TRANSLATION GENERATED OBLIQUE DUAL FRAMES BY ACTIONS OF LOCALLY COMPACT GROUPS

Ph.D. Thesis

by

Sudipta Sarkar

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

JUNE 2023

TRANSLATION GENERATED OBLIQUE DUAL FRAMES BY ACTIONS OF LOCALLY COMPACT GROUPS

A THESIS

submitted in partial fulfillment of the
requirements for the award of the degree

of

DOCTOR OF PHILOSOPHY

by

Sudipta Sarkar

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

JUNE 2023

INDIAN INSTITUTE OF TECHNOLOGY INDORE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **TRANSLATION GENERATED OBLIQUE DUAL FRAMES BY ACTIONS OF LOCALLY COMPACT GROUPS** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DEPARTMENT OF MATHEMATICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2017 to March 2023 under the supervision of Dr. Niraj Kumar Shukla, Associate Professor, Indian Institute of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

17-Mar-2023

Signature of the student with date

(**SUDIPTA SARKAR**)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

17-Mar-2023

Signature of Thesis Supervisor with date

(**DR. NIRAJ KUMAR SHUKLA**)

SUDIPTA SARKAR has successfully given his Ph.D. Oral Examination held on 30-Jun-2023.

30-Jun-2023

Signature of Thesis Supervisor with date

(**DR. NIRAJ KUMAR SHUKLA**)

ACKNOWLEDGEMENTS

First and foremost, I convey my gratitude to the omnipotent, omniscient, and omnipresent God for choosing this path for me and accomplishing this journey. I am always blessed with the grace of the holy trio: Thakur Sri Ramakrishna, Ma Sarada Devi, and Swami Vivekananda. When I encounter darkness on my walk, the merciful God always illuminates my path.

I owe deep gratitude to my thesis supervisor Dr. Niraj Kumar Shukla, whose constant encouragement always inspires me to dive deep inside the ocean of mathematics. His unwavering reinforcement propelled me whenever I got stuck into a problem, whether in mathematics or in various circumstances of life. From research problem finding to article writing, he teaches me every inch like a parent teaches his toddler how to walk. I am proud to be his student. He constantly assesses my abilities and shortcomings and instructs me on how to improve. He has not only encouraged me to develop my potential but also taught me important life skills such as effective communication, organization, and presentation. He also pushed me to excel in every aspect of my life.

It is my parents who have been my greatest supporters throughout the journey. Whenever I was in trouble, they were right there to bail me out. While being around my parents, I never worry about the uncertainty of my destination. My parents are incredibly important to me, and I owe a great debt of gratitude to them for giving me the foundation upon which I have built my entire life, including my vision and core competencies.

I am grateful to Safique Sir and Aquil Sir (faculty of IITI), members of my Ph.D. committee, for their insightful ideas and comments during the annual evaluation of my research progress. I gratefully acknowledge CSIR India for providing me with financial support to conduct research at IIT Indore. I would also like to express my gratitude to Professor Suhas S. Joshi, the Director of IIT Indore, for allowing me to conduct this research on the campus.

My Ph.D. journey would not have been accomplished smoothly without the support of Anupam Da (former faculty of IITI). I learnt the very basics of Fourier analysis from him. He always encouraged me to do challenging problems. Whenever I used to stuck on problems, he always encouraged me to continue trying. I owe an incredible amount of gratitude to him. Another individual for whom I have a great deal of gratitude is Professor Alessio Martini (Politecnico di Torino, Italy). During a conference (DMHA 16) at IISER Bhopal, he taught me the very basics of the Nilpotent Lie group. My Ph.D. progress seminar was attended by Professor Biswaranjan Behera and Professor Rudra P. Sarkar (ISI Kolkata), and they suggested me with very insightful remarks regarding my study. Many thanks go to them. If I don't mention Bibekananda Da (faculty of IITI) for his assistance and comprehensive suggestions in a variety of areas, the list will be lacking someone important and will be considered incomplete. I am very much thankful to him.

I am thankful to Gyan Da for his continuous support throughout the journey. From my bad times to good times, he always helped me a lot. I miss Prince Bhaiya and his great companionship. His advice always provides me immense strength in my journey. Special thanks to Nitin Bhaiya for organizing unforgettable vacations, feeding us excellent cuisine, and providing insightful, intellectual and personal pieces of advice. After leaving IIT Indore, I will miss POD-1A 514, the witnessed of a lot of happy and challenging times, and my lab-mates Rupsha Di and Vineeta Ma'am, whose company in the lab have always encouraged me to work even in hard times. I will always miss them. I also miss Vibhuti Ma'am's handmade foods; thank you Ma'am for such delicious dishes.

During the Covid-19 times when I was quarantined for months, my friends Santanu and Niranjan Mj always kept me positive by regularly conversing over the phone. My warm thanks go to them. My sister Sumana provides me immense strength and motivation. I am also thankful to her. Another beautiful person I met, Sudhanand Sir, always encouraged us in traveling, trekking, climbing, jungle safari, pilgrimage, etc. To me, he is a gem; I always miss him. The list will be incomplete if I don't mention the one and only one Swaraj Da; his personal guidance always motivates me in my journey. Thank you, Swaraj Da. I am also thankful to Mr. Jitendra Verma for his constant support.

I would like to express my appreciation and love to my fellow mates: Ashwani, Shreyas, Neha, Navneet, and Sahil for their support along this journey. For me, the time spent with you guys is irreplaceable, and I shall miss having fun with you all in the future.

Dedicated to . . .

My caring father ***Ranjit Sarkar*** *and loving mother* ***Latika Sarkar*** *whose tireless efforts, unwavering support, and constant prayers throughout the day and night made me able to get such success and honor. . .*

LIST OF PUBLICATIONS BASED ON THE THESIS

1. **S. Sarkar, N. K. Shukla**, *Translation generated oblique dual frames on locally compact groups*, **Linear Multilinear Algebra**, (2023), `doi:10.1080/03081087.2023.2173718`, 32 pages.
2. **S. Sarkar, N. K. Shukla**, *Characterizations of extra-invariant spaces under the left translations on a Lie group*, **Advances in Operator Theory,** (2023), https://doi.org/10.1007/s43036-023-00273-x.
3. **S. Sarkar, N. K. Shukla**, *Subspace dual and orthogonal frames by action of an abelian group*, submitted.
4. **S. Sarkar, N. K. Shukla**, *Reproducing formula associated to translation generated systems on nilpotent Lie groups*, `arXiv:2301.03152`.
5. **S. Sarkar, N. K. Shukla**, *A characterization of MG dual frames using infimum cosine angle*, `arXiv:2301.07448`.
6. **S. Sarkar, S. Kalra, N. K. Shukla**, *An application of the supremum cosine angle between multiplication invariant spaces in* $L^2(X;\mathcal{H})$, `arXiv:2211.15238`.

ABSTRACT

KEYWORDS: Biorthogonal dual; Dual frame; Dual integrable representation; Fiberization; Frame; Gramian; Heisenberg group; Infimum cosine angle; Locally compact group; Multiplication invariant space; Nilpotent Lie group; Oblique dual; Orthogonal frame; Reproducing formula; Riesz Basis; Shift-invariant space; Translation-invariant space; Unitary Representation; Zak transform

For a second countable locally compact group Γ, let ρ be a unitary representation of Γ acting on a separable Hilbert space $\mathcal{H}$. Also for a collection of functions $\{\varphi_t : t \in \mathcal{N}\}$ in $\mathcal{H}$, where $\mathcal{N}$ is a σ-finite measure space, considering the continuous frame of orbit: $\{\rho(\gamma)\varphi_t : \gamma \in \Gamma, t \in \mathcal{N}\}$, we discuss the various dual frames of the same form, i.e., $\{\rho(\gamma)\psi_t : \gamma \in \Gamma, t \in \mathcal{N}\}$ for some $\psi_t \in \mathcal{H}$. We provide various necessary and sufficient conditions for the characterizations of dual frames. In particular, we concentrate on the context of translation generated systems in $\mathcal{H} = L^2(\mathscr{G})$, where translations are from closed subgroup Γ of the locally compact group $\mathscr{G}$. Our characterization results are based on the Zak transform. When $\mathscr{G}$ becomes locally compact abelian (denoted by $\mathcal{G}$), we discuss the same using the fiberization map. At the end, we discuss our characterizations for the SI/Z nilpotent Lie group $\mathscr{G}$ (denoted by G), which is considered to be a high degree of non-abelian structure.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS iii

ABSTRACT vii

Chapter 1 INTRODUCTION 1

1.1 Oblique duals for a frame 2

1.2 Motivation 4

1.3 Translation-invariant spaces 5

1.4 Reproducing formula for nilpotent Lie groups 8

1.5 Structure of the thesis 10

Chapter 2 MULTIPLICATION GENERATED OBLIQUE DUAL FRAMES IN $L^2(X;\mathcal{H})$ 13

2.1 Multiplication invariant spaces 13

2.2 Characterization results 15

2.3 Uniqueness of multiplication generated duals 29

Chapter 3 CONSTRUCTION OF OBLIQUE DUAL FRAMES IN $L^2(X;\mathcal{H})$ 35

3.1 Infimum cosine angles and oblique duals 35

3.2 Riesz basis and its associated dual 42

Chapter 4 DUAL FRAMES BY THE ACTION OF AN ABELIAN GROUP 47

4.1 Translation generated duals of a frame 47

4.2 Examples 55

4.3 Duals for a continuous Gabor frame 58

4.4 Duals associated with the orbit of a representation 60

4.5 Existence of oblique duals and infimum cosine angle 62

Chapter 5 SUBSPACE DUAL AND ORTHOGONAL FRAMES BY ACTION OF AN ABELIAN GROUP 67

5.1 Subspace dual of a frame by a discrete abelian group action 70

5.2 Translation generated biorthogonal system and Riesz basis 87

5.3 Orbit generated by the action of an abelian subgroup 92

5.4 Applications 97

Chapter 6 REPRODUCING FROMULA FOR SI/Z LIE GROUP 101

6.1 Plancherel transform for SI/Z nilpotent Lie group 101

6.2 Translation-invariant spaces 107

6.3 Reproducing formulas associated with continuous frames 108

6.4 Reproducing formulas by the action of discrete translations 116

6.5 Gabor system and Heisenberg group 122

6.6 Extra invariances on Lie group 124

Chapter 7 SUMMARY AND FUTURE DIRECTIONS 133

BIBLIOGRAPHY 135

INDEX 141

CHAPTER 1

INTRODUCTION

One of the primary goals of signal analysis is to represent a signal/vector in terms of the fundamental building blocks, often known as energy blocks. In particular, we are interested in the following expression of a signal f:

$$f = \sum_{i \in I} c_i f_i, \tag{1.0.1}$$

where $f_i' s$ are the fundamental building blocks. If the collection $\{f_i\}_{i \in I}$ is an orthonormal basis then c_i's are predetermined. However, there are situations when additional desired features are required to limit the projected noise level during transmission, which is not possible using an orthonormal basis as the scalars c_i's are fixed. To overcome these issues, the concept of frames was proposed by Duffin and Schaeffer in the study of non-harmonic Fourier analysis in 1952 [32]. Frames generalize the concept of bases so that we have significantly more flexibility in the building of f_i's and plentiful options for c_i's. For a frame $\{f_i\}_{i \in I}$, every element in a Hilbert space $\mathcal{H}$ has a representation as a linear combination of the frame elements, i.e., there exist coefficients $\{c_i(f)\}_{i \in I}$ such that (1.0.1) holds. The coefficients c_i have the form: $c_i(f) = \langle f, g_i \rangle$ for some $g_i \in \mathcal{H}$, by the Riesz representation theorem. The collection $\{g_i\}_{i \in I}$ is called the *dual frame* which has numerous uses in noise reduction and data reconstructions. Additionally, the theory of dual frames has been applied to the investigation and construction of oversampled filter banks and error correction codes [21].

Recently, the conventional idea of frames has been extended to *subspace frames*, i.e., frames on subspaces. In this case, the frame decomposition (1.0.1) becomes $f = \sum_{i \in I} c_i(f) f_i$ for $f \in \mathcal{X}$, where $\mathcal{X}$ is a closed subspaces of $\mathcal{H}$ (see [37, 43, 44, 46]). The vectors $\{f_i\}_{i \in I}$ and $\{g_i\}_{i \in I}$ are still required to be frames, but only for subspace $\mathcal{X}$ and a possibly different subspace S, respectively, such that $\mathcal{H} = \mathcal{X} \oplus S^{\perp}$, where $S^{\perp}$ denotes the orthogonal complement of S in $\mathcal{H}$. Then $\{g_i\}_{i \in I}$ is known as *oblique dual frame*. By choosing $S = \mathcal{X} = \mathcal{H}$, we recover the conventional dual frames.

Frames are not necessarily linearly independent and have the feature that any vector in the space can be expanded in terms of the elements from it. The recent surge in interest in frames is due to the fact that they are useful in studying wavelet expansions and possess robustness qualities [30,41,69,70]. Dual frames are valuable construction tools for series expansion in a Hilbert space, where the series coefficients may not be unique. Recently, numerous linkages between frame theory and signal processing techniques have been identified and developed. Wavelet frames and Gabor frames are utilized in quantum mechanics and numerous other branches of theoretical physics.

1.1. Oblique duals for a frame

Ali, Antoine, and Gazeau [4] and Kaiser [53] all came up with the idea of a continuous frame on their own. By sampling the continuous frames, we can get discrete frames that have wide applications due to their computational simplicity. The discretization problem for continuous frames was studied by many authors, including Freeman and Speegle [35].

Let $\mathcal{H}$ be a complex Hilbert space, and let $(\mathscr{M}, \sum_{\mathscr{M}}, \mu_{\mathscr{M}})$ be a measure space, where $\Sigma_{\mathscr{M}}$ denotes σ-algebra and $\mu_{\mathscr{M}}$ the non-negative measure. A family of vectors $\{f_k\}_{k\in\mathscr{M}}$ in $\mathcal{H}$ is called a *continuous frame* (simply call as, *frame*) for $\mathcal{H}$ with respect to $(\mathscr{M}, \sum_{\mathscr{M}}, \mu_{\mathscr{M}})$ if $k \mapsto f_k$ is weakly measurable, i.e., the map $k \mapsto \langle f, f_k\rangle$ is measurable for each $f \in \mathcal{H}$, and there exist constants $0 < A \leqslant B$, called *frame bounds*, such that

$$A\|f\|^2 \leqslant \int_{\mathscr{M}} |\langle f, f_k\rangle|^2 \, d\mu_{\mathscr{M}}(k) \leqslant B\|f\|^2 \text{ for all } f \in \mathcal{H}. \tag{1.1.1}$$

The frame $\{f_k\}_{k\in\mathcal{M}}$ is called *tight frame* when $A = B$. If we choose $A = B = 1$, it is *Parseval frame* (also known as *coherent state*). When the inequality on the right hand side in (1.1.1) holds, the family $\{f_k\}_{k\in\mathscr{M}}$ is called *Bessel* with bound B. In case of countable set $\mathscr{M}$ and counting measure $\mu_{\mathscr{M}}$, the continuous frame reduces to the usual discrete frame. Here, it can be noted that the above weak measurability condition can be replaced by strong measurability in view of the Pettis' lemma since we consider separable Hilbert spaces.

Given a Bessel family $\mathcal{X} = \{f_k\}_{k\in\mathscr{M}}$ with respect to the measure space $(\mathscr{M}, \sum_{\mathscr{M}}, \mu_{\mathscr{M}})$ in a Hilbert space $\mathcal{H}$, we define a bounded linear operator $T_{\mathcal{X}} : \mathcal{H} \to L^2(\mathscr{M}, \mu_{\mathscr{M}})$, known as *analysis operator*, by

$$T_{\mathcal{X}}(h)(k) = \langle h, f_k \rangle \text{ for all } k \in \mathscr{M},\ h \in \mathcal{H},$$

and its adjoint operator $T_{\mathcal{X}}^* : L^2(\mathscr{M}, \mu_{\mathscr{M}}) \to \mathcal{H}$, known as *synthesis operator*, by

$$\langle T_{\mathcal{X}}^* \varphi, \psi \rangle = \int_{\mathscr{M}} \varphi(k) \langle f_k, \psi \rangle \, d\mu_{\mathscr{M}}(k) \text{ for all } \psi \in \mathcal{H},\ \varphi \in L^2(\mathscr{M}, \mu_{\mathscr{M}}),$$

in the weak sense. Then, $S_{\mathcal{X}} := T_{\mathcal{X}}^* T_{\mathcal{X}} : \mathcal{H} \to \mathcal{H}$ is a *frame operator* given by

$$S_{\mathcal{X}} f = \int_{\mathscr{M}} \langle f, f_k \rangle f_k \, d\mu_{\mathscr{M}}(k) \text{ for all } f \in \mathcal{H}_{\mathcal{X}},$$

while the other side composition $T_{\mathcal{X}} T_{\mathcal{X}}^* : L^2(\mathscr{M}, \mu_{\mathscr{M}}) \to L^2(\mathscr{M}, \mu_{\mathscr{M}})$ is known as *Gramian operator*.

At this juncture, it can be noted that the Bessel family $\mathcal{X} = \{f_k\}_{k \in \mathscr{M}}$ is a continuous frame for $\mathcal{H}_{\mathcal{X}} := \overline{\operatorname{span}}\{f_k\}_{k \in \mathscr{M}} \subset \mathcal{H}$ with bounds $0 < A \leqslant B$ if and only if the frame operator $S_{\mathcal{X}}$ on $\mathcal{H}_{\mathcal{X}}$ is positive, bounded and invertible with $A I_{\mathcal{H}_{\mathcal{X}}} \leqslant S_{\mathcal{X}}\big|_{\mathcal{H}_{\mathcal{X}}} \leqslant B I_{\mathcal{H}_{\mathcal{X}}}$, where $I_{\mathcal{H}_{\mathcal{X}}}$ denotes the identity operator on $\mathcal{H}$ which is restricted on $\mathcal{H}_{\mathcal{X}}$. The inverse of frame operator $S_{\mathcal{X}}$ on $\mathcal{H}_{\mathcal{X}}$ satisfies $\frac{1}{B} I_{\mathcal{H}_{\mathcal{X}}} \leqslant (S_{\mathcal{X}}\big|_{\mathcal{H}_{\mathcal{X}}})^{-1} \leqslant \frac{1}{A} I_{\mathcal{H}_{\mathcal{X}}}$ and the family $\tilde{\mathcal{X}} := \{(S_{\mathcal{X}}\big|_{\mathcal{H}_{\mathcal{X}}})^{-1} f_k\}_{k \in \mathscr{M}}$ is also a continuous frame for $\mathcal{H}_{\mathcal{X}}$, known as *canonical dual frame for* $\mathcal{X}$ *in* $\mathcal{H}_{\mathcal{X}}$, which satisfies the following reproducing formula for all $f \in \mathcal{H}_{\mathcal{X}}$ in the weak sense:

$$f = \int_{\mathscr{M}} \langle f, (S_{\mathcal{X}}\big|_{\mathcal{H}_{\mathcal{X}}})^{-1} f_k \rangle f_k \, d\mu_{\mathscr{M}}(k) = \int_{\mathscr{M}} \langle f, f_k \rangle (S_{\mathcal{X}}\big|_{\mathcal{H}_{\mathcal{X}}})^{-1} f_k \, d\mu_{\mathscr{M}}(k). \tag{1.1.2}$$

That is, we have $\mathcal{H}_{\tilde{\mathcal{X}}} := \overline{\operatorname{span}}(\tilde{\mathcal{X}}) = \overline{\operatorname{span}}(\mathcal{X}) = \mathcal{H}_{\mathcal{X}}$, and $T_{\mathcal{X}}^* T_{\tilde{\mathcal{X}}}\big|_{\mathcal{H}_{\mathcal{X}}} = T_{\tilde{\mathcal{X}}}^* T_{\mathcal{X}}\big|_{\mathcal{H}_{\mathcal{X}}} = I_{\mathcal{H}_{\mathcal{X}}}$. Since the frame operator $S_{\mathcal{X}}$ on $\mathcal{H}$ need not be invertible, we call the family $\tilde{X}^{\dagger} := \{S^{\dagger} f_k\}_{k \in \mathscr{M}}$ as *canonical dual frame for* $\mathcal{X}$ *in* $\mathcal{H}$, where $\dagger$ is the pseudo-inverse of the bounded operator $S_{\mathcal{X}}$ with closed range. The reproducing formula (1.1.2) gives an idea to find a new Bessel family, say $\{g_k\}_{k \in \mathscr{M}} =: \mathcal{Y}$ in $\mathcal{H}$, such that the following decomposition formula holds (weak sense) for a given Bessel family $\{f_k\}_{k \in \mathscr{M}}$ in $\mathcal{H}_{\mathcal{X}}$,

$$f = \int_{\mathscr{M}} \langle f, g_k \rangle f_k \, d\mu_{\mathscr{M}}(k) \text{ for all } f \in \mathcal{H}_{\mathcal{X}}, \text{ i.e., } T_{\mathcal{X}}^* T_{\mathcal{Y}}\big|_{\mathcal{H}_{\mathcal{X}}} = I_{\mathcal{H}_{\mathcal{X}}},$$

where $\mathcal{Y}$ need not be a subset of $\mathcal{H}_{\mathcal{X}}$.

When the duals of a frame are considered outside of the space $\mathcal{H}_{\mathcal{X}}$, one of the benefits is that it is sometime possible for the dual function to have improved localization properties in both the time and frequency domains. This is one of the advantages of defining alternate, oblique, type-I, and type-II duals. The following definitions will be helpful to extend the idea for continuous frames, including the classical definitions of discrete setup [44, 46].

Definition 1.1.1. Let $\mathcal{X} = \{f_k\}_{k\in\mathscr{M}}$ and $\mathcal{Y} = \{g_k\}_{k\in\mathscr{M}}$ be two families in $\mathcal{H}$ and let $\mathcal{H}_{\mathcal{X}} = \overline{\operatorname{span}}(\mathcal{X})$ such that the family $\mathcal{X}$ is a continuous frame for $\mathcal{H}_{\mathcal{X}}$ and $\mathcal{Y}$ is Bessel with respect to the measure space $(\mathscr{M}, \sum_{\mathscr{M}}, \mu_{\mathscr{M}})$. We say the family $\mathcal{Y}$ is

(a) an *alternate dual* for $\mathcal{X}$ if $T_{\mathcal{X}}^* T_{\mathcal{Y}}\big|_{\mathcal{H}_{\mathcal{X}}} = I_{\mathcal{H}_{\mathcal{X}}}$.

(b) an *oblique dual* for $\mathcal{X}$ if $T_{\mathcal{X}}^* T_{\mathcal{Y}}\big|_{\mathcal{H}_{\mathcal{X}}} = I_{\mathcal{H}_{\mathcal{X}}}$, $\mathcal{Y}$ is a continuous frame for $\mathcal{H}_{\mathcal{Y}}$, and $T_{\mathcal{Y}}^* T_{\mathcal{X}}\big|_{\mathcal{H}_{\mathcal{Y}}} = I_{\mathcal{H}_{\mathcal{Y}}}$, where $\mathcal{H}_{\mathcal{Y}} = \overline{\operatorname{span}}(\mathcal{Y})$.

(c) a *type-I dual* for $\mathcal{X}$ if $T_{\mathcal{X}}^* T_{\mathcal{Y}}\big|_{\mathcal{H}_{\mathcal{X}}} = I_{\mathcal{H}_{\mathcal{X}}}$, and $\operatorname{range}(T_{\mathcal{Y}}^*) \subset \operatorname{range}(T_{\mathcal{X}}^*)$.

(d) a *type-II dual* for $\mathcal{X}$ if $T_{\mathcal{X}}^* T_{\mathcal{Y}}\big|_{\mathcal{H}_{\mathcal{X}}} = I_{\mathcal{H}_{\mathcal{X}}}$, and $\operatorname{range}(T_{\mathcal{Y}}) \subset \operatorname{range}(T_{\mathcal{X}})$.

(e) a *dual frame* for $\mathcal{X}$ if $T_{\mathcal{X}}^* T_{\mathcal{Y}}\big|_{\mathcal{H}_{\mathcal{X}}} = I_{\mathcal{H}_{\mathcal{X}}}$, and $\mathcal{H}_{\mathcal{X}} = \mathcal{H}$.

Here, $I_{\mathcal{H}_{\mathcal{X}}}$ denotes the identity operator on $\mathcal{H}$ which is restricted on $\mathcal{H}_{\mathcal{X}}$. The operator $T_{\mathcal{X}}^* T_{\mathcal{Y}}$ is known as the *mixed frame operator*. When the linear operator $T_{\mathcal{X}}^* T_{\mathcal{Y}}$ is thought of as a matrix, it is often referred to as the *mixed dual Gramian* of $\mathcal{X}$ and $\mathcal{Y}$. We also call type-I (type-II) dual as *dual of type-I (type-II)*.

Note that the canonical dual frame $\tilde{\mathcal{X}}$ is a special case of (a), (b), (c), and (d). The Definition (e) is the usual concept of the dual frame in a Hilbert space, and the notions (a), (b), (c), and (d) are identically the same in this case. Additionally, observe that the oblique dual is a special case of alternate dual while type-I and type-II duals are special cases of oblique dual.

1.2. Motivation

Let ρ be a unitary representation of a locally compact group $\mathscr{G}$ on a separable Hilbert space $\mathcal{H}$. Then for a set of vectors $\mathscr{A}$ in $\mathcal{H}$, one of the most attractive research problems in harmonic analysis lies towards the investigation of Bessel, Riesz basis, or frame properties of the orbit $E(\mathscr{A}) = \{\rho(x)\varphi : \varphi \in \mathscr{A}, x \in \mathscr{G}\}$ in $\mathcal{H}$ [9, 10, 12, 19, 49, 50, 63]. The problem is intricately linked to a number of different aspects of functional analysis such as wavelets, frames and harmonic analysis, time-frequency analysis, spectral theory, etc. Many researchers have explored the topics of frame theory for such systems, including

Bownik et al. [19] and Iverson [49]. The next level of discussion is made with two basic questions:

(Q1) *What are the necessary and sufficient conditions to build a dual frame pair $E(\mathscr{A})$ and $E(\mathscr{A}')$ for various sets of generators $\mathscr{A}$ and $\mathscr{A}'$?*

(Q2) *When is the dual frame unique ?*

In this dissertation, we attempt to address the aforementioned two questions. This research seeks to establish a connection between the continuous and discrete theories of translation-invariant systems. When this is accomplished, the "unified strategy" proposed in [63] will be implemented and applied to a considerably broader range of scenarios. In addition to this, the new theory will incorporate intermediate stages, and it will do so in the context of a large number of square-integrable functions on locally compact groups. Particularly, by studying Gabor systems as a particular case, it is possible to obtain the standard conclusions for describing both the discrete and continuous systems (Section 4.3).

1.3. Translation-invariant spaces

Both **(Q1)** and **(Q2)** are the subject of an extensive investigation in the context of shift-invariant spaces where they were first introduced. According to the representation theory, the shift-invariant spaces are derived from the action of $\mathbb{Z}^n$ on $L^2(\mathbb{R}^n)$ by the shift operator $L_k : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by $L_k f(x) = f(x-k)$ for $k \in \mathbb{Z}^n, x \in \mathbb{R}^n$. The orbits associated with this action are $\{L_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$. They are used extensively in Gabor systems, multiresolution analysis, signal analysis, spline systems, approximation theory, and wavelets. The subspaces that are invariant under this representation are referred to as *shift-invariant* or *SI* for short. A closed subspace V of $L^2(\mathbb{R}^n)$ is called shift-invariant if $L_k f \in V$ for all $f \in V$, $k \in \mathbb{Z}^n$.

One of the most attractive areas of research in harmonic analysis is to reproduce a function from a given set of functions via a formula known as *reproducing formula* [51]. Gabor, wavelet, and shift-generated systems are notable platforms for studying such formulas in the Euclidean and locally compact abelian (LCA) group setup due to their wide use in various areas: time-frequency analysis, mathematical physics, quantum mechanics, quantum optics, etc. [13, 26, 30, 37, 41, 42, 46]. In general, the researchers have considered the system $\{\varphi_i(\cdot - k) : k \in \mathbb{Z}^n, i \in I\}$ having countable functions in $L^2(\mathbb{R}^n)$ and tried to characterize ψ_i's in $L^2(\mathbb{R}^n)$ such that the following reproducing formula

$$f = \sum_{i \in I, k \in \mathbb{Z}^n} \langle f, \psi_i(\cdot - k)\rangle \varphi_i(\cdot - k) \text{ for all } f \in \overline{\operatorname{span}}\{\varphi_i(\cdot - k)\}_{i,k},$$

holds true provided both the systems $\{\varphi_i(\cdot - k)\}_{i,k}$ and $\{\psi_i(\cdot - k)\}_{i,k}$ are Bessel. The system $\{\psi_i(\cdot - k)\}_{i,k}$ is known as an *alternate dual* for $\{\varphi_i(\cdot - k)\}_{i,k}$, where $\{\varphi_i(\cdot - k)\}_{i,k}$ is a frame sequence satisfying the above formula. It can be noted that the above stable decomposition of f allows the flexibility of choosing different types of duals for the frame $\{\varphi_i(\cdot - k)\}_{i,k}$. When the frame sequence $\{\varphi_i(\cdot - k)\}_{i,k}$ becomes Riesz basis, the choice of ψ_i is unique and the system $\{\psi_i(\cdot - k)\}_{i,k}$ is *biorthogonal dual* to $\{\varphi_i(\cdot - k)\}_{i,k}$. In general, the researchers consider $\overline{\operatorname{span}}\{\varphi_i(\cdot - k)\}_{i,k} = \overline{\operatorname{span}}\{\psi_i(\cdot - k)\}_{i,k}$ but sometimes choosing ψ_i's outside the space $\overline{\operatorname{span}}\{\varphi_i(\cdot - k)\}_{i,k}$ provides better localization properties in both the time and frequency domains. Such kind of requirements motivates researchers to define various duals for a frame, like oblique dual, type-I and type-II duals, etc. We refer [37, 43, 44, 46] for more details.

The discussion on the characterization of the dual frame was initiated by Ron and Shen [61, Section 4]. Later on, Bownik's revolutionary paper [16] appeared. But they did not classify duals into further categories. The classification of type-I and type-II dual first time appeared through [36]. In this paper, Gabardo and Han categorized the duals for the Gabor system. At the same time, Christensen and Eldar popularize the concepts of the oblique dual frame for a given frame sequence $\{f_k\}_{k \in I}$ of a subspace V on a shift-invariant space [26, 33]. Later, they generalized the same for multi-generators in [25]. The classification of oblique, type-I, and type-II and their existence and uniqueness appeared in [46]. Following that, Heil et al. [44] classified those types of duals for the Hilbert spaces. In this line of research, our focus will be to investigate the characterizations of duals for the translation generated continuous frame systems by the action of locally compact groups.

Throughout the dissertation, let us assume a second countable locally compact group $\mathscr{G}$ (not necessarily abelian) and a closed abelian subgroup Γ of $\mathscr{G}$, and consider a Γ-*translation generated (*Γ*-TG)* system $\mathcal{E}^\Gamma(\mathscr{A})$ and its associated Γ-translation invariant (Γ-TI) space $\mathcal{S}^\Gamma(\mathscr{A})$ for a family of functions $\mathscr{A} \subseteq L^2(\mathscr{G})$ by the action of Γ, i.e.,

$$(1.3.1) \qquad \mathcal{E}^\Gamma(\mathscr{A}) := \{L_\gamma \varphi : \gamma \in \Gamma, \varphi \in \mathscr{A}\}, \text{ and } \mathcal{S}^\Gamma(\mathscr{A}) := \overline{\operatorname{span}}\{L_\gamma \varphi : \gamma \in \Gamma, \varphi \in \mathscr{A}\},$$

where for $\eta \in \mathscr{G}$, the *left translation* L_η on $L^2(\mathscr{G})$ is defined by

$$(L_\eta f)(\gamma) = f(\eta^{-1}\gamma), \ \gamma \in \mathscr{G}.$$

By Γ-*translation invariant (*Γ-*TI)* space V, we mean $L_\xi f \in V$ for all $f \in V$ and $\xi \in \Gamma$, where V is a closed subspace of $L^2(\mathscr{G})$. In this scenario, the main goal of this dissertation is to provide a compendious study of duals for a continuous frame $\mathcal{E}^\Gamma(\mathscr{A})$ of $\mathcal{S}^\Gamma(\mathscr{A})$. The study of frames for Γ-TG system was initiated by Iverson [49] followed by Bownik and Iverson [19].

Our aim is to characterize a collection $\mathscr{A}'$ in $L^2(\mathscr{G})$ such that the Γ-TG system $\mathcal{E}^\Gamma(\mathscr{A}')$ satisfies the following reproducing formula:

$$f = \int_{\psi \in \mathscr{A}'} \int_{\varphi \in \mathscr{A}} \int_{\gamma \in \Gamma} < f, L_\gamma \psi > L_\gamma \varphi \, d\mu_\Gamma \, d\mu_{\mathscr{A}} \, d\mu_{\mathscr{A}'} \text{ for all } f \in \mathcal{S}^\Gamma(\mathscr{A}), \tag{1.3.2}$$

which is defined weakly in terms of the Pettis integral, where $\mu_\Gamma, \mu_{\mathscr{A}}$ and $\mu_{\mathscr{A}'}$ denote the corresponding measures. We call $\mathcal{E}^\Gamma(\mathscr{A}')$ to be a Γ-*TG dual* for a continuous frame $\mathcal{E}^\Gamma(\mathscr{A})$ of $\mathcal{S}^\Gamma(\mathscr{A})$. It can be noted that the above stable decomposition of f allows the flexibility of choosing different types of duals for a continuous frame $\mathcal{E}^\Gamma(\mathscr{A})$. For developing standard dual results authors consider $\mathcal{S}^\Gamma(\mathscr{A}) = \mathcal{S}^\Gamma(\mathscr{A}')$ in general [19], but sometimes choosing a Γ-TG dual $\mathcal{E}^\Gamma(\mathscr{A}')$ outside the space $S^\Gamma(\mathscr{A})$ provides better opportunities. For this, different types of duals for a discrete frame have been discussed by many authors, including Bownik [17], Gabardo and Han [36], Han and Larson [43], Heil, Koo, and Lim [44], and Hemmat and Gabardo [37, 46]. To unify such results related to the duals for a frame, we study alternate (oblique) Γ-TG dual, Γ-TG dual of type-I and type-II, and Γ-TG dual frame for a continuous frame $\mathcal{E}^\Gamma(\mathscr{A})$ of $\mathcal{S}^\Gamma(\mathscr{A})$ in $L^2(\mathscr{G})$, by the action of a closed abelian subgroup Γ of $\mathscr{G}$. We obtain characterizations of these duals in terms of the Zak transform for the pair $(\mathscr{G}, \Gamma)$. Further, we characterize the uniqueness of these duals in terms of the Gramian or dual Gramian operators, which become a discrete frame or Riesz sequence in $L^2(\mathscr{G})$. One of the benefits of such study on the pair $(\mathscr{G}, \Gamma)$ is to access the various number of previously inaccessible pairs, like $(\mathbb{R}^n, \mathbb{Z}^m)$, $(\mathbb{R}^n, \mathbb{R}^m)$, $(\mathcal{G}, \Lambda)$, $(\mathbb{Q}_p, \mathbb{Z}_p)$, etc., where $n \geqslant m$, Λ (not necessarily co-compact, i.e., $\mathcal{G}/\Lambda$-compact, or uniform lattice) is a closed subgroup of the second countable locally compact abelian (LCA) group $\mathcal{G}$, and $\mathbb{Z}_p$ is the p-adic integer in the p-adic number $\mathbb{Q}_p$. Such advantages become possible due to the involvement of the Zak transform, which was independently introduced by

Iverson [49] and Barbieri, Hernández, and Paternostro [47]. In the case of the pair $(\mathcal{G}, \Lambda)$ of LCA groups, we provide the characterizations of these duals in terms of the fiberization operators. Along with the counterexamples using the fiberization, we also illustrate the results for the abelian and non-abelian groups.

In this continuation, we generalize the concept of alternate dual and name it $\mathcal{K}$-subspace dual for a subspace $\mathcal{K}$ in $\mathcal{H}$ (Chapter 5). Further, we provide a detailed study of $\mathcal{S}^{\Gamma}(\mathscr{A})$-subspace duals of a Bessel family/frame $\mathcal{E}^{\Gamma}(\mathscr{A})$ in $L^2(\mathscr{G})$ due to its wide uses. Our results have so many predecessors related to the work on subspace and alternate duals and orthogonal Bessel pair [19, 25, 26, 30, 39–41, 44, 46, 54, 73]. The purpose of this study is devoted to characterizing a pair of orthogonal frames and subspace dual of a Bessel family/frame generated by the Γ-TG system $\mathcal{E}^{\Gamma}(\mathscr{A})$ in $L^2(\mathscr{G})$. When $\mathcal{E}^{\Gamma}(\mathscr{A})$ is a Riesz basis, then there is an associated biorthogonal system, which forms a unique dual and it is called *biorthogonal dual*. A brief study of the biorthogonal system with discrete translations is discussed. We characterize such results using the Zak transform $\mathcal{Z}$ for the pair $(\mathscr{G}, \Gamma)$ defined in (4.1.1). For the case of a locally compact abelian group $\mathcal{G}$, we use the fiberization map $\mathscr{T}$, which unifies the classical results related to the orthogonal and duals of a Bessel family/frame associated with a TI space. This study of orthogonal frames enables us to discuss dual for the super Hilbert space $\oplus^N L^2(\mathscr{G})$. In the past, the Zak transform was mostly used for the case of Gabor systems; however, we now apply it for the translation-invariant systems. This study expands beyond the previously explored realm of locally compact abelian groups with discrete translation to include non-abelian groups with continuous translations.

1.4. Reproducing formula for nilpotent Lie groups

The discussion of frames for shift-invariant spaces on connected, simply connected nilpotent Lie group was started by Ajita et al. [29] and later on by Barberi et al. for the Heisenberg group [12].

Our main goal is to describe the reproducing formulas associated with the translation generated continuous frame in $L^2(G)$. In particular, we assume G to be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and Z be the center of G. Then G is an *SI/Z group* if almost all of its irreducible representations are square-integrable modulo the center Z. An irreducible representation π of G is called *square integrable modulo the center* (SI/Z) if it satisfies the condition

$$\int_{G/Z} |\langle \pi(g)u, v\rangle|^2 \, d\dot{g} < \infty \text{ for all } u, v.$$

Indeed, our work is a continuation of a chain of research carried out by Currey et al. [29] for G, and Barbieri et al. [12] for $\mathbb{H}^d$. The novelty of the current study is in two folds:

(i) For the first time, it encompasses the non-abelian setup of the nilpotent Lie group, which is considered to be a high degree of non-abelian structure.

(ii) It includes non-discrete translations as well.

We briefly start by describing left translation generated systems in $L^2(G)$ as follows. For a sequence of functions $\mathscr{A} = \{\varphi_k : k \in I\}$ in $L^2(G)$ and a subset Λ of G, we define Λ-*translation generated* (Λ-*TG*) *system* $\mathcal{E}^\Lambda(\mathscr{A})$ and its associated Λ-*translation invariant* (Λ-*TI*) *space* $\mathcal{S}^\Lambda(\mathscr{A})$ by the action of Λ from (1.3.1) $\mathcal{E}^\Lambda(\mathscr{A}) := \{L_\lambda\varphi : \lambda \in \Lambda, \varphi \in \mathscr{A}\}$, and $\mathcal{S}^\Lambda(\mathscr{A}) := \overline{\text{span}}\mathcal{E}^\Lambda(\mathscr{A})$. For an integer lattice Λ_0 in the center of G, we particularly consider $\Lambda = \{\lambda_1\lambda_0 : \lambda_i \in \Lambda_i, \ i = 0, 1\}$, where Λ_1 is a subset (not necessarily discrete) of G.

Due to the wide use of continuous frames, we discuss the reproducing formulas for $\mathcal{E}^\Lambda(\mathscr{A})$ associated with the general set Λ_1, not necessarily discrete. The current study provides a compendious study of duals for a continuous frame $\mathcal{E}^\Lambda(\mathscr{A})$ of $\mathcal{S}^\Lambda(\mathscr{A})$. We provide point-wise characterizations of alternate duals and their subcategories, like oblique, type-I, and type-II duals for the continuous frame $\mathcal{E}^\Lambda(\mathscr{A})$ and show that the global properties of duals can be transmitted into locally. The point-wise characterization results in terms of the fibers will also depend upon Λ_1 unlike the Euclidean case [46].

We can get results for discrete frames by sampling the continuous frames which have wide applications due to their computational simplicity. Further, for the discrete Λ_1, we discuss reproducing formulas associated with the singly generated Λ-TG systems $\mathcal{E}^\Lambda(\varphi)$ and $\mathcal{E}^\Lambda(\psi)$ for $\varphi, \psi \in L^2(G)$ having biorthogonal property.

At this juncture, we point out that the Plancherel transform is a standard tool for such study in the Euclidean and LCA group setup. Unlike the Euclidean and LCA group setup, the dual space of the nilpotent Lie groups replaces the frequency domain, and the Plancherel transform of a function is operator-valued. Therefore the technique used in the Euclidean and LCA groups is restrained.

To illustrate the current work for the Heisenberg group $\mathbb{H}^d$, we first note that $\mathbb{H}^d$ is an SI/Z nilpotent Lie group identified with $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$. The *d-dimensional Heisenberg group*, denoted by $\mathbb{H}^d$, is an example of SI/Z group. The group $\mathbb{H}^d$ can be identified with $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ under the group operation $(x, y, w) \cdot (x', y', w') = (x + x', y + y', w + w' + x \cdot y)$, $x, x', y, y' \in \mathbb{R}^d$, $w, w' \in \mathbb{R}$, where "$\cdot$" stands for $\mathbb{R}^d$ scalar product. Using the fiberization map on $L^2(\mathbb{H}^d)$ associated with the Schrödinger representations, we can convert all results into the Heisenberg group $\mathbb{H}^d$ setup from the SI/Z nilpotent Lie group G in a natural way. For this case, the integer lattice is $\Lambda_0 = \mathbb{Z}$, Λ_1 is a discrete set of the form $A\mathbb{Z}^d \times B\mathbb{Z}^d$, where $A, B \in GL(d, \mathbb{R})$ with $AB^t \in \mathbb{Z}$, and $N \in \mathbb{N}$. As a consequence of our results for the Heisenberg group, a reproducing formula associated with the orthonormal Gabor systems of $L^2(\mathbb{R}^d)$ is obtained.

1.5. Structure of the thesis

The structure of the thesis is as follows:

The results of Chapter 2 and Chapter 4 are from our published material **S. Sarkar, N. K. Shukla**, *Translation generated oblique dual frames on locally compact groups*, **Linear Multilinear Algebra**, (2023), doi:10.1080/03081087.2023.2173718, 32 pages. The Section 6.6 of Chapter 6 is a part of our published material **S. Sarkar, N. K. Shukla**, *Characterizations of extra-invariant spaces under the left translations on a Lie group*, **Advances in Operator Theory,** (2023), https://doi.org/10.1007/s43036-023-00273-x.

Chapter 2 and **Chapter 3** offer abstract machinery tools for the dual frames in measure-theoretic abstraction as a preparation for the next chapters.

In **Chapter 2**, we characterize alternate (oblique) duals and duals of type-I and type-II for a frame in multiplication invariant spaces on $L^2(X; \mathcal{H})$ corresponding to the pointwise conditions in $\mathcal{H}$. This contains discussions on Plancherel transform on $L^2(X)$ corresponding to a Parseval determining set in the measure-theoretic setup and then providing a characterization result for the type-II dual. The results present a unified theory connecting the discrete problems with a continuous setup. Besides, we characterize these duals' uniqueness using the Gramian/dual Gramian operators, which become a discrete frame/Riesz basis for the associated range space.

In **Chapter 3**, we discuss the construction of the dual frames and their uniqueness for the multiplication generated frames on $L^2(X;\mathcal{H})$ where X is a σ-finite measure. A necessary and sufficient condition of such duals associated with the infimum cosine angle is obtained.

In **Chapter 4**, we obtain characterizations of alternate (oblique) Γ-TG duals and Γ-TG duals of type-I in terms of the Zak transform for the pair $(\mathscr{G},\Gamma)$ and its uniqueness using the Gramian/dual Gramian operators. In the case of a discrete abelian subgroup Γ of $\mathscr{G}$, Γ-TG duals of type-II and their uniqueness are characterized in terms of the Zak transform. When $\mathscr{G}$ becomes an abelian group $\mathcal{G}$, the fiberization map is used to characterize these duals by the action of its closed subgroup Λ. In Section 4.2, we provide prototype examples on $\mathbb{R}^n$, $\mathbb{Q}_p$ along with counterexamples of duals. In conclusion, we provide the characterizations of the duals for Gabor systems in Section 4.3. In Section 4.4, we additionally take into account the oblique dual frame characterizations and their uniqueness associated with the orbit generated by dual integrable representations of LCA groups. At the end, the construction techniques of new dual frames and their uniqueness associated with the infimum cosine angle for the Γ-TG frame are also discussed in Section 4.5.

By an action of a closed abelian subgroup Γ of $\mathscr{G}$ on a collection of functions $\mathscr{A}$ in $L^2(\mathscr{G})$, we study $\mathcal{S}^\Gamma(\mathscr{A})$-subspace orthogonal and duals to a Bessel family/frame $\mathcal{E}^\Gamma(\mathscr{A})$ in **Chapter 5**, and obtain characterization results in terms of the Zak transform for the pair $(\mathscr{G},\Gamma)$ and the Gramian operator. As an application, we study such subspace dual and orthogonal frames for singly generated systems in Subsection 5.1.1 . An investigation of a translation generated biorthogonal system and dual of a Riesz basis is carried out in Section 5.2. In Section 5.3, we address the theory for a collection of generators indexed by σ-finite measure space (need not be countable) by the action of any closed abelian subgroup Γ of $\mathscr{G}$ which unifies the broader class of continuous frames as well. This chapter ends with an illustration of our results for the various potential applications such as splines, Gabor systems, p-adic fields $\mathbb{Q}_p$, etc.

The **Chapter 6**, starts with a brief discussions about the Plancherel transform for the SI/Z nilpotent Lie group (Section 6.1). Section 6.3 is devoted to construct reproducing formulas associated with the continuous frames generated by the non-discrete translation of multiple functions using the range function. Employing the Plancherel transform followed by periodization, we discuss various reproducing formulas for the singly generated discrete systems $\mathcal{E}^{\Lambda}(\varphi)$ and $\mathcal{E}^{\Lambda}(\psi)$ having biorthogonal property in Section 6.4. We establish the proof of our main results Theorem 6.6.2, 6.6.7, 6.6.8 and 6.6.10 in Section 6.6.1 by involving the range function associated with a $\Lambda_1\Lambda_0$-invarinat space.

In the context of a connected, simply connected nilpotent Lie group, whose representations are square-integrable modulo the center, we find characterization results of extra-invariant spaces under the left translations associated with the range functions. Consequently, the theory is valid for the Heisenberg group $\mathbb{H}^d$, a 2-step nilpotent Lie group.

Finally, **Chapter 7** deals with some concluding remarks and provides directions for future studies.

CHAPTER 2

MULTIPLICATION GENERATED OBLIQUE DUAL FRAMES IN $L^2(X;\mathcal{H})$

In this chapter, we discuss duals of the multiplication generated systems as a measure-theoretic abstraction in $L^2(X;\mathcal{H})$ using the range functions. The results present a unified theory connecting the discrete problems with a continuous setup. Besides, we characterize the uniqueness of these duals using the Gramian/dual Gramian operators, which become a discrete frame/Riesz basis for the associated range space.

2.1. Multiplication invariant spaces

Throughout this chapter, we fix separable Hilbert space $\mathcal{H}$, and a positive, σ-finite and complete measure space (X,μ_X) such that $L^2(X)$ is separable. Now we define multiplication operator on $L^2(X;\mathcal{H})$, where

$$L^2(X;\mathcal{H}) = \left\{\varphi \mid \varphi : X \to \mathcal{H} \text{ is measurable such that } \int_X \|\varphi(x)\|^2 \, d\mu_X(x) < \infty\right\},$$

is a Hilbert space with the inner product

$$\langle \varphi, \psi\rangle = \int_X \langle \varphi(x), \psi(x)\rangle \, d\mu_X(x) \text{ for } \varphi, \psi \in L^2(X;\mathcal{H}).$$

Definition 2.1.1. For $\phi \in L^\infty(X)$, the operator M_ϕ on $L^2(X;\mathcal{H})$ is defined by

$$(M_\phi f)(x) = \phi(x) f(x) \ \text{ a.e. } \ x \in X, \ \ f \in L^2(X;\mathcal{H}),$$

is known as the *multiplication operator.*

This chapter is a part of the following manuscripts:

S. Sarkar, N. K. Shukla, *Translation generated oblique dual frames on locally compact groups*, **Linear Multilinear Algebra**, (2023), doi:10.1080/03081087.2023.2173718, 32 pages.

S. Sarkar, N. K. Shukla, *A characterization of MG dual frames using infimum cosine angle*, arXiv:2301.07448.

The operator M_ϕ is a bounded linear operator on $L^2(X;\mathcal{H})$ satisfying $\|M_\phi\| = \|\phi\|_{L^\infty}$ provided X is a σ-finite measure space. If X is not a σ-finite measure space, then $\|M_\phi\|$ need not be the same as $\|\phi\|_{L^\infty}$ [27, Theorem 1.5].

Next, we define the determining set introduced by Bownik and Ross [20].

Definition 2.1.2. A set $\mathfrak{D} \subset L^\infty(X)$ is called a *determining set for* $L^1(X)$ if for any non-zero f in $L^1(X)$, there exists a $g \in \mathfrak{D}$ such that $\int_X f(x)g(x)\, d\mu_X(x) \neq 0$.

The determining set is a basis like family of functions in $L^\infty(X)$. For example, if $\mathcal{G}$ is a locally compact abelian group, the dual group $\widehat{\mathcal{G}}$ (collection of all continuous homomorphisms from $\mathcal{G}$ to $\mathbb{T}$) is a determining set for $L^1(\mathcal{G})$, and it is an orthonormal basis for $L^2(\mathcal{G})$ when $\mathcal{G}$ is compact.

Now we define multiplication invariant spaces on $L^2(X;\mathcal{H})$ associated with the determining set $\mathfrak{D} \subset L^\infty(X)$.

Definition 2.1.3. Let V be a closed subspace of $L^2(X;\mathcal{H})$ and $\mathfrak{D}$ be a determining set for $L^1(X)$. We say V is a *multiplication invariant (MI) space* corresponding to $\mathfrak{D}$ if

$$M_\phi f \in V \text{ for all } \phi \in \mathfrak{D} \text{ and } f \in V.$$

Given a family $\mathcal{A} \subset L^2(X;\mathcal{H})$ and a determining set $\mathfrak{D} \subset L^\infty(X)$ for $L^1(X)$, we define *multiplication generated (MG)* system $E_{\mathfrak{D}}(\mathcal{A})$ and its associated MI space $S_{\mathfrak{D}}(\mathcal{A})$ as follows:

$$E_{\mathfrak{D}}(\mathcal{A}) := \{M_\phi \varphi : \phi \in \mathfrak{D}, \varphi \in \mathcal{A}\} \text{ and } \; S_{\mathfrak{D}}(\mathcal{A}) := \overline{\operatorname{span}} E_{\mathfrak{D}}(\mathcal{A}).$$

For the characterization of MI spaces, the range function plays a crucial role. The history of the range function traces back to the work of [19,20,45,49,71]. A *range function* on X is a mapping $J : X \to \{\text{closed subspaces of } \mathcal{H}\}$. Further, we say J is *measurable* if for any $u, v \in \mathcal{H}$ the mapping $x \mapsto \langle P_J(x)u, v\rangle$ is measurable on X, where for $x \in X$, the orthogonal projection $P_J(x) : \mathcal{H} \to \mathcal{H}$ projects onto $J(x)$. Next, we define a closed subspace V_J corresponding to the projection-valued map J as follows:

$$V_J := \left\{\varphi \in L^2(X;\mathcal{H}) : \varphi(x) \in J(x) \text{ for a.e. } x \in X\right\}. \tag{2.1.1}$$

The following result is a restatement of [20, Theorem 2.4].

Theorem 2.1.4. *Let (X, μ) be a σ-finite measure space, $\mathfrak{D}$ be a determining set for $L^1(X)$ and $\mathcal{H}$ be a separable Hilbert space. Then the following hold:*

(i) *If J is a measurable range function then $M_\phi V_J \subset V_J$ for $\phi \in \mathfrak{D}$, where V_J is defined in (2.1.1).*

(ii) *The mapping $J \mapsto V_J$ from the collection of all measurable range functions which are equal a.e. to the set of all MI subspaces of $L^2(X;\mathcal{H})$ is a bijection.*

(iii) *For a collection of functions $\mathcal{A}$ in $L^2(X;\mathcal{H})$ and for a countable dense subset $\mathcal{A}_0 \subset \mathcal{A}$, consider the range function $J(x) = \overline{\operatorname{span}}\{\varphi(x) : \varphi \in \mathcal{A}_0\}$ (defined a.e.). Then, $V_J = S_{\mathfrak{D}}(\mathcal{A}_0) = S_{\mathfrak{D}}(\mathcal{A})$.*

At the end of this subsection, we define Parseval determining set for $L^1(X)$ which is a special kind of determining set, introduced by Iverson [49]. It is a measure-theoretic abstraction of characters for an abelian group $\mathcal{G}$ satisfying the Plancherel formula:

$$\int_{\widehat{\mathcal{G}}} \left| \int_{\mathcal{G}} f(x)\overline{\beta(x)}\, d\mu_{\mathcal{G}}(x) \right|^2 d\mu_{\widehat{\mathcal{G}}}(\beta) = \int_{\widehat{\mathcal{G}}} \left| \widehat{f}(\beta) \right|^2 d\mu_{\widehat{\mathcal{G}}}(\beta) = \int_{\mathcal{G}} |f(x)|^2\, d\mu_{\mathcal{G}}(x) \text{ for } f \in L^1(\mathcal{G}). \tag{2.1.2}$$

Definition 2.1.5. Let $(\mathcal{M}, \mu_{\mathcal{M}})$ be a measure space. A set $\mathcal{D} = \{g_s \in L^\infty(X) : s \in \mathcal{M}\}$ is said to be a *Parseval determining set* for $L^1(X)$ if for each $f \in L^1(X)$, $s \mapsto \int_X f(x)\overline{g_s(x)}\, d\mu_X(x)$ is measurable on $\mathcal{M}$ and

$$\int_{\mathcal{M}} \left| \int_X f(x)\overline{g_s(x)}\, d\mu_X(x) \right|^2 d\mu_{\mathcal{M}}(s) = \int_X |f(x)|^2\, d\mu_X(x).$$

2.2. Characterization results

Given a Parseval determining set $\mathcal{D} = \{g_s \in L^\infty(X) : s \in \mathcal{M}\}$ for $L^1(X)$ (see Definition 2.1.5), and a family of functions $\mathcal{A} = \{\varphi_t : t \in \mathcal{N}\}$ having a countable dense subset $\mathcal{A}_0$ in $L^2(X;\mathcal{H})$, we recall the MG system $E_{\mathcal{D}}(\mathcal{A})$ and its associated MI space $S_{\mathcal{D}}(\mathcal{A})$ in $L^2(X;\mathcal{H})$ associated with $\mathcal{D}$, given by:

$$E_{\mathcal{D}}(\mathcal{A}) := \{M_{g_s}\varphi_t(\cdot) = g_s(\cdot)\varphi_t(\cdot) : s \in \mathcal{M}, t \in \mathcal{N}\} \text{ and } S_{\mathcal{D}}(\mathcal{A}) := \overline{\operatorname{span}} E_{\mathcal{D}}(\mathcal{A}),$$

where $(\mathcal{M}, \mu_{\mathcal{M}})$ and $(\mathcal{N}, \mu_{\mathcal{N}})$ are σ-finite measure spaces. For a.e. $x \in X$, we define the range function $J_{\mathcal{A}}(x)$ and a set $\mathcal{A}(x)$ as follows:

$$J_{\mathcal{A}}(x) := \overline{\operatorname{span}}\{\eta(x) : \eta \in \mathcal{A}_0\}, \text{ and } \mathcal{A}(x) := \{\eta(x) : \eta \in \mathcal{A}\}. \tag{2.2.1}$$

Now in the sense of Definition 1.1.1, we define alternate (oblique) duals and its particular cases for a continuous frame $E_{\mathcal{D}}(\mathcal{A})$ of $S_{\mathcal{D}}(\mathcal{A})$ in $L^2(X;\mathcal{H})$.

Definition 2.2.1. Given $\mathcal{A}, \mathcal{A}' \subset L^2(X;\mathcal{H})$, assume that the MG system $E_{\mathcal{D}}(\mathcal{A})$ is a continuous frame for the MI space $S_{\mathcal{D}}(\mathcal{A})$ over $\mathcal{M} \times \mathcal{N}$ and $E_{\mathcal{D}}(\mathcal{A}')$ is Bessel in $L^2(X;\mathcal{H})$ over $\mathcal{M} \times \mathcal{N}$. We call

(i) $E_{\mathcal{D}}(\mathcal{A}')$ an *alternate MG-dual* for $E_{\mathcal{D}}(\mathcal{A})$ if it is an alternate dual for $E_{\mathcal{D}}(\mathcal{A})$ in the sense of Definition 1.1.1 (a).

(ii) $E_{\mathcal{D}}(\mathcal{A}')$ an *oblique MG-dual* for $E_{\mathcal{D}}(\mathcal{A})$ if it is an oblique dual for $E_{\mathcal{D}}(\mathcal{A})$ in the sense of Definition 1.1.1 (b).

(iii) $E_{\mathcal{D}}(\mathcal{A}')$ a *type-I MG-dual* for $E_{\mathcal{D}}(\mathcal{A})$ if it is a type-I dual for $E_{\mathcal{D}}(\mathcal{A})$ in the sense of Definition 1.1.1 (c).

(iv) $E_{\mathcal{D}}(\mathcal{A}')$ a *type-II MG-dual* for $E_{\mathcal{D}}(\mathcal{A})$ if it is a type-II dual for $E_{\mathcal{D}}(\mathcal{A})$ in the sense of Definition 1.1.1 (d).

(v) $E_{\mathcal{D}}(\mathcal{A}')$ an *MG-dual frame* for $E_{\mathcal{D}}(\mathcal{A})$ if it is a dual frame for $E_{\mathcal{D}}(\mathcal{A})$ in the sense of Definition 1.1.1 (e).

In general, we write *MG dual of type-I (type-II)* instead of type-I (type-II) MG dual.

It is worthwhile to mention that the cannonical dual of an MG system $E_{\mathcal{D}}(\mathcal{A})$ is itself an MG system $E_{\mathcal{D}}(\tilde{\mathcal{A}})$ for any collection $\mathcal{A} = \{\varphi_t\}_{t \in \mathcal{N}}$ in $L^2(X;\mathcal{H})$, where $\tilde{\mathcal{A}} := \{(S_{E_{\mathcal{D}}(\mathcal{A})}|_{S_{\mathcal{D}}(\mathcal{A})})^{-1}\varphi_t : t \in \mathcal{N}\}$ [19]. Further note that for a.e. $x \in X$, $\tilde{\mathcal{A}}(x)$ is the canonical dual for $\mathcal{A}(x)$, where

$$\left[(S_{E_{\mathcal{D}}(\mathcal{A})}|_{S_{\mathcal{D}}(\mathcal{A})})^{-1}\varphi_t\right](x) = (S_{\mathcal{A}(x)}|_{J_{\mathcal{A}}(x)})^{-1}(\varphi_t(x)) \text{ for a.e. } x \in X. \tag{2.2.2}$$

In this chapter, we will characterize alternate, oblique, type-I and type-II MG duals corresponding to the point-wise conditions in $\mathcal{H}$. For this, we begin with defining Plancherel transform associated with the Parseval determining set. This is a variant of Fourier transform introduced by Bownik and Iverson [19].

Definition 2.2.2. For $f \in L^1(X) \cap L^2(X)$, we define $\mathcal{F}f \in L^2(\mathcal{M})$ corresponding to the Parseval determining set $\mathcal{D} = \{g_s \in L^\infty(X) : s \in \mathcal{M}\}$ by

$$(\mathcal{F}f)(s) = \int_X f(x)\overline{g_s(x)}\, d\mu_X(x) \text{ a.e. } s \in \mathcal{M}. \tag{2.2.3}$$

The *Plancherel transform* associated with Parseval determining set $\mathcal{D}$ has a unique extension $\mathcal{F} : L^2(X) \to L^2(\mathcal{M})$, which is linear and isometry.

