# An Analogue of Ramanujan's formula for $\zeta(2m+1)$

**M.Sc. Thesis**

*by*


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**DEPARTMENT OF MATHEMATICS**

**INDIAN INSTITUTE OF TECHNOLOGY INDORE**


**JUNE 2023**

**An Analogue of Ramanujan's formula for $\zeta(2m+1)$**

**A THESIS**

*Submitted in partial fulfillment of the requirements for the award of the degree of*

**Master of Science**

*by*


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**JUNE** 2023

**INDIAN INSTITUTE OF TECHNOLOGY INDORE**

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"An Analogue of Ramanujan's formula for $\zeta(2m+1)$"** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF MATHEMATICS**, **Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2022 to June 2023 under the supervision of **Dr. Bibekananda Maji**, Assistant Professor, Department of Mathematics, IIT Indore. The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

Shubhdeep Das 05-06-2023

Signature of the student with date

(**Shubhdeep Das**)

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Bibekananda Maji 06/06/23

Signature of Thesis Supervisor with date

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**Shubhdeep Das** has successfully given his M.Sc. Oral Examination held on 5th June, 2023.

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*Dedicated to my family*

*"I have to remind myself that some birds aren't meant to be caged. Their feathers are just too bright. And when they fly away, the part of you that knows it was a sin to lock them up does rejoice."*
–***Shawshank Redemption(Movie)***

# Acknowledgements

I am incredibly grateful to my supervisor Dr. Bibekananda Maji for his assistance and guidance throughout my M.Sc. period. Without his support my M.Sc journey could not be completed. His way of observing the problem and breaking it down motivates me to look at problems in different way. I am completing M.Sc with much more maturity in mathematics as compared to when I came and his role is magnificent in this.

I extend my thanks to my PSPC members Dr. Charitha Cherugondi, Dr. Vinay Kumar Gupta as well as DPGC convener Dr. Vijay Kumar Sohani and HOD Dr. Niraj Kumar Shukla. I am also thankful to all our teachers for their kind and valuable suggestion.

I am thankful to Dr. Pramod Eyyuni and Dr. Prashanta Garaine for motivating me to strive for more mathematical maturity. I am also thankful to Phd scholars Archit Agrawal and Meghali Garg for clearing my doubts related to thesis, academics and life. Despite having busy schedules they always took out time to help me.

I would like to thank my family as they are the reason of my existence. Their support is always there for me and it makes me feel blessed. Words can't describe their love for me and my love for them.

At last I want to thank some lovely people I met here and will be in my hearts forever Sanjit, Mansi, Asik, Ritik, Aslam, Nikhil, Anushka, Lokendra, Adarsh, Sajin, Ritwik, Dr. Naresh Pati, Nehal Ahmed, Debdeep, Aniruddha, amazing juniors and many others.

# Abstract

Understanding the nature of the particular values of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ has always been a difficult problem in number theory. Euler gave an elegant formula for even zeta values which implies that even zeta values are not only irrational but also transcendental. Nevertheless, the nature of odd zeta values remains open. An interesting formula for odd zeta values can be found in page 319 of the lost notebook of Ramanujan. In the same page, Ramanujan mentioned that the formula for odd zeta values can be obtained from one of his partial fraction decompositions of cotangent function and cotangent hyperbolic function. In this thesis, we study a few partial fraction decompositions for trigonometric functions given by Ramanujan. Utilizing these partial fraction decompositions and Cauchy integration technique, we establish Dirichlet character analogue of the formula for odd zeta values.

# Contents

# CHAPTER 1

## Introduction

For $\Re(s) > 1$, the Riemann zeta function $\zeta(s)$ is defined as the infinite series $\sum_{n=1}^{\infty} n^{-s}$. Euler, in 1734, gave a direct formula for $\zeta(2m)$. For any natural number $m$, Euler showed that

$$\zeta(2m) = c_m \pi^{2m}, \tag{1.1}$$

where $c_m = (-1)^{m+1} \frac{2^{2m-1} B_{2m}}{(2m)!}$ and $B_i$ denotes the $i$-th Bernoulli number. Since all Bernoulli numbers are rationals, from (1.1) we can conclude that $\zeta(2m)$ are transcendental in nature. However, we do not know till date any such explicit expression for $\zeta(2m+1)$. We also do not know much about their algebraic nature. Unexpectedly, Roger Apréy [1, 2] in 1959, showed the irrationality nature of $\zeta(3)$, but nothing much is known about its algebraic nature. Zudilin [27], in 2001, came up with a brilliant idea to show that at least one of $\zeta(2m+1)$, for $2 \leq m \leq 5$, is irrational. Quite amazingly, Rivoal [25], Ball and Rivoal [3] showed that there are infinitely many odd numbers for which $\zeta(s)$ gives irrational values, but their result does not say much about the nature of a special odd zeta value. These are the best known results till date.

Interestingly, Ramanujan gave the following identity for $\zeta(2m+1)$ in his lost notebook [24, p. 319, Entry (28)]. For any non-zero integer $m$, we have

$$\mathcal{F}_m(\alpha) = (-1)^m \mathcal{F}_m(\beta) + \sum_{j=0}^{m+1} \frac{(-1)^{j-1} B_{2j} B_{2m+2-2j}}{(2j)!(2m+2-2j)!} \alpha^{m+1-j} \beta^j. \tag{1.2}$$

where $\mathcal{F}_m(x) = (4x)^{-m} \left\{ \frac{1}{2}\zeta(2m+1) + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2xn}-1} \right\}$ and $\alpha\beta = \pi^2$ with $\alpha, \beta > 0$. The above identity can also be found in his second notebook [23, p. 173, Ch. 14, Entry 21(i)]. At a first glance, it may look the above identity is not as elegant as of Euler's identity (1.1) and also it does not provide information whether odd zeta values are algebraic or transcendental. But it is indeed a remarkable formula as it gives transformation formula for the Eisenstien series. One can read more about its significance in a survey article by Berndt and Straub [8]. Over the time, many mathematicians extended this formula in different directions. To know more readers can see [4, 5, 6, 7, 8, 9, 10, 13, 14, 15].

