Discrete theory of sampling for Ramanujan spaces

M.Sc. Thesis

by

Asik Doctor

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY INDORE
JUNE 2023

Discrete theory of sampling for Ramanujan spaces

A THESIS

Submitted in partial fulfillment of the requirements for the award of the degree of

Master of Science

by

Asik Doctor

(Roll No.2003141001)

Under the guidance of

Dr. Niraj Kumar Shukla

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

June 2023

INDIAN INSTITUTE OF TECHNOLOGY INDORE

CANDIDATE'S DECLARATION

I, hereby, declare that the work which is being presented in the thesis entitled **"Discrete theory of sampling for Ramanujan space"** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF MATHEMATICS**, **Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2022 to June 2023 under the supervision of **Dr. Niraj Kumar Shukla**, Associate Professor, Department of Mathematics, IIT Indore. The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

Asik Doctor 02-06-2023

Signature of the student with date

(**Asik Doctor**)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

7/6/2023

Signature of Thesis Supervisor with date

(**Dr. Niraj Kumar Shukla**)

Asik Doctor has successfully given his M.Sc. Oral Examination held on 5th June, 2023.

7/6/2023

Signature of supervisor of M.Sc Thesis

Date:

Signature of Convener,DPGC

Date: 7 June 2023

Signature of PSPC Member

Prof. Sk. Safique Ahmad

Date: 07/06/2023

Signature of PSPC Member

Dr. Vijay Kumar Sohani

Date: 7 June 2023

Dedicated to my Family

We are what our thoughts have made us;

So take care about what you think.

Thoughts live; They travel far.

–***Swami Vivekananda***

Acknowledgements

I am incredibly grateful to my supervisor, Dr. Niraj Kumar Shukla, for his assistance and guidance throughout my entire M.Sc. course. He has shown deep concern for my overall well-being during these challenging times. His friendly nature, enthusiasm, and supportive attitude have encouraged me to successfully complete this thesis. His confidence and trust in me always push me to excel in every aspect. Additionally, he has instilled in me a spirit of adventure and propelled me towards my research interests. Working with him has been an honor, and I express my heartfelt thanks to him. I am grateful to the Almighty for bestowing blessings upon me. I extend my sincere gratitude to my PSPC members, Prof. Safique Ahmed, Dr. Vijay Kumar Sohani, as well as DPGC convener Dr. Vijay Kumar Sohani and HOD Dr. Niraj Kumar Shukla, for their continuous support. I am also thankful to all our teachers for their kind and valuable suggestions. I would like to thank department's research scholar, Sahil Kalra, for generously sharing his valuable time. His kind and supportive nature made my M.Sc. project journey smooth, and I have learned a lot under his mentorship.

This acknowledgment note would be incomplete without expressing gratitude to my family members. Their unwavering support and faith have steadily boosted my confidence from within. They have given me the strength to pursue my interests even in difficult times. Lastly, I would like to thank my friends for always being there for me. My heart is filled with gratitude towards each and every person who has supported me. I remain indebted and grateful to all of you.

Abstract

This thesis explores the application of discrete sampling in the Ramanujan space, a fundamental concept in signal processing. Discrete sampling involves extracting essential information from large discrete-time signals using a limited number of samples. The Ramanujan space, generated by complex exponential functions, offers unique properties for signal representation and analysis. The thesis focuses on developing a finite sampling approach using reproducing kernel Hilbert spaces (RKHS) to efficiently reconstruct signals in the Ramanujan space. The research investigates the theoretical foundations of discrete sampling in this context, presenting analyses and experimental results to validate the proposed method's accuracy and computational efficiency. The outcomes have implications in various signal processing domains such as communications, image/audio processing, and data compression. The organization of the thesis encompasses foundational concepts, an overview of the Ramanujan space, and the discrete sampling procedure utilizing RKHS methodology. This research contributes to the advancement of robust signal processing algorithms and efficient data manipulation techniques.

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CHAPTER 1

Introduction

The field of signal processing plays a crucial role in various scientific and technological domains, ranging from telecommunications to image and audio processing. One fundamental concept in signal processing is sampling, which involves converting continuous-time signals into discrete-time representations. Sampling allows us to analyze and process signals efficiently while preserving important information. The process of reconstruction, on the other hand, aims to convert a discrete-time signal back into a continuous-time signal.

In many cases, the sequences involved in discrete-time signals can be very large, making them challenging to handle efficiently. To overcome this challenge, we employ the discrete sampling process. Discrete sampling refers to the process of recovering a very large discrete-time signal from a subset of that signal. This subset contains a finite number of samples, which is significantly smaller than the original sequence. By carefully selecting and preserving the essential information in these sampled points, we can reconstruct the complete large sequence with a reasonable level of accuracy.

In this thesis, we look into the realm of discrete sampling and its application in the Ramanujan space. For a given integer q, the Ramanujan space, denoted by S_q, is generated by a set of complex exponential functions $w_q^{k\cdot} = e^{\frac{-2\pi i k \cdot}{q}}, 1 \leq k \leq q$ and $(k, q) = 1$. These functions form an basis for S_q and possess remarkable properties that make them highly suitable for signal representation and analysis. The Ramanujan space has garnered significant attention due to its unique properties and its connection to various areas of mathematics and signal processing.

The History of Ramanujan spaces is associated with the Ramanujan sum. The motivation behind Ramanujan's introduction of these sums was to study and express various arithmetic functions in terms of linear combinations of the these sums. Ramanujan sums are defined as follows: For a positive integer q and an integer n, the Ramanujan sum $c_q(n)$ is given by the formula:

$$c_q(n) = \sum_{\substack{1 \leq k \leq q \\ \gcd(k,q)=1}} e^{2\pi i \frac{kn}{q}}.$$

Ramanujan sums have connections to various areas of mathematics, including number theory, modular forms, and signal processing. They have been used to study properties of multiplicative functions, investigate congruence properties of partitions, and explore the distribution of prime numbers. Additionally, Ramanujan sums find applications in digital signal processing and communication theory, where they are used to analyze and manipulate periodic signals.

Our primary objective in this thesis is to explore the discrete sampling process within the Ramanujan space. We aim to develop a finite sampling approach that allows us to extract essential information from signals represented in S_q using a limited number of samples. This has practical implications, as it enables efficient signal processing while reducing the data acquisition and storage requirements. To achieve our goal, we will leverage the concepts of reproducing kernel Hilbert spaces (RKHS). The RKHS provides a powerful framework for studying signal spaces. By combining these concepts with the unique properties of the Ramanujan space, we can develop a discrete sampling method tailored to this specific context. Throughout this thesis, we will present theoretical analyses, and experimental results to validate the effectiveness of our proposed discrete sampling approach in the Ramanujan space.

By advancing our understanding of discrete sampling in the Ramanujan space, this research opens up new avenues for signal processing applications. The outcomes of this thesis have the potential to contribute to the development of more efficient and robust signal processing algorithms, with implications in fields such as communications, image and audio processing, and data compression.

The organization of the thesis is as follows: The second chapter will focus on providing the foundational definitions and concepts necessary for our study. We will cover essential topics that form the building blocks of our research. Moving on to the third chapter, we will introduce the Ramanujan space, which serves as the main space for our discrete sampling process. Here, we will provide a concise overview of the Ramanujan space, highlighting its key properties and characteristic. We shall go into the sampling procedure for reproducing kernel Hilbert spaces (RKHS) in the fourth chapter. With the use of this methodology, we will be able to prove two crucial theorems about the discrete sampling procedure in the Ramanujan space. These theorems will offer important perspectives and theoretical underpinnings for our work. By systematically progressing through these chapters, we aim to develop a comprehensive understanding of the basic definitions, the Ramanujan space, and the discrete sampling process using RKHS method. This will pave the way for further exploration and analysis in subsequent sections of our thesis.

In our thesis, we primarily utilize papers [1],[5],[4],[2],[8],[3],[10] that are relevant to our topic. Other papers are read to develop conceptual understanding.

CHAPTER 2

Sampling theory for Hilbert spaces

The objective of this chapter is to introduce key definitions pertaining to sampling theory and establish the fundamental concepts of sampling. To lay the foundation, we begin with an overview of the essential aspects of Hilbert spaces.

Definition 2.1. Let V be a vector space over the field F. An inner product V is a mapping $\langle .,. \rangle : V \times V \to F$ that satisfies the following properties-

(i) **Conjugate symmetry:** $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$ for all $\alpha, \beta \in V$, where $\overline{\langle \beta, \alpha \rangle}$ is the conjugate of the complex number $\langle \beta, \alpha \rangle$.

(ii) **Linearity:** $\langle a\alpha + b\beta, \gamma \rangle = a\langle \alpha, \beta \rangle + b\langle \alpha, \gamma \rangle$ for all $\alpha, \beta, \gamma \in V$ and all $a, b \in F$.

(iii) **Positive-definiteness:** $\langle \alpha, \alpha \rangle > 0$ for all $\alpha \in V$ with $\alpha \neq 0$.

Definition 2.2. A Hilbert space is a complete inner product space, which means it is a vector space equipped with an inner product space and is also complete with respect to the induced norm.

In addition to being a complete metric space, a Hilbert space is characterized by the property that every Cauchy sequence in the space converges to a unique limit within the space.

Now we provide some examples of Hilbert space.

Example 2.1. [9] The space $\mathbb{R}^n$ equipped with the standard inner product is a Hilbert space. The inner product between two vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ is defined as $\langle x, y\rangle = x_1y_1 + x_2y_2 + \ldots + x_ny_n$. The completeness property of $\mathbb{R}^n$ guarantees that any Cauchy sequence of vectors in $\mathbb{R}^n$ converges to a unique limit within $\mathbb{R}^n$.

Example 2.2. [9] The space $L^2[0, 1]$, which consists of all square integrable functions on the interval $[0, 1]$, is a Hilbert space. The inner product between two functions f and g is defined as $\langle f, g\rangle = \int_0^1 f(x)g(x), dx$. The completeness property of L^2-space ensures that any Cauchy sequence of functions in $L^2[0, 1]$ converges to a unique limit within $L^2[0, 1]$.

Definition 2.3. Let $\mathbb{H}$ be a Hilbert space. A set $\{\phi_n^*\}_{n\in\mathbb{N}}$ is said to be a bi-orthonormal set of the given $\{\phi_n\}_{n\in\mathbb{N}}$ for the Hilbert space $\mathbb{H}$ if they satisfy the following conditions:

$$\langle \phi_k, \phi_n^*\rangle = \begin{cases} 0, \text{ if } k \neq n, \\ 1, \text{ if } k = n. \end{cases}$$

Here is an example to illustrate the concept of a bi-orthonormal set [5] :

Example 2.3. Let $\mathbb{H}$ be the Hilbert space of square-integrable functions on the interval $[0, 1]$ with the inner product defined as $\langle f, g\rangle = \int_0^1 f(x)g(x), dx$. Consider the following functions:

Let $\phi_n(x) = \sqrt{2}\sin(2\pi nx)$ for $n \geq 1$. These functions are orthonormal because:

$$
\begin{aligned}
\langle \phi_k, \phi_n \rangle &= \int_0^1 \phi_k(x)\phi_n(x)dx \\
&= \int_0^1 \left(\sqrt{2}\sin(2\pi kx)\right)\left(\sqrt{2}\sin(2\pi nx)\right)dx \\
&= 2\int_0^1 \sin(2\pi kx)\sin(2\pi nx)dx \\
&= \delta_{kn}
\end{aligned}
$$

where δ_{kn} is the Kronecker delta, which equals 1 if $k = n$ and 0 otherwise. This shows that ϕ_n is an orthonormal set.

To find the corresponding bi-orthonormal set ϕ_n^*, we need to find functions $\phi_n^*(x)$ such that $\langle \phi_k, \phi_n^* \rangle = \delta_{kn}$. In this case, we can take $\phi_n^*(x) = \sqrt{2}\sin(2\pi nx)$. Thus, the sets $\{\phi_n\}$ and $\{\phi_n^*\}$ defined as $\phi_n(x) = \sqrt{2}\sin(2\pi nx)$ and $\phi_n^*(x) = \sqrt{2}\sin(2\pi nx)$ form a bi-orthonormal set in the space $L^2[0,1]$ with the integration norm.

Riesz's representation theorem is very useful in describing the dual vector space to any space which contains the compactly supported continuous functions as a dense subspace. Riesz's representation theorem is also very useful in reproducing kernel Hilbert space. Before delving into Riesz's representation theorem, let us establish the definition of a functional [9], which holds significant importance in the realm of reproducing kernel Hilbert spaces.

Definition 2.4. Let X be a vector space and K be a scalar field of X. Then a functional f is an operator with domain in vector space X and range in the scalar field K; thus,

$$f : \mathcal{D}(f) \to K,$$

where $K = \mathbb{R}$ if X is real and $K = \mathbb{C}$ if X is complex.

For example, the norm $\|.\| : X \to \mathbb{R}$ on a normed space $(X, \|.\|)$ is a functional on X. Now, we provide the statement of Riesz's representation theorem[9].

