Introduction to Coupled Fractional Fourier Transform

M.Sc. Thesis

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"Introduction to Coupled Fractional Fourier Transform"** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF MATHEMATICS**, **Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2022 to June 2023 under the supervision of **Dr. Santanu Manna**, Assistant Professor, Department of Mathematics, IIT Indore.

The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

05/06/23

Signature of the student with date
(**Mansi Yadav**)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

05/06/2023

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Abstract

This thesis includes an introduction to the Coupled Fractional Fourier Transform (CFrFT) and its various properties. The CFrFT is a generalized version of the Fractional Fourier transform (FrFT) that extends FrFT to two-dimension. The thesis begins with a comprehensive review of the Fourier transforms (FT) and fractional Fourier transforms, providing a solid foundation for the study of the CFrFT. The mathematical framework of the CFrFT is then introduced, including the definition, properties, and some important theorems with examples.

The main focus of the thesis is to investigate the properties of the CFrFT, including its linearity, symmetry, conjugation, time-frequency shift, etc. Additionally, certain properties related to its convolution operator, like commutativity, associativity have been explored. Moreover, relation between convolution of the product of CFrFTs of two functions with CFrFT of the product of these functions has also been investigated. The results of this thesis contribute to a deeper understanding of the CFrFT and its properties. The findings may be of interest to researchers and practitioners in the fields of electrical engineering and mathematics.

This thesis contains four chapters.

Chapter 1 contains a brief introduction to integral transforms and Fourier transform. The definition and inversion formula for FT, as well as some examples has been included. A brief overview of features of FT such as linearity, time shift, and so on has been included. This chapter concludes with the limitations of FT.

Chapter 2 contains the definition and inversion formula for FrFT and some properties of FrFT. The convolution theorem and Parseval's relation are included. This chapter also include FrFT of a function with different angles.

In chapter 3, the introduction to CFrFT, its definition and inversion formula are introduced along with example. Several properties of CFrFT like linearity, symmetry, conjugation, time-frequency shift, etc. are derived with detailed proofs. This chapter also contains theorems and properties related to the convolution operator for CFrFT. This is the main contribution chapter in our thesis.

Contents

List of Figures

Chapter 1

Introduction

Various natural phenomena in physics, quantum mechanics, sciences, etc. are modeled using differential equations, and integral equations. One such mathematical tool which has been widely used to tackle these problems is integral transform. In recent years, a lot of researchers throughout the globe have worked on integral transformations and their applications in various fields of science and technology. Integral transforms such as Fourier transforms, Laplace transforms, Wavelet transforms, etc. are widely used to handle problems related to solving complicated differential equations and integral equations.

An integral transformation refers to a mathematical process where in a function is integrated with respect to a parameter, resulting in the creation of a new function.

Integral Transform: The integral transform of a function $h(t)$ defined in $a \leq t \leq b$ is denoted by $\mathcal{I}h(t) = H(\omega)$, and is defined by

$$\mathcal{I}h(t) = H(\omega) = \int_a^b K(t,\omega)h(t)\,dt, \tag{1.1}$$

where, $K(t,\omega)$ is known as the *kernel* of the integral transform.

Most of the properties like linearity, duality, symmetry, etc. of any integral transform depends on its kernel.

Integral transformation enable the transfer of function from one domain to another, leading to a situation where the process of manipulation and problem solving becomes notably easier compared to the initial domain. The inverse of the integral transform can then be used to map the solution back to the original domain. It has been extensively employed as a mathematical technique to address problems in diverse fields such as physics, applied mathematics, and mechanics.

One such integral transform is Fourier transform. Before going into the Fourier transform, we must initially define $\mathcal{L}^p(\mathbb{R})$.

1.1 The $\mathcal{L}^p(\mathbb{R})$ Space

$\mathcal{L}^p$ spaces are a family of function spaces that are used to study the properties of functions that are integrable to some power. In this dissertation, we will be focused on $\mathcal{L}^p(\mathbb{R})$. $(\mathcal{L}^p(\mathbb{R}), ||.||)$ forms a normed linear space where norm of any function $g \in \mathbb{R}$ is given by

$$||g||_p = \begin{cases} \left(\int_{-\infty}^{\infty} |g|^p \, dx\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ ess\,Sup|g(x)|, & p = \infty. \end{cases} \tag{1.2}$$

This number $||g||_p$ is known as $\mathcal{L}^p$ norm. $\mathcal{L}^p$ spaces are Banach spaces, that means they are complete normed vector spaces. Infact, for $p = 2$, they form a Hilbert space. Our work in this thesis is concentrated on $\mathcal{L}^1(\mathbb{R})$ and $\mathcal{L}^2(\mathbb{R})$.

1.2 The Fourier Transform

The Fourier Transform is a fundamental technique with numerous applications in the fields of signal and image processing, systems for communication, and filtering technologies [1, 2]. It enables us to decompose a signal into its constituent frequencies and analyze its frequency content. The FT has led to significant advancements in the fields of engineering, physics, and mathematics. It decomposes a function into sine and cosine components.

Definition:

For a given function $h(t)$, its Fourier transform, $\mathcal{F}(h(t))$, denoted by $\hat{h}(\omega)$, is defined as

$$\mathcal{F}(h(t)) = H(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-i\omega t}\, dt, \tag{1.3}$$

provided the integral exists.

Now, we will see the Fourier transform of some functions.

Example 1: Consider characteristic function χ_a(t)

$$\chi_a(t) \quad = \quad \begin{cases} 1, & -a < t < a \\ 0, & \text{otherwise} \end{cases}. \tag{1.4}$$

Then, the Fourier transform of $\chi_a(t)$ is given by

$$\hat{\chi}_a(\omega) \quad = \quad \int_{-\infty}^{\infty} \chi_a(t) e^{-i\omega t}\, dt = \int_{-a}^{a} e^{-i\omega t}\, dt = \frac{2}{\omega} \sin(\omega a).$$

Hence, the Fourier transform of $\chi_a(t)$ is given by $\hat{\chi}_a(\omega) = \frac{2}{\omega}\sin(\omega a)$.

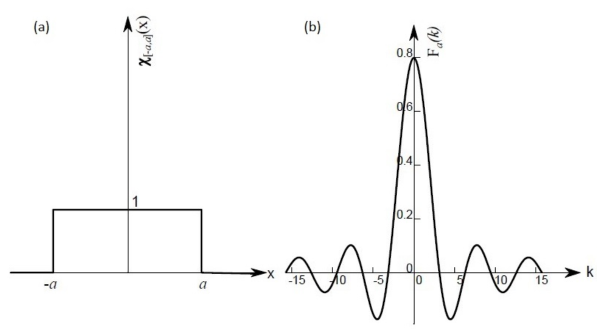

Figure 1.1: (a) $\chi_{[-a,a]}(t)$ and (b) its Fourier transform

Source: https://link.springer.com/book/10.1007/978-1-4612-0097-0

One should notice here that $\chi_{[-a,a]}(t) \in \mathcal{L}^1(\mathbb{R})$ but $\hat{\chi}_{[-a,a]}(\omega) \notin \mathcal{L}^1(\mathbb{R})$. This function $\chi_{[-a,a]}(t)$ is frequently referred to as a rectangular pulse or gate function in science and engineering. It can also be used to analyze the statistical properties of signals, such as noise and interference, in communication systems.

Example 2: Consider function $\psi(t)= t^2 e^{-ct^2}$, for $c > 0$. Then, the FT of $\psi(t)$ is given by $\hat{\psi}(\omega) = \frac{ie^{-\frac{\omega^2}{4c}}}{2\sqrt{2}c^{\frac{3}{2}}}$.

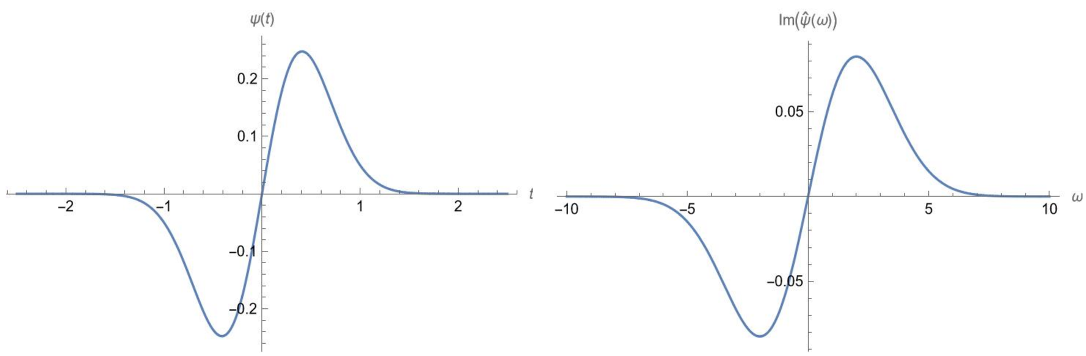

Figure 1.2: $\psi(t)$ and its Fourier transform with $c = 3$.

Inversion Formula:

For a given function $\hat{\phi}(\omega)$, its inverse Fourier transform is defined as

$$\phi(t) = (\mathcal{F}^{-1}\hat{\phi}(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t}\hat{\phi}(\omega)\, d\omega, \quad t \in \mathbb{R}. \tag{1.5}$$

If $\phi(t)$ is continuous, then eq (1.5) holds for every $t \in \mathbb{R}$.

1.2.1 Basic Properties of Fourier Transform

If $\psi(t)$ and $\phi(t)$ are two functions with FT $\hat{\psi}(\omega)$ and $\hat{\phi}(\omega)$ respectively, and α, β are in $\mathbb{C}$. Then we have the following results.

Basic Properties of FT		
Property	Function	Fourier transform
(i) Linearity	$\alpha\psi(t) + \beta\phi(t)$	$\alpha\hat{\psi}(\omega) + \beta\hat{\phi}(\omega)$
(ii) Scaling	$\phi(at)$	$\frac{1}{a}\hat{\phi}(\frac{\omega}{a})$
(iii) Time Shift	$\phi(t-a)$	$e^{-i\omega a}\hat{\phi}(\omega)$
(iv) Frequency Shift	$e^{iat}\phi(t)$	$\hat{\phi}(\omega - a)$
(v) Conjugation	$\overline{\phi(t)}$	$\overline{\hat{\phi}(-\omega)}$

For the proofs of above properties one can take the help of [6].