Employing the Parseval determining set $\mathcal{D}$ for $L^1(X)$ and isometry of $\mathcal{F}$, we have the *Plancherel's relation* and *Parseval's formula*

$$\|\mathcal{F}f\|_{L^2(\mathcal{M})} = \|f\|_{L^2(X)} \text{ and } \langle \mathcal{F}f, \mathcal{F}h\rangle_{L^2(\mathcal{M})} = \langle f, h\rangle_{L^2(X)} \text{ for all } f, h \in L^2(X), \tag{2.2.4}$$

respectively.

We need the following result in the sequel to step ahead.

Lemma 2.2.3. *For two Bessel families $E_{\mathcal{D}}(\mathcal{A})$ and $E_{\mathcal{D}}(\mathcal{A}')$ over $\mathcal{M} \times \mathcal{N}$ in $L^2(X;\mathcal{H})$, we have the following for all $f, g \in L^2(X;\mathcal{H})$:*

$$\left\langle T_{E_{\mathcal{D}}(\mathcal{A}')}f, T_{E_{\mathcal{D}}(\mathcal{A})}g \right\rangle = \int_X \langle T_{\mathcal{A}'(x)}f(x), T_{\mathcal{A}(x)}g(x)\rangle \, d\mu_X(x),$$

where $T_{E_{\mathcal{D}}(\mathcal{A})}$ and $T_{E_{\mathcal{D}}(\mathcal{A}')}$ are the analysis operators associated with the Bessel families $E_{\mathcal{D}}(\mathcal{A})$ and $E_{\mathcal{D}}(\mathcal{A}')$, respectively. Moreover, we obtain

$$\int_{\mathcal{M}} \int_{\mathcal{N}} \langle f, M_{g_s}\psi_t\rangle \overline{\langle g, M_{g_s}\varphi_t\rangle} \, d\mu_{\mathcal{N}}(t) \, d\mu_{\mathcal{M}}(s) = \int_X \int_{\mathcal{N}} \langle f(x), \psi_t(x)\rangle \overline{\langle g(x), \varphi_t(x)\rangle} \, d\mu_{\mathcal{N}}(t) \, d\mu_X(x).$$

Proof. For $f, g \in L^2(X;\mathcal{H})$, the analysis operators $T_{E_{\mathcal{D}}(\mathcal{A})}$ and $T_{E_{\mathcal{D}}(\mathcal{A}')}$ satisfy

$$(T_{E_{\mathcal{D}}(\mathcal{A})}g)(s,t) = \langle g, M_{g_s}\varphi_t\rangle \text{ and } (T_{E_{\mathcal{D}}(\mathcal{A}')}f)(s,t) = \langle f, M_{g_s}\psi_t\rangle \text{ for all } (s,t) \in \mathcal{M} \times \mathcal{N},$$

and then we compute the following:

$$\begin{aligned}
&\left\langle T_{E_{\mathcal{D}}(\mathcal{A}')}f, T_{E_{\mathcal{D}}(\mathcal{A})}g \right\rangle = \int_{\mathcal{M}} \int_{\mathcal{N}} \langle f, M_{g_s}\psi_t\rangle \overline{\langle g, M_{g_s}\varphi_t\rangle} \, d\mu_{\mathcal{N}}(t) \, d\mu_{\mathcal{M}}(s) \\
&= \int_{\mathcal{M}} \int_{\mathcal{N}} \left(\int_X \langle f(x), g_s(x)\psi_t(x)\rangle \, d\mu_X(x) \right) \times \overline{\left(\int_X \langle g, g_s(x)\varphi_t(x)\rangle \, d\mu_X(x) \right)} \, d\mu_{\mathcal{N}}(t) \, d\mu_{\mathcal{M}}(s) \\
&= \int_{\mathcal{M}} \int_{\mathcal{N}} \left(\int_X \langle f(x), \psi_t(x)\rangle \overline{g_s(x)} \, d\mu_X(x) \right) \times \overline{\left(\int_X \langle g(x), \varphi_t(x)\rangle \overline{g_s(x)} \, d\mu_X(x) \right)} \, d\mu_{\mathcal{N}}(t) \, d\mu_{\mathcal{M}}(s).
\end{aligned}$$

Choosing $F_{\psi_t}(x) = \langle f(x), \psi_t(x)\rangle$ and $G_{\varphi_t}(x) = \langle g(x), \varphi_t(x)\rangle$ for $x \in X$ and $t \in \mathcal{N}$, we have

$$
\begin{aligned}
\left\langle T_{E_{\mathcal{D}}(\mathcal{A}')} f, T_{E_{\mathcal{D}}(\mathcal{A})} g\right\rangle &= \int_{\mathcal{M}} \int_{\mathcal{N}} \mathcal{F}F_{\psi_t}(s)\overline{\mathcal{F}G_{\varphi_t}(s)}\ d\mu_{\mathcal{N}}(t)\ d\mu_{\mathcal{M}}(s) \\
&= \int_{\mathcal{N}} \int_{\mathcal{M}} \mathcal{F}F_{\psi_t}(s)\overline{\mathcal{F}G_{\varphi_t}(s)}\ d\mu_{\mathcal{M}}(s)\ d\mu_{\mathcal{N}}(t) \\
&= \int_{\mathcal{N}} \langle \mathcal{F}F_{\psi_t}, \mathcal{F}G_{\varphi_t}\rangle\ d\mu_{\mathcal{N}}(t) = \int_{\mathcal{N}} \langle F_{\psi_t}, G_{\varphi_t}\rangle\ d\mu_{\mathcal{N}}(t) \\
&= \int_{\mathcal{N}} \int_X F_{\psi_t}(x)\overline{G_{\varphi_t}(x)}\ d\mu_X(x)\ d\mu_{\mathcal{N}}(t) \\
&= \int_{\mathcal{N}} \int_X \langle f(x), \psi_t(x)\rangle\ \overline{\langle g(x), \varphi_t(x)\rangle}\ d\mu_X(x)\ d\mu_{\mathcal{N}}(t),
\end{aligned}
$$

using Parseval's formula (2.2.4) on $L^2(X)$. Note that, $F_{\psi_t}, G_{\varphi_t} \in L^2(X;\mathcal{H})$ in the above calculations by the fact that $E_{\mathcal{D}}(\mathcal{A})$ is Bessel with bound B if and only if $\mathcal{A}(x)$ is Bessel with bound B for a.e. $x \in X$, and the following estimate

$$
\begin{aligned}
\int_X \int_{\mathcal{N}} \left|F_{\psi_t}(x)\overline{G_{\varphi_t}(x)}\right|\ d\mu_X(x)\ d\mu_{\mathcal{N}}(t) &\leqslant \left(\int_X \int_{\mathcal{N}} |\langle f(x), \psi_t(x)\rangle|^2\ \ d\mu_X(x)\ d\mu_{\mathcal{N}}(t)\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_X \int_{\mathcal{N}} |\langle g(x), \varphi_t(x)\rangle|^2\ \ d\mu_X(x)\ d\mu_{\mathcal{N}}(t)\right)^{\frac{1}{2}} \\
&\leqslant \sqrt{BB'}\|f\|\|g\|,
\end{aligned}
$$

using Cauchy-Schwarz inequality, where we assume $E_{\mathcal{D}}(\mathcal{A})$ and $E_{\mathcal{D}}(\mathcal{A}')$ are Bessel systems with bounds B and B', respectively. Therefore using Fubini's theorem over $\mathcal{N} \times X$, we get

$$
\begin{aligned}
\left\langle T_{E_{\mathcal{D}}(\mathcal{A}')} f, T_{E_{\mathcal{D}}(\mathcal{A})} g\right\rangle &= \int_{\mathcal{N}} \int_X \langle f(x), \psi_t(x)\rangle\ \overline{\langle g(x), \varphi_t(x)\rangle}\ d\mu_X(x)\ d\mu_{\mathcal{N}}(t) \\
&= \int_X \int_{\mathcal{N}} T_{\mathcal{A}'(x)}(f(x))(t)\overline{T_{\mathcal{A}(x)}(g(x))(t)}\ d\mu_{\mathcal{N}}(t)\ d\mu_X(x) \\
&= \int_X \langle T_{\mathcal{A}'(x)} f(x), T_{\mathcal{A}(x)} g(x)\rangle\ d\mu_X(x),
\end{aligned}
$$

where $T_{\mathcal{A}'(x)}(f(x))(t) = \langle f(x), \psi_t(x)\rangle$ and $T_{\mathcal{A}(x)}(g(x))(t) = \langle g(x), \varphi_t(x)\rangle$ for $t \in \mathcal{N}$. □

Now, we state an abstract version of the results developed by many authors [17, Theorem 7.3], [46, Theorem 5] and [44, Theorem 4.2] for discrete frames on Euclidean spaces. The following results characterize the alternate (oblique) MG-duals, MG-duals of type-I, and MG-dual frames associated with the range function $J_{\mathcal{A}}(x)$ for a.e. $x \in X$.

Theorem 2.2.4. *Let $\mathcal{A} = \{\varphi_t : t \in \mathcal{N}\}$ be a collection of functions in $L^2(X;\mathcal{H})$ such that there exists a countable dense subset $\mathcal{A}_0$ of $\mathcal{A}$ and $J_{\mathcal{A}}(x)$ is defined for a.e. $x \in X$ by (2.2.1), where (X, μ_X) and $(\mathcal{N}, \mu_{\mathcal{N}})$ are σ-finite measure spaces. For a Parseval determining set $\mathcal{D} = \{g_s\}_{s\in\mathcal{M}}$ for $L^1(X)$, let the MG system $E_{\mathcal{D}}(\mathcal{A})$ be a continuous frame for the MI space $S_{\mathcal{D}}(\mathcal{A})$ over $\mathcal{M} \times \mathcal{N}$, where $(\mathcal{M}, \mu_{\mathcal{M}})$ is a σ-finite measure space. If $\mathcal{A}' = \{\psi_t : t \in \mathcal{N}\}$ is a collection of functions in $L^2(X;\mathcal{H})$ such that the MG system $E_{\mathcal{D}}(\mathcal{A}')$ is Bessel over $\mathcal{M} \times \mathcal{N}$ in $L^2(X;\mathcal{H})$, then the following hold:*

(i) *$E_{\mathcal{D}}(\mathcal{A}')$ is an alternate MG-dual for $E_{\mathcal{D}}(\mathcal{A})$ in $L^2(X;\mathcal{H})$ if and only if for a.e. $x \in X$, the system $\mathcal{A}'(x)$ is an alternate dual for the frame $\mathcal{A}(x)$ of $J_{\mathcal{A}}(x)$ in $\mathcal{H}$. Equivalently, $T^*_{E_{\mathcal{D}}(\mathcal{A})} T_{E_{\mathcal{D}}(\mathcal{A}')}\big|_{S_{\mathcal{D}}(\mathcal{A})} = I_{S_{\mathcal{D}}(\mathcal{A})}$ if and only if $T^*_{\mathcal{A}(x)} T_{\mathcal{A}'(x)}\big|_{J_{\mathcal{A}}(x)} = I_{J_{\mathcal{A}}(x)}$ for a.e. $x \in X$.*

(ii) *$E_{\mathcal{D}}(\mathcal{A}')$ is an oblique MG-dual for $E_{\mathcal{D}}(\mathcal{A})$ if and only if for a.e. $x \in X$, $\mathcal{A}'(x)$ is an oblique dual for the frame $\mathcal{A}(x)$ of $J_{\mathcal{A}}(x)$.*

(iii) *$E_{\mathcal{D}}(\mathcal{A}')$ is an MG-dual of type-I for $E_{\mathcal{D}}(\mathcal{A})$ if and only if for a.e. $x \in X$, $\mathcal{A}'(x)$ is a type-I dual for the frame $\mathcal{A}(x)$ of $J_{\mathcal{A}}(x)$.*

(iv) *$E_{\mathcal{D}}(\mathcal{A}')$ is an MG-dual frame for $E_{\mathcal{D}}(\mathcal{A})$ if and only if for a.e. $x \in X$, $\mathcal{A}'(x)$ is a dual frame for $\mathcal{A}(x)$ of $J_{\mathcal{A}}(x)$.*

Proof. (i) Assume that the MG system $E_{\mathcal{D}}(\mathcal{A}')$ is an alternate MG-dual for $E_{\mathcal{D}}(\mathcal{A})$ in $L^2(X;\mathcal{H})$, i.e.,

$$T^*_{E_{\mathcal{D}}(\mathcal{A})} T_{E_{\mathcal{D}}(\mathcal{A}')}\big|_{S_{\mathcal{D}}(\mathcal{A})} = I_{S_{\mathcal{D}}(\mathcal{A})}. \tag{2.2.5}$$

By Lemma 2.2.3, we get the the following for all $f, g \in S_{\mathcal{D}}(\mathcal{A})$:

$$\int_X \langle T^*_{\mathcal{A}(x)} T_{\mathcal{A}'(x)} f(x), g(x)\rangle \, d\mu_X(x) = \int_X \langle f(x), g(x)\rangle \, d\mu_X(x). \tag{2.2.6}$$

We have to show $T^*_{\mathcal{A}(x)} T_{\mathcal{A}'(x)}\big|_{J_{\mathcal{A}}(x)} = I_{J_{\mathcal{A}}(x)}$ for a.e. $x \in X$. For this, let $\{x_n\}_{n\in\mathbb{N}}$ be a countable dense subset of $\mathcal{H}$ and let $P_{J_{\mathcal{A}}}(x)$ be an orthogonal projection onto $J_{\mathcal{A}}(x)$ for a.e. $x \in X$. Clearly for a.e. $x \in X$, $\{P_{J_{\mathcal{A}}}(x)x_n\}_{n\in\mathbb{N}}$ is dense in $J_{\mathcal{A}}(x)$. Next, for each $m, n \in \mathbb{N}$, we define a set $S_{m,n}$ as follows:

$$S_{m,n} = \Big\{x \in X : \rho_{m,n}(x) := \langle T^*_{\mathcal{A}(x)} T_{\mathcal{A}'(x)} P_{J_{\mathcal{A}}}(x)x_m, P_{J_{\mathcal{A}}}(x)x_n\rangle - \langle P_{J_{\mathcal{A}}}(x)x_m, P_{J_{\mathcal{A}}}(x)x_n\rangle \neq 0\Big\}.$$

Now we assume on the contrary $T^*_{\mathcal{A}(x)}T_{\mathcal{A}'(x)}\big|_{J_{\mathcal{A}}(x)} \neq I_{J_{\mathcal{A}(x)}}$ on a Borel measurable subset Y of X having positive measure. Then, there are $m_0, n_0 \in \mathbb{N}$ such that $S_{m_0,n_0} \cap Y$ is a Borel measurable subset of X having positive measure, and hence either real or imaginary parts of $\rho_{m_0,n_0}(x)$ are strictly positive or negative for a.e. $x \in S_{m_0,n_0} \cap Y$. First, we assume that the real part of $\rho_{m_0,n_0}(x)$ is strictly positive on $S_{m_0,n_0} \cap Y$. By choosing a Borel measurable subset S of $S_{m_0,n_0} \cap Y$ having positive measure, we define functions h_1 and h_2 as follows:

$$h_1(x) = \begin{cases} P_{J_{\mathcal{A}}}(x)x_{m_0} & \text{for } x \in S, \\ 0 & \text{for } x \in X\backslash S, \end{cases} \quad \text{and} \quad h_2(x) = \begin{cases} P_{J_{\mathcal{A}}}(x)x_{n_0} & \text{for } x \in S, \\ 0 & \text{for } x \in X\backslash S. \end{cases}$$

Then, we have $h_1(x), h_2(x) \in J_{\mathcal{A}}(x)$ for a.e. $x \in X$ since $\{P_{J_{\mathcal{A}}}(x)x_n\}_{n\in\mathbb{N}}$ is dense in $J_{\mathcal{A}}(x)$, and hence we get $h_1, h_2 \in S_{\mathcal{D}}(\mathcal{A})$ using Theorem 2.1.4. Therefore, using $f = h_1$, $g = h_2$ in (2.2.6), we obtain $\int_S \rho_{m_0,n_0}(x)\, d\mu_X(x) = 0$, which is a contradiction since the measure of S is positive and the real part of $\rho_{m_0,n_0}(x)$ is strictly positive on S. Other cases follow in a similar way.

Conversely, suppose $T^*_{\mathcal{A}(x)}T_{\mathcal{A}'(x)}\big|_{J_{\mathcal{A}}(x)} = I_{J_{\mathcal{A}(x)}}$ for a.e. $x \in X$. Then we have the result (2.2.5), follows from the computation

$$\begin{aligned} \left\langle T^*_{E_{\mathcal{D}}(\mathcal{A})}T_{E_{\mathcal{D}}(\mathcal{A}')}f, g\right\rangle &= \int_X \left\langle T_{\mathcal{A}'(x)}f(x), T_{\mathcal{A}(x)}g(x)\right\rangle\, d\mu_X(x) \\ &= \int_X \left\langle T^*_{\mathcal{A}(x)}T_{\mathcal{A}'(x)}f(x), g(x)\right\rangle\, d\mu_X(x) \\ &= \int_X \langle f(x), g(x)\rangle\, d\mu_X(x) \end{aligned}$$

for all $f, g \in S_{\mathcal{D}}(\mathcal{A})$ by Lemma 2.2.3.

(ii) From the part (i), $T^*_{\mathcal{A}(x)}T_{\mathcal{A}'(x)}\big|_{J_{\mathcal{A}}(x)} = I_{J_{\mathcal{A}(x)}}$ for a.e. $x \in X$ is equivalent to (2.2.5), i.e., $T^*_{E_{\mathcal{D}}(\mathcal{A})}T_{E_{\mathcal{D}}(\mathcal{A}')}\big|_{S_{\mathcal{D}}(\mathcal{A})} = I_{S_{\mathcal{D}}(\mathcal{A})}$. Similarly, we can prove $T^*_{\mathcal{A}'(x)}T_{\mathcal{A}(x)}\big|_{J_{\mathcal{A}'(x)}} = I_{J_{\mathcal{A}'(x)}}$ for a.e. $x \in X$ is equivalent to

$$T^*_{E_{\mathcal{D}}(\mathcal{A}')}T_{E_{\mathcal{D}}(\mathcal{A})}\big|_{S_{\mathcal{D}}(\mathcal{A}')} = I_{S_{\mathcal{D}}(\mathcal{A}')}.$$

Therefore, we have the result by observing Definition 2.2.1.

(iii) Due to the part (i), we need to show range $T^*_{E_{\mathcal{D}}(\mathcal{A}')} \subseteq$ range $T^*_{E_{\mathcal{D}}(\mathcal{A})}$ if and only if range $T^*_{\mathcal{A}'(x)} \subseteq$ range $T^*_{\mathcal{A}(x)}$. Equivalently,

$$\left[T^*_{E_{\mathcal{D}}(\mathcal{A}')}(L^2(\mathcal{M} \times \mathcal{N}))\right] \cap S_{\mathcal{D}}(\mathcal{A}') \subseteq \left[T^*_{E_{\mathcal{D}}(\mathcal{A})}(L^2(\mathcal{M} \times \mathcal{N}))\right] \cap S_{\mathcal{D}}(\mathcal{A}),$$

if and only if for a.e. $x \in X$, $\left[T^*_{\mathcal{A}'(x)}(L^2(\mathcal{N}))\right] \cap J_{\mathcal{A}'}(x) \subseteq \left[T^*_{\mathcal{A}(x)}(L^2(\mathcal{N}))\right] \cap J_{\mathcal{A}}(x)$. It is enough to verify only on generators φ_t, ψ_t for $t \in \mathcal{N}$. Therefore, the result follows by noting that $M_{g_s}\psi_t \in S_{\mathcal{D}}(\mathcal{A})$ for $t \in \mathcal{N}$ and $s \in \mathcal{M}$ if and only if for a.e. $x \in X$, $\psi_{t'}(x) \in J_{\mathcal{A}}(x)$ for $t' \in \mathcal{N}$ in view of Theorem 2.1.4.

(iv) follows easily. □

Next, we will assure the existence of an oblique (type-II) MG-dual in $L^2(X;\mathcal{H})$ while the canonical dual always exists (see Theorem 2.2.6).

We first ensure the commutativity of the multiplication operator M_ϕ with the orthogonal projection on a closed subspace of $L^2(X;\mathcal{H})$. For the sake of completion we provide a proof of the following lemma. The techniques of the proof follow from [27, Proposition 3.7].

Lemma 2.2.5. *Let $\phi \in L^\infty(X)$ and W be a closed subspace of $L^2(X;\mathcal{H})$. Then, the multiplication operator M_ϕ commutes with P_W if and only if W is invariant under both the operators M_ϕ and M_ϕ^*, where P_W is an orthogonal projection on W and M_ϕ^* is the adjoint of bounded linear operator M_ϕ.*

Proof. For each $\phi \in L^\infty(X)$, suppose $M_\phi P_W = P_W M_\phi$. We need to show that $M_\phi f$ and $M_\phi^* g$ are members of W for each $f, g \in W$. For this first note that we can write $L^2(X;\mathcal{H}) = W \oplus W^\perp$, and hence the bounded linear operator M_ϕ can be represented as $M_\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A : W \to W$, $B : W^\perp \to W$, $C : W^\perp \to W$ and $D : W^\perp \to W^\perp$ are bounded linear operators [27]. The orthogonal projection P_W can also be represented in the form of matrix as follows: $P_W = \begin{pmatrix} I_W & 0 \\ 0 & 0 \end{pmatrix}$, where I_W is an identity map on W. Now by employing the matrix representations of M_ϕ and P_W on $M_\phi P_W = P_W M_\phi$, we get $B = 0, C = 0$, and hence we have $M_\phi = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ and also $M_\phi^* = \begin{pmatrix} A^* & 0 \\ 0 & D^* \end{pmatrix}$, where A^* and D^* are the adjoint operators of A and D, respectively. Therefore for any $f, g \in W$, we get $M_\phi \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} Af \\ 0 \end{pmatrix}$ and $M_\phi^* \begin{pmatrix} g \\ 0 \end{pmatrix} = \begin{pmatrix} A^*g \\ 0 \end{pmatrix}$ which are members of W. Thus, W is invariant under both M_ϕ and M_ϕ^*.

Conversely, assume that the closed subspace W is invariant under both the operators M_ϕ and M_ϕ^*. Then observe that $W^\perp$ is also invariant under both operators M_ϕ and M_ϕ^*, follows by observing $\langle g, M_\phi h\rangle = \langle M_\phi^* g, h\rangle = 0$ and $\langle M_\phi^* h, g\rangle = \langle h, M_\phi g\rangle = 0$ for all $h \in W^\perp$ and $g \in W$. Now for $h \in L^2(X;\mathcal{H})$, we have $P_W h \in W$ and then $M_\phi(P_W h) \in W$ since W is invariant under M_ϕ. Therefore, we have

$$(M_\phi P_W)h = P_W(M_\phi P_W)h = (P_W M_\phi P_W)h.$$

Thus, we get $P_W M_\phi P_W = M_\phi P_W$. Similarly, due to the invariant property of $W^\perp$ under M_ϕ, we have

$$(I - P_W)M_\phi(I - P_W) = M_\phi(I - P_W),$$

where I is the identity operator on $L^2(X;\mathcal{H})$ which gives $P_W M_\phi = P_W M_\phi P_W$ by cancellation property. Hence, we get $P_W M_\phi = M_\phi P_W$ since $P_W M_\phi P_W = M_\phi P_W$. □

The following result is a slight modification of [44, Theorem 1.5].

Theorem 2.2.6. *Let $\mathcal{A} = \{\varphi_t : t \in \mathcal{N}\}$ be a collection of functions in $L^2(X;\mathcal{H})$ and let $\mathcal{D}$ be a Parseval determining set for $L^1(X)$ such that for all $g \in \mathcal{D}$, we have $\overline{g} \in \mathcal{D}$, where $\overline{g}(x) = \overline{g(x)}$ for a.e. $x \in X$, (X, μ_X) and $(\mathcal{N}, \mu_\mathcal{N})$ are σ-finite measure spaces. If the MG system $E_\mathcal{D}(\mathcal{A})$ is a continuous frame for the MI space $S_\mathcal{D}(\mathcal{A})$, then the following are equivalent:*

(i) *$L^2(X;\mathcal{H}) = S_\mathcal{D}(\mathcal{A}) \oplus W^\perp$, for some closed subspace W of $L^2(X;\mathcal{H})$ with $M_\phi W \subset W$ for all $\phi \in \mathcal{D}$, where $\oplus$ denotes the direct sum of closed subspaces whose intersection is zero.*

(ii) *There is a family $\mathcal{A}' = \{\psi_t\}_{t\in\mathcal{N}}$ in W such that $W = S_\mathcal{D}(\mathcal{A}')$, and $E_\mathcal{D}(\mathcal{A}')$ is an MG-dual of type-II for $E_\mathcal{D}(\mathcal{A})$.*

(iii) *There is a family $\mathcal{A}' = \{\psi_t\}_{t\in\mathcal{N}}$ in W such that $W = S_\mathcal{D}(\mathcal{A}')$, and $E_\mathcal{D}(\mathcal{A}')$ is an oblique MG-dual for $E_\mathcal{D}(\mathcal{A})$.*

Proof. (i)⇒(ii). Let us assume (i). To prove (ii), we need to construct a family $\mathcal{A}' = \{\psi_t\}_{t\in\mathcal{N}}$ in W such that $E_\mathcal{D}(\mathcal{A}')$ is a frame for W, and $E_\mathcal{D}(\mathcal{A}')$ is an MG-dual of type-II for $E_\mathcal{D}(\mathcal{A})$. Since $L^2(X;\mathcal{H}) = S_\mathcal{D}(\mathcal{A}) \oplus W^\perp$, the orthogonal projection $\mathfrak{P} := P_W|_{S_\mathcal{D}(\mathcal{A})} : S_\mathcal{D}(\mathcal{A}) \to W$ is an isomorphism [15, 27], where P_W is an orthogonal projection from $L^2(X;\mathcal{H})$ onto W. Hence, the collection $\mathfrak{P}(E_\mathcal{D}(\mathcal{A}))$ is a continuous frame for W. Therefore by using Lemma 2.2.5, we have

$$\mathfrak{P}E_\mathcal{D}(\mathcal{A}) = P_W(E_\mathcal{D}(\mathcal{A})) = E_\mathcal{D}(P_W\mathcal{A}) = E_\mathcal{D}(\mathfrak{P}\mathcal{A}),$$

since W is invariant under M_{g_s} and $M_{g_s}^* = M_{\overline{g_s}}$ for all $g_s \in \mathcal{D}$ by noting that $\mathcal{D}$ consists of both g_s and $\overline{g_s}$. Thus, the family $E_{\mathcal{D}}(\mathfrak{P}\mathcal{A})$ is also a continuous frame for W.

Now we assume $E_{\mathcal{D}}(\tilde{\mathcal{A}}_{\mathfrak{P}})$ is the canonical dual frame for $E_{\mathcal{D}}(\mathfrak{P}\mathcal{A})$ in W, where $\tilde{\mathcal{A}}_{\mathfrak{P}} := \{\tilde{\varphi}_t\}_{t\in\mathcal{N}}$ in $L^2(X;\mathcal{H})$. Then, $E_{\mathcal{D}}(\tilde{\mathcal{A}}_{\mathfrak{P}})$ is an alternate MG-dual for $E_{\mathcal{D}}(\mathcal{A})$ since $T_{E_{\mathcal{D}}(\mathcal{A})}T^*_{E_{\mathcal{D}}(\tilde{\mathcal{A}}_{\mathfrak{P}})}\big|_{S_{\mathcal{D}}(\mathcal{A})} = I_{S_{\mathcal{D}}(\mathcal{A})}$, follows by observing

$$\begin{aligned}\int_{\mathcal{M}}\int_{\mathcal{N}}\langle f, M_{g_s}\tilde{\varphi}_t\rangle M_{g_s}\varphi_t \, d\mu_{\mathcal{M}}(s)\, d\mu_{\mathcal{N}}(t) &= \mathfrak{P}^{-1}\left(\int_{\mathcal{M}}\int_{\mathcal{N}}\langle f, P_W M_{g_s}\tilde{\varphi}_t\rangle \mathfrak{P}M_{g_s}\varphi_t \, d\mu_{\mathcal{M}}(s)\, d\mu_{\mathcal{N}}(t)\right)\\ &= \mathfrak{P}^{-1}\left(\int_{\mathcal{M}}\int_{\mathcal{N}}\langle P_W f, M_{g_s}\tilde{\varphi}_t\rangle \mathfrak{P}M_{g_s}\varphi_t \, d\mu_{\mathcal{M}}(s)\, d\mu_{\mathcal{N}}(t)\right)\\ &= \mathfrak{P}^{-1}\left(\int_{\mathcal{M}}\int_{\mathcal{N}}\langle \mathfrak{P} f, M_{g_s}\tilde{\varphi}_t\rangle \mathfrak{P}M_{g_s}\varphi_t \, d\mu_{\mathcal{M}}(s)\, d\mu_{\mathcal{N}}(t)\right)\\ &= \mathfrak{P}^{-1}\mathfrak{P}f = f \text{ for } f \in S_{\mathcal{D}}(\mathcal{A}).\end{aligned}$$

Next the relation $S_{E_{\mathcal{D}}(\mathfrak{P}\mathcal{A})} = T^*_{E_{\mathcal{D}}(\mathfrak{P}\mathcal{A})}T_{E_{\mathcal{D}}(\mathfrak{P}\mathcal{A})} = \mathfrak{P}T^*_{E_{\mathcal{D}}(\mathcal{A})}T_{E_{\mathcal{D}}(\mathcal{A})}\mathfrak{P}^* = \mathfrak{P}S_{E_{\mathcal{D}}(\mathcal{A})}\mathfrak{P}^*$, since $T^*_{E_{\mathcal{D}}(\mathfrak{P}\mathcal{A})} = \mathfrak{P}T^*_{E_{\mathcal{D}}(\mathcal{A})}$, implies

$$\begin{aligned}T^*_{E_{\mathcal{D}}(\tilde{\mathcal{A}}_{\mathfrak{P}})} &= (S_{E_{\mathcal{D}}(\mathfrak{P}\mathcal{A})}\big|_{S_{\mathcal{D}}(\mathfrak{P}\mathcal{A})})^{-1}T^*_{E_{\mathcal{D}}(\mathfrak{P}\mathcal{A})} = (\mathfrak{P}^*)^{-1}(S_{E_{\mathcal{D}}(\mathcal{A})}\big|_{S_{\mathcal{D}}(\mathcal{A})})^{-1}\mathfrak{P}^{-1}\mathfrak{P}T^*_{E_{\mathcal{D}}(\mathcal{A})}\\ &= (\mathfrak{P}^*)^{-1}(S_{E_{\mathcal{D}}(\mathcal{A})}\big|_{S_{\mathcal{D}}(\mathcal{A})})^{-1}T^*_{E_{\mathcal{D}}(\mathcal{A})},\end{aligned}$$

and hence Ker $T^*_{E_{\mathcal{D}}(\tilde{\mathcal{A}}_{\mathfrak{P}})} =$ Ker $T^*_{E_{\mathcal{D}}(\mathcal{A})}$. Thus, we obtain range $T_{E_{\mathcal{D}}(\tilde{\mathcal{A}}_{\mathfrak{P}})} =$ range $T_{E_{\mathcal{D}}(\mathcal{A})}$. Therefore, (ii) follows.

(ii)$\Rightarrow$(iii) follows easily.

(iii)$\Rightarrow$(i). Suppose (iii) holds. Since the family $E_{\mathcal{D}}(\mathcal{A}')$ is a continuous frame for $S_{\mathcal{D}}(\mathcal{A}') = W$, we have range $T^*_{E_{\mathcal{D}}(\mathcal{A}')} = W$, and hence, we get $(T^*_{E_{\mathcal{D}}(\mathcal{A})}T_{E_{\mathcal{D}}(\mathcal{A}')})^2 = T^*_{E_{\mathcal{D}}(\mathcal{A})}T_{E_{\mathcal{D}}(\mathcal{A}')}$,

$$\text{range}(T^*_{E_{\mathcal{D}}(\mathcal{A})}T_{E_{\mathcal{D}}(\mathcal{A}')}) = S_{\mathcal{D}}(\mathcal{A}) \text{ and } \text{Ker}(T^*_{E_{\mathcal{D}}(\mathcal{A})}T_{E_{\mathcal{D}}(\mathcal{A}')}) = W^{\perp}.$$

Therefore, we have $L^2(X;\mathcal{H}) = S_{\mathcal{D}}(\mathcal{A}) \oplus W^{\perp}$ [27, Proposition 3.2 (c)]. Thus (i) follows. □

We need the following result for the characterization of MG-duals of type-II in order to move on to the next step. It is a measure-theoretic abstraction of [46, Lemma 1].

Lemma 2.2.7. *Let $\mathcal{A} = \{\varphi_t\}_{t\in\mathcal{N}}$ be a family of functions in $L^2(X;\mathcal{H})$ such that the collection $E_{\mathcal{D}}(\mathcal{A})$ is Bessel over $\mathcal{M}\times\mathcal{N}$ in $L^2(X;\mathcal{H})$. For each $t \in \mathcal{N}$ and $\varphi_t \in \mathcal{A}$, let p_{φ_t} be a complex valued measurable function over $X \times \mathcal{N}$ such that $\|\{p_{\varphi_t}(x)\}_{(x,t)\in X\times\mathcal{N}}\|_{L^2(X;L^2(\mathcal{N}))} < \infty$. Then following statements are equivalent for all $f \in S_{\mathcal{D}}(\mathcal{A})$:*

(i) $\displaystyle\int_{t\in\mathcal{N}}\int_{x\in X}\langle f(x),\varphi_t(x)\rangle\, p_{\varphi_t}(x)\; d\mu_X(x)\; d\mu_{\mathcal{N}}(t)=0.$

(ii) *For a.e.* $x\in X$, $\displaystyle\int_{t\in\mathcal{N}}\langle f(x),\varphi_t(x)\rangle\, p_{\varphi_t}(x)\; d\mu_{\mathcal{N}}(t)=0.$

In particular, it holds for all $f\in L^2(X;\mathcal{H})$.

Proof. For (ii) $\Longrightarrow$ (i), first note that $E_{\mathcal{D}}(\mathcal{A})$ is Bessel with bound B if and only if for a.e. $x\in X$, $\mathcal{A}(x)$ is so with same bound. Then for $f\in S_{\mathcal{D}}(\mathcal{A})$, we have $f(x)\in J_{\mathcal{A}}(x)$ for a.e. $x\in X$ from Theorem 2.1.4, and hence by Cauchy-Schwarz inequality, we get

(2.2.7)

$$\begin{aligned}&\int_{x\in X}\int_{t\in\mathcal{N}}|\langle f(x),\varphi_t(x)\rangle p_{\varphi_t}(x)|\; d\mu_{\mathcal{N}}(t)\; d\mu_X(x)\\
&\leqslant\left(\int_{x\in X}\int_{t\in\mathcal{N}}|\langle f(x),\varphi_t(x)\rangle|^2\; d\mu_{\mathcal{N}}(t)\; d\mu_X(x)\right)^{1/2}\left(\int_{x\in X}\int_{t\in\mathcal{N}}|p_{\varphi_t}(x)|^2\; d\mu_{\mathcal{N}}(t)\; d\mu_X(x)\right)^{1/2}\\
&\leqslant\left(B\int_{x\in X}\|f(x)\|^2\; d\mu_X(x)\right)^{1/2}\left(\int_{x\in X}\int_{t\in\mathcal{N}}|p_{\varphi_t}(x)|^2\; d\mu_{\mathcal{N}}(t)\; d\mu_X(x)\right)^{1/2}<\infty,\end{aligned}$$

since $f\in L^2(X;\mathcal{H})$ and $\|\{p_{\varphi_t}(x)\}_{(x,t)\in X\times\mathcal{N}}\|_{L^2(X;L^2(\mathcal{N}))}<\infty$. Using Fubini's theorem, we obtain

$$\int_{x\in X}\int_{t\in\mathcal{N}}\langle f(x),\varphi_t(x)\rangle\, p_{\varphi_t}(x)\; d\mu_{\mathcal{N}}(t)\; d\mu_X(x)=\int_{t\in\mathcal{N}}\int_{x\in X}\langle f(x),\varphi_t(x)\rangle p_{\varphi_t}(x)\; d\mu_X(x)\; d\mu_{\mathcal{N}}(t).$$

Thus, (i) holds true.

For (i) $\Longrightarrow$ (ii), let $\{x_n\}_{n\in\mathbb{N}}$ be a countable dense subset of $\mathcal{H}$. Note that for a.e. $x\in X$, $\{P_{J_{\mathcal{A}}}(x)x_n\}_{n\in\mathbb{N}}$ is dense in $J_{\mathcal{A}}(x)$. For each $n\in\mathbb{N}$, we define

$$S_n=\left\{x\in X:\Delta_n(x):=\int_{\mathcal{N}}\langle P_{J_{\mathcal{A}}}(x)x_n,\varphi_t(x)\rangle p_{\varphi_t}(x)\; d\mu_N(t)\neq 0\right\}.$$

If (ii) is false, there exists $n_0\in\mathbb{N}$ and a Borel measurable set Y in X having positive measure such that the measure of $S_{n_0}\cap Y$ is positive. There are four possible inequalities for $\Delta_{n_0}(x)$, viz., real part of $\Delta_{n_0}(x)>0$, real part of $\Delta_{n_0}(x)<0$, imaginary part of $\Delta_{n_0}(x)>0$, and imaginary part of $\Delta_{n_0}(x)(x)<0$. For the case of real part of $\Delta_{n_0}(x)>0$, we choose a Borel measurable set S in $S_{n_0}\cap Y$ having positive measure and then define $f\in S_{\mathcal{D}}(\mathcal{A})$ by

$$f(x)=\begin{cases}P_{J_{\mathcal{A}}}(x)x_{n_0} & \text{for } x\in S,\\ 0 & \text{for } x\in X\backslash S.\end{cases}$$

Hence, we arrive on a contradiction $\int_S \Delta_{n_0}(x)\, d\mu_X(x) = 0$, since the measure of S is positive and real part of $\Delta_{n_0}(x) > 0$. Other cases follow easily. $\square$

Due to the applications of the inverse of $\mathcal{F}$ for further nurture, we require surjectivity of $\mathcal{F}$ defined in (2.2.3). In general, it is not surjective. For example, in the case of countable sets X and $\mathcal{M}$ equipped with counting measures, $\mathcal{F}$ need not be surjective as it becomes an analysis operator for the Parseval frame $\mathcal{D}$. In general, a Parseval frame for $L^2(X)$ consisting of functions in $L^\infty(X)$ need not be a Parseval determining set for $L^1(X)$. Indeed, for any $f \in L^1([0,1])\backslash L^2([0,1])$ there is an orthonormal basis $\mathcal{D} = \{g_s\}_{s\in\mathbb{N}} \subseteq C([0,1])$ for $L^2([0,1])$ such that $\mathcal{D}$ is not a Parseval determining set [49]. Also, observe that $\mathcal{D} = \widehat{\mathcal{G}}$ is a Parseval determining set for $L^1(\mathcal{G})$ in view of (2.1.2) but it need not be an orthonormal basis for $L^2(\mathcal{G})$. The set $\mathcal{D} = \widehat{\mathcal{G}}$ becomes an orthonormal basis when $\mathcal{G}$ is compact [48]. As per our requirement and these discussions, let us define an orthonormal Parseval determining set and $\mathcal{F}^{-1}$ as follows:

Definition 2.2.8. A set $\mathscr{D} = \{g_m\}_{m\in\mathcal{M}} \subseteq L^\infty(X) \cap L^2(X)$ is said to be an *orthonormal Parseval determining set* for $L^2(X)$ if it is a Parseval determining set for $L^1(X)$ as well as orthonormal basis for $L^2(X)$, where $\mathcal{M}$ is a countable set having counting measure.

Lemma 2.2.9. *For an orthonormal Parseval determining set $\mathscr{D} = \{g_m\}_{m\in\mathcal{M}}$, the map $\mathcal{F} : L^2(X) \to \ell^2(\mathcal{M})$ associated with $\mathscr{D}$ defined by (2.2.3) is surjective (and hence, isomorphism), where $\mu_X < \infty$ and $\mathcal{M}$ is a countable set having counting measure. Moreover, the map $\mathcal{F}^{-1} : \ell^2(\mathcal{M}) \to L^2(X)$ defined by*

$$(\mathcal{F}^{-1}(c))(x) := \sum_{m\in\mathcal{M}} c(m) g_m(x), \;\; c = \{c(m)\}_{m\in\mathcal{M}} \in \ell^2(\mathcal{M}), \;\; x \in X$$

satisfies the following:

$$\|\mathcal{F}^{-1}c\| = \|c\| \text{ and } \langle \mathcal{F}^{-1}c, \mathcal{F}^{-1}d\rangle = \langle c, d\rangle \;\text{ for all } c, d \in \ell^2(\mathcal{M}),$$

where the above series is interpreted as its limit in $L^2(X)$.

Proof. The range of $\mathcal{F}$ contains all the functions $c = \{c(m)\}_{m\in\mathcal{M}}$ in $\ell^2(\mathcal{M})$ such that $c(m) = 0$, except for finitely many terms, i.e., the range of $\mathcal{F}$ is dense in $\ell^2(\mathcal{M})$. Since $\mathcal{F}$ is an isometry it is a closed range. Hence, the isomorphism follows. Note that the operator $\mathcal{F}$ is isometry and $\mathcal{F}^{-1}$ is the adjoint of $\mathcal{F}$ using the formula $\langle \mathcal{F}f, c\rangle = \langle f, \mathcal{F}^* c\rangle$

and from the following estimate

$$\langle \mathcal{F}f, c\rangle = \sum_{m\in\mathcal{M}} \mathcal{F}f(m)\overline{c(m)} = \int_X f(x)\overline{\left(\sum_{m\in\mathcal{M}} g_m(x)c(m)\right)}\, d\mu_X(x),$$

the remaining part follows. □

We state an abstract version of a result developed in [46, Theorem 5]. The following result characterizes MG-duals of type-II associated with the range function $J_{\mathcal{A}}(x)$ for a.e. $x \in X$.

Theorem 2.2.10. *In addition to the hypotheses of Theorem 2.2.4, assume that the Parseval determining set $\mathcal{D}$ becomes an orthonormal Parseval determining set $\mathscr{D} = \{g_m\}_{m\in\mathcal{M}}$ for $L^2(X)$, where $\mu_X < \infty$ and $\mathcal{M}$ is the countable set equipped with counting measure. Then the following are equivalent:*

(i) *$E_{\mathscr{D}}(\mathcal{A}')$ is an MG-dual of type-II for $E_{\mathscr{D}}(\mathcal{A})$ in $L^2(X;\mathcal{H})$.*

(ii) *For a.e. $x \in X$, $\mathcal{A}'(x)$ is a dual of type-II for the frame $\mathcal{A}(x)$ of $J_{\mathcal{A}}(x)$ in $\mathcal{H}$.*

Proof. For (i) $\Longrightarrow$ (ii), let $T^*_{E_{\mathscr{D}}(\mathcal{A})}T_{E_{\mathscr{D}}(\mathcal{A}')}\big|_{S_{\mathscr{D}}(\mathcal{A})} = I_{S_{\mathscr{D}}(\mathcal{A})}$ and range $T_{E_{\mathscr{D}}(\mathcal{A}')} \subseteq$ range $T_{E_{\mathscr{D}}(\mathcal{A})}$. Then, the analysis operator $T_{E_{\mathscr{D}}(\mathcal{A}')}$ is restricted on $S_{\mathscr{D}}(\mathcal{A})$ and range $T_{E_{\mathscr{D}}(\mathcal{A}')} = T_{E_{\mathscr{D}}(\mathcal{A}')}(S_{\mathscr{D}}(\mathcal{A}))$. Therefore due to Theorem 2.2.4 (i), it is enough to show

$$\text{range } T_{\mathcal{A}'(x)} \subseteq \text{range } T_{\mathcal{A}(x)}, \text{ i.e., } T_{\mathcal{A}(x)}(\mathcal{H})^{\perp} \subseteq T_{\mathcal{A}'(x)}(J_{\mathcal{A}}(x))^{\perp} \text{ in } L^2(\mathcal{N})$$

since $T^*_{\mathcal{A}(x)}T_{\mathcal{A}'(x)}\big|_{J_{\mathcal{A}}(x)} = I_{J_{\mathcal{A}}(x)}$ for a.e. $x \in X$. For this, let $\mathfrak{p} := \{\mathfrak{p}_t\}_{t\in\mathcal{N}} \in \left[T_{\mathcal{A}(x)}(\mathcal{H})\right]^{\perp}$ in $L^2(\mathcal{N})$. Then for $f \in L^2(X;\mathcal{H})$, we have $f(x) \in \mathcal{H}$ a.e. $x \in X$ and hence

$$\langle T_{\mathcal{A}(x)}f(x), \mathfrak{p}\rangle = \int_{\mathcal{N}} \langle f(x), \varphi_t(x)\rangle \overline{\mathfrak{p}_t}\, d\mu_{\mathcal{N}}(t) = 0 \text{ a.e. } x \in X.$$

For a fixed $m \in \mathcal{M}$, we choose $p_{\varphi_t}(x) = g_m(x)\mathfrak{p}_t$ in Lemma 2.2.7, where $g_m \in \mathscr{D} \subset L^2(X)$, and then by Fubini's theorem (see (2.2.7)) we get the following

$$\int_X \int_{\mathcal{N}} \langle f(x), \varphi_t(x)\rangle \overline{g_m(x)\mathfrak{p}_t}\, d\mu_{\mathcal{N}}(t)\, d\mu_X(x) = 0, \text{ i.e., } \int_{\mathcal{N}} \langle f, M_{g_m}\varphi_t\rangle \overline{\mathfrak{p}_t}\, d\mu_{\mathcal{N}}(t) = 0,$$

since $\{p_{\varphi_t}\} \in L^2(X; L^2(\mathcal{N}))$ and $E_{\mathscr{D}}(\mathcal{A})$ is a Bessel family in $L^2(X;\mathcal{H})$. Therefore for any $c = \{c_m\}_{m\in\mathcal{M}} \in \ell^2(\mathcal{M})$, we can write

$$\langle T_{E_{\mathcal{D}}(\mathcal{A})}f, \{c_m\mathfrak{p}_t\}_{m\in\mathcal{M}, t\in\mathcal{N}}\rangle = \sum_{m\in\mathcal{M}} \int_{\mathcal{N}} \langle f, M_{g_m}\varphi_t\rangle \overline{c_m\mathfrak{p}_t}\, d\mu_{\mathcal{N}}(t) = 0.$$

Thus, the system $\{c_m\mathfrak{p}_t\}_{m\in\mathcal{M},t\in\mathcal{N}} \in [T_{E_{\mathscr{D}}(\mathcal{A})}(L^2(X;\mathcal{H}))]^\perp \cap L^2(\mathcal{M}\times\mathcal{N})$. Now by using the assumption $[T_{E_{\mathscr{D}}(\mathcal{A})}(L^2(X;\mathcal{H}))]^\perp \subseteq [T_{E_{\mathscr{D}}(\mathcal{A}')}(S_{\mathscr{D}}(\mathcal{A}))]^\perp$ in $L^2(\mathcal{M}\times\mathcal{N})$, we get $\{c_m\mathfrak{p}_t\}_{m\in\mathcal{M},t\in\mathcal{N}} \in [T_{E_{\mathscr{D}}(\mathcal{A})}(S_{\mathscr{D}}(\mathcal{A}))]^\perp$ and then for $g \in S_{\mathscr{D}}(\mathcal{A})$, we have

$$0 = \sum_{m\in\mathcal{M}} \int_{\mathcal{N}} \langle g, M_{g_m}\psi_t\rangle \overline{c_m\mathfrak{p}_t}\, d\mu_{\mathcal{N}}(t) = \int_X \int_{\mathcal{N}} \langle g(x), \psi_t(x)\rangle \left(\sum_{m\in\mathcal{M}} \overline{g_m(x)c_m\mathfrak{p}_t}\right) d\mu_{\mathcal{N}}(t)\, d\mu_X(x),$$

due to Fubini's theorem. We can write the following using Lemma 2.2.9

$$\int_{\mathcal{N}} \int_X \langle g(x), \psi_t(x)\rangle\, \overline{\mathfrak{p}_t\left(\mathcal{F}^{-1}(c)\right)(x)}\, d\mu_{\mathcal{N}}(t)\, d\mu_X(x) = 0.$$

Therefore by observing $\{\mathfrak{p}_t\}_{t\in\mathcal{N}} \in L^2(\mathcal{N})$, $\{c_m\}_{m\in\mathcal{M}} \in \ell^2(\mathcal{M})$, and

$$\|\{\mathfrak{p}_t\left(\mathcal{F}^{-1}(c)\right)(x)\}_{t\in\mathcal{N},x\in X}\|_{L^2(\mathcal{N}\times X)} = \|c\|_{\ell^2(\mathcal{M})}\|\{\mathfrak{p}_t\}_{t\in\mathcal{N}}\|_{L^2(\mathcal{N})},$$

we have

$$\int_{\mathcal{N}} \langle g(x), \psi_t(x)\rangle \overline{\mathfrak{p}_t\left(\mathcal{F}^{-1}(c)\right)(x)}\, d\mu_{\mathcal{N}}(t) = 0 \text{ for a.e. } x \in X,$$

using Lemma 2.2.7. Thus for a.e. $x \in X$, we get $\{\mathfrak{p}_t\}_{t\in\mathcal{N}} \in \left[T_{\mathcal{A}'(x)}(J_{\mathcal{A}}(x))\right]^\perp$ since $\{c_m\}_{m\in\mathcal{M}} \in \ell^2(\mathcal{M})$ is an arbitrary element and $g(x) \in J_{\mathcal{A}}(x)$.

For (ii) $\implies$ (i), let $T^*_{\mathcal{A}(x)}T_{\mathcal{A}'(x)}\big|_{J_{\mathcal{A}}(x)} = I_{J_{\mathcal{A}}(x)}$ and range $T_{\mathcal{A}'(x)} \subseteq$ range $T_{\mathcal{A}(x)}$ for a.e. $x \in X$. Then for a.e. $x \in X$, the analysis operator $T_{\mathcal{A}'(x)}$ is restricted on $J_{\mathcal{A}}(x)$ and range $T_{\mathcal{A}'(x)} = T_{\mathcal{A}'(x)}(J_{\mathcal{A}}(x))$. It suffices to show

range $T_{E_{\mathscr{D}}(\mathcal{A}')} \subseteq$ range $T_{E_{\mathscr{D}}(\mathcal{A})}$, i.e., $T_{E_{\mathscr{D}}(\mathcal{A})}(L^2(X;\mathcal{H}))^\perp \subseteq T_{E_{\mathscr{D}}(\mathcal{A}')}(S_{\mathscr{D}}(\mathcal{A}))^\perp$ in $L^2(\mathcal{M}\times\mathcal{N})$,

since $T^*_{E_{\mathscr{D}}(\mathcal{A})}T_{E_{\mathscr{D}}(\mathcal{A}')}\big|_{S_{\mathscr{D}}(\mathcal{A})} = I_{S_{\mathscr{D}}(\mathcal{A})}$. For this, let $\mathfrak{F} := (f_{\varphi_t}(m))_{(m,t)\in\mathcal{M}\times\mathcal{N}} \in [T_{E_{\mathscr{D}}(\mathcal{A})}(L^2(X;\mathcal{H}))]^\perp$ in $L^2(\mathcal{M}\times\mathcal{N})$. Then, we get the following for all $f \in L^2(X;\mathcal{H})$

$$\begin{aligned} 0 &= \sum_{m\in\mathcal{M}} \int_{\mathcal{N}} \langle f, M_{g_m}\varphi_t\rangle \overline{f_{\varphi_t}(m)}\, d\mu_{\mathcal{N}}(t) = \sum_{m\in\mathcal{M}} \int_{\mathcal{N}} \int_X \langle f(x), \varphi_t(x)\rangle \overline{g_m(x) f_{\varphi_t}(m)}\, d\mu_X(x)\, d\mu_{\mathcal{N}}(t) \\ &= \int_{\mathcal{N}} \int_X \langle f(x), \varphi_t(x)\rangle \overline{\left(\sum_{m\in\mathcal{M}} f_{\varphi_t}(m) g_m(x)\right)}\, d\mu_X(x)\, d\mu_{\mathcal{N}}(t) \\ &= \int_{\mathcal{N}} \int_X \langle f(x), \varphi_t(x)\rangle \overline{(\mathcal{F}^{-1}(f_{\varphi_t})(x))}\, d\mu_X(x)\, d\mu_{\mathcal{N}}(t), \end{aligned}$$

using Fubini's theorem and Lemma 2.2.9, and hence by Lemma 2.2.7, we obtain

$$\int_{\mathcal{N}} \langle f(x), \varphi_t(x)\rangle \overline{(\mathcal{F}^{-1}(f_{\varphi_t})(x))}\, d\mu_{\mathcal{N}}(t) = 0 \text{ a.e. } x \in X,$$

since $\|\{\mathcal{F}^{-1}(f_{\varphi_t})(x)\}_{(x,t)}\|_{L^2(X;L^2(\mathcal{N}))} = \|\{f_{\varphi_t}\}_{t\in\mathcal{N}}\|_{L^2(\mathcal{N})} < \infty$. Thus, for a.e. $x \in X$ we get

$$\{\mathcal{F}^{-1}(f_{\varphi_t})(x)\}_{t\in\mathcal{N}} \in \left[T_{\mathcal{A}(x)}(\mathcal{H})\right]^{\perp},$$

which implies $\{\mathcal{F}^{-1}(f_{\varphi_t})(x)\}_{t\in\mathcal{N}} \in \left[T_{\mathcal{A}'(x)}(J_{\mathcal{A}}(x))\right]^{\perp}$ in $L^2(\mathcal{N})$ due to our assumption. That means for a.e. $x \in X$ and $g(x) \in J_{\mathcal{A}}(x)$, we have

$$\int_{\mathcal{N}} \langle g(x), \psi_t(x)\rangle \overline{(\mathcal{F}^{-1}(f_{\varphi_t})(x))}\, d\mu_{\mathcal{N}}(t) = 0,$$

and hence using Lemma 2.2.7, we obtain $\mathfrak{F} = (f_{\varphi_t}(m))_{(m,t)\in\mathcal{M}\times\mathcal{N}} \in [T_{E_{\mathscr{D}}(\mathcal{A}')}(S_{\mathscr{D}}(\mathcal{A}))]^{\perp}$, i.e.,

$$\sum_{m\in\mathcal{M}} \int_{\mathcal{N}} \langle g, M_{g_m}\psi_t\rangle \overline{f_{\varphi_t}(m)}\, d\mu_{\mathcal{N}}(t) = 0 \text{ for all } g \in S_{\mathscr{D}}(\mathcal{A}),$$

in view of Theorem 2.1.4 since $E_{\mathscr{D}}(\mathcal{A})$ is a Bessel system in $L^2(X;\mathcal{H})$. □

Remark 2.2.11. As a consequence of the above results for $\mathcal{A} = \{\varphi\}$ and $\mathcal{A}' = \{\psi\}$, the MG system $E_{\mathcal{D}}(\mathcal{A}')$ is an alternate MG-dual for $E_{\mathcal{D}}(\mathcal{A}')$ if and only if for a.e. $x \in X$ such that $J_{\mathcal{A}}(x) \neq 0$, we have $\langle\varphi(x), \psi(x)\rangle = 1$. In addition if $\mathcal{A} = \mathcal{A}'$, $E_{\mathcal{D}}(\mathcal{A})$ is an alternate MG-dual of itself if and only if $\|\varphi(x)\| = 1$ for a.e. $x \in X$ such that $J_{\mathcal{A}}(x) \neq 0$. This generalizes the results [26, Theorem 4.1] and [44, Corollary 4.6] for the set theoretic abstraction.

Example 2.2.12. Let $\mathcal{A} = \{\varphi\}$ and $\mathcal{A}' = \{\psi\}$ be two collections of function in $L^2([0,1];\ell^2(\mathbb{Z}))$ such that

$$\varphi = \{\chi_{[0,1/2]}(. - k)\}_{k\in\mathbb{Z}} \text{ and } \psi = \{\chi_{[0,1/2]\cup[1,3/2]}(. - k)\}_{k\in\mathbb{Z}}.$$

Choose $\mathcal{D} = \{e^{2\pi ik\cdot}\}_{k\in\mathbb{Z}}$, which is a Parseval determining set for $L^1([0,1])$. This follows by noting

$$\sum_{k\in\mathbb{Z}} \left|\int_0^1 f(x)e^{-2\pi ikx}\, dx\right|^2 = \sum_{k\in\mathbb{Z}} \left|\widehat{f}(k)\right|^2 = \|\widehat{f}\|^2 = \int_0^1 |f(x)|^2\, dx \text{ for } f \in L^2([0,1]).$$

Also $\|f\|_2 = \infty$ and $\|\widehat{f}\|_2 = \infty$ for $f \in L^1([0,1])\backslash L^2([0,1])$.

Here the MG systems are $E_{\mathcal{D}}(\mathcal{A}) = \{e^{2\pi ik\cdot}\varphi : k \in \mathbb{Z}\}$ and $E_{\mathcal{D}}(\mathcal{A}') = \{e^{2\pi ik\cdot}\psi : k \in \mathbb{Z}\}$. For a.e. $x \in [0,1/2]$ the range function $J_{\mathcal{A}}(x) = \operatorname{span}\{\varphi(x)\} = \operatorname{span}\{(\ldots, 0, 0, 1, 0, 0, \ldots)\} \neq 0$, and the value of

$$\langle\varphi(x), \psi(x)\rangle = \sum_{k\in\mathbb{Z}} \chi_{[0,1/2]}(x-k)\chi_{[0,1/2]\cup[1,3/2]}(x-k) = 1 \text{ on } [0,1/2].$$

Applying Remark 2.2.11 we say $E_{\mathcal{D}}(\mathcal{A}')$ is an alternate MG-dual for $E_{\mathcal{D}}(\mathcal{A})$.

Indeed, this singly generated MG system $E_{\mathcal{D}}(\mathcal{A})$ is a Parseval frame (known as, *Coherent State*) for $S_{\mathcal{D}}(A)$ whose associated canonical dual is unique alternate MG-dual. Adopting the idea of this characterization associated with $J_{\mathcal{A}}(x)$, next we concentrate on the canonical dual when it becomes unique MG-dual for a discrete frame. Such duals are associated with Riesz sequences which are always discrete [20].

2.3. Uniqueness of multiplication generated duals

For a countable sequence $\{x_i\}$ in $\mathcal{H}$, if there are $0 < C_1 \leqslant C_2 < \infty$ such that for all $\{c_i\} \in \ell^2$ with $c_i \neq 0$ for only finitely many i,

$$C_1 \left\|\{c_i\}\right\|_{\ell^2}^2 \leqslant \left\|\sum_i c_i x_i\right\|_{\mathcal{H}}^2 \leqslant C_2 \left\|\{c_i\}\right\|_{\ell^2}^2,$$

$\{x_i\}$ is known as *Riesz sequence* in $\mathcal{H}$ with bounds C_1 and C_2. It is known as *Riesz basis* if $\overline{\text{span}}\{x_i\} = \mathcal{H}$. Note that a sequence $\{x_i\}$ in $\mathcal{H}$ is a Riesz basis for $\mathcal{H}$ if and only if it is a frame for $\mathcal{H}$ and it has a unique dual frame. In this section, our goal is to find characterizations when alternate (oblique) MG-dual/ MG-dual of type-I and type-II/ MG-dual frame for $E_{\mathcal{D}}(\mathcal{A})$ admits unique MG-dual.