A formula for $\zeta(4m+3)$ given by Lerch can be obtained from (1.2) by substituting $\alpha = \pi = \beta$ and substituting $m$ to be $2m+1$. Thus, for $m \in \mathbb{N}$, we get

$$\begin{aligned} \zeta(4m+3) =& \pi^{4m+3} 2^{4m+2} \sum_{j=0}^{2m+2} (-1)^j \frac{B_{2j}}{(2j)!} \frac{B_{4m+4+2j}}{(4m+4-2j)!} \\ & - 2 \sum_{n=1}^{\infty} \frac{1}{n^{4m+3}(e^{2\pi n}-1)} \end{aligned} \tag{1.3}$$

Ramanujan mentioned that (1.2) can be obtained using a partial fraction expansion of $\cot(\pi x) \coth(\pi y)$, but there was an error. The partial fraction decomposition offered by Ramanujan [23, p. 171, Ch. 14, Entry 19(i)], [24, p. 318, Entry (21)] is the following identity:

$$\pi^2 xy \cot(\pi x) \coth(\pi y) = 1 + 2\pi xy \sum_{n=1}^{\infty} \frac{n \coth\left(\frac{\pi n x}{y}\right)}{n^2 + y^2} - 2\pi xy \sum_{n=1}^{\infty} \frac{n \coth\left(\frac{\pi n y}{x}\right)}{n^2 - x^2}, \tag{1.4}$$

where both the infinite series diverges individually. R. Sitaramachandrarao [26] established a corrected form of the above partial fraction decomposition formula given as:

$$\pi^2 xy\cot(\pi x))\coth(\pi y) = 1 + \frac{\pi^2}{3}(y^2 - x^2) - 2\pi xy \sum_{m=1}^{\infty} \left( \frac{y^2 \coth\left(\frac{\pi m x}{y}\right)}{m(m^2 + y^2)} + \frac{x^2 \coth\left(\frac{\pi m y}{x}\right)}{m(m^2 - x^2)} \right). \tag{1.5}$$

The identity (1.2) can be obtained from (1.5) by substituting $\pi x$ by $\sqrt{w\alpha}$ and $\pi y$ by $\sqrt{w\beta}$ and then comparing the coefficient of $w^m$, for $m \geq 1$. Grosswald [16] in 1972 gave an noteworthy extension of the Ramanujan's formula (1.2). For any non-zero integer $m$, define $\sigma_m(n) = \sum_{d|n} d^m$ and $F_m(z) = \sum_{n=1}^{\infty} \sigma_{-m}(n) e^{2\pi i n z}$ for $z \in \mathbb{H}$. Then we have

$$F_{2m+1}(z) - z^{2m} F_{2m+1}\left(-\frac{1}{z}\right) = \frac{1}{2}\zeta(2m+1)(z^{2m} - 1) + \frac{(2\pi i)^{2m+1}}{2z} \sum_{l=0}^{m+1} z^{2m+2-2l} \frac{B_{2l}}{(2l)!} \frac{B_{2m+2-2l}}{(2m+2-2l)!}. \tag{1.6}$$

The identity (1.2) can be obtained from (1.6) by substituting $z = \frac{i\beta}{\pi}$, $\alpha\beta = \pi^2$ where $\alpha, \beta > 0$ . Recently, Chourasiya, Jamal and Maji [11] obtained a Ramanujan-type identity for $\zeta(2m+1)$. The identity is as follows. For $m \in \mathbb{Z} - \{0\}$, we have

$$\mathcal{G}_m(\alpha) = (-1)^m \mathcal{G}_m(\beta) + \sum_{l=1}^{m} (-1)^{l-1} (2^{2l} - 1)(2^{2m+2-2l} - 1) \frac{B_{2l}}{(2l)!} \frac{B_{2m+2-2l}}{(2m+2-2l)!} \alpha^{m+1-l} \beta^l, \tag{1.7}$$

where

$$\mathcal{G}_m(x) = (4x)^{-m} \left( \frac{1}{2}\zeta(2m+1)(1 - 2^{-2m-1}) - \sum_{n=0}^{\infty} \frac{(2n+1)^{-2m-1}}{e^{2(2n+1)x} + 1} \right)$$

and $\alpha\beta = \frac{\pi^2}{4}$ with $\alpha, \beta > 0$. To obtain (1.7) Chourasiya et al. [11] used the following partial fraction decomposition given by Ramanujan [23, p. 171, Ch.

14, Entry 19(iii)]:

$$\frac{\pi}{4}\tan\left(\frac{\pi x}{2}\right)\tanh\left(\frac{\pi y}{2}\right) = y^2\sum_{k=0}^{\infty}\frac{\tanh\left((2k+1)\frac{\pi x}{2y}\right)}{(2k+1)\{(2k+1)^2+y^2\}} + x^2\sum_{k=0}^{\infty}\frac{\tanh\left((2k+1)\frac{\pi y}{2x}\right)}{(2k+1)\{(2k+1)^2-x^2\}}, \tag{1.8}$$

where $x, y \in \mathbb{C}$ such that $\Re(\frac{y}{x}) \neq 0$. Chourasiya et al. [11] also gave a more general form of (1.7), namely, for Dirichlet $L$-function $L(\chi, s)$. For any prime number $\ell$, define

$$a_n = \begin{cases} 1, & \textit{if } \gcd(n,\ell)=1, \\ 1-\ell, & \textit{if } \gcd(n,\ell)=\ell. \end{cases} \tag{1.9}$$

Then for $m \in \mathbb{Z}\backslash\{0\}$, we have

$$\mathcal{H}_m(\alpha)+(-1)^{m+1}\mathcal{H}_m(\beta) = \sum_{j=1}^{\infty}(-1)^{j-1}(\ell^{2j}-1)(\ell^{2m+2-2j}-1)\frac{B_{2j}}{(2j)!}\frac{B_{2m+2-2j}}{(2m+2-2j)!}\alpha^{m+1-j}\beta^j, \tag{1.10}$$

where

$$\mathcal{H}_m(x) = (4x)^{-m}\left(\frac{\ell-1}{2}L(\chi_1, 2m+1) - \sum_{n=1}^{\infty}a_n\left(\sum_{d|n}\frac{\chi_1(d)}{d^{2m+1}}\right)e^{-2nx}\right),$$

where $\alpha, \beta > 0$ such that $\alpha\beta = \frac{\pi^2}{\ell^2}$, $\chi_1$ denotes the principal Dirichlet character modulo $\ell$.

In the second notebook [23, p. 171, Ch. 14, Entry 19(iv),19(v)] Ramanujan provided the following partial fraction decompositions:

$$\frac{\pi}{4}\sec\left(\frac{\pi x}{2}\right)\operatorname{sech}\left(\frac{\pi y}{2}\right) = \sum_{l=0}^{\infty}(-1)^l(2l+1)\left\{\frac{\operatorname{sech}\left(\frac{(2l+1)\pi x}{2y}\right)}{(2l+1)^2+y^2} + \frac{\operatorname{sech}\left(\frac{(2l+1)\pi y}{2x}\right)}{(2l+1)^2-x^2}\right\}, \tag{1.11}$$

$$\frac{\pi}{4}\cot\left(\frac{\pi x}{2}\right)\operatorname{sech}\left(\frac{\pi y}{2}\right) = \frac{1}{2x} - y\sum_{l=0}^{\infty}(-1)^l\frac{\coth\frac{(2l+1)\pi x}{2y}}{(2l+1)^2+y^2} - x\sum_{l=1}^{\infty}\frac{\operatorname{sech}\frac{l\pi y}{x}}{(2l)^2-x^2}. \tag{1.12}$$

In this thesis, we analyse these two partial fractional decompositions.