1.2.2 Convolution

Definition: If $\phi(t)$ and $\psi(t)$ are two functions, then their convolution is denoted by $(\phi * \psi)(t)$ and is given by

$$(\phi * \psi)(t) = \int_{-\infty}^{\infty} \phi(\tau)\psi(t-\tau)\, d\tau, \quad \text{for} \quad t \in \mathbb{R}, \tag{1.6}$$

provided the integral eq (1.6) exists.

Properties of Convolution:

For the functions $f(t)$, $\phi(t)$ and $\psi(t)$, the following properties can be given.

(i) *Commutative:* $\phi * \psi = \psi * \phi$

(ii) *Associative:* $\phi * (\psi * f) = (\phi * \psi) * f$

(iii) *Distributive:* $(af + b\psi) * \phi = a(f * \phi) + b(\psi * \phi)$, where $a, b \in \mathbb{C}$

Convolution Theorem:

Let $g(t)$ be a time domain function with $\hat{g}(\omega)$ as its frequency domain FT. This output function $\hat{g}(\omega)$ is frequently multiplied with other frequency domain functions, such as $\hat{h}(\omega)$. The problem now is to tackle which time domain function will have Fourier transform as the product of $\hat{g}(\omega)$ and $\hat{h}(\omega)$? The convolution theorem provides an answer to this topic.

Statement: If $\phi(t)$ and $\psi(t)$ are two functions with FT $\hat{\phi}(\omega)$ and $\hat{\psi}(\omega)$, then

$$\mathcal{F}(\phi * \psi)(t) = \hat{\phi}(\omega)\hat{\psi}(\omega). \tag{1.7}$$

It is observed follow from the convolution theorem that if $\phi(t)$ and $\psi(t)$ are two functions with FT $\hat{\phi}(\omega)$ and $\hat{\psi}(\omega)$, then

$$\int_{-\infty}^{\infty} \phi(t)\overline{\psi(t)}\, dt = \int_{-\infty}^{\infty} \hat{\phi}(\omega)\overline{\hat{\psi}(\omega)}\, d\omega \quad \text{(Parseval's relation)}, \tag{1.8}$$

$$\int_{-\infty}^{\infty} |\phi(t)|^2\, dx = \int_{-\infty}^{\infty} |\hat{\phi}(\omega)|^2\, d\omega \quad \text{(Plancherel's relation)}. \tag{1.9}$$

1.3 Limitations of Fourier Transform

We conclude this chapter by listing few limitations of FT which lead to further development of wavelet transforms, Gabor transforms, fractional Fourier transforms, etc.

- The Fourier transform approach allows us to explore problems in either the frequency domain or time (space) domain, rather than both domains at the same time.

- The Fourier Transform is beneficial when evaluating stationary data but it is unsuitable for analyzing non-stationary information.

To overcome limitations of FT, FrFT was introduced. FrFT has been demonstrated to be a valuable technique for managing differential and integral equations. Namias [12] explained the generalized operational calculus associated with FrFT and proposed approaches to using FrFT to handle ordinary and partial differential equations with quadratic terms in the variables.

The advantages of working with FrFT over the usual Fourier transform to tackle differential equations are carefully examined by McBride et al. [13]. Kerr [14] studied the theory of FrFT on $\mathcal{L}^2$ as well as used it to resolve some partial differential equations. Applications of FrFT on a generalized wave equation and generalized nth-order linear nonhomogeneous ordinary differential equations are also presented by Prasad et al. [3]. FrFT have several applications in the fields of optics, identifying patterns, encoding messages, and processing signals and images [19, 20, 21, 22, 23, 24]. R Iwai and H Yoshimura in their work [29] displayed how applying FrFT can decrease processing time for fingerprint data while simultaneously improving matching accuracy. There are various other uses of FrFT like in radar systems and in cryptography which has been examined by various researchers like Sun, Djurovic, Amein, Youssef, Ran, Cusamario, etc. in their works [36, 37, 38, 39, 40]. Other applications of FrFT include pattern recognition data compression, tomography, etc.

The CFrFT is a novel extension of the two-dimensional FrFT introduced by Zayed [8]. Unlike the previous extensions of the FrFT, the kernel used in the CFrFT is not a tensor product of one-dimensional functions. However, it depends on the coupling of two angles to produce a new set of transformation parameters. Zayed [8] also deduced several of its features, including its inversion formula, convolution structure and Poisson summation formula. Using the kernel of CFrFT, Kamalakkannan et al. [10] introduced short-time CFrFT and derived its inversion formula and properties. In 2022, Kamalakkannan et al. [9] provided extension of some of the properties of CFrFT to $\mathcal{L}^2(\mathbb{R}^2)$ and showed that CFrFT satisfies additive property.

The CFrFT is a novel integral transformation and few of its basic properties like symmetry, linearity, conjugation, time-frequency shift, etc. are still unknown. In this thesis, we present these properties with detailed proofs. Few results related to convolution operator and product theorems are also addressed.

Although the focus of this thesis is on CFrFT, references to works on FrFT are included because they may have the potential to serve as applications of CFrFT.

Chapter 2

The Fractional Fourier Transform

The Fourier transform has been a vital and irreplaceable tool in the realm of signal analysis and processing for investigating and manipulating information in the domain of frequencies. However, the traditional Fourier transform operates on the assumption that the signal remains stationary throughout its entire duration. This assumption is often limiting when dealing with non-stationary signals, such as those encountered in time-varying systems or communication channels. The FT also treats the time and frequency domain as an orthogonal system, neglecting the vital role that phase information plays in signal analysis.

To overcome these limitations, a more versatile transform known as the fractional Fourier transform (FrFT) was introduced by V. Namias [12] in the 1980s . The fractional Fourier transform extends the concept of the traditional Fourier transform by introducing a parameter called the fractional order. This parameter allows a smooth transition between the time and frequency domains, providing a richer representation of signals that incorporates both magnitude and phase information. By varying the fractional order, we can selectively emphasize different aspects of a signal, revealing hidden characteristics that may not be apparent in the standard Fourier domain.

2.1 Fractional Operations

Several of the significant conceptual advancements are based on the process of going from the total of an entity to its fractions. Like generalizing concepts of addition and multiplication from whole numbers to fractional numbers. Similarly, the generalization of the derivative operator to fractional powers lead to development of important differential equations like the fractional Fornberg-Whitham equation, time-fractional Sawada-Kotera equation, Riesz fractional Sine-Gordon equation, etc.

We can certainly find nth-order transform of any integral transformation $\mathcal{I}$, by simply operating $\mathcal{I}$, n-times. That is,

$$\mathcal{I}^n g(t) = \underbrace{\mathcal{I} \circ \ldots \circ \mathcal{I}}_{n\text{-}times} g(t), \tag{2.1}$$

where, $n \in \mathbb{N}$ and $\circ$ denotes function composition operation. For negative natural numbers, say m, $\mathcal{I}^m$ is defined as

$$\mathcal{I}^m g(t) = \underbrace{\mathcal{I}^{-1} \circ \ldots \circ \mathcal{I}^{-1}}_{m\text{-}times} g(t), \tag{2.2}$$

where, $\mathcal{I}^{-1}$ denotes inverse of integral transformation.

It is important to understand, what is a fractional transform, and how can we convert an integral transform to a fractional transform?
Let $\mathcal{I}$ denotes an integral transform as defined in eq (1.1). Now consider the following new transform

$$\mathcal{I}^\alpha g(t) = G^\alpha(\omega), \quad \text{where,} \quad \alpha \in \mathbb{R}. \tag{2.3}$$

When $\mathcal{I}^\alpha$ satisfies

$$\mathcal{I}^0 g(t) = g(\omega) \tag{2.4}$$

$$\text{and} \quad \mathcal{I}^1 g(t) = G(\omega), \tag{2.5}$$

then $\mathcal{I}^\alpha$ is called "α-order fractional $\mathcal{I}$ transform". Moreover the parameter α is known as the "fractional order."

2.2 FrFT Representation in Time and Frequency Domain

Almeida [15], Ozaktas et al. [25] have formulated the FrFT time-frequency representation. The FrFT is a type of time-frequency representation widely used in the signal-processing field. While representing the FT operator in time-frequency representation, a plane having two perpendicular axes that correspond to time and frequency is used. The FT operator can be viewed as a shift in the signal's representation corresponding to an anti-clockwise rotation of the axis by an angle $\frac{\pi}{2}$, whereas, FrFT of angle α, $\{\mathcal{F}^{\alpha}\}$ rotates the input signal corresponding to the initial coordinates (x, y) anticlockwise to the set of coordinates (ω, ξ) with an angle α in the time-frequency plane, as indicated in the figure 2.1.

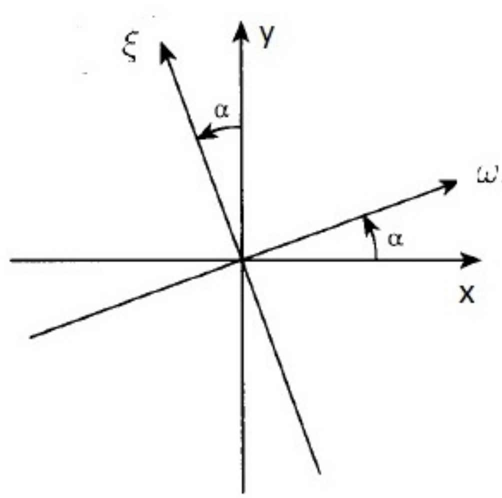

Figure 2.1: Time-frequency representation

Let $\alpha = \frac{k\pi}{2}$ in eq (2.6), here $k \in \mathbb{R}$, then the FrFT with angle α is also referred as FrFT of k-th order. Mathematically, this means that k-th order FrFT is simply k-th power of FT.

Observations:

- When $k = 4l,\ l \in \mathbb{Z}$, then k-th order FrFT reduces to identity operator, i.e.,
$$\mathcal{F}^{\alpha} = \mathcal{F}^{2l\pi} = \mathcal{F}^{0} = \mathcal{I},$$
where, $\mathcal{I}$ denotes identity operator.

- When $k = 1$, then FrFT of angle $\alpha = \frac{\pi}{2}$ reduces to ordinary FT, i.e.,
$$\mathcal{F}^{\alpha} = \mathcal{F}^{\frac{\pi}{2}} = \mathcal{F}.$$

- When $k = 2$, then FrFT of angle $\alpha = \pi$ reduces to reflection operator, i.e.,
$$\mathcal{F}^{\alpha} = \mathcal{F}^{\pi} = \mathcal{R},$$
where, $\mathcal{R}$ denotes reflection operator, i.e., $\mathcal{R}\phi(x) = \phi(-x)$.

