n = \begin{pmatrix} 1 & 0 \\ nx & 1 \end{pmatrix}.$$

If $v = (-1, 0)'$ then for $x = -1$, frame $\{\phi_n^N\}_{n=1}^N \subset \mathbb{R}^3$ is defined by

$$\phi_n^N = (R_x)^n v = \begin{pmatrix} -1 \\ n \end{pmatrix}.$$

This is a dynamical frame of size N, and $\{\phi_n\}_{n=1}^N = \left\{ \begin{pmatrix} -1 \\ n \end{pmatrix} \right\}_{n=1}^N$ be dynamical frame for $\ell^2(\mathbb{Z}_2)$. Now take $N = 12$ then dynamical frame will be

$$\{\phi_n\}_{n=1}^{12} = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \cdots, \begin{pmatrix} -1 \\ 12 \end{pmatrix} \right\}$$

Then Analysis dynamical matrix T^* and synthesis dynamical matrix T are

$$T^* = \begin{pmatrix} -1 & 1 \\ -1 & 2 \\ \vdots & \vdots \\ -1 & 12 \end{pmatrix}$$

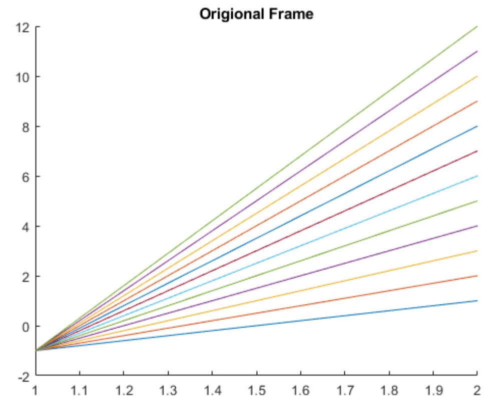
and

$$T = \begin{pmatrix} -1 & -1 & \cdots & -1 \\ 1 & 2 & \cdots & 12 \end{pmatrix}.$$

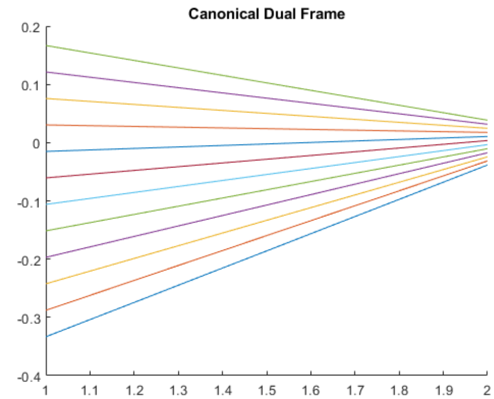
Then dynamical frame matrix is

$$S = TT^* = \begin{pmatrix} 12 & -78 \\ -78 & 650 \end{pmatrix}, \text{ and } S^{-1} = \begin{pmatrix} \frac{25}{66} & \frac{1}{22} \\ \frac{1}{22} & \frac{1}{143} \end{pmatrix}.$$

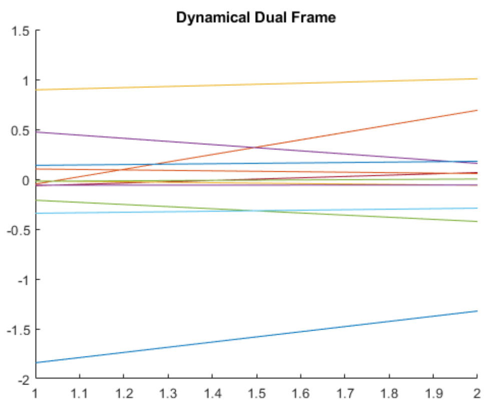
Now canonical dynamical dual frame of this dynamical frame is $\{\psi_n\}_{n=1}^{12} = \{S^{-1}\phi_n\}_{n=1}^{12}$.