Theorem 2.1. *Let $\mathbb{H}$ be a Hilbert space. Then for any bounded linear functional f defined on Hilbert space $\mathbb{H}$ can be represented f in terms of the inner product as follows:*

$$f(x) = \langle x, z \rangle,$$

where z is a vector that depends on f, uniquely determined by f, and has a norm given by $\|f\| = \|z\|$.

Now we provide the definition of reproducing kernel Hilbert spaces [5].

Definition 2.5. Let E be an arbitrary set and $\mathbb{H}$ be a Hilbert space of real valued functions on E. Then $\mathbb{H}$ is called *reproducing kernel Hilbert space* if and only if the pointwise evaluation functional over the Hilbert space of functions $\mathbb{H}$ is a linear functional that evaluates each function at a point x,

$$L_x : f \to f(x), \forall f \in \mathbb{H},$$

is continuous at every f in H or equivalently, L_x is a bounded operator on $\mathbb{H}$ that is, there exist $B_x > 0$ such that

$$|L_x(f)| = |f(x)| \leq B_x \|f\|_{\mathbb{H}}$$

for all $f \in \mathbb{H}$.

If a Hilbert space $\mathbb{H}$ has finite dimension, then all the linear functional on $\mathbb{H}$ are guaranteed to be bounded. According to the Riesz representation theorem, for every $x \in E$, there exists a unique element K_x in $\mathbb{H}$ that possesses the reproducing property. This property is expressed as follows:

For any function f in $\mathbb{H}$, the evaluation of f at x is equivalent to the inner product between f and K_x in $\mathbb{H}$. In other words, it can be written as:

$$f(x) = L_x(f) = \langle f, K_x \rangle_{\mathbb{H}}. \tag{2.1}$$

This relation holds for all functions $f \in \mathbb{H}$. Alternatively, the reproducing property of the kernel K_x implies that it can reconstruct the value of a function f at any point x by performing the inner product between f and the kernel's K_x at that specific point. In essence, the kernel serves as a mechanism for extracting the function's value at any given location. By considering equation (2.1), we can observe that $K_x \in \mathbb{H}$, is a function defined on E. Consequently, we have the relationship

$$K_x(y) = L_y(K_x) = \langle K_x, K_y \rangle_{\mathbb{H}},$$

where $K_y \in \mathbb{H}$.

Definition 2.6. The reproducing kernel of $\mathbb{H}$ can be defined as a function $K : E \times E \to \mathbb{R}$ given by the expression

$$K(x, y) = \langle K_x, K_y \rangle_{\mathbb{H}}.$$

Some properties of Reproducing kernel Hilbert spaces

(i) Let $\mathbb{H}$ be a reproducing kernel Hilbert space and $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis. Then the reproducing kernel is given by

$$K(x, y) = \sum_{n\in\mathbb{N}} \phi_n(x)\overline{\phi_n(y)}. \tag{2.2}$$

By examining Equation (2.2), it is evident that the reproducing kernel satisfies the property of conjugate symmetry, given by

$$K(x, y) = \overline{K(y, x)}.$$

(ii) Let $\mathbb{H}$ be a reproducing kernel Hilbert space. If $\{\phi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for reproducing kernel Hilbert space $\mathbb{H}$ with a biorthonormal basis $\{\phi^*\}_{n\in\mathbb{N}}$. Then the reproducing kernel can be expressed as follows:

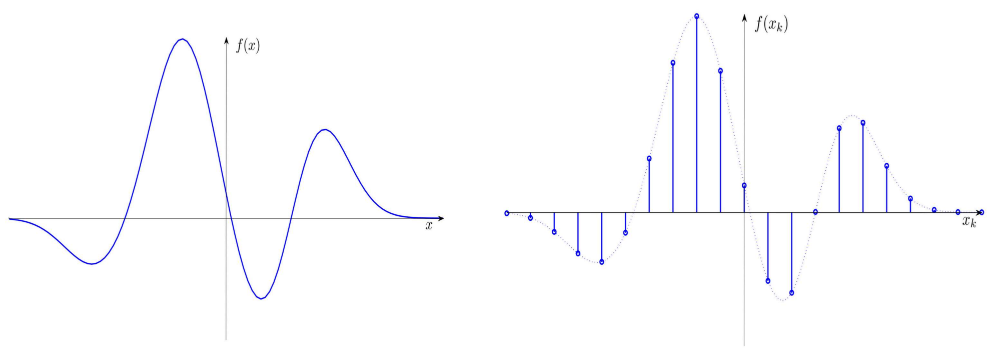
$$K(x, y) = \sum_{n\in\mathbb{N}} \phi_n(x)\overline{\phi_n^*(x)}.$$

2.1 Sampling and reconstruction of a signal

We read the papers [5], [6], [8] to understand the process of sampling. Now let's explore the sampling process provided below.

Sampling is the essential procedure of capturing measurements or observations of a continuous-time signal at specific time intervals. It involves converting the continuous-time signal into a discrete-time signal by regularly measuring it at discrete time points. This conversion is typically performed using an analog-to-digital converter (ADC) that samples the signal and converts it into a digital format suitable for processing on a computer or other digital signal processing devices.

Imagine a continuous signal represented by a smooth waveform, such as a sine



(a) continuous function (b) discrete function

Figure 2.1: Sampling process

wave. To sample this signal, we need to capture discrete points along the waveform at specific intervals. Starting at the beginning of the signal, we place a sampling point or marker on the waveform. This point represents the value of the signal at that particular moment.

On the other hand, reconstruction refers to the process of converting a discrete-time signal back into a continuous-time signal. This is accomplished using a digital-to-analog converter (DAC) that transforms the digital signal into an analog format, allowing it to be reproduced through speakers or other output devices as a continuous-time waveform. The Shannon Sampling Theorem, a fundamental principle in signal processing, establishes the minimum sampling rate required to accurately represent a continuous-time signal in the discrete-time domain. It provides guidance on how to choose an appropriate sampling rate to avoid aliasing and faithfully reconstruct the original continuous-time signal from its discrete samples. By adhering to the sampling theorem, signal processing systems can achieve accurate and faithful representation of continuous-time signals in the digital domain.

Suppose that a function f is defined for every point of some domain E and has a series representation [5] of the form

$$f(t) = \sum_{n \in \mathbb{N}} f(\lambda_n) S_n(t),$$

in which $\{\lambda_n\}$ is a collection of points of D, and $\{S_n\}$ is some set of suitable expansion functions. Such an expansion is called a sampling series, and the first thing we notice about it, the property that gives it its name, is that f is represented in its entirety in terms of its values, that is, its samples, at just a discrete subset of its domain.

Now we provided some terminologies which we will use quite offense in the coming section.

In signal processing, the term *sampling set* refers to the set of time instants at which a continuous-time signal is sampled to produce a discrete-time signal. In other words, the *sampling set* is the sequence of time instants at which the analog signal is measured or sampled. we call each member of the *sampling set* as *sampling point*. The *samples* are the discrete data in the form of functional values of on a discrete set or typically some average data that close to functional values.

We now provide an example to illustrate the above definitions.

Example 2.4. [8] Let $\mathbb{H}$ be a Hilbert space quadratic polynomials. If we consider the unknown quadratic polynomial $p(x) \in \mathbb{H}$ and sample it at the values $x = 0, 1, 2$, obtaining $p(0) = 2, p(1) = 1, p(2) = 4$, we can determine the expression for $p(x)$. We apply the Lagrange formula to reconstruct the polynomial $p(x)$:

$$\begin{aligned} p(x) &= \frac{(x-1)(x-2)}{(0-1)(0-2)}f(0) + \frac{(x-0)(x-2)}{(1-0)(1-2)}f(1) + \frac{(x-0)(x-1)}{(2-0)(2-1)}f(2) \\ &= \left(\frac{1}{2}x^2 - \frac{3}{2}x + 1\right)2 + (-x^2 + 2x) + \left(\frac{1}{2}x^2 - \frac{1}{2}x\right)4 \\ &= 2x^2 - 3x + 2. \end{aligned}$$

In this thesis, our focus lies on a particular aspect of sampling known as discrete sampling.

2.2 Discrete theory of Sampling

We read the papers [3], [13] to understand the process of discrete sampling. Now let's explore the discrete sampling process provided below.

In signal processing, discrete sampling is a technique used to address the challenge of handling very large discrete-time signals. When a discrete-time signal consists of a large number of data points, it can become computationally intensive and inefficient to process and analyze the signal in its entirety. Moreover, it also offers a solution by reducing the number of samples while preserving essential information. The process of discrete sampling involves selecting a subset of samples from the original signal. Instead of processing the entire sequence, we strategically choose specific samples that capture the significant characteristics of the signal. These selected samples serve as representatives of the original signal, allowing for efficient analysis and processing.

By carefully choosing the sampling points, we can reconstruct the complete large sequence from this reduced set of samples. This reconstruction process involves using interpolation or extrapolation techniques to estimate the values of the signal at the unsampled points. With the reconstructed signal, we can perform various signal processing tasks, such as filtering, analysis, or feature extraction, while minimizing computational complexity and resource requirements.

Furthermore, it also offers several benefits in signal processing. It allows us to represent and manipulate signals in a more manageable and computationally efficient manner. By reducing the number of samples, we can decrease storage requirements, transmission bandwidth, and processing time. Moreover, discrete sampling enables the application of signal processing algorithms on limited resources, such as embedded systems or real-time applications, where processing large amounts of data may not be feasible.

In this thesis, we develop a discrete sampling formula specifically tailored for the Ramanujan space, which is an important part of the sampling process. However, we utilize the principles of the sampling theorem within the framework of the reproducing kernel Hilbert space. The details and explanation of the discrete sampling process are extensively described and analyzed. Sampling is the process of converting a continuous-time signal into a discrete-time signal by taking measurements or observations at regular time intervals. It involves capturing a countable number of data points from the continuous signal. On the other hand, discrete sampling refers to the specific scenario where a discrete signal, which is already in a discrete-time format, can be accurately reconstructed or recovered from a subset of its data points.

In the process, the goal is to determine the conditions under which a discrete signal can be completely recovered from a reduced set of data points. This implies that by having access to a limited number of samples, we aim to reconstruct the entire signal with minimal loss of information. This is a significant problem in signal processing and has practical applications in various fields such as data compression and transmission. By understanding the principles and techniques of discrete sampling, we can explore methods to efficiently represent and reconstruct discrete signals, allowing us to make the most of limited data sets while preserving the essential characteristics of the original signal.

CHAPTER 3

Ramanujan spaces and its associated properties

The Ramanujan space, named after the renowned Indian mathematician Srinivasa Ramanujan, is a closed subspace of Hilbert space $\mathbb{C}^q$ and every closed subspace of a Hilbert space is again a Hilbert space. Hence the Ramanujan space is a Hilbert space. Let us explore the definition of the Ramanujan space[4].

Definition 3.1. For given integer q, the space generated by the set

$$\{w_q^{k\cdot} = e^{\frac{-2\pi i k.}{q}} : 1 \leq k \leq q, (k, q) = 1\}$$

is called the Ramanujan space, which is denoted by the symbol S_q. Mathematically,

$$S_q = \text{span}\{w_q^{k\cdot} : 1 \leq k \leq q, (k, q) = 1\}.$$

Any function (signal) $x(n) \in S_q$ can be written as

$$x(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} a_k w_q^{kn} = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} a_k e^{\frac{-2\pi i n k}{q}}.$$

In this thesis, the background of the Ramanujan space is comprehensively explained. We see the crucial periodicity properties associated with space S_q.

3.1 Background of Ramanujan spaces

A trigonometric summation known as the Ramanujan sum ([1],[4],[2]) was first established in 1918 by renowned Indian mathematician Srinivasa Ramanujan. This sum has the form

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} w_q^{kn},$$

where $w_q = e^{\frac{-2\pi i}{q}}$ and (k, q) denotes the gcd of k and q. As a result, the sum exceeds those k that are co-prime to q. For example, if $q = 8$, then $k \in \{1, 3, 5, 7\}$ so that

$$c_8(n) = e^{\frac{-2\pi in}{8}} + e^{\frac{-6\pi in}{8}} + e^{\frac{-10\pi in}{8}} + e^{\frac{-14\pi in}{8}}.$$

The objective of this section is to establish the definition of the space S_q. To begin, we will define the $q \times q$ integer circulant matrix[1] based on $c_q(n)$ as depicted below:

$$B_q = \begin{bmatrix} c_q(0) & c_q(q-1) & \cdots & c_q(1) \\ c_q(1) & c_q(0) & \cdots & c_q(2) \\ \vdots & \vdots & \ddots & \vdots \\ c_q(q-1) & c_q(q-2) & \cdots & c_q(0) \end{bmatrix}$$

Each column is created by circularly shifting the previous column downward and each row is essentially a right circular shift of the one before it. There are several obvious, basic characteristics of this matrix:

(i) The $0th$ row is the time-reversed Ramanujan sum $c_q(q - n)$.