- When $k = 3$, then FrFT of angle $\alpha = \frac{3\pi}{2}$ reduces to inverse FT, ie.,
$$\mathcal{F}^{\alpha} = \mathcal{F}^{\frac{3\pi}{2}} = \mathcal{F}^{-1}.$$

For detailed explanation of the above properties one can take the help of reference [31].

2.3 Definition and Inversion Formula for FrFT

The integral expression for FrFT was introduced by Namais in 1980 [12] using the fact that $e^{\frac{in\pi}{2}}$ are the eigenvalues of FT with corresponding eigenfunctions $e^{-\frac{1}{2}x^2}h_n(x)$, where $h_n(x)$ are the Hermite functions of order n. He generalized FT to FrFT with an angle α in a way that eigenvalues of FrFT are $e^{in\alpha}$ and corresponding eigenfunctions are $e^{-\frac{1}{2}x^2}h_n(x)$.

Definition:

One-dimensional FrFT [3], with an angle α, $\{\mathcal{F}^{\alpha}\}$ of $\psi(x)$ is defined as
$$(\mathcal{F}^{\alpha}\psi)(\omega) = \hat{\psi}^{\alpha}(\omega) = \int_{-\infty}^{\infty} K^{\alpha}(x,\omega)\psi(x)dx, \tag{2.6}$$

where, the kernel of the transform is

$$K^\alpha(x,\omega) = \begin{cases} C^\alpha e^{i(x^2+\omega^2)\frac{\cot\alpha}{2} - ix\omega\csc\alpha}, & \text{if } \alpha \neq n\pi, \\ \frac{1}{\sqrt{2\pi}} e^{-ix\omega}, & \text{if } \alpha = \frac{\pi}{2}, \\ \delta(x-\omega), & \text{if } \alpha = 2n\pi, \\ \delta(x+\omega), & \text{if } \alpha = 2(n+1)\pi, \quad n \in \mathbb{Z}, \end{cases} \tag{2.7}$$

and $C^\alpha = \sqrt{\frac{1-i\cot\alpha}{2\pi}}$.

Inversion Formula:

The inversion formula for FrFT associated with with an angle α is given by [3]

$$\psi(x) = \int_{-\infty}^{\infty} \overline{K^\alpha(x,\omega)} \hat{\psi}^\alpha(\omega)\, d\omega, \tag{2.8}$$

where,

$$\overline{K^\alpha(x,\omega)} = \overline{C^\alpha} e^{-i(x^2+\omega^2)\frac{cot\alpha}{2} + ix\omega\csc\alpha} = K^{-\alpha}(x,\omega), \tag{2.9}$$

and

$$\overline{C^\alpha} = \sqrt{\frac{1+i\cot\alpha}{2\pi}} = C^{-\alpha}. \tag{2.10}$$

It should be noted that the FrFT with an angle α has its inverse as the FrFT with an angle $-\alpha$.

Example: Consider the function $\phi(x) = x^4 e^{-x^2}$, then its FrFT of order α, is given by

$$\begin{aligned} \hat{\phi}^\alpha(\omega) &= \int_{-\infty}^{\infty} K^\alpha(x,\omega)\phi(x)dx \\ &= e^{\frac{\omega^2(i+2\cot\alpha)}{2(2i+\cot\alpha)}} \frac{\sqrt{1-i\cot\alpha}\left(12-3\cot^2\alpha-12\omega^2\csc^2\alpha+\omega^4\csc^4\alpha+6i\cot\alpha\left(-2+\omega^2\csc^2\alpha\right)\right)}{\sqrt{2-i\cot\alpha}(2i+\cot\alpha)^4}. \end{aligned}$$

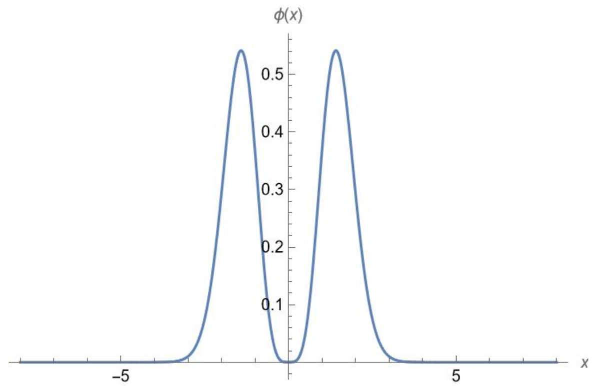

Figure 2.2: Graph of $\phi(x)$

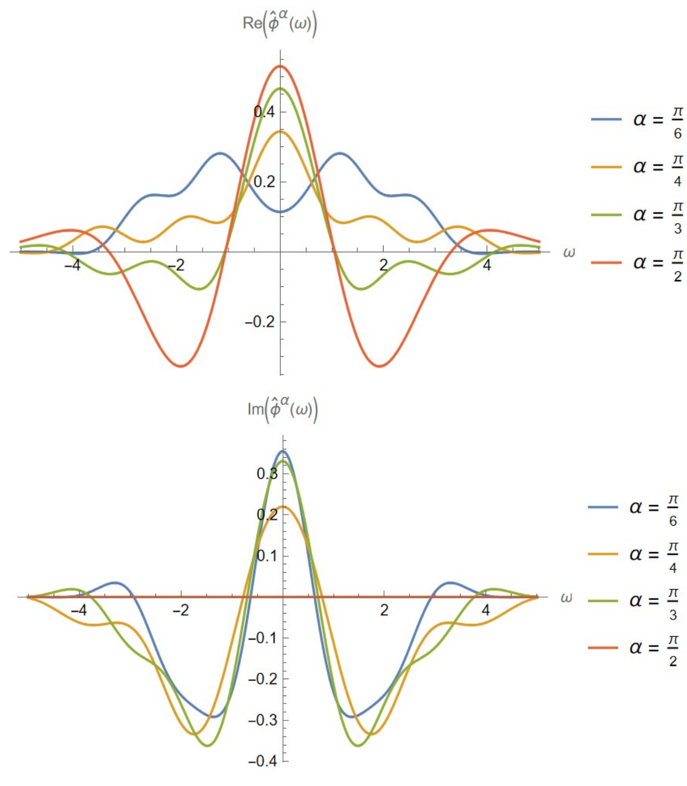

Figure 2.3: FrFT of $\phi(x)$ with angles $\alpha = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$

2.4 Basic Properties of FrFT

If $g(x)$ and $h(x)$ are two functions having FrFT with an angle α, as $\mathcal{F}^\alpha g(\omega)$ and $\mathcal{F}^\alpha h(\omega)$ respectively, and a_1, a_2 are in $\mathbb{C}$. Then we have the following results.

Basic Properties of FrFT [12]		
Property	Function	FrFT
(i) Linearity	$a_1 g(x) + a_2 h(x)$	$a_1 \mathcal{F}^\alpha g(\omega) + a_2 \mathcal{F}^\alpha h(\omega)$
(ii) Translation	$g(x-c)$	$e^{\frac{i}{2} c \sin\alpha\cos\alpha - ic\omega\sin\alpha} \mathcal{F}^\alpha g(\omega - c\cos\alpha)$
(iii) Frequency Shift	$e^{ikx} g(x)$	$e^{-i\frac{k^2}{2}\sin\alpha\cos\alpha + ik\omega\cos\alpha} \mathcal{F}^\alpha g(\omega - k\sin\alpha)$
(iv) Reflection	$g(-x)$	$\mathcal{F}^\alpha g(-\omega)$

2.5 Convolution

In 1997, Almeida in his research [27] presented an extension of convolution theorems for FT and examined the FrFT of product and convolution of two functions. However, his conclusions were not pleasing as a result of the product and convolution theorems for FT. So, in 1998, Zayed [28] presented a new convolution structure for FrFT. In this section, we will present Zayed's convolution structure for FrFT and its properties.

Definition:
Let $\psi(x)$ be a function, define $\tilde{\psi}(x) = \psi(x) e^{i\frac{\cot\alpha}{2}x^2}$ and $\bar{\psi}(x) = \psi(x) e^{-i\frac{\cot\alpha}{2}x^2}$. The convolution operation $\circledast$ for any two functions $\psi(x)$ and $\phi(x)$ is defined as

$$h(x) = (\psi \circledast \phi)(x) = C^\alpha e^{-i\frac{\cot\alpha}{2}x^2} (\tilde{\psi} * \tilde{\phi})(x), \tag{2.11}$$

where, $*$ denotes the usual convolution defined in eq (1.6).

2.5.1 Convolution Theorem

Theorem [32] Let $\psi(x)$ and $\phi(x)$ be two functions with FrFT $\psi^\alpha(\omega)$ and $\phi^\alpha(\omega)$ respectively and $h(x) = (\psi \circledast \phi)(x)$, then

$$H^\alpha(\omega) = \psi^\alpha(\omega) \phi^\alpha(\omega) e^{-i\frac{\cot\alpha}{2}\omega^2}, \tag{2.12}$$

where $H^\alpha(\omega)$ denotes FrFT of $h(x)$.

2.6 Parseval's Relation

Let $\phi(x)$ and $\psi(x)$ be two functions with FrFT $\phi^\alpha(\omega)$ and $\psi^\alpha(\omega)$ respectively, then

$$\int_{-\infty}^{\infty} \phi(x)\overline{\psi(x)}\,dx = \int_{-\infty}^{\infty} \phi^\alpha(\omega)\overline{\psi^\alpha(\omega)}\,d\omega, \tag{2.13}$$

$$\int_{-\infty}^{\infty} |\psi(x)|^2\,dx = \int_{-\infty}^{\infty} |\psi^\alpha(\omega)|^2\,d\omega. \tag{2.14}$$

Chapter 3

The Coupled Fractional Fourier Transform

3.1 Introduction

In chapter 3, we learned about the Fractional Fourier Transform (FrFT). It has proven to be a powerful mathematical tool in signal processing and communication systems. It provides a way to analyze signals in the frequency-time domain, allowing for a more precise analysis of non-stationary signals. However, in some cases, the FrFT may not be sufficient to accurately analyze signals that have complex time-frequency behaviors.