# CHAPTER 2

## Preliminaries from Number Theory

### 2.1 Dirichlet characters

Let $N, q \in \mathbb{N}$. A *Dirichlet character modulo* $q$ is a homomorphism $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ satisfying the following properties:

$(i)\chi(a+q) = \chi(a)$, $\forall\ a \in \mathbb{N}$,

$(ii)\chi(ab) = \chi(a)\chi(b)$, $\forall\ a, b \in \mathbb{N}$.

The character $\chi_1 = \chi_{1,q}$ defined by

$$\chi_1(m) = \begin{cases} 1, & if\ (m,q) = 1, \\ 0, & elsewhere, \end{cases} \tag{2.1}$$

is known as the *principal character modulo* $q$. The *conductor* $c_\chi$ of a Dirichlet character $\chi$ is the least positive integer $c$ such that $\chi$ can be generated from $(\mathbb{Z}/c\mathbb{Z})^*$. We say a character is *primitive* if its conductor is equal to its period. If $\chi(-1) = 1$ we say character $\chi$ is *even* character and if $\chi(-1) = -1$ we say it to be of *odd* character.

## 2.2 Bernoulli polynomials

**Definition 2.1.** *The generating function for the Bernoulli polynomial $B_m(x)$ is defined as*

$$\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}.$$

The *mth* Bernoulli number is given by $B_m(0) = B_m$.

**Remark 2.1.** $B_{2m+1} = 0$ *for all* $m \in \mathbb{N}$.

**Definition 2.2.** *A character analogue of Bernoulli number $B_{m,\chi}$ is defined by*

$$\sum_{n=1}^{q} \chi(n)\frac{te^{nt}}{e^{qt} - 1} = \sum_{m=0}^{\infty} B_{m,\chi}\frac{t^m}{m!}, \tag{2.2}$$

*where $\chi$ is a primitive character modulo $q$.*

Note that if $\chi$ is odd, $B_{n,\chi} = 0$ for every even $n$. In general, $B_{n,\chi}$=0, for $n \equiv 1$ modulo 2, with the single exception of $B_{1,1} = \frac{1}{2}$.

**Definition 2.3.** *The sequence of Euler polynomials $E_m(x)$ satisfy the following generating function*

$$\frac{te^{xt}}{e^t + 1} = \sum_{m=0}^{\infty} E_m(x)\frac{t^m}{m!} \tag{2.3}$$

*The mth Euler number $E_m$ is defined as $2^n E_m(1/2)$.*

**Remark 2.2.** *It can be shown that $E_{2m+1} = 0$ for all $m \in \mathbb{N} \cup \{0\}$.*

## 2.3 Gamma function

**Definition 2.4.** *Let $s \in \mathbb{C}$. This function is defined as*

$$\Gamma(s) := \int_0^{\infty} x^{s-1}e^{-x}\,dx, \quad \Re(s) > 0. \tag{2.4}$$

**Remark 2.3.** $\Gamma(s + 1) = s\Gamma(s)$.

**Proposition 2.1.** $\Gamma(s)$ *is never zero for any* $s \in \mathbb{C}$.

**Proposition 2.2.** *The singularities of* $\Gamma(s)$ *occurs at negative integers including* 0 *and are simple in nature. Let* $m \in \mathbb{N} \cup \{0\}$*, then the residue at* $-m$ *is* $\frac{(-1)^m}{m!}$.

**Lemma 2.3.** *The Gamma function satisfies the below property*

$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}} \Gamma(s) \Gamma\left(s + \frac{1}{2}\right). \tag{2.5}$$

## 2.4 Mellin Transform

**Definition 2.5.** *The Mellin transform of* $g(t)$ *is defined by*

$$M(g; s) := \int_0^\infty g(t) t^{s-1} dt.$$

If $|t^{s-1} g(t)| = t^{\sigma-1}|g(t)|$ is integrable on $(0, \infty)$ for $c < \sigma < d$, then $M(g; s)$ can be defined and is absolutely convergent on $c < \Re(s) < d$ and uniformly on its compact subsets. Thus, it is holomorphic in $c < \Re(s) < d$.

**Proposition 2.4.** *The Gamma function is the Mellin transform of* $e^{-t}$*, that is,*

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \quad \textit{for} \quad \Re(s) > 0.$$

### 2.4.1 Inverse Mellin transform

**Definition 2.6.** *If* $M(g, s)$ *is analytic in the strip* $c < \Re(s) < d$*, and if for any* $c < \Re(s) = a < d$ *as* $|\Im(s)| \to \infty$*,* $M(g, s)$ *tends to zero uniformly. Then, the inverse Mellin transform is defined as*

$$g(t) := \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} M(g; s) t^{-s} ds.$$

**Example 2.1.** *For* $\Re(t) > 0$*, one has*

$$e^{-t} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(s) t^{-s}\, ds, \quad \textit{for any} \quad a > 0.$$

# CHAPTER 3

## Well known results on $\zeta(s)$ and $L(s,\chi)$

One can easily show that the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ converges uniformly and absolutely if $\sigma \geq \sigma_0 > 1$, where $s = \sigma + it$. By Weierstrass's theorem, $\zeta(s)$ is holomorphic for $\sigma > 1$.

**Definition 3.1.** *Let $\chi(m)$ be a character modulo $q$. Then, for $\Re(s) > 1$, the Dirichlet L-function is defined by*

$$L(s,\chi) := \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}. \tag{3.1}$$

Since, the character $\chi(m)$ is bounded by 1, the above $L$-function converges absolutely and uniformly if $\sigma \geq \sigma_0 > 1$. Therefore, $L(s,\chi)$ is holomorphic for $\sigma > 1$.

## 3.1 Functional equation

Riemann proved that $\zeta(s)$ satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin\left(\frac{\pi s}{2}\right) \zeta(1-s). \tag{3.2}$$

The Gauss sum associated to the character $\chi$ modulo $q$ is defined as

$$\tau(\chi) := \sum_{r=1}^{q} \chi(r) e^{\frac{2\pi i r}{q}}. \tag{3.3}$$

Let

$$a := \frac{1-\chi(-1)}{2} = \begin{cases} 0, \textit{if } \chi \textit{ is even}, \\ 1, \textit{if } \chi \textit{ is odd}. \end{cases} \tag{3.4}$$

Now if $\chi$ is any primitive Dirichlet character modulo $q$, then for any $s \in \mathbb{C}$,

$$L(s,\chi) = \xi(\chi) 2^s \pi^{s-1} q^{\frac{1}{2}-s} \sin\left(\frac{\pi}{2}(s+a)\right) \Gamma(1-s) L(1-s,\bar{\chi}), \tag{3.5}$$

where

$$\xi(\chi) = \frac{\tau(\chi)}{i^a \sqrt{q}} \tag{3.6}$$

is an algebraic number of absolute value 1.