Throughout this section, we assume $\mathcal{M}$ and $\mathcal{N}$ are countable sets having counting measures, and the measure of X is finite. For two countable families $\mathcal{A} = \{\varphi_n\}_{n\in\mathcal{N}}$ and $\mathcal{A}' = \{\psi_n\}_{n\in\mathcal{N}}$ in $L^2(X;\mathcal{H})$ and an orthonormal Parseval determining set $\mathscr{D} = \{g_m\}_{m\in\mathcal{M}}$ for $L^2(X)$, we observe that $E_{\mathscr{D}}(\mathcal{A}')$ is an alternate MG-dual for $E_{\mathscr{D}}(\mathcal{A})$ other than the canonical dual $E_{\mathscr{D}}(\tilde{\mathcal{A}})$ if and only if the following fact holds for all $f, g \in S_{\mathscr{D}}(\mathcal{A})$:

$$\sum_{m\in\mathcal{M}} \sum_{n\in\mathcal{N}} \langle f, M_{g_m}\eta_n\rangle \langle M_{g_m}\varphi_n, g\rangle = 0, \tag{2.3.1}$$

where $\eta_n = \psi_n - (S_{E_{\mathscr{D}}(\mathcal{A})}\big|_{S_{\mathscr{D}}(\mathcal{A})})^{-1}\varphi_n$, if and only if for a.e. $x \in X$, $\mathcal{A}'(x)$ is an alternate dual for $\mathcal{A}(x)$ other than the canonical dual $\tilde{\mathcal{A}}(x)$ if and only if for a.e. $x \in X$, $\sum_{n\in\mathcal{N}}\langle f(x), \eta_n(x)\rangle\langle\varphi_n(x), g(x)\rangle = 0$, where $\eta_n(x) = \psi_n(x) - (S_{\mathcal{A}(x)}\big|_{J_{\mathcal{A}}(x)})^{-1}\varphi_n(x)$ for a.e. $x \in X$ from (2.2.2). Observe that $E_{\mathscr{D}}(\mathcal{A}'')$ is Bessel in $L^2(X;\mathcal{H})$, where $\mathcal{A}'' := \{\eta_n : n \in \mathcal{N}\}$, and hence the system $\mathcal{A}''(x)$ is Bessel in $\mathcal{H}$ for a.e. $x \in X$.

Next we mention characterization results for the uniqueness of alternate (oblique) MG-dual, MG-dual of type-I and type-II, and MG-dual frame for $E_{\mathcal{D}}(\mathcal{A})$ which are an abstract version of some results of [36,37,42,44,46]. These results are also connected with the Gramian and the dual Gramian operators associated with $\mathcal{A}(x) = \{\varphi_n(x) : n \in \mathcal{N}\}$. Given a Bessel sequence $\mathcal{A}(x)$ in $J_{\mathcal{A}}(x) = \overline{\text{span}}\{\varphi_n(x) : n \in \mathcal{N}\}$ for a.e. $x \in X$, the operator $G(x) = \Theta(x)\Theta^*(x)$ on $\ell^2(\mathcal{N})$ is the *Gramian* corresponding to $\mathcal{A}(x)$, while its dual operator $\Theta(x)^*\Theta(x) =: \tilde{G}(x)$ on $J_{\mathcal{A}}(x)$ is the *dual Gramian operator*, where $\Theta(x) : J_{\mathcal{A}}(x) \to \ell^2(\mathcal{N})$ and its adjoint $\Theta^*(x) : \ell^2(\mathcal{N}) \to J_{\mathcal{A}}(x)$ are given as follows:

$$\Theta(x)h = \{\langle h, \varphi_n(x)\rangle\}_{n\in\mathcal{N}}, \text{ and } \Theta^*(x)(c) = \sum_{n\in\mathcal{N}} c_n\varphi_n(x)$$

for $h \in J_{\mathcal{A}}(x)$ and $c = \{c_n\}_{n\in\mathcal{N}}$ having finitely many non-zero terms. Further, we can obtain easily the following associated matrix for a.e. $x \in X$:

(2.3.2)

$$G(x) = [\langle\varphi_i(x), \varphi_j(x)\rangle]_{i,j\in\mathcal{N}} \text{ and } \tilde{G}(x) = \left[\sum_{i\in\mathcal{N}}\langle e_k(x), \varphi_i(x)\rangle\overline{\langle e_l(x), \varphi_i(x)\rangle}\right]_{k,l\in\mathcal{J}},$$

where $\{e_k(x)\}_{k\in\mathcal{J}}$ ($\mathcal{J}$-countable index set) is the standard orthonormal basis for $\mathcal{H}$. The entries of matrix $\tilde{G}(x)$ are well defined if $\{\langle e_k(x), \varphi_i(x)\rangle\}_{i\in\mathcal{N}} \in \ell^2(\mathcal{N})$ for $k \in \mathcal{J}$ and a.e. $x \in X$. For a.e. $x \in X$, $J_{\mathcal{A}}(x) = 0$ is equivalent to $G(x) = 0$ as well as $\tilde{G}(x) = 0$.

The following result is a measure-theoretic abstraction of [36, Theorem 2.3] and [46, Theorem 6].

Theorem 2.3.1. *Let (X, μ_X) be a finite measure space, and $\mathcal{M}, \mathcal{N}$ are two countable sets having counting measures. For an orthonormal Parseval determining set $\mathscr{D} = \{g_m\}_{m\in\mathcal{M}}$ for $L^2(X)$, let $\mathcal{A} = \{\varphi_n : n \in \mathcal{N}\}$ be a countable family in $L^2(X;\mathcal{H})$ such that $E_{\mathscr{D}}(\mathcal{A})$ is a frame for $S_{\mathscr{D}}(\mathcal{A})$ over $\mathcal{M} \times \mathcal{N}$ with bounds A and B, and for a.e. $x \in X$, $J_{\mathcal{A}}(x)$ defined by $J_{\mathcal{A}}(x) = \overline{\text{span}}\{\varphi_n(x) : n \in \mathcal{N}\}$ is non-zero. Then, the following are equivalent:*

(i) *An MG-dual of type-I for $E_{\mathscr{D}}(\mathcal{A})$ is the only canonical dual frame in $S_{\mathscr{D}}(\mathcal{A})$ with bounds $1/B$ and $1/A$.*
(ii) *$E_{\mathscr{D}}(\mathcal{A})$ is a Riesz basis for $S_{\mathscr{D}}(\mathcal{A})$ with some bounds C_1 and C_2.*
(iii) *For a.e. $x \in X$, the dual of type-I for $\mathcal{A}(x)$ is the only canonical dual frame in $J_{\mathcal{A}}(x)$ with bounds $1/B$ and $1/A$.*
(iv) *For a.e. $x \in X$, $\mathcal{A}(x)$ is a Riesz basis for $J_{\mathcal{A}}(x)$ with bounds C_1 and C_2.*
(v) *For a.e. $x \in X$, the synthesis operator $\Theta^*(x)$ (Gramian operator $G(x)$) associated with $\mathcal{A}(x)$ is injective.*

(vi) *For a.e.* $x \in X$, *the Gramian operator* $G(x)$ *associated with* $\mathcal{A}(x)$ *satisfies*

$$C_1 I_{\ell^2(\mathcal{N})} \leqslant G(x) \leqslant C_2 I_{\ell^2(\mathcal{N})}.$$

(vii) *For* $f = \{f_{\varphi_n}\}_{n\in\mathcal{N}}$ *in* $\ell^2(\mathcal{N}; L^2(X))$, *we have*

$$C_1 \|f\|^2_{\ell^2(\mathcal{N};L^2(X))} \leqslant \left\| \left\{ \sum_{n\in\mathcal{N}} f_{\varphi_n}(x)\varphi_n(x) \right\}_{x\in X} \right\|^2_{L^2(X;\mathcal{H})} \leqslant C_2 \|f\|^2_{\ell^2(\mathcal{N};L^2(X))}.$$

(viii) *For any Riesz sequence* $D \subset L^\infty(X)$ *in* $L^2(X)$ *with bounds* c_1 *and* c_2, $E_D(\mathcal{A})$ *is a Riesz basis for* $S_D(\mathcal{A})$ *with bounds* c_1C_1 *and* c_2C_2.

Moreover, $J_{\mathcal{A}}(x)$ *need not be non-zero, then (i) is equivalent to either* $J_{\mathcal{A}}(x) = 0$ *or (iv) for a.e.* $x \in X$.

Proof. The equivalence of (i) and (ii) follows by $S_{\mathscr{D}}(\mathcal{A}) = \text{range } T^*_{E_{\mathscr{D}}(\mathcal{A})} = \text{range } T^*_{E_{\mathscr{D}}(\tilde{\mathcal{A}})} = S_{\mathscr{D}}(\tilde{\mathcal{A}})$ from Theorems 3.6.2 and 6.3.1, and Theorem 1.2 of [44]. Similarly, (iii) and (iv) are equivalent since $J_{\mathcal{A}}(x) = \text{range } T^*_{\mathcal{A}(x)} = \text{range } T^*_{\tilde{\mathcal{A}}(x)} = J_{\tilde{\mathcal{A}}}(x)$ for a.e. $x \in X$. Employing Theorem 2.3 of [49], (ii), (iv), (vii) and (viii) are equivalent.

Further the equivalence between (iv) and (vi) follows by the definition of Riesz basis and from the property of Gramian operator $G(x) = \Theta(x)\Theta^*(x)$,

$$\langle G(x)c, c\rangle = \langle \Theta^*(x)c, \Theta^*(x)c\rangle = \left\| \sum_{n\in\mathcal{N}} c_n\varphi_n(x) \right\|^2_{\ell^2(\mathcal{N})}$$

for a.e. $x \in X$ and for all $c = \{c_n\}_{n\in\mathcal{N}} \in \ell^2(\mathcal{N})$.

Next for (iv) $\Longleftrightarrow$ (v), assume (iv) is true. Then (v) follows by observing Ker $G(x) =$ Ker $\Theta^*(x)$, and Ker $\Theta^*(x) = \{\{c_n\}_{n\in\mathcal{N}} \in \ell^2(\mathcal{N}) : \sum_{n\in\mathcal{N}} c_n\varphi_n(x) = 0\} = \{0\}$ for a.e. $x \in X$, since $C_1 \sum_{n\in\mathcal{N}} |c_n|^2 \leqslant \|\sum_{n\in\mathcal{N}} c_n\varphi_n(x)\|^2$ for some $C_1 > 0$. Conversely, assume (v) holds. Since for a.e. $x \in X$, $\mathcal{A}(x)$ is a frame for $J_{\mathcal{A}}(x)$, the operator $\Theta^*(x)$ is surjective, and hence $\Theta^*(x)$ is an isomorphism for a.e. $x \in X$. If we fix an orthonormal basis $\{e_n\}_{n\in\mathcal{N}}$ for $\ell^2(\mathcal{N})$, we have $\Theta^*(x)e_n = \varphi_n(x)$ for all $n \in \mathcal{N}$ and for a.e. $x \in X$. Therefore the result (iv) follows by observing the isomorphism of $\Theta^*(x)$ and an orthonormal basis $\{e_n\}_{n\in\mathcal{N}}$ for $\ell^2(\mathcal{N})$.

Moreover part follows by observing the equivalence between (i) to (vi) for a Borel measurable subset Y of X having positive measure such that $J_{\mathcal{A}}(x) \neq 0$ for a.e. $x \in Y$. □

The following result establishes the uniqueness of MG duals of type-II in $L^2(X; \mathcal{H})$. It is an abstraction of [36, Theorem 2.3] and [46, Theorem 7] in measure-theoretic setup.

Theorem 2.3.2. *Under the standing hypotheses mentioned in Theorem 2.3.1, the following are equivalent:*

(i) *An MG-dual of type-II for $E_{\mathscr{D}}(\mathcal{A})$ is the only canonical dual frame in $S_{\mathscr{D}}(\mathcal{A})$ with bounds $1/B$ and $1/A$.*

(ii) *$E_{\mathscr{D}}(\mathcal{A})$ is a frame for $S_{\mathscr{D}}(\mathcal{A})$ with bounds A and B, and $S_{\mathscr{D}}(\mathcal{A}) = L^2(X;\mathcal{H})$.*

(iii) *For a.e. $x \in X$, the dual of type-II for $\mathcal{A}(x)$ is the only canonical dual frame in $J_{\mathcal{A}}(x)$ with bounds $1/B$ and $1/A$.*

(iv) *For a.e. $x \in X$, $\mathcal{A}(x)$ is a frame for $J_{\mathcal{A}}(x)$ with bounds A and B and $J_{\mathcal{A}}(x) = \mathcal{H}$.*

(v) *For a.e. $x \in X$, the analysis operator $\Theta(x)$ (dual Gramian operator $\tilde{G}(x)$) associated with $\mathcal{A}(x)$ is injective.*

(vi) *For a.e $x \in X$, the dual Gramian operator $\tilde{G}(x)$ associated with $\mathcal{A}(x)$ satisfies $AI_{\mathcal{H}} \leqslant \tilde{G}(x) \leqslant BI_{\mathcal{H}}$ and $\mathcal{H} = J_{\mathcal{A}}(x)$.*

Moreover, $J_{\mathcal{A}}(x)$ need not be non-zero, then (i) is equivalent to either $J_{\mathcal{A}}(x) = 0$ or (iv) for a.e. $x \in X$.

Proof. Using Proposition 7.4 of [44], (i) and (ii) as well as (iii) and (iv) are equivalent. Also the equivalence between (iv) and (vi) is obvious by noting the definition of frame and the property of dual Gramian operator $\tilde{G}(x) = \Theta^*(x)\Theta(x)$ given below

$$\langle \tilde{G}(x)h, h\rangle = \langle \Theta(x)h, \Theta(x)h\rangle = \sum_{n\in\mathcal{N}} |\langle h, \varphi_n(x)\rangle|^2$$

for a.e. $x \in X$ and $h \in \mathcal{H}$. Next for (iv) $\Longleftrightarrow$ (v), assume (iv) holds. Then for a.e. $x \in X$, we have Ker $\Theta(x) = \{h \in \mathcal{H} : \langle h, \varphi_n(x)\rangle = 0 \text{ for all } n \in \mathcal{N}\} = J_{\mathcal{A}}(x)^{\perp} = \mathcal{H}^{\perp}$, and also Ker $\tilde{G}(x) =$ Ker $\Theta(x) = J_{\mathcal{A}}(x)^{\perp}$. Thus (v) follows while the converse part (iv) is obvious by using the fact Ker $\Theta(x) = J_{\mathcal{A}}(x)^{\perp}$.

Now it suffices to show (i) $\Longleftrightarrow$ (iv). For (i) $\Rightarrow$ (iv), assume on the contrary (iv) does not hold. Since $E_{\mathscr{D}}(\mathcal{A})$ is a frame for $S_{\mathscr{D}}(\mathcal{A})$, therefore there is a Borel measurable set $Y \subseteq X$ with $\mu_X(Y) > 0$ such that for a.e. $x \in Y$, $\mathcal{A}(x)$ is a frame for $J_{\mathcal{A}}(x)$ and $J_{\mathcal{A}}(x)$ is a non-trivial proper subspace of $\mathcal{H}$, i.e., $J_{\mathcal{A}}(x)^{\perp} \cap \mathcal{H} \neq \{0\}$ as $J_{\mathcal{A}}(x) \neq \{0\}$.

Fix an orthonormal basis $\{u_j\}$ for $\mathcal{H}$ and let $P_{J_{\mathcal{A}}(x)^{\perp}}(x)$ be an orthogonal projection from $\mathcal{H}$ onto $J_{\mathcal{A}}(x)^{\perp}$. Then for some Borel measurable set $Y_1 \subset Y$ with $\mu_X(Y_1) > 0$, there exists j_0 such that $\rho_{j_0}(x) := P_{J_{\mathcal{A}}(x)^{\perp}}(x)u_{j_0} \neq 0$ for a.e. $x \in Y_1$. Further, we can also choose some Borel measurable set $Y_2 \subset Y_1$ with $\mu_X(Y_2) > 0$ such that there is a function H in

$S_{\mathscr{D}}(\mathcal{A})$ so that for a.e. $x \in Y_2$, $c_n(x) = \langle H(x), \varphi_n(x)\rangle \neq 0$ for some $n \in \mathcal{N}$. For each $n \in \mathcal{N}$ if we define $\beta_n \in L^2(X;\mathcal{H})$ by

$$\beta_n(x) = \begin{cases} \overline{c_n(x)}\rho_{j_0}(x) & \text{for } x \in Y_2, \\ 0, & \text{otherwise,} \end{cases}$$

then for a.e. $x \in X \backslash Y_2$, we have $\sum_{n\in\mathcal{N}} |\langle \beta_n(x), \{c_n\}_{n\in\mathcal{N}}\rangle|^2 = 0$ while for a.e. $x \in Y_2$ and $u \in \mathcal{H}$, we have

$$\sum_{n\in\mathcal{N}} |\langle \beta_n(x), u\rangle|^2 = \sum_{n\in\mathcal{N}} |c_n(x)|^2 |\langle \rho_{j_0}(x), u\rangle|^2 \leqslant \|u\|^2 \sum_{n\in\mathcal{N}} |c_n(x)|^2 \leqslant B\|u\|^2\|H\|^2,$$

since $\mathcal{A}(x)$ is Bessel with bound B. Therefore, the system $\mathfrak{B}(x) := \{\beta_n(x) : n \in \mathcal{N}\}$ is Bessel with bound $B\|H\|^2$ for a.e. $x \in X$, and hence the family $E_{\mathscr{D}}(\mathfrak{B})$ is also Bessel in $L^2(X;\mathcal{H})$ with same bound. Therefore for $f \in L^2(X;\mathcal{H})$ and $n \in \mathcal{N}$, the calculation

$$\langle f(x), \beta_n(x)\rangle = \begin{cases} \langle \Xi(x), \varphi_n(x)\rangle \langle f(x), \rho_{j_0}(x)\rangle & \text{for } x \in Y_2, \\ 0 & \text{for } x \in X\backslash Y_2, \end{cases}$$

implies range $T_{\mathfrak{B}(x)} \subseteq$ range $T_{\mathcal{A}(x)}$, and so, we have range $T_{E_{\mathscr{D}}(\mathfrak{B})} \subseteq$ range $T_{E_{\mathscr{D}}(\mathcal{A})}$ using Theorem 2.2.10. Thus by using $\eta_n = \beta_n$ mentioned in (2.3.1) for each n, the system $\{M_{g_m}(\eta_n + (S_{E_{\mathscr{D}}(\mathcal{A})}|_{S_{\mathscr{D}}(\mathcal{A})})^{-1}\varphi_n) : m \in \mathcal{M}, n \in \mathcal{N}\}$ is another MG-dual of type-II for $E_{\mathcal{D}}(\mathcal{A})$ other than $E_{\mathcal{D}}(\tilde{\mathcal{A}})$, follows by noting $\langle f(x), \rho_{j_0}(x)\rangle = 0$ for a.e. $x \in Y_2$, and

$$\sum_{m\in\mathcal{M}} \sum_{n\in\mathcal{N}} \langle f, M_{g_m}\beta_n\rangle \overline{\langle g, M_{g_m}\varphi_n\rangle} = \int_{Y_2} \sum_{n\in\mathcal{N}} c_n(x)\langle f(x), \rho_{j_0}(x)\rangle \overline{\langle g(x), \varphi_n(x)\rangle}\, d\mu_X(x)$$

for all $f, g \in S_{\mathscr{D}}(\mathcal{A})$ from Lemma 2.2.3. Thus we arrive on a contradiction.

Conversely for (iv) $\Rightarrow$ (i), assume (iv) holds. Then due to the equivalence of (iii) and (iv), dual of type-II for $\mathcal{A}(x)$ is the only canonical dual. Now by proceeding with the contradiction of (i), assume that there is a countable family $\mathcal{A}' = \{\psi_n\}_{n\in\mathcal{N}} \neq \tilde{\mathcal{A}}$ in $L^2(X;\mathcal{H})$ such that $E_{\mathscr{D}}(\mathcal{A}')$ is another MG dual of type-II for $E_{\mathscr{D}}(\mathcal{A})$. Then the system $E_{\mathscr{D}}(\mathcal{A}'')$ is Bessel by (2.3.1), where $\mathcal{A}'' = \{\eta_n = \psi_n - (S_{E_{\mathscr{D}}(\mathcal{A})}|_{S_{\mathscr{D}}(\mathcal{A})})^{-1}\varphi_n\}_{n\in\mathcal{N}}$, and hence for a.e. $x \in X$, $\mathcal{A}''(x)$ is Bessel in $\mathcal{H} = J_{\mathcal{A}}(x)$. Using Theorem 2.2.10, the system $\mathcal{A}'(x)$ is another type-II dual for $\mathcal{A}(x)$ for a.e. $x \in X$. Thus the result follows. Moreover part follows quickly by illustrating the equivalence of (i) to (vi) on a Borel measurable subset X where $J_{\mathcal{A}}(x)$ is non-zero a.e. $x \in X$. □

The following result summarized Theorem 2.3.1 and Theorem 2.3.2 for the uniqueness of alternate duals in $L^2(X;\mathcal{H})$. This follows by observing that the canonical dual is both type-I and type-II dual.

Theorem 2.3.3. *Under the standing hypotheses mentioned in Theorem 2.3.1, the following are equivalent:*

(i) *An alternate MG-dual for $E_{\mathscr{D}}(\mathcal{A})$ is the only canonical dual frame in $S_{\mathscr{D}}(\mathcal{A})$ with bounds $1/B$ and $1/A$.*

(ii) *$E_{\mathscr{D}}(\mathcal{A})$ is a Riesz basis for $S_{\mathscr{D}}(\mathcal{A})$ with some bounds C_1 and C_2 and $S_{\mathscr{D}}(\mathcal{A}) = L^2(X;\mathcal{H})$.*

(iii) *For a.e. $x \in X$, the system $\mathcal{A}(x)$ is a Riesz basis for $J_{\mathcal{A}}(x)$ with bounds C_1 and C_2 and $J_{\mathcal{A}}(x) = \mathcal{H}$.*

(iv) *For a.e. $x \in X$, the alternate dual for $\mathcal{A}(x)$ is the only canonical dual frame $J_{\mathcal{A}}(x)$ with bounds $1/B$ and $1/A$.*

(v) *For a.e. $x \in X$, the Gramian $G(x)$ and dual Gramian $\tilde{G}(x)$ operators associated with $\mathcal{A}(x)$ are injective.*

(vi) *For a.e. $x \in X$, the Gramian $G(x)$ and dual Gramian $\tilde{G}(x)$ operators associated with $\mathcal{A}(x)$ satisfy $C_1 I_{\ell^2(\mathcal{N})} \leqslant G(x) \leqslant C_2 I_{\ell^2(\mathcal{N})}$ and $AI_{\mathcal{H}} \leqslant \tilde{G}(x) \leqslant BI_{\mathcal{H}}$.*

(vii) *For $f = \{f_{\varphi_n}\}_{n\in\mathcal{N}}$ in $\ell^2(\mathcal{N}; L^2(X))$, we have*

$$C_1 \|f\|^2_{\ell^2(\mathcal{N};L^2(X))} \leqslant \left\| \left\{ \sum_{n\in\mathcal{N}} f_{\varphi_n}(x)\varphi_n(x) \right\}_{x\in X} \right\|^2_{L^2(X;\mathcal{H})} \leqslant C_2 \|f\|^2_{\ell^2(\mathcal{N};L^2(X))}.$$

(viii) *For any Riesz sequence $D \subset L^\infty(X)$ in $L^2(X)$ with bounds c_1 and c_2, $E_D(\mathcal{A})$ is a Riesz basis for $S_D(\mathcal{A})$ with bounds $c_1 C_1$ and $c_2 C_2$.*

Till now, we have characterized various duals with their global and local behavior. Next we are going to construct multiplication generated oblique dual frames in a multiplication invariant space. In this construction technique, we find the connection of the infimum cosine angle between subspaces with oblique dual. The infimum cosine angle is directly connected with the oblique projections and it further relates to oblique dual. In the next chapter, we will talk about this briefly.

CHAPTER 3

CONSTRUCTION OF OBLIQUE DUAL FRAMES IN $L^2(X;\mathcal{H})$

This chapter discusses the construction of dual frames and their uniqueness for a multiplication generated frame on $L^2(X;\mathcal{H})$, where X is a σ-finite measure. Various necessary and sufficient conditions of such duals associated with the infimum cosine angle are obtained [66].

3.1. Infimum cosine angles and oblique duals

In this section, our goal is to find characterization results for the MG duals $E_{\mathcal{D}}(\mathcal{A}')$ of a frame $E_{\mathcal{D}}(\mathcal{A})$ associated with the infimum cosine angles between the closed subspaces $S_{\mathcal{D}}(\mathcal{A}')$ and $S_{\mathcal{D}}(\mathcal{A})$ of $L^2(X;\mathcal{H})$, for some finite collections of functions $\mathcal{A}, \mathcal{A}'$ in $L^2(X;\mathcal{H})$. The infimum cosine angle between two closed subspaces of Hilbert spaces [1] is defined as follows:

Definition 3.1.1. Let V and W be closed subspaces of $\mathcal{H}$. The *infimum cosine angle* between V and W of $\mathcal{H}$ is defined by

$$R(V,W) = \inf_{v \in V \setminus \{0\}} \frac{\|P_W v\|}{\|v\|},$$

where P_W is the projection on W.

In general, $R(V,W) \neq R(W,V)$. If $R(V,W) > 0$ and $R(W,V) > 0$ then $R(V,W) = R(W,V)$, and hence we can decompose the Hilbert space as $\mathcal{H} = V \oplus W^{\perp}$ (not necessarily orthogonal direct sum), means, $\mathcal{H} = V + W^{\perp}$ and $V \bigcap W^{\perp} = 0$ [26]. In addition, if the following reproducing formula holds :

$$f = \sum_{k \in I} \langle f, f_k \rangle g_k \text{ for all } f \in V,$$

This chapter is a part of the following manuscript:

S. Sarkar, N. K. Shukla, *A characterization of MG dual frames using infimum cosine angle*, arXiv:2301.07448.

where $\{f_k\}_{k\in I}$ and $\{g_k\}_{k\in I}$ are Bessel sequences in $\mathcal{H}$ and $W = \overline{\text{span}}\{f_k\}$, then $\{f_k\}_{k\in I}$ is an oblique dual frame for $\{g_k\}_{k\in I}$ on W, and $\{g_k\}_{k\in I}$ is an oblique dual frame for $\{f_k\}_{k\in I}$ on V [26, Lemma 3.1]. Furthermore, $\{g_k\}_{k\in I}$ and $\{P_V f_k\}_{k\in I}$ are dual frames in V and $\{f_k\}_{k\in I}$ and $\{P_W g_k\}_{k\in I}$ are dual frames in W. This decomposition is important to recover data from a given set of samples. Tang in [72] studied the infimum cosine angles in connection with oblique projections that leads to oblique dual frames, followed by Kim et al. for the different contexts in [55, 56]. Further, Christensen and Eldar in [26], and Kim et al. in [57] developed a connection of the infimum cosine angle with oblique dual frames for shift-invariant (SI) spaces in $L^2(\mathbb{R}^n)$. An existence of Riesz basis using infimum cosine angle for the theory of multiresolution analysis in $L^2(\mathbb{R}^n)$ was discussed by Bownik and Garrigós in [18]. We aim to continue the work in the context of set-theoretic abstraction.

Now we provide a characterization of an alternate dual associated with the Gramian operators. Recalling the *Gramian* and *dual Gramian operators* from (2.3.2) as follows :

$$G_{\mathcal{A}}(x) = T_{\mathcal{A}}(x)T^*_{\mathcal{A}}(x) \text{ and } \tilde{G}_{\mathcal{A}}(x) = T^*_{\mathcal{A}}(x)T_{\mathcal{A}}(x) \text{ a. e. } x \in X,$$

where $T_{\mathcal{A}}(x)$ and $T^*_{\mathcal{A}}(x)$ are the analysis and synthesis operators corresponding to $\mathcal{A}(x) = \{\varphi_i(x)\}_{i=1}^r$. For $\mathcal{A} = \{\varphi_i\}_{i=1}^r$ and $\mathcal{A}' = \{\psi_i\}_{i=1}^r$, the operator $G_{\mathcal{A},\mathcal{A}'}(x) = [\langle \varphi_j(x), \psi_i(x)\rangle]_{i,j\in\{1,2,\ldots,r\}}$ is known as the *mixed Gramian operator* for a.e. $x \in X$. The following result is a generalization of [57, Theorem 4.1] and [46, Theorem 5(a)].

Proposition 3.1.2. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two finite collections of functions having same cardinality such that $E_{\mathcal{D}}(\mathcal{A})$ and $E_{\mathcal{D}}(\mathcal{A}')$ are Bessel. Let us assume $E_{\mathcal{D}}(\mathcal{A})$ be a frame for $S_{\mathcal{D}}(\mathcal{A})$. Then the system $E_{\mathcal{D}}(\mathcal{A}')$ is an alternate MG-dual for $E_{\mathcal{D}}(\mathcal{A})$ if and only if for a.e. $x \in X$ the system $\mathcal{A}'(x) = \{\psi(x) : \psi \in \mathcal{A}'\}$ is an alternate dual for $\mathcal{A}(x) = \{\varphi(x) : \varphi \in \mathcal{A}\}$, equivalently, the Gramian $G_{\mathcal{A}}(x)$ and mixed Gramian $G_{\mathcal{A},\mathcal{A}'}(x)$ operators satisfy the following relation:*

$$G_{\mathcal{A}}(x)G_{\mathcal{A},\mathcal{A}'}(x) = G_{\mathcal{A}}(x) \textit{ for a.e. } x \in X.$$

The following result is a measure-theoretic abstraction of [57, Theorem 4.1] for oblique dual frames associated with the rank of the mixed Gramian operator and the dimension of range functions.

Proposition 3.1.3. *In addition to the assumptions of Proposition 3.1.2, let us assume $E_{\mathcal{D}}(\mathcal{A})$ and $E_{\mathcal{D}}(\mathcal{A}')$ be frames for $S_{\mathcal{D}}(\mathcal{A})$ and $S_{\mathcal{D}}(\mathcal{A}')$, respectively, such that $E_{\mathcal{D}}(\mathcal{A}')$ is an* *alternate MG-dual for $E_{\mathcal{D}}(\mathcal{A})$ and*

$$\text{(3.1.1)} \qquad \operatorname{rank}\ G_{\mathcal{A},\mathcal{A}'}(x) = \dim J_{\mathcal{A}}(x) = \dim J_{\mathcal{A}'}(x) \text{ a.e. } x \in X,$$

where $J_{\mathcal{A}}(x) = \operatorname{span}\{f(x) : f \in \mathcal{A}\}$ and $J_{\mathcal{A}'}(x) = \operatorname{span}\{g(x) : g \in \mathcal{A}'\}$. Then $E_{\mathcal{D}}(\mathcal{A}')$ is an oblique MG-dual for $E_{\mathcal{D}}(\mathcal{A})$.

Proof. Observing the proof of Proposition 3.1.2, we get $\mathcal{A}'(x)$ is an alternate dual for $\mathcal{A}(x)$ for a.e. $x \in X$ since $E_{\mathcal{D}}(\mathcal{A}')$ is an alternate MG-dual for $E_{\mathcal{D}}(\mathcal{A})$. Then for a.e. $x \in X$, we can write $\varphi(x) = \sum_{i=1}^{r}\langle \varphi(x), \psi_i(x)\rangle \varphi_i(x)$ for each $\varphi \in S_{\mathcal{D}}(\mathcal{A})$. Further note that $P(x) := P_{J_{\mathcal{A}'(x)}}|_{J_{\mathcal{A}}(x)} : J_{\mathcal{A}(x)} \to J_{\mathcal{A}'(x)}$ is invertible in view of [57, Lemma 3.1] and relation (3.1.1). Therefore for $k = 1, 2, \ldots, r$ and a.e. $x \in X$, we get

$$\begin{aligned}\langle P(x)\varphi(x), g_k(x)\rangle = \langle P_{J_{\mathcal{A}'(x)}}\varphi(x), g_k(x)\rangle &= \langle \varphi(x), P_{J_{\mathcal{A}'(x)}} g_k(x)\rangle = \langle \varphi(x), g_k(x)\rangle \\ &= \left\langle \langle \sum_{i=1}^{r} \varphi(x), \psi_i(x)\rangle \varphi_i(x), g_k(x) \right\rangle \\ &= \left\langle \varphi(x), \sum_{i=1}^{r} \langle g_k(x), \varphi_i(x)\rangle \psi_i(x) \right\rangle \\ &= \left\langle P(x)\varphi(x), \sum_{i=1}^{r} \langle g_k(x), \varphi_i(x)\rangle \psi_i(x) \right\rangle,\end{aligned}$$

and hence $g_k(x) = \sum_{i=1}^{r}\langle g_k(x), \varphi_i(x)\rangle \psi_i(x)$ since $P(x)$ is invertible. Therefore the result holds by noting Proposition 3.1.2. □

The next result tells that the space $L^2(X; \mathcal{H})$ can be decomposed with the help of $S_{\mathcal{D}}(\mathcal{A})$ and $S_{\mathcal{D}}(\mathcal{A}')$ using the rank condition (3.1.1). We use the concept of angle between two MI subspaces and their point-wise characterizations for its proof. From Definition 3.1.1, note that

$$(P_{S_{\mathcal{D}}(\mathcal{A})}|_{S_{\mathcal{D}}(\mathcal{A}')} f)(x) = (P_{S_{\mathcal{D}}(\mathcal{A})} P_{S_{\mathcal{D}}(\mathcal{A}')} f)(x) = P_{S_{\mathcal{D}}(\mathcal{A}')}(x) P_{S_{\mathcal{D}}(\mathcal{A})}(x) f(x) = P_{S_{\mathcal{D}}(\mathcal{A})(x)}|_{S_{\mathcal{D}}(\mathcal{A}')(x)} f(x),$$

and by [19, Theorem 4.1 (iii)], we have

$$\inf\left\{\frac{\|P_{S_{\mathcal{D}}(\mathcal{A}')} f\|}{\|f\|} : f \in S_{\mathcal{D}}(\mathcal{A})\backslash\{0\}\right\} = \underset{x \in X}{\text{ess-inf}} \left\{\frac{\|P_{S_{\mathcal{D}}(\mathcal{A}')(x)} w\|}{\|w\|} : w \in J_{\mathcal{A}}(x)\backslash\{0\}\right\}.$$

Thus if we define $\sigma(S_{\mathcal{D}}(\mathcal{A})) := \{x \in X : J_{\mathcal{A}}(x) \neq 0\}$, then

(3.1.2)

$$R(S_{\mathcal{D}}(\mathcal{A}), S_{\mathcal{D}}(\mathcal{A}')) = \begin{cases} \text{ess-inf}_{x\in\sigma(S_{\mathcal{D}}(\mathcal{A}))} R(J_{\mathcal{A}}(x), J_{\mathcal{A}'}(x)) \text{ if } \mu_X(\sigma(S_{\mathcal{D}}(\mathcal{A}))) > 0, \\ 1, \text{otherwise.} \end{cases}.$$

Proposition 3.1.4. *In addition to the assumptions of Proposition 3.1.2, the following statements are equivalent:*

(i) *For a.e. $x \in X$, the relation (3.1.1) holds, i.e.,* $\text{rank } G_{\mathcal{A},\mathcal{A}'}(x) = \dim J_{\mathcal{A}}(x) = \dim J_{\mathcal{A}'}(x)$ *a.e. $x \in X$, and there exists a constant $C > 0$ such that*

$$\|(G_{\mathcal{A}}(x))^{1/2} G_{\mathcal{A},\mathcal{A}'}(x)^{\dagger} (G_{\mathcal{A}'}(x))^{1/2}\| \leqslant C \text{ a.e. } x \in \{x \in X : J_{\mathcal{A}}(x) \neq 0\} := \sigma(S_{\mathcal{D}}(\mathcal{A})),$$

where $G_{\mathcal{A},\mathcal{A}'}(x)^{\dagger}$ denotes the pseudo inverse of $G_{\mathcal{A},\mathcal{A}'}(x)$.

(ii) $L^2(X;\mathcal{H}) = S_{\mathcal{D}}(\mathcal{A}) \oplus S_{\mathcal{D}}(\mathcal{A}')^{\perp}$.

(ii) $L^2(X;\mathcal{H}) = S_{\mathcal{D}}(\mathcal{A}') \oplus S_{\mathcal{D}}(\mathcal{A})^{\perp}$.

(iv) $R(S_{\mathcal{D}}(\mathcal{A}), S_{\mathcal{D}}(\mathcal{A}')) > 0$ *and* $R(S_{\mathcal{D}}(\mathcal{A}'), S_{\mathcal{D}}(\mathcal{A})) > 0$.

Proof. The result can be established easily following the steps of [19, Theorem 4.18] and [57, Theorem 3.8]. □

At the end of this section, we provide a method to construct alternate (oblique) duals, which is an abstraction version of [57, Lemma 5.1].

Proposition 3.1.5. *For a σ-finite measure space (X, μ_X) with $\mu_X < \infty$, consider the assumptions of Proposition 3.1.2 and assume $E_{\mathcal{D}}(\mathcal{A})$ to be a frame for $S_{\mathcal{D}}(\mathcal{A})$. Define a class of functions $\tilde{\mathcal{A}}' = \{h_i\}_{i=1}^r$ associated with $\mathcal{A}' = \{\psi_i\}_{i=1}^r \subset L^2(X;\mathcal{H})$ by*

$$h_i(x) = \begin{cases} \sum_{j=1}^r \overline{G_{\mathcal{A},\mathcal{A}'}(x)_{i,j}^{\dagger}}\ \psi_j(x) \text{ if } x \in \sigma(S_{\mathcal{D}}(\mathcal{A})), \\ 0, \text{ otherwise.} \end{cases} \tag{3.1.3}$$

Then, $E_{\mathcal{D}}(\tilde{\mathcal{A}}')$ is an alternate (oblique) MG-dual for $E_{\mathcal{D}}(\mathcal{A})$ if the Proposition 3.1.4 (i) rank condition holds and there exists a $C > 0$ such that $\|G_{\mathcal{A},\mathcal{A}'}(x)^{\dagger}\| \leqslant C$ a.e. $x \in \sigma(S_{\mathcal{D}}(\mathcal{A}))$.

Proof. Note that $G_{\tilde{\mathcal{A}}'}(x) = G_{\mathcal{A},\mathcal{A}'}(x)^{\dagger}G_{\mathcal{A}'}(x)(G_{\mathcal{A},\mathcal{A}'}(x)^{\dagger})^{*}$ a.e. $x \in \sigma(S_{\mathcal{D}}(\mathcal{A}))$ and $G_{\tilde{\mathcal{A}}'}(x) = 0$, otherwise, $\|G_{\tilde{\mathcal{A}}'}(x)\|$ is bounded above due to Bessel property of $\mathcal{A}'(x)$. By the Proposition 3.1.2, we need to verify $G_{\mathcal{A}}(x)G_{\mathcal{A},\tilde{\mathcal{A}}'}(x) = G_{\mathcal{A}}(x)$ which follows from the same techniques of proof from [57, Lemma 5.3]. □

Now we provide our first main result which is a measure theoretic abstraction of [57, Theorem 4.10] using range function.

Theorem 3.1.6. *Let (X, μ_X) and $(\mathcal{M}, \mu_{\mathcal{M}})$ be σ-finite measure spaces such that $\mu_X < \infty$, and the set $\mathcal{D} = \{\phi_s \in L^{\infty}(X) : s \in \mathcal{M}\}$ is Parseval determining set for $L^1(X)$. For the finite collections of functions $\mathcal{A} = \{\varphi_i\}_{i=1}^{m}$ and $\mathcal{B} = \{\psi_i\}_{i=1}^{n}$ in $L^2(X;\mathcal{H})$, and for a.e. $x \in X$, assume the range functions $J_{\mathcal{A}}(x) = \operatorname{span}\{\varphi_i(x) : i = 1, 2, \ldots, m\}$ and $J_{\mathcal{B}}(x) = \operatorname{span}\{\psi_i(x) : i = 1, 2, \ldots, n\}$ associated with the MI spaces $S_{\mathcal{D}}(\mathcal{A})$ and $S_{\mathcal{D}}(\mathcal{B})$, respectively. Then the following are equivalent:*

(i) *There exist $\mathcal{A}' = \{\varphi_i'\}_{i=1}^{r}$ and $\mathcal{B}' = \{\psi_i'\}_{i=1}^{r}$ in $L^2(X;\mathcal{H})$ such that $E_{\mathcal{D}}(\mathcal{A}')$ and $E_{\mathcal{D}}(\mathcal{B}')$ are continuous frames for $S_{\mathcal{D}}(\mathcal{A})$ and $S_{\mathcal{D}}(\mathcal{B})$, respectively, and they are oblique duals to each other, i.e., the following reproducing formulas hold for $g \in S_{\mathcal{D}}(\mathcal{A})$ and $h \in S_{\mathcal{D}}(\mathcal{B})$:*

$$g = \sum_{i=1}^{r}\int_{\mathcal{M}} \langle g, M_{\phi_s}\psi_i'\rangle M_{\phi_s}\varphi_i' \, d\mu_{\mathcal{M}}(s) \text{ and } h = \sum_{i=1}^{r}\int_{\mathcal{M}} \langle h, M_{\phi_s}\varphi_i'\rangle M_{\phi_s}\psi_i' \, d\mu_{\mathcal{M}}(s). \tag{3.1.4}$$

(ii) *The infimum cosine angles of $S_{\mathcal{D}}(\mathcal{A})$ and $S_{\mathcal{D}}(\mathcal{B})$ are greater than zero, i.e.,*

$$R(S_{\mathcal{D}}(\mathcal{A}), S_{\mathcal{D}}(\mathcal{B})) > 0 \text{ and } R(S_{\mathcal{D}}(\mathcal{B}), S_{\mathcal{D}}(\mathcal{A})) > 0.$$

(iii) *There exist collections of functions $\{\varphi_i'\}_{i=1}^{r}$ and $\{\psi_i'\}_{i=1}^{r}$ in $L^2(X;\mathcal{H})$ such that for a.e. $x \in X$, the systems $\{\varphi_i'(x)\}_{i=1}^{r}$ and $\{\psi_i'(x)\}_{i=1}^{r}$ are finite frames for $J_{\mathcal{A}}(x)$ and $J_{\mathcal{B}}(x)$, respectively, and they are oblique duals, i.e., they satisfy the following reproducing formulas for $u \in J_{\mathcal{A}}(x)$ and $v \in J_{\mathcal{B}}(x)$:*

$$u = \sum_{i=1}^{r}\langle u, \psi_i'(x)\rangle\varphi_i'(x) \text{ and } v = \sum_{i=1}^{r}\langle v, \varphi_i'(x)\rangle\psi_i'(x) \text{ a.e. } x \in X. \tag{3.1.5}$$

(iv) *For a.e. $x \in X$, the infimum cosine angles of $J_{\mathcal{A}}(x)$ and $J_{\mathcal{B}}(x)$ are greater than zero, i.e.,*

$$R(J_{\mathcal{A}}(x), J_{\mathcal{B}}(x)) > 0 \text{ and } R(J_{\mathcal{B}}(x), J_{\mathcal{A}}(x)) > 0.$$

Proof. (ii) $\implies$ (i). Assume that (ii) holds, then we have rank $G_{\mathcal{A},\mathcal{B}}(x) = \dim J_{\mathcal{A}}(x) = \dim J_{\mathcal{B}}(x)$ a.e. $x \in X$ by Proposition 3.1.4 (iv). Considering the projection $P(x) := P_{J_{\mathcal{A}}(x)|_{\mathcal{B}(x)}} : J_{\mathcal{B}(x)} \to J_{\mathcal{A}(x)}$, we have $G_{\mathcal{A},\mathcal{B}}(x) = T_{\mathcal{B}}(x)P(x)T^*_{\mathcal{A}}(x)$, and $P(x)$ is invertible by [57, Lemma 3.1]. Then the length of $S_{\mathcal{D}}(\mathcal{A})$ = length $S_{\mathcal{D}}(\mathcal{B})$ since $S_{\mathcal{D}}(\mathcal{A})$ and $S_{\mathcal{D}}(\mathscr{B})$ are finitely generated. Let r be the common length of $S_{\mathcal{D}}(\mathcal{A})$ and $S_{\mathcal{D}}(\mathcal{B})$. Then using [20, Theorem 2.6], there exist $\mathcal{A}^{\#} = \{\varphi_i^{\#}\}_{i=1}^r$ and $\mathscr{K} = \{\psi_i^{\#}\}_{i=1}^r$ such that $S_{\mathcal{D}}(\mathcal{A}^{\#}) = S_{\mathcal{D}}(\mathcal{A})$ and $S_{\mathcal{D}}(\mathcal{B}^{\#}) = S_{\mathcal{D}}(\mathcal{B})$. Hence $R(S_{\mathcal{D}}(\mathcal{A}^{\#}), S_{\mathcal{D}}(\mathcal{B}^{\#})) > 0$ and $R(S_{\mathcal{D}}(\mathcal{B}^{\#}), S_{\mathcal{D}}(\mathcal{A}^{\#})) > 0$. Further, applying Proposition 3.1.4 (iv), there exists a positive constant C such that $\|G_{\mathcal{A}^{\#}}(x)^{1/2} G_{\mathcal{A}^{\#},\mathcal{B}^{\#}}(x)^{\dagger} G_{\mathcal{B}^{\#}}(x)^{1/2}\| \leqslant C$ a.e. $x \in \sigma(S_{\mathcal{D}}(\mathcal{A}))$.

For the class of functions $\mathcal{A}^{\#} = \{\varphi_i^{\#}\}_{i=1}^r$, define the new class of functions $\mathcal{A}' = \{\varphi_i'\}_{i=1}^r$ by

$$\varphi_i'(x) = \sum_{j=1}^{r} \overline{((G_{\mathcal{A}^{\#}}(x)^{\dagger})^{1/2})}_{i,j} \varphi_j^{\#}(x) \text{ for a.e. } x \in X, \quad \text{and for each } i \in \{1, 2, \ldots, r\}.$$

Applying the singular value decomposition of the positive semidefinite matrix $G_{\mathcal{A}^{\#}}(x)$ for a.e. $x \in X$,

$$G_{\mathcal{A}^{\#}}(x) = Q(x)D(x)Q(x)^*,$$

where the diagonal entries of $D(x)$ are the non-zero eigenvalues of $G_{\mathcal{A}^{\#}}(x)$ and $Q(x)$ is unitary. Also, note that

$$\|\varphi_i'(x)\|^2 = (G_{\mathcal{A}^{\#}}(x)^{\dagger})^{1/2} G_{\mathcal{A}^{\#}}(x) (G_{\mathcal{A}^{\#}}(x)^{\dagger})^{1/2})_{ii} = 0 \text{ or } 1.$$

For each $i \in \{1, 2, \ldots, r\}$, $\|\varphi_i'\|^2 = \int_X \|\varphi_i'(x)\|^2 \, d\mu(X) < \infty$, hence $\varphi_i' \in L^2(X; \mathcal{H})$. Also

$$G_{\mathcal{A}'}(x) = (G_{\mathcal{A}^{\#}}(x)^{\dagger})^{1/2} G_{\mathcal{A}^{\#}}(x) (G_{\mathcal{A}^{\#}}(x)^{\dagger})^{1/2} = G_{\mathcal{A}^{\#}}(x)^{\dagger} G_{\mathcal{A}^{\#}}(x) \;\text{ a. e. } x \in X.$$

The eigenvalues of $G_{\mathcal{A}'}(x)$ are 0 or 1. Thus $E_{\mathcal{D}}(\mathcal{A}')$ is a frame for $S_{\mathcal{D}}(\mathcal{A}^{\#})$. Now we will show $S_{\mathcal{D}}(\mathcal{A}') = S_{\mathcal{D}}(\mathcal{A}^{\#})$. It is clear that $S_{\mathcal{D}}(\mathcal{A}')(x) \subset S_{\mathcal{D}}(\mathcal{A}^{\#})(x)$ for a.e. $x \in X$. Also,

$$\dim J_{\mathcal{A}'}(x) = \text{rank } \; G_{\mathcal{A}'}(x) = \text{rank } \; G_{\mathcal{A}^{\#}}(x) = \dim \; J_{\mathcal{A}^{\#}}(x).$$

Hence $J_{\mathcal{A}'}(x) = J_{\mathcal{A}^{\#}}(x)$ a.e. $x \in X$, i.e., $S_{\mathcal{D}}(\mathcal{A}') = S_{\mathcal{D}}(\mathcal{A}^{\#})$ [49, Proposition 2.2 (iii)]. The class $E_{\mathcal{D}}(\mathcal{A}')$ is a tight frame for $S_{\mathcal{D}}(\mathcal{A}^{\#})$. In a similar way, we can show that there exists a collection $\mathcal{B}' = \{\psi_i'\}_{i=1}^r$ such that $E_{\mathcal{D}}(\mathcal{B}')$ is a tight frame for $S_{\mathcal{D}}(\mathcal{B}^{\#})$, and also we have

$$G_{\mathcal{A}',\mathcal{B}'}(x) = (G_{\mathcal{B}^{\#}}(x)^{\dagger})^{1/2} G_{\mathcal{A}^{\#},\mathcal{B}^{\#}}(x) (G_{\mathcal{A}^{\#}}(x)^{\dagger})^{1/2}.$$

Since $P(x)$ is invertible a.e. $G_{\mathcal{A}',\mathcal{B}'}(x)^\dagger = (G_{\mathcal{A}^\#}(x))^{1/2}G_{\mathcal{A}^\#,\mathcal{B}^\#}(x)^\dagger(G_{\mathcal{B}^\#}(x))^{1/2}$. Now $\|G_{\mathcal{A}',\mathcal{B}'}(x)^\dagger\| = \|(G_{\mathcal{A}^\#}(x))^{1/2}G_{\mathcal{A}^\#,\mathcal{B}^\#}(x)^\dagger(G_{\mathcal{B}^\#}(x))^{1/2}\| \leqslant C$ a.e. $x \in \sigma(S_{\mathcal{D}}(\mathcal{A}^\#))$. By the Proposition 3.1.5, $E_{\mathcal{D}}(\mathcal{B}')$ is an oblique MG-dual for $E_{\mathcal{D}}(\mathcal{A}')$.

(i)$\Longrightarrow$(ii). Define a map $\Xi : L^2(X;\mathcal{H}) \to S_{\mathcal{D}}(\mathcal{A})$ by $\Xi f = \sum_{i=1}^r \int_{\mathcal{M}} \langle f, M_\phi \psi_i' \rangle M_\phi \varphi_i' \, d\mu_{\mathcal{M}}(s)$. Then Ξ is not necessarily an orthogonal projection. Therefor $L^2(X;\mathcal{H})$ cab be decomposed as $L^2(X;\mathcal{H}) = \text{range } \Xi \oplus \text{Ker } \Xi = S_{\mathcal{D}}(\mathcal{A}) \oplus \text{Ker } \Xi$. We now show $\text{Ker } \Xi = S_{\mathcal{D}}(\mathcal{B})^\perp$. Let $\varphi \in \text{Ker}\,\Xi$. Then $\varphi = f - \Xi f$, $f \in L^2(X;\mathcal{H})$. For $\psi \in S_{\mathcal{D}}(\mathcal{B})$,

$$\begin{aligned}\langle \varphi, \psi \rangle = \langle f - \Xi f, \psi \rangle = \langle f, \psi \rangle - \langle \Xi f, \psi \rangle &= \langle f, \psi \rangle - \sum_{i=1}^r \int_{\mathcal{M}} \langle f, M_\phi \psi_i' \rangle \langle M_\phi \varphi_i', \psi \rangle \, d\mu_{\mathcal{M}}(s) \\ &= \langle f, \psi \rangle - \left\langle f, \sum_{i=1}^r \int_{\mathcal{M}} \langle \psi, M_\phi \varphi_i' \rangle M_\phi \psi_i' \, d\mu_{\mathcal{M}}(s) \right\rangle \\ &= \langle f, \psi \rangle - \langle f, \psi \rangle = 0.\end{aligned}$$

Hence $\varphi \in S_{\mathcal{D}}(\mathcal{B})^\perp$. On the other side, if $\varphi \in S_{\mathcal{D}}(\mathcal{B})^\perp$ then $\Xi\varphi = 0$.

(i)$\Longleftrightarrow$(iii). Since $E_{\mathcal{D}}(\mathcal{A}')$ and $E_{\mathcal{D}}(\mathcal{B}')$ are frames for $S_{\mathcal{D}}(\mathcal{A})$ and $S_{\mathcal{D}}(\mathcal{B})$, respectively, the systems $\mathcal{A}'(x) = \{\varphi_i'(x) : i = 1, \dots, r\}$ and $\mathcal{B}'(x) = \{\psi_i'(x) : i = 1, \dots, r\}$ are frames for $J_{\mathcal{A}}(x)$ and $J_{\mathcal{B}}(x)$, respectively, for a.e. $x \in X$ [49, Theorem 2.10] . The rest part of the result follows by Proposition 3.1.2.

In a similar way, the converse part follows.

(iv)$\Longrightarrow$(iii). Assume (iv) holds, i.e., there exist frames $\{\varphi_i'(x)\}_{i=1}^r$ and $\{\psi_i'(x)\}_{i=1}^r$ for $J_{\mathcal{A}}(x)$ and $J_{\mathcal{B}}(x)$, respectively. We need to show that $R(J_{\mathcal{A}}(x), J_{\mathcal{B}}(x)) > 0$ and $R(J_{\mathcal{B}}(x), J_{\mathcal{A}}(x)) > 0$, which is equivalent to $J_{\mathcal{A}}(x) \oplus J_{\mathcal{B}}(x)^\perp = \mathcal{H}$ [26, Lemma 2.1]. For this define a map, $\Xi : \mathcal{H} \to J_{\mathcal{A}}(x)$ by $\Xi(f) = \sum_{i=1}^r \langle f, \psi_i'(x) \rangle \varphi_i'(x)$. Then Ξ need not be an orthogonal projection. Hence

$$\mathcal{H} = \text{range}\,\Xi \ \oplus \ \text{Ker}\,\Xi = J_{\mathcal{A}}(x) \oplus \text{Ker}\,\Xi.$$

Our aim is to prove $\text{Ker}\,\Xi = J_{\mathcal{B}}(x)^\perp$. Let $u \in \text{Ker}\,\Xi$. Then $u = f - \Xi f$ for some f. Let $h \in J_{\mathcal{B}}(x)$. Writing

$$\begin{aligned}\langle u, h \rangle = \langle f - \Xi f, h \rangle = \langle f, h \rangle - \left\langle \sum_{i=1}^r \langle f, \psi_i'(x) \rangle \varphi_i'(x), h \right\rangle &= \langle f, h \rangle - \sum_{i=1}^r \langle f, \psi_i'(x) \rangle \langle \varphi_i'(x), h \rangle \\ &= \langle f, h \rangle - \left\langle f, \sum_{i=1}^r \langle h, \varphi_i'(x) \rangle \psi_i'(x) \right\rangle = \langle f, h \rangle - \langle f, h \rangle = 0,\end{aligned}$$

we have $u \in J_{\mathcal{B}}(x)^\perp$, and if $u \in J_{\mathcal{B}}(x)^\perp$, then $u \in \text{Ker}\,\Xi$.

(ii)$\Longrightarrow$ (iv). If $R(S_{\mathcal{D}}(\mathcal{A}), S_{\mathcal{D}}(\mathcal{B})) > 0$, we have $R(J_{\mathcal{A}}(x), J_{\mathcal{B}}(x)) > 0$ for a.e. $x \in X$. The remaining part follows easily. $\square$

3.2. Riesz basis and its associated dual

The equations (3.1.4) and (3.1.5) explore the various possibilities of obtaining oblique dual frames in the global and local setups, respectively. These duals and associated reproducing formulas are not necessarily unique. But when $E_{\mathcal{D}}(\mathcal{A})$ is a Riesz basis then the dual is always unique. The following main result discusses the uniqueness of reproducing formula, which is a measure-theoretic abstraction of [72, Corrollary 2.4] and [18, Proposition 2.13].

Theorem 3.2.1. *Let (X, μ_X) be a σ-finite measure space with $\mu_X < \infty$, and let $\mathscr{V}$ and $\mathscr{W}$ be multiplication invariant subspaces of $L^2(X;\mathcal{H})$ corresponding to an orthonormal basis $\mathscr{D}$ of $L^2(X)$. For a finite collection of functions $\mathcal{A} = \{\varphi_i\}_{i=1}^r$, assume $E_{\mathscr{D}}(\mathcal{A})$ is a Riesz basis for $\mathscr{V}$. Then the following holds:*

(i) ***Global setup:*** *If there exists $\mathcal{A}' = \{\varphi'_i\}_{i=1}^r$ in $L^2(X;\mathcal{H})$ such that $E_{\mathscr{D}}(\mathcal{A}')$ is a Riesz basis for $\mathscr{W}$ satisfying the following biorthogonality condition*

$$\langle M_\phi \varphi_i, M_{\phi'} \varphi'_{i'} \rangle = \delta_{i,i'} \delta_{\phi,\phi'}, \quad i, i' = 1, 2, \cdots, r; \quad \phi, \phi' \in \mathscr{D}, \tag{3.2.1}$$

then the infimum cosine angles of $\mathscr{V}$ and $\mathscr{W}$ are greater than zero, i.e.,

$$R(\mathscr{V}, \mathscr{W}) > 0 \text{ and } R(\mathscr{W}, \mathscr{V}) > 0. \tag{3.2.2}$$

Conversely if (3.2.2) holds true, then there exists $\mathcal{A}' = \{\varphi'_i\}_{i=1}^r$ in $L^2(X;\mathcal{H})$ such that $E_{\mathscr{D}}(\mathcal{A}')$ is a Riesz basis for $\mathscr{W}$ satisfying the biorthogonality condition (3.2.1). Moreover, the following reproducing formulas hold:

$$f = \sum_{\phi \in \mathscr{D}} \sum_{i=1}^r \langle f, M_\phi \varphi'_i \rangle M_\phi \varphi_i \text{ for all } f \in \mathscr{V}, \text{ and } g = \sum_{\phi \in \mathscr{D}} \sum_{i=1}^r \langle g, M_\phi \varphi_i \rangle M_\phi \varphi'_i \text{ for all } g \in \mathscr{W}.$$

(ii) ***Local setup:*** *If there exists $\mathcal{A}' = \{\varphi'_i\}_{i=1}^r$ in $L^2(X;\mathcal{H})$ such that for a.e. $x \in X$, $\{\varphi'_i(x)\}_{i=1}^r$ is a Riesz sequence in $\mathcal{H}$ satisfying the following biorthogonality condition*

$$\langle \varphi_i(x), \varphi'_{i'}(x) \rangle = \delta_{i,i'}, \quad i, i' = 1, 2, \ldots, r, \tag{3.2.3}$$

then the infimum cosine angles of $J_{\mathcal{A}}(x) = \operatorname{span}\{\varphi_i(x)\}_{i=1}^r$ *and* $J_{\mathcal{A}'}(x) = \operatorname{span}\{\varphi_i'(x)\}_{i=1}^r$ *are greater than zero, i.e.,*

$$R(J_{\mathcal{A}}(x), J_{\mathcal{A}'}(x)) > 0 \text{ and } R(J_{\mathcal{A}'}(x), J_{\mathcal{A}}(x)) > 0 \text{ a.e. } x \in X. \tag{3.2.4}$$

Conversely if (3.2.4) holds, there exists $\mathcal{A}' = \{\varphi_i'\}_{i=1}^r$ *in* $L^2(X;\mathcal{H})$ *such that for a.e.* $x \in X$, $\{\varphi_i'(x)\}_{i=1}^r$ *is a Riesz sequence in* $\mathcal{H}$ *satisfying the biorthogonality condition (3.2.3). Moreover, the following reproducing formulas hold for* $u \in J_{\mathcal{A}}(x)$, *and* $v \in J_{\mathcal{A}'}(x)$*:*

$$u = \sum_{i=1}^r \langle u, \varphi_i'(x)\rangle \varphi_i(x), \ \text{ and } v = \sum_{i=1}^r \langle v, \varphi_i(x)\rangle \varphi_i'(x) \text{ for a.e. } x \in X.$$

Before moving towards the proof of Theorem 3.2.1, we need the concept of supremum cosine angle [68]. For two subspaces V and W of a Hilbert space $\mathcal{H}$, the supremum cosine angle between them is:

$$S(V, W) = \sup_{v \in V\setminus\{0\}} \frac{\|P_W v\|}{\|v\|}.$$

The correlation between supremum and infimum cosine angle is related with the following: $R(V, W) = \sqrt{1 - S(V, W^{\perp})^2}$. One of the main uses of supremum cosine angle is to determine whether the addition of two closed subspaces is again closed or not. The sum of two closed subspaces V and W is again closed and $V \bigcap W = \{0\}$ if and only if $S(V, W) < 1$ [72, Theorem 2.1].