To address this issue, researchers have introduced an extension of the FrFT, known as the Coupled Fractional Fourier Transform (CFrFT). The CFrFT is a type of 2D FrFT $\mathcal{F}_{\gamma,\theta}$ which depends on two coupled angles $\xi = \frac{\gamma+\theta}{2}$ and $\eta = \frac{\gamma-\theta}{2}$ as transformation parameters. It can act as a handy tool in several applications in signal processing and optics.

In this chapter, we have introduced the CFrFT and provided a comprehensive overview of its properties. In this chapter we have started by introducing the definitions of the CFrFT and with some examples. We also have introduced some mathematical properties of the CFrFT, including the inversion formula, and convolution theorem.

In recent years, many scientists have been attempting to expand FrFT to higher dimensions. Due to the fact that input data like photos, videos, etc. have dimensions higher than one, these extensions are quite important in practice.

Earlier, the extensions to n-dimensional space were made using a tensor product of n copies of a one-dimensional transform. But Zayed [8] derived a novel strategy for bringing FrFT into the second dimension. He did not use the standard tensor product scheme; instead, he extended using eigenfunctions.

As we have already seen in chapter 1, the eigenfunctions of the Fourier transform are Hermite functions of order n with eigenvalues of i^n. Similarly, Hermite functions of order n, with different eigenvalues λ_n are the eigenfunctions of the FrFT of order θ, where, $0 \le \theta \le 1$. The eigenvalues of the FrFT approach i^n as θ approaches to 1.

To define the novel transform CFrFT the Hermite functions of two complex variables are used as eigenfunctions.

3.2 Complex Hermite Polynomials

These polynomials are a natural extension of the classical Hermite polynomials, which are typically defined over the real line. However, contrary to the real Hermite polynomials, the complex Hermite polynomials are defined across the complex plane and are capable of capturing the behavior of complex-valued functions.

In this section, a brief introduction to complex Hermite polynomials is given. For more information of complex Hermite polynomials one can read [11].

Definition: The complex Hermite polynomials for two variables x and y is defined as

$$H_{m,n}(x,y) = \sum_{k=0}^{m \wedge n} (-1)^k k! \begin{pmatrix} m \\ k \end{pmatrix} \begin{pmatrix} n \\ k \end{pmatrix} x^{m-k} y^{n-k}. \tag{3.1}$$

where, m and n are non-negative integers and $m \wedge n = min\{m, n\}$.

The expression representing the generating function of complex Hermite polynomials is provided as follows

$$\sum_{m,n=0}^{\infty} H_{m,n}(x,y) \frac{t^m}{m!} \frac{s^n}{n!} = \mathrm{e}^{tx+ssy-ts}, \tag{3.2}$$

and the orthogonality relation fulfilled by them is given as

$$\frac{1}{\pi} \int_{\mathbb{R}^2} H_{m_1,n_1}(x+iy, x-iy) \bar{H}_{m_2,n_2}(x+iy, x-iy) \mathrm{e}^{-x^2-y^2} \, dx \, dy = m_1! n_1! \delta_{m_1,m_2} \delta_{n_1,n_2}. \tag{3.3}$$

where, δ_{m_1,m_2} represents the Kronecker delta function.

3.3 Definition and Inversion Formula for CFrFT

Definition:

Let $\gamma, \theta \in \mathbb{R}$ such that $\gamma + \theta \neq 2n\pi$, where $n \in \mathbb{Z}$, then for a given function $f \in \mathcal{L}^1(\mathbb{R}^2)$, its CFrFt is given by

$$\mathcal{F}_f^{\gamma,\theta}(x,y,s,t) = \int_{\mathbb{R}^2} f(x,y) K_{\gamma,\theta}(x,y,s,t) \, dx \, dy, \tag{3.4}$$

where,

$$\begin{aligned} K_{\gamma,\theta}(x,\, y,\, s,\, t) &= d(\xi) \exp\left\{-a(\xi)\left(x^2+y^2+s^2+t^2\right)\right. \\ &\quad \left. + b(\xi,\eta)(sx+ty) + c(\xi,\eta)(tx-sy)\right\}, \end{aligned} \tag{3.5}$$

$$\left. \begin{array}{ll} a = a(\xi) = \frac{i\cot\xi}{2}, & b = b(\xi,\eta) = \frac{i\cos\eta}{\sin\xi}, \\ c = c(\xi,\eta) = \frac{i\sin\eta}{\sin\xi}, & d = d(\xi) = \frac{ie^{-i\xi}}{2\pi\sin\xi}, \end{array} \right\} \tag{3.6}$$

and

$$\xi = \frac{\gamma+\theta}{2} \quad \text{and} \quad \eta = \frac{\gamma-\theta}{2}. \tag{3.7}$$

- The CFrFT is also known as 2D FrFT.
- When there is no ambiguity regarding kernel, we will denote $\mathcal{F}_f^{\gamma,\theta}(x,y,s,t)$ by $\mathcal{F}^{\gamma,\theta}(f)(s,t)$.

Remark 1:

In a special case when $\gamma = \theta$, then $\xi = \gamma$ and $\eta = 0$, which implies,

$$\mathcal{F}_f^{\gamma}(x,y,s,t) = \int_{\mathbb{R}^2} f(x,y) d(\gamma) \exp\left\{-a(\gamma)\left(x^2+y^2+s^2+t^2\right) + b(\gamma)(sx+ty)\right\} \tag{3.8}$$

where,

$$\left.\begin{aligned} a(\gamma) &= \tfrac{i\cot\gamma}{2}, \\ b(\gamma) &= i\csc\gamma, \\ d(\gamma) &= \tfrac{ie^{-i\gamma}}{2\pi\sin\gamma}. \end{aligned}\right\} \tag{3.9}$$

Now, it is easy to observe from above equations that when $\gamma = \theta$, then the CFrFT can be expressed as tensor combination of two 1-D FrFTs.

Remark 2:

If $\gamma = \theta = \frac{\pi}{2}$, then the CFrFT simplifies to the traditional two dimensional FT.

Inversion Formula [8]**:**

The inversion formula for the CFrFT is defined as follows:

Statement: If $f \in \mathcal{L}^1(\mathbb{R}^2)$ and its CFrFT $\mathcal{F}_f^{\gamma,\theta} \in \mathcal{L}^1(\mathbb{R}^2)$ then the inverse CFrFT of $\mathcal{F}_f^{\gamma,\theta}$ is defined as

$$\begin{aligned} f(x,y) = d(-\xi) \int_{\mathbb{R}^2} \mathcal{F}_f^{\gamma,\theta}(x,y,s,t) d(\xi) \exp\{ & a(\xi)\left(x^2+y^2+s^2+t^2\right) \\ & -b(\xi,\eta)(sx+ty) - c(\xi,\eta)(tx-sy)\}\, ds\, dt. \end{aligned} \tag{3.10}$$

As,

$$\left.\begin{aligned} a(-\xi) &= -a(\xi), \\ b(-\xi,-\eta) &= -b(\xi,\eta), \\ c(-\xi,-\eta) &= c(\xi,\eta). \end{aligned}\right\} \tag{3.11}$$

Thus,

$$\begin{aligned} K_{-\gamma,-\theta}(s,t,x,y) = d(-\xi)\exp\{ & a(\xi)\left(x^2+y^2+s^2+t^2\right) \\ & -b(\xi,\eta)(sx+ty) - c(\xi,\eta)(tx-sy)\}. \end{aligned} \tag{3.12}$$

It is worth noting that the above-mentioned inversion formula can be seen as

$$f(x,y) = \int_{\mathbb{R}^2} \mathcal{F}_f^{\gamma,\theta}(s,t) K_{-\gamma,-\theta}(s,t,x,y)\, ds\, dt. \tag{3.13}$$

3.4 Properties of CFrFT

In this section of the chapter, we will investigate the CFrFT's properties that are necessary to analyze coupled signals. These properties include linearity, reflection, conjugation, and time-frequency shift. By understanding these properties, we will be able to efficiently analyze 2D signals and extract meaningful information from them.

1. Linearity:

Let $f(x,y) = \alpha\phi(x,y) + \beta\psi(x,y)$, where, $\phi(x,y)$ and $\psi(x,y)$ are two functions in $\mathcal{L}^1(\mathbb{R}^2)$ and $\alpha, \beta \in \mathbb{C}$, then,

$$\mathcal{F}_f^{\gamma,\theta}(x,y,s,t) = \alpha\,\mathcal{F}_\phi^{\gamma,\theta}(x,y,s,t) + \beta\,\mathcal{F}_\psi^{\gamma,\theta}(x,y,s,t). \tag{3.14}$$

Proof:

$$\begin{aligned}
\mathcal{F}_\phi^{\gamma,\theta}(x,y,s,t) &= \int_{\mathbb{R}^2} f(x,y)K_{\gamma,\theta}(x,y,s,t)\,dx\,dy \\
&= \int_{\mathbb{R}^2} (\alpha\phi(x,y) + \beta\psi(x,y))K_{\gamma,\theta}(x,y,s,t)\,dx\,dy \\
&= \alpha\int_{\mathbb{R}^2} \phi(x,y)K_{\gamma,\theta}(x,y,s,t)\,dx\,dy \\
&\quad + \beta\int_{\mathbb{R}^2} \psi(x,y)K_{\gamma,\theta}(x,y,s,t)\,dx\,dy \\
&= \alpha\,\mathcal{F}_\phi^{\gamma,\theta}(x,y,s,t) + \beta\,\mathcal{F}_\psi^{\gamma,\theta}(x,y,s,t).
\end{aligned} \tag{3.15}$$

The above property shows that CFrFT is a linear transform.

2. Reflection Property:

Let $\phi(x,y)$ be a function with $\mathcal{L}^1(\mathbb{R}^2)$ with CFrFT $\mathcal{F}_\phi^{\gamma,\theta}(x,y,s,t)$, then the CFrFT of its reflection $\bar{\phi} = \phi(-x,-y)$ is given by $\mathcal{F}_\phi^{\gamma,\theta}(x,y,-s,-t)$.