(ii) Since $c_q(q - n) = c_q(n)$, we see that B_q is symmetric.

(iii) The matrix is Toeplitz because it is circulant, meaning that all elements along any line parallel to diagonal are identical.

The Ramanujan space S_q is the column space of B_q. It is obvious that $c_q(n)$ and all of its circularly shifted variants belong to the space S_q. The space $S_q \subset \mathbb{C}^q$

has dimension $\phi(q)$ because, as we shall observe, matrix B_q has rank $\phi(q)$. We know any circulant matrix is diagonalized by the DFT matrix. That is,

$$B_q = W^{-1} A_q W,$$

where W is the $q \times q$ DFT matrix, and

$$A_q = \begin{bmatrix} C_q[0] & 0 & \cdots & 0 \\ 0 & C_q[1] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_q[q-1] \end{bmatrix}.$$

Here $C_q[k]$ are the DFT [1] coefficients of the Ramanujan sum $c_q(n)$, that is,

$$C_q[k] = \sum_{n=0}^{q-1} c_q(n) w_q^{nk} = \begin{cases} q \text{ if } (k,q) = 1, \\ 0 \text{ if } otherwise. \end{cases} \tag{3.1}$$

We can see that $B_q = W^{-1} A_q W$ is identical to $B_q W^* = W^* A_q$ because $W/\sqrt{q}$ is unitary. Using the facts that $c_q(n)$ and $c_q[k]$ are real numbers and $W = W^T$, we can conjugate both sides to obtain

$$B_q W = W A_q$$

the eigenvectors of B_q are represented by the columns of W, and their associated eigenvalues are $C_q[k]$. Since there are $\phi(q)$ nonzero eigenvalues of the circulant matrix B_q, so dimension of the Ramanujan subspace is $\phi(q)$.

The circulant B_q can be factored as

$$B_q = V V^* \tag{3.2}$$

where V is a $q \times \phi(q)$ submatrix of the DFT matrix W that is produced by keeping the co-prime columns, that is, columns with numbers k_i such that $(k_i, q) = 1$. We see that the column space of B_q is the column space of V, and is spanned by the

$\phi(q)$ columns

$$\begin{bmatrix} 1 \\ w^{k_1} \\ w^{2k_1} \\ \vdots \\ w^{(q-1)k_1} \end{bmatrix}, \begin{bmatrix} 1 \\ w^{k_2} \\ w^{2k_2} \\ \vdots \\ w^{(q-1)k_2} \end{bmatrix}, \cdots, \begin{bmatrix} 1 \\ w^{k_{\phi(q)}} \\ w^{2k_{\phi(q)}} \\ \vdots \\ w^{(q-1)k_{\phi(q)}} \end{bmatrix}$$

where $w = e^{\frac{-i2\pi}{q}}$ and $(k_i, q) = 1$.

Therefore, the connection between the DFT matrix and the Ramanujan space S_q can be summed up as follows:

The Ramanujan space $S_q \subset \mathbb{C}^q$ is identical to the space spanned by the $\phi(q)$ columns of the $q \times q$ DFT matrix W, whose column indices k are co-prime to q. So the Ramanujan space S_q is the spanning set of

$$\{w_q^{k.} = e^{-2\pi ik./q} : 1 \leq k \leq q, (k, q) = 1\},$$

where (k, q)= gcd of k and q. Now w_q^k is a primitive qth root of unity if and only if $(k, q) = 1$. In this instance, $w_q^{kn} = 1$ only when n is a multiple of q. Equivalently, if and only if $(k, q) = 1$, $x(n) = w_q^{kn}$ has period q.

Within the space S_q, there exists a distinctive type of basis known as the Ramanujan integer basis. Now, let us provide a detailed explanation of the Ramanujan integer basis in the space S_q.

3.2 Integer basis for the Ramanujan space S_q

The basis $\{w_q^{k.} : 1 \leq k \leq q, (k, q) = 1\}$ obtained from the Discrete Fourier Transform (DFT) matrix is not an integer basis([1]) for the Ramanujan space S_q. In order to find an integer basis for S_q, we need to identify $\phi(q)$ columns from the integer matrix B_q that fulfill this requirement. The consecutive column theorem comes into play, which asserts that a specific set of $\phi(q)$ consecutive columns from the matrix B_q form a linearly independent set. By leveraging the consecutive column theorem, we can select a subset of $\phi(q)$ columns from the matrix B_q in a consecutive manner. These columns are chosen in such a way that they possess the property of being linearly independent. This means that the chosen columns form an integer basis for the Ramanujan space S_q, allowing us to accurately represent any element within S_q using integers.

Theorem 3.1. *[1] Let M be a submatrix of the circulant matrix B_q obtained by retaining any consecutive $\phi(q)$ columns, i.e., $r, r+1, \cdots, r+\phi(q)-1$ for some r. Then submatrix M has rank $\phi(q)$.*

Proof. Based on equation (3.2), the matrix B_q can be expressed as

$$B_q = VV^*,$$

where V is a $q \times \phi(q)$ matrix and V^* denotes its conjugate transpose. Consequently, the submatrix mentioned in the theorem can be represented as

$$M = VV_1,$$

where V_1 is a $\phi(q) \times \phi(q)$ matrix.

Now, let's consider the j-th row of the V_1 matrix, which can be written as:

$$\begin{bmatrix} a_j^r \\ a_j^{r+1} \\ \vdots \\ a_j^{r-\phi(q)+1} \end{bmatrix} = a_i^r \begin{bmatrix} 1 \\ a_j \\ \vdots \\ a_j^{\phi(q)+1} \end{bmatrix}, \tag{3.3}$$

where $a_j = e^{\frac{-2\pi i k_j}{q}}$ with $(k_j, q) = 1$. Considering equation (3.3), the matrix V_1 can be expressed as the product of a diagonal matrix and a Vandermonde matrix. This is possible because the values of a_i are distinct for different rows. Specifically, the matrix V_1 takes the form

$$V_1 = \Lambda V_2,$$

where Λ represents a diagonal matrix with non-zero diagonal elements, and V_2 denotes a $\phi(q) \times \phi(q)$ row-wise Vandermonde matrix.

Consequently, the matrix V_1 is non-singular due to the presence of the non-zero diagonal elements in Λ. Additionally, considering the definition of matrix V, which is a column-wise Vandermonde matrix with $\phi(q)$ distinct columns, it can be inferred that V has a rank of $\phi(q)$. As a result, the product matrix M formed by multiplying V and V_1 also has a rank of $\phi(q)$. □

Now we have seen that when the Ramanujan sum $c_q(n)'s$ are form an integer basis.

Theorem 3.2. [1] *The Ramanujan sum $c_q(n)$ and its $\phi(q) - 1$ consecutive shift, that is, the first $\phi(q)$ columns of the circular matrix B_q constitute an integer basis for the Ramanujan space S_q.*

Proof. According to the theorem mentioned, it can be established that the first $\phi(q)$ columns of the circular matrix B_q exhibit linear independence. This implies that these columns form a linearly independent set. Additionally, the dimension of the Ramanujan space S_q is precisely $\phi(q)$. As a consequence, the Ramanujan sum $c_q(n)$, along with its $\phi(q)$ consecutive shifts, serves as an integer basis for the Ramanujan space S_q. In other words, they form a set of integers that spans the entirety of S_q and are linearly independent, allowing them to represent any element within S_q uniquely. □

In simple terms, the integer basis for the Ramanujan space S_q can be represented by the following set: $\{c_q(n), c_q(n-1), \cdots, c_q(n-\phi(q)+1)\}$. This set of $\phi(q)$ circularly shifted Ramanujan sequences forms an integer basis for S_q. Consequently, any signal $x(n)$ belonging to S_q can be expressed in either of the following equivalent forms:$x(n) = \sum_{i=0}^{\phi(q)-1} b_i c_q(n-i) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} a_k w_q^{kn}$, where b_i represents the coefficients in the integer basis and a_k denotes the coefficients in the complex basis $w_q^{kn} : 1 \leq k \leq q, (k,q) = 1$. Both representations hold true for expressing signals within the Ramanujan space S_q.

3.3 Properties associated with Ramanujan spaces

One of the key properties of the Ramanujan space S_q is that any signal belonging to this space has a period of q. This property can be stated as follows:

Theorem 3.3. *[10] Any signal $x(n)$ in the Ramanujan space S_q has period q unless $x(n) = 0$ for all n.*

Proof. Any signal $x(n) \in S_q$ can be expressed as

$$x(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} a_k e^{\frac{-2\pi i n k}{q}}.$$

Now, let's consider $x(n+q)$:

$$x(n+q) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} a_k e^{\frac{-2\pi i (n+q)k}{q}} = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} a_k e^{\frac{-2\pi i n k}{q} + \frac{-2\pi i q k}{q}} = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} a_k e^{\frac{-2\pi i n k}{q}} = x(n).$$

Since we have shown that $x(n) = x(n+q)$, it remains to prove that the period is not a proper divisor of q. We will use proof by contradiction. Suppose P is the period of $x(n)$, so that $P|q$. Assume that $P < q$, implying $q = Pr$ for some integer $r > 1$. Since $x(n) = x(n+P)$, we can express $x(n)$ uniquely in terms of the P-point DFT as follows:

$$x(n) = \sum_{t=0}^{P-1} \beta_t e^{\frac{-2\pi i t n}{P}}. \tag{3.4}$$

Since $q = Pr$, equation (3.4) can be rewritten as:

$$x(n) = \sum_{t=0}^{P-1} \beta_t e^{\frac{-2\pi i r t n}{q}}. \tag{3.5}$$

Both equations (3.4) and (3.5) can be considered as q-point DFT representations of $x(n)$ for $0 \leq n \leq q-1$. Therefore, the terms of the summation must match. This implies that there exists a pair of k and l such that $k = rl$. Hence, $(k, q) = r > 1$, which contradicts the condition $(k, q) = 1$. Therefore, it can be concluded that $x(n) \in S_q$ has a period of q unless $x(n) = 0$ for all n. □

The second property of the Ramanujan space is its orthogonality for different values of q_i and q_j. Now, we present a theorem that establishes the orthogonality within the Ramanujan space.

Theorem 3.4. [1] *The spaces S_{q_i} and S_{q_j} are orthogonal for $q_i \neq q_j$.*

Proof. Suppose we consider m, which is the least common multiple (lcm) of (q_i, q_j), where q_i and q_j are distinct elements. We can express m as $m = q_i l_i = q_j l_j$ for some integers l_i and l_j. By rearranging terms, we obtain:

$$\sum_{n=0}^{m-1} w_{q_i}^{k_i n} w_{q_j}^{-k_j n} = \sum_{n=0}^{m-1} w_m^{(k_i l_i - k_j l_j)n}$$

This sum evaluates to zero unless $k_i l_i - k_j l_j = mk$ for some $k \in \mathbb{N}$. However, since $1 \leq k_t \leq q_t$, it follows that $k_t l_t < m$. As a result, $|k_i l_i - k_j l_j| < m$, implying that the sum is zero unless $k_i l_i - k_j l_j = 0$. In other words, $k_i m/q_i = k_j m/q_j$ or equivalently $k_i/q_i = k_j/q_j$. Nevertheless, this scenario is not feasible since both k_i and k_j are coprime with q_i and q_j respectively, denoted by $(k_i, q_i) = (k_j, q_j) = 1$. Thus, the assumption $k_i/q_i = k_j/q_j$ cannot hold, leading to a contradiction. □

The elements in the space S_q exhibit the following exquisite autocorrelation properties:

Theorem 3.5. [1] *The circular autocorrelation of the Ramanujan sum $c_q(n)$ is expressed as follows:*

$$r_{qq}(l) = \sum_{n=0}^{q-1} c_q(n) c_q(n-l) = q c_q(l),$$

where the term $(n-l)$ is considered modulo q due to the periodic nature of $c_q(n)$ with period q.

Proof. Given $r_{qq}(l) = \sum_{n=0}^{q-1} c_q(n) c_q(n-l)$. So by (3.1) the DFT of $r_{qq}(l)$ is given by

$$R_{qq}[k] = C_q[k] C_q^*[k] = \begin{cases} q^2 \text{ if } (k,q) = 1 \\ o \text{ otherwise.} \end{cases}$$

Hence $R_{qq}[k] = q C_q[k]$. So $r_{qq}(l) = q c_q(l)$ indeed. □

3.4 Applications

Let us now explore the information gleaned from the representations in Ramanujan space, abbreviated as S_q. In [4], we can estimate the hidden periods of real or complex signals within the Ramanujan space S_q. These hidden periods correspond to non-zero periodic components, denoted as $x_{q_i}(n)$, belonging to the Ramanujan spaces S_{q_i}. The signal representation is given as $x(n) = \sum_{q=1}^{Q} x_q(n)$, where each x_q belongs to its respective Ramanujan space S_q. It is important to note that if q_k is a divisor of q_i, then the sum $x_{q_i}(n) + x_{q_k}(n)$ exhibits a period of q_i. Consequently, the smaller divisor q_k can be disregarded. Thus, we can conclude that a subset of integers, denoted as p_i, can be selected from the set q_i in such a way that p_i is not a proper divisor of p_j if $i \neq j$. This reduced set p_i is considered to be the collection of hidden periods associated with $x(n)$. The process of estimating these hidden periods is commonly known as the Ramanujan dictionary. It is worth mentioning that these methods have found applications in identifying integer periodicity within DNA molecules and protein molecules.