## 3.2 Important properties of $\zeta(s)$ and Ł$(s,\chi)$

**Theorem 3.1.** *$\zeta(s)$ is never zero for $\Re(s) > 1$.*

**Proposition 3.2.** *For $l \in \mathbb{N} \cup \{0\}$,*

$$\zeta(l) = (-1)^l \frac{B_{l+1}}{l+1}. \tag{3.7}$$

*Since $B_{2l+1} = 0$ for all $l \in \mathbb{N}$, $\zeta(s)$ vanishes at $-2l$. These are known as the trivial zeros of $\zeta(s)$.*

**Proposition 3.3.** *Let $\chi$ be a primitive character modulo $q$, with $q > 1$.*

*(i) If $\chi$ is even primitive character, then $\{-2n | n \in \mathbb{N} \cup \{0\}\}$ are the trivial zeros of $L(s,\chi)$.*

*(ii) If $\chi$ is odd primitive character, then $\{-(2n+1) | n \in \mathbb{N} \cup \{0\}\}$ are the trivial zeros of $L(s,\chi)$.*

*Note that all of these trivial zeros are simple.*

**Lemma 3.4.** *Let $s = \sigma + iR \in \mathbb{C}$. Then for any $\sigma_0 \leq \sigma \leq b$, $\exists$ a constant $A(\sigma_0)$, such that*

$$|L(s,\chi)| \ll |R|^{A(\sigma_0)} \tag{3.8}$$

*as $|R| \to \infty$.*

*Proof.* One can find a proof of this result in [18, p. 97, Lemma 5.2] $\square$

# CHAPTER 4

## Main Results

With all the prerequisites mentioned in the earlier chapter, now we are ready to state our main results of the thesis.

The partial fraction decomposition (1.11) produces the following interesting identity.

**Theorem 4.1.** *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. For any $n \in \mathbb{Z}$, we have*

$$\mathcal{F}_n(\alpha) = (-1)^n \mathcal{F}_n(\beta) + \sum_{l=0}^{n-1} (-1)^{l+1-n} \frac{E_{2l}}{(2l)!} \frac{E_{2n-2l-2}}{(2n-2l-2)!} \alpha^l \beta^{n-l-1}, \tag{4.1}$$

*where*

$$\mathcal{F}_n(x) = x^{1-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^{2n-1}\left(e^{\frac{(2m+1)x}{2}} + e^{\frac{-(2m+1)x}{2}}\right)} \frac{2^{2n+1}}{\pi}.$$

**Remark 4.1.** *This identity has been mentioned by Ramanujan in his lost notebook [23, p. 173, Ch. 14, Entry 21(ii)]. Malurkar [20] first proved this identity and later by Nanjundiah [22], and Bruce berndt [7] also gave an alternative proof for this. Note that, in the above identity, for non positive integer $n$, one should* *consider the finite sum involving Euler number's as an empty sum.*

**Remark 4.2.** *Theorem* 4.1 *can also be written in the following form:*

$$\alpha^{1-n}\sum_{m=0}^{\infty}\frac{\chi_4(m)\,\mathrm{sech}\left(\frac{m\alpha}{2}\right)}{m^{2n-1}}\frac{2^{2n}}{\pi}+(-\beta)^{1-n}\sum_{m=0}^{\infty}\frac{\chi_4(m)\,\mathrm{sech}\left(\frac{m\beta}{2}\right)}{m^{2n-1}}\frac{2^{2n}}{\pi}$$
$$=\sum_{l=0}^{n-1}(-1)^{l+1-n}\frac{E_{2l}}{(2l)!}\frac{E_{2n-2l-2}}{(2n-2l-2)!}\alpha^{l}\beta^{n-l-1}, \tag{4.2}$$

*where* $\chi_4$ *denote primitive Dirichlet character modulo* 4.

An application of (4.2) provides an exact evaluation for some infinite series related to the sec-hyperbolic function.

**Corollary 4.2.** *For any* $n\in\mathbb{N}$*, we have*

$$\sum_{m=0}^{\infty}\frac{\chi_4(m)\,\mathrm{sech}\left(\frac{m\pi}{2}\right)}{m^{4n+1}}=\sum_{k=0}^{\infty}\frac{(-1)^k\,\mathrm{sech}\left((2k+1)\frac{\pi}{2}\right)}{(2k+1)^{4n+1}}$$
$$=\frac{\pi^{4n+1}}{2^{4n+3}}\sum_{l=0}^{2n}(-1)^{l}\frac{E_{2l}}{(2l)!}\frac{E_{4n-2l}}{(4n-2l)!}. \tag{4.3}$$

For $n=0$, $n=1$ and $n=2$ of (4.3) we get special values that are mentioned in [23, p. 180] as Entry 25 (*vii*), (*viii*) and (*ix*).

The partial fraction decomposition (1.12) gives us an interesting identity.

**Theorem 4.3.** *Let* $\alpha,\beta>0$ *be such that* $\alpha\beta=\pi^2$*. For* $n\in\mathbb{N}\cup\{0\}$*, we have*

$$\frac{\sqrt{\alpha}}{\alpha^n}\left\{\frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{2n}}+\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{2n}\left(e^{(2m+1)\alpha}-1\right)}\right\}$$
$$=\frac{\sqrt{\beta}}{\beta^n}(-1)^n\left\{\sum_{m=1}^{\infty}\frac{1}{(2m)^{2n}\left(e^{m\beta}+e^{-m\beta}\right)}\right\}$$
$$+\frac{\sqrt{\beta}}{\beta^n}(-1)^n\left\{\frac{1}{4}\sum_{m=0}^{n}(-1)^m\frac{B_{2m}}{(2m)!}\frac{E_{2n-2m}}{(2n-2m)!}(2\alpha)^m\left(\frac{\beta}{2}\right)^{2n-m}\right\}. \tag{4.4}$$