Proof:

$$\begin{aligned}
\mathcal{F}_{\bar{\phi}}^{\gamma,\theta}(x,y,s,t) &= \int_{\mathbb{R}^2} \phi(-x,-y)K_{\gamma,\theta}(x,y,s,t)\,dx\,dy, \\
&= \int_{\mathbb{R}^2} \phi(-x,-y)\big[d(\xi)exp\{-a(\xi)(x^2+y^2+s^2+t^2) \\
&\quad + b(\xi,\eta)(sx+ty) + c(\xi,\eta)(xt-ys)\}\big]\,dx\,dy,
\end{aligned}$$

Substituting $-x = u$ and $-y = v$, we get

$$\begin{aligned}
\mathcal{F}_{\tilde{\phi}}^{\gamma,\theta}(x,y,s,t) &= \int_{\mathbb{R}^2} \phi(u,v)\left[d(\xi)exp\{-a(\xi)(u^2+v^2+s^2+t^2)\right. \\
&\quad +b(\xi,\eta)(u(-s)+v(-t)) \\
&\quad \left.+c(\xi,\eta)(u(-t)-v(-s))\}\right]\,du\,dv \\
&= \int_{\mathbb{R}^2} \phi(u,v)K_{\gamma,\theta}(u,v,-s,-t)\,du\,dv
\end{aligned}$$

Now by changing the variables, we have

$$\begin{aligned}
\mathcal{F}_{\tilde{\phi}}^{\gamma,\theta}(x,y,s,t) &= \int_{\mathbb{R}^2} \phi(x,y)K_{\gamma,\theta}(x,y,-s,-t)\,dx\,dy \\
&= \mathcal{F}_{\phi}^{\gamma,\theta}(x,y,-s,-t). \qquad (3.16)
\end{aligned}$$

Physically, this property claims that the CFrFT of the function with reflection in the input domain coincides with the CFrFT of the original function with reflection in output domain.

3. Conjugation Property:

(i) Let $\phi(x,y)$ be a function with $\mathcal{L}^1(\mathbb{R}^2)$ with CFrFT $\mathcal{F}_{\phi}^{\gamma,\theta}(x,y,s,t)$, then the CFrFT of its conjugation $\phi^* = \phi(x,-y)$ is given by $\mathcal{F}_{\tilde{\phi}}^{\gamma,\theta}(x,y,-s,-t)$, where, $\tilde{\phi}(x,y) = \phi(-x,y)$.

Proof:

Since,

$$\begin{aligned}
\mathcal{F}_{\phi^*}^{\gamma,\theta}(x,y,s,t) &= \int_{\mathbb{R}^2} \phi^* K_{\gamma,\theta}(x,y,s,t)\,dx\,dy \\
&= \int_{\mathbb{R}^2} \phi(x,-y)\left[d(\xi)exp\left\{-a(\xi)(x^2+y^2+s^2+t^2)\right.\right. \\
&\quad \left.\left.+b(\xi,\eta)(sx+ty)+c(\xi,\eta)(xt-ys)\right\}\right]\,dx\,dy \qquad (3.17)
\end{aligned}$$

Substituting $-y = u$, we get

$$\begin{aligned}
\mathcal{F}_{\phi^*}^{\gamma,\theta}(x,y,s,t) &= -\int_{\mathbb{R}^2} \phi(x,u)\left[d(\xi)exp\{-a(\xi)(x^2+u^2+s^2+t^2)\right. \\
&\quad +b(\xi,\eta)((-s(-x))+(-t)(u)) \\
&\quad \left.+c(\xi,\eta)((-t)(-x)-(-s)(u))\}\right]\,dx\,du
\end{aligned}$$

Again substituting $x = -w$, we get

$$\begin{aligned}
\mathcal{F}^{\gamma,\theta}_{\phi^*}(x,y,s,t) &= \int_{\mathbb{R}^2} \phi(-w,u)\Big[d(\xi)exp\{-a(\xi)(w^2+u^2+s^2+t^2) \\
&\quad +b(\xi,\eta)((-s(w))+(-t)(u)) \\
&\quad +c(\xi,\eta)((-t)(w)-(-s)(u))\}\Big]\,dw\,du \\
&= \int_{\mathbb{R}^2} \phi(-w,u)K_{\gamma,\theta}(w,u,-s,-t)\,dw\,du \\
&= \int_{\mathbb{R}^2} \phi(-x,y)K_{\gamma,\theta}(x,y,-s,-t)\,dx\,dy \\
&= \mathcal{F}^{\gamma,\theta}_{\tilde{\phi}}(x,y,-s,-t),
\end{aligned}$$

where, $\tilde{\phi}(x,y) = \phi(-x,y)$.

(ii) If $\mathcal{F}^{\gamma,\theta}_{\phi}(x,y,s,t)$ is the CFrFT of a function $\phi \in \mathcal{L}^1(\mathbb{R}^2)$, then,

$$\left[\mathcal{F}^{\gamma,\theta}_{\phi}(x,y,s,t)\right]^* = e^{2i\xi}\mathcal{F}^{\gamma,\theta}_{\bar{\phi}}(x,y,s,t), \tag{3.18}$$

where, $\bar{\phi}(x,y) = \phi^*(-x,-y)$.

Proof:

Since,

$$\begin{aligned}
\left[\mathcal{F}^{\gamma,\theta}_{\phi}(x,y,s,t)\right]^* &= \int_{\mathbb{R}^2} \phi^*(x,y)K^*_{\gamma,\theta}(x,y,s,t)\,dx\,dy \\
&= \int_{\mathbb{R}^2} \phi^*(x,y)d^*(\xi)exp\left\{\frac{i\cot\xi}{2}(x^2+y^2+s^2+t^2)\right. \\
&- \left.\frac{i\cos\eta}{\sin\xi}(sx+ty) - \frac{i\sin\eta}{\sin\xi}(xt-ys)\right\}\,dx\,dy
\end{aligned}$$

Note that $d^*(\xi) = -\exp(2i\xi)d(\xi)$.

$$\begin{aligned}
\left[\mathcal{F}^{\gamma,\theta}_{\phi}(x,y,s,t)\right]^* = -\int_{\mathbb{R}^2} \phi^*(x,y)d(\xi)exp\{2i\xi\}\exp\{\frac{i\cot\xi}{2}(x^2+y^2+s^2+t^2) \\
-\frac{i\cos\eta}{\sin\xi}(sx+ty) - \frac{i\sin\eta}{\sin\xi}(xt-ys)\}\,dx\,dy
\end{aligned}$$

Substitute $x = -u$ and $y = -w$, then we get

$$\begin{aligned}
\left[\mathcal{F}^{\gamma,\theta}_{\phi}(x,y,s,t)\right]^* &= \int_{\mathbb{R}^2} \phi^*(-u,-w)\exp\{2i\xi\}\exp\{\frac{-i\cot\xi}{2}(u^2+w^2+s^2+t^2) \\
&+ \frac{i\cos\eta}{\sin\xi}(sw+tu) + \frac{i\sin\eta}{\sin\xi}(tw-su)\}\,dt\,dw \\
&= e^{2i\xi}\int_{\mathbb{R}^2} \phi^*(-u,-w)K_{\gamma,\theta}(u,w,s,t)\,dt\,dw \\
&= e^{2i\xi}\int_{\mathbb{R}^2} \phi^*(-x,-y)K_{\gamma,\theta}(x,y,s,t)\,dx\,dy \\
&= e^{2i\xi}\mathcal{F}^{\gamma,\theta}_{\bar{\phi}}(x,y,s,t),
\end{aligned}$$

where, $\bar{\phi}(x,y) = \phi^*(-x,-y)$.

Thus we have,

$$\left[\mathcal{F}_{\phi}^{\gamma,\theta}(x,y,s,t)\right]^* = e^{2i\xi}\mathcal{F}_{\bar{\phi}}^{\gamma,\theta}(x,y,s,t), \tag{3.19}$$

where, $\bar{\phi}(x,y) = \phi^*(-x,-y)$.

The first part of this property investigates CFrFT of the conjugation of the input function, while the conjugation of the output function is covered in the latter part.

5. Joint Time-Frequency Shift Property:

If $\phi(x,y) \in \mathcal{L}^1(\mathbb{R}^2)$, then the CFrFT of its joint time-frequency shifted form defined by $g(x,y) = \phi(x-t_1, y-t_2)e^{i(xk_1+yk_2)}$ is given by

$$e^{a\left(\frac{k_1^2+k_2^2}{b^{*2}}+\frac{2(sk_1+tk_2)}{b^*}\right)-\frac{c}{b^*}(t_2k_1-t_1k_2)}\mathcal{F}_{\tilde{\tilde{\phi}}}^{\gamma,\theta}\left(x+t_1, y+t_2, s+\frac{k_1}{b^*}, t+\frac{k_2}{b^*}\right), \tag{3.20}$$

where, $\tilde{\tilde{\phi}}(x,y) = \phi(x,y)e^{\frac{c}{b^*}\{(y+t_2)k_1-(x+t_1)k_2\}}$.

Proof:

$$\begin{aligned}
\mathcal{F}_{g}^{\gamma,\theta}(x,y,s,t) &= \int_{\mathbb{R}^2}\phi(x-t_1,y-t_2)e^{i(xk_1+yk_2)}K_{\gamma,\theta}(x,y,s,t)\,dx\,dy \\
&= \int_{\mathbb{R}^2}\phi(x-t_1,y-t_2)e^{i(xk_1+yk_2)} \\
&\quad \times \left[d^*e^{-ia^*(x^2+y^2+s^2+t^2)+ib^*(sx+ty)+ic^*(xt-ys)}\right]dx\,dy
\end{aligned}$$

where,

$a^* = \frac{\cot\xi}{2}$, $\quad b^* = \frac{\cos\eta}{\sin\xi}$, $\quad c^* = \frac{\sin\eta}{\sin\xi}$, and $\quad d^* = d(\xi)$.