Proof. (i) **Global Setup:** Suppose $E_{\mathscr{D}}(\mathcal{A})$ and $E_{\mathscr{D}}(\mathcal{A}')$ are Riesz basis for $\mathscr{V}$ and $\mathscr{W}$, with constants A, B and A', B', respectively, and are biorthogonal. By [49, Theorem 2.3], we have $\mathcal{A}(x) = \{\varphi_i(x) : i = 1, 2, \ldots, r\}$ and $\mathcal{A}'(x) = \{\varphi_i'(x) : i = 1, 2, \ldots, r\}$ are Riesz bases for $J_{\mathcal{A}}(x)$ and $J_{\mathcal{A}'}(x)$ for a.e. $x \in X$. It suffices to show $R(\mathscr{V}, \mathscr{W}) = R(\mathscr{W}, \mathscr{V}) > 0$. The dual Riesz basis for $E_{\mathscr{D}}(\mathcal{A})$ in $\mathscr{V}$ is of the form $E_{\mathscr{D}}(\mathcal{A}^\#)$, where $\mathcal{A}^\# = \{\varphi_i^\# : i = 1, 2, \ldots, r\} \subseteq \mathscr{V}$. Therefore the orthogonal projection $P_{\mathscr{V}}$ onto $\mathscr{V}$ can be expressed as

$$P_{\mathscr{V}} f = \sum_{i=1}^r \sum_{\phi \in \mathscr{D}} \langle f, M_\phi \varphi_i^\#\rangle M_\phi \varphi_i = \sum_{i=1}^r \sum_{\phi \in \mathscr{D}} \langle f, M_\phi \varphi_i\rangle M_\phi \varphi_i^\# \text{ for all } f \in L^2(X;\mathcal{H}).$$

Observe that $P_{\mathscr{V}} M_\phi \varphi_i' = \varphi_i^\#$ for all $\phi \in \mathscr{D}$ and $i = 1, 2, \ldots, r$. For $f \in \mathscr{W}\setminus\{0\}$, we have $f = \sum_{i=1}^r \sum_{\phi \in \mathscr{D}} c_i^\phi M_\phi \varphi_i'$, where

$$A' \sum_{i=1}^r \sum_{\phi \in \mathscr{D}} |c_i^\phi|^2 \leqslant \|f\|^2 \leqslant B' \sum_{i=1}^r \sum_{\phi \in \mathscr{D}} |c_i^\phi|^2.$$

Then

$$P_{\mathscr{V}} f = \sum_{i=1}^{r} \sum_{\phi \in \mathscr{D}} c_i^{\phi} M_{\phi} \varphi_i^{\#} \text{ and } \frac{\|P_{\mathscr{V}} f\|^2}{\|f\|^2} \geqslant \frac{B^{-1} \sum_{i,\phi} |c_i^{\phi}|^2}{B' \sum_{i,\phi} |c_i^{\phi}|^2} = \frac{1}{BB'},$$

since $E_{\mathscr{D}}(\mathcal{A}^{\#})$ is a Riesz basis with constants B^{-1}, A^{-1}. Hence $R(\mathscr{V}, \mathscr{W}) \geqslant (BB')^{-1/2}$.

Conversely, Since $R(\mathscr{W}, \mathscr{V}) > 0$, then $R(\mathscr{W}, \mathscr{V})\|f\| \leqslant \|f\|$ for all $f \in \mathscr{V}$. Also since $E_{\mathscr{D}}(\mathcal{A})$ is a Riesz basis for $\mathscr{V}$, then the corresponding projection on $\mathscr{W}$, that is, $\{P_{\mathscr{W}} M_{\phi} \varphi_i : \phi \in \mathscr{D}, i = 1, 2, \ldots, r\}$ is a Riesz basis for $\mathscr{W}$. Since $R(\mathscr{V}, \mathscr{W}) > 0$, we get $\overline{\text{span}}\{P_{\mathscr{W}} M_{\phi} \varphi_i : \phi \in \mathscr{D}, i = 1, 2, \ldots, r\} = \mathscr{W}$, and by [19, Corrollary 5.14] there exists a dual Riesz basis for $\overline{\text{span}}\{P_{\mathscr{W}} M_{\phi} \varphi_i : \phi \in \mathscr{D}, i = 1, 2, \ldots, r\}$ of the multiplication generated form, i.e., $\{M_{\phi} \varphi_i' : \phi \in \mathscr{D}, i = 1, 2, \ldots, r\}$ for some φ_i' in $\mathscr{W}$. Thus we have

$$\begin{aligned}
\langle M_{\phi} \varphi_j, M_{\phi'} \varphi_i' \rangle &= \langle M_{\phi} \varphi_j, P_{\mathscr{W}} M_{\phi'} \varphi_i' \rangle \\
&= \langle P_{\mathscr{W}} M_{\phi} \varphi_j, M_{\phi'} \varphi_i' \rangle \\
&= \delta_{\phi,\phi'} \delta_{i,j}, \text{ where } \phi, \phi' \in \mathscr{D} \text{ and } i, j \in \{1, 2, \ldots, r\}.
\end{aligned}$$

Therefore the result follows.

(ii) **Local Setup:** For a.e. $x \in X$, let $\{\varphi_i(x)\}_{i=1}^r$ and $\{\varphi_i'(x)\}_{i=1}^r$ be Riesz bases for $J_{\mathcal{A}}(x)$ and $J_{\mathcal{A}'}(x)$, respectively, and they are biorthogonal. We now show this is equivalent to

$$J_{\mathcal{A}'}(x) \oplus J_{\mathcal{A}}(x)^{\perp} = \mathcal{H} \text{ and } J_{\mathcal{A}}(x) \oplus J_{\mathcal{A}'}(x)^{\perp} = \mathcal{H} \text{ for a.e. } x \in X. \tag{3.2.5}$$

Since $\{\varphi_i(x)\}_{i=1}^r$ and $\{\varphi_i'(x)\}_{i=1}^r$ are Riesz basis, then $J_{\mathcal{A}}(x) = \{u \in \mathcal{H} : u = \sum_{i=1}^r c_i \varphi_i(x)\}$ and $J_{\mathcal{A}'}(x) = \{v \in \mathcal{H} : v = \sum_{i=1}^r c_i \varphi_i'(x)\}$. Let $h \in J_{\mathcal{A}}(x) \bigcap J_{\mathcal{A}'}(x)^{\perp}$ then

$$h = \sum_{i=1}^{r} \langle h, \varphi_i(x) \rangle \varphi_i'(x) = 0,$$

hence $J_{\mathcal{A}'}(x) \bigcap J_{\mathcal{A}}(x)^{\perp} = \{0\}$. Let $w \in \mathcal{H}$, then $Pw := \sum_{i=1}^r \langle w, \varphi_i(x) \rangle \varphi_i'(x) \in J_{\mathcal{A}'}(x)$. By the biorthogonal property of $\{\varphi_i(x)\}_{i=1}^r$ and $\{\varphi_i'(x)\}_{i=1}^r$, we have $\langle w - Pw, \varphi_i(x) \rangle = 0$ for all $i = 1, \ldots, r$, i.e., $w - Pw \in J_{\mathcal{A}}(x)^{\perp}$. So $w = Pw + (w - Pw) \in J_{\mathcal{A}'}(x) + J_{\mathcal{A}}(x)^{\perp}$ which implies $J_{\mathcal{A}'}(x) + J_{\mathcal{A}}(x)^{\perp} = \mathcal{H}$. Combining, we have $J_{\mathcal{A}'}(x) \oplus J_{\mathcal{A}}(x)^{\perp} = \mathcal{H}$. In a similar way, interchanging the roll of $J_{\mathcal{A}}(x)$ and $J_{\mathcal{A}'}(x)$ in another part of (3.2.5), i.e., $J_{\mathcal{A}}(x) \oplus J_{\mathcal{A}'}(x)^{\perp} = \mathcal{H}$ for a.e. $x \in X$, can be shown.

Since $J_{\mathcal{A}}(x) + J_{\mathcal{A}'}(x)^{\perp}$ is closed and $J_{\mathcal{A}}(x) \bigcap J_{\mathcal{A}'}(x)^{\perp} = \{0\}$, by the [72, Theorem 2.1] the supremum cosine angle

$$S(J_{\mathcal{A}}(x), J_{\mathcal{A}'}(x)^{\perp}) := \sup\{|\langle v, w\rangle| : v \in J_{\mathcal{A}}(x), w \in J_{\mathcal{A}'}(x)^{\perp}, \|v\| = \|w\| = 1\} < 1.$$

Hence $R(J_{\mathcal{A}}(x), J_{\mathcal{A}'}(x)) = \sqrt{1 - S(J_{\mathcal{A}}(x), J_{\mathcal{A}'}(x)^{\perp})^2} > 0$. Interchanging the roll of $J_{\mathcal{A}}(x)$ and $J_{\mathcal{A}'}(x)$ in the above argument $R(J_{\mathcal{A}'}(x), J_{\mathcal{A}}(x)) > 0$.

For converse part, let $R(J_{\mathcal{A}}(x), J_{\mathcal{A}'}(x)) > 0$. Then $S(J_{\mathcal{A}}(x), J_{\mathcal{A}'}(x)^{\perp}) < 1$. Using [72, Theorem 2.1], we have $J_{\mathcal{A}}(x) + J_{\mathcal{A}'}(x)^{\perp}$ is closed and $J_{\mathcal{A}}(x) \bigcap J_{\mathcal{A}'}(x)^{\perp} = \{0\}$. In a similar way when $R(J_{\mathcal{A}'}(x), J_{\mathcal{A}(x)}) > 0$, then we can show $J_{\mathcal{A}'}(x) + J_{\mathcal{A}}(x)^{\perp}$ is closed and $J_{\mathcal{A}'}(x) \bigcap J_{\mathcal{A}}(x)^{\perp} = \{0\}$. Hence

$$J_{\mathcal{A}}(x) + J_{\mathcal{A}'}(x)^{\perp} = (J_{\mathcal{A}}(x) + J_{\mathcal{A}'}(x)^{\perp})^{\perp\perp} = \left(J_{\mathcal{A}}(x)^{\perp} \bigcap J_{\mathcal{A}'}(x)\right)^{\perp} = \mathcal{H}.$$

So $\mathcal{H} = J_{\mathcal{A}}(x) \oplus J_{\mathcal{A}'}(x)^{\perp}$. In a similar way, $\mathcal{H} = J_{\mathcal{A}'}(x) \oplus J_{\mathcal{A}}(x)^{\perp}$.

Assume $J_{\mathcal{A}'}(x) \bigoplus J_{\mathcal{A}}(x)^{\perp} = \mathcal{H}$. Let for a.e. $x \in X$, $\{\varphi_i(x)\}_{i=1}^r$ and $\{h_i(x)\}_{i=1}^r$ be the dual Riesz bases for $J_{\mathcal{A}'}(x)$, i.e., $\langle \varphi_i(x), h_j(x)\rangle = \delta_{i,j}$. Let $S : J_{\mathcal{A}}(x) \to J_{\mathcal{A}}(x)$ be the frame operator, then consider

$$\psi_i(x) := S^{-1}\varphi_i(x), 1 \leqslant i \leqslant r.$$

Now the map $P_{J_{\mathcal{A}(x)}} : \mathcal{H} \to \mathcal{H}$ by $P_{J_{\mathcal{A}}(x)} f := \sum_{i=1}^r \langle f, \varphi_i(x)\rangle \psi_i(x)$ is the orthogonal projection of $\mathcal{H}$ on $J_{\mathcal{A}}(x)$. Consider $\mathscr{P} := P_{J_{\mathcal{A}}(x)}|_{J_{\mathcal{A}'}(x)}$. If $f \in J_{\mathcal{A}'}(x)$ and $\mathscr{P}(f) = 0$, then $f \in J_{\mathcal{A}'}(x) \bigcap J_{\mathcal{A}}(x)^{\perp} = \{0\}$ so $\mathscr{P}$ is injective and $\mathscr{P}(J_{\mathcal{A}'}(x)) = P_{J_{\mathcal{A}}(x)}(J_{\mathcal{A}'}(x)) = P_{J_{\mathcal{A}}(x)}(J_{\mathcal{A}'}(x) + J_{\mathcal{A}}(x)^{\perp}) = P_{J_{\mathcal{A}}(x)}(\mathcal{H}) = J_{\mathcal{A}}(x)$. Hence $\mathscr{P}$ is bounded invertible operator. Define

$$\varphi_i'(x) := \mathscr{P}^{-1}(\varphi_i(x)), 1 \leqslant i \leqslant r.$$

Then $\{\varphi_i'(x)\}_{i=1}^r$ is the required Riesz basis for $J_{\mathcal{A}'}(x)$, satisfying the biorthogonality condition.

□

The development of the theory of duals for a continuous frame on a locally compact group (need not be abelian) translated by its closed abelian subgroup is the novel aspect of general machinery developed for $L^2(X; \mathcal{H})$ in this chapter and the earlier one. In the next chapter, we will explore this topic and we will also demonstrate how our approach towards a measure theoretic abstraction can help us to offer various characterizations of duals for the locally compact groups.

CHAPTER 4

DUAL FRAMES BY THE ACTION OF AN ABELIAN GROUP

In this chapter, we characterize Γ-TG duals of a continuous frame on locally compact group $\mathscr{G}$ by the action of its closed abelian subgroup Γ. We characterize such results using the Zak transform $\mathcal{Z}$ for the pair $(\mathscr{G}, \Gamma)$. When $\mathscr{G}$ becomes an abelian group $\mathcal{G}$, the fiberization map is used to characterize these duals by the action of its closed subgroup Λ. The vast majority of these traditional results for integer shifts were merely dependent on the fiberization map [37, 42, 44, 46]. We are investigating the study of alternate duals in the setup of general locally compact abelian and non-abelian groups in which we do not require Γ to be discrete or Λ to be uniform lattice or co-compact (i.e., $\mathcal{G}/\Lambda$-compact).

4.1. Translation generated duals of a frame

For a second countable locally compact group $\mathscr{G}$ (not necessarily abelian) and a closed abelian subgroup Γ of $\mathscr{G}$, let us recall a Γ-*translation generated* (Γ-*TG*) system $\mathcal{E}^{\Gamma}(\mathscr{A})$ and its associated Γ-translation invariant (Γ-TI) space $\mathcal{S}^{\Gamma}(\mathscr{A})$ from (1.3.1) for a family of functions $\mathscr{A} \subseteq L^2(\mathscr{G})$ by the action of Γ, i.e.,

$$\mathcal{E}^{\Gamma}(\mathscr{A}) := \{L_{\xi}\varphi : \varphi \in \mathscr{A}, \xi \in \Gamma\} \text{ and } \mathcal{S}^{\Gamma}(\mathscr{A}) := \overline{\operatorname{span}}\mathcal{E}^{\Gamma}(\mathscr{A}),$$

where for $\eta \in \mathscr{G}$, the *left translation* L_{η} on $L^2(\mathscr{G})$ is defined by

$$(L_{\eta}f)(\gamma) = f(\eta^{-1}\gamma), \ \gamma \in \mathscr{G}.$$

This chapter is a part of the following manuscripts:

S. Sarkar, N. K. Shukla, *Translation generated oblique dual frames on locally compact groups*, **Linear Multilinear Algebra**, (2023), doi:10.1080/03081087.2023.2173718, 32 pages.

S. Sarkar, N. K. Shukla, *A characterization of MG dual frames using infimum cosine angle*, arXiv:2301.07448.

By Γ-*translation invariant* (Γ-*TI*) space V, we mean $L_\xi f \in V$ for all $f \in V$ and $\xi \in \Gamma$, where V is a closed subspace of $L^2(\mathscr{G})$.

We now define the translation generated dual and its types in $L^2(\mathscr{G})$.

Definition 4.1.1. Let $\mathcal{N}$ be a complete, σ-finite measure space. Suppose $\mathscr{A} = \{\varphi_t\}_{t\in\mathcal{N}}$, $\mathscr{A}' = \{\psi_t\}_{t\in\mathcal{N}}$ are families of functions in $L^2(\mathscr{G})$ such that $\mathcal{E}^\Gamma(\mathscr{A})$ is a continuous frame for $\mathcal{S}^\Gamma(\mathscr{A})$, and $\mathcal{E}^\Gamma(\mathscr{A}')$ is Bessel. We call $\mathcal{E}^\Gamma(\mathscr{A}')$,

(i) an *alternate* Γ-*TG dual* (simply, *alternate TG-dual*) for $\mathcal{E}^\Gamma(\mathscr{A})$ if it is an alternate dual for $\mathcal{E}^\Gamma(\mathscr{A})$ in the sense of Definition 1.1.1 (a).

(ii) an *oblique* Γ-*TG dual* (simply, *oblique TG-dual*) for $\mathcal{E}^\Gamma(\mathscr{A})$ if it is an oblique dual for $\mathcal{E}^\Gamma(\mathscr{A})$ in the sense of Definition 1.1.1 (b).

(iii) a Γ-*TG dual of type-I* (simply, *type-I TG dual*) for $\mathcal{E}^\Gamma(\mathscr{A})$ if it is a *type-I* for $\mathcal{E}^\Gamma(\mathscr{A})$ in the sense of Definition 1.1.1 (c).

(iv) a Γ-*TG dual of type-II* (simply, *type-II TG dual*) for $\mathcal{E}^\Gamma(\mathscr{A})$ if it is a *type-II* for $\mathcal{E}^\Gamma(\mathscr{A})$ in the sense of Definition 1.1.1 (d).

(v) a Γ-*TG dual frame* (simply, *TG dual frame*) for $\mathcal{E}^\Gamma(\mathscr{A})$ if it is a *dual frame* for $\mathcal{E}^\Gamma(\mathscr{A})$ in the sense of Definition 1.1.1 (e).

For the stable decomposition and reproducing formula of a signal/image, we study alternate (oblique) Γ-TG dual, Γ-TG dual of type-I and type-II, and Γ-TG dual frames for a continuous frame $\mathcal{E}^\Gamma(\mathscr{A})$ in $L^2(\mathscr{G})$ by the action of a closed abelian subgroup Γ of $\mathscr{G}$ [see Theorems 4.1.2 - 4.1.6].

We now return to our primary objective to find a possible collection of functions which generates a translation-invariant (TI) space in $L^2(\mathscr{G})$ such that the reproducing formula (1.3.2) is satisfied. The system generated by translations of such collections need not to be a frame for its associated TI space. We start the investigation by discussing the Zak transform and fiberization [47, 49]. By applying these operators, we obtain alternate (oblique) TG-dual's characterization for a continuous frame and its uniqueness in the TI subspaces of $L^2(\mathscr{G})$ along with prototype examples.

4.1.1. Zak transform, fiberization and Γ-TI space

Let $\mathscr{G}$ be a second countable locally compact group (not necessarily abelian) with Haar measure $\mu_{\mathscr{G}}$ such that $\mathscr{G}$ contains a closed abelian subgroup Γ. For $x \in \mathscr{G}$, a right coset of Γ in $\mathscr{G}$ with respect to x is denoted by Γx, and for a function $f : \mathscr{G} \to \mathbb{C}$, we define a complex valued function $f^{\Gamma x}$ on Γ by

$$f^{\Gamma x}(\gamma) = f(\gamma\, \Xi(\Gamma x)),\ \gamma \in \Gamma,$$

where the space of orbits $\Gamma \backslash \mathscr{G} = \{\Gamma x : x \in \mathscr{G}\}$ is the set of all right cosets of Γ in $\mathscr{G}$, and $\Xi : \Gamma\backslash\mathscr{G} \to \mathscr{G}$ is a *Borel section* for the quotient space $\Gamma\backslash\mathscr{G}$ whose existence is guaranteed by [60, Lemma 1.1]. Then for $f^{\Gamma x} \in L^1(\Gamma)$, the Fourier transform is

$$\widehat{f^{\Gamma x}}(\alpha) = \int_{\Gamma} f^{\Gamma x}(\gamma)\alpha(\gamma^{-1})\, d\mu_{\Gamma}(\gamma),\ \alpha \in \widehat{\Gamma}.$$

Therefore, the *Zak transformation* $\mathcal{Z}$ of $f \in L^2(\mathscr{G})$ for the pair $(\mathscr{G}, \Gamma)$ is defined by

$$(\mathcal{Z}f)(\alpha)(\Gamma x) = \widehat{f^{\Gamma x}}(\alpha) \text{ a.e. } \alpha \in \widehat{\Gamma} \text{ and } \Gamma x \in \Gamma\backslash\mathscr{G}, \tag{4.1.1}$$

which is a unitary linear transformation from $L^2(\mathscr{G})$ to $L^2(\widehat{\Gamma}; L^2(\Gamma\backslash\mathscr{G}))$ [49]. Since the space $L^2(\widehat{\Gamma}; L^2(\Gamma\backslash\mathscr{G}))$ can be identified with the space $L^2(\widehat{\Gamma} \times \Gamma\backslash\mathscr{G})$, we can interpret $\mathcal{Z}$ as $\tilde{\mathcal{Z}}$ from $L^2(\mathscr{G})$ to $L^2(\widehat{\Gamma} \times \Gamma\backslash\mathscr{G})$ by $(\tilde{\mathcal{Z}}f)(\alpha, \Gamma x) = (\mathcal{Z}f)(\alpha)(\Gamma x)$. When $\mathscr{G}$ becomes abelian group $\mathcal{G}$, then we also denotes the corresponding Zak transform as $\tilde{\mathcal{Z}}$.

Note that the map $\mathcal{Z}$ is closely associated with the fiberization map $\mathscr{T}$ when $\mathscr{G}$ becomes abelian. For a second countable LCA group $\mathcal{G}$ and its closed subgroup Λ, the *fiberization* $\mathscr{T}$ is a unitary map from $L^2(\mathcal{G})$ to $L^2(\widehat{\mathcal{G}}/\Lambda^{\perp}; L^2(\Lambda^{\perp}))$ given by

$$(\mathscr{T}f)(\beta\Lambda^{\perp})(\omega) = \widehat{f}(\omega\ \zeta(\beta\Lambda^{\perp})),\ \omega \in \Lambda^{\perp},\ \beta \in \widehat{\mathcal{G}}$$

for $f \in L^2(\mathcal{G})$, where $\Lambda^{\perp} := \{\beta \in \widehat{\mathcal{G}} : \beta(\lambda) = 1 \text{ for all } \lambda \in \Lambda\}$, $\Lambda^{\perp}\backslash\widehat{\mathcal{G}} = \widehat{\mathcal{G}}/\Lambda^{\perp}$ and $\zeta : \widehat{\mathcal{G}}/\Lambda^{\perp} \to \widehat{\mathcal{G}}$ is a Borel section which maps compact sets to pre-compact sets. For more details about the Zak transformation we refer [9, 10, 49].

Observe that $\mathcal{Z}$ intertwines the left translation with the multiplication operators, i.e., for $f \in L^2(\mathscr{G})$,

$$(\mathcal{Z}L_{\gamma}f)(\alpha) =(M_{\phi_{\gamma}}\mathcal{Z}f)(\alpha) \text{ for a.e. } \alpha \in \widehat{\Gamma} \text{ and } \gamma \in \Gamma, \tag{4.1.2}$$

where $M_{\phi_{\gamma}}$ is the multiplication operator on $L^2(\widehat{\Gamma}; L^2(\Gamma\backslash\mathscr{G}))$, $\phi_{\gamma}(\alpha) = \overline{\alpha(\gamma)}$ and $\phi_{\gamma} \in L^{\infty}(\widehat{\Gamma})$ for each $\gamma \in \Gamma$. Therefore, our goal can be established by converting the problem of Γ-TI space $\mathcal{S}^{\Gamma}(\mathscr{A})$ into the MI space on $L^2(X; \mathcal{H})$ with the help of $\mathcal{Z}$, where $X = \widehat{\Gamma}$

and $\mathcal{H} = L^2(\Gamma\backslash\mathscr{G})$. Similar situation will arise in case of abelian group $\mathcal{G}$ and its closed subgroup Λ since the map $\mathscr{T}$ satisfies the following intertwining relation for $f \in L^2(\mathcal{G})$:

$$(4.1.3) \qquad (\mathscr{T}L_\lambda f)(\beta\Lambda^\perp) = (M_{\phi_\lambda}\mathscr{T}f)(\beta\Lambda^\perp) \text{ for a.e. } \beta \in \widehat{\mathcal{G}} \text{ and } \lambda \in \Lambda,$$

where $\phi_\lambda \in L^\infty(\widehat{\mathcal{G}}/\Lambda^\perp)$ given by $\phi_\lambda(\beta\Lambda^\perp) = \overline{\beta(\lambda)}$ and the multiplication operator M_{ϕ_λ} is defined on $L^2(\widehat{\mathcal{G}}/\Lambda^\perp; L^2(\Lambda^\perp))$.

Now we modify the Theorem 2.1.4 in the setup of $L^2(\widehat{\Gamma}; L^2(\Gamma\backslash\mathscr{G}))$ using the Zak transform which will provide a classification of all Γ-TI spaces $\mathcal{S}^\Gamma(\mathscr{A})$ in $L^2(\mathscr{G})$. For a family of functions $\mathscr{A} = \{\varphi_t\}_{t\in\mathcal{N}}$ in $L^2(\mathscr{G})$, we recall Γ-translation generated (Γ-TG) system $\mathcal{E}^\Gamma(\mathscr{A})$ by left action of Γ and its associated Γ-translation invariant (Γ-TI) space $\mathcal{S}^\Gamma(\mathscr{A})$, i.e., $\mathcal{E}^\Gamma(\mathscr{A}) := \{L_\gamma\varphi_t : \gamma \in \Gamma, t \in \mathcal{N}\}$ and $\mathcal{S}^\Gamma(\mathscr{A}) := \overline{\operatorname{span}}\{L_\gamma\varphi_t : \gamma \in \Gamma, t \in \mathcal{N}\}$, where $(\mathcal{N}, \mu_\mathcal{N})$ is a complete, σ-finite measure space. In this setup, the range function is

$$J : \widehat{\Gamma} \to \{\text{closed subspaces of } L^2(\Gamma\backslash\mathscr{G})\},$$

and the orthogonal projection for each $\alpha \in \widehat{\Gamma}$ is $P_J(\alpha) : L^2(\Gamma\backslash\mathscr{G}) \to J(\alpha)$, and hence the associated closed subspace V_J given in (2.1.1) can be written as:

$$V_J = \left\{f \in L^2(\mathscr{G}) : (\mathcal{Z}f)(\alpha) \in J(\alpha) \text{ for a.e. } \alpha \in \widehat{\Gamma}\right\}.$$

Using the range function J and associated space V_J we can write Theorem 2.1.4. For Γ-TI space $\mathcal{S}^\Gamma(\mathscr{A})$ in $L^2(\mathscr{G})$, the corresponding range function J is such that, for a.e. $\alpha \in \widehat{\Gamma}$, $J(\alpha)$ is defined by

$$(4.1.4) \qquad J(\alpha) = \overline{\operatorname{span}}\{(\mathcal{Z}f)(\alpha) : f \in \mathscr{A}_0\} =: J_\mathscr{A}(\alpha),$$

for some countable dense subset $\mathscr{A}_0$ of $\mathscr{A}$ in $L^2(\mathscr{G})$ [Denote $\mathcal{Z}\mathscr{A} := \{\mathcal{Z}f : f \in \mathscr{A}\}$]. This follows by noting that the set $\mathcal{D}$ defined by

$$(4.1.5) \qquad \mathcal{D} = \left\{\phi_\gamma \in L^\infty(\widehat{\Gamma}) : \gamma \in \Gamma, \phi_\gamma(\alpha) = \overline{\alpha(\gamma)} \text{ for } \alpha \in \widehat{\Gamma}\right\}$$

is a determining set for $L^1(\widehat{\Gamma})$ since for $f \in L^1(\widehat{\Gamma})$, we have $0 = \int_{\widehat{\Gamma}} f(\alpha)\phi_\gamma(\alpha)\, d\mu_{\widehat{\Gamma}}(\alpha) = \int_{\widehat{\Gamma}} f(\alpha)\overline{\alpha(\gamma)}\, d\mu_{\widehat{\Gamma}}(\alpha)$ for all $\gamma \in \Gamma$ which implies $f = 0$. Note that $\{\phi_\gamma\}_{\gamma\in\Gamma}$ is the collection of all characters on $\widehat{\Gamma}$ using the identification between $\widehat{\widehat{\Gamma}}$ and Γ from Pontryagin Duality theorem.

For the abelian group $\mathcal{G}$ and its closed subgroup Λ, the range function is $J : \widehat{\mathcal{G}}/\Lambda^{\perp} \to$ {closed subspaces of $L^2(\Lambda^{\perp})$} and its associated space V_J is

$$V_J = \left\{ f \in L^2(\mathcal{G}) : (\mathscr{T} f)(\beta\Lambda^{\perp}) \in J(\beta\Lambda^{\perp}) \text{ for a.e. } \beta\Lambda^{\perp} \in \widehat{\mathcal{G}}/\Lambda^{\perp} \right\}.$$

Then, we can write Theorem 2.1.4 for an LCA group, where for the Λ-TI space $\mathcal{S}^{\Lambda}(\mathscr{A})$, the corresponding range function J is given by: for a.e. $\beta\Lambda^{\perp} \in \widehat{\mathcal{G}}/\Lambda^{\perp}$ the space $J(\beta\Lambda^{\perp})$ is defined by

$$J(\beta\Lambda^{\perp}) = \overline{\operatorname{span}} \left\{ (\mathscr{T} f)(\beta\Lambda^{\perp}) : f \in \mathscr{A}_0 \right\} =: J^{\mathcal{G}}_{\mathscr{A}}(\beta\Lambda^{\perp}), \tag{4.1.6}$$

for some countable dense subset $\mathscr{A}_0$ of $\mathscr{A}$ in $L^2(\mathcal{G})$ [Denote $\mathscr{T}\mathscr{A} := \{\mathscr{T} f : f \in \mathscr{A}\}$]. The determining set $\mathcal{D}$ for $L^1(\widehat{\mathcal{G}}/\Lambda^{\perp})$ is given by

$$\mathcal{D} = \left\{ \phi_\lambda \in L^\infty(\widehat{\mathcal{G}}/\Lambda^{\perp}) : \lambda \in \Lambda, \phi_\lambda(\beta\Lambda^{\perp}) = \overline{\beta(\lambda)} \text{ for } \beta\Lambda^{\perp} \in \widehat{\mathcal{G}}/\Lambda^{\perp} \right\} := \mathcal{D}^{\mathcal{G}}. \tag{4.1.7}$$

4.1.2. Characterization of Γ-TG duals for continuous frames

Now, we present the most important outcomes of our characterizations which are the applications of the theory developed in Chapter 2.2.

The following Theorems 4.1.2, 4.1.4, 4.1.5, and 4.1.6 are generalizations to a locally compact group of the results [44, Theorem 4.2] and [46, Theorem 5].

Theorem 4.1.2. *Let $\mathscr{A} = \{\varphi_t\}_{t \in \mathcal{N}}$ be a collection of functions in $L^2(\mathscr{G})$ such that the TG system $\mathcal{E}^{\Gamma}(\mathscr{A})$ is a continuous frame for the TI space $\mathcal{S}^{\Gamma}(\mathscr{A})$, and $\mathscr{A}$ has a countable dense subset $\mathscr{A}_0$ for which $J_{\mathscr{A}}(\alpha)$ is defined by (4.1.4) for each $\alpha \in \widehat{\Gamma}$, where $(\mathcal{N}, \mu_{\mathcal{N}})$ is a complete, σ-finite measure space. If $\mathscr{A}' = \{\psi_t : t \in \mathcal{N}\}$ is a collection of functions in $L^2(\mathscr{G})$ such that the TG system $\mathcal{E}^{\Gamma}(\mathscr{A}')$ is Bessel in $L^2(\mathscr{G})$, then the following are equivalent:*

(i) *$\mathcal{E}^{\Gamma}(\mathscr{A}')$ is an alternate TG-dual for $\mathcal{E}^{\Gamma}(\mathscr{A})$, i.e., for all $f \in \mathcal{S}^{\Gamma}(\mathscr{A})$, we have (in the weak sense)*

$$f = \int_{\mathcal{N}} \int_{\Gamma} \langle f, L_\gamma \psi_t \rangle L_\gamma \varphi_t \, d\mu_\Gamma(\gamma) \, d\mu_{\mathcal{N}}(t).$$

(ii) *For a.e. $\alpha \in \widehat{\Gamma}$, the system $(\mathcal{Z}\mathscr{A}')(\alpha) := \{\mathcal{Z}\psi_t(\alpha) : t \in \mathcal{N}\}$ is an alternate dual for the frame $(\mathcal{Z}\mathscr{A})(\alpha) := \{\mathcal{Z}\varphi_t(\alpha) : t \in \mathcal{N}\}$ of $J_{\mathcal{A}}(\alpha)$, i.e.,*

$$h = \int_{\mathcal{N}} \langle h, \mathcal{Z}\psi_t(\alpha) \rangle \mathcal{Z}\varphi_t(\alpha) \, d\mu_{\mathcal{N}}(t) \text{ for all } h \in J_{\mathscr{A}}(\alpha).$$

When the pair $(\mathscr{G}, \Gamma)$ becomes an abelian pair $(\mathcal{G}, \Lambda)$, let $J^{\mathcal{G}}_{\mathscr{A}}(\beta\Lambda^\perp)$ be defined by (4.1.6), then the above (ii) is equivalent to the following:

(ii') for a.e. $\beta\Lambda^\perp \in \widehat{\mathcal{G}}/\Lambda^\perp$ and $h \in J^{\mathcal{G}}_{\mathscr{A}}(\beta\Lambda^\perp)$,

$$h = \int_{\mathcal{N}} \langle h, \mathscr{T}\psi_t(\beta\Lambda^\perp)\rangle \mathscr{T}\varphi_t(\beta\Lambda^\perp)\, d\mu_{\mathcal{N}}(t),$$

i.e., the system $(\mathscr{T}\mathscr{A}')(\beta\Lambda^\perp) = \{\mathscr{T}\psi_t(\beta\Lambda^\perp) : t \in \mathcal{N}\}$ is an alternate dual for the frame $(\mathscr{T}\mathscr{A})(\beta\Lambda^\perp) = \{\mathscr{T}\varphi_t(\beta\Lambda^\perp) : t \in \mathcal{N}\}$ of $J^{\mathcal{G}}_{\mathscr{A}}(\beta\Lambda^\perp)$.

Proof. For all $f \in \mathcal{S}^\Gamma(\mathscr{A})$, the expression $f = \int_{\mathcal{N}}\int_\Gamma \langle f, L_\gamma\psi_t\rangle L_\gamma\varphi_t\, d\mu_\Gamma(\gamma)\, d\mu_{\mathcal{N}}(t)$ (in the weak sense) is equivalent to the following :

$$\mathcal{Z}f = \int_{\mathcal{N}}\int_\Gamma \langle \mathcal{Z}f, M_{\phi_\gamma}\mathcal{Z}\psi_t\rangle M_{\phi_\gamma}\mathcal{Z}\varphi_t\, d\mu_\Gamma(\gamma)\, d\mu_{\mathcal{N}}(t),$$

follows from the intertwining relation (4.1.2). Therefore from the part (i) of Theorem 2.2.4, the MG system $E_{\mathcal{D}}(\mathcal{Z}\mathscr{A}')$ is an alternate MG-dual for $E_{\mathcal{D}}(\mathcal{Z}\mathscr{A})$ in $L^2(X;\mathcal{H})$ if and only if the system $(\mathcal{Z}\mathscr{A}')(\alpha)$ is an alternate dual for $(\mathcal{Z}\mathscr{A})(\alpha)$ a.e. $\alpha \in \widehat{\Gamma}$. Here, $X = \widehat{\Gamma}$, $\mathcal{H} = L^2(\Gamma\backslash\mathscr{G})$, and the set $\mathcal{D}$ defined by (4.1.5) is a Parseval determining set for $L^1(\widehat{\Gamma})$, follows by noting (2.1.2) and

$$\int_\Gamma \left|\int_{\widehat{\Gamma}} H(\alpha)\phi_\gamma(\alpha)\, d\mu_{\widehat{\Gamma}}(\alpha)\right|^2 d\mu_\Gamma(\gamma) = \int_{\widehat{\widehat{\Gamma}}} \left|\int_{\widehat{\Gamma}} H(\alpha)\overline{\alpha(\gamma)}\, d\mu_{\widehat{\Gamma}}(\alpha)\right|^2 d\mu_\Gamma(\gamma)$$

for $H \in L^1(\widehat{\Gamma})$ due to Pontryagin Duality theorem.

The moreover part follows by replacing $\mathcal{Z}$ with the fiberization $\mathscr{T}$, $\mathcal{D}$ with $\mathcal{D}^{\mathcal{G}}$ defined in (4.1.7), and also choosing $X = \widehat{\mathcal{G}}/\Lambda^\perp$ and $\mathcal{H} = L^2(\Lambda^\perp)$ in the above argument. □

Remark 4.1.3. In case of $\mathscr{A} = \{\varphi\}$ and $\mathscr{A}' = \{\psi\}$, the above result says that the TG system $\mathcal{E}^\Gamma(\mathscr{A}')$ is an alternate TG-dual for $\mathcal{E}^\Gamma(\mathscr{A})$ if and only if for a.e. $\alpha \in \widehat{\Gamma}$ such that $J_{\mathscr{A}}(\alpha) \neq 0$, we have $\langle \mathcal{Z}\varphi(\alpha), \mathcal{Z}\psi(\alpha)\rangle = 1$. In addition if $\mathscr{A} = \mathscr{A}' = \{\varphi\}$, $\mathcal{E}^\Gamma(\mathscr{A})$ is a Parseval frame (coherent state) for $\mathcal{S}^\Gamma(\mathscr{A})$ if and only if $\|\mathcal{Z}\varphi(\alpha)\| = 1$ for a.e. $\alpha \in \widehat{\Gamma}$ such that $J_{\mathscr{A}}(\alpha) \neq 0$. The same can be written for the pair $(\mathcal{G}, \Lambda)$ using a fiberization map.

Theorem 4.1.4. *Under the standing hypotheses mentioned in Theorem 4.1.2, $\mathcal{E}^\Gamma(\mathscr{A}')$ is an oblique TG-dual (TG-dual frame) for $\mathcal{E}^\Gamma(\mathscr{A})$ if and only if for a.e. $\alpha \in \widehat{\Gamma}$, the system $(\mathcal{Z}\mathscr{A}')(\alpha)$ is an oblique dual (dual frame) for the frame $(\mathcal{Z}\mathscr{A})(\alpha)$ of $J_{\mathscr{A}}(\alpha)$.*

Moreover, for an abelian pair $(\mathcal{G}, \Lambda)$, let $J^{\mathcal{G}}_{\mathscr{A}}(\beta\Lambda^{\perp})$ be defined by (4.1.6) then $(\mathscr{T}\mathscr{A}')(\beta\Lambda^{\perp})$ is an oblique dual (dual frame) for the frame $(\mathscr{T}\mathscr{A})(\beta\Lambda^{\perp})$ of $J^{\mathcal{G}}_{\mathscr{A}}(\beta\Lambda^{\perp})$ for a.e. $\beta\Lambda^{\perp} \in \widehat{\mathcal{G}}/\Lambda^{\perp}$.

Proof. This follows on the same line of proof of Theorem 4.1.2 along with parts (ii) and (iv) of Theorem 2.2.4. □

Theorem 4.1.5. *Under the standard assumptions in Theorem 4.1.2, $\mathcal{E}^{\Gamma}(\mathscr{A}')$ is a TG-dual of type-I for $\mathcal{E}^{\Gamma}(\mathscr{A})$ if and only if for a.e. $\alpha \in \widehat{\Gamma}$, $(\mathcal{Z}\mathscr{A}')(\alpha)$ is a type-I dual for the frame $(\mathcal{Z}\mathscr{A})(\alpha)$ of $J_{\mathscr{A}}(\alpha)$.*

Moreover, when the pair $(\mathscr{G}, \Gamma)$ becomes an abelian pair $(\mathcal{G}, \Lambda)$, let $J^{\mathcal{G}}_{\mathscr{A}}(\beta\Lambda^{\perp})$ be defined by (4.1.6). Then, the last statement is equivalent to: for a.e. $\beta\Lambda^{\perp} \in \widehat{\mathcal{G}}/\Lambda^{\perp}$ the system $(\mathscr{T}\mathscr{A}')(\beta\Lambda^{\perp})$ is a dual of type-I for the frame $(\mathscr{T}\ \mathscr{A})(\beta\Lambda^{\perp})$ of $J^{\mathcal{G}}_{\mathscr{A}}(\beta\Lambda^{\perp})$.

Proof. This follows on the same line of proof of Theorem 4.1.2 along with part (iii) of Theorem 2.2.4 and the observation $L_\gamma \psi_t \in \mathcal{S}^{\Gamma}(\mathscr{A})$ if and only if $\mathcal{Z} L_\gamma \psi_t \in S_{\mathcal{D}}(\mathcal{Z}\mathscr{A})$ for all $(\gamma, t) \in \Gamma \times \mathcal{N}$. Similarly, we can proof for the fiberization map. □

Theorem 4.1.6. *In addition to the standing hypotheses mentioned in Theorem 4.1.2, let $\mathcal{N}$ and Γ be two countable families and discrete sets having counting measures, respectively. Then, $\mathcal{E}^{\Gamma}(\mathscr{A}')$ is a TG-dual of type-II for $\mathcal{E}^{\Gamma}(\mathscr{A})$ if and only if for a.e. $\alpha \in \widehat{\Gamma}$, $(\mathcal{Z}\mathscr{A}')(\alpha)$ is a type-II dual for the frame $(\mathcal{Z}\mathscr{A})(\alpha)$ of $J_{\mathscr{A}}(\alpha)$, where $J_{\mathscr{A}}(\alpha) = \overline{\operatorname{span}}\{(\mathcal{Z}\varphi_t)(\alpha) : t \in \mathcal{N}\}$.*

Moreover, when the pair $(\mathscr{G}, \Gamma)$ becomes an abelian pair $(\mathcal{G}, \Lambda)$ and let $J^{\mathcal{G}}_{\mathscr{A}}(\beta\Lambda^{\perp})$ be defined by $J^{\mathcal{G}}_{\mathscr{A}}(\beta\Lambda^{\perp}) = \overline{\operatorname{span}}\{(\mathscr{T}\varphi_t)(\beta\Lambda^{\perp}) : t \in \mathcal{N}\}$, then the above characterization is true just by replacing the system $(\mathcal{Z}\mathscr{A})(\alpha)$ with the system $(\mathscr{T}\mathscr{A})(\beta\Lambda^{\perp})$ as well as $J_{\mathscr{A}}(\alpha)$ with $J^{\mathcal{G}}_{\mathscr{A}}(\beta\Lambda^{\perp})$ for a.e. $\beta\Lambda^{\perp} \in \widehat{\mathcal{G}}/\Lambda^{\perp}$.

Proof. Observe that the Parseval determining set $\mathcal{D}$ for $L^1(\widehat{\Gamma})$ (defined by (4.1.5)) can be identified with $\widehat{\widehat{\Gamma}}$. Therefore, the set $\mathcal{D}$ is an orthonormal Parseval determining set since $\mathcal{D}$ is an orthonormal basis for $L^2(\widehat{\Gamma})$ due to the discrete set Γ. Thus, we have the result on the same line of the proof of Theorem 4.1.2 along with compactness of $\widehat{\Gamma}$, Theorem 2.2.10, and the observation $T_{\mathcal{E}^{\Gamma}(\mathscr{A}')}(\mathcal{S}^{\Gamma}(\mathscr{A}')) \subset T_{\mathcal{E}^{\Gamma}(\mathscr{A})}(\mathcal{S}^{\Gamma}(\mathscr{A}))$ if and only if $T_{E_{\mathcal{D}}(\mathcal{Z}\mathscr{A}')}(S_{\mathcal{D}}(\mathcal{Z}\mathscr{A}')) \subset T_{E_{\mathcal{D}}(\mathcal{Z}\mathscr{A})}(S_{\mathcal{D}}(\mathcal{Z}\mathscr{A}))$. The observation follows by noting $\{\langle f, L_\gamma \psi_t\rangle\}_{\gamma\in\Gamma, t\in\mathcal{N}} \in \operatorname{range} T_{\mathcal{E}^{\Gamma}(\mathscr{A})}$ for all $f \in S_{\mathcal{D}}(\mathscr{A}')$ if and only if there exists $g \in S_{\mathcal{D}}(\mathscr{A})$ such that for all $\gamma \in \Gamma$ and $t \in \mathcal{N}$, we have

$$\langle f, L_\gamma \psi_t\rangle = \langle g, L_\gamma \varphi_t\rangle, \text{ i.e., } \langle \mathcal{Z}f, M_{\phi_\gamma}\mathcal{Z}\psi_t\rangle = \langle \mathcal{Z}g, M_{\phi_\gamma}\mathcal{Z}\varphi_t\rangle.$$

Similarly, we can prove the moreover part. □

Next, we are presenting the characterizations when TG duals for a frame become the canonical dual frame in $L^2(\mathscr{G})$. Note that the system $\mathcal{E}^\Gamma(\tilde{\mathscr{A}}) := \{L_\gamma \tilde{\varphi}_t : \gamma \in \Gamma, \tilde{\varphi}_t = (S_{\mathcal{E}^\Gamma(\mathscr{A})}|_{\mathcal{S}^\Gamma(\mathscr{A})})^{-1}\varphi_t, t \in \mathcal{N}\}$ is the canonical dual for a frame $\mathcal{E}^\Gamma(\mathscr{A})$ of $\mathcal{S}^\Gamma(\mathscr{A})$, follows by the commutativity of the left translation operator L_γ and the frame operator $S_{\mathcal{E}^\Gamma(\mathscr{A})}|_{\mathcal{S}^\Gamma(\mathscr{A})}$.

The following theorem is a generalization of [46, Theorems 6, 7 and 8] for a locally compact group.

Theorem 4.1.7. *Under the standing hypotheses mentioned in Theorem 4.1.6, and for a.e* $\alpha \in \widehat{\Gamma}, J_{\mathscr{A}}(\alpha) \neq \{0\}$, *the following statements hold:*

(i) *An alternate (oblique) TG-dual (TG-dual frame) for $\mathcal{E}^\Gamma(\mathscr{A})$ is the only canonical dual frame of $\mathcal{E}^\Gamma(\mathscr{A})$ if and only if for a.e. $\alpha \in \widehat{\Gamma}$, the system $(\mathcal{Z}\mathscr{A})(\alpha)$ is a Riesz basis for $L^2(\Gamma\backslash\mathscr{G})$.*

(ii) *A TG-dual of type-I for $\mathcal{E}^\Gamma(\mathscr{A})$ is the only canonical dual frame of $\mathcal{E}^\Gamma(\mathscr{A})$ if and only if for a.e. $\alpha \in \widehat{\Gamma}$, the system $(\mathcal{Z}\mathscr{A})(\alpha)$ is a Riesz basis for $J_{\mathscr{A}}(\alpha)$, where $J_{\mathscr{A}}(\alpha) = \overline{\text{span}}\{(\mathcal{Z}\varphi_t)(\alpha) : t \in \mathcal{N}\}$.*

(iii) *A TG-dual of type-II for $\mathcal{E}^\Gamma(\mathscr{A})$ is the only canonical dual frame of $\mathcal{E}^\Gamma(\mathscr{A})$ if and only if for a.e. $\alpha \in \widehat{\Gamma}$, the system $(\mathcal{Z}\mathscr{A})(\alpha)$ is a frame for $L^2(\Gamma\backslash\mathscr{G})$.*

Moreover, in case of abelian pair $(\mathcal{G}, \Lambda)$, the above characterizations are true just by replacing $L^2(\Gamma\backslash\mathscr{G})$ with $L^2(\Lambda^\perp)$, and the system $(\mathcal{Z}\mathscr{A})(\alpha)$ with the system $(\mathscr{T}\mathscr{A})(\beta\Lambda^\perp)$ as well as $J_{\mathscr{A}}(\alpha)$ with $J_{\mathscr{A}}^{\mathcal{G}}(\beta\Lambda^\perp)$ for a.e. $\beta\Lambda^\perp \in \widehat{\mathcal{G}}/\Lambda^\perp$.

Proof. Since the set $\mathcal{D}$ is an orthonormal Parseval determining set as discussed in the proof of Theorem 4.1.6, the results follow by observing Theorems 2.3.1, 2.3.2, and 2.3.3 on the line of Theorems 4.1.2, 4.1.4, 4.1.5, and 4.1.6. □

Remark 4.1.8. (i) For the uniqueness, the remaining results of Theorems 2.3.1, 2.3.2 and 2.3.3 can be transformed for the locally compact group in a similar way.
(ii) Analogous to the Theorem 2.2.6, the existence of oblique TG-dual and TG-dual of type-II can be obtained for the continuous frame $\mathcal{E}^\Gamma(\mathscr{A})$ of $\mathcal{S}^\Gamma(\mathscr{A})$ which will generalize a result of Heil, Koo and Lim [44, Theorem 1.5].

4.2. Examples

In this part, we will show our findings for a variety of experimental setups and present counterexamples.

4.2.1. On the Euclidian space $\mathbb{R}^n$ by the action of integer shifts $\mathbb{Z}^n$

Let $\mathcal{G} = \mathbb{R}^n$ and $\Lambda = \mathbb{Z}^n$. Then, $\widehat{\mathcal{G}} = \mathbb{R}^n$, $\Lambda^\perp = \mathbb{Z}^n$ and the fundamental domain for $\mathbb{Z}^n$ is $\widehat{\mathcal{G}}\backslash\Lambda^\perp = \mathbb{T}^n$. Then the fiberization map is

$$\mathscr{T} : L^2(\mathbb{R}^n) \to L^2(\mathbb{T}^n; \ell^2(\mathbb{Z}^n)) \text{ defined by } \mathscr{T} f(\xi) = \{\widehat{f}(\xi + k)\}_{k \in \mathbb{Z}^n}, \xi \in \mathbb{T}^n.$$

Therefore by considering a countable family $\mathscr{A}$ in $L^2(\mathbb{R}^n)$, we can obtain characterization results of duals for a discrete frame sequence $\mathcal{E}^{\mathbb{Z}^n}(\mathscr{A})$ using $\mathscr{T}$ that provide results of references within [37, 46]. Indeed, our study covers duals for a continuous frame also.

4.2.2. On the p-adic group $\mathbb{Q}_p$ by the action of p-adic integers $\mathbb{Z}_p$

For a prime number p, consider the group of p-adic numbers $\mathbb{Q}_p$ and its closed subgroup $\mathbb{Z}_p$ of p-adic integers. The p-adic group $\mathbb{Q}_p$ is an LCA group and all its proper subgroups H are compact and open, and hence G/H is not compact. Therefore, $\mathbb{Q}_p$ does not have any proper co-compact closed subgroup while the maximum literature for duals on LCA groups was mainly based on the action by closed co-compact subgroups [52]. Note that in our situation we require only a closed subgroup.

Let Ω be a fundamental domain for $\mathbb{Z}_p$ which is a discrete set. Then for $f \in L^1(\mathbb{Q}_p) \bigcap L^2(\mathbb{Q}_p)$, the Zak transform is defined by

$$\tilde{\mathcal{Z}} f(x, y) = \int_{\mathbb{Z}_p} f(y + \xi) e^{-2\pi i x \xi} \, d\mu_{\mathbb{Z}_p}(\xi) \quad \text{for } x, y \in \Omega.$$

Therefore, we can derive characterization results of duals for a continuous frame $\mathcal{E}^{\mathbb{Z}_p}(\mathscr{A})$ using $\tilde{\mathcal{Z}}$ and Theorems 4.1.2-4.1.7.

4.2.3. Counter examples of duals in LCA groups

For a second countable LCA group $\mathcal{G}$ having closed co-compact subgroup Λ, we can write $\widehat{\mathcal{G}} = \Omega \bigoplus \Lambda^\perp$ by the Pontryagin Duality theorem, where Ω is a *fundamental domain*. Then, the system $\{\Omega + \lambda : \lambda \in \Lambda^\perp\}$ is a measurable partition of $\widehat{\mathcal{G}}$. Here we also fix an automorphism A on $\widehat{\mathcal{G}}$ such that $A\Omega \subsetneq \Omega$ throughout the entire section. Now, we provide counter examples using the fiberization, Remarks 4.1.3, 4.1.8, and Theorems 4.1.2-4.1.7.

Example 4.2.1 (**Alternate dual but not frame**). Let $\eta_1, \eta_2 \in L^2(\mathcal{G})$ be two functions such that its Fourier transforms are defined by $\widehat{\eta}_1(\xi) = \chi_{A\Omega}(\xi)$ and $\widehat{\eta}_2(\xi) = \chi_{A\Omega}(\xi) + \sqrt{g(\xi)}\chi_{\Omega\backslash A\Omega}(\xi)$ for $\xi \in \widehat{\mathcal{G}}$, where g is a real valued function on $\widehat{\mathcal{G}}$ such that the restriction map $g\big|_{\Omega\backslash A\Omega} : \Omega\backslash A\Omega \to (0,1]$ is strictly decreasing and onto. Then, we conclude the following:

(i) The system $\mathcal{E}^\Lambda(\{\eta_1\})$ is a Parseval frame for $\mathcal{S}^\Lambda(\{\eta_1\})$ since for a.e. $\xi \in A\Omega$, we have $\|\mathscr{T}\eta_1(\xi)\|^2 = \sum_{\lambda\in\Lambda^\perp} |\widehat{\eta}_1(\xi+\lambda)|^2 = 1$, by observing that the system $\{\Omega+\lambda : \lambda \in \Lambda^\perp\}$ is a measurable partition of $\widehat{\mathcal{G}}$, $A\Omega \subset \Omega$ and $\mu_{\widehat{\mathcal{G}}}(A\Omega \cap (A\Omega+\lambda)) = 0$ for $\lambda \in \Lambda^\perp\backslash\{0\}$.

(ii) $\mathcal{E}^\Lambda(\{\eta_2\})$ is Bessel with bound 2 since for a.e. $\xi \in \Omega$, we have $\|\mathscr{T}\eta_2(\xi)\| = \sum_{\lambda\in\Lambda^\perp} |\widehat{\eta}_2(\xi+\lambda)|^2 \leqslant 2$, by observing $\mu_{\widehat{\mathcal{G}}}(A\Omega \cap (A\Omega+\lambda)) = 0$, $\mu_{\widehat{\mathcal{G}}}((\Omega\backslash A\Omega) \cap ((\Omega\backslash A\Omega)+\lambda)) = 0$ for $\lambda \in \Lambda^\perp\backslash\{0\}$ and also $\mu_{\widehat{\mathcal{G}}}(A\Omega \cap ((\Omega\backslash A\Omega)+\lambda)) = 0$ for $\lambda \in \Lambda^\perp$ since the measurable sets $A\Omega$ and $\Omega\backslash A\Omega$ are disjoint subsets of Ω.

(iii) $\mathcal{E}^\Lambda(\{\eta_2\})$ is an alternate TG-dual for $\mathcal{E}^\Lambda(\{\eta_1\})$ since

$$\langle \mathscr{T}\eta_1(\xi), \mathscr{T}\eta_2(\xi)\rangle = \sum_{\lambda\in\Lambda^\perp} \widehat{\eta}_1(\xi+\lambda)\overline{\widehat{\eta}_2(\xi+\lambda)} = \chi_{A\Omega}(\xi) \text{ for a.e. } \xi.$$

(iv) But $\mathcal{E}^\Lambda(\{\eta_2\})$ is not a frame for $\mathcal{S}^\Lambda(\{\eta_2\})$ since for a.e. $\xi \in \Omega$, no lower bound of $\|\mathscr{T}\eta_2(\xi)\|$ is greater than zero, follows by noting that the infimum of $g\big|_{\Omega\backslash A\Omega}(\Omega\backslash A\Omega)$ is zero and

$$\sum_{\lambda\in\Lambda^\perp} |\widehat{\eta}_2(\xi+\lambda)|^2 = \chi_{A\Omega}(\xi) + g(\xi)\chi_{\Omega\backslash A\Omega}(\xi).$$

Example 4.2.2 (**Alternate but not oblique dual**). Let $\eta_1, \eta_2 \in L^2(\mathcal{G})$ be such that $\widehat{\eta}_1 = \chi_{A\Omega}$ and $\widehat{\eta}_2 = \chi_\Omega$. Then, $\mathcal{E}^\Lambda(\{\eta_i\})$ is a frame for $\mathcal{S}^\Lambda(\{\eta_i\})$ for $i = 1, 2$, and $\mathcal{E}^\Lambda(\{\eta_2\})$ is an alternate TG-dual for $\mathcal{E}^\Lambda(\{\eta_1\})$ since for a.e. $\xi \in \Omega$, $\sum_{\lambda\in\Lambda^\perp} |\widehat{\eta}_1(\xi+\lambda)|^2 = \chi_{A\Omega}(\xi)$, $\sum_{\lambda\in\Lambda^\perp} |\widehat{\eta}_2(\xi+\lambda)|^2 = \chi_\Omega(\xi)$, and $\sum_{\lambda\in\Lambda^\perp} \widehat{\eta}_1(\xi+\lambda)\overline{\widehat{\eta}_2(\xi+\lambda)} = \chi_{A\Omega}(\xi)$. Since $A\Omega \subsetneq \Omega$, the system $\mathcal{E}^\Lambda(\{\eta_1\})$ is not an alternate TG-dual for $\mathcal{E}^\Lambda(\{\eta_2\})$, and hence the system $\mathcal{E}^\Lambda(\{\eta_2\})$ is not an oblique TG-dual for $\mathcal{E}^\Lambda(\{\eta_1\})$. Further note that it is neither type-I nor type-II dual.

Example 4.2.3 (**Oblique but not dual frame**; **Type-I but not Type-II dual**). Let $\mathscr{A} = \{\eta_1, \eta_2\}$ and $\mathscr{A}' = \{\zeta_1, \zeta_2\}$ be two families of functions in $L^2(\mathcal{G})$ such that $\widehat{\eta}_1 = \mathcal{X}_\Omega = 2\,\widehat{\eta}_2$ and $\widehat{\zeta}_1 = \mathcal{X}_\Omega, \widehat{\zeta}_2 = 0$. Then for a discrete set Λ, we get $\sum_{\lambda\in\Lambda^\perp} |\widehat{\eta}_1(\xi+\lambda)|^2 = \chi_\Omega(\xi) = 2\sum_{\lambda\in\Lambda^\perp} \widehat{\eta}_1(\xi+\lambda)\overline{\widehat{\eta}_2(\xi+\lambda)}$ and $\sum_{\lambda\in\Lambda^\perp} |\widehat{\eta}_2(\xi+\lambda)|^2 = \frac{1}{4}\chi_\Omega(\xi)$, and hence the Gramian matrices $G_{\mathscr{A}}(\xi)$ associated with $\mathscr{A}$ is $[\langle \mathscr{T}\eta_i(\xi), \mathscr{T}\eta_j(\xi)\rangle]_{1\leqslant i,j\leqslant 2}$ (in view of (2.3.2)), which will be $\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$. Similarly, the Gramian matrices $G_{\mathscr{A}'}(\xi)$ associated with $\mathscr{A}'$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ for $\xi \in \Omega$. Therefore for $\xi \in \Omega$, we conclude $J_{\mathscr{A}}(\xi) = J_{\mathscr{A}'}(\xi)$ and Ker $G_{\mathscr{A}}(\xi) \neq$Ker $G_{\mathscr{A}'}(\xi)$. That means, we get $\mathcal{S}^{\Lambda}(\mathscr{A}) = \mathcal{S}^{\Lambda}(\mathscr{A}')$, and Ker $T^*_{\mathcal{E}^{\Lambda}(\mathscr{A})} \neq$ Ker $T_{\mathcal{E}^{\Lambda}(\mathscr{A}')}$. Thus, $\mathcal{E}^{\Lambda}(\mathscr{A}')$ is a dual of type-I for $\mathcal{E}^{\Lambda}(\mathscr{A})$ but not type-II dual. Also note that it is oblique but not dual frame since $\mathcal{S}^{\Lambda}(\mathscr{A}) \subsetneq L^2(\mathcal{G})$.

Example 4.2.4 (**Unique Type-I dual but $J_{\mathscr{A}}(x) = 0$ for a measurable set**). Let $\eta_1, \eta_2 \in L^2(\mathcal{G})$ and $\lambda_0 \in \Lambda^{\perp}\backslash\{0\}$ be such that $\widehat{\eta}_1 = \chi_{A\Omega}$ and $\widehat{\eta}_2 = \chi_{(A\Omega+\lambda_0)}$. Then for a.e. $\xi \in \Omega$, we get $\sum_{\lambda\in\Lambda^{\perp}} |\widehat{\eta}_i(\xi+\lambda)|^2 = \chi_{A\Omega}(\xi)$ for $i = 1, 2$ and $\sum_{\lambda\in\Lambda^{\perp}} \widehat{\eta}_1(\xi+\lambda)\overline{\widehat{\eta}_2(\xi+\lambda)} = 0$, and hence the associated Gramian matrix $G(\xi)$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for $\xi \in A\Omega$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for $\xi \in \Omega\backslash A\Omega$. Since for each $\xi \in \Omega$, either $G(\xi)$ is invertible or $G(\xi) = 0$, the frame $\mathcal{E}^{\Lambda}(\{\eta_1, \eta_2\})$ admits unique TG-dual of type-I by Theorem 4.1.7 and Theorem 2.3.1.