Additionally, Ramanujan filter banks [4], which are widely recognized in the field of digital signal processing, offer a valuable tool for generating a time-period plane plot. This plot is particularly effective for tracking localized periodic behavior. A Ramanujan analysis filter bank, denoted as $c_q(n)$ with $1 \leq q \leq N$ (where N represents the number of samples), is employed. Let $x(n)$ be an input signal with a period of P, where $1 \leq P \leq N$. In this case, only the filters $c_q(n)$ with filter index q being a divisor of P can generate non-zero outputs. It is worth noting that Ramanujan filter banks have been successfully applied in the identification of epileptic seizures in patients. Epileptic seizures are characterized by the sudden appearance of periodic waveforms in EEG records. By leveraging the capabilities of the Ramanujan filter bank, these periodic patterns can be effectively detected and analyzed.

In the field of digital signal processing, a signal is represented by a finite sequence of numbers denoted as $x(n)$, where n ranges from 0 to $N-1$. Here, N represents the length of the sequence, which can be quite large. The fundamental question in sampling is to determine the conditions under which the original signal x can be accurately reconstructed or recovered from the sampled values $x(n_i)$.The problem of sampling explores the criteria or constraints that need to be satisfied to enable the recovery of the complete signal x from its sampled values $x(n_i)$. By understanding the principles of sampling, we can establish guidelines and techniques for accurately capturing and reconstructing signals in digital systems. In this thesis, I will elucidate the process of discrete sampling in reproducing kernel Hilbert spaces. We present a sampling formula applicable to signals originating from Ramanujan spaces, accompanied by a selection of examples along with their corresponding sampling plots.

CHAPTER 4

Sampling formula for Ramanujan spaces

The discrete sampling procedure in finite dimensional Hilbert spaces will be covered in detail in this chapter. This sampling procedure is carried out specifically in the context of reproducing kernel Hilbert spaces. During the sampling process, our objective is to determine a set of points where the reproducing kernel exhibits orthogonality. Subsequently, we obtain the expansion of the sampling series. However, it is worth noting that it may not always be feasible to identify a set of points that satisfy the orthogonality condition for the reproducing kernel. In such cases, we will explore alternative approaches to accomplish the sampling process. Furthermore, we will demonstrate the application of this sampling process in the context of Ramanujan spaces, showcasing how it can be effectively employed in this specific domain.

4.1 A general setting for sampling

In [5], the sampling process is described as the procedure of discretizing a continuous-time signal by taking a finite number of samples within a specific time interval. This process involves selecting discrete time points and measuring the signal's amplitude at those points to obtain a sequence of discrete samples.By applying the sampling process to the Ramanujan space, we can discretize the signals within this space by acquiring a finite number of samples at specific time instances. This enables us to represent the continuous-time signals in a discrete form, facilitating further analysis and processing. By understanding and applying the finite sampling process described in [5], along with the principles of the sampling theorem, we can effectively analyze and process signals within the Ramanujan space, while maintaining the integrity and properties of the original signals. The sampling process in [5] is as follows. Let E be a bounded subset of $\mathbb{R}$ or $\mathbb{R}^n$. Suppose that $L^2(E)$ has an orthonormal basis consisting of $\{\phi_n\}_{n\in\mathbb{N}}$. Then every polynomial in the finite-dimensional Hilbert space $\mathbb{H}$ can be expressed as follows:

$$f(t) = \sum_{n=1}^{N} c_n \phi_n(t),$$

where dimension of $\mathbb{H}$ equal to $N \in \mathbb{N}$ and $\{c_n\} \subset \mathbb{C}$. Let g be another function of the same type with coefficients $c'_n \subset \mathbb{C}$. The inner product between f and g can be defined as follows:

$$\langle f, g\rangle = \sum_{n=1}^{N} c_n c'_n. \tag{4.1}$$

Now from (2.2), we have

$$K(t,\lambda) = \sum_{i=1}^{N} \phi_i(t)\overline{\phi_i(\lambda)}.$$

Consider a set of points $\{\lambda_i\}_{i=1}^N$ that is a subset of E and satisfies the property that $\{K(t,\lambda_i)\}_{i=1}^N$ forms an orthogonal set in $\mathbb{H}$. In other words,

$$\langle K(.,\lambda_j), K(.,\lambda_i)\rangle = K(\lambda_i,\lambda_j)\delta_{ij},$$

for all $i, j = 1, 2, ..., N$, where δ_{ij} Kronecker delta function is defined by

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

Since $\{K(t, \lambda_i)\}_{i=1}^{N}$ is an orthogonal set in $\mathbb{H}$ and N is the dimension of $\mathbb{H}$, it follows that $\{K(t, \lambda_i)\}_{i=1}^{N}$ forms a basis for $\mathbb{H}$. Therefore, $\{\frac{K(t,\lambda_i)}{K(\lambda_i,\lambda_i)}\}_{i=1}^{N}$ constitutes an orthonormal basis for $\mathbb{H}$. Consequently, any function f in $\mathbb{H}$ can be represented as a finite sampling series, expressed as:

$$f(t) = \sum_{n=1}^{N} f(\lambda_n) \frac{K(t, \lambda_n)}{K(\lambda_n, \lambda_n)}.$$

The existence of the point set $\{\lambda_i\}$ with the mentioned properties is uncertain. While it may not be orthogonal, it is possible to find $\{\lambda_i\}$ such that $\{K(t, \lambda_i)\}$ forms a spanning set for $\mathbb{H}$. Since $\{K(t, \lambda_i)\}$ will be complete in $\mathbb{H}$ and $\mathbb{H}$ has finite dimension, it implies that $\mathbb{H}$ must possess a unique bi-orthonormal set $\{\psi_j\}_{j=1}^{N}$, which means a set such that:

$$\langle K(., \lambda_i), \psi_j \rangle = \delta_{ij}. \tag{4.2}$$

The expansion of a function $f \in \mathbb{H}$ in the set ψ_j takes the form:

$$f(t) = \sum_{n=1}^{N} b_n \psi_n(t),$$

where the coefficients b_n are given by $b_n = \langle f, K(., \lambda_n) \rangle = f(\lambda_n)$. This leads to the sampling representation:

$$f(t) = \sum_{n=1}^{N} f(\lambda_n) \psi_n(t).$$

To determine the functions ψ_j, we express them as a linear combination of ϕ_k:

$$\psi_j(t) = \sum_{k=1}^{N} c_k^{(j)} \phi_k(t). \tag{4.3}$$

By substituting (4.3) into (4.2) and using (4.4), we obtain:

$$\langle \psi_j, K(., \lambda_i) \rangle = \delta_{ij} = \psi_j(\lambda_i) = \sum_{n=1}^{N} c_n^{(j)} \phi_n(\lambda_i). \tag{4.4}$$

Therefore, we need to solve the set of equations:

$$\sum_{n=1}^{N} c_n^{(j)}\phi_n(\lambda_i) = \delta_{ij}.$$

4.2 Derivation of sampling formula for S_q

We will examine the process of discrete sampling using an exponential basis in the Ramanujan space S_q, following the reproducing rernel method described above. This method provides a framework for representing signals in S_q and allows us to explore the discrete sampling process within this context. An exponential basis elements in the Ramanujan space S_q is $\{w_q^k : 1 \leq k \leq q, (k, q) = 1\}$. First, prove a lemma such that the set $\{\frac{1}{\sqrt{q}} w_q^{k\cdot} : 1 \leq k \leq q, (k, q) = 1\}$ is an orthonormal basis for S_q.

Lemma 4.1. *The set* $\{\frac{1}{\sqrt{q}} w_q^{k\cdot} : 1 \leq k \leq q, (k, q) = 1\}$ *is an orthonormal basis for* S_q.

Proof. Suppose $1 \leq k_1, k_2 \leq q$ and $(k_1, q) = 1$, $(k_2, q) = 1$.

Then

$$\begin{aligned}
\langle w_q^{k_1\cdot}, w_q^{k_2\cdot}\rangle &= \sum_{n=0}^{q-1} w_q^{k_1 n}\overline{w_q^{k_2 n}} \\
&= \sum_{n=0}^{q-1} e^{\frac{-2\pi i k_1 n}{q}} e^{\frac{2\pi i k_2 n}{q}} \\
&= \sum_{n=0}^{q-1} e^{\frac{-2\pi i (k_1 - k_2) n}{q}} \\
&= \frac{1 - e^{\frac{-2\pi i q (k_1 - k_2)}{q}}}{1 - e^{\frac{-2\pi i (k_1 - k_2)}{q}}} = 0.
\end{aligned}$$

Also

$$\begin{aligned}\langle w_q^{k_1\cdot}, w_q^{k_1\cdot}\rangle &= \sum_{n=0}^{q-1} w_q^{k_1 n}\overline{w_q^{k_1 n}}\\ &= \sum_{n=0}^{q-1} e^{\frac{-2\pi i k_1 n}{q}}\overline{e^{\frac{-2\pi i k_1 n}{q}}}\\ &= \sum_{n=0}^{q-1} 1 = q.\end{aligned}$$

Hence $\langle \frac{1}{\sqrt{q}} w_q^{k_1\cdot}, \frac{1}{\sqrt{q}} w_q^{k_1\cdot}\rangle = 1$. Thus the set $\{\frac{1}{\sqrt{q}} w_q^{k\cdot} : 1 \leq k \leq q, (k,q) = 1\}$ is an orthonormal set for S_q. □

Theorem 4.2. *Let q be a very large natural number. Then for any x in the Ramanujan space S_q, we have the following representation*

$$x(n) = \sum_{i=1}^{\phi(q)} x(\lambda_i)\psi_i(n),$$

where $\lambda_i = i - 1$ for $i = 1, 2, \ldots, \phi(q)$, $\{\psi_i\}$ is bi-orthogonal to $k(., \lambda_j)$, that is

$$\langle k(., \lambda_i), \psi_j\rangle = \delta_{ij}.$$

Proof. For the Ramanujan space S_q, an orthonormal basis is given by $\{\frac{1}{\sqrt{q}} w_q^{k\cdot} : 1 \leq k \leq q, (k,q) = 1\}$. Therefore, in terms of the reproducing kernel, it can be expressed as follows:

$$K(n,\lambda) = \sum_{\substack{k=1\\(k,q)=1}}^{q} (\frac{1}{\sqrt{q}} w_q^{kn})\overline{(\frac{1}{\sqrt{q}} w_q^{k\lambda})} = \sum_{\substack{k=1\\(k,q)=1}}^{q} \frac{1}{q} w_q^{k(n-\lambda)}. \tag{4.5}$$

Now, we select arbitrary set of points $\{\lambda_1 = 0, \lambda_2 = 1, \cdots, \lambda_{\phi(q)} = \phi(q) - 1\}$. Then claim that $\{k(., \lambda_i)\}_{i=1}^{\phi(q)}$ spanning set for the Ramanujan space S_q.

Now consider,

$K(n, \lambda_1) = \sum_{\substack{k=1\\(k,q)=1}}^{q} \frac{1}{q} w_q^{k(n-\lambda_1)} = \frac{1}{q} c_q(n - \lambda_1) = \frac{1}{q} c_q(n)$,

$K(n, \lambda_2) = \sum_{\substack{k=1\\(k,q)=1}}^{q} \frac{1}{q} w_q^{k(n-\lambda_2)} = \frac{1}{q} c_q(n - \lambda_2) = \frac{1}{q} c_q(n - 1)$,

$\vdots$

$K(n, \lambda_{\phi(q)}) = \sum_{\substack{k=1\\(k,q)=1}}^{q} \frac{1}{q} w_q^{k(n-\lambda_{\phi(q)})} = \frac{1}{q} c_q(n - \lambda_{\phi(q)}) = \frac{1}{q} c_q(n - \phi(q) + 1)$.