$$
\begin{aligned}
\mathcal{F}_g^{\gamma,\theta}(x,y,s,t) \quad &= \int_{\mathbb{R}^2} \phi(x-t_1, y-t_2)\, d^*\, e^{i(xk_1+yk_2)-ia^*(x^2+y^2+s^2+t^2)} \\
&\quad \times \;\; e^{ib^*(sx+ty)+ic^*(xt-ys)}\, dx\, dy \\
&= \int_{\mathbb{R}^2} \phi(u,v)\, d^*\, e^{-ia^*\left\{(u+t_1)^2+(v+t_2)^2+\left(s+\frac{k_1}{b^*}\right)^2\left(t+\frac{k_2}{b^*}\right)^2\right\}} \\
&\quad \times e^{ib^*\left((u+t_1)(s+\frac{k_1}{b^*})+(v+t_2)(t+\frac{k_2}{b^*})\right)+ic^*\left((u+t_1)(t+\frac{k_2}{b^*})-(v+t_2)(s+\frac{k_1}{b^*})\right)} \\
&\quad \times \;\; e^{ia^*\left(\frac{k_1^2+k_2^2}{b^{*2}}+\frac{2(sk_1+tk_2)}{b^*}\right)-ic^*\left(\frac{(u+t_1)k_2-(v+t_2)k_1}{b^*}\right)}\, du\, dv \\
&= \int_{\mathbb{R}^2} \phi(u,v) K_{\gamma,\theta}\big(u+t_1, v+t_2, s+\frac{k_1}{b^*}, t+\frac{k_2}{b^*}\big) \\
&\quad \times \;\; e^{ia^*\left(\frac{k_1^2+k_2^2}{b^{*2}}+\frac{2(sk_1+tk_2)}{b^*}\right)-ic^*\left(\frac{(u+t_1)k_2-(v+t_2)k_1}{b^*}\right)}\, du\, dv \\
&= e^{a\left(\frac{k_1^2+k_2^2}{b^{*2}}+\frac{2(sk_1+tk_2)}{b^*}\right)-\frac{c}{b^*}(t_2k_1-t_1k_2)} \int_{\mathbb{R}^2} \tilde{\tilde{\phi}}(u,v) \\
&\quad \times \;\; K_{\gamma,\theta}\left(u+t_1, v+t_2, s+\frac{k_1}{b^*}, t+\frac{k_2}{b^*}\right) du\, dv
\end{aligned}
$$

where, $\tilde{\tilde{\phi}}(x,y) = \phi(x,y) e^{\frac{c}{b^*}\{(y+t_2)k_1-(x+t_1)k_2\}}$.

So,

$\mathcal{F}_g^{\gamma,\theta}(x,y,s,t) = e^{a\left(\frac{k_1^2+k_2^2}{b^{*2}}+\frac{2(sk_1+tk_2)}{b^*}\right)-\frac{c}{b^*}(t_2k_1-t_1k_2)} \mathcal{F}_{\tilde{\tilde{\phi}}}^{\gamma,\theta}\left(x+t_1, y+t_2, s+\frac{k_1}{b^*}, t+\frac{k_2}{b^*}\right).$

Remark:

- **Time Shift Property**

  If $\phi(x,y) \in \mathcal{L}^1(\mathbb{R}^2)$, then the CFrFT of its time shifted form denoted by $g(x,y) = \phi(x-t_1, y-t_2)$, is defined by

  $$\mathcal{F}_g^{\gamma,\theta}(x,y,s,t) = \mathcal{F}_\phi^{\gamma,\theta}\big(x+t_1, y+t_2, s, t\big). \tag{3.21}$$

  This property asserts that time-shifting in the input function results in the same amount of time-shift in CFrFT of the function.

- **Frequency Shift Property**

  If $\phi(x,y) \in \mathcal{L}^1(\mathbb{R}^2)$, then the CFrFT of its frequency shifted form denoted by $g(x,y) = \phi(x,y)e^{i(xk_1+yk_2)}$, is defined by

  $$\mathcal{F}_g^{\gamma,\theta}(x,y,s,t) = e^{a\left(\frac{k_1^2+k_2^2}{b^{*2}}+\frac{2(sk_1+tk_2)}{b^*}\right)} \mathcal{F}_{\bar{\bar{\phi}}}^{\gamma,\theta}\big(x+, y, s+\frac{k_1}{b^*}, t+\frac{k_2}{b^*}\big), \tag{3.22}$$

  where, $\bar{\bar{\phi}} = \phi(x,y)e^{\frac{c}{b^*}(yk_1-xk_2)}$.

The proof for both time shift and frequency shift properties can be derived from the proof of joint time-frequency shift property.

3.5 Convolution for CFrFT

The convolution plays a crucial role in integral transforms. In the context of the Fourier transform, for example, the convolution product of two functions in the time domain corresponds to the pointwise multiplication of their Fourier transforms in the frequency domain. This property allows us to express convolution operations in the time domain as simple multiplication operations in the frequency domain, making certain calculations much simpler.

Convolution and product theorems for FrFT have been studied by [27], but since the results were not nice as Fourier transform Zayed [28] later modified the convolution structure that preserved the convolution theorem in Fourier transform.

This section contains both the convolution formula and the corresponding convolution theorems for CFrFT.

3.5.1 Definition

For any function $\phi(x,y)$ define $\tilde{\phi}(x,y) = e^{-a(x^2+y^2)}\phi(x,y)$. Then the convolution for any two functions $\phi(x,y)$ and $\psi(x,y)$ is given as [8]

$$h(x,y) = (\phi \star \psi) = d(\xi)e^{a(x^2+y^2)}(\tilde{\phi} * \tilde{\psi})(x,y), \tag{3.23}$$

where, '$*$' denotes the usual convolution given by

$$(\phi * \psi)(x,y) = \int_{\mathbb{R}^2} \phi(x-\tau, y-\lambda)\psi(\tau,\lambda)\, d\tau\, d\lambda. \tag{3.24}$$

3.5.2 Convolution Theorems

Theorem 3.5.2(i): Let $\phi(x,y)$ and $\psi(x,y)$ be two functions in $\mathcal{L}^p(\mathbb{R}^2)$, $p = 1,2$ and $\mathcal{L}^1(\mathbb{R}^2)$, respectively and $h(x,y) = (\phi \star \psi)(x,y)$. Also, let Φ, Ψ, and

H denotes CFrFT of ϕ, ψ, and h respectively. Then

$$H(s,t) = e^{a(s^2+t^2)}\Phi(s,t)\Psi(s,t). \tag{3.25}$$

This is a more generalized version of the convolution theorem proved by Zayed in [8]. For the proof of the above theorem, one may refer [8, 9].

Remark: The above theorem states that the CFrFT of convolution of two functions (satisfying the above conditions) is equal to the product of individual CFrFT of those functions with some exponential factor multiplication.

Now, we will prove another important theorem related to convolution operator.

Define $\bar{\phi}(x,y) = e^{a(x^2+y^2)}\phi(x,y)$ and an operator '$\circ$' as

$$(\phi \circ \psi)(x,y) = e^{-a(x^2+y^2)}(\bar{\phi} * \bar{\psi})(x,y) \tag{3.26}$$

Theorem 3.5.2(ii): If $\phi(x,y)$ and $\psi(x,y)$ are two functions in $\mathcal{L}^1(\mathbb{R}^2)$, and Φ and Ψ represents CFrFT of ϕ and ψ respectively, then

$$(\Phi \circ \Psi)(s,t) = \frac{1}{d(-\xi)}\mathcal{F}^{\gamma,\theta}\Big(\phi(x,y)\psi(x,y)e^{-a(x^2+y^2)}\Big)(s,t) \tag{3.27}$$

Proof:

Since,

$$(\Phi \circ \Psi)(s,t) = e^{-a(s^2+t^2)}(\bar{\Phi} * \bar{\Psi})(s,t)$$

$$= e^{-a(s^2+t^2)} \int_{\mathbb{R}^2} e^{a(\tau^2+\lambda^2)}\Phi(\tau,\lambda)e^{a((s-\tau)^2+(v-\lambda)^2)}\Psi(s-\tau,t-\lambda)\,d\tau\,d\lambda$$

$$= e^{-a(s^2+t^2)} \int_{\mathbb{R}^2} e^{a(\tau^2+\lambda^2)}\left[\int_{\mathbb{R}^2}\phi(x,y)K_{\gamma,\theta}(x,y,\tau,\lambda)dx\,dy\right]e^{a((s-\tau)^2+(v-\lambda)^2)}$$

$$\times \quad \Psi(s-\tau,t-\lambda)\,d\tau\,d\lambda$$

$$= e^{-a(s^2+t^2)} \int_{\mathbb{R}^2} e^{a(\tau^2+\lambda^2)}$$

$$\times \quad \left[\int_{\mathbb{R}^2}\phi(x,y)d(\xi)e^{-a(x^2+y^2+\tau^2+\lambda^2)+b(\tau x+\lambda y)+c(\lambda x-\tau y)}\,dx\,dy\right]$$

$$\times \quad e^{a((s-\tau)^2+(v-\lambda)^2)}\Psi(s-\tau,t-\lambda)\,d\tau\,d\lambda$$

$$= e^{-a(s^2+t^2)}d(\xi)\int_{\mathbb{R}^2}\phi(x,y)\,dx\,dy$$

$$\times \int_{\mathbb{R}^2}\Psi(s-\tau,t-\lambda)\,e^{a((s-\tau)^2+(v-\lambda)^2)-a(x^2+y^2)+b(\tau x+\lambda y)+c(\lambda x-\tau y)}\,d\tau\,d\lambda$$

Substitute $\tau = s-u$ and $\lambda = t-v$

$$= e^{-a(s^2+t^2)}d(\xi)\int_{\mathbb{R}^2}\phi(x,y)\,dx\,dy$$

$$\times \int_{\mathbb{R}^2}\Psi(u,v)e^{a(u^2+v^2)-a(x^2+y^2)+b\big((s-u)x+(t-v)y\big)}\,e^{c\big((t-v)x-(s-u)y\big)}\,du\,dv$$

$$= \frac{e^{-a(s^2+t^2)}d(\xi)}{d(-\xi)}\int_{\mathbb{R}^2}\phi(x,y)e^{-2a(x^2+y^2)+b(sx+ty)+c(tx-sy)}\,dx\,dy$$

$$\times \int_{\mathbb{R}^2}\Psi(u,v)d(-\xi)e^{a(u^2+v^2)-b(ux+vy)-c(vx-uy)}\,du\,dv$$

$$= \frac{e^{-a(s^2+t^2)}d(\xi)}{d(-\xi)}\int_{\mathbb{R}^2}\phi(x,y)e^{-2a(x^2+y^2)+b(sx+ty)+c(tx-sy)}\psi(x,y)\,dx\,dy$$

$$= \frac{1}{d(-\xi)}\int_{\mathbb{R}^2}e^{-a(x^2+y^2)}\phi(x,y)\psi(x,y)\,d(\xi)e^{-a(x^2+y^2+s^2+t^2)+b(sx+ty)+c(tx-sy)}\,dx\,dy$$

$$= \frac{1}{d(-\xi)}\Phi_{\gamma,\theta}\Big(\phi(x,y)\psi(x,y)e^{-a(x^2+y^2)}\Big)(s,t).$$

Hence,

$$d(-\xi)(\Phi \circ \Psi)(s,t) = \mathcal{F}^{\gamma,\theta}\Big(\phi(x,y)\psi(x,y)e^{-a(x^2+y^2)}\Big)(s,t). \tag{3.28}$$

The following are some more characteristics of the convolution operation:

(a) Commutativity: $\psi(x,y) \star \phi(x,y) = \phi(x,y) \star \psi(x,y)$.