Example 4.2.5 (**Non-unique Type-I dual**). Let $\eta_1, \eta_2 \in L^2(\mathcal{G})$ be such that $\widehat{\eta}_1 = \chi_{A\Omega}$ and $\widehat{\eta}_2 = \chi_{\Omega\backslash A\Omega}$. Then for a.e. $\xi \in \Omega$, we get $\sum_{\lambda\in\Lambda^{\perp}} |\widehat{\eta}_1(\xi+\lambda)|^2 = \chi_{A\Omega}(\xi)$, $\sum_{\lambda\in\Lambda^{\perp}} |\widehat{\eta}_2(\xi+\lambda)|^2 = \chi_{\Omega\backslash A\Omega}(\xi)$, and $\sum_{\lambda\in\Lambda^{\perp}} \widehat{\eta}_1(\xi+\lambda)\overline{\widehat{\eta}_2(\xi+\lambda)} = 0$, and hence the associated Gramian matrix $G(\xi)$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ for $\xi \in A\Omega$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ for $\xi \in \Omega\backslash A\Omega$. Since for each $\xi \in \Omega$, $G(\xi)$ is not invertible, the frame $\mathcal{E}^{\Lambda}(\{\eta_1, \eta_2\})$ admits more than one TG-dual of type-I.

Example 4.2.6 (**Type-II but not Type-I dual**). Let $\eta_1, \eta_2 \in L^2(\mathcal{G})$ and $\lambda_0 \in \Lambda^{\perp}\backslash\{0\}$ be such that $\widehat{\eta}_1 = \chi_{A\Omega}$ and $\widehat{\eta}_2 = \chi_{A\Omega\cup(A\Omega+\lambda_0)}$. Then for a uniform lattice Λ, $\mathcal{E}^{\Lambda}(\{\eta_2\})$ is an oblique TG-dual for $\mathcal{E}^{\Lambda}(\{\eta_1\})$ follows by noting $\sum_{\lambda\in\Lambda^{\perp}} \widehat{\eta}_i(\xi+\lambda)\overline{\widehat{\eta}_j(\xi+\lambda)} = \chi_{A\Omega}(\xi)$ for $1 \leqslant i, j \leqslant 2$. Thus, it is a dual of type-II since for a.e. $\xi \in \Omega$, Ker $T^*_{\{\eta_1\}}(\xi) =$ Ker $T^*_{\{\eta_2\}}(\xi)$ but not type-I dual since $J_{\{\eta_1\}}(\xi) \neq J_{\{\eta_2\}}(\xi)$.

Example 4.2.7 (**Unique TG-dual of Type-II**). Let $\eta_1, \eta_2 \in L^2(\mathcal{G})$ and $\lambda_0 \in \Lambda^{\perp}$ be such that $\widehat{\eta}_1 = \chi_{\Omega}$ and $\widehat{\eta}_2 = \chi_{\Omega+\lambda_0}$. Then, for a uniform lattice Λ, for a.e. $\xi \in \Omega$ and $i = 1, 2$, we get $\sum_{\lambda\in\Lambda^{\perp}} |\widehat{\eta}_i(\xi+\lambda)|^2 = \chi_{\Omega}(\xi)$, and $\sum_{\lambda\in\Lambda^{\perp}} \widehat{\eta}_1(\xi+\lambda)\overline{\widehat{\eta}_2(\xi+\lambda)} = 0$, and hence the system $\mathcal{E}^{\Lambda}(\{\eta_1, \eta_2\})$ is an orthonormal basis for $\mathcal{S}^{\Lambda}(\{\eta_1, \eta_2\})$. Therefore, it is a Riesz basis for $\mathcal{S}^{\Lambda}(\{\eta_1, \eta_2\})$.

Example 4.2.8 (**Non-unique Type-II dual**)**.** For a uniform lattice Λ and $\lambda \in \Lambda^{\perp}$, let $\eta_\lambda \in L^2(\mathcal{G})$ be such that $\widehat{\eta}_\lambda = \chi_{\Omega+B\lambda}$, where B is an automorphism on $\widehat{\mathcal{G}}$ such that $B\Lambda^{\perp} \subsetneq \Lambda^{\perp}$. Then for a.e. $\xi \in \Omega$, the dual Gramian matrix

$$\tilde{G}(\xi) = \left[\sum_{\lambda \in \Lambda^{\perp}} \widehat{\eta}_\lambda(\xi+\lambda_1)\overline{\widehat{\eta}_\lambda(\xi+\lambda_2)}\right]_{\lambda_1,\lambda_2 \in \Lambda^{\perp}} = \left[\sum_{\lambda \in \Lambda^{\perp}} \chi_{\Omega+B\lambda}(\xi+\lambda_1)\chi_{\Omega+B\lambda}(\xi+\lambda_2)\right]_{\lambda_1,\lambda_2 \in \Lambda^{\perp}}$$

is non-zero and non-invertible matrix, follows by noting that all the terms of columns of $\tilde{G}(\xi)$ with respect to λ_2 is identically non-zero when $\lambda_1 \in B\Lambda^{\perp}$, while it is identically zero for $\lambda_1 \notin B\Lambda^{\perp}$. Also note that the Gramian matrix $G(\xi) = [\langle \mathscr{T}\eta_{\lambda_1}(\xi), \mathscr{T}\eta_{\lambda_2}(\xi)\rangle]_{\lambda_1,\lambda_2 \in \Lambda^{\perp}}$ is an identity matrix. Therefore, the frame $\mathcal{E}^{\Lambda}(\{\eta_\lambda\}_{\lambda \in \Lambda^{\perp}})$ admits more than one TG-dual of type-II.

Next, we start the continuous Gabor system due to its importance into various applications. The characterizations established here generalize various results available in the literature including [8, 17, 24, 49, 52].

4.3. Duals for a continuous Gabor frame

For a second countable LCA group $\mathcal{G}$ having closed subgroup Λ and a family $\mathscr{A}$ in $L^2(\mathcal{G})$, let us consider a *Gabor system* (also known as, $(\Lambda, \Lambda^{\perp})$-*translation modulation generated (TMG) system*) $G(\mathscr{A}, \Lambda, \Lambda^{\perp})$ given by

$$G(\mathscr{A}, \Lambda, \Lambda^{\perp}) := \left\{L_\lambda E_\omega \varphi : \lambda \in \Lambda, \omega \in \Lambda^{\perp}, \varphi \in \mathscr{A}\right\},$$

and its associated $(\Lambda, \Lambda^{\perp})$-translation modulation invariant (TMI) space

$$\mathcal{S}(\mathscr{A}, \Lambda, \Lambda^{\perp}) := \overline{\operatorname{span}} G(\mathscr{A}, \Lambda, \Lambda^{\perp}),$$

where for $\omega \in \widehat{\mathcal{G}}$, the *modulation operator* E_ω on $L^2(\mathcal{G})$ is defined by $(E_\omega f)(x) = \omega(x)f(x)$, $x \in \mathcal{G}$, $f \in L^2(\mathcal{G})$. By a *TMI space* V, we mean it is closed subspace of $L^2(\mathcal{G})$ such that $L_\lambda E_\omega f \in V$ for all $f \in V$ and $(\lambda, \omega) \in (\Lambda, \Lambda^{\perp})$.

Since the Zak transform satisfies the relation $(\tilde{\mathcal{Z}} L_\lambda E_\omega f)(\beta, x\Lambda) = M_{\phi_{\lambda,\omega}}(\tilde{\mathcal{Z}} f)(\beta, x\Lambda)$, for $f \in L^2(\mathcal{G})$, $(\lambda, \omega) \in (\Lambda, \Lambda^{\perp})$ and $(\beta, x\lambda) \in (\widehat{\Lambda}, \mathcal{G}/\Lambda)$, where $\phi_{\lambda,\omega}(\beta, x) = \beta(\lambda^{-1})\omega(x)$, therefore the set $\mathcal{D} = \{\phi_{\lambda,\omega} \in L^\infty(\widehat{\Lambda} \times \mathcal{G}/\Lambda)\}_{(\lambda,\omega)\in(\Lambda,\Lambda^{\perp})}$ is a Parseval determining set for $L^1(\widehat{\Lambda} \times \mathcal{G}/\Lambda)$ follows by the isomorphism between groups $(\widehat{\Lambda} \times \mathcal{G}/\Lambda)^\wedge$ and $\Lambda \times \Lambda^{\perp}$ due to Pontryagin Duality theorem.

For a Borel subset $B \subset \widehat{\Lambda} \times \mathcal{G}/\Lambda$, consider $V_B = \{f \in L^2(\mathcal{G}) : (\tilde{\mathcal{Z}} f)(\beta, \Lambda x) = 0 \text{ for a.e. } (\beta, \Lambda x) \notin B\}$. Then the map $B \mapsto V_B$ is a bijection and corresponding to the $(\Lambda, \Lambda^\perp)$-TMI system $\mathcal{S}(\mathscr{A}, \Lambda, \Lambda^\perp) = \mathcal{S}(\mathscr{A}_0, \Lambda, \Lambda^\perp)$, the set B is given by

$$B = \{(\beta, \Lambda x) \in \widehat{\Lambda} \times \mathcal{G}/\Lambda : (\tilde{\mathcal{Z}} f)(\beta, \Lambda x) \neq 0 \text{ for some } \ f \in \mathscr{A}_0\}, \tag{4.3.1}$$

where $\mathscr{A}_0$ is a countable dense subset of $\mathscr{A}$ [17, Theorem 3.1], [23, Theorem 4.1].

Now we state the following characterization results of duals (*alternate (oblique) TMG-dual, TMG-dual of type-I (type-II), TMG-dual frame* in the sense of Definition 1.1.1) for a continuous frame $G(\mathscr{A}, \Lambda, \Lambda^\perp)$ of $(\Lambda, \Lambda^\perp)$-TMI space $\mathcal{S}(\mathscr{A}, \Lambda, \Lambda^\perp)$ in view of Theorems 2.3.1, 2.3.2 and 2.3.3. It is a generalization of [17, Theorem 7.3] and [36, Theorem 2.3] for a locally compact abelian group.

Theorem 4.3.1. *Let $\mathscr{A} = \{\varphi_t\}_{t \in \mathcal{N}}$ be a collection of functions in $L^2(\mathcal{G})$ such that $G(\mathscr{A}, \Lambda, \Lambda^\perp)$ is a continuous frame for $\mathcal{S}(\mathscr{A}, \Lambda, \Lambda^\perp)$, and B is defined by (4.3.1), where $(\mathcal{N}, \mu_{\mathcal{N}})$ is a complete, σ-finite measure space. If $\mathscr{A}' = \{\psi_t\}_{t \in \mathcal{N}}$ is a collection of functions in $L^2(\mathcal{G})$ such that $G(\mathscr{A}, \Lambda, \Lambda^\perp)$ is Bessel in $L^2(\mathcal{G})$, then we have the following:*

(i) *$G(\mathscr{A}', \Lambda, \Lambda^\perp)$ is an alternate (oblique) TMG-dual (dual frame) (dual of type-I) for $G(\mathscr{A}, \Lambda, \Lambda^\perp)$ if and only if for a.e. $(\beta, x\Lambda) \in B$, the system $(\tilde{\mathcal{Z}} \mathscr{A}')(\beta, \Lambda x) := \{\tilde{\mathcal{Z}} \psi_t(\beta, \Lambda x) : t \in \mathcal{N}\}$ is an alternate (oblique) dual (dual frame) (dual of type-I) for the frame $(\tilde{\mathcal{Z}} \mathscr{A})(\beta, \Lambda x) := \{\tilde{\mathcal{Z}} \psi_t(\beta, \Lambda x) : t \in \mathcal{N}\}$.*

(ii) *For a countable family $\mathcal{N}$ and discrete set Λ having counting measures, $G(\mathscr{A}', \Lambda, \Lambda^\perp)$ is a TMG-dual of type-II for $G(\mathscr{A}, \Lambda, \Lambda^\perp)$ if and only if for a.e. $(\beta, x\Lambda) \in B$, $(\tilde{\mathcal{Z}} \mathscr{A}')(\beta, x\Lambda)$ is a dual of type-II for the frame $(\tilde{\mathcal{Z}} \mathscr{A})(\beta, x\Lambda)$.*

Proof. Using the Parseval determining set $\mathcal{D} = \{\phi_{\lambda,\omega} \in L^\infty(\widehat{\Lambda} \times \mathcal{G}/\Lambda)\}_{(\lambda,\omega) \in (\Lambda, \Lambda^\perp)}$ for $L^1(\widehat{\Lambda} \times \mathcal{G}/\Lambda)$, and the intertwining relation $(\tilde{\mathcal{Z}} L_\lambda E_\omega f)(\beta, x\Lambda) = M_{\phi_{\lambda,\omega}}(\tilde{\mathcal{Z}} f)(\beta, x\Lambda)$, we can get the results similar to Theorems 4.1.2-4.1.6. □

Example 4.3.2. Let $\mathcal{G} = \mathbb{R}^n$, $\Lambda = \mathbb{Z}^m$ where $m \leqslant n$, considering $\mathbb{Z}^m$ as a subgroup of $\mathbb{R}^n$ by fixing the first m entries are non-zero and remaining are 0 in n-tuple, then the Zak transform $\tilde{\mathcal{Z}} : L^2(\mathcal{G}) \to L^2(\widehat{\Lambda} \times \Lambda \backslash \mathcal{G})$ takes of the form :

$$\tilde{\mathcal{Z}} : L^2(\mathbb{R}^n) \to L^2([0,1)^m \times [0,1)^m \times \mathbb{R}^{n-m})$$

by $(\tilde{\mathcal{Z}}f)(u,v,x) = \sum_{k\in\mathbb{Z}^m} f(v+k,x)e^{-2\pi k.u}$ for $(u,v,x) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{n-m}$, $f \in L^2(\mathbb{R}^n)$. Then for two countable collections of functions $\mathscr{A} = \{\varphi_t\}_{t\in\mathcal{N}}$, $\mathscr{A}' = \{\psi_t\}_{t\in\mathcal{N}}$ in $L^2(\mathbb{R}^n)$ we can state the Theorem 4.3.1 for the discrete translations in Euclidean setup.

4.4. Duals associated with the orbit of a representation

The present section overwhelms with a deep connection between frame theory and representation theory. Such study in connection with discrete as well as continuous frames has been done by many researchers including Hernández, Šikic, Weiss and Wilson [47], Barbieri and Hernández and Parcet [9], Iverson [49], and Bownik and Iverson [19]. A *representation (unitary)* ρ of a second countable LCA group $\mathcal{G}$ is a homomorphism from $\mathcal{G}$ to $U(\mathcal{H})$, where $U(\mathcal{H})$ denotes collection of the unitary operators on a Hilbert space $\mathcal{H}$, such that the map $x \mapsto \rho(x)v$ is continuous from $\mathcal{G}$ to $\mathcal{H}$ for any $v \in \mathcal{H}$. We simply call $(\rho, \mathcal{G}, \mathcal{H})$ as a *representation of* $\mathcal{G}$. Further, we call $(\rho, \mathcal{G}, \mathcal{H})$ as a *dual integrable representation* if there is a bracket map $[\cdot,\cdot] : \mathcal{H} \times \mathcal{H} \to L^1(\widehat{\mathcal{G}})$ satisfying

$$\langle f, \rho(x)g\rangle = \int_{\widehat{\mathcal{G}}} [f,g](\beta)\overline{\beta(x)}\, d\mu_{\widehat{\mathcal{G}}}(\beta), \;\; x \in \mathcal{G} \text{ and } f, g \in \mathcal{H}.$$

Note that the translation and modulation representations are dual integrable acting on $\mathcal{H} = L^2(\mathcal{G})$ and $\mathcal{H} = L^2(\widehat{\mathcal{G}})$, respectively. Through out this section $(\rho, \mathcal{G}, \mathcal{H})$ denotes a dual integrable representation acting on a separable Hilbert space $\mathcal{H}$.

For a dual integrable representation $(\rho, \mathcal{G}, \mathcal{H})$ and a family $\mathscr{A} = \{\varphi_t\}_{t\in\mathcal{N}} \subseteq \mathcal{H}$, we define its orbit under ρ by

$$\mathcal{O}_{\mathcal{G}}(\mathscr{A}) := \{\rho(x)\varphi : x \in \mathcal{G}, \varphi \in \mathscr{A}\},$$

and its associated ρ-invariant space $\mathcal{S}_{\mathcal{G}}(\mathscr{A}) = \overline{\text{span}}\mathcal{O}_{\mathcal{G}}(\mathscr{A})$. By a *$\rho$-invariant space* we mean a closed subspace V of $\mathcal{H}$ such that $\rho(x)\varphi \in M$ for all $\varphi \in V$ and $x \in \mathcal{G}$. Since we are interested to discuss duals viz. alternate (oblique) dual, dual frame, type-I and type-II duals for the frame $\mathcal{O}_{\mathcal{G}}(\mathscr{A})$ of $\mathcal{S}_{\mathcal{G}}(\mathscr{A})$, we require the isometry $\mathcal{T}$ between $\mathcal{H}$ and $L^2(\widehat{\mathcal{G}}; \ell^2(I))$ defined by

$$\mathcal{T}(f)(\beta) = \left(\frac{[f,\zeta_i](\beta)}{[\zeta_i,\zeta_i]^{\frac{1}{2}}(\beta)} \chi_{\Omega_i}(\beta) \right)_{i\in I} \quad \text{for } f \in \mathcal{H}, \beta \in \widehat{\mathcal{G}},$$

where $\{\zeta_i\}_{i\in I} \subseteq \mathcal{H}$ is the orthogonal generators for ρ, i.e., $\mathcal{H} = \oplus_{i\in I}\, \overline{\text{span}}\{\rho(x)\zeta_i\}_{x\in\mathcal{G}}$, and $\Omega_i = \{\beta \in \widehat{\mathcal{G}} : [\zeta_i,\zeta_i] \neq 0\}$. Then from [47, 49], note that it satisfies the following intertwining properties with representation, i.e., $\mathcal{T}(\rho(x)\varphi) = M_{\phi_x}\mathcal{T}(\varphi)$ for all $\varphi \in \mathcal{H}$ and $x \in \mathcal{G}$, where $\phi_x(\beta) = \beta(x), \beta \in \widehat{\mathcal{G}}$. Therefore, the set $\mathcal{D} = \{\phi_x : x \in \mathcal{G}\}$ is a Parseval determining set for $L^1(\widehat{\mathcal{G}})$ by Pontryagin Duality theorem. Analogous to the previous case, we notice that for $\beta \in \widehat{\mathcal{G}}$, the space $J(\beta)$ associated to the range function $J : \widehat{\mathcal{G}} \to \{\text{closed subspaces of } \ell^2(I\}$ is defined by

$$J(\beta) = \overline{\operatorname{span}}\{(\mathcal{T}f)(\beta) : f \in \mathscr{A}_0\} =: J_{\mathscr{A}}(\beta) \tag{4.4.1}$$

for some countable dense subset $\mathscr{A}_0$ of $\mathscr{A}$ in $\mathcal{H}$.

Now we state the following characterization results of duals (in the sense of Definition 1.1.1) for a continuous frame $\mathcal{O}_{\mathcal{G}}(\mathscr{A})$ of $\mathcal{S}_{\mathcal{G}}(\mathscr{A})$:

Theorem 4.4.1. *Let $(\rho, \mathcal{G}, \mathcal{H})$ be a dual integrable representation and let $\mathscr{A} = \{\varphi_t\}_{t\in\mathcal{N}}$ be a collection of functions in $\mathcal{H}$ such that $\mathcal{O}_{\mathcal{G}}(\mathscr{A})$ is a continuous frame for $\mathcal{S}_{\mathcal{G}}(\mathscr{A})$, and the associated range function is defined by (4.4.1), where $(\mathcal{N}, \mu_{\mathcal{N}})$ is a complete, σ-finite measure space. If $\mathscr{A}' = \{\psi_t\}_{t\in\mathcal{N}}$ is an another collection of functions in $\mathcal{H}$ such that $\mathcal{O}_{\mathcal{G}}(\mathscr{A}')$ is Bessel in $\mathcal{H}$, then we have the following:*

(i) *$\mathcal{O}_{\mathcal{G}}(\mathscr{A}')$ is an alternate dual (oblique dual, dual frame, dual of type-I) for $\mathcal{O}_{\mathcal{G}}(\mathscr{A})$ if and only if for a.e. $\beta \in \widehat{\mathcal{G}}$ the system $(\mathcal{T}\mathscr{A}')(\beta) := \{\mathcal{T}\psi_t(\beta) : t \in \mathcal{N}\}$ is an alternate dual (oblique dual, dual frame, dual of type-I) for the frame $(\mathcal{T}\mathscr{A})(\beta) := \{\mathcal{T}\psi_t(\beta) : t \in \mathcal{N}\}$ of $J_{\mathscr{A}}(\beta)$.*

(ii) *For a countable family $\mathcal{N}$ and a discrete set $\mathcal{G}$ having counting measure, $\mathcal{O}_{\mathcal{G}}(\mathscr{A}')$ is a dual of type-II for $\mathcal{O}_{\mathcal{G}}(\mathscr{A})$ if and only if for a.e. $\beta \in \widehat{\mathcal{G}}$, $(\tilde{\mathcal{Z}}\mathscr{A}')(\beta)$ is a dual of type-II for the frame $(\mathcal{T}\mathscr{A})(\beta)$ of $J_{\mathscr{A}}(\beta)$.*

Proof. By noting the Parseval determining set $\mathcal{D} = \{\phi_x : \phi_x(\beta) = \beta(x), \beta \in \widehat{\mathcal{G}}, x \in \mathcal{G}\}$ for $L^1(\widehat{\mathcal{G}})$, and the intertwining relation $\mathcal{T}(\rho(x)\varphi) = M_{\phi_x}\mathcal{T}(\varphi)$ for all $\varphi \in \mathcal{H}$ and $x \in \mathcal{G}$, the results can be proved in a similar way of Theorems 4.1.2, 4.1.4, 4.1.5, and 4.1.6 using the linear isometry $\mathcal{T} : \mathcal{H} \to L^2(\widehat{\mathcal{G}}; \ell^2(I))$ maps $\mathcal{S}_{\mathcal{G}}(\mathscr{A})$ unitarily onto $S_{\mathcal{D}}(\mathcal{T}\mathscr{A})$, sending $\mathcal{O}_{\mathcal{G}}(\mathscr{A})$ to $E_{\mathcal{D}}(\mathcal{T}\mathscr{A})$. □

Similarly we can prove the following using Theorems 2.3.1, 2.3.2 and 2.3.3 and Theorem 4.1.7.

Theorem 4.4.2. *In addition to hypotheses of the Theorem 4.4.1, let $\mathcal{N}$ and $\mathcal{G}$ be a countable family, and a discrete set having counting measure, respectively, such that for a.e. $\beta \in \widehat{\mathcal{G}}$, $J_{\mathscr{A}}(\beta)$ is defined by $J_{\mathscr{A}}(\beta) = \overline{\operatorname{span}}\{(\mathcal{T}\varphi_t)(\beta) : t \in \mathcal{N}\} \neq \{0\}$. Then, we have*

(i) *an alternate dual (oblique dual, dual frame) of $\mathcal{O}_{\mathcal{G}}(\mathscr{A})$ is the only canonical dual frame of $\mathcal{O}_{\mathcal{G}}(\mathscr{A})$ if and only if for a.e. $\beta \in \widehat{\mathcal{G}}$, $(\mathcal{T}\mathscr{A})(\beta)$ is a Riesz basis for $\mathcal{H}$.*

(ii) *a dual of type-I for $\mathcal{O}_{\mathcal{G}}(\mathscr{A})$ is the only canonical dual frame of $\mathcal{O}_{\mathcal{G}}(\mathscr{A})$ if and only if for a.e. $\beta \in \widehat{\mathcal{G}}$, $(\mathcal{T}\mathscr{A})(\beta)$ is a Riesz basis for $J_{\mathscr{A}}(\beta)$.*

(iii) *a dual of type-II for $\mathcal{O}_{\mathcal{G}}(\mathscr{A})$ is the only canonical dual frame of $\mathcal{O}_{\mathcal{G}}(\mathscr{A})$ if and only if for a.e. $\beta \in \widehat{\mathcal{G}}$, $(\mathcal{T}\mathscr{A})(\beta)$ is a frame for $J_{\mathscr{A}}(\beta)$.*

At the end of this section we discuss dual integrable representations associated with the dilation map in view of the importance of wavelet theory.

Example 4.4.3. For an LCA group $\mathcal{G}$ acting on a σ-finite measure space $(\mathscr{X}, \mu)$ and $g \in \mathcal{G}$, we define a *dilation map* $D_g : \mathcal{G} \to L^2(\mathscr{X})$ by $D_g f(x) = (J_g(x))^{\frac{1}{2}} f(g.x)$, where $J_g(x) = \frac{d\mu(g.x)}{d\mu(x)}$ and "." is an action of $\mathcal{G}$ on $\mathscr{X}$. Then it is a dual integrable unitary representation of $\mathcal{G}$ on $L^2(\mathscr{X}, \mu)$ [47]. Now for a collection $\mathscr{A} \subset L^2(\mathscr{X})$, if we consider the system $\mathcal{O}_{\mathcal{G}}(\mathscr{A}) = \{D_g\varphi : g \in \mathcal{G}, \varphi \in \mathscr{A}\}$, then we can discuss the local dual frame property applying Theorem 4.4.1 and their uniqueness from Theorem 4.4.2 in terms of suitable Zak transform.

In the next section, we are going to provide new oblique dual frames for given TI spaces.

4.5. Existence of oblique duals and infimum cosine angle

The following result is a generalization of [57, Theorem 4.10] for the locally compact group.

Theorem 4.5.1. *Let $\mathscr{G}$ be a locally compact group having a discrete abelian subgroup Γ, then for the finite collections of functions $\mathscr{A} = \{\varphi_i\}_{i=1}^m$ and $\mathscr{B} = \{\psi_i\}_{i=1}^n$ in $L^2(\mathscr{G})$, and for a.e. $\alpha \in \widehat{\Gamma}$, assume the range functions $J_{\mathscr{A}}(\alpha) = \operatorname{span}\{\mathcal{Z}\varphi_i(\alpha) : i = 1, 2, \ldots, m\}$ and $J_{\mathscr{B}}(\alpha) = \operatorname{span}\{\mathcal{Z}\psi_i(\alpha) : i = 1, 2, \ldots, n\}$ associated with Γ-TI spaces $\mathcal{S}^{\Gamma}(\mathscr{A})$ and $\mathcal{S}^{\Gamma}(\mathscr{B})$, respectively. Then the following are equivalent:*

(i) *There exist* $\mathscr{A}' = \{\varphi_i'\}_{i=1}^r$ *and* $\mathscr{B}' = \{\psi_i'\}_{i=1}^r$ *in* $L^2(\mathscr{G})$ *such that* $\mathcal{E}^\Gamma(\mathscr{A}')$ *and* $\mathcal{E}^\Gamma(\mathscr{B}')$ *are continuous frames for* $\mathcal{S}^\Gamma(\mathscr{A})$ *and* $\mathcal{S}^\Gamma(\mathscr{B})$*, respectively, satisfying the following reproducing formulas for* $g \in \mathcal{S}^\Gamma(\mathscr{A})$ *and* $h \in \mathcal{S}^\Gamma(\mathscr{B})$*:*

$$g = \sum_{i=1}^{r} \int_\Gamma \langle g, L_\gamma g_i' \rangle L_\gamma \varphi_i' \, d\mu_\Gamma(\gamma), \text{ and } h = \sum_{i=1}^{r} \int_\Gamma \langle h, L_\gamma \varphi_i' \rangle L_\gamma \psi_i' \, d\mu_\Gamma(\gamma).$$

(ii) *The infimum cosine angles of* $\mathcal{S}^\Gamma(\mathscr{A})$ *and* $\mathcal{S}^\Gamma(\mathscr{B})$ *are greater than zero, i.e.,*

$$R(\mathcal{S}^\Gamma(\mathscr{A}), \mathcal{S}^\Gamma(\mathscr{B})) > 0 \text{ and } R(\mathcal{S}^\Gamma(\mathscr{B}), \mathcal{S}^\Gamma(\mathscr{A})) > 0.$$

(iii) *There exist collections of functions* $\{\varphi_i'\}_{i=1}^r$ *and* $\{\psi_i'\}_{i=1}^r$ *in* $L^2(\mathscr{G})$ *such that for a.e.* $\alpha \in \widehat{\Gamma}$*, the systems* $\{\mathcal{Z}\varphi_i'(\alpha)\}_{i=1}^r$ *and* $\{\mathcal{Z}\psi_i'(\alpha)\}_{i=1}^r$ *are finite frames for* $J_\mathscr{A}(\alpha)$ *and* $J_\mathscr{B}(\alpha)$*, respectively, satisfying the following reproducing formulas for* $u \in J_\mathscr{A}(\alpha)$ *and* $v \in J_\mathscr{B}(\alpha)$*:*

$$u = \sum_{i=1}^{r} \langle u, \mathcal{Z}\psi_i'(\alpha) \rangle \mathcal{Z}\varphi_i'(\alpha), \text{ and } v = \sum_{i=1}^{r} \langle v, \mathcal{Z}\varphi_i'(\alpha) \rangle \mathcal{Z}\psi_i'(\alpha) \text{ a.e. } \alpha \in \widehat{\Gamma}.$$

(iv) *For a.e.* $\alpha \in \widehat{\Gamma}$*, the infimum cosine angles of* $J_\mathscr{A}(\alpha)$ *and* $J_\mathscr{B}(\alpha)$ *are greater than zero, i.e.,*

$$R(J_\mathscr{A}(\alpha), J_\mathscr{B}(\alpha)) > 0 \text{ and } R(J_\mathscr{B}(\alpha), J_\mathscr{A}(\alpha)) > 0.$$

Proof. Since $\mathcal{Z}$ is an unitary operator, $R(\mathcal{S}^\Gamma(\mathscr{A}), \mathcal{S}^\Gamma(\mathscr{A}')) = R(\mathcal{Z}\mathcal{S}^\Gamma(\mathscr{A}), \mathcal{Z}\mathcal{S}^\Gamma(\mathscr{A}'))$ using Definition 3.1.1. Hence we have the desired result using Theorem 3.1.6. □

Next we state the following result which is a generalization to the locally compact group in case of Riesz basis [18, Proposition 2.13].

Theorem 4.5.2. *Let* $\mathscr{G}$ *be a locally compact group having a discrete abelian subgroup* Γ *and* $\mathscr{V}, \mathscr{W}$ *be* Γ*-TI subspaces of* $L^2(\mathscr{G})$*. For the finite collection of functions* $\mathscr{A} = \{\varphi_i\}_{i=1}^r$*, assume* $\mathcal{E}^\Gamma(\mathscr{A})$ *is a Riesz basis for* $\mathscr{V}$*. Then the following holds:*

(i) ***Global setup:*** *If there exists* $\mathscr{A}' = \{\varphi_i'\}_{i=1}^r$ *in* $L^2(\mathscr{G})$ *such that* $\mathcal{E}^\Gamma(\mathscr{A}')$ *is a Riesz basis for* $\mathscr{W}$ *satisfying the biorthogonality condition* $\langle L_\gamma \varphi_i, L_{\gamma'} f_{i'}' \rangle = \delta_{i,i'}\delta_{\gamma,\gamma'},\ i, i' = 1, 2, \ldots, r;\ \gamma, \gamma' \in \Gamma$*, then the infimum cosine angles of* $\mathscr{V}$ *and* $\mathscr{W}$ *are greater than zero, i.e.,*

$$R(\mathscr{V}, \mathscr{W}) > 0 \text{ and } R(\mathscr{W}, \mathscr{V}) > 0. \tag{4.5.1}$$

Conversely if (4.5.1) holds true, then there exists $\mathscr{A}' = \{\varphi_i'\}_{i=1}^r$ *in* $L^2(\mathscr{G})$ *such that* $\mathcal{E}^\Gamma(\mathscr{A}')$ *is a Riesz basis for* $\mathscr{W}$ *satisfying the biorthogonality condition. Moreover, the following reproducing formulas hold:*

$$f = \sum_{\gamma\in\Gamma}\sum_{i=1}^{r}\langle f, L_\gamma\varphi_i'\rangle L_{\gamma'}\varphi_i \text{ for all } f \in \mathscr{V}, \text{ and } g = \sum_{\gamma\in\Gamma}\sum_{i=1}^{r}\langle g, L_\gamma\varphi_i\rangle L_\gamma\varphi_i' \text{ for all } g \in \mathscr{W}.$$

(ii) ***Local setup:*** *If there exists* $\mathscr{A}' = \{\varphi_i'\}_{i=1}^r$ *in* $L^2(\mathscr{G})$ *such that for a.e.* $\alpha \in \widehat{\Gamma}$, $\{\mathcal{Z}\varphi_i'(\alpha)\}_{i=1}^r$ *is a Riesz sequence in* $L^2(\Gamma\backslash\mathscr{G})$ *satisfying the following biorthogonality condition*

$$\langle \mathcal{Z}\varphi_i(\alpha), \mathcal{Z}\varphi_{i'}(\alpha)\rangle = \delta_{i,i'},\ i, i' = 1, 2, \ldots, r, \text{ a.e. } \alpha \in \widehat{\Gamma}, \tag{4.5.2}$$

then the infimum cosine angles of $J_{\mathscr{A}}(\alpha) = \operatorname{span}\{\mathcal{Z}\varphi_i(\alpha)\}_{i=1}^r$ *and* $J_{\mathscr{A}'}(\alpha) = \operatorname{span}\{\mathcal{Z}\varphi_i'(\alpha)\}_{i=1}^r$ *are greater than zero, i.e.,*

$$R(J_{\mathscr{A}}(\alpha), J_{\mathscr{A}'}(\alpha)) > 0 \text{ and } R(J_{\mathscr{A}'}(\alpha), J_{\mathscr{A}}(\alpha)) > 0 \text{ a.e. } \alpha \in \widehat{\Gamma}. \tag{4.5.3}$$

Conversely if (4.5.3) holds, there exists $\mathscr{A}' = \{\varphi_i'\}_{i=1}^r$ *in* $L^2(\mathscr{G})$ *such that for a.e.* $\alpha \in \widehat{\Gamma}$, $\{\mathcal{Z}\varphi_i'(\alpha)\}_{i=1}^r$ *is a Riesz sequence in* $L^2(\Gamma\backslash\mathscr{G})$ *satisfying the biorthogonality condition (4.5.2). Moreover, the following reproducing formulas hold for* $u \in J_{\mathscr{A}}(\alpha)$*, and* $v \in J_{\mathscr{A}'}(\alpha)$*:*

$$u = \sum_{i=1}^{r}\langle u, \mathcal{Z}\varphi_i'(\alpha)\rangle \mathcal{Z}\varphi_i(\alpha), \text{ and } v = \sum_{i=1}^{r}\langle v, \mathcal{Z}\varphi_i(\alpha)\rangle \mathcal{Z}\varphi_i'(\alpha) \text{ for a.e. } \alpha \in \widehat{\Gamma}.$$

The similar results can be deduced for locally compact abelian group $\mathcal{G}$ using the fiberization map $\mathscr{T}$.

Example 4.5.3. Let $\mathcal{G} = \mathbb{R}^n$, $\Lambda = \mathbb{R}^m$ where $m \leqslant n$, considering $\mathbb{R}^m$ as a subgroup of $\mathbb{R}^n$ by fixing the first m entries are non-zero and remaining are 0 in n-tuple, then the Zak transform $\tilde{\mathcal{Z}} : L^2(\mathcal{G}) \to L^2(\widehat{\Lambda}; L^2(\Lambda\backslash\mathcal{G}))$ takes of the form :

$$\tilde{\mathcal{Z}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^m \times L^2(\mathbb{R}^{n-m}))$$

by $(\tilde{\mathcal{Z}}f)(u)(y) = \int_{\mathbb{R}^m} f(x, y)e^{-2\pi i u.x}\,dx$ for $u \in \mathbb{R}^m, y \in \mathbb{R}^{n-m}$. Then for the finite collections of functions $\mathscr{A} = \{\varphi_i\}_{i=1}^m$ and $\mathscr{B} = \{\psi_i\}_{i=1}^n$ in $L^2(\mathbb{R}^n)$ we can state the Theorem 4.5.1 for the continuous translations in Euclidean setup. Similarly, Theorem 4.5.2 can be stated.

Example 4.5.4. Recall Example 4.2.2, the Gramian matrices corresponding to the system $\eta_1 = \chi_{A\Omega}$ and $\eta_2 = \chi_\Omega$, where $A\Omega \subsetneq \Omega$ are

$$G_{\eta_1}(\alpha) = \begin{cases} I_2 \text{ for } \alpha \in A\Omega, \\ 0, \text{ otherwise}, \end{cases} \quad \text{and } G_{\eta_2}(\alpha) = \begin{cases} I_2 \text{ for } \alpha \in \Omega, \\ 0, \text{otherwise}, \end{cases}$$

$$G_{\eta_1,\eta_2}(\alpha) = \begin{cases} I_2 \text{ for } \alpha \in A\Omega, \\ 0, \text{otherwise}. \end{cases}$$

$R(J_{\eta_1}(\alpha), J_{\eta_2}(\alpha)) = \|G_{\eta_1}(\alpha)G_{\eta_1,\eta_2}(\alpha)^\dagger G_{\eta_2}(\alpha)\| > 0$ only when $\alpha \in A\Omega$. Hence the dual is not oblique.

Example 4.5.5. When $A\Omega = \Omega$, then $R(J_{\eta_1}(\alpha), J_{\eta_2}(\alpha)) > 0$ for all $\alpha \in \widehat{\Gamma}$. Hence the dual is oblique in this case.

We have discussed various duals for the translation generated systems in $L^2(\mathscr{G})$. Orthogonality of frames is a key concept to generalize these duals for the super Hilbert space $L^2(\mathscr{G}) \oplus \cdots \oplus L^2(\mathscr{G})$ (N-copies) or $\oplus^N L^2(\mathscr{G})$ [43]. In the next chapter, we characterize orthogonal frame pairs and generalize the dual frames for the super Hilbert spaces.

CHAPTER 5

SUBSPACE DUAL AND ORTHOGONAL FRAMES BY ACTION OF AN ABELIAN GROUP

In this chapter, we discuss subspace duals of a frame of translates by an action of a closed abelian subgroup Γ of a locally compact group $\mathscr{G}$. These subspace duals are not required to lie in the space generated by the frame. We characterize translation generated subspace duals of a frame/Riesz basis involving the Zak transform for the pair $(\mathscr{G}, \Gamma)$. We continue our discussion on the orthogonality of two translation generated Bessel pair using the Zak transform, which allows us to explore the duals of super-frames. As an example, we extend our findings to splines, Gabor systems, p-adic fields $\mathbb{Q}_p$, locally compact abelian groups through the ways of the fiberization [64].

5.0.1. Orbit generated by a representation of a locally compact group

Let $\mathscr{G}$ be a second countable locally compact group with a Haar measure $\mu_{\mathscr{G}}$ and $\mathcal{K}$ be a closed subspace of a separable Hilbert space $\mathcal{H}$. By a *unitary representation* π of $\mathscr{G}$, we mean it is a strongly continuous group homomorphism $\pi : \mathscr{G} \to \mathscr{U}(\mathcal{H})$, where $\mathscr{U}(\mathcal{H}) = \{U : U \text{ is a unitary operator on } \mathcal{H}\}$. Then for a σ-finite measure space $\mathcal{N}$ with measure $\mu_{\mathcal{N}}$, and a family of functions $\mathscr{A} = \{\varphi_t\}_{t\in\mathcal{N}}$ in $\mathcal{H}$, an *orbit* $\mathcal{E}(\mathscr{A})$ generated by a unitary representation $(\pi, \mathcal{H})$ of $\mathscr{G}$ given by

$$\mathcal{E}(\mathscr{A}) := \{\pi(x)\varphi_t\}_{x\in\mathscr{G}, t\in\mathcal{N}}$$

is said to be a *continuous $\mathcal{K}$-subspace frame* (simply, call as *$\mathcal{K}$-subspace frame*) if the map $(x,t) \mapsto \langle f, \pi(x)\varphi_t\rangle$ from $(\mathscr{G} \times \mathcal{N})$ to $\mathbb{C}$ is measurable and there exist $0 < A \leqslant B < \infty$ such that

$$A\|f\|^2 \leqslant \int_{\mathcal{N}}\int_{\mathscr{G}} |\langle f, \pi(x)\varphi_t\rangle|^2 \, d\mu_{\mathscr{G}}(x) \, d\mu_{\mathcal{N}}(t) \leqslant B\|f\|^2 \text{ for all } f \in \mathcal{K}. \tag{5.0.1}$$

This chapter is a part of the following preprint:

S. Sarkar, N. K. Shukla, *Subspace dual and orthogonal frames by action of an abelian group*, submitted.

If $\mathcal{K} = \mathcal{H}$, then the orbit $\mathcal{E}(\mathscr{A})$ is a *frame for* $\mathcal{H}$, and it is *Bessel* in $\mathcal{H}$ when only upper bound holds in (5.0.1), and *complete* when $\overline{\text{span}}\mathcal{E}(\mathscr{A}) = \mathcal{H}$.

Definition 5.0.1. For two Bessel families $\mathcal{E}(\mathscr{A})$ and $\{g_{x,t}\}_{x\in\mathscr{G},t\in\mathcal{N}}$ in $\mathcal{H}$, if they satisfy the following reproducing formula:

$$\int_{\mathcal{N}}\int_{\mathscr{G}}\langle f, g_{x,t}\rangle\pi(x)\varphi_t \, d\mu_{\mathscr{G}}(x) \, d\mu_{\mathcal{N}}(t) = f \text{ for all } f \in \mathcal{K} \subset \mathcal{H}, \tag{5.0.2}$$

then $\{g_{x,t}\}_{x\in\mathscr{G},t\in\mathcal{N}}$ is called a $\mathcal{K}$-*subspace dual* to the orbit $\mathcal{E}(\mathscr{A})$.

Note that $\mathcal{E}(\mathscr{A})$ need not be a $\mathcal{K}$-subspace dual to $\{g_{x,t}\}_{x\in\mathscr{G},t\in\mathcal{N}}$, but the family $\{g_{x,t} + h_{x,t}\}_{x\in\mathscr{G},t\in\mathcal{N}}$ is a $\mathcal{K}$-subspace dual to $\mathcal{E}(\mathscr{A})$ provided

$$\int_{\mathcal{N}}\int_{\mathscr{G}}\langle f, h_{x,t}\rangle\pi(x)\varphi_t \, d\mu_{\mathscr{G}}(x) \, d\mu_{\mathcal{N}}(t) = 0 \text{ for all } f \in \mathcal{K}, \tag{5.0.3}$$

where $\{h_{x,t}\}_{x\in\mathscr{G},t\in\mathcal{N}}$ is Bessel in $\mathcal{H}$. Such $\{h_{x,t}\}_{x\in\mathscr{G},t\in\mathcal{N}}$ satisfying (5.0.3) is known as $\mathcal{K}$-*subspace orthogonal* to the orbit $\mathcal{E}(\mathscr{A})$.

Every frame or a Bessel family is associated with an analysis operator, the range of which carries out a lot of information of a signal/image or function. Given a Bessel family $\mathcal{E}(\mathscr{A})$ in $\mathcal{H}$ we define a bounded linear operator $T_{\mathcal{E}(\mathscr{A})} : \mathcal{H} \to L^2(\mathscr{G} \times \mathcal{N})$, known as *analysis operator*, by

$$T_{\mathcal{E}(\mathscr{A})}(f)(x,t) = \langle f, \pi(x)\varphi_t\rangle \text{ for all } (x,t) \in \mathscr{G} \times \mathcal{N}, \text{ and } f \in \mathcal{H}, \tag{5.0.4}$$

and its adjoint operator $T^*_{\mathcal{E}(\mathscr{A})} : L^2(\mathscr{G} \times \mathcal{N}) \to \mathcal{H}$, known as *synthesis operator*, by

$$T^*_{\mathcal{E}(\mathscr{A})}\psi = \int_{\mathcal{N}}\int_{\mathscr{G}}\psi(x,t)\pi(x)\varphi_t \, d\mu_{\mathscr{G}}(x) \, d\mu_{\mathcal{N}}(t) \text{ for all } \psi \in L^2(\mathscr{G} \times \mathcal{N}), \tag{5.0.5}$$

in the weak sense. For two Bessel families $\mathcal{E}(\mathscr{A})$ and $\mathcal{Y} := \{g_{x,t}\}_{x\in\mathscr{G},t\in\mathcal{N}}$ in $\mathcal{H}$, the operator $T^*_{\mathcal{E}(\mathscr{A})}T_{\mathcal{Y}} : \mathcal{H} \to \mathcal{H}$ given by $f \mapsto \int_{\mathcal{N}}\int_{\mathscr{G}}\langle f, g_{x,t}\rangle\pi(x)\varphi_t \, d\mu_{\mathscr{G}}(x) \, d\mu_{\mathcal{N}}(t)$ is known as *mixed dual Gramian*.

Definition 5.0.2. Let $\mathcal{E}(\mathscr{A})$ and $\mathcal{Y}$ be frames for $\overline{\text{span}}\mathcal{E}(\mathscr{A}) = \mathcal{S}(\mathscr{A}) \subseteq \mathcal{H}$, and $\overline{\text{span}}\mathcal{Y}$, respectively.

(i) If $T^*_{\mathcal{E}(\mathscr{A})}T_{\mathcal{Y}} = I_{\mathcal{S}(\mathscr{A})}$ on $\mathcal{S}(\mathscr{A}) = \overline{\text{span}}\mathcal{Y}$ then $\mathcal{E}(\mathscr{A})$ and $\mathcal{Y}$ are *dual frame* to each other, where $I_{\mathcal{S}(\mathscr{A})}$ is an identity operator on $\mathcal{S}(\mathscr{A})$.

(ii) If $T^*_{\mathcal{E}(\mathscr{A})}T_{\mathcal{Y}} = 0$ on $\mathcal{S}(\mathscr{A}) = \overline{\text{span}}\mathcal{Y}$, $\mathcal{E}(\mathscr{A})$ and $\mathcal{Y}$ are *orthogonal frame pair*.

Orthogonal frame pair plays a prominent role to generalize dual frames for super Hilbert spaces [73]. For more details on orthogonal Bessel families, we refer [19, 39, 40, 54, 73] . Next for the case of $\mathcal{K}$-subspace dual (5.0.2) and $\mathcal{K}$-subspace orthogonal (5.0.3) to the orbit $\mathcal{E}(\mathscr{A})$, we can write $T^*_{\mathcal{E}(\mathscr{A})}T_{\mathcal{Y}}\big|_{\mathcal{K}} = I_{\mathcal{K}}$ and $T^*_{\mathcal{E}(\mathscr{A})}T_{\mathcal{Z}}\big|_{\mathcal{K}} = 0$, respectively, where $\mathcal{Z} = \{h_{x,t}\}_{x\in\mathcal{G},t\in\mathcal{N}}$ is Bessel family in $\mathcal{H}$. In particular, if the orbit $\mathcal{E}(\mathscr{A})$ is a $\mathcal{K}$-subspace frame for $\mathcal{K} = \overline{\text{span}}\mathcal{E}(\mathscr{A})$, and $T^*_{\mathcal{E}(\mathscr{A})}T_{\mathcal{Y}}\big|_{\mathcal{K}} = I_{\mathcal{K}}$, then $\mathcal{Y}$ is an *alternate dual* to the orbit $\mathcal{E}(\mathscr{A})$. We refer [26, 44, 46] for more details on alternate duals.

5.0.2. Transformation of the orbit to a translation generated system

For a sequence $\mathscr{A} = \{\varphi_i\}_i$ in $\mathcal{H}$, there is a correspondence between the orbit $\mathcal{E}(\mathscr{A})$ and a translation-invariant system generated by a sequence of functions $\{f_i\}_i$ in $L^2(\mathscr{G})$ with the action of Γ, where Γ is a closed abelian subgroup of a locally compact group $\mathscr{G}$ [19] . Infact, there is a unitary map $U : \overline{\text{span}}\mathcal{E}(\mathscr{A}) \to \overline{\text{span}}\{\pi_L(x)f_i\}_{x\in\Gamma,i}$ such that $U\pi(x)\varphi_i = \pi_L(x)f_i$, for all i and $x \in \Gamma$, where the *left regular representation* $\pi_L : \mathscr{G} \to \mathscr{U}(L^2(\mathscr{G}))$ is defined by

$$[\pi_L(x)]f(y) = f(y^{-1}x) =: (L_x f)(y) \text{ for all } f \in L^2(\mathscr{G}) \text{ and } x, y \in \mathscr{G}.$$

The left regular representation π_L is unitary. Using the information that has been provided so far, we build a translation-invariant system that is indexed by a σ-finite measure space called $\mathcal{N}$ in order to cover the extensive class of $\mathcal{E}(\mathscr{A})$.

For a closed abelian subgroup Γ of a second countable locally compact group $\mathscr{G}$ (not necessarily abelian), we recall, the Γ-*translation generated (TG)* system $\mathcal{E}^\Gamma(\mathscr{A})$ and its associated Γ-translation invariant space $\mathcal{S}^\Gamma(\mathscr{A})$ from (1.3.1):

$$\mathcal{E}^\Gamma(\mathscr{A}) := \{L_\gamma\varphi : \gamma \in \Gamma, \varphi \in \mathscr{A}\}, \text{ and } \mathcal{S}^\Gamma(\mathscr{A}) := \overline{\text{span}}\{L_\gamma\varphi : \gamma \in \Gamma, \varphi \in \mathscr{A}\}.$$

By a Γ-*translation invariant (*Γ*-TI)* space V, we mean $L_\xi f \in V$ for all $f \in V$ and $\xi \in \Gamma$, where V is a closed subspace of $L^2(\mathscr{G})$. For $\mathscr{A} = \{\varphi\}$, we denote $\mathcal{E}^\Gamma(\mathscr{A})$ and $\mathcal{S}^\Gamma(\mathscr{A})$ by $\mathcal{E}^\Gamma(\varphi)$ and $\mathcal{S}^\Gamma(\varphi)$, respectively. In this scenario, our main goal is to provide a detailed study of $\mathcal{S}^\Gamma(\mathscr{A})$-subspace duals of a Bessel family/frame $\mathcal{E}^\Gamma(\mathscr{A})$ in $L^2(\mathscr{G})$ due to its wide use in the various areas like harmonic analysis, mathematical physics, etc. Our results have so many predecessors related to the work on subspace and alternate duals, orthogonal Bessel pair, etc. [19, 25, 26, 39, 40, 44, 46, 54, 73]. The purpose of this section is devoted to characterize a pair of orthogonal frames, and subspace dual of a Bessel family/frame generated by the Γ-TG system $\mathcal{E}^{\Gamma}(\mathscr{A})$ in $L^2(\mathscr{G})$. When $\mathcal{E}^{\Gamma}(\mathscr{A})$ is a Riesz basis then there is a associated biorthogonal system, which forms a unique dual and it is called *biorthogonal dual*. A brief study of biorthogonal system with discrete translation is discussed here. We characterize such results using the Zak transform $\mathcal{Z}$ for the pair $(\mathscr{G}, \Gamma)$ defined by (4.1.1). For the case of locally compact abelian group $\mathcal{G}$, we use the fiberization map $\mathscr{T}$ which unifies the classical results related to the orthogonal and duals of a Bessel family/frame associated with a TI space. This study of frames for their orthogonality also enable us to discuss dual for the super Hilbert space $\oplus^N L^2(\mathscr{G})$.

5.1. Subspace dual of a frame by a discrete abelian group action

Throughout the section we assume that Γ is a closed discrete abelian subgroup of a second countable locally compact group $\mathscr{G}$. In this section we study $\mathcal{S}^{\Gamma}(\mathscr{A})$-subspace duals of a Bessel/frame sequence $\mathcal{E}^{\Gamma}(\mathscr{A})$ in $L^2(\mathscr{G})$ in terms of the Zak transform for the pair $(\mathscr{G}, \Gamma)$. Such study on the pair $(\mathscr{G}, \Gamma)$ allows to access the various number of previously inaccessible pairs, like $(\mathbb{R}^n, \mathbb{Z}^m)$, $(\mathbb{Z}^n, \mathbb{Z}^m)$, $(\mathbb{Z}_N^n, \mathbb{Z}_N^m)$, etc., where $n \geqslant m$ and $\mathbb{Z}_N$ is a group modulo N. In the setup of discrete group Γ, the Zak transform $\mathcal{Z}$ for the pair $(\mathscr{G}, \Gamma)$ can be rewritten from (4.1.1) as follows, for $f \in L^1(\mathscr{G}) \cap L^2(\mathscr{G})$,

$$(\mathcal{Z}f)(\alpha, \Gamma x) = \widehat{f^{\Gamma x}}(\alpha) = \sum_{\gamma \in \Gamma} f^{\Gamma x}(\gamma)\alpha(\gamma^{-1}) \text{ for } \alpha \in \widehat{\Gamma} \text{ and } \Gamma x \in \Gamma \backslash \mathscr{G}.$$

In the present section we discuss subspace dual and orthogonal frames for Γ-TI spaces generated by a countable number of functions $\mathscr{A} = \{\varphi_t : t \in \mathcal{N}\}$ in $L^2(\mathscr{G})$, where $\mathcal{N}$ is a σ-finite measure space having counting measure. We refer [14, 25, 54, 73] regarding the orthogonality and dual frame related results of a frame in the Euclidean spaces and LCA groups using Fourier transform. We begin with the notion of matrix elements for the left regular representation [34, Section 5.2].

Definition 5.1.1. For $\varphi, \psi \in L^2(\mathscr{G})$, let $\mathcal{W}_{\varphi}\psi : \Gamma \to \mathbb{C}$ be a function defined by

$$(\mathcal{W}_{\varphi}\psi)(\gamma) = \langle \psi, \pi_L(\gamma)\varphi \rangle = \langle \psi, L_{\gamma}\varphi \rangle, \ \gamma \in \Gamma.$$

Then $\mathcal{M}_{\varphi}\psi$ is known as a *matrix element* of the left regular representation π_L associated with φ and ψ.

In the sequel we require the discrete-time Fourier transform of $\mathcal{W}_\varphi\psi$ at a point of $\widehat{\Gamma}$. Recall, a *discrete-time Fourier transform* $\widehat{z}(\alpha)$ of a sequence $z = (z(\gamma)) \in \ell^2(\Gamma)$ at a point $\alpha \in \widehat{\Gamma}$, defined by $\widehat{z}(\alpha) = \sum_{\gamma\in\Gamma} z(\gamma)\overline{\alpha(\gamma)}$. The convergence of series is interpreted as its limit in $L^2(\widehat{\Gamma})$. Next we describe the discrete-time Fourier transform of $\mathcal{W}_\varphi\psi$ in terms of the Zak transform associated with the right cosets in $\Gamma\backslash\mathscr{G}$.

Lemma 5.1.2. *Let $\varphi, \psi \in L^2(\mathscr{G})$ be such that the associated matrix element $\mathcal{W}_\varphi\psi$ is a member of $\ell^2(\Gamma)$. Then the discrete-time Fourier transform of $\mathcal{W}_\varphi\psi$ at $\alpha \in \widehat{\Gamma}$ is*

$$\widehat{(\mathcal{W}_\varphi\psi)}(\alpha) = [\mathcal{Z}\psi, \mathcal{Z}\varphi](\alpha),$$

provided $[\mathcal{Z}\psi, \mathcal{Z}\varphi](\cdot) \in L^2(\widehat{\Gamma})$, where the complex valued function $[\mathcal{Z}\psi, \mathcal{Z}\varphi](\cdot)$ on $\widehat{\Gamma}$ is given by

$$[\mathcal{Z}\psi, \mathcal{Z}\varphi](\alpha) := \int_{\Gamma\backslash\mathscr{G}} \mathcal{Z}\psi(\alpha, \Gamma x)\ \overline{\mathcal{Z}\varphi(\alpha, \Gamma x)}\ d\mu_{\Gamma\backslash\mathscr{G}}(\Gamma x) \text{ for } \alpha \in \widehat{\Gamma}. \tag{5.1.1}$$

Moreover, for a Bessel sequence $\mathcal{E}^\Gamma(\varphi) = \{L_\gamma\varphi : \gamma \in \Gamma\}$ in $L^2(\mathscr{G})$, $\mathcal{W}_\varphi\psi$ and $[\mathcal{Z}\psi, \mathcal{Z}\varphi](\cdot)$ are members of $\ell^2(\Gamma)$ and $L^2(\widehat{\Gamma})$, respectively, and hence the above result holds true.

Proof. Since $\mathcal{Z}$ is unitary, the discrete-time Fourier transform of $\{\mathcal{W}_\varphi\psi(\gamma)\}_{\gamma\in\Gamma} \in \ell^2(\Gamma)$ at $\alpha \in \widehat{\Gamma}$ is,

$$\widehat{(\mathcal{W}_\varphi\psi)}(\alpha) = \sum_{\gamma\in\Gamma}\langle\psi, L_\gamma\varphi\rangle\alpha(\gamma^{-1}) = \sum_{\gamma\in\Gamma}\langle\mathcal{Z}\psi, \mathcal{Z}(L_\gamma\varphi)\rangle_{L^2(\widehat{\Gamma}\times\Gamma\backslash\mathscr{G})}\ \alpha(\gamma^{-1}).$$

Employing the intertwining property of Zak transform $\mathcal{Z}$ on the left translation with modulation, i.e., for $f \in L^1(\mathscr{G}) \cap L^2(\mathscr{G})$, $\gamma \in \Gamma$, $\alpha \in \widehat{\Gamma}$, $x \in \mathscr{G}$, and using (4.1.1),

$$\begin{aligned}\mathcal{Z}(L_\gamma f)(\alpha, \Gamma x) &= \sum_{\lambda\in\Gamma}(L_\gamma f)^{\Gamma x}(\lambda)\alpha(\lambda^{-1}) = \sum_{\lambda\in\Gamma} f((\gamma^{-1}\lambda)\Xi(\Gamma x))\alpha(\lambda^{-1}) \\ &= \sum_{\lambda\in\Gamma} f^{\Gamma x}(\gamma^{-1}\lambda)\alpha(\lambda^{-1}) = \alpha(\gamma^{-1})\mathcal{Z}f(\alpha, \Gamma x).\end{aligned}$$

Then we obtain

$$\widehat{(\mathcal{W}_\varphi\psi)}(\alpha) = \sum_{\gamma\in\Gamma}\alpha(\gamma^{-1})\int_{\widehat{\Gamma}}\beta(\gamma)\langle\mathcal{Z}\psi(\beta), \mathcal{Z}\varphi(\beta)\rangle_{L^2(\Gamma\backslash\mathscr{G})}\ d\mu_{\widehat{\Gamma}}(\beta) = \sum_{\gamma\in\Gamma}\alpha(\gamma^{-1})\zeta(\gamma), \tag{5.1.2}$$

where for $\gamma \in \Gamma$, the function $\zeta(\gamma) := \int_{\widehat{\Gamma}}\beta(\gamma)\langle\mathcal{Z}\psi(\beta), \mathcal{Z}\varphi(\beta)\rangle\ d\mu_{\widehat{\Gamma}}(\beta)$ is identical with $\mathcal{W}_\varphi\psi(\gamma)$. The sequence $\{\zeta(\gamma)\}_{\gamma\in\Gamma} \in \ell^2(\Gamma)$ since $\{\mathcal{W}_\varphi\psi(\gamma)\}_{\gamma\in\Gamma} \in \ell^2(\Gamma)$. Further, we can write (5.1.2) for $\alpha \in \widehat{\Gamma}$ as follows:

$$
\begin{aligned}
\widehat{(\mathcal{W}_\varphi\psi)}(\alpha) &= \sum_{\gamma\in\Gamma} \alpha(\gamma^{-1})\zeta(\gamma) = \sum_{\gamma\in\Gamma} \left(\int_{\widehat{\Gamma}} \beta(\gamma) \langle \mathcal{Z}\psi(\beta), \mathcal{Z}\varphi(\beta) \rangle \, d\mu_{\widehat{\Gamma}}(\beta) \right) \alpha(\gamma^{-1}) \\
&= \sum_{\gamma\in\Gamma} \left(\int_{\widehat{\Gamma}} \beta(\gamma) \left(\int_{\Gamma\backslash\mathscr{G}} \mathcal{Z}\psi(\beta, \Gamma x)\overline{\mathcal{Z}\varphi(\beta, \Gamma x)} \, d\mu_{\Gamma\backslash\mathscr{G}}(\Gamma x) \right) d\mu_{\widehat{\Gamma}}(\beta) \right) \alpha(\gamma^{-1}) \\
&= \sum_{\gamma\in\Gamma} \left(\int_{\widehat{\Gamma}} [\mathcal{Z}\psi, \mathcal{Z}\varphi](\beta)\beta(\gamma) \, d\mu_{\widehat{\Gamma}}(\beta) \right) \alpha(\gamma^{-1}),
\end{aligned}
$$

where for $\beta \in \widehat{\Gamma}$, $[\mathcal{Z}\psi, \mathcal{Z}\varphi](\beta)$ is defined by $[\mathcal{Z}\psi, \mathcal{Z}\varphi](\beta) = \int_{\Gamma\backslash\mathscr{G}} \mathcal{Z}\psi(\beta, \Gamma x)\overline{\mathcal{Z}\varphi(\beta, \Gamma x)} \, d\mu_{\Gamma\backslash\mathscr{G}}(\Gamma x)$. Also by identifying Γ to $\widehat{\widehat{\Gamma}}$ as $\gamma \mapsto \widehat{\gamma}$ and noting that $\widehat{\widehat{\Gamma}}$ is an orthonormal basis for $L^2(\widehat{\Gamma})$, we can write

$$
\begin{aligned}
\widehat{(\mathcal{W}_\varphi\psi)}(\alpha) &= \sum_{\gamma\in\Gamma} \left(\int_{\widehat{\Gamma}} [\mathcal{Z}\psi, \mathcal{Z}\varphi](\beta)\widehat{\gamma}(\beta) \, d\mu_{\widehat{\Gamma}}(\beta) \right) \overline{\widehat{\gamma}(\alpha)} \\
&= \sum_{\gamma\in\Gamma} \left\langle [\mathcal{Z}\psi, \mathcal{Z}\varphi](\cdot), \overline{\widehat{\gamma}}(\cdot) \right\rangle \overline{\widehat{\gamma}(\alpha)} = [\mathcal{Z}\psi, \mathcal{Z}\varphi](\alpha),
\end{aligned}
$$

provided $[\mathcal{Z}\psi, \mathcal{Z}\varphi](\cdot) \in L^2(\widehat{\Gamma})$.