Then by theorem (3.2), the Ramanujan sum $c_q(n)$ and its $\phi(q) - 1$ consecutive circular shifts constitute an integer basis for the Ramanujan space S_q. Thus the set $\{K(n,\lambda_i)\}_{i=1}^{\phi(q)}$ is a spanning set for S_q. So S_q being finite dimensional it must then possess a unique biorthonormal set $\{\psi_j\}_{j=1}^{\phi(q)}$, say; that is, a set such that

$$\langle K(.,\lambda_i),\psi_j\rangle = \delta_{ij}.$$

Then the bi-orthonormal basis element is defined by

$$\psi_j(n) = \sum_{\substack{k=1\\(k,q)=1}}^{q} c_k^{(j)} \frac{1}{\sqrt{q}} w_q^{kn}, \tag{4.6}$$

where $j = 1, 2, \cdots, \phi(q)$. The representation of $x \in S_q$ in the set $\{\psi_j\}_{j=1}^{\phi(q)}$ takes the form:

$$x(n) = \sum_{i=1}^{\phi(q)} b_i \psi_i(n),$$

where $b_i = \langle x, K(.,\lambda_i)\rangle = x(\lambda_i)$. This gives us the sampling representation:

$$x(n) = \sum_{i=1}^{\phi(q)} x(\lambda_i)\psi_i(n). \tag{4.7}$$

□

We now present the technique for obtaining a biorthonormal basis. We begin by considering the matrix Δ given by:

$$\Delta = \begin{bmatrix} w_q^{1\lambda_1} & \cdots & w_q^{k\lambda_1} \\ w_q^{1\lambda_2} & \cdots & w_q^{k\lambda_2} \\ \vdots & \ddots & \vdots \\ w_q^{1\lambda_{\phi(q)}} & \cdots & w_q^{k\lambda_{\phi(q)}} \end{bmatrix}$$

where k is the largest value such that $(k,q) = 1$, and Δ is a $\phi(q) \times \phi(q)$ square matrix. Alternatively, we can write Δ as a Vandermonde matrix:

$$\Delta = \begin{bmatrix} 1 & \cdots & 1 \\ w_q^{11} & \cdots & w_q^{k1} \\ \vdots & \ddots & \vdots \\ w_q^{1(\phi(q)-1)} & \cdots & w_q^{k(\phi(q)-1)} \end{bmatrix}$$

Since each column of Δ has distinct elements, it is a non-singular matrix.

Next, from equation (4.4), we have:

$$\psi_j(\lambda_i) = \sum_{\substack{k=1\\(k,q)=1}}^{q} c_k^{(j)} \frac{1}{\sqrt{q}} w_q^{kn}(\lambda_i) = \delta_{ij}. \tag{4.8}$$

Here, $j = 1, 2, \cdots, \phi(q)$ and $i = 1, 2, \cdots, \phi(q)$. The coefficients $c_n^{(j)}$ are the unknown variables for fixed j.

For each j, equation (4.8) gives us a system of linear equations with $\phi(q)$ unknowns $\{c_1^{(j)}, c_2^{(j)}, \cdots, c_{\phi(q)}^{(j)}\}$. The coefficient matrix of this system is Δ. Since Δ is non-singular, the system of linear equations has a unique solution.

Now, let's provide a couple of examples based on the above theorem.

Example 4.1. In this example we provide a sampling formula for the Ramanujan space S_5 with dimension $\phi(5) = 4$. The basis of space S_5 is given by

$$\{w_5^{k\cdot} : 1 \leq k \leq 5, (k, 5) = 1\}.$$

Also $\{\frac{1}{\sqrt{5}} w_5^{k\cdot} : 1 \leq k \leq 5, (k, 5) = 1\}$ is orthonormal basis for S_5. Then from equation (4.5) reproducing kernel is given by

$$K(n, \lambda) = \sum_{\substack{k=1\\(k,5)=1}}^{5} \frac{1}{5} w_5^{k(n-\lambda)}.$$

Here, if we choose set of points where $\{\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2, \lambda_4 = 3\}$.

$$K(n, \lambda_1) = \sum_{\substack{k=1\\(k,5)=1}}^{5} \frac{1}{5} w_5^{k(n-\lambda_1)} = \frac{1}{5} c_5(n - \lambda_1) = \frac{1}{5} c_5(n),$$

$$K(n, \lambda_2) = \sum_{\substack{k=1\\(k,5)=1}}^{5} \frac{1}{5} w_5^{k(n-\lambda_2)} = \frac{1}{5} c_5(n - \lambda_2) = \frac{1}{5} c_5(n - 1),$$

$$K(n, \lambda_3) = \sum_{\substack{k=1\\(k,5)=1}}^{5} \frac{1}{5} w_5^{k(n-\lambda_3)} = \frac{1}{5} c_5(n - \lambda_3) = \frac{1}{5} c_5(n - 2),$$

$$K(n, \lambda_4) = \sum_{\substack{k=1\\(k,5)=1}}^{5} \frac{1}{5} w_5^{k(n-\lambda_4)} = \frac{1}{5} c_5(n - \lambda_4) = \frac{1}{5} c_5(n - 3).$$

The Ramanujan sum $c_5(n)$ and its $\phi(5) - 1 = 3$ consecutive circular shifts constitute an integer basis for Ramanujan space S_5. Thus the set $\{K(n, \lambda_i)\}_{i=1}^{4}$ is an spanning set for S_5.

Now we can see the singularity of Δ matrix, where

$$\Delta = \begin{bmatrix} w_5^{1\lambda_1} & w_5^{2\lambda_1} & w_5^{3\lambda_1} & w_5^{4\lambda_1} \\ w_5^{1\lambda_2} & w_5^{2\lambda_2} & w_5^{3\lambda_2} & w_5^{4\lambda_2} \\ w_5^{1\lambda_3} & w_5^{2\lambda_3} & w_5^{3\lambda_3} & w_5^{4\lambda_3} \\ w_5^{1\lambda_4} & w_5^{2\lambda_4} & w_5^{3\lambda_4} & w_5^{4\lambda_4} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ w_5^{11} & w_5^{21} & w_5^{31} & w_5^{41} \\ w_5^{12} & w_5^{22} & w_5^{32} & w_5^{42} \\ w_5^{13} & w_5^{23} & w_5^{33} & w_5^{43} \end{bmatrix}$$

This is a Vandermonde matrix and each column has different elements, the matrix Δ is a non-singular matrix. Now from equation (4.8) we have

$$\psi_j(\lambda_i) = \sum_{\substack{n=1 \\ (n,5)=1}}^{5} c_n^{(j)} (\frac{1}{\sqrt{5}} w_5^{n\lambda_i}) = \delta_{ij}, \tag{4.9}$$

for $i = 1, 2, 3, 4$, $j = 1, 2, 3, 4$ and $\{c_n^{(j)}\}$ being the coefficients of ψ_j. For $j = 1$, we get a system of equations

$$\psi_1(\lambda_i) = \sum_{\substack{n=1 \\ (n,5)=1}}^{5} c_n^{(1)} (\frac{1}{\sqrt{5}} w_5^{n\lambda_i}) = \delta_{i1},$$

for $i = 1, 2, 3, 4$. Equivalently, the above system of equations can be written as

$$AX = b,$$

where

$$A = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ w_5^{11} & w_5^{21} & w_5^{31} & w_5^{41} \\ w_5^{12} & w_5^{22} & w_5^{32} & w_5^{42} \\ w_5^{13} & w_5^{23} & w_5^{33} & w_5^{43} \end{bmatrix}$$

That is,

$$A = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.3090 - 0.9511i & -0.8090 - 0.5878i & -0.8090 + 0.5878i & 0.3090 + 0.9511i \\ -0.8090 - 0.5878i & 0.3090 + 0.9511i & 0.3090 - 0.9511i & -0.8090 + 0.5878i \\ -0.8090 + 0.5878i & 0.3090 - 0.9511i & 0.3090 + 0.9511i & -0.8090 - 0.5878i \end{bmatrix},$$

$$X = \begin{bmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_3^{(1)} \\ c_4^{(1)} \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving above system of linear equation, we get

$$c_1^{(1)} = 0.3090 + 0.4253i, c_2^{(1)} = 0.8090 + 0.2629i, c_3^{(1)} = 0.8090 - 0.2629i,$$
$$c_4^{(1)} = 0.3090 - 0.4253i.$$

Similarly, for $j = 2, 3, 4$ we get system of linear equations and solving those system of linear equations we have

$$c_1^{(2)} = 0.8506i, c_2^{(2)} = 0.5257i, c_3^{(2)} = 0.5257i, c_4^{(2)} = 0.8506i,$$

$$c_1^{(3)} = -0.5 + 0.6882i, c_2^{(3)} = 0.5 - 0.1625i, c_3^{(3)} = 0.5 + 0.1625i, c_4^{(3)} = -0.5 - 0.6882i,$$

$$c_1^{(4)} = -0.5 + 0.1625i, c_2^{(4)} = 0.5 + 0.6882i, c_3^{(4)} = 0.5 - 0.6882i, c_4^{(4)} = -0.5 - 0.1625i$$

respectively. Therefore from equation (4.6) the bi-orthonormal basis is given by

$$\psi_1(n) = (0.3090 + 0.4253i)(\frac{1}{\sqrt{5}} w_5^{1n}) + (0.8090 + 0.2629i)(\frac{1}{\sqrt{5}} w_5^{2n})$$
$$+ (0.8090 - 0.2629i)(\frac{1}{\sqrt{5}} w_5^{3n}) + (0.3090 - 0.4253i)(\frac{1}{\sqrt{5}} w_5^{4n}),$$

$$\psi_2(n) = (0.8506i)(\frac{1}{\sqrt{5}} w_5^{1n}) + (0.5257i)(\frac{1}{\sqrt{5}} w_5^{2n}) + (0.5257i)(\frac{1}{\sqrt{5}} w_5^{3n}) + (0.8506i)(\frac{1}{\sqrt{5}} w_5^{4n}),$$

$$\psi_3(n) = (-0.5 + 0.6882i)(\frac{1}{\sqrt{5}} w_5^{1n}) + (0.5 - 0.1625i)(\frac{1}{\sqrt{5}} w_5^{2n}) + (0.5 + 0.1625i)(\frac{1}{\sqrt{5}} w_5^{3n})$$
$$+ (-0.5 - 0.6882i)(\frac{1}{\sqrt{5}} w_5^{4n}),$$

$$\psi_4(n) = (-0.5 + 0.1625i)(\frac{1}{\sqrt{5}} w_5^{1n}) + (0.5 + 0.6882i)(\frac{1}{\sqrt{5}} w_5^{2n}) + (0.5 - 0.6882i)(\frac{1}{\sqrt{5}} w_5^{3n})$$
$$+ (-0.5 - 0.1625i)(\frac{1}{\sqrt{5}} w_5^{4n}).$$

Hence from (4.7) we have the sampling representation

$$x(n) = \sum_{i=1}^{4} x(\lambda_i)\psi_i(n).$$

Now we can take a signal $x(n) = 2cos^3(\frac{2\pi n}{5}) + 6cos(\frac{2\pi n}{5})sin^2(\frac{2\pi n}{5})$, which is belong to the Ramanujan space S_5. At the sampling points $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2, \lambda_4 = 3,$

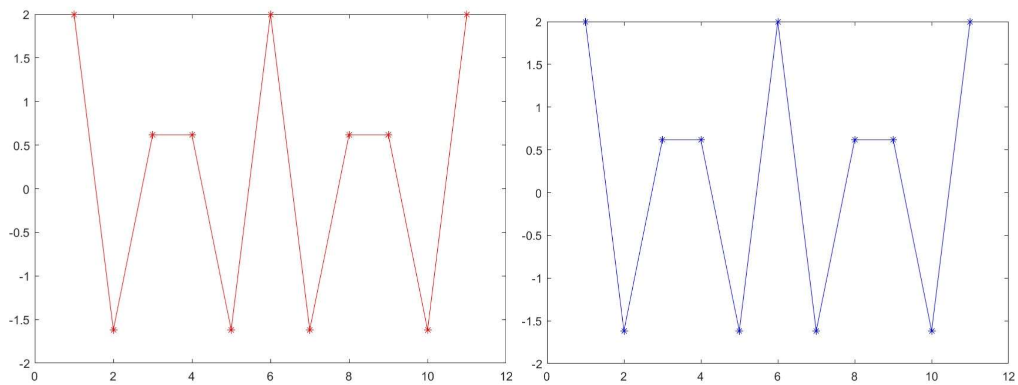
samples are respectively

$$x(\lambda_1) = x(0) = 2, x(\lambda_2) = x(1) = -1.6180,$$

$$x(\lambda_3) = x(2) = 0.6180, x(\lambda_4) = x(3) = 0.6180.$$

Then the sampling representation for this signal is given by

$$x(n) = 2\psi_1(n) + (-1.6180)\psi_2(n) + (0.6180)\psi_3(n) + (0.6180)\psi_4(n).$$



(a) Original function (b) Reconstruction

Figure 4.1: Sampling process in Ramanujan space

Example 4.2. In this example we provide sampling formula for the Ramanujan space S_{24} with dimension $\phi(24) = 8$. The basis of S_{24} is given by

$$\{w_{24}^{k\cdot} : 1 \leq k \leq 24, (k, 24) = 1\}.$$

Also $\{\frac{1}{\sqrt{24}} w_{24}^{k\cdot} : 1 \leq k \leq 24, (k, 24) = 1\}$ is an orthonormal basis for S_{24}. Then from (4.5), the reproducing kernel is given by

$$k(n, \lambda) = \sum_{\substack{k=1 \\ (k,24)=1}}^{24} \frac{1}{24} w_2 4^{k(n-\lambda)}.$$

Here we choose the set of points $\{\lambda_1 = 0, \lambda_2 = 1, \cdots, \lambda_8 = 7\}$.

$$k(n, \lambda_1) = \sum_{\substack{k=1 \\ (k,24)=1}}^{24} \frac{1}{24} w_{24}^{k(n-\lambda_1)} = \frac{1}{24} c_{24}(n - \lambda_1) = \frac{1}{24} c_{24}(n),$$

$$k(n,\lambda_2) = \sum_{\substack{k=1\\(k,24)=1}}^{24} \frac{1}{24} w_{24}^{k(n-\lambda_2)} = \frac{1}{24} c_{24}(n-\lambda_2) = \frac{1}{24} c_{24}(n-1)$$

$$\vdots$$

$$k(n,\lambda_8) = \sum_{\substack{k=1\\(k,24)=1}}^{24} \frac{1}{24} w_{24}^{k(n-\lambda_8)} = \frac{1}{24} c_{24}(n-\lambda_8) = \frac{1}{24} c_{24}(n-7).$$

The Ramanujan sum $c_{24}(n)$ and its $\phi(24)-1=7$ consecutive circular shifts constitute an integer basis for Ramanujan space S_{24}. Thus the set $\{k(n,\lambda_i)\}_{i=1}^{8}$ is an spanning set for S_{24}.