**Remark 4.3.** *An alternate form of the above theorem is the following:*

$$\begin{aligned}\alpha^{-n+\frac{1}{2}}&\left(\frac{1}{2}L(2n,\chi_4)+\sum_{m=1}^{\infty}\frac{\chi_4(m)}{m^{2n}(e^{\alpha m}-1)}\right)\\&=\frac{(-1)^n\beta^{-n+\frac{1}{2}}}{2^{2n+1}}\sum_{m=1}^{\infty}\frac{1}{m^{2n}\cosh\beta m}\\&+\frac{1}{4}\sum_{m=0}^{n}\frac{(-1)^m}{2^{2m}}\frac{E_{2m}}{(2m)!}\frac{B_{2n-2m}}{(2n-2m)!}\alpha^{n-m}\beta^{m+\frac{1}{2}},\end{aligned}\tag{4.5}$$

*where $\chi_4$ is primitive character modulo 4.*

**Remark 4.4.** *Chowla [12, Eq. (1.2)] gave the first published proof for the Theorem* 4.3*. Berndt [7, Eq. (3.20)] also gave a proof but unfortunately there was an error in the Equation (3.20); replacing $\left(\frac{\beta}{8}\right)^k$ by $\frac{\beta^{k+\frac{1}{2}}}{2^{-4k}}$ at the end of* (3.20)*, we can obtain Theorem* (4.3)*.*

Quite interestingly, we observe that Theorem 4.1 can also be obtained using contour integration technique and during the process we derive a more generalized form of it. Here we give the generalized version of Theorem 4.1.

**Theorem 4.4.** *Let $q \in \mathbb{N}$ and $\alpha,\beta > 0$ with $\alpha\beta = \frac{4\pi^2}{q^2}$. Let $\chi$ be the primitive Dirichlet character modulo $q$. For any $r \in \mathbb{Z}$ , we have*

$$\begin{aligned}\alpha^{-r}\sum_{n=1}^{\infty}\chi(n)\sigma_{-1-2r}(n)e^{-n\alpha}-(-1)^{r+a}\xi^2(\chi)\beta^{-r}\sum_{n=1}^{\infty}\bar{\chi}(n)\sigma_{-1-2r}(n)e^{-n\beta}\\=\sum_{l=0}^{2r+1}\frac{(-1)^l}{l!}L(-l,\chi)L(2r+1-l,\chi)\alpha^{l-r},\end{aligned}\tag{4.6}$$

*where $a = 0$ or $1$ accordingly as $\chi$ is even or odd.*

**Remark 4.5.** *When $r$ is a negative integer, the finite sum in the right hand side of* (4.6) *must be considered as an empty sum.*

Depending on the nature of the primitive Dirichlet character $\chi$, we derive interesting corollaries of Theorem 4.4.

**Corollary 4.5.** *If $\chi$ is even, then we have*

$$\alpha^{1-r}\sum_{n=1}^{\infty}\chi(n)\sigma_{1-2r}(n)e^{-n\alpha}+(-1)^r\xi^2(\chi)\beta^{1-r}\sum_{n=1}^{\infty}\bar{\chi}(n)\sigma_{1-2r}(n)e^{-n\beta}$$
$$=\frac{\tau(\chi)}{2}(-1)^{r-1}\sum_{l=0}^{r-1}(-1)^l\frac{B_{2l+2,\chi}}{(2l+2)!}\frac{B_{2r-2l,\chi}}{(2r-2l)!}\alpha^{l+2}\beta^{r-l}, \tag{4.7}$$

where $B_{n,\chi}$ denotes as the generalized Bernoulli number defined in (2.2).

**Corollary 4.6.** *If $\chi$ is odd, then we have*

$$\alpha^{1-r}\sum_{n=1}^{\infty}\chi(n)\sigma_{1-2r}(n)e^{-n\alpha}-(-1)^r\xi^2(\chi)\beta^{1-r}\sum_{n=1}^{\infty}\bar{\chi}(n)\sigma_{1-2r}(n)e^{-n\beta}$$
$$=\frac{\tau(\chi)}{2}(-1)^{r-1}\left(\frac{q}{2\pi i}\right)\sum_{l=0}^{r-1}(-1)^l\frac{B_{2l+1,\chi}}{(2l+1)!}\frac{B_{2r-2l-1,\chi}}{(2r-2l-1)!}\alpha^{l+1}\beta^{r-l}. \tag{4.8}$$

**Remark 4.6.** *In particular, when $\chi$ is a non principal character modulo* 4, *we can derive Theorem* 4.1.

# CHAPTER 5

# Preliminaries

In this chapter, we present a few results that will be needed in deducing our main conclusions. First, we state a result which gives behaviour of $\Gamma(s)$ as $|s| \to \infty$. For $s = c + iR$ with $\alpha \leq c \leq \beta$,

$$|\Gamma(c+iR)| = \sqrt{2\pi}|R|^{c-1/2}e^{-\pi|R|/2}\left(1 + \mathcal{O}\left(\frac{1}{|R|}\right)\right), \quad \text{as } |R| \to \infty. \tag{5.1}$$

One can find this result in [17, p. 492, A.7 (A.34)].

**Lemma 5.1.** *For any $r \in \mathbb{Z}$, one has*

$$\nabla(s,\chi) = (-1)^{r+a}\xi^2(\chi)\nabla(-s-2r,\bar{\chi}), \tag{5.2}$$

*where $\nabla(s,\chi) = \Gamma(s)L(s,\chi)L(s+2r+1,\chi)$.*

*Proof.* Employing the functional equation (3.5) for $L(s,\chi)$, and the identity (2.3), Lemma (5.1) can be obtained.

□

**Lemma 5.2.** *Let $\chi$ be any primitive Dirichlet character modulo $q$. Then for*

$\Re(s) > \max\{1, -2r\}$, *one has*

$$\sum_{m=1}^{\infty} \frac{\chi(m)\sigma_{-1-2r}(m)}{m^s} = L(s,\chi)L(s+1+2r,\chi). \tag{5.3}$$

*Proof.* For $\Re(s) > \max\{1, -2r\}$, we can write

$$\begin{aligned} L(s,\chi)L(s+1+2r,\chi) &= \left(\sum_{m_1}^{\infty} \frac{\chi(m_1)}{m_1^{s+1+2r}}\right)\left(\sum_{m_2}^{\infty} \frac{\chi(m_2)}{m_2^s}\right) \\ &= \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} \sum_{d|m} \frac{1}{d^{2r+1}} \\ &= \sum_{m=1}^{\infty} \frac{\chi(m)\sigma_{-1-2r}(m)}{m^s}. \end{aligned} \tag{5.4}$$

In the second last step, we have used completely multiplicative nature of the Dirichlet character. □

**Lemma 5.3.** *Let $n \in \mathbb{N}$ and $\chi$ be a character of conductor $L$ with $\chi(-1) = (-1)^n$. Then we have*

$$\begin{aligned} L(n,\chi) &= (-1)^{n-1}\frac{\tau(\chi)}{2}\left(\frac{2\pi i}{L}\right)^n \frac{B_{n,\bar{\chi}}}{n!} \\ L(1-n,\chi) &= -\frac{B_{n,\chi}}{n}. \end{aligned} \tag{5.5}$$

*Proof.* Proof of it can be found in [21]. □

For a special case of $\chi$ we have an interesting identity involving Euler numbers which we state below.