(b) Associativity: $\psi(x,y) \star \{\phi(x,y) \star h(x,y)\} = \big(\psi(x,y) \star \phi(x,y)\big) \star h(x,y)$.

Proof (a):

According to the definition of '$\star$', it is enough to show that

$$(\tilde{\psi} * \tilde{\phi})(x,y) = (\tilde{\phi} * \tilde{\psi})(x,y). \tag{3.29}$$

As,

$$\begin{aligned}(\tilde{\psi} * \tilde{\phi})(x,y) &= \int_{\mathbb{R}^2} \tilde{\psi}(x-\tau, y-\lambda)\tilde{\phi}(\tau,\lambda)\, d\tau\, d\lambda \qquad (3.30)\\ &= \int_{\mathbb{R}^2} e^{-a\big((x-\tau)^2+(y-\lambda)^2\big)}\psi(x-\tau, y-\lambda)e^{-a(\tau^2+\lambda^2)}\phi(\tau,\lambda)\, d\tau\, d\lambda,\end{aligned}$$

then substituting $x-\tau = u$ and $y-\lambda = v$, one can obtain

$$\begin{aligned}(\tilde{\psi} * \tilde{\phi})(x,y) &= \int_{\mathbb{R}^2} e^{-a(u^2+v^2)}\psi(u,v)e^{-a\big((x-u)^2+(y-v)^2\big)}\phi(x-u, y-v)\, du\, dv\\ &= (\tilde{\phi} * \tilde{\psi})(x,y). \qquad (3.31)\end{aligned}$$

Hence, the convolution operator is commutative.

Proof (b): We have,

$$\begin{aligned}&\psi(x,y) \star \{\phi(x,y) \star h(x,y)\} = d(\xi)e^{a(x^2+y^2)}\big[\tilde{\psi}(x,y) * \{\phi(x,y) \tilde{\star}\, h(x,y)\}\big]\\ &= d(\xi)e^{a(x^2+y^2)} \int_{\mathbb{R}^2} \tilde{\psi}(x-\tau, y-\lambda)(\phi \tilde{\star}\, h)(\tau,\lambda)\, d\tau\, d\lambda\\ &= d(\xi)e^{a(x^2+y^2)} \int_{\mathbb{R}^2} [e^{-a\big((x-\tau)^2+(y-\lambda)^2\big)}\psi(x-\tau, y-\lambda)\, d(\xi)\\ &\times \int_{\mathbb{R}^2} e^{-a\big((\tau-\zeta)^2+(\lambda-\upsilon)^2\big)}\phi(\tau-\zeta, \lambda-\upsilon)e^{-a(\zeta^2+\upsilon^2)}h(\zeta,\upsilon)\, d\zeta\, d\upsilon]\, d\tau\, d\lambda\end{aligned}$$

Let $\quad \tau-\zeta = u \quad$ and $\quad \lambda-\upsilon = v$

$$\begin{aligned}&= d(\xi)e^{a(x^2+y^2)} \int_{\mathbb{R}^2} [e^{-a\big((x-\tau-u)^2+(y-\lambda)^2\big)}\psi(x-\tau, y-\lambda)\, d(\xi)\\ &\times \int_{\mathbb{R}^2} e^{-a(u^2+v^2)}\phi(u,v)e^{-a(\zeta^2+\upsilon^2)}h(\zeta,\upsilon)\, d\zeta\, d\upsilon]\, du\, dv\end{aligned}$$

Now changing the order of integration, one can obtain

$$\begin{aligned}&= d(\xi)e^{a(x^2+y^2)} \int_{\mathbb{R}^2} \Big[d(\xi)e^{-a\big((x-u-\zeta)^2+(y-v-\upsilon)^2\big)}\psi(x-u-\zeta, y-v-\upsilon)\\ &\times\, e^{-a(u^2+v^2)}\phi(u,v)\, du\, dv\Big]\, e^{-a(\zeta^2+\upsilon^2)}h(\zeta,\upsilon)\, d\zeta\, d\upsilon\\ &= d(\xi)e^{a(x^2+y^2)}\big[(\psi \tilde{\star}\, \phi)(x,y) * \tilde{h}(x,y)\big]\\ &= (\psi \star \phi) \star h.\end{aligned}$$

Hence, the convolution operator satisfy associativity property as well.

Thus, we can conclude that $(\mathcal{C}, \star)$ is a commutative semi-group since the convolution operator '$\star$' satisfies the conditions of commutativity and associativity, where, $\mathcal{C}$ represents the set of all coupled fractional Fourier transformable functions.

Proposition: Let $f,\ f_1,\ f_2 \in \mathcal{L}^p(\mathbb{R}^2),\ p = 1,\ 2$ and $g \in \mathcal{L}^1(\mathbb{R}^2),\ \alpha \in \mathbb{C}$, then

(i) $\mathcal{F}^{\gamma,\theta}\big[(f_1 + f_2) \star g\big] = \mathcal{F}^{\gamma,\theta}(f_1 \star g) + \mathcal{F}^{\gamma,\theta}(f_2 \star g)$.

(ii) $\mathcal{F}^{\gamma,\theta}\{\alpha(f \star g)\} = \mathcal{F}^{\gamma,\theta}\{(\alpha f) \star g\} = \mathcal{F}^{\gamma,\theta}\{f \star (\alpha g)\}$.

Proof (i):

$$\begin{aligned}
\mathcal{F}^{\gamma,\theta}\big[(f_1 + f_2) \star g\big] &= e^{a(s^2+t^2)}\big[\mathcal{F}^{\gamma,\theta}(f_1 + f_2).\mathcal{F}^{\gamma,\theta}g\big] &(3.32)\\
&= e^{a(s^2+t^2)}\big[\{\mathcal{F}^{\gamma,\theta}(f_1) + \mathcal{F}^{\gamma,\theta}(f_2)\}.\mathcal{F}^{\gamma,\theta}g\big] &(3.33)\\
&= e^{a(s^2+t^2)}\big[\mathcal{F}^{\gamma,\theta}(f_1)\mathcal{F}^{\gamma,\theta}g + \mathcal{F}^{\gamma,\theta}(f_2)\mathcal{F}^{\gamma,\theta}g\big] &(3.34)\\
&= e^{a(s^2+t^2)}\mathcal{F}^{\gamma,\theta}(f_1)\mathcal{F}^{\gamma,\theta}g + e^{a(s^2+t^2)}\mathcal{F}^{\gamma,\theta}(f_2)\mathcal{F}^{\gamma,\theta}g\\
&= \mathcal{F}^{\gamma,\theta}(f_1 \star g) + \mathcal{F}^{\gamma,\theta}(f_2 \star g). &(3.35)
\end{aligned}$$

Note that, convolution theorem has been used to obtain eq (3.32) and eq (3.33) is deduced from eq (3.32) by using linearity property of CFrFT.

Proof (ii):

$$\begin{aligned}
\mathcal{F}^{\gamma,\theta}\{\alpha(f \star g)\} &= \alpha\mathcal{F}^{\gamma,\theta}(f \star g)\\
&= \alpha\{e^{a(s^2+t^2)}\mathcal{F}^{\gamma,\theta}(f)\mathcal{F}^{\gamma,\theta}(g)\}\\
&= e^{a(s^2+t^2)}\mathcal{F}^{\gamma,\theta}(\alpha f)\mathcal{F}^{\gamma,\theta}(g)\\
&= \mathcal{F}^{\gamma,\theta}\{(\alpha f) \star g\}.
\end{aligned}$$

Similarly, the other part can be proved. Notice that, convolution theorem and linearity property for CFrFT have been used again to prove the above result.

3.6 Additional Properties of the CFrFT

In this section, a few more properties of the CFrFT have been introduced.

a. Parseval's Relation: If $\phi \in \mathcal{L}^1(\mathbb{R}^2) \cap \mathcal{L}^2(\mathbb{R}^2)$, then $\|\mathcal{F}_{\phi}^{\gamma,\theta}\| = \|\phi\|$.

b. General Parseval's Relation: If $\phi, \psi \in \mathcal{L}^1(\mathbb{R}^2) \cap \mathcal{L}^2(\mathbb{R}^2)$, then $\langle \mathcal{F}_{\phi}^{\gamma,\theta}, \mathcal{F}_{\psi}^{\gamma,\theta} \rangle = \langle \phi, \psi \rangle$.
For the proofs of Parseval's and General Parseval's relation, the reader may refer [9].

c. If $\phi(x,y)$ and $\psi(x,y)$ are any two functions in $\mathcal{L}^1(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} \psi(x,y) \mathcal{F}^{\gamma,\theta} \phi(x,y)\, dx\, dy = \int_{\mathbb{R}^2} \phi(s,t) \mathcal{F}^{-\gamma,-\theta} \psi(s,t)\, ds\, dt,$$

provided both the integrals exist.

Proof:

$$\begin{aligned}
\int_{\mathbb{R}^2} \psi(x,y) \mathcal{F}^{\gamma,\theta} \phi(x,y)\, dx\, dy &= \int_{\mathbb{R}^2} \psi(x,y) \left(\int_{\mathbb{R}^2} \phi(s,t) K_{\gamma,\theta}(s,t,x,y)\, ds\, dt \right) dx\, dy \\
&= \int_{\mathbb{R}^2} \phi(s,t) \left(\int_{\mathbb{R}^2} \psi(x,y) K_{-\gamma,-\theta}(x,y,s,t)\, dx\, dy \right) ds\, dt \\
&= \int_{\mathbb{R}^2} \phi(s,t) \mathcal{F}^{-\gamma,-\theta} \psi(s,t)\, ds\, dt. \qquad (3.36)
\end{aligned}$$

Here, we used information that $K_{\gamma,\theta}(s,t,x,y) = K_{-\gamma,-\theta}(x,y,s,t)$.

Remark:

The difference between this result and the duality theorem for the Fourier transform and FrFT is mostly due to the change in angle direction during the computation of the CFrFT.