Now we see the singularity of Δ matrix, where

$$\Delta = \begin{bmatrix} w_{24}^{1\lambda_1} & w_{24}^{5\lambda_1} & w_{24}^{7\lambda_1} & w_{24}^{11\lambda_1} & w_{24}^{13\lambda_1} & w_{24}^{17\lambda_1} & w_{24}^{19\lambda_1} & w_{24}^{23\lambda_1} \\ w_{24}^{1\lambda_2} & w_{24}^{5\lambda_2} & w_{24}^{7\lambda_2} & w_{24}^{11\lambda_2} & w_{24}^{13\lambda_2} & w_{24}^{17\lambda_2} & w_{24}^{19\lambda_2} & w_{24}^{23\lambda_2} \\ w_{24}^{1\lambda_3} & w_{24}^{5\lambda_3} & w_{24}^{7\lambda_3} & w_{24}^{11\lambda_3} & w_{24}^{13\lambda_3} & w_{24}^{17\lambda_3} & w_{24}^{19\lambda_3} & w_{24}^{23\lambda_3} \\ w_{24}^{1\lambda_4} & w_{24}^{5\lambda_4} & w_{24}^{7\lambda_4} & w_{24}^{11\lambda_4} & w_{24}^{13\lambda_4} & w_{24}^{17\lambda_4} & w_{24}^{19\lambda_4} & w_{24}^{23\lambda_4} \\ w_{24}^{1\lambda_5} & w_{24}^{5\lambda_5} & w_{24}^{7\lambda_5} & w_{24}^{11\lambda_5} & w_{24}^{13\lambda_5} & w_{24}^{17\lambda_5} & w_{24}^{19\lambda_5} & w_{24}^{23\lambda_5} \\ w_{24}^{1\lambda_6} & w_{24}^{5\lambda_6} & w_{24}^{7\lambda_6} & w_{24}^{11\lambda_6} & w_{24}^{13\lambda_6} & w_{24}^{17\lambda_6} & w_{24}^{19\lambda_6} & w_{24}^{23\lambda_6} \\ w_{24}^{1\lambda_7} & w_{24}^{5\lambda_7} & w_{24}^{7\lambda_7} & w_{24}^{11\lambda_7} & w_{24}^{13\lambda_7} & w_{24}^{17\lambda_7} & w_{24}^{19\lambda_7} & w_{24}^{23\lambda_7} \\ w_{24}^{1\lambda_8} & w_{24}^{5\lambda_8} & w_{24}^{8\lambda_7} & w_{24}^{11\lambda_8} & w_{24}^{13\lambda_8} & w_{24}^{17\lambda_8} & w_{24}^{19\lambda_8} & w_{24}^{23\lambda_8} \end{bmatrix}$$

This is a Vandermonde matrix and each column has different elements, the matrix Δ is a non-singular matrix. Now from equation (4.8), we have

$$\psi_j(\lambda_i) = \sum_{\substack{n=1\\(n,24)=1}}^{24} c_n^{(j)} \left(\frac{1}{\sqrt{24}} w_{24}^{n\lambda_i}\right) = \delta_{ij},$$

where $i=1,2,\cdots,8$, $j=1,2,\cdots,8$ and $\{c_n^{(j)}\}$ coefficients of ψ_j. For $j=1$, we get a system of equations

$$\psi_1(\lambda_i) = \sum_{\substack{n=1\\(n,24)=1}}^{24} c_n^{(1)} \left(\frac{1}{\sqrt{24}} w_{24}^{n\lambda_i}\right) = \delta_{i1},$$

where $i=1,2,\cdots,8$. Equivalently, this system of equations can be written as

$$AX = b,$$

where

$$A = \frac{1}{\sqrt{24}} \begin{bmatrix} w_{24}^{1\lambda_1} & w_{24}^{5\lambda_1} & w_{24}^{7\lambda_1} & w_{24}^{11\lambda_1} & w_{24}^{13\lambda_1} & w_{24}^{17\lambda_1} & w_{24}^{19\lambda_1} & w_{24}^{23\lambda_1} \\ w_{24}^{1\lambda_2} & w_{24}^{5\lambda_2} & w_{24}^{7\lambda_2} & w_{24}^{11\lambda_2} & w_{24}^{13\lambda_2} & w_{24}^{17\lambda_2} & w_{24}^{19\lambda_2} & w_{24}^{23\lambda_2} \\ w_{24}^{1\lambda_3} & w_{24}^{5\lambda_3} & w_{24}^{7\lambda_3} & w_{24}^{11\lambda_3} & w_{24}^{13\lambda_3} & w_{24}^{17\lambda_3} & w_{24}^{19\lambda_3} & w_{24}^{23\lambda_3} \\ w_{24}^{1\lambda_4} & w_{24}^{5\lambda_4} & w_{24}^{7\lambda_4} & w_{24}^{11\lambda_4} & w_{24}^{13\lambda_4} & w_{24}^{17\lambda_4} & w_{24}^{19\lambda_4} & w_{24}^{23\lambda_4} \\ w_{24}^{1\lambda_5} & w_{24}^{5\lambda_5} & w_{24}^{7\lambda_5} & w_{24}^{11\lambda_5} & w_{24}^{13\lambda_5} & w_{24}^{17\lambda_5} & w_{24}^{19\lambda_5} & w_{24}^{23\lambda_5} \\ w_{24}^{1\lambda_6} & w_{24}^{5\lambda_6} & w_{24}^{7\lambda_6} & w_{24}^{11\lambda_6} & w_{24}^{13\lambda_6} & w_{24}^{17\lambda_6} & w_{24}^{19\lambda_6} & w_{24}^{23\lambda_6} \\ w_{24}^{1\lambda_7} & w_{24}^{5\lambda_7} & w_{24}^{7\lambda_7} & w_{24}^{11\lambda_7} & w_{24}^{13\lambda_7} & w_{24}^{17\lambda_7} & w_{24}^{19\lambda_7} & w_{24}^{23\lambda_7} \\ w_{24}^{1\lambda_8} & w_{24}^{5\lambda_8} & w_{24}^{8\lambda_7} & w_{24}^{11\lambda_8} & w_{24}^{13\lambda_8} & w_{24}^{17\lambda_8} & w_{24}^{19\lambda_8} & w_{24}^{23\lambda_8} \end{bmatrix},$$

$$X = \begin{bmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_3^{(1)} \\ c_4^{(1)} \\ c_5^{(1)} \\ c_6^{(1)} \\ c_7^{(1)} \\ c_8^{(1)} \end{bmatrix}, \ b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving the above system of linear equations, we get

$$c_1^{(1)} = 0.6124 - 0.3536i, c_2^{(1)} = 0.6124 + 0.3536i,$$

$$c_3^{(1)} = 0.6124 - 0.3536i, c_4^{(1)} = 0.6124 + 0.3536i,$$

$$c_5^{(1)} = 0.6124 - 0.3536i, c_6^{(1)} = 0.6124 + 0.3536i,$$

$$c_7^{(1)} = 0.6124 - 0.3536i, c_8^{(1)} = 0.6124 + 0.3536i.$$

After solving the system of equation corresponding to $j = 2$, we get

$$c_1^{(2)} = 0.6830 - 0.1830i, c_2^{(2)} = -0.1830 + 0.6830i,$$

$$c_3^{(2)} = 0.1830 + 0.6830i, c_4^{(2)} = -0.6830 - 0.1830i,$$

$$c_5^{(2)} = -0.6830 + 0.1830i, c_6^{(2)} = 0.1830 - 0.6830i,$$

$$c_7^{(2)} = -0.1830 - 0.6830i, c_7^{(2)} = 0.6830 + 0.1830i.$$

After solving the system of equation corresponding to $j = 3$, we get

$$c_1^{(3)} = 0.7071 - 0.0000i, c_2^{(3)} = -0.7071 + 0.0000i,$$

$$c_3^{(3)} = -0.7071 + 0.0000i, c_4^{(3)} = 0.7071 - 0.0000i,$$

$$c_5^{(3)} = 0.7071 - 0.0000i, c_6^{(3)} = -0.7071 + 0.0000i,$$

$$c_7^{(3)} = -0.7071 + 0.0000i, c_8^{(3)} = 0.7071 - 0.0000i.$$

After solving the system of equation corresponding to $j = 4$, we get

$$c_1^{(4)} = 0.6830 + 0.1830i, c_2^{(4)} = -0.1830 - 0.6830i,$$

$$c_3^{(4)} = 0.1830 - 0.6830i, c_4^{(4)} = -0.6830 + 0.1830i,$$

$$c_5^{(4)} = -0.6830 - 0.1830i, c_6^{(4)} = 0.1830 + 0.6830i,$$

$$c_7^{(4)} = -0.1830 + 0.6830i, c_8^{(4)} = 0.6830 - 0.1830i.$$

After solving the system of equation corresponding to $j = 5$, we get

$$c_1^{(5)} = 0.0000 + 0.7071i, c_2^{(5)} = -0.0000 - 0.7071i,$$

$$c_3^{(5)} = 0.0000 + 0.7071i, c_4^{(5)} = -0.0000 - 0.7071i,$$

$$c_5^{(5)} = 0.0000 + 0.7071i, c_6^{(5)} = -0.0000 - 0.7071i,$$

$$c_7^{(5)} = 0.0000 + 0.7071i, c_8^{(5)} - 0.0000 - 0.7071i.$$

After solving the system of equation corresponding to $j = 6$, we get

$$c_1^{(6)} = -0.1830 + 0.6830i, c_2^{(6)} = 0.6830 - 0.1830i,$$

$$c_3^{(6)} = -0.6830 - 0.1830i, c_4^{(6)} = 0.1830 + 0.6830i,$$

$$c_5^{(6)} = 0.1830 - 0.6830i, c_6^{(6)} = -0.6830 + 0.1830i,$$

$$c_7^{(6)} = 0.6830 + 0.1830i, c_8^{(6)} = -0.1830 - 0.6830i.$$

After solving the system of equation corresponding to $j = 7$, we get

$$c_1^{(7)} = -0.3536 + 0.6124i, c_2^{(7)} = 0.3536 + 0.6124i,$$

$$c_3^{(7)} = 0.3536 - 0.6124i, c_4^{(7)} = -0.3536 - 0.6124i,$$

$$c_5^{(7)} = -0.3536 + 0.6124i, c_6^{(7)} = 0.3536 + 0.6124i,$$

$$c_7^{(7)} = 0.3536 - 0.6124i, c_8^{(7)} = -0.3536 - 0.6124i.$$

After solving the system of equation corresponding to $j = 8$, we get

$$c_1^{(8)} = -0.5000 + 0.5000i, c_2^{(8)} = -0.5000 + 0.5000i,$$

$$c_3^{(8)} = 0.5000 + 0.5000i, c_4^{(8)} = 0.5000 + 0.5000i,$$

$$c_5^{(8)} = 0.5000 - 0.5000i, c_6^{(8)} = 0.5000 - 0.5000i,$$

$$c_7^{(8)} = -0.5000 - 0.5000i, c_8^{(8)} - 0.5000 - 0.5000i.$$

Therefore from equation (4.6) the bi-orthonormal basis is given by

$$\begin{aligned}
\psi_1(n) = &(0.6124 - 0.3536i)(\frac{1}{\sqrt{24}}w_{24}^{1n}) + (0.6124 + 0.3536i)(\frac{1}{\sqrt{24}}w_{24}^{5n}) \\
&+ (0.6124 - 0.3536i)(\frac{1}{\sqrt{24}}w_{24}^{7n}) + (0.6124 + 0.3536i)(\frac{1}{\sqrt{24}}w_{24}^{11n}) \\
&+ (0.6124 - 0.3536i)(\frac{1}{\sqrt{24}}w_{24}^{13n}) + (0.6124 + 0.3536i)(\frac{1}{\sqrt{24}}w_{24}^{17n}) \\
&+ (0.6124 - 0.3536i)(\frac{1}{\sqrt{24}}w_{24}^{19n}) + (0.6124 + 0.3536i)(\frac{1}{\sqrt{24}}w_{24}^{23n}).
\end{aligned}$$