**Lemma 5.4.** *If $\chi_4$ denote the primitive Dirichlet character modulo 4, then we have*

$$\frac{1}{2}\frac{E_{2m}}{(2m)!} = -\frac{B_{2m+1,\chi_4}}{(2m+1)!}, \tag{5.6}$$

*where $E_m$ and $B_{n,\chi}$ denote as the Euler number and generalized Bernoulli number, respectively.*

*Proof.* Proof of it can be found in [21]. □

# CHAPTER 6

## Proof of main results

*Proof of Theorem* 4.1. Let $\alpha, \beta > 0$ satisfying $\alpha\beta = \pi^2$. Substituting $\frac{\pi x}{2} = \sqrt{w\alpha}$ and $\frac{\pi y}{2} = \sqrt{w\beta}$ in (1.11), we get

$$\frac{\pi}{4}\sec\left(\sqrt{w\alpha}\right)\operatorname{sech}\left(\sqrt{w\beta}\right) = \sum_{l=0}^{\infty}(-1)^l(2l+1)\left\{\frac{\alpha\operatorname{sech}\left(\frac{(2l+1)\alpha}{2}\right)}{\alpha(2l+1)^2+w} + \frac{\beta\operatorname{sech}\left(\frac{(2l+1)\beta}{2}\right)}{\beta(2l+1)^2-w}\right\}. \tag{6.1}$$

Now using the definition of sech, we can write

$$\begin{aligned}
&\sum_{l=0}^{\infty}\frac{(-1)^l(2l+1)\alpha}{\alpha(2l+1)^2+4w}\operatorname{sech}\left(\frac{(2l+1)\alpha}{2}\right)\\
&=\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l+1)}\frac{1}{\left(1+\frac{4w}{\alpha(2l+1)^2}\right)}\left\{\frac{2}{e^{\frac{(2l+1)\alpha}{2}}+e^{\frac{-(2l+1)\alpha}{2}}}\right\}\\
&=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^l}{(2l+1)}\left(\frac{-4w}{\alpha(2l+1)^2}\right)^m\left\{\frac{2}{e^{\frac{(2l+1)\alpha}{2}}+e^{\frac{-(2l+1)\alpha}{2}}}\right\},
\end{aligned} \tag{6.2}$$

for $|w| < \frac{\alpha}{4}$. Similarly, $|w| < \frac{\beta}{4}$, one can show that

$$\sum_{l=0}^{\infty} \frac{(-1)^l (2l+1)\beta}{\beta(2l+1)^2 - 4w} \operatorname{sech}\left(\frac{(2l+1)\beta}{2}\right)$$
$$= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^l}{(2l+1)} \left(\frac{4w}{\beta(2l+1)^2}\right)^m \left\{\frac{2}{e^{\frac{(2l+1)\beta}{2}} + e^{\frac{-(2l+1)\beta}{2}}}\right\}. \tag{6.3}$$

In view of (6.2) and (6.3), we collect the coefficient of $w^n$ from the right hand side of (6.1):

$$2^{2n+1}(-\alpha)^{-n} \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)^{2n+1}} \left\{\frac{1}{e^{\frac{(2n+1)\alpha}{2}} + e^{\frac{-(2n+1)\alpha}{2}}}\right\}$$
$$+ 2^{2n+1}\beta^{-n} \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)^{2n+1}} \left\{\frac{1}{e^{\frac{(2n+1)\beta}{2}} + e^{\frac{-(2n+1)\beta}{2}}}\right\}. \tag{6.4}$$

Now let us look at the expansion of the Laurent series for $\sec z$ and $\operatorname{sech} z$ around $z = 0$. For $0 < |z| < \frac{\pi}{2}$,

$$\operatorname{sech} z = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!} = \sum_{n=0}^{\infty} E_{2n} \frac{z^{2n}}{(2n)!},$$
$$\sec z = \operatorname{sech} iz = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{z^{2n}}{(2n)!}.$$

Use the above series expansions in the left-hand side of (6.1), then it becomes

$$\frac{\pi}{4} \left(\sum_{m=0}^{\infty} (-1)^m \frac{E_{2m}}{(2m)!} w^m \alpha^m\right) \left(\sum_{m=0}^{\infty} \frac{E_{2m}}{(2m)!} w^m \beta^m\right) \tag{6.5}$$
$$= \frac{\pi}{4} \sum_{m=0}^{\infty} \sum_{l=0}^{m} (-1)^l \frac{E_{2l}}{(2l)!} \frac{E_{2m-2l}}{(2m-2l)!} \alpha^l \beta^{m-l} w^m. \tag{6.6}$$

Now equating coefficients of $w^n$ in (6.4) and (6.6), we arrive at

$$2^{2n+1}(-\alpha)^{-n} \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)^{2n+1}} \left\{\frac{1}{e^{\frac{(2n+1)\alpha}{2}} + e^{\frac{-(2n+1)\alpha}{2}}}\right\}$$
$$+ 2^{2n+1}\beta^{-n} \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)^{2n+1}} \left\{\frac{1}{e^{\frac{(2n+1)\beta}{2}} + e^{\frac{-(2n+1)\beta}{2}}}\right\}$$
$$= \frac{\pi}{4} \sum_{l=0}^{n} (-1)^k \frac{E_{2l}}{(2l)!} \frac{E_{2n-2l}}{(2n-2l)!} \alpha^l \beta^{n-l}. \tag{6.7}$$

After simplifying one can derive Theorem 4.1. □

*Proof of Corollary* 4.3. Putting $\alpha = \pi = \beta$ and substituting $n$ by $2n+1$ in Theorem 4.1, one can obtain (4.3). □

*Proof of Theorem* 4.3. Replacing $\frac{\pi x}{2} = \sqrt{w\alpha}$ and $\frac{\pi y}{2} = \sqrt{w\beta}$ with $\alpha\beta = \pi^2$ in (1.12), we get

$$\frac{\pi}{4}\cot\left(\sqrt{w\alpha}\right)\operatorname{sech}\left(\sqrt{w\beta}\right) = \frac{\pi}{4\sqrt{w\alpha}} - \frac{2\sqrt{w\beta}}{\pi}\sum_{l=0}^{\infty}(-1)^l \frac{\coth\frac{(2l+1)\alpha}{2}}{(2l+1)^2 + \frac{4w\beta}{\pi^2}}$$
$$- \frac{2\sqrt{w\alpha}}{\pi}\sum_{l=1}^{\infty}\frac{\operatorname{sech} l\beta}{(2l)^2 - \frac{4w\alpha}{\pi^2}}. \tag{6.8}$$