For the moreover part, assume that the Γ-TG system $\mathcal{E}^\Gamma(\varphi)$ is Bessel. Then for all $f \in L^2(\mathscr{G})$ we have the inequality $\sum_{\gamma\in\Gamma} | < f, L_\gamma\varphi > |^2 \leqslant B\|f\|^2$ for some constant $B > 0$, and hence by choosing $f = \psi$, we get $\mathcal{W}_\varphi\psi$ as a member of $\ell^2(\Gamma)$. Also the Bessel property of $\mathcal{E}^\Gamma(\varphi)$ implies $[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha) \leqslant B$ a.e. $\alpha \in \widehat{\Gamma}$ [49], and hence using the Cauchy-Schwarz inequality

$$
\begin{aligned}
\int_{\widehat{\Gamma}} \left| [\mathcal{Z}\psi, \mathcal{Z}\varphi](\alpha) \right|^2 d\mu_{\widehat{\Gamma}}(\alpha) &= \int_{\widehat{\Gamma}} \left| \int_{\Gamma\backslash\mathscr{G}} \mathcal{Z}\psi(\alpha, \Gamma x)\overline{\mathcal{Z}\varphi(\alpha, \Gamma x)} \, d\mu_{\Gamma\backslash\mathscr{G}}(\Gamma x) \right|^2 d\mu_{\widehat{\Gamma}}(\alpha) \\
&\leqslant \int_{\widehat{\Gamma}} \left(\int_{\Gamma\backslash\mathscr{G}} \left| \mathcal{Z}\psi(\alpha, \Gamma x) \right|^2 d\mu_{\Gamma\backslash\mathscr{G}}(\Gamma x) \right) \left(\int_{\Gamma\backslash\mathscr{G}} \left| \mathcal{Z}\varphi(\alpha, \Gamma x) \right|^2 d\mu_{\Gamma\backslash\mathscr{G}}(\Gamma x) \right) d\mu_{\widehat{\Gamma}}(\alpha) \\
&= \int_{\widehat{\Gamma}} [\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha)[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha) \, d\mu_{\widehat{\Gamma}}(\alpha) \\
&\leqslant B \int_{\widehat{\Gamma}} [\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha) \, d\mu_{\widehat{\Gamma}}(\alpha) = B\|\mathcal{Z}\psi\|^2 = B\|\psi\|^2.
\end{aligned}
$$

□

When the locally compact group $\mathscr{G}$ becomes abelian, *denoted by* $\mathcal{G}$, the groups $\widehat{\mathcal{G}}/\Lambda^\perp$ and $\widehat{\mathcal{G}/\Lambda}$ are topologically isomorphic to $\widehat{\Lambda}$ and $\Lambda^\perp$, respectively [34], where Λ is a closed discrete subgroup of $\mathcal{G}$. Instead of the Zak transform $\mathcal{Z}$ for the pair $(\mathscr{G}, \Gamma)$, we will use the fiberization map $\mathscr{T} : L^2(\mathcal{G}) \to L^2(\widehat{\mathcal{G}}/\Lambda^\perp; L^2(\Lambda^\perp))$ for the pair $(\mathcal{G}, \Lambda)$ which is unitarily defined by

$$(\mathscr{T} f)(\omega\Lambda^\perp)(\xi) = \widehat{f}(\Theta(w\Lambda^\perp)\xi),$$

for $f \in L^2(\mathcal{G})$ $w\Lambda^\perp \in \widehat{\mathscr{G}}/\Lambda^\perp$ and $\xi \in \Lambda^\perp$, where the Borel section $\Theta : \widehat{\mathcal{G}}/\Lambda^\perp \to \widehat{\mathcal{G}}$ sends compact sets to pre-compact sets.

For any $\varphi, \psi \in L^2(\mathcal{G})$, next we obtain a relation between $[\mathcal{Z}\psi, \mathcal{Z}\varphi](\cdot)$ and $[\mathscr{T}\psi, \mathscr{T}\varphi](\cdot)$ in the setup of locally compact abelian (LCA) group $\mathcal{G}$ and its closed discrete subgroup Λ, where

$$(5.1.3) \quad [\mathscr{T}\psi, \mathscr{T}\varphi](\omega\Lambda^\perp) := \int_{\Lambda^\perp} \mathscr{T}\psi(\omega\Lambda^\perp)(\xi)\overline{\mathscr{T}\varphi(\omega\Lambda^\perp)(\xi)}\, d\mu_{\Lambda^\perp}(\xi) \;\text{ for } \omega\Lambda^\perp \in \widehat{\mathcal{G}}/\Lambda^\perp,$$

which is a reminiscence of [49]. Since $[\mathcal{Z}\psi, \mathcal{Z}\varphi](\cdot) \in L^1(\widehat{\Lambda})$ from (5.1.2) the Fourier transform $\mathcal{F}$ of $[\mathcal{Z}\psi, \mathcal{Z}\varphi](\cdot)$ at $\lambda \in \Lambda$ can be written as

$$\begin{aligned}
\mathcal{F}[\mathcal{Z}\psi, \mathcal{Z}\varphi](\lambda) &= \int_{\widehat{\Lambda}} [\mathcal{Z}\psi, \mathcal{Z}\varphi](\beta)\overline{\beta(\lambda)}\, d\mu_{\widehat{\Lambda}}(\beta) \\
&= \int_{\widehat{\Lambda}} \left(\int_{\Lambda\backslash\mathcal{G}} \mathcal{Z}\psi(\beta, \Lambda x)\overline{\mathcal{Z}\varphi(\beta, \Lambda x)}\, d\mu_{\Lambda\backslash\mathcal{G}} \right) \overline{\beta(\lambda)}\, d\mu_{\widehat{\Lambda}}(\beta) \\
&= \int_{\widehat{\Lambda}} \langle \mathcal{Z}\psi(\beta), \mathcal{Z}\varphi(\beta)\rangle \overline{\beta(\lambda)}\, d\mu_{\widehat{\Lambda}}(\beta) \\
&= \int_{\widehat{\Lambda}} \langle \mathcal{Z}(L_\lambda\psi)(\beta), \mathcal{Z}\varphi(\beta)\rangle\, d\mu_{\widehat{\Lambda}}(\beta) \\
&= \langle L_\lambda\psi, \varphi\rangle.
\end{aligned}$$

Since $[\mathscr{T}\psi, \mathscr{T}\varphi](\cdot) \in L^1(\widehat{\mathcal{G}}/\Lambda^\perp)$, from the similar calculations of (5.1.2), and the groups $\widehat{\mathcal{G}}/\Lambda^\perp$ and $\widehat{\Lambda}$ are topologically isomorphic, the Fourier transform $\mathcal{F}$ of $[\mathscr{T}\psi, \mathscr{T}\varphi](\cdot)$ at $\lambda \in \Lambda$ can be written as follows:

$$\begin{aligned}
\mathcal{F}[\mathscr{T}\psi, \mathscr{T}\varphi](\lambda) &= \int_{\widehat{\Lambda}} [\mathscr{T}\psi, \mathscr{T}\varphi](\omega\Lambda^\perp)\overline{\omega(\lambda)}\, d\mu_{\widehat{\Lambda}}(\omega|_\Lambda) \\
&= \int_{\widehat{\Lambda}} \left(\int_{\Lambda^\perp} \mathscr{T}\psi(\omega\Lambda^\perp)(\xi)\overline{\mathscr{T}\varphi(\omega\Lambda^\perp)(\xi)}\, d\mu_{\Lambda^\perp}(\xi) \right) \overline{\omega(\lambda)}\, d\mu_{\widehat{\Lambda}}(\omega|_\Lambda) \\
&= \int_{\widehat{\Lambda}} \langle \mathscr{T}\psi(\omega\Lambda^\perp), \mathscr{T}\varphi(\omega\Lambda^\perp)\rangle \overline{\omega(\lambda)}\, d\mu_{\widehat{\Lambda}}(\omega|_\Lambda) \\
&= \int_{\widehat{\mathcal{G}}/\Lambda^\perp} \langle \mathscr{T}\psi(\omega\Lambda^\perp), \mathscr{T}\varphi(\omega\Lambda^\perp)\rangle \overline{\omega(\lambda)}\, d\mu_{\widehat{\mathcal{G}}/\Lambda^\perp}(w\Lambda^\perp).
\end{aligned}$$

Employing the unitary property of the fiberization map $\mathscr{T}$, we have

$$\mathcal{F}[\mathscr{T}\psi, \mathscr{T}\varphi](\lambda) = \int_{\widehat{\mathcal{G}}/\Lambda^{\perp}} \langle \mathscr{T} L_{\lambda}\psi(\omega\Lambda^{\perp}), \mathscr{T}\varphi(\omega\Lambda^{\perp})\rangle \, d\mu_{\widehat{\mathcal{G}}/\Lambda^{\perp}}(\omega\Lambda^{\perp}) = \langle L_{\lambda}\psi, \varphi\rangle,$$

since the fiberization map $\mathscr{T}$ intertwines left translation with modulation, i.e.,

$$\mathscr{T} L_{\lambda} f(\omega\Lambda^{\perp})(\xi) = \widehat{(L_{\lambda} f)}(\Theta(\omega\Lambda^{\perp})\xi) = \Theta(\omega\Lambda^{\perp})(\lambda^{-1})\xi(\lambda^{-1})\mathscr{T} f(\omega\Lambda^{\perp})(\xi) = \omega(\lambda^{-1})\mathscr{T} f(\omega\Lambda^{\perp})(\xi),$$

as Fourier transform intertwines left translation with modulation and $\Theta : \widehat{\mathcal{G}}/\Lambda^{\perp} \to \widehat{\mathcal{G}}$ is a Borel section $\Theta(\omega\Lambda^{\perp}) = \omega\eta$ for some $\eta \in \Lambda^{\perp}$ and $\eta(\lambda^{-1}) = \xi(\lambda^{-1}) = 1$. Therefore for all $\lambda \in \Lambda$, we have $\mathcal{F}[\mathcal{Z}\psi, \mathcal{Z}\varphi](\lambda) = \mathcal{F}[\mathscr{T}\psi, \mathscr{T}\varphi](\lambda)$ which implies

$$[\mathcal{Z}\psi, \mathcal{Z}\varphi](\omega|_{\Lambda}) = [\mathscr{T}\psi, \mathscr{T}\varphi](\omega\Lambda^{\perp}) \text{ a.e. } \omega \in \widehat{\mathcal{G}}, \tag{5.1.4}$$

since the Fourier transform $\mathcal{F} : L^1(\widehat{\Lambda}) \to C_0(\Lambda)$ is injective. Thus by using the relation (5.1.4), we state the following result analogous to Lemma 5.1.2 for the case of an LCA group $\mathcal{G}$ and its closed discrete subgroup Λ in terms of the fiberization. In particular the same result can be realized for the case of uniform lattice Λ. By a *uniform lattice* Λ, we mean it is a closed discrete subgroup of an LCA group $\mathcal{G}$ such that $\mathcal{G}/\Lambda$ is compact.

Lemma 5.1.3. *Let $\mathcal{G}$ be a locally compact abelian group and Λ be a closed discrete subgroup of $\mathcal{G}$. If $\varphi, \psi \in L^2(\mathcal{G})$ such that the matrix element $\mathcal{W}_{\varphi}\psi$ is a member of $\ell^2(\Lambda)$, then the discrete-time Fourier transform of $\mathcal{W}_{\varphi}\psi$ in terms of the fiberization for the abelian pair $(\mathcal{G}, \Lambda)$ is*

$$\widehat{(\mathcal{W}_{\varphi}\psi)}(\omega|_{\Lambda}) = [\mathscr{T}\psi, \mathscr{T}\varphi](\omega\Lambda^{\perp}), \; \omega\Lambda^{\perp} \in \widehat{\mathcal{G}}/\Lambda^{\perp},$$

provided $[\mathscr{T}\psi, \mathscr{T}\varphi](\cdot) \in L^2(\widehat{\mathcal{G}}/\Lambda^{\perp})$, where the complex valued function $[\mathscr{T}\psi, \mathscr{T}\varphi](\cdot)$ on $\widehat{\mathcal{G}}/\Lambda^{\perp}$ is given by (5.1.3). Moreover, for a Bessel sequence $\mathcal{E}^{\Lambda}(\varphi)$ in $L^2(\mathcal{G})$, $\mathcal{W}_{\varphi}\psi$ and $[\mathscr{T}\psi, \mathscr{T}\varphi](\cdot)$ are members of $\ell^2(\Lambda)$ and $L^2(\widehat{\mathcal{G}}/\Lambda^{\perp})$, respectively, and hence the above result holds true.

The following result plays an important role to study the duals of a Γ-TG system using the Zak transform. As an additional point of reference, it expresses the transition from the role of Γ to $\widehat{\Gamma}$.

Lemma 5.1.4. *Let φ and ψ be two functions in $L^2(\mathscr{G})$ be such that the corresponding Γ-TG systems defined as in (1.3.1), $\mathcal{E}^{\Gamma}(\varphi) = \{L_{\gamma}\varphi : \gamma \in \Gamma\}$ and $\mathcal{E}^{\Gamma}(\psi) = \{L_{\gamma}\psi : \gamma \in \Gamma\}$*

are Bessel. Then for all $f, g \in L^2(\mathscr{G})$, we have

$$\sum_{\gamma\in\Gamma} \langle f, L_\gamma\varphi\rangle\langle L_\gamma\psi, g\rangle = \int_{\widehat{\Gamma}} [\mathcal{Z}f, \mathcal{Z}\varphi](\alpha)[\mathcal{Z}\psi, \mathcal{Z}g](\alpha)\ d\mu_{\widehat{\Gamma}}(\alpha).$$

Moreover, when the pair $(\mathscr{G}, \Gamma)$ is an abelian pair $(\mathcal{G}, \Lambda)$ for $f, g \in L^2(\mathcal{G})$ and a.e. $\omega\Lambda^\perp \in \widehat{\mathcal{G}}/\Lambda^\perp$,

$$\sum_{\lambda\in\Lambda} \langle f, L_\lambda\varphi\rangle\langle L_\lambda\psi, g\rangle = \int_{\widehat{\mathcal{G}}/\Lambda^\perp} [\mathscr{T}f, \mathscr{T}\varphi](\omega\Lambda^\perp)[\mathscr{T}\psi, \mathscr{T}g](\omega\Lambda^\perp)\ d\mu_{\widehat{\mathcal{G}}/\Lambda^\perp}(\omega\Lambda^\perp),$$

in terms of the fiberization $\mathscr{T}$.

Proof. For all $f \in L^2(\mathscr{G})$ and from (5.1.1),

$$\langle f, L_\gamma\varphi\rangle = \langle \mathcal{Z}f, \mathcal{Z}L_\gamma\varphi\rangle = \int_{\widehat{\Gamma}} \langle \mathcal{Z}f(\alpha), \mathcal{Z}\varphi(\alpha)\rangle\alpha(\gamma)\ d\mu_{\widehat{\Gamma}}(\alpha) = \int_{\widehat{\Gamma}} [\mathcal{Z}f, \mathcal{Z}\varphi](\alpha)\alpha(\gamma)\ d\mu_{\widehat{\Gamma}}(\alpha).$$

Since the Γ-TG system $\mathcal{E}^\Gamma(\varphi)$ is Bessel, we have $[\mathcal{Z}f, \mathcal{Z}\varphi](\cdot) \in L^2(\widehat{\Gamma})$ from Lemma 5.1.2, and hence using the inverse Fourier transform $[\mathcal{Z}f, \mathcal{Z}\varphi]^\vee(\gamma)$ at $\gamma \in \Gamma$, the above expression can be written as follows:

$$\langle f, L_\gamma\varphi\rangle = [\mathcal{Z}f, \mathcal{Z}\varphi]^\vee(\gamma).$$

Similarly, we have $\langle L_\gamma\psi, g\rangle = [\mathcal{Z}\psi, \mathcal{Z}g]^\vee(\gamma)$ for $\gamma \in \Gamma$ and $g \in L^2(\mathscr{G})$. Further, note that the sequences $\{[\mathcal{Z}f, \mathcal{Z}\varphi]^\vee(\gamma)\}_{\gamma\in\Gamma}$ and $\{[\mathcal{Z}\psi, \mathcal{Z}g]^\vee(\gamma)\}_{\gamma\in\Gamma}$ are members of $\ell^2(\Gamma)$ follow from Lemma 5.1.2. Hence the result follows by observing the Parseval formula on $\ell^2(\Gamma)$ in the following calculation,

$$\begin{aligned}\sum_{\gamma\in\Gamma} \langle f, L_\gamma\varphi\rangle\langle L_\gamma\psi, g\rangle &= \sum_{\gamma\in\Gamma} \left([\mathcal{Z}f, \mathcal{Z}\varphi]^\vee(\gamma)\right)\left([\mathcal{Z}\psi, \mathcal{Z}g]^\vee(\gamma)\right) = \langle [\mathcal{Z}f, \mathcal{Z}\varphi]^\vee, [\mathcal{Z}g, \mathcal{Z}\psi]^\vee\rangle_{\ell^2(\Gamma)} \\ &= \langle [\mathcal{Z}f, \mathcal{Z}\varphi](\cdot), [\mathcal{Z}g, \mathcal{Z}\psi](\cdot)\rangle_{L^2(\widehat{\Gamma})} = \int_{\widehat{\Gamma}} [\mathcal{Z}f, \mathcal{Z}\varphi](\alpha)[\mathcal{Z}\psi, \mathcal{Z}g](\alpha)\ d\mu_{\widehat{\Gamma}}(\alpha).\end{aligned}$$

The moreover part follows from the same argument as above by substituting the Zak transform $\mathcal{Z}$ for the pair $(\mathscr{G}, \Gamma)$ with the fiberization $\mathscr{T}$ for the pair $(\mathcal{G}, \Lambda)$ by the Lemma 5.1.3. □

The next result connects analysis and synthesis operators with the pre-Gramian operator in terms of the Zak transform for the pair $(\mathscr{G}, \Gamma)$. We recall Γ-TG system $\mathcal{E}^\Gamma(\mathscr{A}) = \{L_\gamma\varphi\}_{\gamma\in\Gamma,\varphi\in\mathscr{A}}$ and its associated Γ-TI space $\mathcal{S}^\Gamma(\mathscr{A}) = \overline{\operatorname{span}}\mathcal{E}^\Gamma(\mathscr{A})$ from (1.3.1), for a countable collection $\mathscr{A} = \{\varphi_t : t \in \mathcal{N}\}$ in $L^2(\mathscr{G})$. If $\mathcal{E}^\Gamma(\mathscr{A})$ is a Bessel family in $L^2(\mathscr{G})$, then from (5.0.4), the associated analysis operator $T_{\mathcal{E}^\Gamma(\mathscr{A})} : L^2(\mathscr{G}) \to \ell^2(\Gamma \times \mathcal{N})$ is defined by $f \mapsto \{\langle f, L_\gamma\varphi_t\rangle\}_{\gamma\in\Gamma,t\in\mathcal{N}}$ and the synthesis operator is $T^*_{\mathcal{E}^\Gamma(\mathscr{A})} : \ell^2(\Gamma \times \mathcal{N}) \to L^2(\mathscr{G})$

defined by $\{h_t(\gamma)\}_{t\in\mathcal{N},\gamma\in\Gamma} \mapsto \sum_{t\in\mathcal{N}}\sum_{\gamma\in\Gamma} h_t(\gamma)L_\gamma\varphi_t$ from (5.0.5). Since $\mathcal{E}^\Gamma(\mathscr{A})$ is Bessel in $L^2(\mathscr{G})$, the system $\{\mathcal{Z}\varphi_t(\alpha) = \{\mathcal{Z}\varphi_t(\alpha,\Gamma x)\}_{\Gamma x\in\Gamma\backslash\mathscr{G}}\}_{t\in\mathcal{N}}$ is also Bessel in $L^2(\Gamma\backslash\mathscr{G})$ for a.e. $\alpha\in\widehat{\Gamma}$ [49], and hence the associated *pre-Gramian operator* $\mathfrak{J}_{\mathscr{A}}(\alpha)$ corresponding to the Bessel system $\mathcal{E}^\Gamma(\mathscr{A})$ is defined by $\mathfrak{J}_{\mathscr{A}}(\alpha):\ell^2(\mathcal{N})\to L^2(\Gamma\backslash\mathscr{G})$, $\eta=\{\eta_t\}_{t\in\mathcal{N}}\mapsto \{\sum_{t\in\mathcal{N}}\eta_t\mathcal{Z}\varphi_t(\alpha,\Gamma x)\}_{\Gamma x\in\Gamma\backslash\mathscr{G}}$, which is a well defined bounded linear operator due the Bessel system $\mathcal{Z}\mathscr{A}(\alpha)$. Further, we define its adjoint operator $\mathfrak{J}_{\mathscr{A}}(\alpha)^*: L^2(\Gamma\backslash\mathscr{G})\to\ell^2(\mathcal{N})$ by $\nu\mapsto\{\langle\nu,\mathcal{Z}\varphi_t(\alpha)\rangle\}_{t\in\mathcal{N}}$. Recall the *Gramian operator* $G_{\mathscr{A}}(\alpha)=\mathfrak{J}_{\mathscr{A}}(\alpha)^*\mathfrak{J}_{\mathscr{A}}(\alpha)$ from $\ell^2(\mathcal{N})$ to $\ell^2(\mathcal{N})$ is also bounded linear operator for a.e. $\alpha\in\widehat{\Gamma}$. For two Bessel systems $\mathcal{E}^\Gamma(\mathscr{A})$ and $\mathcal{E}^\Gamma(\mathscr{A}')$, the associated *mixed dual-Gramian operator* $\tilde{G}_{\mathscr{A},\mathscr{A}'}(\alpha)=\mathfrak{J}_{\mathscr{A}}(\alpha)\mathfrak{J}_{\mathscr{A}'}(\alpha)^*$: $L^2(\Gamma\backslash\mathscr{G})\to L^2(\Gamma\backslash\mathscr{G})$ is defined by $\langle\tilde{G}_{\mathscr{A},\mathscr{A}'}(\alpha)v_1,v_2\rangle=\sum_{t\in\mathcal{N}}\langle v_1,\mathcal{Z}\psi_t(\alpha)\rangle\,\overline{\langle v_2,\mathcal{Z}\varphi_t(\alpha)\rangle}$ for a.e. $\alpha\in\widehat{\Gamma}$, where $v_1,v_2\in L^2(\Gamma\backslash\mathscr{G})$ and $\mathscr{A}'=\{\psi_t: t\in\mathcal{N}\}\subset L^2(\mathscr{G})$. This terminology and the following proposition can be deduced using the fiberization map $\mathscr{T}$ for the abelian pair $(\mathcal{G},\Lambda)$.

Proposition 5.1.5. *Let $\mathscr{A}=\{\varphi_t\}_{t\in\mathcal{N}}$ and $\mathscr{A}'=\{\psi_t\}_{t\in\mathcal{N}}$ be two countable collections of functions in $L^2(\mathscr{G})$ such that $\mathcal{E}^\Gamma(\mathscr{A})$ and $\mathcal{E}^\Gamma(\mathscr{A}')$ are Bessel. Then the following are true:*

(i) *For each $t\in\mathcal{N}$ and $f\in L^2(\mathscr{G})$, the Fourier transform of $(T_{\mathcal{E}^\Gamma(\mathscr{A})}f)_t$ is given by*

$$\widehat{(T_{\mathcal{E}^\Gamma(\mathscr{A})}f)}_t(\alpha)=[\mathcal{Z}f,\mathcal{Z}\varphi_t](\alpha),\ \ \textit{and}\ \left\{\widehat{(T_{\mathcal{E}^\Gamma(\mathscr{A})}f)}_t(\alpha)\right\}_{t\in\mathcal{N}}=\mathfrak{J}_{\mathscr{A}}(\alpha)^*\{(\mathcal{Z}f)(\alpha,\Gamma x)\}_{\Gamma x\in\Gamma\backslash\mathscr{G}},$$

for a.e. $\alpha\in\widehat{\Gamma}$, where $(T_{\mathcal{E}^\Gamma(\mathscr{A})}f)_t=\{\langle f,L_\gamma\varphi_t\rangle\}_{\gamma\in\Gamma}$.

(ii) *For $h=\{h_t(\gamma)\}_{t\in\mathcal{N},\gamma\in\Gamma}\in\ell^2(\Gamma\times\mathcal{N})$, the Zak transform of $(T^*_{\mathcal{E}^\Gamma(\mathscr{A})}h)$ at $(\alpha,\Gamma x)\in\widehat{\Gamma}\times\Gamma\backslash\mathscr{G}$ is $\left[\mathcal{Z}(T^*_{\mathcal{E}^\Gamma(\mathscr{A})}h)\right](\alpha,\Gamma x)=\sum_{t\in\mathcal{N}}\widehat{h}_t(\alpha)\ \mathcal{Z}\varphi_t(\alpha,\Gamma x)$.*

Moreover, we have

$$\left\{\left[\mathcal{Z}(T^*_{\mathcal{E}^\Gamma(\mathscr{A})}h)\right](\alpha,\Gamma x)\right\}_{\Gamma x\in\Gamma\backslash\mathscr{G}}=\mathfrak{J}_{\mathscr{A}}(\alpha)\left\{\widehat{h}_t(\alpha)\right\}_{t\in\mathcal{N}}\ \textit{for a.e. } \alpha\in\widehat{\Gamma}.$$

(iii) *For $f\in L^2(\mathscr{G})$ and a.e. $\alpha\in\widehat{\Gamma}$,*

$$\begin{aligned}\left\{\mathcal{Z}\left(T^*_{\mathcal{E}^\Gamma(\mathscr{A})}T_{\mathcal{E}^\Gamma(\mathscr{A}')}f\right)(\alpha,\Gamma x)\right\}_{\Gamma x\in\Gamma\backslash\mathscr{G}}&=\mathfrak{J}_{\mathscr{A}}(\alpha)\mathfrak{J}_{\mathscr{A}'}(\alpha)^*\left\{\mathcal{Z}f(\alpha,\Gamma x)\right\}_{\Gamma x\in\Gamma\backslash\mathscr{G}}\\&=\tilde{G}_{\mathscr{A},\mathscr{A}'}(\alpha)\left\{(\mathcal{Z}f)(\alpha,\Gamma x)\right\}_{\Gamma x\in\Gamma\backslash\mathscr{G}}.\end{aligned}$$

Proof. (i) The Fourier transform of $(T_{\mathcal{E}^\Gamma(\mathscr{A})}f)_t$ at $\alpha\in\widehat{\Gamma}$, $\widehat{(T_{\mathcal{E}^\Gamma(\mathscr{A})}f)}_t(\alpha)=[\mathcal{Z}f,\mathcal{Z}\varphi_t](\alpha)$, follows by Lemma 5.1.2 and $(T_{\mathcal{E}^\Gamma(\mathscr{A})}f)_t=\{\langle f,L_\gamma\varphi_t\rangle\}_{\gamma\in\Gamma}=\{(\mathcal{W}_{\varphi_t}f)(\gamma)\}_{\gamma\in\Gamma}$ for each $t\in\mathcal{N}$

and $f \in L^2(\mathscr{G})$. Further,

$$\left\{(\widehat{T_{\mathcal{E}^\Gamma(\mathscr{A})}f})_t(\alpha)\right\}_{t\in\mathcal{N}} = \{[\mathcal{Z}f, \mathcal{Z}\varphi_t](\alpha)\}_{t\in\mathcal{N}} = \left\{\int_{\Gamma\backslash\mathscr{G}} (\mathcal{Z}f)(\alpha, \Gamma x)\overline{\mathcal{Z}\varphi_t(\alpha, \Gamma x)}\, d\mu_{\Gamma\backslash\mathscr{G}}(\Gamma x)\right\}_{t\in\mathcal{N}}$$
$$= \{\langle \mathcal{Z}f(\alpha), \mathcal{Z}\varphi_t(\alpha)\rangle\}_{t\in\mathcal{N}} = \mathfrak{J}_{\mathscr{A}}(\alpha)^*\{(\mathcal{Z}f)(\alpha, \Gamma x)\}_{\Gamma x\in\Gamma\backslash\mathscr{G}} \text{ a.e. } \alpha \in \widehat{\Gamma}.$$

(ii) Let $h = \{h_t(\gamma)\}_{t\in\mathcal{N},\gamma\in\Gamma} \in \ell^2(\Gamma \times \mathcal{N})$. Employing the Zak transform on the synthesis operator $(T^*_{\mathcal{E}^\Gamma(\mathscr{A})}h)$ at $(\alpha, \Gamma x) \in \widehat{\Gamma}\times\Gamma\backslash\mathscr{G}$ and the discrete-Fourier transform on the sequence $\{h_t(\gamma)\}_{\gamma\in\Gamma}$ at $\alpha \in \widehat{\Gamma}$, we obtain

$$\mathcal{Z}(T^*_{\mathcal{E}^\Gamma(\mathscr{A})}h)(\alpha, \Gamma x) = \mathcal{Z}\left(\sum_{t\in\mathcal{N}}\sum_{\gamma\in\Gamma} h_t(\gamma)L_\gamma\varphi_t\right)(\alpha, \Gamma x) = \left(\sum_{t\in\mathcal{N}}\sum_{\gamma\in\Gamma} h_t(\gamma)\mathcal{Z}(L_\gamma\varphi_t)\right)(\alpha, \Gamma x)$$
$$= \sum_{t\in\mathcal{N}}\left(\sum_{\gamma\in\Gamma} h_t(\gamma)\overline{\alpha(\gamma)}\right)\mathcal{Z}\varphi_t(\alpha, \Gamma x) = \sum_{t\in\mathcal{N}} \widehat{h}_t(\alpha)\mathcal{Z}\varphi_t(\alpha, \Gamma x).$$

Then in terms of pre-Gramian operator for a.e. $\alpha \in \widehat{\Gamma}$, we get

$$\left\{\mathcal{Z}(T^*_{\mathcal{E}^\Gamma(\mathscr{A})}h)(\alpha, \Gamma x)\right\}_{\Gamma x\in\Gamma\backslash\mathscr{G}} = \left\{\sum_{t\in\mathcal{N}} \widehat{h}_t(\alpha)\mathcal{Z}\varphi_t(\alpha, \Gamma x)\right\}_{\Gamma x\in\Gamma\backslash\mathscr{G}} = \mathfrak{J}_{\mathscr{A}}(\alpha)\left\{\widehat{h}_t(\alpha)\right\}_{t\in\mathcal{N}}.$$

(iii) From the above (i) and (ii) parts and for a.e. $\alpha \in \widehat{\Gamma}$, we get the following by combining both the analysis and synthesis operators for $f \in L^2(\mathscr{G})$:

$$\left\{\mathcal{Z}(T^*_{\mathcal{E}^\Gamma(\mathscr{A})}T_{\mathcal{E}^\Gamma(\mathscr{A}')}f)(\alpha, \Gamma x)\right\}_{\Gamma x\in\Gamma\backslash\mathscr{G}} = \mathfrak{J}_{\mathscr{A}}(\alpha)\left\{(\widehat{T_{\mathcal{E}^\Gamma(\mathscr{A}')}f})_t(\alpha)\right\}_{t\in\mathcal{N}}$$
$$= \mathfrak{J}_{\mathscr{A}}(\alpha)\mathfrak{J}_{\mathscr{A}'}(\alpha)^*\left\{\mathcal{Z}f(\alpha, \Gamma x)\right\}_{\Gamma x\in\Gamma\backslash\mathscr{G}}.$$

□

Now, we state main results of this section to characterize subspace orthogonal and dual to a Bessel family having multiple generators. Theorem 5.1.6 is a successor of the results of [25,73] studied for $L^2(\mathbb{R}^n)$. Theorem 5.1.8 characterizes orthogonal frames $\mathcal{E}^\Gamma(\mathscr{A})$ and $\mathcal{E}^\Gamma(\mathscr{A}')$ in terms of pre-Gramian and mixed-dual Gramian operators for locally compact groups by action of its abelian subgroup. The result has so many predecessors by action of integer translates in $L^2(\mathbb{R}^n)$ and uniform lattices in $L^2(\mathcal{G})$ [39,40,54,73], where $\mathcal{G}$ is an LCA group.

Theorem 5.1.6. *For a σ-finite measure space $\mathcal{N}$ having counting measure, consider two sequences of functions $\mathscr{A} = \{\varphi_t\}_{t\in\mathcal{N}}$ and $\mathscr{A}' = \{\psi_t\}_{t\in\mathcal{N}}$ in $L^2(\mathscr{G})$ such that the Γ-TG systems $\mathcal{E}^\Gamma(\mathscr{A})$ and $\mathcal{E}^\Gamma(\mathscr{A}')$ are Bessel. Then the following hold:*

(i) $\mathcal{E}^\Gamma(\mathscr{A}')$ *is an* $\mathcal{S}^\Gamma(\mathscr{A})$*-subspace dual to* $\mathcal{E}^\Gamma(\mathscr{A})$ *if and only if for all* $t' \in \mathcal{N}$*, we have*

$$(\mathcal{Z}\varphi_{t'})(\alpha, \Gamma x) = \sum_{t\in\mathcal{N}} [\mathcal{Z}\varphi_{t'}, \mathcal{Z}\psi_t](\alpha)\ (\mathcal{Z}\varphi_t)(\alpha, \Gamma x) \text{ for a.e. } \alpha \in \widehat{\Gamma},\ \Gamma x \in \Gamma\backslash\mathscr{G}.$$

Equivalently, for $t' \in \mathcal{N}$*,* $\{(\mathcal{Z}\varphi_{t'})(\alpha, \Gamma x)\}_{\Gamma x\in\Gamma\backslash\mathscr{G}} = \mathfrak{J}_{\mathscr{A}}(\alpha)\,\{[\mathcal{Z}\varphi_{t'}, \mathcal{Z}\psi_t](\alpha)\}_{t\in\mathcal{N}}$ *for a.e.* $\alpha \in \widehat{\Gamma}$*.*

(ii) $\mathcal{E}^\Gamma(\mathscr{A}')$ *is an* $\mathcal{S}^\Gamma(\mathscr{A})$*-subspace orthogonal to* $\mathcal{E}^\Gamma(\mathscr{A})$ *if and only if for* $f \in \mathcal{S}^\Gamma(\mathscr{A})$ *and* $g \in L^2(\mathscr{G})$*,*

$$\sum_{t\in\mathcal{N}} [\mathcal{Z}f, \mathcal{Z}\psi_t](\alpha)[\mathcal{Z}\varphi_t, \mathcal{Z}g](\alpha) = 0 = \left\langle \tilde{G}_{\mathscr{A},\mathscr{A}'}(\alpha)(\mathcal{Z}f(\alpha)), \mathcal{Z}g(\alpha)\right\rangle \text{ a.e. } \alpha \in \widehat{\Gamma}.$$

Moreover, when the pair $(\mathscr{G}, \Gamma)$ *is an abelian pair* $(\mathcal{G}, \Lambda)$*, then (i) and (ii) become (i') and (ii') as follows:*

(i') $\mathcal{E}^\Lambda(\mathscr{A}')$ *is an* $\mathcal{S}^\Lambda(\mathscr{A})$*-subspace dual to* $\mathcal{E}^\Lambda(\mathscr{A})$ *if and only if for all* $t' \in \mathcal{N}$*,*

$$(\mathscr{T}\varphi_{t'})(\omega\Lambda^\perp)(\xi) = \sum_{t\in\mathcal{N}} [\mathscr{T}\varphi_{t'}, \mathscr{T}\psi_t](\omega\Lambda^\perp)\ \mathscr{T}\varphi_t(\omega\Lambda^\perp)(\xi)$$

for a.e. $\omega\Lambda^\perp \in \widehat{\mathcal{G}}/\Lambda^\perp$ *and* $\xi \in \Lambda^\perp$*.*

(ii') $\mathcal{E}^\Lambda(\mathscr{A}')$ *is an* $\mathcal{S}^\Lambda(\mathscr{A})$*-subspace orthogonal to* $\mathcal{E}^\Lambda(\mathscr{A})$ *if and only if for all* $f \in \mathcal{S}^\Lambda(\mathscr{A})$ *and* $g \in L^2(\mathcal{G})$ *,*

$$\sum_{t\in\mathcal{N}} [\mathscr{T}f, \mathscr{T}\psi_t](\omega\Lambda^\perp)[\mathscr{T}\varphi_t, \mathscr{T}g](\omega\Lambda^\perp) = 0$$

for a.e. $\omega\Lambda^\perp \in \widehat{\mathcal{G}}/\Lambda^\perp$*.*

Proof. (i) Let $\mathcal{E}^\Gamma(\mathscr{A}')$ be an $\mathcal{S}^\Gamma(\mathscr{A})$-subspace dual to $\mathcal{E}^\Gamma(\mathscr{A})$. Then for $f \in \mathcal{S}^\Gamma(\mathscr{A})$, we can write $f = \sum_{t\in\mathcal{N}}\sum_{\gamma\in\Gamma}\langle f, L_\gamma\psi_t\rangle L_\gamma\varphi_t$. By choosing $f = L_\eta\varphi_{t'}$ for $\eta \in \Gamma$ and $t' \in \mathcal{N}$, and applying the Zak transformation $\mathcal{Z}$ on both the sides, we have $\mathcal{Z}(L_\eta\varphi_{t'})(\alpha, \Gamma x) = \mathcal{Z}\left(\sum_{t\in\mathcal{N}}\sum_{\gamma\in\Gamma}\langle L_\eta\varphi_{t'}, L_\gamma\psi_t\rangle L_\gamma\varphi_t\right)(\alpha, \Gamma x)$ for a.e. $\alpha \in \widehat{\Gamma}$ and $\Gamma x \in \Gamma\backslash\mathscr{G}$. Thus we get the result by noting $\mathcal{Z}(L_\eta\varphi_{t'})(\alpha, \Gamma x) = \alpha(\eta^{-1})\mathcal{Z}\varphi_{t'}(\alpha, \Gamma x)$, and Proposition 5.1.5 (iii),

$$\begin{aligned}
\mathcal{Z}(T^*_{\mathcal{E}^\Gamma(\mathscr{A})}T_{\mathcal{E}^\Gamma(A')}L_\eta\varphi_{t'})(\alpha, \Gamma x) &= \mathfrak{J}_{\mathscr{A}}(\alpha)\mathfrak{J}_{\mathscr{A}'}(\alpha)^*\,\{\mathcal{Z}L_\eta\varphi_{t'}(\alpha, \Gamma x)\}_{\Gamma x\in\Gamma\backslash\mathscr{G}}\\
&= \sum_{t\in\mathcal{N}} [\mathcal{Z}(L_\eta\varphi_{t'}), \mathcal{Z}\psi_t](\alpha)\mathcal{Z}\varphi_t(\alpha, \Gamma x)\\
&= \alpha(\eta^{-1})\sum_{t\in\mathcal{N}} [\mathcal{Z}\varphi_{t'}, \mathcal{Z}\psi_t](\alpha)\mathcal{Z}\varphi_t(\alpha, \Gamma x).
\end{aligned}$$

Conversely, assume $(\mathcal{Z}\varphi_{t'})(\alpha, \Gamma x) = \sum_{t\in\mathcal{N}}[\mathcal{Z}\varphi_{t'}, \mathcal{Z}\psi_t](\alpha)\ (\mathcal{Z}\varphi_t)(\alpha, \Gamma x)$ a.e. $\alpha \in \widehat{\Gamma}$, $\Gamma x \in \Gamma\backslash\mathscr{G}$ and $t' \in \mathcal{N}$. Then, we have $L_\eta\varphi_{t'} = \sum_{t\in\mathcal{N}}\sum_{\gamma\in\Gamma}\langle L_\eta\varphi_{t'}, L_\gamma\psi_t\rangle L_\gamma\varphi_t$ in view of the above calculations for $\eta \in \Gamma$ and $t' \in \mathcal{N}$. Therefore for $f \in \text{span}\mathcal{E}^\Gamma(\mathscr{A})$, we can write $f = \sum_{t\in\mathcal{N}}\sum_{\gamma\in\Gamma}\langle f, L_\gamma\psi_t\rangle L_\gamma\varphi_t$ which is also valid for all $f \in \mathcal{S}^\Gamma(\mathscr{A})$ since the function $f \mapsto \sum_{t\in\mathcal{N}}\sum_{\gamma\in\Gamma}\langle f, L_\gamma\psi_t\rangle L_\gamma\varphi_t$ from $\mathcal{S}^\Gamma(\mathscr{A})$ to $L^2(\mathscr{G})$ is continuous due to the Bessel systems $\mathcal{E}^\Gamma(\mathscr{A})$ and $\mathcal{E}^\Gamma(\mathscr{A}')$. Thus the result holds.

For the equivalent part, $\{[\mathcal{Z}\varphi_{t'}, \mathcal{Z}\psi_t](\alpha)\}_{t\in\mathcal{N}}$ is a members of $\ell^2(\mathcal{N})$ for a.e. $\alpha \in \widehat{\Gamma}$, follows from

$$\sum_{t\in\mathcal{N}}\int_{\widehat{\Gamma}} |[\mathcal{Z}\varphi_{t'}, \mathcal{Z}\psi_t](\alpha)|^2 \ d\mu_{\widehat{\Gamma}}(\alpha) = \sum_{t\in\mathcal{N}}\int_{\widehat{\Gamma}} \left|(\widehat{\mathcal{W}_{\psi_t}\varphi_{t'}})(\alpha)\right|^2 \ d\mu_{\widehat{\Gamma}}(\alpha) = \sum_{t\in\mathcal{N}}\sum_{\gamma\in\Gamma} |(\mathcal{W}_{\psi_t}\varphi_{t'})(\gamma)|^2$$

$$= \sum_{t\in\mathcal{N}}\sum_{\gamma\in\Gamma} |\langle\varphi_{t'}, L_\gamma\psi_t\rangle|^2 \leqslant B\|\varphi_{t'}\|^2$$

for some $B > 0$, since $\mathcal{E}^\Gamma(\mathscr{A}')$ is Bessel. Using the definition of $\mathfrak{J}_{\mathscr{A}}(\alpha)$ we get the result.
(ii) Suppose $\mathcal{E}^\Gamma(\mathscr{A}')$ is an $\mathcal{S}^\Gamma(\mathscr{A})$-subspace orthogonal to $\mathcal{E}^\Gamma(\mathscr{A})$, then for $f \in \mathcal{S}^\Gamma(\mathscr{A})$ and $g \in L^2(\mathscr{G})$, we have $\sum_{t\in\mathcal{N}}\sum_{\gamma\in\Gamma}\langle f, L_\gamma\psi\rangle\langle L_\gamma\varphi, g\rangle = 0$ from (5.0.3), and hence we can get the following easily

$$\sum_{t\in\mathcal{N}}\int_{\widehat{\Gamma}}[\mathcal{Z}f, \mathcal{Z}\psi_t](\alpha)[\mathcal{Z}\varphi_t, \mathcal{Z}g](\alpha) \ d\mu_{\widehat{\Gamma}}(\alpha) = 0 \tag{5.1.5}$$

by considering countable functions in Lemma 5.1.4 . Therefore for a.e. $\alpha \in \widehat{\Gamma}$, we need to prove $\sum_{t\in\mathcal{N}}[\mathcal{Z}f, \mathcal{Z}\psi_t](\alpha)[\mathcal{Z}\varphi_t, \mathcal{Z}g](\alpha) = 0$, i.e., $\sum_{t\in\mathcal{N}}\langle\mathcal{Z}f(\alpha), \mathcal{Z}\psi_t(\alpha)\rangle \langle\mathcal{Z}\varphi_t(\alpha), \mathcal{Z}g(\alpha)\rangle = 0$ for a.e. α. For this, let $(e_i)_{i\in\mathbb{Z}}$ be an orthonormal basis for $L^2(\Gamma\backslash\mathscr{G})$ and $P(\alpha)$ be an orthogonal projection of $L^2(\Gamma\backslash\mathscr{G})$ on $\overline{\text{span}}\{\{\mathcal{Z}\varphi_t(\alpha, \Gamma x)\}_{\Gamma x\in\Gamma\backslash\mathscr{G}} : t \in \mathcal{N}\}$ for a.e. $\alpha \in \widehat{\Gamma}$. Assume on the contrary, there exists $i_0 \in \mathbb{Z}$ such that

$$h(\alpha) = \sum_{t\in\mathcal{N}}\langle P(\alpha)e_{i_0}, \mathcal{Z}\varphi_t(\alpha)\rangle \ \langle\mathcal{Z}\psi_t(\alpha), \mathcal{Z}g(\alpha)\rangle \neq 0,$$

on a measurable set $E \subseteq \widehat{\Gamma}$ with $\mu_{\widehat{\Gamma}}(E) > 0$. Then one of the four sets must have positive measure:

$$E_1 = \{\alpha \in E : \text{Re } h(\alpha) > 0\}, \qquad E_3 = \{\alpha \in E : \text{Im } h(\alpha) > 0\},$$

$$E_2 = \{\alpha \in E : \text{Re } h(\alpha) < 0\}, \qquad E_4 = \{\alpha \in E : \text{Im } h(\alpha) < 0\}.$$

Suppose $\mu_{\widehat{\Gamma}}(E_1) > 0$, and choosing $f \in \mathcal{S}^\Gamma(\mathscr{A})$ such that for all $\Gamma x \in \Gamma\backslash\mathscr{G}$, $\mathcal{Z}f(\alpha, \Gamma x) = P(\alpha)e_{i_0}$ for a.e. $\alpha \in E_1$ and zero for other α's. Then the estimate

$$\text{Re}\left\{\sum_{t\in\mathcal{N}}\left(\int_{\widehat{\Gamma}}[\mathcal{Z}f, \mathcal{Z}\psi_t](\alpha)[\mathcal{Z}\varphi_t, \mathcal{Z}g](\alpha) \ d\mu_{\widehat{\Gamma}}(\alpha)\right)\right\} > 0$$

due to $\mu_{\widehat{\Gamma}}(E_1) > 0$, which contradicts the fact that the integration is zero by (5.1.5). Similarly, we can discuss for the other sets E_2, E_3 and E_4, and will arrive on the same conclusion. The converse part follows immediately by Lemma 5.1.4. The remaining part follows easily from the definition of $\tilde{G}_{\mathscr{A},\mathscr{A}'}(\alpha)$ for a.e. $\alpha \in \widehat{\Gamma}$. □

The following immediate consequence can be observed easily by Theorem 5.1.6.

Corollary 5.1.7. *For $t, t' \in \mathcal{N}$, let $[\mathcal{Z}\varphi_{t'}, \mathcal{Z}\psi_t](\alpha) = \delta_{t,t'}$ for a.e. $\alpha \in \widehat{\Gamma}$ along with the assumptions of Theorem 5.1.6. Then $\mathcal{E}^{\Gamma}(\mathscr{A}')$ is an $\mathcal{S}^{\Gamma}(\mathscr{A})$-subspace dual to $\mathcal{E}^{\Gamma}(\mathscr{A})$.*

The following result describes few more properties of orthogonal frames using mixed dual-Gramian operator.

Theorem 5.1.8. *Let $\mathscr{A} = \{\varphi_t\}_{t\in\mathcal{N}}$ and $\mathscr{A}' = \{\psi_t\}_{t\in\mathcal{N}}$ be two sequences of functions in $L^2(\mathscr{G})$ such that $\mathcal{E}^{\Gamma}(\mathscr{A})$ and $\mathcal{E}^{\Gamma}(\mathscr{A}')$ are $\mathcal{S}^{\Gamma}(\mathscr{A})$ and $\mathcal{S}^{\Gamma}(\mathscr{A}')$-subspace frames, respectively. If $\mathcal{S}^{\Gamma}(\mathscr{A}) = \mathcal{S}^{\Gamma}(\mathscr{A}')$, then the following are equivalent:*

(i) *$\mathcal{E}^{\Gamma}(\mathscr{A})$ and $\mathcal{E}^{\Gamma}(\mathscr{A}')$ are orthogonal pair.*

(ii) *$\mathfrak{J}_{\mathscr{A}}(\alpha)\mathfrak{J}_{\mathscr{A}'}(\alpha)^*\mathfrak{J}_{\mathscr{A}'}(\alpha) = 0$ for a.e. $\alpha \in \widehat{\Gamma}$.*

(iii) *$G_{\mathscr{A}}(\alpha)G_{\mathscr{A}'}(\alpha) = 0$ for a.e. $\alpha \in \widehat{\Gamma}$.*

Additionally, when $\mathcal{S}^{\Gamma}(\mathscr{A}) = \mathcal{S}^{\Gamma}(\mathscr{A}') = L^2(\mathscr{G})$, then $\mathcal{E}^{\Gamma}(\mathscr{A})$ and $\mathcal{E}^{\Gamma}(\mathscr{A}')$ are orthogonal pair if and only if $\tilde{G}_{\mathscr{A},\mathscr{A}'}(\alpha) = 0$ for a.e. $\alpha \in \widehat{\Gamma}$.

We proceed by decomposing any Γ-TI space as an orthogonal direct sum of $\mathcal{S}^{\Gamma}(\varphi_i)$'s up to countable, where $\mathcal{E}^{\Gamma}(\varphi_i)$ is a frame for $\mathcal{S}^{\Gamma}(\varphi_i)$ with bounds $A = 1$ and $B = 1$, for each i. Our procedure is motivated by DeBoor, DeVore, and Ron [31] and Bownik [16]. Their analysis relied on the Fourier transform, whereas ours based on the Zak transform.

Proposition 5.1.9. *For $\varphi \in L^2(\mathscr{G})$, a function $f \in S^{\Gamma}(\varphi)$ if and only if for a.e. $\alpha \in \widehat{\Gamma}$, and $\Gamma x \in \Gamma\backslash\mathscr{G}$, $\mathcal{Z}f(\alpha, \Gamma x) = \mathfrak{m}(\alpha)\mathcal{Z}\varphi(\alpha, \Gamma x)$, where $\mathfrak{m}$ is a member of the weighted space $L^2\left(\widehat{\Gamma}, [\mathcal{Z}\varphi, \mathcal{Z}\varphi]\right)$.*

Moreover, if V is a Γ-TI subspace of $L^2(\mathscr{G})$ then there are at most countably many φ_n's in V such that $f \in V$ can be decomposed as follows:

$$\mathcal{Z}f(\alpha, \Gamma x) = \sum_{n\in\mathbb{N}} \mathfrak{m}_n(\alpha)\mathcal{Z}\varphi_n(\alpha, \Gamma x) \text{ for all } \Gamma x \in \Gamma\backslash\mathscr{G} \text{ and } \alpha \in \widehat{\Gamma}, \tag{5.1.6}$$

where $\mathfrak{m}_n \in L^2(\widehat{\Gamma} \cap \Omega_{\varphi_n})$ and $\Omega_{\varphi_n} = \{\alpha \in \widehat{\Gamma} : [\mathcal{Z}\varphi_n, \mathcal{Z}\varphi_n] \neq 0\}$.

Proof. For $f \in \text{span}\mathcal{E}^\Gamma(\varphi)$, a representation of f is of the form $f = \sum_{\gamma\in\Gamma} c_\gamma L_\gamma \varphi$, where only finitely many c_γ's are non-zero, and hence by applying the Zak transform on both the sides, we obtain,

$$(\mathcal{Z}f)(\alpha, \Gamma x) = \sum_{\gamma\in\Gamma} c_\gamma \mathcal{Z}\varphi(\alpha, \Gamma x)\overline{\alpha(\gamma)} = \mathcal{Z}\varphi(\alpha, \Gamma x)\sum_{\gamma\in\Gamma} c_\gamma\overline{\alpha(\gamma)} = \mathfrak{m}(\alpha)\mathcal{Z}\varphi(\alpha, \Gamma x)$$

for a.e. $\alpha \in \widehat{\Gamma}$, and $\Gamma x \in \Gamma\backslash\mathscr{G}$, where $\mathfrak{m} : \widehat{\Gamma} \mapsto \mathbb{C}$, $\mathfrak{m}(\alpha) := \sum_{\gamma\in\Gamma} c_\gamma\overline{\alpha(\gamma)}$. Conversely, we can recover $f \in \text{span}\mathcal{E}^\Gamma(\varphi)$ from the above relation. It only remains to generalize it for $f \in S^\Gamma(\varphi)$. For this define an operator $\mathscr{U} : \text{span}\mathcal{E}^\Gamma(\varphi) \to \mathcal{P}$ by $\mathscr{U}f = \mathfrak{m}$, where $\mathcal{P}$ is the collection of all trigonometric polynomials, which is an isometry and onto, follows by

$$\begin{aligned}\|f\|^2 &= \int_{\widehat{\Gamma}}\int_{\Gamma\backslash\mathscr{G}} |\mathfrak{m}(\alpha)\mathcal{Z}\varphi(\alpha, \Gamma x)|^2\, d\mu_{\Gamma\backslash\mathscr{G}}(\Gamma x)\, d\mu_{\widehat{\Gamma}}(\alpha) \\ &= \int_{\widehat{\Gamma}} |\mathfrak{m}(\alpha)|^2[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)\, d\mu_{\widehat{\Gamma}}(\alpha) = \|\mathfrak{m}\|^2_{L^2(\widehat{\Gamma}, [\mathcal{Z}\varphi, \mathcal{Z}\varphi])}. \end{aligned} \tag{5.1.7}$$

Therefore, there exists a unique isometry $\tilde{\mathscr{U}} : S^\Gamma(\varphi) \to \overline{\mathcal{P}} = L^2(\widehat{\Gamma}, [\mathcal{Z}\varphi, \mathcal{Z}\varphi])$. The moreover part follows by observing orthogonal projections P_n's on $\mathcal{S}^\Gamma(\varphi_n)$ and the following calculation for every $f \in V$:

$$\mathcal{Z}f(\alpha, \Gamma x) = \sum_{n\in\mathbb{N}} \mathcal{Z}(P_n f)(\alpha, \Gamma x) = \sum_{n\in\mathbb{N}} \mathfrak{m}_n(\alpha)\mathcal{Z}\varphi_n(\alpha, \Gamma x),\ \mathfrak{m}_n \in L^2(\widehat{\Gamma} \cap \Omega_{\varphi_n})$$

a.e. $\alpha \in \widehat{\Gamma}$ and $\Gamma x \in \Gamma\backslash\mathscr{G}$. Thus the result follows. □

Proof of Theorem 5.1.8. The Γ-TG systems $\mathcal{E}^\Gamma(\mathscr{A})$ and $\mathcal{E}^\Gamma(\mathscr{A}')$ are orthogonal if and only if $T^*_{\mathcal{E}^\Gamma(\mathscr{A})}T_{\mathcal{E}^\Gamma(\mathscr{A}')}f = 0$ for $f \in \mathcal{S}^\Gamma(\mathscr{A})$. Equivalently,

$$\begin{aligned}0 = \|\mathcal{Z}(T^*_{\mathcal{E}^\Gamma(\mathscr{A})}T_{\mathcal{E}^\Gamma(\mathscr{A}')})f\|^2 &= \int_{\widehat{\Gamma}}[\mathcal{Z}(T^*_{\mathcal{E}^\Gamma(\mathscr{A})}T_{\mathcal{E}^\Gamma(\mathscr{A}')})f, \mathcal{Z}(T^*_{\mathcal{E}^\Gamma(\mathscr{A})}T_{\mathcal{E}^\Gamma(\mathscr{A}')})f](\alpha)\, d\mu_{\widehat{\Gamma}}(\alpha) \\ = \int_{\Gamma\backslash\mathscr{G}}\int_{\widehat{\Gamma}} |\mathcal{Z}T^*_{\mathcal{E}^\Gamma(\mathscr{A})}T_{\mathcal{E}^\Gamma(\mathscr{A}')}f(\alpha, \Gamma x)|^2\, d\mu_{\widehat{\Gamma}}(\alpha)\, d\mu_{\Gamma\backslash\mathscr{G}}(\Gamma x).\end{aligned}$$

Further, it is equivalent to

$$\{\mathcal{Z}(T^*_{\mathcal{E}^\Gamma(\mathscr{A})}T_{\mathcal{E}^\Gamma(\mathscr{A}')})f(\alpha, \Gamma x)\}_{\Gamma x\in\Gamma\backslash\mathscr{G}} = 0 \text{ for a.e. } \alpha \in \widehat{\Gamma}.$$

Since $\left\{\mathcal{Z}(T^*_{\mathcal{E}^\Gamma(\mathscr{A})}T_{\mathcal{E}^\Gamma(\mathscr{A}')}f)(\alpha, \Gamma x)\right\}_{\Gamma x\in\Gamma\backslash\mathscr{G}} = \mathfrak{J}_{\mathscr{A}}(\alpha)\mathfrak{J}^*_{\mathscr{A}'}(\alpha)\left\{\mathcal{Z}f(\alpha, \Gamma x)\right\}_{\Gamma x\in\Gamma\backslash\mathscr{G}}$ by Proposition 5.1.5, and also from Proposition 5.1.9, $f \in \mathcal{S}^\Gamma(\mathscr{A}')$ if and only if $\{\mathcal{Z}f(\alpha, \Gamma x)\}_{\Gamma x\in\Gamma\backslash\mathscr{G}} = \{\sum_{t\in\mathcal{N}} \mathfrak{m}_t(\alpha)\mathcal{Z}\psi_t(\alpha, \Gamma x)\}_{\Gamma x\in\Gamma\backslash\mathscr{G}} = \mathfrak{J}_{\mathscr{A}'}(\alpha)\left\{\mathfrak{m}_t(\alpha)\right\}_{t\in\mathcal{N}}$ for a.e. $\alpha \in \widehat{\Gamma}$. Therefore we get

$$\left\{\mathcal{Z}T^*_{\mathcal{E}^\Gamma(\mathscr{A})}T_{\mathcal{E}^\Gamma(\mathscr{A}')}f(\alpha, \Gamma x)\right\}_{\Gamma x\in\Gamma\backslash\mathscr{G}} = \mathfrak{J}_{\mathscr{A}}(\alpha)\mathfrak{J}_{\mathscr{A}'}(\alpha)^*\mathfrak{J}_{\mathscr{A}'}(\alpha)\left\{\mathfrak{m}_t(\alpha)\right\}_{t\in\mathcal{N}} \text{ a.e. } \alpha \in \widehat{\Gamma}.$$

Thus (i) is equivalent to (ii) follows by observing that f is an arbitrary member of $\mathcal{S}^\Gamma(\mathscr{A})$. The equivalence of (ii) and (iii) follows immediately by just observing frame property of $\mathcal{E}^\Gamma(\mathscr{A})$.