Now, we are going to conclude this chapter by computing CFrFT of a function. The computation of integral is done via Mathematica.

Example: Find CFrFT of $\psi(x,y) = \begin{cases} x+y, & -r \le x \le r,\ -R \le y \le R \\ 0, & \text{otherwise} \end{cases}$.

Clearly, one can verify that $\psi(x,y) \in \mathcal{L}^1(\mathbb{R}^2)$. Hence CFrFT of $\psi(x,y)$ is given by

$$\mathcal{F}_{\psi}^{\gamma,\theta}(x,y,s,t) = \int_{\mathbb{R}^2} \psi(x,y) K_{\gamma,\theta}(x,y,s,t)\, dx\, dy \tag{3.37}$$

$$= \int_{-R}^{R}\int_{-r}^{r} (x+y) d(\xi) \exp\{-a(x^2+y^2+s^2+t^2) + iBx + iCy\}\, dx\, dy,$$

where, $\bar{b} = -ib$, $\bar{c} = -ic$, $B = \bar{b}s + \bar{c}t$ and $C = \bar{b}t - \bar{c}s$.

Hence,

$$B = \frac{\cos\eta}{\sin\xi}s + \frac{\sin\eta}{\sin\xi}t = \frac{s\,\cos\eta + t\,\sin\eta}{\sin\xi}, \tag{3.38}$$

$$C = \frac{\cos\eta}{\sin\xi}t - \frac{\sin\eta}{\sin\xi}s = \frac{t\,\cos\eta - s\,\sin\eta}{\sin\xi}. \tag{3.39}$$

Therefore,

$$\begin{aligned}
&\mathcal{F}_{\psi}^{\gamma,\theta}(x,y,s,t) = \\
&\quad \frac{d(\xi)\ e^{s^2+t^2}}{8a^2} e^{-\frac{B^2+C^2+4iaCR+4a^2\left(r^2+R^2\right)}{4a}} \sqrt{\pi}\Bigg(2\sqrt{a}e^{\frac{C^2}{4a}+ar^2}\left(-1+e^{2iCR}\right) \\
&\quad \times\ \ \mathrm{Erf}\left[\frac{-iB-2ar}{2\sqrt{a}}\right] - 2\sqrt{a}\ e^{\frac{C^2}{4a}+ar^2}\left(-1+e^{2iCR}\right) Erf\left[\frac{-iB+2ar}{2\sqrt{a}}\right] \\
&\quad +ie^{-iBr+R(iC+aR)}\Bigg(-2\sqrt{a}\ e^{\frac{B^2}{4a}}\left(-1+e^{2iBr}\right) - (B+C)e^{r(iB+ar)} \\
&\quad \times\ \ \sqrt{\pi}\,\mathrm{Erfi}\left[\frac{B-2iar}{2\sqrt{a}}\right] + (B+C)e^{r(iB+ar)}\sqrt{\pi}Erfi\left[\frac{B+2iar}{2\sqrt{a}}\right]\Bigg) \\
&\qquad\qquad \times\Bigg(\mathrm{Erfi}\left[\frac{B-2iaR}{2\sqrt{a}}\right] - \mathrm{Erfi}\left[\frac{B+2iaR}{2\sqrt{a}}\right]\Bigg)\Bigg).
\end{aligned} \tag{3.40}$$

Special case: If both the angles γ and θ are $\frac{\pi}{2}$, then we have $\xi = \frac{\pi}{2}$ and $\eta = 0$, therefore, $a = 0$, $B = s$, $C = t$, and $d = \frac{-i}{2\pi}$.

So,

$$\mathcal{F}_{\psi}^{\frac{\pi}{2},\frac{\pi}{2}}(x,y,s,t) = -\frac{2\big[R\,s\,t\cos(Rt)\sin(rs) + \big(r\,s\,t\cos(rs) - (s+t)\sin(rs)\big)\sin(Rt)\big]}{\pi\, s^2t^2}.$$

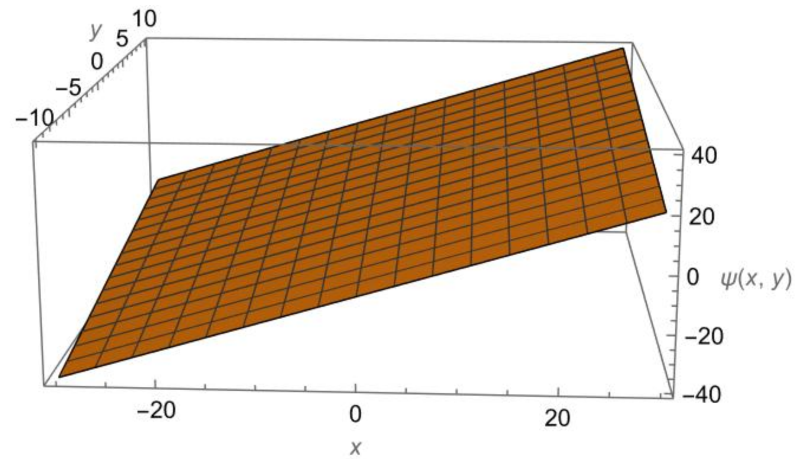

Figure 3.1: Graphical representation of $\psi(x, y)$

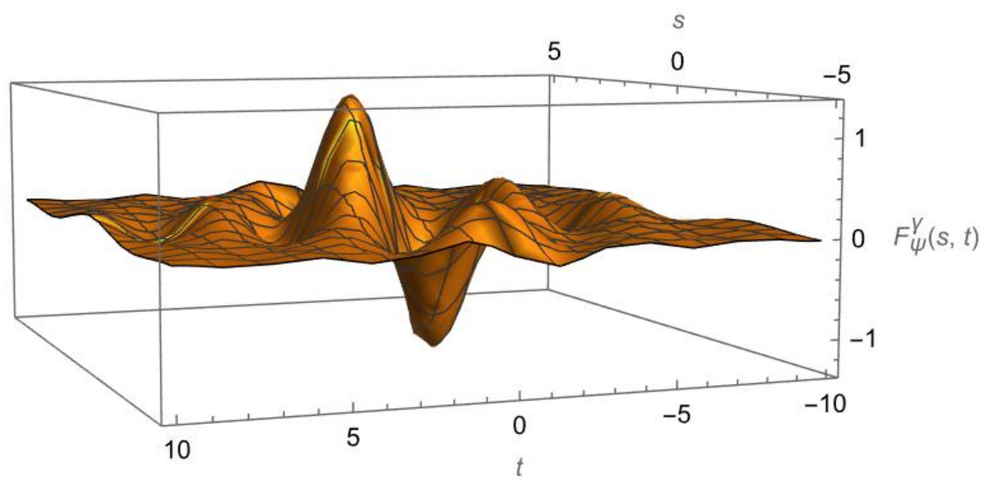

Figure 3.2: CFrFT of $\psi(x, y)$ when $r = \frac{\pi}{2} = R$

Conclusions

This thesis entitled "Introduction to Coupled Fractional Fourier Transform" is devoted to the study of the CFrFT and its properties. The CFrFT is a new generalization of FrFT to 2D, introduced by Zayed [8]. Zayed used the idea of generalizing eigenfunctions of FrFT to define the kernel for the CFrFT instead of using the tensor product of 1D kernels of FrFT.

In this thesis, derivation of several properties of the CFrFT are presented, some of which are given below.

If $\phi(x,y)$ and $\psi(x,y)$ are two functions having CFrFT, as $\mathcal{F}_{\phi}^{\gamma,\theta}(x,y,s,t)$ and $\mathcal{F}_{\psi}^{\gamma,\theta}(x,y,s,t)$, respectively, and α, β are in $\mathbb{C}$. Then we have the following results.

Properties of CFrFT		
Property	Function	CFrFT
(i) Linearity	$\alpha\phi(x,y) + \beta h(x,y)$	$\alpha\mathcal{F}_{\phi}^{\gamma,\theta}(x,y,s,t) + \beta\mathcal{F}_{\psi}^{\gamma,\theta}(x,y,s,t)$
(ii) Reflection	$\phi(-x,-y)$	$\mathcal{F}_{\phi}^{\gamma,\theta}(x,y,-s,-t)$
(iii) Conjugation	$\phi(x,-y)$	$\mathcal{F}_{\tilde{\phi}}^{\gamma,\theta}(x,y,-s,-t)$, where, $\tilde{\phi}(x,y) = \phi(-x,y)$
(iv) Time Shift	$\phi(x-t_1, y-t_2)$	$\mathcal{F}_{\phi}^{\gamma,\theta}\left(x+t_1, y+t_2, s, t\right)$
(v) Frequency Shift	$\phi(x,y)e^{i(xk_1+yk_2)}$	$e^{a\left(\frac{k_1^2+k_2^2}{b^{*2}} + \frac{2(sk_1+tk_2)}{b^*}\right)}\mathcal{F}_{\bar{\bar{\phi}}}^{\gamma,\theta}\left(x,y,s+\frac{k_1}{b^*},t+\frac{k_2}{b^*}\right)$, where, $\bar{\bar{\phi}} = \phi(x,y)e^{\frac{c}{b^*}(yk_1-xk_2)}$

The relation between conjugation of the CFrFT of a function with the CFrFT of the function is also examined in this thesis.

Convolution and product theorems plays a crucial role in the theory of integral transforms. The convolution structure and theorem for the CFrFT are given in [27] and [28]. Moreover, a new result similar to product theorem for the CFrFT is given. Several other properties like commutativity, associativity and others are also examined for convolution operator.

This thesis also contain a result examining the duality property for the CFrFT. Overall, the explored properties including linearity, reflection, conjugation and shift properties, will make the CFrFT a versatile transform with a wide range of applications. As further research continues in this field, the CFrFT is likely to find even more applications and contribute to advancements in various areas of signal processing and analysis.

Future Scopes

- Even though several properties of the CFrFT have been investigated but still a few properties like energy conservation, scaling, etc. as well as properties related to the inverse CFrFT are yet to be explored.

- The conditions on the domain of the CFrFT which ensures the preservation of continuity, differentiability and other properties of the input function need to be investigated.

- Applications of the CFrFT on two-dimensional differential equations can be explored.

In the future, the CFRFT is expected to play a significant role in signal processing and related fields. Collaborations between researchers in signal processing, mathematics, communication systems, and other domains will facilitate the exploration and application of CFrFT in new ways.

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