For $|w| < \frac{\alpha}{4}$, we can write

$$\frac{2\sqrt{w\beta}}{\pi}\sum_{l=0}^{\infty}(-1)^l \frac{\coth\frac{(2l+1)\alpha}{2}}{(2l+1)^2 + \frac{4w\beta}{\pi^2}}$$
$$= \frac{2\sqrt{w\beta}}{\pi}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^l}{(2l+1)^2}\left(\frac{-4w}{(2l+1)^2\alpha}\right)^m\left(1 + \frac{2}{e^{(2l+1)\alpha} - 1}\right). \tag{6.9}$$

Similarly, for $|w| < \beta$, we can write

$$\frac{2\sqrt{w\alpha}}{\pi}\sum_{l=1}^{\infty}\frac{\operatorname{sech} l\beta}{(2l)^2 - \frac{4w\alpha}{\pi^2}} = \frac{2\sqrt{w\alpha}}{\pi}\sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(2l)^2}\left(\frac{4w}{(2l)^2\beta}\right)^m\left(\frac{2}{e^{l\beta} + e^{-l\beta}}\right). \tag{6.10}$$

In view of (6.9) and (6.10), the coefficient of $w^{n-\frac{1}{2}}$ for the right-hand side expression of (6.8) will be

$$2^{2n}\alpha^{\frac{1}{2}-n}(-1)^n\left\{\frac{1}{2}\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l+1)^{2n}} + \sum_{l=0}^{\infty}\frac{(-1)^l}{(2l+1)^{2n}\left(e^{(2l+1)\alpha} - 1\right)}\right\}$$
$$- 2^{2n}\beta^{\frac{1}{2}-n}\left\{\sum_{l=1}^{\infty}\frac{1}{(2l)^{2n}\left(e^{l\beta} + e^{-l\beta}\right)}\right\}. \tag{6.11}$$

Now let us look at the Laurent series expansions for $\cot z$ and $\operatorname{sech} z$ around $z = 0$. For $0 < |z| < \frac{\pi}{2}$,

$$\cot z = \sum_{n=0}^{\infty}(-1)^n\frac{B_{2n}}{(2n)!}2^{2n}z^{2n-1},$$
$$\operatorname{sech} z = \sum_{n=0}^{\infty}E_n\frac{z^n}{n!} = \sum_{n=0}^{\infty}E_{2n}\frac{z^{2n}}{(2n)!}. \tag{6.12}$$

Use above series expansions to see that

$$
\begin{aligned}
&\frac{\pi}{4}\cot\left(\sqrt{w\alpha}\right)\operatorname{sech}\left(\sqrt{w\beta}\right)\\
&=\frac{\pi}{4}\left(\sum_{i=0}^{\infty}(-1)^i\frac{B_{2i}}{(2i)!}2^{2i}w^{i-\frac{1}{2}}\alpha^{i-\frac{1}{2}}\right)\left(\sum_{n=0}^{\infty}\frac{E_{2n}}{(2n)!}w^{n-\frac{1}{2}}\beta^{n-\frac{1}{2}}\right).\\
&=\frac{\pi}{4}\sum_{m=0}^{\infty}\sum_{l=0}^{m}(-1)^l2^{2l}\frac{B_{2l}}{(2l)!}\frac{E_{2m-2l}}{(2m-2l)!}w^{m-\frac{1}{2}}\alpha^{m-\frac{1}{2}}\beta^{m-l}. \qquad (6.13)
\end{aligned}
$$

Now equating coefficients of $w^{n-\frac{1}{2}}$ in (6.11) and (6.13), we arrive at

$$
\begin{aligned}
&\frac{\pi}{4}\sum_{l=0}^{n}(-1)^l2^{2l}\frac{B_{2l}}{(2l)!}\frac{E_{2n-2l}}{(2n-2l)!}\alpha^{l-\frac{1}{2}}\beta^{n-l}\\
&=2^{2n}\alpha^{\frac{1}{2}-n}(-1)^n\left\{\frac{1}{2}\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l+1)^{2n}}+\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l+1)^{2n}\left(e^{(2l+1)\alpha}-1\right)}\right\}\\
&-2^{2n}\beta^{\frac{1}{2}-n}\left\{\sum_{l=1}^{\infty}\frac{1}{(2l)^{2n}\left(e^{l\beta}+e^{-l\beta}\right)}\right\}.
\end{aligned}
$$

Simplifying further, we complete the proof of Theorem 4.3. $\square$

*Proof of Theorem* 4.4 . We use (2.1) and Lemma 5.2 to see that, for $\Re(s)=\gamma>\max\{1,-2r\}$,

$$
\begin{aligned}
\sum_{l=1}^{\infty}\chi(l)\sigma_{-1-2r}(l)e^{-l\alpha}&=\sum_{l=1}^{\infty}\chi(l)\sigma_{-1-2r}(l)\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s)(l\alpha)^{-s}ds\\
&=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s)\sum_{l=1}^{\infty}\frac{\chi(l)\sigma_{-1-2r}(l)}{l^s}\alpha^{-s}ds\\
&=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s)L(s,\chi)L(s+2r+1,\chi)\alpha^{-s}ds. \qquad (6.14)
\end{aligned}
$$

Now we have to calculate the following line integral:

$$
I_k:=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s)L(s,\chi)L(s+2r+1,\chi)\alpha^{-s}ds.
$$

We define our integrand function as

$$
g(s):=\Gamma(s)L(s,\chi)L(s+2r+1,\chi)\alpha^{-s}.
$$

First, we shall look for the poles of this integrand function. We know about the trivial zeros of $L(s,\chi)$ from Proposition (3.3). Hence in general, $L(s,\chi)$ has simple zeros at $-2n-a$, $n\in\mathbb{N}\cup\{0\}$, where $a=0$ and $a=1$ according to even and odd character, respectively. Now let us build an appropriate rectangular contour $\mathfrak{C}$

comprising of the vertices $\gamma - iP$, $\gamma + iP$, $\lambda - iP$, $\lambda + iP$ with positive orientation. Here $P > 0$ is chosen to be a big positive number and $\gamma > \max\{1, -2r\}$, and $\lambda$ is cleverly selected in the open interval $\min\{-1, -2r-2\} < \lambda < \min\{0, -2r-1\}$ in order to get all the poles of $g(s)$ inside the contour $\mathfrak{C}$ . Now making use of the Cauchy's residue theorem, we observe that

$$\frac{1}{2\pi i}\left(\int_{\gamma-iP}^{\gamma+iP} + \int_{\gamma+iP}^{\lambda+iP} + \int_{\lambda+iP}^{\lambda-iP} + \int_{\lambda-iP}^{\gamma-iP}\right) g(s)ds = \sum_{l=1}^{r} R_{-l} \tag{6.15}$$

where $R_p$ denotes the residue at the pole $s = p$. We now assess the residual term $R_{p-l}$, i.e.,

$$R_{-l} = \frac{(-1)^l}{l!} L(-l,\chi)L(2r-l+1,\chi)\alpha^l. \tag{6.16}$$

Hence, summing all the residues, we get

$$R = \sum_{l=0}^{2r+1} \frac{(-1)^l}{l!} L(-l,\chi)L(2r-l+1,\chi)\alpha^l.$$

Now using Stirling's formula (5.1) for $\Gamma(s)$ and estimate (3.4) for $L(s,\chi)$, the contribution of the horizontal integrations can be shown to be vanishing as $P$ goes to infinity. Thus, letting $P \to \infty$ in (6.15), we arrive at