When $\mathcal{S}^\Gamma(\mathscr{A}) = L^2(\mathscr{G})$, $\mathfrak{J}_{\mathscr{A}'}(\alpha)^*$ has bounded inverse on the range of $\mathfrak{J}_{\mathscr{A}'}(\alpha)$, and hence the result follows. ☐

Next, we observe that a new orthogonal pair can be constructed from the given orthogonal pair by involving Γ-periodic functions. A function $f : \mathscr{G} \to \mathbb{C}$ is said to be Γ-*periodic* if $f(x+\gamma) = f(x)$ for all $\gamma \in \Gamma$, and $x \in \mathscr{G}$.

Proposition 5.1.10. *Under the assumptions of Theorem 5.1.6, let $\mathcal{E}^\Gamma(\mathscr{A}')$ be an $\mathcal{S}^\Gamma(\mathscr{A})$-subspace orthogonal to $\mathcal{E}^\Gamma(\mathscr{A})$. If h is a Γ-periodic function on $\mathscr{G}$, $\mathcal{E}^\Gamma(h\mathscr{A}')$ is also $\mathcal{S}^\Gamma(\mathscr{A})$-subspace orthogonal to $\mathcal{E}^\Gamma(\mathscr{A})$, where $h\mathscr{A}' = \{h\psi : \psi \in \mathscr{A}', (h\psi)(x) = h(x)\psi(x), x \in \mathscr{G}\}$.*

Proof. The result follows by observing $\sum_{t\in\mathcal{N}}\sum_{\gamma\in\Gamma}\langle f, \psi_t(x-\gamma)\rangle\varphi_t(x-\gamma) = 0$, and

$$\sum_{t\in\mathcal{N}}\sum_{\gamma\in\Gamma}\langle f, (h\psi_t)(x-\gamma)\rangle\varphi_t(x-\gamma) = \overline{h(x)}\sum_{t\in\mathcal{N}}\sum_{\gamma\in\Gamma}\langle f, \psi_t(x-\gamma)\rangle\varphi_t(x-\gamma)$$

for $x \in \mathscr{G}$ and $f \in L^2(\mathscr{G})$. ☐

5.1.1. Application to singly generated system

For a function $\varphi \in L^2(\mathscr{G})$, we recall the Γ-TG system $\mathcal{E}^\Gamma(\varphi)$ and its associated Γ-TI space $\mathcal{S}^\Gamma(\varphi)$ from (1.3.1). The following consequences of Theorem 5.1.6 state about $\mathcal{S}^\Gamma(\varphi)$-subspace duals/ orthogonal to $\mathcal{E}^\Gamma(\varphi)$. The pedigree of our results traces back to the seminal works of many articles including [24–26, 44] for $L^2(\mathbb{R}^n)$ by the action of integer translations, and [51, 52] for LCA group setup.

Corollary 5.1.11. *Let φ and ψ be two functions in $L^2(\mathscr{G})$ such that the corresponding Γ-TG systems $\mathcal{E}^\Gamma(\varphi)$ and $\mathcal{E}^\Gamma(\psi)$ are Bessel. Assume a measurable set Ω_φ defined by $\Omega_\varphi := \left\{\alpha \in \widehat{\Gamma} : [\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha) \neq 0\right\}$. Then, the following are true:*

(i) *$\mathcal{E}^\Gamma(\psi)$ is an $\mathcal{S}^\Gamma(\varphi)$-subspace dual to $\mathcal{E}^\Gamma(\varphi)$ if and only if $[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha) = 1$ a.e. $\alpha \in \Omega_\varphi$.*

(ii) *$\mathcal{E}^\Gamma(\psi)$ is an $\mathcal{S}^\Gamma(\varphi)$-subspace orthogonal to $\mathcal{E}^\Gamma(\varphi)$ if and only if $[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha) = 0$ a.e. $\alpha \in \Omega_\varphi$. In this case, we have $[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha) = 0$ for a.e. $\alpha \in \widehat{\Gamma}$, which implies $\mathcal{E}^\Gamma(\varphi)$ is also an $\mathcal{S}^\Gamma(\psi)$-subspace orthogonal to $\mathcal{E}^\Gamma(\psi)$.*

Moreover, the same can be deduced in terms of the fiberization $\mathscr{T}$ for an abelian pair $(\mathcal{G}, \Lambda)$.

Proof. The result follows easily by choosing $\varphi_t = \varphi$, and $\psi_t = \psi$ for every t in Theorem 5.1.6 and $f = g = \varphi_t$.

Next assume that $[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha) = 0$ for a.e. $\alpha \in \Omega_\varphi$. Then first note that $[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha) = 0$ on a.e. $\widehat{\Gamma}\backslash\Omega_\varphi$, and hence using the Cauchy-Schwarz inequality in the following estimate: for a.e. $\alpha \in \widehat{\Gamma}$,

$$\begin{aligned}
|[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha)| &\leqslant \int_{\Gamma\backslash\mathscr{G}} \left|\mathcal{Z}\varphi(\alpha, \Gamma x)\overline{\mathcal{Z}\psi(\alpha, \Gamma x)}\right| \, d\mu_{\Gamma\backslash\mathscr{G}}(\Gamma x) \\
&\leqslant \left(\int_{\Gamma\backslash\mathscr{G}} |\mathcal{Z}\varphi(\alpha, \Gamma x)|^2 \, d\mu_{\Gamma\backslash\mathscr{G}}(\Gamma x)\right)^{1/2} \left(\int_{\Gamma\backslash\mathscr{G}} |\mathcal{Z}\psi(\alpha, \Gamma x)|^2 \, d\mu_{\Gamma\backslash\mathscr{G}}(\Gamma x)\right)^{1/2} \\
&= ([\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha))^{1/2}([\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha))^{1/2}, \qquad (5.1.8)
\end{aligned}$$

we get $[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha) = 0$ on a.e. $\widehat{\Gamma}\backslash\Omega_\varphi$. Thus we have $[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha) = 0$ for a.e. $\alpha \in \widehat{\Gamma}$.

The moreover part follows from the same argument as above by replacing the Zak transform $\mathcal{Z}$ for the pair $(\mathscr{G}, \Gamma)$ with the fiberization $\mathscr{T}$ for the pair $(\mathcal{G}, \Lambda)$ □

The part (ii) in Corollary 5.1.11 motivates to elaborate more regarding the symmetry of φ and ψ. For part (i), a counter example is provided in Example 5.1.13. We provide various necessary and sufficient conditions on the Γ-TG systems to become orthogonal pairs.

Theorem 5.1.12. *Let φ and ψ be two functions in $L^2(\mathscr{G})$ such that the Γ-TG systems $\mathcal{E}^\Gamma(\varphi)$ and $\mathcal{E}^\Gamma(\psi)$ are Bessel. Then the following are true:*

(i) *Assume $[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)[\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha) = 0$ for a.e. $\alpha \in \widehat{\Gamma}$. Then $[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha) = 0$ a.e. α, and hence $\mathcal{E}^\Gamma(\psi)$ is an $\mathcal{S}^\Gamma(\varphi)$-subspace orthogonal to $\mathcal{E}^\Gamma(\varphi)$.*

(ii) *If $\mathcal{E}^\Gamma(\psi)$ is an $\mathcal{S}^\Gamma(\varphi)$-subspace orthogonal to $\mathcal{E}^\Gamma(\varphi)$, then $\mathcal{E}^\Gamma(\tilde{\psi})$ is also so for all $\tilde{\psi} \in \mathcal{S}^\Gamma(\psi)$.*

(iii) *If $\mathcal{S}^\Gamma(\psi) = \mathcal{S}^\Gamma(\varphi)$ and $\mathcal{E}^\Gamma(\psi)$ is an $\mathcal{S}^\Gamma(\varphi)$-subspace orthogonal to $\mathcal{E}^\Gamma(\varphi)$, then $\mathcal{E}^\Gamma(\varphi)$ is also an $\mathcal{S}^\Gamma(\varphi)$-subspace orthogonal to $\mathcal{E}^\Gamma(\psi)$, and hence $\mathcal{E}^\Gamma(\psi)$ and $\mathcal{E}^\Gamma(\varphi)$ are orthogonal pair.*

(iv) *Assume that $\mathcal{S}^\Gamma(\psi) = \mathcal{S}^\Gamma(\varphi)$. Then $\mathcal{E}^\Gamma(\psi)$ and $\mathcal{E}^\Gamma(\varphi)$ are orthogonal pair if and only if for a.e. $\alpha \in \widehat{\Gamma}$,*

$$[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)[\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha) = 0.$$

(v) *If the functions* φ *and* ψ *satisfy* $(\operatorname{supp} \mathcal{Z}\varphi) \cap (\operatorname{supp} \mathcal{Z}\psi) = 0$ *a.e., then* $\mathcal{E}^{\Gamma}(\psi)$ *is an* $\mathcal{S}^{\Gamma}(\varphi)$*-subspace orthogonal to* $\mathcal{E}^{\Gamma}(\varphi)$*, where* $\operatorname{supp} \mathcal{Z}\varphi$ *denotes the support of* $\mathcal{Z}\varphi$ *by considering the map* $\mathcal{Z}\varphi : \widehat{\Gamma} \to L^2(\Gamma\backslash\mathscr{G})$.

Proof. (i) The expression $[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha) = 0$ follows by observing the estimate from (5.1.8) for a.e. $\alpha \in \widehat{\Gamma}$,

$$|[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha)| \leqslant \int_{\Gamma\backslash\mathscr{G}} \left|\mathcal{Z}\varphi(\alpha, \Gamma x)\overline{\mathcal{Z}\psi(\alpha, \Gamma x)}\right| \, d\mu_{\Gamma\backslash\mathscr{G}}(\Gamma x) \leqslant ([\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha))^{1/2}([\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha))^{1/2}$$

using Cauchy-Schwarz inequality. From Corollary 5.1.11, $\mathcal{E}^{\Gamma}(\psi)$ is an $\mathcal{S}^{\Gamma}(\varphi)$-subspace orthogonal to $\mathcal{E}^{\Gamma}(\varphi)$.

(ii) For $\tilde{\psi} \in S^{\Gamma}(\psi)$, we can write $\mathcal{Z}\tilde{\psi}(\alpha, \Gamma x) = \mathfrak{m}(\alpha)\mathcal{Z}\psi(\alpha, \Gamma x)$ for a.e. $\alpha \in \widehat{\Gamma}$, and $\Gamma x \in \Gamma\backslash\mathscr{G}$ due to Proposition 5.1.9, where $\mathfrak{m}$ is a member of the weighted space $L^2(\widehat{\Gamma}, [\mathcal{Z}\psi, \mathcal{Z}\psi])$. Then, we get $[\mathcal{Z}\varphi, \mathcal{Z}\tilde{\psi}](\alpha) = \overline{\mathfrak{m}(\alpha)}[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha)$, and hence the result follows since the system $\mathcal{E}^{\Gamma}(\psi)$ is an $\mathcal{S}^{\Gamma}(\varphi)$-subspace orthogonal to $\mathcal{E}^{\Gamma}(\varphi)$, equivalently, $[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha) = 0$ a.e. $\alpha \in \widehat{\Gamma}$ by Corollary 5.1.11.

(iii) This follows easily by Corollary 5.1.11.

(iv) It is enough to show the orthogonality of the Bessel pair $\mathcal{E}^{\Gamma}(\varphi)$ and $\mathcal{E}^{\Gamma}(\psi)$ implies the expression $[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)[\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha)$ becomes zero for a.e. α in view of Corollary 5.1.11 and part (i). For this, we proceed similar to the part (i) of Corollary 5.1.11 by considering $\sum_{\gamma\in\Gamma}\langle f, L_\gamma\psi\rangle L_\gamma\varphi = 0$ for all $f \in \mathcal{S}^{\Gamma}(\varphi) = S^{\Gamma}(\psi)$. Then we get $[\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha)\mathcal{Z}\varphi(\alpha, \Gamma x) = 0$ by choosing $f = \psi$ and hence, we obtain either $[\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha) = 0$ or $[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha) = 0$ for a.e. $\alpha \in \widehat{\Gamma}$. This proves the result.

(v) Observe that the set $\{\alpha \in \widehat{\Gamma} : \{\mathcal{Z}\varphi(\alpha, \Gamma x)\}_{\Gamma x\in\Gamma\backslash\mathscr{G}} \neq 0\}$ is same as the set $\{\alpha \in \widehat{\Gamma} : \|\{\mathcal{Z}\varphi(\alpha, \Gamma x)\}_{\Gamma x\in\Gamma\backslash\mathscr{G}}\| \neq 0\}$ which is further equal to $\{\alpha \in \widehat{\Gamma} : [\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha) \neq 0\}$. Therefore, the support of $\mathcal{Z}\varphi$ is same as the support of $[\mathcal{Z}\varphi, \mathcal{Z}\varphi]$. Hence $(\operatorname{supp}\ \mathcal{Z}\varphi) \cap (\operatorname{supp}\ \mathcal{Z}\psi) = 0$ a.e. implies $(\operatorname{supp}\ [\mathcal{Z}\varphi, \mathcal{Z}\varphi]) \cap (\operatorname{supp}\ [\mathcal{Z}\psi, \mathcal{Z}\psi]) = 0$ a.e. Thus, we get $[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)[\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha) = 0$ for a.e. $\alpha \in \widehat{\Gamma}$. Now from part (i), the result follows. □

Now we provide some examples to illustrate our results.

Example 5.1.13. For a second countable LCA group $\mathcal{G}$ having uniform lattice Λ, we can write $\widehat{\mathcal{G}} = \Omega \oplus \Lambda^{\perp}$ due to the Pontryagin Duality theorem, where Ω is a *fundamental domain*. Then, the system $\{\Omega + \lambda : \lambda \in \Lambda^{\perp}\}$ is a measurable partition of $\widehat{\mathcal{G}}$. Note that Ω is a Borel section of $\widehat{\mathcal{G}}/\Lambda^{\perp}$. Let $\eta_1, \eta_2 \in L^2(\mathcal{G})$ be such that $\widehat{\eta_1} = \chi_{\Omega_1}$ and $\widehat{\eta_2} = \chi_{\Omega_2}$, where

$\mu_{\widehat{\mathcal{G}}}(\Omega_1 \cap \Omega_2) = 0$ and for each $i = 1, 2$, the system $\{\Omega_i + \lambda : \lambda \in \Lambda^\perp\}$ is a measurable partition of $\widehat{\mathcal{G}}$. Then for each $i = 1, 2$, the system $\mathcal{E}^\Lambda(\eta_i)$ is an $\mathcal{S}^\Lambda(\eta_i)$-subspace frame since $[\mathscr{T}\eta_i, \mathscr{T}\eta_i](\xi) = 1$ for a.e. $\xi \in \Omega_i$. Further note that $\mathcal{E}^\Lambda(\eta_2)$ is an $\mathcal{S}^\Lambda(\eta_1)$-subspace orthogonal to $\mathcal{E}^\Lambda(\eta_1)$ since for a.e. $\xi \in \Omega_1$, $[\mathscr{T}\eta_2, \mathscr{T}\eta_1](\xi) = \sum_{\lambda \in \Lambda^\perp} \widehat{\eta}_2(\xi+\lambda)\overline{\widehat{\eta}_1(\xi+\lambda)} = 0$. Similarly, $\mathcal{E}^\Lambda(\eta_1)$ is also an $\mathcal{S}^\Lambda(\eta_2)$-subspace orthogonal to $\mathcal{E}^\Lambda(\eta_2)$.

Example 5.1.14. First we recall Example 5.1.13 and also fix an automorphism A on $\widehat{\mathcal{G}}$ such that $A\Omega \subsetneq \Omega$. Let $\mathscr{A} = \{\eta_1, \eta_2\} \subset L^2(\mathcal{G})$ be such that $\widehat{\eta}_1 = \chi_{A\Omega}$ and $\widehat{\eta}_2 = \chi_{\Omega \backslash A\Omega}$. Then the associated Gramian matrix $G_{\mathscr{A}}(\xi)$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ for $\xi \in A\Omega$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ for $\xi \in \Omega \backslash A\Omega$ by noting $\sum_{\lambda \in \Lambda^\perp} |\widehat{\eta}_1(\xi+\lambda)|^2 = \chi_{A\Omega}(\xi)$, $\sum_{k \in \Lambda^\perp} |\widehat{\eta}_2(\xi+\lambda)|^2 = \chi_{\Omega \backslash A\Omega}(\xi)$, and $\sum_{\lambda \in \Lambda^\perp} \widehat{\eta}_1(\xi+\lambda)\overline{\widehat{\eta}_2(\xi+\lambda)} = 0$.

Further, let $\mathscr{A}' = \{\zeta_1, \zeta_2\} \in L^2(\mathcal{G})$ be such that $\widehat{\zeta}_1 = \chi_{\Omega \backslash A\Omega}$ and $\widehat{\zeta}_2 = \chi_{A\Omega}$ for a.e. $\xi \in \Omega$. Then the associated Gramian matrix $G_{\mathscr{A}'}(\xi)$ is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ for $\xi \in A\Omega$, and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ for $\xi \in \Omega \backslash A\Omega$. Since $G_{\mathscr{A}}(\xi) G_{\mathscr{A}'}(\xi) = 0$ for a.e. $\xi \in \Omega$, the systems $\mathcal{E}^\Lambda(\mathscr{A})$ and $\mathcal{E}^\Lambda(\mathscr{A}')$ are orthogonal pair by Theorem 5.1.8. From the Corollary 5.1.7, note that $\mathcal{E}^\Lambda(\mathscr{A})$ and $\mathcal{E}^\Lambda(\mathscr{A}')$ are $\mathcal{S}^\Lambda(\mathscr{A})$ and $\mathcal{S}^\Lambda(\mathscr{A}')$-subspace dual to itself, respectively.

Example 5.1.15. Let $\psi \in L^2(\mathscr{G})$ be such that $\mathcal{E}^\Gamma(\psi)$ is an $\mathcal{S}^\Gamma(\psi)$-subspace frame. Assume that $\varphi, \tilde{\varphi} \in L^2(\mathscr{G})$ which are defined in terms of the Zak transform for a.e. $\alpha \in \widehat{\Gamma}$,

$$(\mathcal{Z}\varphi)(\alpha, \Gamma x) = \mathfrak{m}(\alpha)(\mathcal{Z}\psi)(\alpha, \Gamma x) \text{ and } (\mathcal{Z}\tilde{\varphi})(\alpha, \Gamma x) = \tilde{\mathfrak{m}}(\alpha)(\mathcal{Z}\psi)(\alpha, \Gamma x) \text{ for all } \Gamma x \in \Gamma \backslash \mathscr{G},$$

where $\mathfrak{m}, \tilde{\mathfrak{m}} \in L^2\left(\widehat{\Gamma}, [\mathcal{Z}\psi, \mathcal{Z}\psi]\right)$ are bounded functions. Then $\mathcal{E}^\Gamma(\varphi)$ is an $\mathcal{S}^\Gamma(\varphi)$-subspace frame as $\mathcal{S}^\Gamma(\varphi) = \mathcal{S}^\Gamma(\psi)$ and $[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha) = |\mathfrak{m}(\alpha)|^2 [\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha)$ for a.e. $\alpha \in \widehat{\Gamma}$. Similarly, $\mathcal{E}^\Gamma(\tilde{\varphi})$ is also an $S^\Gamma(\tilde{\varphi})$-subspace frame. Also note that

$$[\mathcal{Z}\varphi, \mathcal{Z}\tilde{\varphi}](\alpha) = \mathfrak{m}(\alpha)\overline{\tilde{\mathfrak{m}}(\alpha)}[\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha) \text{ for a.e. } \alpha \in \widehat{\Gamma}.$$

By the Corollary 5.1.11, $\mathcal{E}^\Gamma(\varphi)$ is an $\mathcal{S}^\Gamma(\tilde{\varphi})$-subspace dual to $\mathcal{E}^\Gamma(\tilde{\varphi})$ if and only if

$$\mathfrak{m}(\alpha)\overline{\tilde{\mathfrak{m}}(\alpha)} = \frac{1}{[\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha)} \text{ on } \left\{\alpha \in \widehat{\Gamma} : [\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha) \neq 0\right\}.$$

In this case, both $\mathcal{E}^\Gamma(\varphi)$ and $\mathcal{E}^\Gamma(\tilde{\varphi})$ are dual frames to each other. The condition gives various choices of subspace dual frames. Also, $\mathcal{E}^\Gamma(\varphi)$ is an $\mathcal{S}^\Gamma(\tilde{\varphi})$-subspace orthogonal to $\mathcal{E}^\Gamma(\tilde{\varphi})$ if and only if $\mathfrak{m}(\alpha)\overline{\tilde{\mathfrak{m}}(\alpha)} = 0$ a.e. $\alpha \in \widehat{\Gamma}$. Then $\mathcal{E}^\Gamma(\varphi)$ and $\mathcal{E}^\Gamma(\tilde{\varphi})$ are orthogonal pair.

The Example 5.1.15 also concludes that there is a unique $\varphi \in \mathcal{S}^{\Gamma}(\psi)$ such that $\mathcal{E}^{\Gamma}(\varphi)$ is an $\mathcal{S}^{\Gamma}(\psi)$-subspace dual frame to $\mathcal{E}^{\Gamma}(\psi)$, where $\varphi = S^{\dagger}\psi$ (pseudo inverse). Since $[\mathcal{Z}\psi, \mathcal{Z}\varphi](\alpha) = \overline{\mathfrak{m}(\alpha)}[\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha) = 1$ a.e. on Ω_{ψ}, the function $\mathfrak{m}$ is unique except on the set $\{\alpha \in \widehat{\Gamma} : [\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha) = 0\}$ as various choices of $\mathfrak{m}$ is possible on this set. But note that whatever choices we have for $\mathfrak{m}$, we always have $\mathcal{Z}\varphi(\alpha, \Gamma x) = \mathfrak{m}(\alpha)\mathcal{Z}\psi(\alpha, \Gamma x) = 0$ on $\left\{\alpha \in \widehat{\Gamma} : [\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha) = 0\right\}$. So $\mathcal{Z}\varphi$ is uniquely defined and hence φ is so.

The next result discusses an existence of an $\mathcal{S}^{\Gamma}(\varphi)$-subspace dual to a frame $\mathcal{E}^{\Gamma}(\varphi)$ and provide a condition to get unique dual (upto a scalar multiplication). This theorem generalizes a result provided for the case of $L^2(\mathbb{R}^n)$ [24, Theorem 4.3].

Theorem 5.1.16. *Let $\varphi, \psi \in L^2(\mathscr{G})$ be such that $\mathcal{E}^{\Gamma}(\varphi)$ and $\mathcal{E}^{\Gamma}(\psi)$ are $\mathcal{S}^{\Gamma}(\varphi)$ and $\mathcal{S}^{\Gamma}(\psi)$-subspace frames, respectively. If for some positive constant C, the expression $|[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha)| \geqslant C$ holds for a.e. $\alpha \in \Omega_{\varphi}$ (defined in Corollary 5.1.11), then there exists a $\tilde{\psi} \in \mathcal{S}^{\Gamma}(\psi)$ such that $\mathcal{E}^{\Gamma}(\tilde{\psi})$ is an $\mathcal{S}^{\Gamma}(\varphi)$-subspace dual to $\mathcal{E}^{\Gamma}(\varphi)$. Moreover, $\tilde{\psi} \in \mathcal{S}^{\Gamma}(\psi)$ is unique if and only if $\Omega_{\varphi} = \Omega_{\psi}$ a.e. In particular, the $\tilde{\psi} \in \mathcal{S}^{\Gamma}(\psi)$ is unique and satisfies the following relation for a.e. $\alpha \in \Omega_{\varphi}$:*

$$[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha)\ (\mathcal{Z}\tilde{\psi})(\alpha, \Gamma x) = (\mathcal{Z}\psi)(\alpha, \Gamma x)\ \chi_{\Omega_{\varphi}}(\alpha) \text{ for all } \Gamma x \in \Gamma\backslash\mathscr{G}.$$

Proof. Firstly note that for a.e. $\alpha \in \Omega_{\varphi}$, $C \leqslant |[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha)| \leqslant \sqrt{BB'}$, follows from the estimate (5.1.8) as $\mathcal{E}^{\Gamma}(\varphi)$ and $\mathcal{E}^{\Gamma}(\psi)$ are Bessel sequences with bounds B and B', respectively [49]. Further note that any function $\tilde{\psi} \in \mathcal{S}^{\Gamma}(\psi)$ if and only if for a.e. $\alpha \in \widehat{\Gamma}$, we have $\mathcal{Z}\tilde{\psi}(\alpha, \Gamma x) = \mathfrak{m}(\alpha)\mathcal{Z}\psi(\alpha, \Gamma x)$ for all $\Gamma x \in \Gamma\backslash\mathscr{G}$ where $\mathfrak{m} \in L^2(\widehat{\Gamma}, [\mathcal{Z}\psi, \mathcal{Z}\psi])$ from Proposition 5.1.9. From Corollary 5.1.11, additionally note that $\mathcal{E}^{\Gamma}(\tilde{\psi})$ is an $\mathcal{S}^{\Gamma}(\varphi)$-subspace dual to $\mathcal{E}^{\Gamma}(\varphi)$ if and only if for a.e. $\alpha \in \Omega_{\varphi}$, $1 = [\mathcal{Z}\varphi, \mathcal{Z}\tilde{\psi}](\alpha) = \overline{\mathfrak{m}(\alpha)}[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha)$. Hence, $\mathfrak{m}$ is both bounded above and bounded below on Ω_{φ}. Extending this to an arbitrary function in $L^2(\widehat{\Gamma})$, will produce a function $\tilde{\psi} \in S^{\Gamma}(\psi)$ such that $\mathcal{E}^{\Gamma}(\tilde{\psi})$ is an $\mathcal{S}^{\Gamma}(\varphi)$-subspace dual to $\mathcal{E}^{\Gamma}(\varphi)$.

Since $|[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha)| \geqslant C$ for a.e. $\alpha \in \Omega_{\varphi}$, we have

$$\left\{\alpha \in \widehat{\Gamma} : [\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha) = 0\right\} \subseteq \left\{\alpha \in \widehat{\Gamma} : [\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha) = 0\right\}.$$

When the equality holds on the above sets, we get $\mathcal{Z}\tilde{\psi}(\alpha) = 0$ on $\left\{\alpha \in \widehat{\Gamma} : [\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha) = 0\right\}$ always, whatever $\mathfrak{m}$ is considered. In this case there exists unique $\tilde{\psi}$, which fulfils the requirements. Otherwise for the case

$\left\{\alpha \in \widehat{\Gamma} : [\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha) = 0\right\} \subsetneq \left\{\alpha \in \widehat{\Gamma} : [\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)(\alpha) = 0\right\}$, various choices will lead to various $\mathcal{Z}\tilde{\psi}(\alpha)$ as it is non-zero, by considering $\mathfrak{m}$ on

$$\left\{\alpha \in \widehat{\Gamma} : [\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)(\alpha) = 0\right\} \setminus \left\{\alpha \in \widehat{\Gamma} : [\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha) = 0\right\}.$$

Thus the various $\tilde{\psi}$ is possible. Hence the result follows.

□

In this section, we have discussed $\mathcal{S}^{\Gamma}(\mathscr{A})$-subspace duals of a frame/Bessel family $\mathcal{E}^{\Gamma}(\mathscr{A})$ in $L^2(\mathscr{G})$, and realized that we can obtain various duals of a frame/Bessel family (for instance, Example 5.1.15). Theorem 5.1.16 motivates to discuss about unique dual. It is well known that the unique dual can be obtained when the frame/Bessel family $\mathcal{E}^{\Gamma}(\mathscr{A})$ becomes Riesz basis for $L^2(\mathscr{G})$, known as *dual basis* or *biorthogonal basis.* We refer [24, 74] for more details on Riesz basis and biorthogonal basis. Next, we study biorthogonal systems and Riesz basis generated by translations in $L^2(\mathscr{G})$.

5.2. Translation generated biorthogonal system and Riesz basis

Recall that for non-zero functions $\varphi, \psi \in L^2(\mathscr{G})$, the Γ-TG systems $\mathcal{E}^{\Gamma}(\varphi)$ and $\mathcal{E}^{\Gamma}(\psi)$ are said to be *biorthogonal* if $\langle L_\gamma \varphi, L_{\gamma'}\psi\rangle = \delta_{\gamma,\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. *Throughout the section,* we assume Γ to be a discrete abelian subgroup of $\mathscr{G}$. The following result characterizes biorthogonal systems in terms of the Zak transform and describes whether a translation generated system is linearly independent or not.

The system $\mathcal{E}^{\Gamma}(\varphi)$ is *linearly independent* if $\sum_{\gamma\in\Gamma} c_\gamma L_\gamma \varphi = 0$ for some $\{c_\gamma\}_{\gamma\in\Gamma} \in \ell^2(\Gamma)$ implies $c_\gamma = 0$ for all γ.

Theorem 5.2.1. *For non-zero functions $\varphi, \psi \in L^2(\mathscr{G})$, the Γ-TG systems $\mathcal{E}^{\Gamma}(\varphi)$ and $\mathcal{E}^{\Gamma}(\psi)$ are biorthogonal if and only if for a.e. $\alpha \in \widehat{\Gamma}$, $[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha) = 1$. In this case, the following hold:*

(i) *The systems $\mathcal{E}^{\Gamma}(\varphi)$ and $\mathcal{E}^{\Gamma}(\psi)$ are linearly independent.*

(ii) *$\mathcal{E}^{\Gamma}(\psi)$ is an $\mathcal{S}^{\Gamma}(\varphi)$-subspace dual to $\mathcal{E}^{\Gamma}(\varphi)$ for compactly supported φ and ψ.*

Proof. Firstly observe that the biorthogonal relation between $\mathcal{E}^\Gamma(\varphi)$ and $\mathcal{E}^\Gamma(\psi)$ is $\langle L_\gamma\varphi, \psi\rangle = \delta_{\gamma,0}$ for $\gamma \in \Gamma$. Now,

$$\begin{aligned}\delta_{\gamma,0} = \langle \mathcal{Z}L_\gamma\varphi, \mathcal{Z}\psi\rangle_{L^2(\widehat{\Gamma};L^2(\Gamma\backslash\mathscr{G}))} &= \int_{\widehat{\Gamma}}\int_{\Gamma\backslash\mathscr{G}} \mathcal{Z}\varphi(\alpha,\Gamma x)\overline{\mathcal{Z}\psi(\alpha,\Gamma x)\alpha(\gamma)}\, d\mu_{\Gamma\backslash\mathscr{G}}(\Gamma x)\, d\mu_{\widehat{\Gamma}}(\alpha)\\ &= \int_{\widehat{\Gamma}} [\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha)\overline{\alpha(\gamma)}\, d\mu_{\widehat{\Gamma}}(\alpha).\end{aligned}$$

Thus the result follows from the uniqueness of the Fourier coefficients.

For the remaining part of (i), let $\{c_\gamma\}_{\gamma\in\Gamma} \in \ell^2(\Gamma)$ be such that $\sum_{\gamma\in\Gamma} c_\gamma L_\gamma\varphi = 0$. Then for each $\gamma' \in \Gamma$, $0 = \langle 0, L_{\gamma'}\psi\rangle = \langle\sum_{\gamma\in\Gamma} c_\gamma L_\gamma\varphi, L_{\gamma'}\psi\rangle = \sum_{\gamma\in\Gamma} c_\gamma\langle L_\gamma\varphi, L_{\gamma'}\psi\rangle = c_{\gamma'}$ by the biorthogonal relation between $\mathcal{E}^\Gamma(\varphi)$ and $\mathcal{E}^\Gamma(\psi)$, and hence all c_γ's are zero. Thus $\mathcal{E}^\Gamma(\varphi)$ is linearly independent. Similarly, $\mathcal{E}^\Gamma(\psi)$ is also linearly independent.

(ii) Due to $\langle\varphi, L_\gamma\psi\rangle = \delta_{\gamma,0}$ for $\gamma \in \Gamma$, we have $f = \sum_{\gamma\in\Gamma_1}\langle f, L_\gamma\psi\rangle L_\gamma\varphi$ for all $f \in \operatorname{span}\mathcal{E}^\Gamma(\varphi)$ and Γ_1 is a finite subset of Γ, which holds for all $f \in \mathcal{S}^\Gamma(\varphi)$ in view of compactly supported functions φ and ψ, and the continuity of the function $f \mapsto \sum_{\gamma\in\Gamma}\langle f, L_\gamma\psi\rangle L_\gamma\varphi$. Note that $\mathcal{E}^\Gamma(\varphi)$ and $\mathcal{E}^\Gamma(\psi)$ are Bessel families since $\{[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)\}_{\alpha\in\widehat{\Gamma}}$ and $\{[\mathcal{Z}\psi, \mathcal{Z}\psi](\alpha)\}_{\alpha\in\widehat{\Gamma}}$ are bounded sets for a.e. $\alpha \in \widehat{\Gamma}$ [49]. The boundedness of $\{[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)\}_{\alpha\in\widehat{\Gamma}}$ follows by observing the continuity of the function $\alpha \mapsto [\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)$ from the compact set $\widehat{\Gamma}$ to $\mathbb{R}$. Indeed, $[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)$ is a polynomial for a.e. $\alpha \in \widehat{\Gamma}$, that can be realised by writing it in the form of Fourier series expansion where only finitely many Fourier coefficients are non-zero in view of the compact support of φ. □

Corollary 5.2.2. *Let $\varphi \in L^2(\mathscr{G})$ be such that for a.e. $\alpha \in \widehat{\Gamma}$, $C \leqslant [\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha) \leqslant D$ for some constants $0 < C \leqslant D < \infty$. Then there is a $\psi \in L^2(\mathscr{G})$ such that $\mathcal{E}^\Gamma(\varphi)$ and $\mathcal{E}^\Gamma(\psi)$ are biorthogonal systems. Moreover, $\mathcal{E}^\Gamma(\varphi)$ is linearly independent.*

Proof. For $\varphi \in L^2(\mathscr{G})$, choose $\psi \in L^2(\mathscr{G})$ satisfying $\mathcal{Z}\varphi(\alpha,\Gamma x) = \mathcal{Z}\psi(\alpha,\Gamma x)[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)$, for a.e. $\alpha \in \widehat{\Gamma}$ and $\Gamma x \in \Gamma\backslash\mathscr{G}$. Then $\mathcal{E}^\Gamma(\varphi)$ and $\mathcal{E}^\Gamma(\psi)$ are biorthogonal systems in view of Theorem 5.2.1 since $[\mathcal{Z}\varphi, \mathcal{Z}\psi](\alpha) = 1$ for a.e. $\alpha \in \widehat{\Gamma}$. The moreover part follows by Theorem 5.2.1. □

Following the concept of Corollary 5.2.2, we state a characterization result for the existence of a generator to make a biorthogonal system on locally compact group. It is a reminiscence of a result developed for group frames in [47, Theorem 6.1].

Theorem 5.2.3. *For a non-zero function $\varphi \in L^2(\mathscr{G})$, there exists a function $\psi \in \mathcal{S}^\Gamma(\varphi)$ such that $\mathcal{E}^\Gamma(\varphi)$ and $\mathcal{E}^\Gamma(\psi)$ are biorthogonal if and only if $\{1/[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)\}_{\alpha\in\widehat{\Gamma}} \in L^1(\widehat{\Gamma})$.*

For this first we prove the following lemma which provides an isometric isomorphism between $\mathcal{S}^\Gamma(\varphi)$ and the weighted Hilbert space $L^2(\widehat{\Gamma}, [\mathcal{Z}\varphi, \mathcal{Z}\varphi])$, which intertwines the left translation with modulation.

Lemma 5.2.4. *For a non-zero function $\varphi \in L^2(\mathscr{G})$, define an operator $\mathfrak{T}_\varphi : \mathcal{S}^\Gamma(\varphi) \to L^2(\widehat{\Gamma}, [\mathcal{Z}\varphi, \mathcal{Z}\varphi])$ by*

$$\mathfrak{T}_\varphi f(\alpha) = \frac{[\mathcal{Z}f, \mathcal{Z}\varphi](\alpha)}{[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)} \chi_{\Omega_\varphi}(\alpha) \text{ for } f \in \mathcal{S}^\Gamma(\varphi) \quad \text{and a.e. } \alpha \in \widehat{\Gamma},$$

where $\Omega_\varphi = \{\alpha \in \widehat{\Gamma} : [\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha) \neq 0\}$. Then, the operator $\mathfrak{T}_\varphi$ is an isometric isomorphism.

Proof. The operator $\mathfrak{T}_\varphi$ is well defined since

$$\int_{\widehat{\Gamma}} |\mathfrak{T}_\varphi f(\alpha)|^2 [\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)\, d\mu_{\widehat{\Gamma}}(\alpha) = \int_{\Omega_\varphi} |[\mathcal{Z}f, \mathcal{Z}\varphi](\alpha)|^2\ d\mu_{\widehat{\Gamma}}(\alpha) \leqslant \|f\|^2$$

for $f \in \mathcal{S}^\Gamma(\varphi)$ using (5.1.8). For the isometry of $\mathfrak{T}_\varphi$, it suffices to verify $\|\mathfrak{T}_\varphi f\| = \|f\|$ for all $f \in \operatorname{span}\mathcal{E}^\Gamma(\varphi)$ since $\operatorname{span}\mathcal{E}^\Gamma(\varphi)$ is dense in $\mathcal{S}^\Gamma(\varphi)$. By writing $f = \sum_{\gamma\in\Gamma} c_\gamma L_\gamma \varphi$ (only finitely many c_γ's are non-zero), we have

$$\begin{aligned}\int_{\widehat{\Gamma}} |\mathfrak{T}_\varphi f(\alpha)|^2 [\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)\, d\mu_{\widehat{\Gamma}}(\alpha) &= \int_{\widehat{\Gamma}} \Big|\sum_{\gamma\in\Gamma} c_\gamma \alpha(\gamma)\Big|^2 [\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)\, d\mu_{\widehat{\Gamma}}(\alpha)\\ &= \int_{\widehat{\Gamma}} \Big[\mathcal{Z}\Big(\sum_{\gamma\in\Gamma} c_\gamma L_\gamma\varphi\Big), \mathcal{Z}\Big(\sum_{\gamma\in\Gamma} c_\gamma L_\gamma\varphi\Big)\Big](\alpha)\, d\mu_{\widehat{\Gamma}}(\alpha)\\ &= \int_{\widehat{\Gamma}} [\mathcal{Z}f, \mathcal{Z}f](\alpha)\, d\mu_{\widehat{\Gamma}}(\alpha) = \|f\|^2.\end{aligned}$$

Next for the subjectivity of $\mathfrak{T}_\varphi$, we can proceed by assuming a non-zero element $\eta \in L^2(\widehat{\Gamma}, [\mathcal{Z}\varphi, \mathcal{Z}\varphi])$ such that $\eta \perp \mathfrak{T}_\varphi(\mathcal{S}^\Gamma(\varphi))$, which leads to a contradiction. □

Proof of Theorem 5.2.3. Assume that there exists a $\psi \in \mathcal{S}^\Gamma(\varphi)$ such that $\mathcal{E}^\Gamma(\varphi)$ and $\mathcal{E}^\Gamma(\psi)$ are biorthogonal. Then $\mathfrak{T}_\varphi\psi \in L^1(\widehat{\Gamma}, [\mathcal{Z}\varphi, \mathcal{Z}\varphi])$ follows by

$$\begin{aligned}\int_{\widehat{\Gamma}} |\mathfrak{T}_\varphi\psi(\alpha)|\, [\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)\, d\mu_{\widehat{\Gamma}}(\alpha) \leqslant \left(\int_{\widehat{\Gamma}} |\mathfrak{T}_\varphi\psi(\alpha)|^2 [\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)\, d\mu_{\widehat{\Gamma}}(\alpha)\right)^{\frac{1}{2}} \times\\ \left(\int_{\widehat{\Gamma}} [\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)\, d\mu_{\widehat{\Gamma}}(\alpha)\right)^{\frac{1}{2}}.\end{aligned}$$

Since $\mathfrak{T}_\varphi$ is an isometric isomorphism from Lemma 5.2.4, then from the assumptions we have $\delta_{\gamma,0} = \langle L_\gamma\psi, \varphi\rangle = \langle \mathfrak{T}_\varphi(L_\gamma\psi), \mathfrak{T}_\varphi(\varphi)\rangle = \int_{\widehat{\Gamma}} \mathfrak{T}_\varphi(\psi)(\alpha)[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)\overline{\alpha(\gamma)}\, d\mu_{\widehat{\Gamma}}(\alpha)$ for $\gamma \in \Gamma$, and hence by the Fourier expansion $\mathfrak{T}_\varphi(\psi)(\alpha)[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha) = 1$ a.e. $\alpha \in \widehat{\Gamma}$. Thus $[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha) > 0$ a.e. $\alpha \in \widehat{\Gamma}$ and also $\{1/[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)\}_{\alpha\in\widehat{\Gamma}} \in L^1(\widehat{\Gamma})$ using Lemma 5.2.4, $\int_{\widehat{\Gamma}} \frac{1}{[\mathcal{Z}\varphi,\mathcal{Z}\varphi](\alpha)}\, d\mu_{\widehat{\Gamma}}(\alpha) = \int_{\widehat{\Gamma}} |\mathfrak{T}_\varphi(\psi)(\alpha)|^2[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)\, d\mu_{\widehat{\Gamma}}(\alpha) = \|\psi\|^2$.

Conversely, suppose $\{1/[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)\}_{\alpha\in\widehat{\Gamma}} \in L^1(\widehat{\Gamma})$. Then $\{1/[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)\}_{\alpha\in\widehat{\Gamma}}$ is a member of the weighted space $L^2(\widehat{\Gamma}, [\mathcal{Z}\varphi, \mathcal{Z}\varphi])$, and hence $\psi := \mathfrak{T}_\varphi^{-1}(\{1/[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)\}_{\alpha\in\widehat{\Gamma}})$ is an element of $\mathcal{S}^\Gamma(\varphi)$ by Lemma 5.2.4. Therefore for $\gamma \in \Gamma$,

$$
\begin{aligned}
\langle L_\gamma\varphi, \psi\rangle &= \langle \mathfrak{T}_\varphi(L_\gamma\varphi), \{1/[\mathcal{Z}\varphi, \mathcal{Z}\varphi](\alpha)\}_{\alpha\in\widehat{\Gamma}}\rangle \\
&= \int_{\widehat{\Gamma}} \alpha(\gamma)\, d\mu_{\widehat{\Gamma}}(\alpha) \\
&= \delta_{\gamma,0},
\end{aligned}
$$

since $\widehat{\widehat{\Gamma}}$ is an orthonormal basis for $L^2(\widehat{\Gamma})$. This proves the result. □

Our next goal is to fix a function $\varphi \in L^2(\mathscr{G})$ and to find $\psi \in L^2(\mathscr{G})$ such that $\mathcal{E}^\Gamma(\psi)$ is an $\mathcal{S}^\Gamma(\varphi)$-subspace dual to $\mathcal{E}^\Gamma(\varphi)$ by following the idea of Theorem 5.2.1. To find such ψ, we will assume $\varphi \in L^2(\mathscr{G})$ with compact support such that $\mathcal{E}^\Gamma(\varphi)$ is an $\mathcal{S}^\Gamma(\varphi)$-subspace Riesz basis in the next result. By an *$\mathcal{S}^\Gamma(\varphi)$-subspace Riesz basis*, we mean $\mathcal{E}^\Gamma(\varphi)$ is an $\mathcal{S}^\Gamma(\varphi)$-subspace frame and $\mathcal{E}^\Gamma(\varphi)$ is linearly independent. Equivalently, there are $0 < A \leqslant B < \infty$ such that $A\sum_{\gamma\in\Gamma}|c_\gamma|^2 \leqslant \|\sum_{\gamma\in\Gamma} c_\gamma L_\gamma\varphi\|^2 \leqslant B\sum_{\gamma\in\Gamma}|c_\gamma|^2$ for some sequence $\{c_\gamma\}_{\gamma\in\Gamma} \in \ell^2(\Gamma)$ having finitely many non-zero terms. We refer [20, 49, 74] for more details.

Theorem 5.2.5. *Let $\varphi \in L^2(\mathscr{G})$ be a function with compact support such that $\mathcal{E}^\Gamma(\varphi)$ is an $\mathcal{S}^\Gamma(\varphi)$-subspace Riesz basis. If there exists a function $\psi \in L^2(\mathscr{G})$ such that $\mathcal{E}^\Gamma(\psi)$ is biorthogonal to $\mathcal{E}^\Gamma(\varphi)$, $\mathcal{E}^\Gamma(\psi)$ is an $\mathcal{S}^\Gamma(\varphi)$-subspace dual to $\mathcal{E}^\Gamma(\varphi)$. Moreover, $\mathcal{E}^\Gamma(\psi)$ is also an $\mathcal{S}^\Gamma(\psi)$-subspace Riesz basis.*

Proof. Firstly note that for any $f \in \operatorname{span}\mathcal{E}^\Gamma(\varphi)$, we can write $f = \sum_{\gamma\in\Gamma'}\langle f, L_\gamma\psi\rangle L_\gamma\varphi$ due to the biorthogonality of $\mathcal{E}^\Gamma(\psi)$ and $\mathcal{E}^\Gamma(\varphi)$ for some finite set Γ' in Γ. We need to show the expression for all $f \in \mathcal{S}^\Gamma(\varphi)$. For this, let $f \in \mathcal{S}^\Gamma(\varphi)$, then there is an element $g \in \operatorname{span}\mathcal{E}^\Gamma(\varphi)$ such that $\|f-g\| < \epsilon$ for $\epsilon > 0$. By writing $g = \sum_{\gamma\in\Gamma_1}\langle f, L_\gamma\psi\rangle L_\gamma\varphi$, where Γ_1 is a finite subset of Γ, we have $f - \sum_{\gamma\in\Gamma_1}\langle f, L_\gamma\psi\rangle L_\gamma\varphi = (f-g) + \sum_{\gamma\in\Gamma_1}\langle (g-f), L_\gamma\psi\rangle L_\gamma\varphi$, and by taking norm on both the sides, we obtain

$$\begin{aligned}\|f-\sum_{\gamma\in\Gamma_1}\langle f,L_\gamma\psi\rangle L_\gamma\varphi\| &\leqslant \|f-g\|+\|\sum_{\gamma\in\Gamma_1}\langle (g-f),L_\gamma\psi\rangle L_\gamma\varphi\| \\ &\leqslant \|f-g\|+\sqrt{B}\Big(\sum_{\gamma\in\Gamma_1}|\langle (g-f),L_\gamma\psi\rangle|^2\Big)^{1/2} \\ &\leqslant (1+\sqrt{B'B})\|f-g\|<(1+\sqrt{B'B})\epsilon,\end{aligned}$$

for some $B,B'>0$ since $\mathcal{E}^\Gamma(\varphi)$ is an $\mathcal{S}^\Gamma(\varphi)$-subspace Riesz basis. The last inequality holds true provided $\sum_{\gamma\in\Gamma}|\langle f,L_\gamma\psi\rangle|^2\leqslant B'\|f\|^2$ for all $f\in\mathcal{S}^\Gamma(\varphi)$. For this, let $f\in\mathcal{S}^\Gamma(\varphi)$. Then there is a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\text{span}\mathcal{E}^\Gamma(\varphi)$ such that $\lim_{n\to\infty}\|f_n-f\|=0$, and also by Cauchy-Schwarz inequality we have $\lim_{n\to\infty}\langle f_n,L_\gamma\psi\rangle=\langle f,L_\gamma\psi\rangle$ for every $\gamma\in\Gamma$. Hence for any finite set Γ_1 of Γ, we have

$$\sum_{\gamma\in\Gamma_1}|\langle f,L_\gamma\psi\rangle|^2=\sum_{\gamma\in\Gamma_1}\lim_{n\to\infty}|\langle f_n,L_\gamma\psi\rangle|^2=\lim_{n\to\infty}\sum_{\gamma\in\Gamma_1}|\langle f_n,L_\gamma\psi\rangle|^2\leqslant B'\lim_{n\to\infty}\|f_n\|^2=B'\|f\|^2,$$

provided $\sum_{\gamma\in\Gamma}|\langle f,L_\gamma\psi\rangle|^2\leqslant B'\|f\|^2$ for all $f\in\text{span}\mathcal{E}^\Gamma(\varphi)$. To show, this we proceed as follows:
By writing $f\in\text{span}\mathcal{E}^\Gamma(\varphi)$ in the form $f=\sum_{\gamma\in\Gamma}c_\gamma L_\gamma\varphi$ with finitely many non-zeros $\{c_\gamma\}_{\gamma\in\Gamma}\in\ell^2$, we get $\|f\|^2=\|\mathcal{Z}f\|^2=\int_{\widehat{\Gamma}}|\widehat{c}(\alpha)|^2[\mathcal{Z}\varphi,\mathcal{Z}\varphi](\alpha)\,d\mu_{\widehat{\Gamma}}(\alpha)$ by following the steps of (5.1.7). Since $\mathcal{E}^\Gamma(\varphi)$ is a $\mathcal{S}^\Gamma(\varphi)$-subspace Riesz basis, we have $C\leqslant[\mathcal{Z}\varphi,\mathcal{Z}\varphi](\alpha)\leqslant D$ for a.e. $\alpha\in\widehat{\Gamma}$ and some $0<C\leqslant D<\infty$ [49, Remark 5.6], and hence $C\int_{\widehat{\Gamma}}|\widehat{c}(\alpha)|^2\,d\mu_{\widehat{\Gamma}}(\alpha)\leqslant\|f\|^2\leqslant D\int_{\widehat{\Gamma}}|\widehat{c}(\gamma)|^2\,d\mu_{\widehat{\Gamma}}(\alpha)$. Thus, we get $\int_{\widehat{\Gamma}}|\widehat{c}(\alpha)|^2\,d\mu_{\widehat{\Gamma}}(\alpha)\leqslant\frac{1}{C}\|f\|^2$. Further, due to the Biorthogonality of the sets $\mathcal{E}^\Gamma(\varphi)$ and $\mathcal{E}^\Gamma(\psi)$ and Parseval's formula, we write $\int_{\widehat{\Gamma}}|\widehat{c}(\alpha)|^2\,d\mu_{\widehat{\Gamma}}(\alpha)=\sum_{\gamma\in\Gamma}|c_\gamma|^2=\sum_{\gamma\in\Gamma}|\langle f,L_\gamma\psi\rangle|^2$ which gives the inequality $\sum_{\gamma\in\Gamma}|\langle f,L_\gamma\psi\rangle|^2\leqslant B'\|f\|^2$ for $f\in\text{span}\mathcal{E}^\Gamma(\varphi)$ and $B'=\frac{1}{C}$. Thus the result follows. □

In the following example we construct various biorthogonal systems using Theorem 5.2.1.

Example 5.2.6. First we recall Example 5.1.15 and assume that $\mathcal{E}^\Gamma(\psi)$ is an $\mathcal{S}^\Gamma(\psi)$-subspace Riesz basis. The functions $\varphi,\tilde{\varphi}\in L^2(\mathscr{G})$ are defined by $(\mathcal{Z}\varphi)(\alpha,\Gamma x)=\mathfrak{m}(\alpha)(\mathcal{Z}\psi)(\alpha,\Gamma x)$ and $(\mathcal{Z}\tilde{\varphi})(\alpha,\Gamma x)=\tilde{\mathfrak{m}}(\alpha)(\mathcal{Z}\psi)(\alpha,\Gamma x)$ for all $\Gamma x\in\Gamma\backslash\mathscr{G}$, and a.e. $\alpha\in\widehat{\Gamma}$, where $\mathfrak{m},\tilde{\mathfrak{m}}\in L^2\left(\widehat{\Gamma},[\mathcal{Z}\psi,\mathcal{Z}\psi]\right)$. Then in view of Theorem 5.2.1, $\mathcal{E}^\Gamma(\varphi)$ and $\mathcal{E}^\Gamma(\tilde{\varphi})$ are biorthogonal if and only if $\mathfrak{m}(\alpha)\overline{\tilde{\mathfrak{m}}(\alpha)}=\frac{1}{[\mathcal{Z}\psi,\mathcal{Z}\psi](\alpha)}$ for a.e. $\alpha\in\widehat{\Gamma}$, follows from the calculations of Example 5.1.15.

5.3. Orbit generated by the action of an abelian subgroup

The purpose of this section is devoted to characterize a pair of orthogonal frames and subspace dual of a Bessel family/frame $\mathcal{E}^{\Gamma}(\mathscr{A}) = \{L_{\gamma}\varphi_t : \gamma \in \Gamma, t \in \mathcal{N}\}$ in $L^2(\mathscr{G})$, where the group Γ is a closed abelian (need not be discrete) subgroup of $\mathscr{G}$, $\mathcal{N}$ is a σ-finite measure space (need not be countable), and $\mathscr{A} = \{\varphi_t\}_{t\in\mathcal{N}} \subset L^2(\mathscr{G})$. We characterize such results using the Zak transform $\mathcal{Z}$ for the pair $(\mathscr{G}, \Gamma)$ defined by (4.1.1). When $\mathscr{G}$ becomes an abelian group $\mathcal{G}$, the fiberization map is also used which unifies the classical results related to the orthogonal and duals of a Bessel family/frame associated with a TI space.

5.3.1. Orthogonal and dual frames' characterization using the range function

Now, we are going to discuss our main result, which is connected to the subspace orthogonal and duals of a Bessel family associated with the range function in terms of the Zak transform. It includes certain results of [19] which contains an alternative strategy for proving the result.

Theorem 5.3.1. *Let $(\mathcal{N}, \mu_{\mathcal{N}})$ be a complete, σ-finite measure space and let $\mathscr{A} = \{\varphi_t\}_{t\in\mathcal{N}}$ and $\mathscr{A}' = \{\psi_t\}_{t\in\mathcal{N}}$ be two collections of functions in $L^2(\mathscr{G})$ such that the Γ-TG systems $\mathcal{E}^{\Gamma}(\mathscr{A})$ and $\mathcal{E}^{\Gamma}(\mathscr{A}')$ are Bessel. Assume $\mathscr{A}$ has a countable dense subset $\mathscr{A}_0$ for which $J_{\mathcal{A}}(\alpha) = \overline{\operatorname{span}}\{(\mathcal{Z}f)(\alpha) : f \in \mathscr{A}_0\}$ a.e. $\alpha \in \widehat{\Gamma}$. Then the following hold true:*

- (i) *$\mathcal{E}^{\Gamma}(\mathscr{A}')$ is an $\mathcal{S}^{\Gamma}(\mathscr{A})$-subspace dual to $\mathcal{E}^{\Gamma}(\mathscr{A})$ if and only if the system $\mathcal{Z}\mathscr{A}'(\alpha) = \{\mathcal{Z}\psi(\alpha) : \psi \in \mathscr{A}'\}$ is a $J_{\mathscr{A}}(\alpha)$-subspace dual to $\mathcal{Z}\mathscr{A}(\alpha) = \{\mathcal{Z}\varphi(\alpha) : \varphi \in \mathscr{A}\}$ for a.e. $\alpha \in \widehat{\Gamma}$.*
- (ii) *$\mathcal{E}^{\Gamma}(\mathscr{A}')$ is an $\mathcal{S}^{\Gamma}(\mathscr{A})$-subspace orthogonal to $\mathcal{E}^{\Gamma}(\mathscr{A})$ if and only if the system $\mathcal{Z}\mathscr{A}'(\alpha)$ is a $J_{\mathscr{A}}(\alpha)$-subspace orthogonal to $\mathcal{Z}\mathscr{A}(\alpha)$ for a.e. $\alpha \in \widehat{\Gamma}$.*

When the pair $(\mathscr{G}, \Gamma)$ is an abelian pair $(\mathcal{G}, \Lambda)$, let $J_{\mathscr{A}}(\alpha) = \overline{\operatorname{span}}\{(\mathscr{T}f)(\beta\Lambda^{\perp}) : f \in \mathscr{A}_0\}$ for a.e. $\beta\Lambda^{\perp} \in \widehat{\mathcal{G}}/\Lambda^{\perp}$. Then (i) and (ii) become (i') and (ii') as follows:

- (i') *$\mathcal{E}^{\Lambda}(\mathscr{A}')$ is an $\mathcal{S}^{\Lambda}(\mathscr{A})$-subspace dual to $\mathcal{E}^{\Lambda}(\mathscr{A})$ if and only if the system $\mathscr{T}\mathscr{A}'(\beta\Lambda^{\perp}) = \{\mathscr{T}\psi(\beta\Lambda^{\perp}) : \psi \in \mathscr{A}'\}$ is a $J_{\mathscr{A}}(\beta\Lambda^{\perp})$-subspace dual to $\mathscr{T}\mathscr{A}(\beta\Lambda^{\perp}) = \{\mathscr{T}\varphi(\beta\Lambda^{\perp}) : \varphi \in \mathscr{A}\}$ for a.e. $\beta\Lambda^{\perp} \in \widehat{\mathcal{G}}/\Lambda^{\perp}$.*
- (ii') *$\mathcal{E}^{\Lambda}(\mathscr{A}')$ is an $\mathcal{S}^{\Lambda}(\mathscr{A})$-subspace orthogonal to $\mathcal{E}^{\Lambda}(\mathscr{A})$ if and only if the system $\mathscr{T}\mathscr{A}'(\beta\Lambda^{\perp})$ is a $J_{\mathscr{A}}(\beta\Lambda^{\perp})$-subspace orthogonal for a.e. $\beta\Lambda^{\perp} \in \widehat{\mathcal{G}}/\Lambda^{\perp}$.*

Before proceeding for a proof of Theorem 5.3.1, we first establish the following result to express the change of role of Γ to $\widehat{\Gamma}$ in terms of the Zak transform $\mathcal{Z}$.