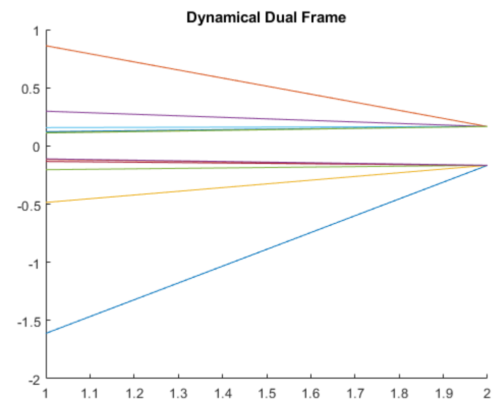
(a) Graph of frame $\{\phi_n\}_{n=1}^{12}$



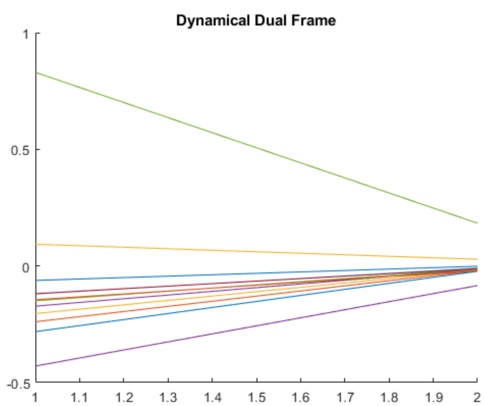
(b) Graph of frame $\{\psi_n\}_{n=1}^{12}$



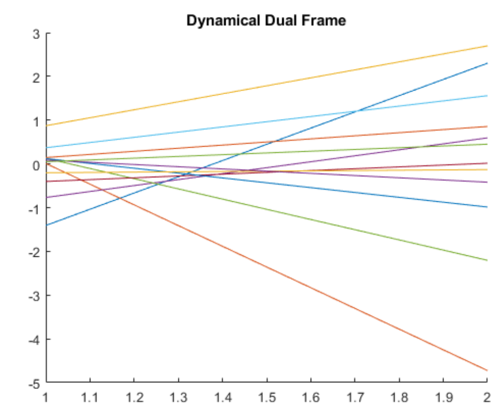
(c) For $b = (1, -1, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0)'$



(d) For $b = (1, 3, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0)'$



(e) For $b = (-4, 3, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0)'$



(f) For $b = (5, 7, 8, 0, 0, 0, 0, 0, 0, 0, 0, 0)'$

Figure 3.2: These are 5-different duals of frame $\{\phi_n^N\}_{n=1}^{12} \subset \ell^2(\mathbb{Z}_2)$ for given d-suitable vectors b.

CHAPTER 4

Application of dynamical sampling in P.D.E.

In the previous chapter, we have already introduced the background of this project work. Now we are ready to mention some application of dynamical sampling of this thesis.

Depending on what is given and what is not given there are three types of problems,

(i) **Time-space trade-off in sampling:**[4] In this case operator A is known, $f = \eta = 0$ and (x_i, t_i) points are given at these points values $\{u()\}$ are also given, and we have to sample u_0.

(ii) **System identification:**[2] In this case $f = \eta = 0$ is given and we have to recover operator A and u_0 from $\{u(x_i, t_i)\}$.

(iii) **Source identification:**[2] Identify the function f driving the solution u from space time samples of u.

4.1 Reconstruction and stability of signal in the presence of noise

First application of dynamical sampling A. Aldroubi described in his paper "Dynamical sampling: Time–space trade-off [4]".

Time-space trade-off in sampling: In this research, Akram Aldroubi addresses the challenge of spatiotemporal sampling, which involves reconstructing the initial state of an evolving process based on samples taken at various time intervals. His focus lies on achieving a trade-off between spatial and temporal samples that preserves all the information without any loss. Aldroubi demonstrates that, specifically for a certain category of signals, it is feasible to recover the initial state using a reduced number of measuring devices that are activated more frequently. He introduces multiple algorithms designed for this type of recovery and evaluates their resilience to noise.

4.2 Solution of abstract initial value problem

In the paper "Recovery of Rapidly Decaying Source Terms from Dynamical Samples in Evolution Equations [5]" by Akram Aldroubi, dynamical sampling is used to solve inverse problems in initial value problems (IVP) for evolution equations. This problem is third kind of dynamical sampling problem, i.e. source identification problem.