$$\begin{aligned}
\psi_2(n) = &(0.6830 - 0.1830i)(\frac{1}{\sqrt{24}}w_{24}^{1n}) + (-0.1830 + 0.6830)(\frac{1}{\sqrt{24}}w_{24}^{5n}) \\
&+ (0.1830 + 0.6830)(\frac{1}{\sqrt{24}}w_{24}^{7n}) + (-0.6830 - 0.1830)(\frac{1}{\sqrt{24}}w_{24}^{11n}) \\
&+ (-0.6830 + 0.1830)(\frac{1}{\sqrt{24}}w_{24}^{13n}) + (0.1830 - 0.6830)(\frac{1}{\sqrt{24}}w_{24}^{17n}) \\
&+ (-0.1830 - 0.6830)(\frac{1}{\sqrt{24}}w_{24}^{19n}) + (0.6830 + 0.1830i)(\frac{1}{\sqrt{24}}w_{24}^{23n}).
\end{aligned}$$

$$\begin{aligned}
\psi_3(n) = &(0.7071 - 0.0000i)(\frac{1}{\sqrt{24}}w_{24}^{1n}) + (-0.7071 + 0.0000i)(\frac{1}{\sqrt{24}}w_{24}^{5n}) \\
&+ (-0.7071 + 0.0000i)(\frac{1}{\sqrt{24}}w_{24}^{7n}) + (0.7071 - 0.0000i)(\frac{1}{\sqrt{24}}w_{24}^{11n}) \\
&+ (0.7071 + 0.0000i)(\frac{1}{\sqrt{24}}w_{24}^{13n}) + (-0.7071 - 0.0000i)(\frac{1}{\sqrt{24}}w_{24}^{17n}) \\
&+ (-0.7071 - 0.0000i)(\frac{1}{\sqrt{24}}w_{24}^{19n}) + (0.7071 + 0.0000i)(\frac{1}{\sqrt{24}}w_{24}^{23n}).
\end{aligned}$$

$$
\begin{aligned}
\psi_4(n) = &(0.6830 + 0.1830i)(\frac{1}{\sqrt{24}}w_{24}^{1n}) + (-0.1830 - 0.6830)(\frac{1}{\sqrt{24}}w_{24}^{5n})\\
&+ (0.1830 - 0.6830)(\frac{1}{\sqrt{24}}w_{24}^{7n}) + (-0.6830 + 0.1830)(\frac{1}{\sqrt{24}}w_{24}^{11n})\\
&+ (-0.6830 - 0.1830)(\frac{1}{\sqrt{24}}w_{24}^{13n}) + (0.1830 + 0.6830)(\frac{1}{\sqrt{24}}w_{24}^{17n})\\
&+ (-0.1830 + 0.6830)(\frac{1}{\sqrt{24}}w_{24}^{19n}) + (0.6830 - 0.1830i)(\frac{1}{\sqrt{24}}w_{24}^{23n}).\\
\psi_5(n) = &(0.0000 + 0.7071i)(\frac{1}{\sqrt{24}}w_{24}^{1n}) + (0.0000 - 0.7071i)(\frac{1}{\sqrt{24}}w_{24}^{5n})\\
&+ (0.0000 + 0.7071i)(\frac{1}{\sqrt{24}}w_{24}^{7n}) + (0.0000 - 0.7071i)(\frac{1}{\sqrt{24}}w_{24}^{11n})\\
&+ (0.0000 + 0.7071i)(\frac{1}{\sqrt{24}}w_{24}^{13n}) + (0.0000 - 0.7071i)(\frac{1}{\sqrt{24}}w_{24}^{17n})\\
&+ (0.0000 + 0.7071i)(\frac{1}{\sqrt{24}}w_{24}^{19n}) + (0.0000 - 0.7071i)(\frac{1}{\sqrt{24}}w_{24}^{23n}).\\
\psi_6(n) = &(-0.1830 + 0.6830i)(\frac{1}{\sqrt{24}}w_{24}^{1n}) + (0.6830 - 0.1830)(\frac{1}{\sqrt{24}}w_{24}^{5n})\\
&+ (-0.6830 - 0.1830)(\frac{1}{\sqrt{24}}w_{24}^{7n}) + (0.1830 + 0.6830)(\frac{1}{\sqrt{24}}w_{24}^{11n})\\
&+ (0.1830 - 0.6830)(\frac{1}{\sqrt{24}}w_{24}^{13n}) + (-0.6830 + 0.1830)(\frac{1}{\sqrt{24}}w_{24}^{17n})\\
&+ (0.6830 + 0.1830)(\frac{1}{\sqrt{24}}w_{24}^{19n}) + (-0.1830 - 0.6830i)(\frac{1}{\sqrt{24}}w_{24}^{23n}).\\
\psi_8(n) = &(-0.3536 + 0.6124i)(\frac{1}{\sqrt{24}}w_{24}^{1n}) + (0.3536 + 0.6124i)(\frac{1}{\sqrt{24}}w_{24}^{5n})\\
&+ (0.3536 - 0.6124i)(\frac{1}{\sqrt{24}}w_{24}^{7n}) + (-0.3536 - 0.6124i)(\frac{1}{\sqrt{24}}w_{24}^{11n})\\
&+ (-0.3536 + 0.6124i)(\frac{1}{\sqrt{24}}w_{24}^{13n}) + (0.3536 + 0.6124i)(\frac{1}{\sqrt{24}}w_{24}^{17n})\\
&+ (0.35360.6124i)(\frac{1}{\sqrt{24}}w_{24}^{19n}) + (-0.3536 - 0.6124i)(\frac{1}{\sqrt{24}}w_{24}^{23n}).
\end{aligned}
$$

Hence the sampling representation is given by

$$x(n) = \sum_{i=1}^{8} x(\lambda_i)\psi_i(n).$$

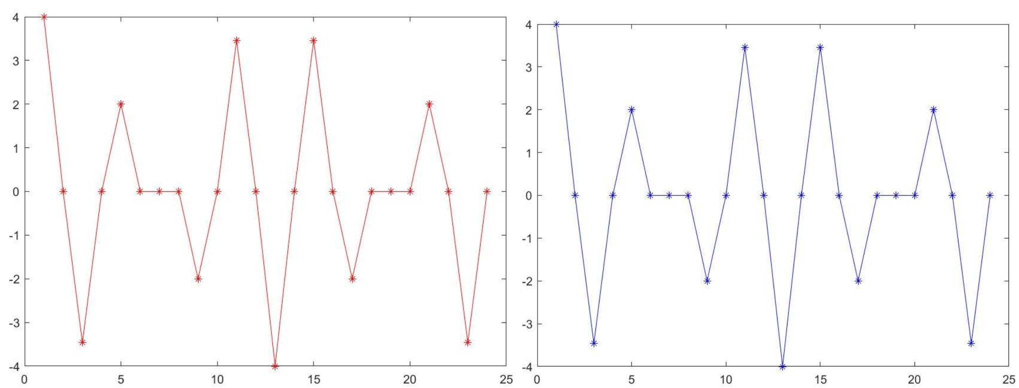
We take a signal $x(n) = 2cos(\frac{10\pi n}{24}) + 2cos(\frac{14\pi n}{24})$, which belongs to the Ramanujan space S_7. At the sampling points $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2, \lambda_4 = 3, \lambda_5 = 4, \lambda_6 =$

5, $\lambda_7 = 6, \lambda_8 = 7$ samples are given by,

$$x(\lambda_1) = x(0) = 4.0000, x(\lambda_2) = x(1) = -0.0000,$$
$$x(\lambda_3) = x(2) = -3.4641, x(\lambda_4) = x(3) = -0.0000,$$
$$x(\lambda_5) = x(4) = 2.0000, x(\lambda_6) = x(5) = -0.0000,$$
$$x(\lambda_7) = x(6) = -0.0000, x(\lambda_8) = x(7) = -0.0000.$$

Then the sampling representation for this signal is given by

$$x(n) = 4\psi_1(n) + (0)\psi_2(n) + (-3.4641)\psi_3(n) + (0)\psi_4(n)$$
$$+ (2)\psi_5(n) + (0)\psi_6(n) + (0)\psi_7(n) + (0)\psi_8(n).$$



(a) Original function (b) Reconstruction

4.3 Analysis of the Sampling process on the Ramanujan space

The first example illustrates the disadvantage of the discrete sampling process in the context of the Ramanujan space S_q when q is a prime number. In this case, it is necessary to have $q - 1$ sampling points to fully recover the signal. This requirement of a large number of sampling points can be time-consuming and resource-intensive, especially for signals with a high value of q. Thus, the disadvantage lies in the increased complexity and computational burden associated with the sampling process.

However, the second example highlights an advantage of the discrete sampling process in the Ramanujan space S_q when q is not prime. In this scenario, the dimension of S_q is $\phi(q)$, which is less than or equal to $\frac{q}{2}$. This implies that a smaller number of sampling points are needed to recover the signal compared to the case when q is not prime. Consequently, the advantage lies in the potential reduction of the sampling requirement, leading to a more efficient and streamlined signal recovery process.

Overall, the advantages and disadvantages of the discrete sampling process in the Ramanujan space S_q depend on the specific characteristics of q, such as its primality and parity. Understanding these properties enables us to assess the sampling requirements and optimize the process accordingly, considering both the computational resources and the desired signal reconstruction accuracy.

4.4 Sampling process in Ramanujan space with respect to integer basis

Ramanujan space S_q is column space of B_q matrix. Now, we determine $\phi(q)$ columns of the integer matrix B_q which can serve as an integer basis for S_q. An arbitrary set of $\phi_(q)$ columns of B_q may not be independent despite the rank of B_q being $\phi_(q)$. Therefore the Ramanujan sum $c_q(n)$ and its $\phi(q) - 1$ consecutive circular shifts constitute an integer basis for the Ramanujan space S_q.

Lemma 4.3. [1] *For $q = 2^m$ where m is any positive ineteger, the Ramanujan sum c_q can be written as*

$$c_q(n) = \begin{cases} 2^{m-1}, \ \textit{for } n = 0, \\ -2^{m-1}, \ \textit{for } n = q/2 = 2^{m-1}, \\ 0 \quad \textit{otherwise.} \end{cases} \tag{4.10}$$

in the fundamental range $0 \le n \le q-1$.

Proof. First note that $(2^m, k) = 1$ is equivalent to $(2, k) = 1$, so that

$$c_{2^m}(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{2^m} w_{2^m}^{kn} = \sum_{i=0}^{2^m-1} \sum_{\substack{l=1 \\ (l,2)=1}}^{2} w_{2^m}^{(i2+l)n},$$

where we have decomposed k as $k = 2i+l$, use the fact that $(2, p) = 1$ is equivalent to $(l, 2) = 1$. We therefore have

$$c_{2^m}(n) = \left(\sum_{i=0}^{2^{m-1}-1} w_{2^{m-1}}^{in}\right)(w_{2^m}^{n}).$$

The first summation is

$$\sum_{i=0}^{2^{m-1}-1} w_{2^{m-1}}^{ni} = \begin{cases} 0 \text{ if } 2^{m-1} \nmid n \\ 2^{m-1} \text{ if } 2^{m-1} | n \end{cases}$$

Only when $n = 2^{m-1}r$ for integer r,

$$w_{2^m}^{n} = \begin{cases} 1 \text{ if } n = 0 \\ -1 \text{ if } n = 2^{m-1} \end{cases}$$

Hence for $q = 2^m$,

$$c_q(n) = \begin{cases} 2^{m-1}, \text{ for } n = 0, \\ -2^{m-1}, \text{ for } n = q/2 = 2^{m-1}, \\ 0 \text{ otherwise.} \end{cases}$$

□

Given that $\phi(q) = 2^{m-1}$, we have $c_q(n) = 0$ for $1 \le n \le \phi(q) - 1$. This implies that the set $\{c_q(n), c_q(n-1), \cdots, c_q(n-\phi(q)+1)\}$ forms an orthogonal basis for the Ramanujan space S_q.