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} g(s)ds = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} g(s)ds + R. \tag{6.17}$$

Now we would like to simplify the integral

$$J_k := \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} \Gamma(s)L(s,\chi)L(s+2r+1,\chi)\alpha^{-s}ds. \tag{6.18}$$

Applying Lemma 5.1, we obtain

$$J_k = \frac{1}{2\pi i}\int_{(\lambda)} (-1)^{r+a}\xi^2(\chi)\left(\frac{2\pi}{q}\right)^{2s+2r} \Gamma(-s-2r)L(1-s,\bar{\chi})L(-s-2r,\bar{\chi})\alpha^{-s}ds, \tag{6.19}$$

where $(\gamma)$ denotes the straight path from $\gamma - i\infty$ to $\gamma + i\infty$. To write this integral in form of an infinite series, replace $s$ by $-s-2r$, and we obtain

$$\begin{aligned} J_k &= (-1)^{r+a}\xi^2(\chi)\left(\frac{4\pi^2}{(\alpha q)^2}\right)^{-r} \frac{1}{2\pi i}\int_{(-\lambda-2r)} \Gamma(s)L(s+2r+1,\bar{\chi})L(s,\bar{\chi})\left(\frac{4\pi^2}{\alpha q^2}\right)^{-s} ds \\ &= (-1)^{r+a}\xi^2(\chi)\left(\frac{q^2\alpha^2}{4\pi^2}\right)^{r} \sum_{n=1}^{\infty} \bar{\chi}(n)\sigma_{-1-2r}(n)e^{\frac{-n4\pi^2}{\alpha q^2}}. \end{aligned} \tag{6.20}$$

To obtain the last equality, we have used (6.14) and the fact that $-\lambda - 2r >$

$\max\{1, -2r\}$ as we have considered $\lambda < \min\{0, -2r-1\}$. Substituting $\alpha\beta = \frac{4\pi^2}{q^2}$, we get

$$J_k = (-1)^{r+a}\xi^2(\chi)\left(\frac{\alpha}{\beta}\right)^r \sum_{n=1}^{\infty} \bar{\chi}(n)\sigma_{-1-2r}(n)e^{-n\beta}. \tag{6.21}$$

Substituting (6.21) in (6.17) and together with (6.14), and collecting residue terms the result follows. □

*Proof of Corollary* 4.5 . If $\chi$ is an even character modulo $q$, then substituting $a = 0$ in Theorem 4.4, we get

$$\begin{aligned} \alpha^{1-r}\sum_{l=1}^{\infty}\chi(l)\sigma_{1-2r}(l)e^{-l\alpha} + (-1)^r\xi^2(\chi)\beta^{1-r}\sum_{l=1}^{\infty}\bar{\chi}(l)\sigma_{1-2r}(l)e^{-l\beta} \\ = -\sum_{l=0}^{r-1}\frac{1}{(2l+1)!}L(-(2l+1),\chi)L(2r-2l,\chi)\alpha^{2l+2-2r}, \end{aligned} \tag{6.22}$$

Finally, using Lemma 5.3 and 5.4, and simplifying we get the desired result. □

*Proof of Corollary* 4.6. If $\chi$ is odd character modulo $q$, then substituting $a = 1$ in Theorem 4.4, we get

$$\begin{aligned} \alpha^{1-r}\sum_{l=1}^{\infty}\chi(l)\sigma_{1-2r}(l)e^{-l\alpha} - (-1)^r\xi^2(\chi)\beta^{1-r}\sum_{l=1}^{\infty}\bar{\chi}(l)\sigma_{1-2r}(l)e^{-l\beta} \\ = \sum_{l=0}^{r-1}\frac{1}{(2l)!}L(-2l,\chi)L(2r-2l-1,\chi)\alpha^{2l+1-r}. \end{aligned} \tag{6.23}$$

Finally, using Lemma 5.3 and 5.4 and simplifying we get the desired result. □

# CHAPTER 7

## Concluding Remarks

Table 7.1: Verification of Theorem 4.4: To obtain this numerical data, we used Mathematica software.

| $q$ | $r$ | $\chi_{x,q}$ | $\alpha$ | $\beta$ | Left-hand side | Right-hand side |
|---|---|---|---|---|---|---|
| 3 | 10 | $\chi_{2,3}$ | $\pi$ | $\frac{4\pi}{9}$ | 0.006715783766287... | 0.006715783766287.. |
| 3 | 10 | $\chi_{2,3}$ | $\frac{\pi}{2}$ | $\frac{8\pi}{9}$ | 0.001818862168468... | 0.001818862168468... |
| 3 | 10 | $\chi_{2,3}$ | 1 | $\frac{4\pi^2}{9}$ | 0.244728419776567.. | 0.244728419776568... |
| 3 | 20 | $\chi_{2,3}$ | $\pi$ | $\frac{4\pi}{9}$ | 0.002384485442980364... | 0.002384485442980369... |
| 4 | 10 | $\chi_{2,4}$ | $\pi$ | $\frac{\pi}{4}$ | 4.226525632874012... | 4.226525632874011... |
| 4 | 20 | $\chi_{2,4}$ | $\pi$ | $\frac{\pi}{4}$ | 47.32434875687574... | 47.32434875687568... |
| 5 | 10 | $\chi_{3,5}$ | $\pi$ | $\frac{4\pi}{25}$ | $-160.14199212537..$ | $-160.141992125369...$ |
| 5 | 20 | $\chi_{3,5}$ | $\pi$ | $\frac{4\pi}{25}$ | $-155527.4059713702...$ | $-155527.4059713692...$ |
| 5 | 10 | $\chi_{3,5}$ | 1 | $\frac{4\pi^2}{25}$ | 0.200811020360... | 0.2008110203605503... |

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