The basic idea behind dynamical sampling is to select a set of sampling points that capture the essential dynamics of the system under consideration. In the context of IVPs, this means choosing a set of initial conditions that result in a solution that exhibits the desired behavior.

Problem setting of this IVP:[5] Let us consider the following abstract initial value problem:

$$\begin{cases} \frac{\partial}{\partial t} u(t) = Au(t) + f(t) + \eta(t) \\ u(0) = u_0. \end{cases} \quad t \in \mathbb{R}_+, u_0 \in \mathcal{H}$$

It could be heat equation or some other IVP. And where $\eta : \mathbb{R}_+ \to \mathcal{H}$ is a Lipschitz continuous function and models a background source term. And $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is a generator of a strongly continuous semi-group $T : \mathbb{R}_+ \to \mathcal{B}(H)$, f could be relevant source at the driving the system, and u_0 is initial condition. This is the general setting of the problem.

The burst-like forcing term f is of the form

$$f(t) = \sum_{j=1}^{N} f_j \delta(t - t_j),$$

For some unknowns $N \in \mathbb{N}$, $0 = t_0 < t_1 < t_2 < \cdots < t_N$, and $f_j \in \mathcal{H}$.

Visual things of the problem: Suppose you are in the ocean and you have waves, but they are kind of going around, and so this is the η, which you don't know, but you know it's kind of going somewhat slowly. Then, you have these functions that show those are little fish that kind of jump, and so they form little waves. You also have those recording devices bobbing onto the sea, recording things. The question is, from these recording devices, can you actually figure out when these jumps occur and where?

Image of jumping fish and using some devices we are trying to predict that when will this fish occur jump and what will be its location?

And that is essentially what this problem is all about. So, of course, you can take it as it is, and we have other options for more general things. This is the idea, and here we are going to show that you don't know these things that belong to some subspace of $\mathcal{H}$. So, I am going to think of it as elevation. Let us not worry about that, and we know something about η.

About the problem: We want to find the condition or design in this case, the structure of samples or sampling design, if you like, a subset of vectors in $\mathcal{H}$, and a time sampling beta that allows us to recover or approximate this established f from noisy samples. We want to design this vector and find the beta such that it will allow us to essentially recover f in some good way and get estimates on it.

Main problem:[5] Find conditions on the semi-group $T(t) = A^t$, and a countable sets $G \subset \mathcal{H}$ and $B \in \mathbb{R}_+$ that allow one to stably and accurately approximate $f(t) = \sum_{j=1}^N f_j \delta(t - t_j)$, from the noisy samples

$$\langle u(n\beta), g\rangle + v(n\beta, g),\ n \in \mathbb{N}, g \in G$$

We will assume that the background source is Lipschitz with Lipschitz constant M, and $\sup_{n,g} |v(n\beta, g)| < \infty$.

CHAPTER 5

Conclusion and Future Plan

5.1 Conclusion

In conclusion, this thesis has addressed the formulation and study of dynamical sampling, exploring its connections to frames in a separable Hilbert space. We have specifically examined various scenarios, such as when the evolution operator has a diagonalizable or Jordan form, and provided solutions to the dynamical sampling problem for different types of evolution operators, including translation invariant, Fourier multipliers, and convolution operators. Furthermore, we have presented a characterization of dynamical frames and established that redundant frames possess an infinite number of dual frames. In addition, we have delved into the practical applications of dynamical sampling, showcasing its effectiveness in solving initial value problems and sampling noisy signals. Through this research, we have contributed to the understanding of dynamical sampling and its utility in diverse areas.

5.2 Future Plan

As a future work, we extend our results(Theorem 2.11, 2.12 and 2.14) in infinite dimensional separable Hilbert space. In another direction we will also try to find out conditions on wavelet frame $\{\Psi\}_{j,k\in\mathbb{Z}} = \{2^{j/2}\phi(2^j. - k)\}_{j,k\in\mathbb{Z}}$ such that

$$\{\Psi_{j,k}\}_{j,k\in\mathbb{Z}} = \{X^n\phi\}_{n=1}^{\infty},$$

where $X : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is an evolution operator and $\phi \in L^2(\mathbb{R})$. Moreover, we also plan to explore similar problems using other function systems such as Gabor, wave-packet, and shearlet.

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