The Discrete Fourier Transform (DFT) of $c_q(n)$ is denoted as $C_q[k]$, and it is defined as follows:

$$C_q[k] = \begin{cases} q, \text{ if } (k, q) = 1, \\ 0, \text{ otherwise.} \end{cases}$$

Using Parseval's relation for DFTs, we can establish that:

$$\sum_{n=0}^{q-1} c_q^2(n) = \sum_{k=0}^{q-1} C_q^2[k]/q = q^2\phi(q)/q = q\phi(q).$$

Since $q = 2^m$, we can substitute this value into the above equation, yielding $\alpha^2 = \sum_{n=0}^{q-1} c_q^2(n) = 2^{2m-1}$. Therefore, the set $\{\frac{1}{\alpha}c_q(n), \frac{1}{\alpha}c_q(n-1), \cdots, \frac{1}{\alpha}c_q(n-\phi(q)+1)\}$ forms an orthonormal basis for the Ramanujan space S_q, where $\alpha = \sqrt{2^{2m-1}}$. Based on the lemma provided earlier, we deduce the following theorem.:

Theorem 4.4. *Let $q = 2^m$, m being a very large natural number. Then for any x in the Ramanujan space S_q, we have the following representation*

$$x(n) = \sum_{i=1}^{\phi(q)} x(\lambda_i)\psi_i(n),$$

where $\lambda_i = i - 1$ for $i = 1, 2, \cdots, \phi(q) - 1$, $\{\psi_i\}$ is bi-orthogonal to reproducing kernel, that is $\langle k(., \lambda_i), \psi_j\rangle = \delta_{ij}$.

Proof. In Ramanujan space S_q, orthonormal basis is $\{c_q(n-l)\}_{l=0}^{\phi(q)-1}$. Then by equation reproducing kernel is given by

$$\begin{aligned} K(m,n) &= \sum_{l=0}^{\phi(q)-1} \frac{1}{\alpha}c_q(m-l)\overline{\frac{1}{\alpha}c_q(n-l)} \\ &= \sum_{l=0}^{\phi(q)-1} \frac{1}{\alpha}c_q(m-l)\frac{1}{\alpha}c_q(n-l), \end{aligned}$$

as $\{c_q(n-l)\}_{l=0}^{\phi(q)-1}$ is integer basis for Ramanujan space S_q. Now, we select arbitrary set of points $\{\lambda_1 = 0, \lambda_2 = 1, \cdots, \lambda_{\phi(q)} = \phi(q)-1\}$. Now, we claim that $\{k(., \lambda_i)\}_{i=1}^{\phi(q)}$ spanning set for the Ramanujan space S_q.

Now consider,

$$\begin{aligned} K(m,\lambda_1) =& K(m,0) \\ =& \sum_{l=0}^{\phi(q)-1} \frac{1}{\alpha}c_q(m-l)\frac{1}{\alpha}c_q(0-l). \end{aligned}$$

In this last sum $c_q(-l) = c_q(q-l)$ because $c_q(n)$ has q periods. Also from (4.10), we obtain

$$K(m,\lambda_1) = K(m,0) = \frac{1}{\alpha}c_q(m)\frac{1}{\alpha}c_q(0).$$

Similarly, we have

$$K(m, \lambda_2) = K(m, 1) = \frac{1}{\alpha} c_q(m-1) \frac{1}{\alpha} c_q(0),$$

$$\vdots$$

$$K(m, \lambda_{\phi(q)}) = K(m, \phi(q)-1) = \frac{1}{\alpha} c_q(m-\phi(q)+1) \frac{1}{\alpha} c_q(0).$$

The Ramanujan sum $c_q(n)$ and its $\phi(q)-1$ consecutive circular shifts constitute an integer basis for the Ramanujan space S_q. Thus the set $\{K(m, \lambda_i)\}_{i=1}^{\phi(q)}$ is a spanning set for S_q. Now S_q is finite dimensional it must then possess a unique bi-orthonormal set $\{\psi_j\}_{j=1}^{\phi(q)}$, say; that is, a set such that

$$\langle K(., \lambda_i), \psi_j \rangle = \delta_{ij}.$$

where the functions ψ_j's are given by

$$\psi_j(n) = \sum_{l=0}^{\phi(q)-1} a_l^{(j)} \frac{1}{\alpha} c_q(n-l) \tag{4.11}$$

where $j = 1, 2, \cdots, \phi(q)$. The expansion for $x \in S_q$ in the set $\{\psi_j\}_{j=1}^{\phi(q)}$ is of the form

$$x(n) = \sum_{i=1}^{\phi(q)} b_i \psi_i(n)$$

in which $b_i = \langle x, K(., \lambda_i) \rangle = x(\lambda_i)$, so we have the sampling representation

$$x(n) = \sum_{i=1}^{\phi(q)} x(\lambda_i) \psi_i(n).$$

□

We now present the technique for obtaining a bi-orthonormal basis. Let us examine the singularity of the matrix:

$$\Delta = \begin{bmatrix} c_q(\lambda_1) & c_q(\lambda_1 - 1) & \cdots & c_q(\lambda_1 - \phi(q)+1) \\ c_q(\lambda_2) & c_q(\lambda_2 - 1) & \cdots & c_q(\lambda_2 - \phi(q)+1) \\ \vdots & \vdots & \ddots & \vdots \\ c_q(\lambda_{\phi(q)}) & c_q(\lambda_{\phi(q)} - 1) & \cdots & c_q(\lambda_{\phi(q)} - \phi(q)+1) \end{bmatrix}.$$

This matrix has dimensions $\phi(q) \times \phi(q)$.

By rearranging the terms, we can express Δ as:

$$\Delta = \begin{bmatrix} c_q(0) & c_q(-1) & \cdots & c_q(-\phi(q)+1) \\ c_q(1) & c_q(1-1) & \cdots & c_q(1-\phi(q)+1) \\ \vdots & \vdots & \ddots & \vdots \\ c_q(\phi(q)-1) & c_q(\phi(q)-1) & \cdots & c_q(\phi(q)-1-\phi(q)+1) \end{bmatrix}.$$

According to the definition (4.10), $c_q(n)$ can be expressed as 2^{m-1} for $n = 0$ and 0 for $0 < n \leq \phi(q) - 1$. Therefore, Δ simplifies to:

$$\Delta = \begin{bmatrix} 2^{m-1} & 0 & \cdots & 0 \\ 0 & 2^{m-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2^{m-1} \end{bmatrix}$$

This is a diagonal matrix with non-zero elements on the principal diagonal. Hence, Δ is a non-singular matrix. By solving the system of linear equations derived from equation (4.4), we can determine the coefficients of the bi-orthonormal basis. The equation can be written as:

$$\psi_j(\lambda_i) = \sum_{l=0}^{\phi(q)-1} a_l^{(j)} \frac{1}{\alpha} c_q(\lambda_i - l) = \delta_{ij}, \tag{4.12}$$

Here, $j = 1, 2, \ldots, \phi(q)$, $i = 1, 2, \ldots, \phi(q)$, and $a_l^{(j)}$ are the coefficients of ψ_j. For a fixed j, each i from 1 to $\phi(q)$ represents a system of linear equations with $\phi(q)$ unknown variables $\{a_1^{(j)}, a_2^{(j)}, \ldots, a_{\phi(q)}^{(j)}\}$. The matrix Δ represents the coefficients of the system of linear equations. Since Δ is non-singular, the system of linear equations has a unique solution.

Now, we provide an example based on the above theorem.

Example 4.3. Let us consider $q = 2^3$. We find the sampling formula for the Ramanujan space S_{2^3} with dimension 2^2. Then the Ramanujan sum c_{2^3} and its $(2^2 - 1)$ consecutive circular shifts constitute an integer basis for the Ramanujan space S_{2^3}. From (4.10) we know that

$$c_{2^3}(n) = \begin{cases} 2^2, \text{ for } n = 0, \\ -2^2 \text{ for } n = 2^3/2 = 2^2, \\ 0 \text{ otherwise} \end{cases}$$

in the fundamental range $0 \leq n \leq 2^3 - 1$.

Here the set $\{c_{2^3}(n), c_{2^3}(n-1), \cdots, c_{2^3}(n-2^2+1)\}$ is an orthogonal basis for the Ramanujan space S_{2^3}. Also $\alpha^2 = \sum_{n=0}^{2^3-1} c_{2^3}^2(n) = 2^{2*3-1} = 2^5$. Hence the set $\{\frac{1}{\alpha}c_{2^3}(n), \frac{1}{\alpha}c_{2^3}(n-1), \cdots, \frac{1}{\alpha}c_{2^3}(n-2^2+1)\}$ is an orthonormal basis for the Ramanujan space S_{2^3}. We choose the set of points $\{\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2, \lambda_4 = 3\}$ as sampling points such that

$$\begin{aligned} K(m,\lambda_1) = K(m,0) &= \frac{1}{\alpha}c_{2^3}(m)\frac{1}{\alpha}c_{2^3}(0), \\ K(m,\lambda_2) = K(m,1) &= \frac{1}{\alpha}c_{2^3}(m-1)\frac{1}{\alpha}c_{2^3}(0), \\ K(m,\lambda_3) = K(m,2) &= \frac{1}{\alpha}c_{2^3}(m-2)\frac{1}{\alpha}c_{2^3}(0), \\ K(m,\lambda_4) = K(m,3) &= \frac{1}{\alpha}c_{2^3}(m-3)\frac{1}{\alpha}c_{2^3}(0). \end{aligned}$$

Also the Ramanujan sum c_{2^3} and its $2^2 - 1$ consecutive circular shifts constitute an integer basis for the Ramanujan space S_{2^3}. Hence the set $\{K(m,\lambda_i)\}_{i=1}^{\phi(2^3)}$ is a spanning set for S_{2^3}. From equation (4.11) the biorthonormal basis functions are defined by

$$\psi_j(n) = \sum_{l=0}^{2^2-1} a_l^{(j)} \frac{1}{\alpha} c_{2^3}(n-l),$$

where $j = 1, 2, \cdots, 2^2$. Here the matrix Δ can be written as

$$\Delta = \begin{bmatrix} 2^2 & 0 & 0 & 0 \\ 0 & 2^2 & 0 & 0 \\ 0 & 0 & 2^2 & 0 \\ 0 & 0 & 0 & 2^2 \end{bmatrix}$$

which is a diagonal matrix with non-zero principal diagonal elements. Hence the matrix Δ is a non-singular matrix. Now from equation (4.12) we have

$$\psi_j(\lambda_i) = \sum_{l=0}^{2^2-1} a_l^{(j)} c_{2^3}(\lambda_i - l) = \delta_{ij},$$

where $j = 1, 2, \cdots, 2^2$, $i = 1, 2, \cdots, 2^2$, and $\{a_l^{(j)}\}$ coefficients of ψ_j. For $j = 1$, we get a system of equations

$$\psi_1(\lambda_i) = \sum_{l=0}^{3} a_l^{(1)} c_{2^3}(\lambda_i - l) = \delta_{i1},$$

where $i = 1, 2, 3, 4$. Equivalently, this system of equations can be written as

$$AX = b,$$

where

$$A = \frac{1}{\alpha}\begin{bmatrix} 2^2 & 0 & 0 & 0 \\ 0 & 2^2 & 0 & 0 \\ 0 & 0 & 2^2 & 0 \\ 0 & 0 & 0 & 2^2 \end{bmatrix}, \ x = \begin{bmatrix} a_0^{(1)} \\ a_1^{(1)} \\ a_2^{(1)} \\ a_3^{(1)} \end{bmatrix}, \ b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving the above system of linear equations, we get

$$a_0^{(1)} = 1.4142, a_1^{(1)} = 0, a_2^{(2)} = 0, a_3^{(3)} = 0.$$

Similarly, after solving the system of linear equations corresponding to $j = 2, 3, 4$, we get

$$a_0^{(2)} = 0, a_1^{(2)} = 1.4142, a_2^{(2)} = 0, a_3^{(2)} = 0,$$

$$a_0^{(3)} = 0, a_1^{(3)} = 0, a_2^{(3)} = 1.4142, a_3^{(3)} = 0,$$

$$a_0^{(4)} = 0, a_1^{(4)} = 0, a_2^{(4)} = 0, a_3^{(4)} = 1.4142,$$

respectively. Therefore from equation (4.11) the bi-orthonormal basis functions are given by

$$\psi_1(n) = 1.4142(\frac{1}{\alpha}c_{2^3}(n)), \psi_2(n) = 1.4142(\frac{1}{\alpha}c_{2^3}(n-1)),$$
$$\psi_3(n) = 1.4142(\frac{1}{\alpha}c_{2^3}(n-2)), \psi_4(n) = 1.4142(\frac{1}{\alpha}c_{2^3}(n-3)),$$

respectively. Hence the sampling representation is given by

$$x(n) = \sum_{i=1}^{4} x(\lambda_i)\psi_i(n).$$

CHAPTER 5

Future plan

In the future, we will focus on developing algorithms for the discrete sampling process in the Ramanujan space. Our goal is to improve the existing methods and incorporate explicit error estimation techniques to ensure more accurate signal reconstruction. Additionally, we plan to explore the discrete sampling process within polynomial spaces that exist in the Ramanujan space. This investigation will provide valuable insights into the properties and applications of these polynomial spaces, expanding the possibilities for signal processing techniques within the Ramanujan framework. These efforts will contribute to advancing efficient and reliable signal processing algorithms and expanding our understanding of Ramanujan space analysis and its practical implications.

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