Proposition 5.3.2. *Assuming the hypotheses of Theorem 5.3.1, the following holds for all $f, g \in L^2(\mathscr{G})$:*

$$\int_{\mathcal{N}}\int_{\Gamma}\langle f, L_\gamma\varphi_t\rangle\langle L_\gamma\psi_t, g\rangle d\mu_{\mathcal{N}}(t)d\mu_{\Gamma}(\gamma) = \int_{\mathcal{N}}\int_{\widehat{\Gamma}}\langle \mathcal{Z}f(\alpha), \mathcal{Z}\varphi_t(\alpha)\rangle\langle \mathcal{Z}\psi_t(\alpha), \mathcal{Z}g(\alpha)\rangle d\mu_{\widehat{\Gamma}}(\alpha)d\mu_{\mathcal{N}}(t).$$

Proof. Applying the Zak transform,

$$\int_{\mathcal{N}}\int_{\Gamma}\langle f, L_\gamma\varphi_t\rangle\,\langle L_\gamma\psi_t, g\rangle\; d\mu_{\Gamma}(\gamma)\; d\mu_{\mathcal{N}}(t) = \int_{\mathcal{N}}\int_{\Gamma}\langle \mathcal{Z}f, \mathcal{Z}(L_\gamma\varphi_t)\rangle\,\langle \mathcal{Z}(L_\gamma\psi_t), \mathcal{Z}g\rangle\; d\mu_{\Gamma}(\gamma)\; d\mu_{\mathcal{N}}(t)$$

$$= \int_{\mathcal{N}}\int_{\Gamma}\left(\int_{\widehat{\Gamma}}\zeta_t(\alpha)\alpha(\gamma)\; d\mu_{\widehat{\Gamma}}(\alpha)\right)\overline{\left(\int_{\widehat{\Gamma}}\eta_t(\alpha)\alpha(\gamma)\; d\mu_{\widehat{\Gamma}}(\alpha)\right)}d\mu_{\Gamma}(\gamma)\; d\mu_{\mathcal{N}}(t), \tag{5.3.1}$$

where $\zeta_t(\alpha) = \langle \mathcal{Z}f(\alpha), \mathcal{Z}\varphi_t(\alpha)\rangle$ and $\eta_t(\alpha) = \langle \mathcal{Z}g(\alpha), \mathcal{Z}\psi_t(\alpha)\rangle$ for each $t \in \mathcal{N}$. The functions ζ_t and η_t are in $L^1(\widehat{\Gamma})$ due to Cauchy-Schwarz inequality and

$$\int_{\widehat{\Gamma}}|\zeta_t(\alpha)|\; d\mu_{\widehat{\Gamma}}(\alpha) = \int_{\widehat{\Gamma}}\left|\int_{\Gamma\backslash\mathscr{G}}\mathcal{Z}f(\alpha)(\Gamma x)\overline{\mathcal{Z}\varphi_t(\alpha)(\Gamma x)}\; d\mu_{\Gamma\backslash\mathscr{G}}(\Gamma x)\right| d\mu_{\widehat{\Gamma}}(\alpha)$$
$$\leqslant \left(\int_{\widehat{\Gamma}}\int_{\Gamma\backslash\mathscr{G}}|\mathcal{Z}f(\alpha)(\Gamma x)|^2\; d\mu_{\Gamma\backslash\mathscr{G}}(\Gamma x)\; d\mu_{\widehat{\Gamma}}(\alpha)\right)^{1/2}\left(\int_{\widehat{\Gamma}}\int_{\Gamma\backslash\mathscr{G}}|\mathcal{Z}\varphi_t(\alpha)(\Gamma x)|^2\; d\mu_{\Gamma\backslash\mathscr{G}}(\Gamma x)\; d\mu_{\widehat{\Gamma}}(\alpha)\right)^{1/2}$$
$$= \|\mathcal{Z}f\|\|\mathcal{Z}\varphi_t\| = \|f\|\|\varphi_t\| < \infty.$$

Similarly, $\eta_t \in L^1(\widehat{\Gamma})$. Then for each $t \in \mathcal{N}$, the inverse Fourier transform $\check{\zeta}_t$ and $\check{\eta}_t$ of ζ_t and η_t, respectively, are members of $L^2(\Gamma)$, where

$$\check{\zeta}_t(\gamma) = \int_{\widehat{\Gamma}}\zeta_t(\alpha)\alpha(\gamma)\; d\mu_{\widehat{\Gamma}}(\alpha) \text{ and } \check{\eta}_t(\gamma) = \int_{\widehat{\Gamma}}\eta_t(\alpha)\alpha(\gamma)\; d\mu_{\widehat{\Gamma}}(\alpha).$$

This follows by observing the Bessel property of $\mathcal{E}^{\Gamma}(\mathscr{A})$ and calculations

$$\infty > \int_{\mathcal{N}}\int_{\Gamma}|\langle f, L_\gamma\varphi_t\rangle|^2\; d\mu_{\Gamma}(\gamma) = \int_{\mathcal{N}}\int_{\Gamma}\left|\int_{\widehat{\Gamma}}\langle \mathcal{Z}f(\alpha), \mathcal{Z}(L_\gamma\varphi_t)(\alpha)\rangle\; d\mu_{\widehat{\Gamma}}(\alpha)\right|^2\; d\mu_{\Gamma}(\gamma)\; d\mu_{\mathcal{N}}(t)$$
$$= \int_{\mathcal{N}}\int_{\Gamma}\left|\int_{\widehat{\Gamma}}\langle \mathcal{Z}f(\alpha), \mathcal{Z}\varphi_t(\alpha)\rangle\alpha(\gamma)\; d\mu_{\widehat{\Gamma}}(\alpha)\right|^2 d\mu_{\Gamma}(\gamma)\; d\mu_{\mathcal{N}}(t) = \int_{\mathcal{N}}\int_{\Gamma}|\check{\zeta}_t(\gamma)|^2\; d\mu_{\Gamma}(\gamma)\; d\mu_{\mathcal{N}}(t).$$

Similarly, we have $\check{\eta}_t \in L^2(\Gamma)$. Therefore, the equation (5.3.1) is equal to the following

$$\int_{\mathcal{N}}\int_{\Gamma}\check{\zeta}_t(\gamma)\overline{\check{\eta}_t(\gamma)}\; d\mu_{\Gamma}(\gamma)\; d\mu_{\mathcal{N}}(t) = \int_{\mathcal{N}}\int_{\widehat{\Gamma}}\eta_t(\alpha)\overline{\zeta_t(\alpha)}\; d\mu_{\widehat{\Gamma}}(\alpha)\; d\mu_{\mathcal{N}}(t).$$

Thus the result follows. ∎

Proof. (i) Firstly assume $\mathcal{E}^\Gamma(\mathscr{A}')$ is an $\mathcal{S}^\Gamma(\mathscr{A})$-subspace dual to $\mathcal{E}^\Gamma(\mathscr{A})$. Then for $f \in \mathcal{S}^\Gamma(\mathscr{A})$ and $g \in L^2(\mathscr{G})$, we have

$$\int_{t\in\mathcal{N}}\int_{\gamma\in\Gamma}\langle f, L_\gamma\psi_t\rangle\langle L_\gamma\varphi_t, g\rangle\, d\mu_\Gamma(\gamma)\, d\mu_{\mathcal{N}}(t) = \langle f, g\rangle.$$

Equivalently, we get the following by applying Proposition 5.3.2 and the Zak transform $\mathcal{Z}$,

$$\int_{\mathcal{N}}\int_{\widehat{\Gamma}}\langle \mathcal{Z}f(\alpha), \mathcal{Z}\varphi_t(\alpha)\rangle\langle \mathcal{Z}\psi_t(\alpha), \mathcal{Z}g(\alpha)\rangle\ d\mu_{\widehat{\Gamma}}(\alpha)\, d\mu_{\mathcal{N}}(t) = \int_{\widehat{\Gamma}}\langle \mathcal{Z}f(\alpha), \mathcal{Z}g(\alpha)\rangle\, d\mu_{\widehat{\Gamma}}(\alpha).$$

To get the result for a.e. $\alpha \in \widehat{\Gamma}$, we show $\int_{\mathcal{N}}\langle \mathcal{Z}f(\alpha), \mathcal{Z}\varphi_t(\alpha)\rangle\langle \mathcal{Z}\psi_t(\alpha), \mathcal{Z}g(\alpha)\rangle\ d\mu_{\mathcal{N}}(t) = \langle \mathcal{Z}f(\alpha), \mathcal{Z}g(\alpha)\rangle$ for $f \in \mathcal{S}^\Gamma(\mathscr{A})$ and $g \in L^2(\mathscr{G})$. On the contrary of this we assume a Borel measurable subset Y in $\widehat{\Gamma}$ having positive measure such that the equality does not hold on Y. Then, there are $m_0, n_0 \in \mathbb{N}$ such that $S_{m_0,n_0} \cap Y$ is a Borel measurable subset of $\widehat{\Gamma}$ having positive measure, where for each $m, n \in \mathbb{N}$, the set $S_{m,n}$ is

$$S_{m,n} = \left\{\alpha \in \widehat{\Gamma} : \rho_{m,n}(\alpha) := \int_{\mathcal{N}}\langle P_{J_{\mathscr{A}}}(\alpha)x_m, \mathcal{Z}\varphi_t(\alpha)\rangle\langle \mathcal{Z}\psi_t(\alpha), x_n\rangle\ d\mu_{\mathcal{N}}(t) - \langle P_{J_{\mathscr{A}}}(\alpha)x_m, x_n\rangle \neq 0\right\},$$

when $P_{J_{\mathscr{A}}}(\alpha)$ is an orthogonal projection onto $J_{\mathscr{A}}(\alpha)$ for a.e. $\alpha \in \widehat{\Gamma}$ and $\{x_n\}_{n\in\mathbb{N}}$ is a countable dense subset of $L^2(\Gamma\backslash\mathscr{G})$. Clearly, $\{P_{J_{\mathscr{A}}}(\alpha)x_n\}_{n\in\mathbb{N}}$ is dense in $J_{\mathscr{A}}(\alpha)$ for a.e. $\alpha \in \widehat{\Gamma}$. Hence, either real or imaginary parts of $\rho_{m_0,n_0}(\alpha)$ are strictly positive or negative for a.e. $\alpha \in S_{m_0,n_0} \cap Y$. By adopting the standard techniques, first we assume the real part of $\rho_{m_0,n_0}(\alpha)$ is strictly positive on $S_{m_0,n_0} \cap Y$. By choosing a Borel measurable subset S of $S_{m_0,n_0} \cap Y$ having positive measure, we define functions h_1 and h_2 as follows: $h_1(\alpha) = \begin{cases} P_{J_{\mathscr{A}}}(\alpha)x_{m_0} & \text{for } \alpha \in S, \\ 0 & \text{for } \alpha \in \widehat{\Gamma}\backslash S, \end{cases}$ and $h_2(\alpha) = \begin{cases} P_{J_{\mathscr{A}}}(\alpha)x_{n_0} & \text{for } \alpha \in S, \\ 0 & \text{for } \alpha \in \widehat{\Gamma}\backslash S. \end{cases}$
Then, we have $h_1(\alpha)$ and $h_2(\alpha) \in J_{\mathscr{A}}(\alpha)$ for a.e. $\alpha \in \widehat{\Gamma}$ since $\{P_{J_{\mathscr{A}}}(\alpha)x_n\}_{n\in\mathbb{N}}$ is dense in $J_{\mathscr{A}}(\alpha)$. Hence we get $h_1, h_2 \in \mathcal{S}^\Gamma(\mathscr{A})$ which gives $\int_S \rho_{m_0,n_0}(\alpha)\, d\mu_{\widehat{\Gamma}}(\alpha) = 0$. We arrive on a contradiction since the measure of S is positive and the real part of $\rho_{m_0,n_0}(\alpha)$ is strictly positive on S. Other cases follow in a similar way. Thus the result follows.

The converse part follows easily by the Proposition 5.3.2.

(ii) For $f \in \mathcal{S}^\Gamma(\mathscr{A})$ and $g \in L^2(\mathscr{G})$, first assume $\int_{t\in\mathcal{N}}\int_{\gamma\in\Gamma}\langle f, L_\gamma\psi_t\rangle\langle L_\gamma\varphi_t, g\rangle\, d\mu_\Gamma(\gamma)\, d\mu_{\mathcal{N}}(t) = 0$, which is equivalent to $\int_{\mathcal{N}}\int_{\widehat{\Gamma}}\langle \mathcal{Z}f(\alpha), \mathcal{Z}\varphi_t(\alpha)\rangle\ \langle \mathcal{Z}\psi_t(\alpha), \mathcal{Z}g(\alpha)\rangle\ d\mu_{\widehat{\Gamma}}(\alpha)\, d\mu_{\mathcal{N}}(t) = 0$ from Proposition 5.3.2. To get the result for a.e. $\alpha \in \widehat{\Gamma}$, we need to show

$$\int_{\mathcal{N}}\langle \mathcal{Z}f(\alpha), \mathcal{Z}\varphi_t(\alpha)\rangle\langle \mathcal{Z}\psi_t(\alpha), \mathcal{Z}g(\alpha)\rangle\ d\mu_{\mathcal{N}}(t) = 0 \text{ for } f \in \mathcal{S}^\Gamma(\mathscr{A}),\ g \in L^2(\mathscr{G}).$$

For this, let $(e_i)_{i\in\mathbb{Z}}$ be an orthonormal basis for $L^2(\Gamma\backslash\mathscr{G})$ and $P_{J_{\mathscr{A}}}(\alpha)$ is an orthogonal projection onto $J_{\mathscr{A}}(\alpha)$ for a.e. $\alpha \in \widehat{\Gamma}$. Assume on the contrary, there exists $i_0 \in \mathbb{Z}$ such that $h(\alpha) = \int_{\mathcal{N}} \langle P(\alpha)e_{i_0}, \mathcal{Z}\varphi_t(\alpha)\rangle \overline{\langle \mathcal{Z}\psi_t(\alpha), \mathcal{Z}g(\alpha)\rangle}\, d\mu_{\mathcal{N}}(t) \neq 0$ on a measurable set $E \subseteq \widehat{\Gamma}$ with $\mu_{\widehat{\Gamma}}(E) > 0$. The rest of the proof follows in the similar manner of Theorem 5.1.6 (ii). The converse part follows immediately by Proposition 5.3.2. □

In case of $\mathcal{S}^\Gamma(\mathscr{A}) = \mathcal{S}^\Gamma(\mathscr{A}')$ in Theorem 5.3.1, we get $J_{\mathscr{A}'}(\alpha) = J_{\mathscr{A}}(\alpha)$ a.e. $\alpha \in \widehat{\Gamma}$, follows by observing the bijection $J \mapsto V_J$ and $V_J = S^\Gamma(\mathscr{A}) = S^\Gamma(\mathscr{A}')$. Then we have following result.

Corollary 5.3.3. *Under the hypotheses mentioned in Theorem 5.3.1 let $\mathcal{S}^\Gamma(\mathscr{A}) = \mathcal{S}^\Gamma(\mathscr{A}')$. Then,*

(i) *$\mathcal{E}^\Gamma(\mathscr{A}')$ and $\mathcal{E}^\Gamma(\mathscr{A})$ are dual frames to each other if and only if for a.e. $\alpha \in \widehat{\Gamma}$, the system $\mathcal{Z}\mathscr{A}'(\alpha) = \{\mathcal{Z}\psi(\alpha) : \psi \in \mathscr{A}\}$ and $\mathcal{Z}\mathscr{A}(\alpha) = \{\mathcal{Z}\varphi(\alpha) : \varphi \in \mathscr{A}\}$ are dual to each other.*

(ii) *$\mathcal{E}^\Gamma(\mathscr{A}')$ and $\mathcal{E}^\Gamma(\mathscr{A})$ are orthogonal pair if and only if for a.e. $\alpha \in \widehat{\Gamma}$, the system $\mathcal{Z}\mathscr{A}'(\alpha)$ and $\mathcal{Z}\mathscr{A}(\alpha)$ are orthogonal pair.*

5.3.2. Super dual frames

Orthogonality is a fundamental idea that plays a significant role in the discussion of the dual frame property of super-frames in orthogonal direct sum of Hilbert spaces. This concept was first presented by Han and Larson [43] and Balan [11], who developed it further. This notion is further generalized in the context of TI and Gabor systems [58,59]. By a *super Hilbert space* $L^2(\mathscr{G}) \oplus \cdots \oplus L^2(\mathscr{G})$ (N-copies) or $\oplus^N L^2(\mathscr{G})$, we mean it is a collection of functions of the form

$$\oplus^N L^2(\mathscr{G}) := \{\oplus_{n=1}^N f^{(n)} := (f^{(1)}, f^{(2)}, \ldots, f^{(N)}) : f^{(n)} \in L^2(\mathscr{G}), 1 \leqslant n \leqslant N\}, \tag{5.3.2}$$

with the inner product $\langle \oplus_{n=1}^N f^{(n)}, \oplus_{n=1}^N g^{(n)}\rangle = \sum_{n=1}^N \langle f^{(n)}, g^{(n)}\rangle$. Indeed, $\oplus^N L^2(\mathscr{G})$ is nothing but the Hilbert space $L^2(\mathscr{G} \times \mathbb{Z}_N)$, where $\mathbb{Z}_N$ is an abelian group with modulo N. Analogous to the classical trend, we state the following characterization result for (super) dual frames of translates in the super Hilbert space $\oplus^N L^2(\mathscr{G})$.

Theorem 5.3.4. *Let $N \in \mathbb{N}$ and $\mathcal{N}$ be a σ-finite measure space with counting measure. For $1 \leqslant n \leqslant N$, let $\{\varphi_t^{(n)}\}_{t\in\mathcal{N}}$ and $\{\psi_t^{(n)}\}_{t\in\mathcal{N}}$ be two collections of functions in $L^2(\mathscr{G})$ such that $\{L_\gamma\varphi_t^{(n)}\}_{t\in\mathcal{N},\gamma\in\Gamma}$ and $\{L_\gamma\psi_t^{(n)}\}_{t\in\mathcal{N},\gamma\in\Gamma}$ are Bessel.*

For each $\gamma \in \Gamma$, define the translation operator $\mathcal{L}_\gamma := \oplus^N L_\gamma$ which acts on an element $\oplus_{n=1}^N f^{(n)}$ by

$$\mathcal{L}_\gamma(\oplus_{n=1}^N f^{(n)}) := \oplus_{n=1}^N L_\gamma f^{(n)}.$$

Then, $\{\mathcal{L}_\gamma(\oplus_{n=1}^N\varphi_t^{(n)})\}_{t\in\mathcal{N},\gamma\in\Gamma}$ and $\{\mathcal{L}_\gamma(\oplus_{n=1}^N\psi_t^{(n)})\}_{t\in\mathcal{N},\gamma\in\Gamma}$ are (super) dual frames in $\oplus^N L^2(\mathscr{G})$ if and only if for a.e. $\alpha \in \widehat{\Gamma}$, the following holds:

(i) *The systems $\left\{\{\mathcal{Z}\varphi_t^{(n)}(\alpha,\Gamma x)\}_{\Gamma x\in\Gamma\backslash\mathscr{G}}\right\}_{t\in\mathcal{N}}$ and $\left\{\{\mathcal{Z}\psi_t^{(n)}(\alpha,\Gamma x)\}_{\Gamma x\in\Gamma\backslash\mathscr{G}}\right\}_{t\in\mathcal{N}}$ are dual frames in $L^2(\Gamma\backslash\mathscr{G})$ for $1 \leqslant n \leqslant N$.*

(ii) *For $1 \leqslant n_1 \neq n_2 \leqslant N$, $\left\{\{\mathcal{Z}\varphi_t^{(n_1)}(\alpha,\Gamma x)\}_{\Gamma x\in\Gamma\backslash\mathscr{G}}\right\}_{t\in\mathcal{N}}$ and $\left\{\{\mathcal{Z}\psi_t^{(n_2)}(\alpha,\Gamma x)\}_{\Gamma x\in\Gamma\backslash\mathscr{G}} : t \in \mathcal{N}\right\}$ form an orthogonal pair.*

Proof. Assume the systems $\{\mathcal{L}_\gamma(\oplus_{n=1}^N\varphi_t^{(n)})\}_{t\in\mathcal{N},\gamma\in\Gamma}$ and $\{\mathcal{L}_\gamma(\oplus_{n=1}^N\psi_t^{(n)})\}_{t\in\mathcal{N},\gamma\in\Gamma}$ are (super) dual frames in $\oplus^N L^2(\mathscr{G})$. Then for each $1 \leqslant n \leqslant N$, (i) follows by just applying the orthogonal projection P_n on it and Corollary 5.3.3 . For the part (ii), let $1 \leqslant n_1 \neq n_2 \leqslant N$ and $h \in \oplus^N L^2(\mathscr{G})$. Then, we have $P_{n_1}(P_{n_2}h) = 0$, where $P_{n_1}(P_{n_2}h)$ is equal to

$$\begin{aligned}&\int_{\mathcal{N}}\int_\Gamma \langle P_{n_2}h, P_{n_2}^*\mathcal{L}_\gamma(\oplus_{n=1}^N\varphi_t^{(n)})\rangle P_{n_1}(\mathcal{L}_\gamma \oplus_{n=1}^N \psi_t^{(n)})\, d\mu_\Gamma(\gamma)\, d\mu_{\mathcal{N}}(t)\\ &\qquad = \int_{\mathcal{N}}\int_\Gamma \langle P_{n_2}h, L_\gamma\varphi_t^{(n_2)}\rangle L_\gamma\psi_t^{(n_1)}\, d\mu_\Gamma(\gamma)\, d\mu_{\mathcal{N}}(t).\end{aligned}$$

Hence, $\mathcal{E}^\Gamma(\{\varphi_t^{(n_2)}\}_{t\in\mathcal{N}})$ and $\mathcal{E}^\Gamma(\{\psi_t^{(n_1)}\}_{t\in\mathcal{N}})$ are an orthogonal pair. Therefore (ii) follows.

Conversely, let us assume (i) and (ii) hold. Then notice that both $\{\mathcal{L}_\gamma(\oplus_{n=1}^N\varphi_t^{(n)})\}_{t\in\mathcal{N},\gamma\in\Gamma}$ and $\{\mathcal{L}_\gamma(\oplus_{n=1}^N\psi_t^{(n)})\}_{t\in\mathcal{N},\gamma\in\Gamma}$ are Bessel families in $\oplus^N L^2(\mathscr{G})$, follows by the below calculations for $h \in L^2(\mathscr{G})^N$ using the Bessel property of $\{L_\gamma\varphi_t^{(n)}\}_{t\in\mathcal{N},\gamma\in\Gamma}$ with Bessel bound $B^{(n)}$:

$$\begin{aligned}\int_{\mathcal{N}}\int_\Gamma |\langle h, \mathcal{L}_\gamma(\oplus_{n=1}^N\varphi_t^{(n)})\rangle|^2\, d\mu_\Gamma(\gamma)\, d\mu_{\mathcal{N}}(t) &= \int_{\mathcal{N}}\int_\Gamma |\langle \oplus_{n=1}^N P_n h, \mathcal{L}_\gamma(\oplus_{n=1}^N\varphi_t^{(n)})\rangle|^2\, d\mu_\Gamma(\gamma)\, d\mu_{\mathcal{N}}(t)\\ = \int_{\mathcal{N}}\int_\Gamma \Big|\sum_{n=1}^N \langle P_n h, L_\gamma\varphi_t^{(n)}\rangle\Big|^2\, d\mu_\Gamma(\gamma)\, d\mu_{\mathcal{N}}(t) &\leqslant C\|h\|^2 \sum_{n=1}^N B^{(n)}\end{aligned}$$

for some constant $C > 0$ (similarly, for $\{L_\gamma\psi_t^{(n)}\}_{t\in\mathcal{N},\gamma\in\Gamma}$). Thus we have the result using Theorem 5.3.3, by just looking the reproducing formula for each $h \in \oplus^N L^2(\mathscr{G})$ and writing

$h = \oplus_{n=1}^{N} P_n h$ in the below calculations:

$$\begin{aligned}
&\int_{\mathcal{N}}\int_{\Gamma}\langle h, \mathcal{L}_\gamma(\oplus_{n=1}^{N}\psi_t^{(n)})\rangle \mathcal{L}_\gamma(\oplus_{n=1}^{N}\varphi_t^{(n)})\ d\mu_\Gamma(\gamma)\ d\mu_{\mathcal{N}}(t)\\
&= \int_{\mathcal{N}}\int_{\Gamma}\sum_{n=1}^{N}\langle P_n h, L_\gamma\psi_t^{(n)}\rangle \mathcal{L}_\gamma(\oplus_{n=1}^{N}\varphi_t^{(n)})\ d\mu_\Gamma(\gamma)\ d\mu_{\mathcal{N}}(t)\\
&= \int_{\mathcal{N}}\int_{\Gamma}\sum_{n=1}^{N}\langle P_n h, L_\gamma\psi_t^{(n)}\rangle L_\gamma\varphi_t^{(1)}\ d\mu_\Gamma(\gamma)\ d\mu_{\mathcal{N}}(t) \oplus \dots\\
&\qquad\qquad \oplus \int_{\mathcal{N}}\int_{\Gamma}\sum_{n=1}^{N}\langle P_n h, L_\gamma\psi_t^{(n)}\rangle L_\gamma\varphi_t^{(N)}\ d\mu_\Gamma(\gamma)\ d\mu_{\mathcal{N}}(t)\\
&= P_1 h \oplus \cdots \oplus P_N h = h.
\end{aligned}$$

□

5.4. Applications

In this section, we explore how our findings can be put to use. Since there is always an attraction of researches to find various properties of Gabor systems (see [8, 17, 23, 24, 43, 49, 52]) and references therein, firstly we focus on the Gabor system.

5.4.1. Gabor System

Let $\mathcal{G}$ be a second countable LCA group having a closed subgroup Λ. Then for a family of functions $\mathscr{A} = \{\varphi_t : t \in \mathcal{N}\}$ in $L^2(\mathcal{G})$, a *Gabor system* $G(\mathscr{A}, \Lambda, \Lambda^\perp)$ is

$$G(\mathscr{A}, \Lambda, \Lambda^\perp) := \left\{L_\lambda E_\omega \varphi_t : \lambda \in \Lambda, \omega \in \Lambda^\perp, t \in \mathcal{N}\right\},$$

where $\mathcal{N}$ is a σ-finite measure space, and for $\omega \in \widehat{\mathcal{G}}$, the *modulation operator* E_ω on $L^2(\mathcal{G})$ is defined by $(E_\omega f)(x) = \omega(x) f(x)$, $x \in \mathcal{G},\ f \in L^2(\mathcal{G})$. We denote $\mathcal{S}(\mathscr{A}, \Lambda, \Lambda^\perp) := \overline{\text{span}} G(\mathscr{A}, \Lambda, \Lambda^\perp)$.

In case of $\mathcal{N}$ having counting measure and discrete subgroup Λ, the following result is established for the pair $(\mathcal{G}, \Lambda)$ by observing the Gabor system $G(\mathscr{A}, \Lambda, \Lambda^\perp)$ as a Λ-TG system $\mathcal{E}^\Lambda(\tilde{\mathscr{A}})$, where $\tilde{\mathscr{A}} = \{E_\omega \varphi : \varphi \in \mathscr{A}, \omega \in \Lambda^\perp\}$. The similar results can be deduced for the case of uniform lattice Λ in $\mathcal{G}$, in particular, $\mathbb{Z}^m$ in $\mathbb{R}^m$.

Theorem 5.4.1. *Let $\mathscr{A} = \{\varphi_t\}_{t\in\mathcal{N}}$ and $\mathscr{A}' = \{\psi_t\}_{t\in\mathcal{N}}$ be sequences in $L^2(\mathcal{G})$ such that the Gabor systems $G(\mathscr{A}, \Lambda, \Lambda^\perp)$ and $G(\mathscr{A}', \Lambda, \Lambda^\perp)$ are Bessel, where Λ is a closed discrete* *subgroup of an LCA group $\mathcal{G}$. Then $G(\mathscr{A}', \Lambda, \Lambda^{\perp})$ is an $\mathcal{S}(\mathscr{A}, \Lambda, \Lambda^{\perp})$-subspace dual to $G(\mathscr{A}, \Lambda, \Lambda^{\perp}))$ if and only if for all $t' \in \mathcal{N}$,*

$$\mathcal{Z}\varphi_{t'}(\beta, x\Lambda) = \sum_{t\in\mathcal{N}} [\mathcal{Z}\varphi_{t'}, \mathcal{Z}\psi_t](\beta)\mathcal{Z}\varphi_t(\beta, x\Lambda) \textit{ for } x\Lambda \in \mathcal{G}/\Lambda \textit{ and a.e. } \beta \in \widehat{\Lambda}.$$

In particular, $G(\{\psi\}, \Lambda, \Lambda^{\perp})$ is an $\mathcal{S}(\{\varphi\}, \Lambda, \Lambda^{\perp})$-subspace dual (orthogonal) to $G(\{\varphi\}, \Lambda, \Lambda^{\perp})$ if and only if $[\mathcal{Z}\varphi, \mathcal{Z}\psi](\beta) = 1$ ($[\mathcal{Z}\varphi, \mathcal{Z}\psi](\beta) = 0$) for a.e. $\beta \in \Omega_{\varphi}$.

Proof. This follows from Theorem 5.1.6 due to the relation $[\mathcal{Z}(E_\omega\varphi_t), \mathcal{Z}(E_\omega\psi_t)](\beta) = [\mathcal{Z}\varphi_t, \mathcal{Z}\psi_t](\beta)$ for a.e. $\beta \in \widehat{\Lambda}$ and $t, t' \in \mathcal{N}$ since the Zak transform satisfies the formula $\mathcal{Z}(E_\omega f)(\beta, x\Lambda) = \omega(x)\mathcal{Z}f(\beta, x\Lambda)$ for $\omega \in \Lambda^{\perp}$, $(\beta, x\Lambda) \in (\widehat{\Lambda}, \mathcal{G}/\Lambda)$ and $f \in L^2(\mathcal{G})$. The remaining part follows by Corollary 5.1.11. □

For an arbitrary closed subgroup Λ and σ-finite measure space $\mathcal{N}$, the following result can be deduced for the set B defined by $B := \{(\beta, x\Lambda) \in \widehat{\Lambda} \times \mathcal{G}/\Lambda : \mathcal{Z}f(\beta, x\Lambda) \neq 0 \text{ for some } f \in \mathscr{A}_0\}$, where for a given $\mathscr{A}$ in $L^2(\mathcal{G})$, the family of functions $\mathscr{A}_0 \subseteq \mathscr{A}$ is a countable dense subset of $\mathscr{A}$ (see [17, 23, 49]). The associated range function $J_{\mathscr{A}}(\beta, x\Lambda) = \mathbb{C}$.

Theorem 5.4.2. *Let $\mathscr{A} = \{\varphi_t\}_{t\in\mathcal{N}}$ and $\mathscr{A}' = \{\psi_t\}_{t\in\mathcal{N}}$ be two collections of functions in $L^2(\mathcal{G})$ such that $G(\mathscr{A}, \Lambda, \Lambda^{\perp})$ and $G(\mathscr{A}', \Lambda, \Lambda^{\perp})$ are Bessel, where $(\mathcal{N}, \mu_{\mathcal{N}})$ is a complete, σ-finite measure space. Then $G(\mathscr{A}', \Lambda, \Lambda^{\perp})$ is an $\mathcal{S}(\mathscr{A}, \Lambda, \Lambda^{\perp})$-subspace orthogonal to $G(\mathscr{A}, \Lambda, \Lambda^{\perp})$ if and only if for a.e. $(\beta, x\Lambda) \in B$, the system $\mathcal{Z}\mathscr{A}'(\beta, x\Lambda)$ is a $J_{\mathscr{A}}(\beta, x\Lambda)$-subspace orthogonal to $\mathcal{Z}\mathscr{A}(\beta, x\Lambda)$.*

5.4.2. Splines on LCA groups

Fix $N \in \mathbb{N}$. For an LCA group $\mathcal{G}$ with the uniform lattice Λ and the associated fundamental domain $\mho$, the *weighted B-spline of order N* is defined by $B_N = \varphi_1\chi_{\mho} * \cdots * \varphi_N\chi_{\mho}$, where $\varphi_i \in L^2(\mho)$ for $1 \leqslant i \leqslant N$. Then the system $\mathcal{E}^{\Lambda}(B_N)$ is Bessel [24]. Similarly, the system $\mathcal{E}^{\Lambda}(B'_N)$ is also Bessel, where $B'_N = \psi_1\chi_{\mho} * \cdots * \psi_N\chi_{\mho}$ for $\psi_i \in L^2(\mho)$ with $1 \leqslant i \leqslant N$. Therefore for a.e. $\xi \in \Omega$ (fundamental domain associated with $\Lambda^{\perp}$ in $\widehat{\mathcal{G}}$), we have the following similar to Example 5.1.13:

$$[\mathscr{T}B_N, \mathscr{T}B'_N](\xi) = \sum_{\lambda\in\Lambda^{\perp}} \widehat{B_N}(\xi+\lambda)\overline{\widehat{B'_N}(\xi+\lambda)} = \sum_{\lambda\in\Lambda^{\perp}} \Big(\prod_{j=1}^{N} \widehat{(\varphi_j\chi_{\mho})}(\xi+\lambda)\overline{\widehat{(\psi_j\chi_{\mho})}(\xi+\lambda)}\Big).$$

By the Corollary 5.1.11, the subspace orthogonal and duals for the system $\mathcal{E}^{\Lambda}(B'_N)$ associated with $\mathcal{E}^{\Lambda}(B_N)$ can be described by assigning the values on the above expression either

0 or 1, respectively. Due to the importance of splines in numerous applications, both the Euclidean and the LCA group setups conduct an in-depth research for the Gabor and Wavelet systems [24, 52].

5.4.3. For other setups

In the scenario of p-adic numbers $\mathbb{Q}_p$ and Heisenberg groups, we may examine our results for the subspace orthogonality and duality of Bessel families. Considering that the Zak transform plays a significant role in describing our results, such as, Theorems 5.1.6, 5.1.8, 5.1.16, 5.2.3, and 5.3.1, Corollary 5.1.11, we describe below the Zak transform in these setups (see [8, 49]) and references therein.

5.4.3.1. *p-adic numbers $\mathbb{Q}_p$.* Recall Subsection 4.2.2, for a prime number p, the locally compact field of p-adic numbers $\mathbb{Q}_p$ is $\{\sum_{j=m}^{\infty} c_j p^j : m \in \mathbb{Z}, c_j \in \{0, 1, \ldots, p-1\}\}$ in which the associated with the p-adic norm is $|x|_p = p^{-m}$ for $x = \sum_{j=m}^{\infty} c_j p^j, c_m \neq 0$. Indeed, it is an LCA group. The p-adic integers $\mathbb{Z}_p$ defined by $\{\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leqslant 1\}\}$ is a compact open subgroup of $\mathbb{Q}_p$. In this setup, the Zak transform is given by $\mathcal{Z}f(x, y) = \int_{\mathbb{Z}_p} f(y + \xi) e^{-2\pi i x \xi} \, d\mu_{\mathbb{Z}_p}(\xi)$ for $f \in L^1(\mathbb{Q}_p) \cap L^2(\mathbb{Q}_p)$ and $x, y \in \Omega$ which can be extended from $L^2(\mathbb{Q}_p)$ to $\ell^2(\Omega \times \Omega)$, where Ω is the fundamental domain.

5.4.3.2. *Semidirect product of LCA groups.* For LCA groups Γ_1 and Γ_2, let us consider a locally compact group $\mathscr{G}_\tau = \Gamma_1 \rtimes_\tau \Gamma_2$ by the semidirect product of Γ_1 and Γ_2 with the binary operation $(\gamma_1, \gamma_2).(\gamma_1', \gamma_2') = (\gamma_1\gamma_1', \gamma_2\tau_{\gamma_1}(\gamma_2))$, where $\gamma_1 \mapsto \tau_{\gamma_1}$ is a group homomorphism from Γ_1 to the set of all automorphisms on Γ_2, such that $(\gamma_1, \gamma_2) \mapsto \tau_{\gamma_1}(\gamma_2)$ from $\Gamma_1 \times \Gamma_2 \mapsto \Gamma_2$ is continuous. Then the Zak transform $\mathcal{Z}$ is defined by $\mathcal{Z}f(\gamma_1, w) = \int_{\Gamma_2} f(\gamma_1, \gamma_2)\overline{w(\gamma_2)}\delta(\gamma_1) \, d\mu_{\Gamma_2}(\gamma_2)$ for $f \in L^1(\Gamma_1 \times_\tau \Gamma_2)$, $(\gamma_1, \omega) \in \Gamma_1 \rtimes_{\hat{\tau}} \widehat{\Gamma_2}$, where δ is a positive homeomorphism on Γ_1 given by $d\mu_{\Gamma_2}(\gamma_2) = \delta(\gamma_1) d\mu_{\Gamma_2}(\tau_{\gamma_1}(\gamma_2))$. It can be extended from $L^2(\Gamma_1 \times_\tau \Gamma_2)$ to $L^2(\Gamma_1 \rtimes_{\hat{\tau}} \widehat{\Gamma_2})$.

Till now, we have discussed the dual frames and their types for the locally compact groups translated by their closed abelian subgroups. Next, we are going to discuss these duals in the case of connected, simply connected nilpotent Lie group which is considered to be a high degree of non-abelian structure. Unlike the previous chapters, our translations are from the non-abelian subgroup. This type of discussion was started by Currey et al. [29].

CHAPTER 6

REPRODUCING FROMULA FOR SI/Z LIE GROUP

In this chapter, we discuss characterization results for reproducing formulas associated with the left translation generated systems in $L^2(G)$, where G is a connected, simply connected nilpotent Lie group whose irreducible unitary representations are square-integrable modulo the center. Unlike the previous study of discrete frames on the nilpotent Lie groups [29], the current research occurs within the setup of continuous frames, which means the resulting reproducing formulas are given in terms of integral representations instead of discrete sums. As a consequence of our results for the Heisenberg group, a reproducing formula associated with the orthonormal Gabor systems of $L^2(\mathbb{R}^d)$ is obtained [65].

6.1. Plancherel transform for SI/Z nilpotent Lie group

Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. We identify G with $\mathfrak{g} \cong \mathbb{R}^n$ due to the analytic diffeomorphism of the exponential map $\exp : \mathfrak{g} \to G$, where $n = \dim \mathfrak{g}$. To choose a basis for the Lie algebra $\mathfrak{g}$, we consider the Jordon-Hölder series $(0) \subseteq \mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \ldots \subseteq \mathfrak{g}_n = \mathfrak{g}$ of ideals of $\mathfrak{g}$ such that $\dim \mathfrak{g}_j = j$ for $j = 0, 1, \ldots, n$ satisfying $\mathrm{ad}(X)\mathfrak{g}_j \subseteq \mathfrak{g}_{j-1}$ for $j = 1, \ldots, n$ and for all $X \in \mathfrak{g}$, where for $X, Y \in \mathfrak{g}$, $\mathrm{ad}(X)(Y) = [X, Y]$, the Lie bracket of X and Y. Now we pick $X_j \in \mathfrak{g}_j \backslash \mathfrak{g}_{j-1}$ for each $j = 1, 2, \ldots, n$ such that the collection $\{X_1, X_2, \ldots, X_n\}$ is a Jordan-Hölder basis. The map $\mathbb{R}^n \longrightarrow \mathfrak{g} \longrightarrow G$ defined by $(x_1, x_2, \ldots, x_n) \mapsto \sum_{j=1}^n x_j X_j \mapsto \exp(\sum_{j=1}^n x_j X_j)$

This chapter is a part of the following manuscripts:

S. Sarkar, N. K. Shukla, *Reproducing formulas associated to translation generated systems on nilpotent Lie groups*, arXiv:2301.03152.

S. Sarkar, N. K. Shukla, *Characterizations of extra-invariant spaces under the left translations on a Lie group*, **Advances in Operator Theory,** (2023), https://doi.org/10.1007/s43036-023-00273-x.

is a diffeomorphism, and hence the Lebesgue measure on $\mathbb{R}^n$ can be realized as a Haar measure on G [28].

Note that the center $\mathfrak{z}$ of the Lie algebra $\mathfrak{g}$ is non-trivial, and it maps to the center $Z := \exp \mathfrak{z}$ of G. The Lie group G acts on $\mathfrak{g}$ and $\mathfrak{g}^*$ by the *adjoint action* $\exp(Ad(x)X) := x\exp(X)x^{-1}$ and *co-adjoint action* $(Ad^*(x)\ell)(X) = \ell(Ad(x^{-1})X)$, respectively, for $x \in G, X \in \mathfrak{g}$, and $\ell \in \mathfrak{g}^*$. The $\mathfrak{g}^*$ denotes the vector space of all real-valued linear functionals on $\mathfrak{g}$. For $\ell \in \mathfrak{g}^*$ the *stabilizer* $R_\ell = \{x \in G : (Ad^*x)\ell = \ell\}$ is a Lie group with the associated Lie algebra $r_\ell := \{X \in \mathfrak{g} : \ell[Y, X] = 0 \text{ for all } Y \in \mathfrak{g}\}$.

Our aim is to discuss Kirilov Theory [28] to define the Plancherel transform for SI/Z group. Given any $\ell \in \mathfrak{g}^*$, there exists a subalgebra $\mathfrak{h}_\ell$ (known as *polarizing* or *maximal subordinate subalgebra*) of $\mathfrak{g}$ which is maximal with respect to the property $\ell[\mathfrak{h}_\ell, \mathfrak{h}_\ell] = 0$. Then the map $\mathcal{X}_\ell : \exp(\mathfrak{h}_\ell) \to \mathbb{T}$ defined by $\mathcal{X}_\ell(\exp X) = e^{2\pi i \ell(X)}$, $X \in \mathfrak{h}_\ell$ is a character on $\exp(\mathfrak{h}_\ell)$, and hence the representations induced from $\mathcal{X}_l$, $\pi_\ell := \mathrm{ind}_{\exp \mathfrak{h}_\ell}^G \mathcal{X}_\ell$, have the following properties:

(i) π_ℓ is an irreducible unitary representation of G.

(ii) Suppose ${\mathfrak{h}'}_\ell$ is another subalgebra which is maximal with respect to the property $\ell[\mathfrak{h}'_\ell, \mathfrak{h}'_\ell] = 0$, then $\mathrm{ind}_{\exp \mathfrak{h}_\ell}^G \mathcal{X}_\ell \cong \mathrm{ind}_{\exp(\mathfrak{h}'_\ell)}^G \mathcal{X}_{\ell'}$.

(iii) $\pi_{\ell_1} \cong \pi_{\ell_2}$ if and only if ℓ_1 and ℓ_2 lie in the same co-adjoint orbit.

(iv) Suppose π is a irreducible unitary representation of G, then there exists $\ell \in \mathfrak{g}^*$ such that $\pi \cong \pi_\ell$.

Therefore there exists a bijection $\mathfrak{k}^* : \mathfrak{g}^*/Ad^*(G) \to \widehat{G}$ which is also a Borel isomorphism, where $\widehat{G}$ is the collection of all irreducible unitary representations of G.

For an irreducible representation $\pi \in \widehat{G}$, let O_π denote coadjoint orbit corresponding to the equivalence class of π. Then the orbital characterization for the SI/Z representation is:

π is square integrable modulo the center if and only if for $\ell \in O_\pi, r_\ell = \mathfrak{z}$ and $O_\pi = \ell + \mathfrak{z}^\perp$.

If $SI/Z \neq \phi$ then SI/Z=$\widehat{G}_{\max}$, where the Borel subset $\widehat{G}_{\max} \subseteq \widehat{G}$ corresponds to coadjoint orbits of maximal dimension which is co-null for Plancherel measure class. Hence, when G is an SI/Z group, $\widehat{G}_{\max}$ is parameterized by a subset of $\mathfrak{z}^*$. If $\pi \in \widehat{G}_{\max}$, then dim $O_\pi = n - \dim \mathfrak{z}$, since O_π is symplectic manifold, it is of even dimension, say, $\dim O_\pi = 2d$.

By Schurs' Lemma, the restriction of π on Z is a character and hence it is a unique element $\sigma = \sigma_\pi \in \mathfrak{z}^*$ (say) and $\pi(z) = e^{2\pi i\langle\sigma,\log z\rangle}I$, where I is the identity operator. It shows that $O_\pi = \{l \in \mathfrak{g}^* : l|_{\mathfrak{z}} = \sigma\}$ and $\pi \mapsto \sigma_\pi$ is injective.

Let G be an SI/Z group and $\mathcal{W} = \{\sigma \in \mathfrak{z}^* : \mathbf{Pf}(\sigma) \neq 0\}$ be a cross section for the coadjoint orbits of maximal dimension, where the *Pfaffian determinant* $\mathbf{Pf} : \mathfrak{z}^* \longrightarrow \mathbb{R}$ is given by

$$\ell \mapsto \sqrt{\left|\det(\ell[X_i, X_j])_{i,j=r\ldots n}\right|}.$$

Then, for a fixed $\sigma \in \mathcal{W}$, $p(\sigma) = \sum_{j=1}^{d} \mathfrak{g}_j(\sigma\big|_{\mathfrak{g}_j})$ is a maximal subordinate subalgebra for σ and the corresponding induced representation π_σ is realized naturally on $L^2(\mathbb{R}^d)$, where $n = r + 2d$ for some d. For each $\varphi \in L^1(G) \cap L^2(G)$, the Fourier transform of φ given by

$$\widehat{\varphi}(\sigma) = \int_G \varphi(x)\pi_\sigma(x)\; dx,\;\; \sigma \in \mathcal{W},$$

defines a Hilbert-Schmidt operator on $L^2(\mathbb{R}^d)$ with the inner product $\langle A, B\rangle_{\mathcal{H}S} = \mathrm{tr}(B^*A)$. This space is denoted by $\mathcal{H}S(L^2(\mathbb{R}^d))$. When $d\sigma$ is suitable normalized, then

$$\|\varphi\|^2 = \int_{\mathcal{W}} \|\widehat{\varphi}(\sigma)\|^2_{\mathcal{H}S(L^2(\mathbb{R}^d))} |\mathbf{Pf}(\sigma)|\; d(\sigma).$$

The Fourier transform can be extended unitarily as $\mathcal{F}$- the *Plancherel transform*,

$$\mathcal{F} : L^2(G) \to L^2(\mathfrak{z}^*, \mathcal{H}S(L^2(\mathbb{R}^d)), |\mathbf{Pf}(\sigma)|d\sigma)),\;\; \mathcal{F}f = \widehat{f}.$$

Note that the Plancherel transform $\mathcal{F}$ satisfies the relation

$$\mathcal{F}(L_\lambda f)(\sigma) = \pi_\sigma(\lambda)\mathcal{F}f(\sigma) \text{ for } \lambda \in G, \text{a. e. } \sigma \in \mathfrak{z}^*, \text{ and } f \in L^2(G),$$

where the left translation operator L_λ on $L^2(G)$ is given by $L_\lambda f(x) = f(\lambda^{-1}x)$.

6.1.1. Plancherel transformation followed by a periodization

Throughout the next, let us assume that G be an SI/Z nilpotent Lie group with center Z. From the Section 6.1, we consider the center Z identified with $\mathbb{R}^r$ $(r < n)$ for a chosen (ordered) basis $\{X_1, X_2, \ldots, X_n\}$ of the corresponding Lie algebra $\mathfrak{g}$, as follows:

$$Z = \exp \mathbb{R}X_1 \exp \mathbb{R}X_2 \ldots \exp \mathbb{R}X_r.$$

Also we write a set $\mathcal{X}$ identified with $\mathbb{R}^{2d}$ $(n = r + 2d)$ as follows:

$$\mathcal{X} = \exp \mathbb{R}X_{r+1} \exp \mathbb{R}X_{r+2} \ldots \exp \mathbb{R}X_n.$$

The elements $y = (y_1, y_2, \ldots, y_r) \in \mathbb{R}^r$ and $x = (x_{r+1}, \ldots, x_n) \in \mathbb{R}^{2d}$ are identified by

$$y = \exp y_1 X_1 \exp y_2 X_2 \ldots \exp y_r X_r, \text{ and } x = \exp x_{r+1} X_{r+1} \exp x_{r+2} X_{r+2} \ldots \exp x_n X_n,$$

which can be observed from the homeomorphism between $\mathbb{R}^r \times \mathbb{R}^{2d}$ and G given by

$$(y_1, y_2, \ldots, y_r, x_{r+1}, \ldots, x_n) \mapsto \exp y_1 X_1 \exp y_2 X_2 \ldots \\ \ldots \exp y_r X_r \exp x_{r+1} X_{r+1} \exp x_{r+2} X_{r+2} \ldots \exp x_n X_n.$$

Further assume that Λ_0 is a uniform lattice in Z, means, it is a discrete closed subgroup of Z such that Z/Λ_0 is compact. Then we have $\widehat{Z/\Lambda_0} \cong {\Lambda_0}^{\perp}$ and $\widehat{Z}/{\Lambda_0}^{\perp} \cong \widehat{\Lambda_0}$ since the center Z becomes a locally compact abelian group. The dual group of Z, denote by $\widehat{Z}$, is also identical with $\mathbb{R}^r$. The group $\widehat{Z}$ consists of continuous homomorphisms from Z to $\mathbb{T}$, and the *annihilator* ${\Lambda_0}^{\perp}$ is defined by

$${\Lambda_0}^{\perp} = \{\lambda^* \in \widehat{Z} : \lambda^*(\lambda) = 1 \text{ for all } \ \lambda \in \Lambda_0\}.$$

The set $\widehat{Z}$ can be tiled by Σ with the tiling partner ${\Lambda_0}^{\perp}$, where Σ is *measurable section* of $\widehat{Z}/{\Lambda_0}^{\perp}$ having finite measure. The set Σ is a tiling set of $\widehat{Z}$, means, the collection $\{\Sigma + \lambda^* : \lambda^* \in {\Lambda_0}^{\perp}\}$ is a measurable partition of $\widehat{Z}$.

Definition 6.1.1. For a measure space (X, μ), a countable set $\{\Omega_j\}_j$ of subsets of X is *tiling* of X if $\mu(X \backslash \bigcup_j \Omega_j) = 0$, and $\mu(\Omega_i \cap \Omega_j) = 0$, when $i \neq j$. A set T is *tiling partner* of Ω for X if there is a set Ω in X such that the collection $\{\Omega + x : x \in T\}$ is a tiling of X.

In this section, we discuss the Plancherel transform followed by a periodization named $\mathscr{F}$, which is an operator-valued linear isometry (similar to the Fiberization). It is well known but for the sake of completion, we provide its proof with the approach of the composition of unitary maps. The map $\mathscr{F}$ intertwines left translation with a representation $\tilde{\pi}$. For $g \in G$ and $h \in L^2(\Sigma, \ell^2(\Lambda_0^{\perp}, \mathcal{HS}(L^2(\mathbb{R}^d)))$, the representation $\tilde{\pi}$ is given by

$$\tilde{\pi}(g)h(\sigma) = \tilde{\pi}_\sigma(g)h(\sigma) \ \text{ a.e. } \ \sigma \in \Sigma.$$

The associated representation $\tilde{\pi}_\sigma(g)$ on $\ell^2({\Lambda_0}^{\perp}, \mathcal{HS}(L^2(\mathbb{R}^d))$ is given by

$$\tilde{\pi}_\sigma(g)z(\lambda^*) = \pi_{\sigma+\lambda^*}(g) \circ z(\lambda^*), \ \ \lambda^* \in {\Lambda_0}^{\perp},$$

where the sequence $(z(\lambda^*))$ lies in $\ell^2({\Lambda_0}^{\perp}, \mathcal{HS}(L^2(\mathbb{R}^d))$, "$\circ$" denotes the composition of operators in $\mathcal{HS}(L^2(\mathbb{R}^d))$ and $\pi_{\sigma+\lambda^*}(g)$ is the Hilbert-Schmidt operator defined on $L^2(\mathbb{R}^d)$.

The following Proposition was developed in [29] but for the sake of simplicity we write a simplified proof.

Proposition 6.1.2. *(i) There is a unitary map* $\mathscr{F} : L^2(G) \to L^2(\Sigma, \ell^2({\Lambda_0}^{\perp}, \mathcal{H}S(L^2(\mathbb{R}^d)))$ *given by*

$$\mathscr{F} f(\sigma)(\lambda^*) = \mathcal{F} f(\sigma + \lambda^*)|\mathbf{Pf}(\sigma + \lambda^*)|^{1/2}, \ f \in L^2(G), \lambda^* \in {\Lambda_0}^{\perp} \text{ and a.e. } \sigma \in \Sigma.$$

(ii) The map $\mathscr{F}$ *satisfies the intertwining property of left translation with the representation* $\tilde{\pi}$*. For* $\lambda \in \Lambda = \Lambda_1\Lambda_0 = \{\lambda_1\lambda_0 : \lambda_1 \in \Lambda_1, \lambda_0 \in \Lambda_0\}$*, where*

$$\Lambda_1 \subseteq \exp \mathbb{R}X_{r+1} \ldots \exp \mathbb{R}X_n \text{ and } \Lambda_0 = \exp \mathbb{Z}X_1 \ldots \exp \mathbb{Z}X_r$$

is the integer lattice in G*,*

$$\mathscr{F}(L_\lambda f)(\sigma) = e^{2\pi i\langle\sigma,\lambda_0\rangle}\tilde{\pi}_\sigma(\lambda_1)\mathscr{F} f(\sigma). \tag{6.1.1}$$

Proof. (i) Since the set $\mathcal{W}$ is Zariski open in $\mathfrak{z}^*$ and $\mathbf{Pf}(\sigma)$ is non-vanishing on $\mathcal{W}$, the map

$$\mathscr{U}_2 : L^2(\mathcal{W}, \mathcal{H}S(L^2(\mathbb{R}^d)), |\mathbf{Pf}(\sigma)|\, d\sigma) \to L^2(\mathfrak{z}^*, \mathcal{H}S(L^2(\mathbb{R}^d))), \ h \mapsto h|\mathbf{Pf}(\sigma)|^{1/2}$$

is unitary. Further note that ${\Lambda_0}^{\perp}$ is tiling partner of Σ for $\widehat{Z} \cong \mathfrak{z}^*$, we can define a periodization map

$$\mathscr{U}_3 : L^2(\mathfrak{z}^*, \mathcal{H}S(L^2(\mathbb{R}^d))) \to L^2\left(\Sigma, \ell^2({\Lambda_0}^{\perp}, \mathcal{H}S(L^2(\mathbb{R}^d)))\right), \ h \mapsto (h(\cdot + \lambda^*))_{\lambda^* \in {\Lambda_0}^{\perp}}.$$

It is also unitary by identifying the linear dual $\mathfrak{z}^*$ of $\mathfrak{z}$ with $\mathbb{R}^r$. Therefore we get a sequence of unitary maps as follows:

$$L^2(G) \xrightarrow{\mathscr{U}_1} L^2\left(\mathcal{W}, \mathcal{H}S(L^2(\mathbb{R}^d)), |\mathbf{Pf}(w)|dw\right) \xrightarrow{\mathscr{U}_2} L^2(\mathfrak{z}^*, \mathcal{H}S(L^2(\mathbb{R}^d))) \xrightarrow{\mathscr{U}_3} L^2\left(\Sigma, \ell^2({\Lambda_0}^{\perp}, \mathcal{H}S(L^2(\mathbb{R}^d)))\right),$$

where the first unitary map $\mathscr{U}_1$ is the usual Plancherel transform $\mathcal{F}$.

For $h \in L^2\left(\mathcal{W}, \mathcal{H}S(L^2(\mathbb{R}^d)), |\mathbf{Pf}(\sigma)|d\sigma\right)$, we observe

$$(\mathscr{U}_2\mathscr{U}_1)h(\sigma) = (\mathscr{U}_1 h)(\sigma)|\mathbf{Pf}(\sigma)|^{1/2} = \mathcal{F}h(\sigma)|\mathbf{Pf}(\sigma)|^{1/2} \text{ a.e. } \sigma \in \mathcal{W}$$

and then for a.e. $\sigma \in \Sigma$, $\lambda^* \in {\Lambda_0}^{\perp}$ and $f \in L^2(G)$, we have

$$(\mathscr{U}_3\mathscr{U}_2\mathscr{U}_1)f(\sigma)(\lambda^*) = (\mathscr{U}_3(\mathscr{U}_2\mathscr{U}_1)f(\sigma))(\lambda^*) = (\mathscr{U}_2\mathscr{U}_1)f(\sigma+\lambda^*) = \mathcal{F}f(\sigma+\lambda^*)|\mathbf{Pf}(\sigma+\lambda^*)|^{1/2}.$$

Thus the result follows by choosing $\mathscr{F} = \mathscr{U}_3\mathscr{U}_2\mathscr{U}_1$.

(ii) For $\lambda^* \in {\Lambda_0}^{\perp}$ and a.e. $\sigma \in \Sigma$, we get

$$\begin{aligned}\mathscr{F}(L_{\lambda_1\lambda_0}f)(\sigma)(\lambda^*) =& \mathcal{F}(L_{\lambda_1\lambda_0}f)(\sigma+\lambda^*)|\mathbf{Pf}(\sigma+\lambda^*)|^{\frac{1}{2}} = \pi_{\sigma+\lambda^*}(\lambda_1\lambda_0)\mathcal{F}f(\sigma+\lambda^*)|\mathbf{Pf}(\sigma+\lambda^*)|^{\frac{1}{2}} \\ =& e^{2\pi i\langle\sigma,\lambda_0\rangle}\pi_{\sigma+\lambda^*}(\lambda)\mathcal{F}f(\sigma+\lambda^*)|\mathbf{Pf}(\sigma+\lambda^*)|^{\frac{1}{2}} \\ =& e^{2\pi i\langle\sigma,\lambda_0\rangle}\left(\tilde{\pi}_\sigma(\lambda_1)\mathscr{F}f(\sigma)\right)(\lambda^*),\end{aligned}$$

since $\mathcal{F}(L_\lambda f)(\sigma) = \pi_\sigma(\lambda)\mathcal{F}f(\sigma)$. □

Example 6.1.3 (Heisenberg group)**.** Let G be a d-dimensional Heisenberg group denoted by $\mathbb{H}^d$ with a Lie algebra $\mathfrak{g}$ having a basis $\{X_1, X_2, \ldots, X_d, Y_1, Y_2, \ldots, Y_d, W\}$. Its associated Lie brackets are:

(i) $[X_i, X_j] = [Y_i, Y_j] = 0, 1 \leqslant i, j \leqslant d$;

(ii) $[W, X_i] = [W, Y_i] = 0, 1 \leqslant i \leqslant d$;

(iii) $[X_i, Y_j] = \delta_{i,j}W, 1 \leqslant i, j \leqslant d$.

So the center of the Heigenberg group becomes: $Z = \exp \mathbb{R}W$ and non-center part $\mathcal{X} = \exp \mathbb{R}X_1 \ldots \exp \mathbb{R}X_d \exp \mathbb{R}Y_1 \ldots \exp \mathbb{R}Y_d$. $\mathbb{H}^d$ is the d-dimensional Heisenberg group identified with $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ and group operation:

$$(x, y, w) \cdot (x', y', w') = (x + x', y + y', w + w' + xy).$$

It is an SI/Z group, and when $\mathfrak{z}^*$ is identified with $\mathbb{R}$, then $\mathcal{W} = \mathbb{R}^* = \mathbb{R}\backslash\{0\}$, $|\mathbf{Pf}(\lambda)| = |\lambda|^d$. So its irreducible unitary representations π of $\mathbb{H}^d$ are indexed by $\lambda \in \mathbb{R}^*$, upto a set of measure-zero. For $\sigma \in \mathbb{R}^* = \mathbb{R}\backslash\{0\}$ and $u = (x, y, z) \in \mathbb{H}^d$, the Schrödinger representations $\pi_\sigma(u)$ on $L^2(\mathbb{R}^d)$ is given below for $f \in L^2(\mathbb{R}^d)$,

$$\pi_\sigma(u)f(x') = \pi_\sigma(x, y, z)f(x') = e^{2\pi i\sigma z}e^{-2\pi i\sigma y.x'}f(x' - x),\ x, y, x' \in \mathbb{R}^d \text{ and } z \in \mathbb{R}.$$

For $\varphi \in L^1(\mathbb{H}^d) \cap L^2(\mathbb{H}^d)$, the Fourier transform is defined by:

$$\mathcal{F}\varphi(\sigma) = \int_{\mathbb{H}^d} \varphi(x)\pi_\sigma(x)\,dx,\ \sigma \in \mathbb{R}^*,$$

and the fiberization map $\mathscr{F} : L^2(\mathbb{H}^d) \to L^2(\mathbb{T}; \ell^2(\mathbb{Z}, \mathcal{HS}(L^2(\mathbb{R}^d))))$ is given by

$$\mathscr{F}(\varphi)(\alpha)(m) = |\alpha + m|^{\frac{d}{2}}\mathcal{F}\varphi(\alpha + m).$$

Consider $\Lambda_1 = \mathbb{R}^d \times \mathbb{R}^d \times \{0\}$ and $\Lambda_0 = \{(0, 0)\} \times \mathbb{Z}$, then the set $\Lambda = \Lambda_1\Lambda_0$ can be identified with $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{Z}$. The associated representation $\tilde{\pi}_\alpha$, which is defined by

$$\tilde{\pi}_\alpha(\lambda_1)z(m) := \pi_{\alpha+m}(\lambda_1)\circ z(m), \text{where } (z(m)) \in \ell^2(\mathbb{Z}, \mathcal{HS}(L^2(\mathbb{R}^d))).$$

6.2. Translation-invariant spaces

We briefly start by describing the left translation generated systems in $L^2(G)$ as follows:

Definition 6.2.1. Let Λ_0 be a uniform lattice in the center Z of G and Λ_1 be a discrete set lying outside the center Z. A closed subspace W of $L^2(G)$ is said to be $\Lambda_1\Lambda_0$-*invariant* if

$$L_{\lambda_1\lambda_0} f \in W \text{ for all } \lambda_1 \in \Lambda_1, \lambda_0 \in \Lambda_0 \text{ and } f \in W,$$

where for each $y \in G$, $L_y f(x) = f(y^{-1}x)$ for $x \in G$ and $f \in L^2(G)$.

Next we discuss the range function J for a $\Lambda_1\Lambda_0$-invariant space. The *range function* is a mapping

$$J : \Sigma \to \left\{\text{closed subspaces of } \ell^2({\Lambda_0}^{\perp}, \mathcal{HS}(L^2(\mathbb{R}^d)))\right\}.$$

It is measurable if the projection map $P(\sigma) : L^2(G) \to J(\sigma)$ is weakly measurable, i.e., for each $a, b \in \ell^2({\Lambda_0}^{\perp}, \mathcal{HS}(L^2(\mathbb{R}^d))$ and $\sigma \mapsto \langle P(\sigma)a, b\rangle$ is measurable.

The space W can be expressed as follows: $W = \{\varphi \in L^2(G) : \mathscr{F}\varphi(\sigma) \in J(\sigma)$ for a.e. $\sigma \in \Sigma\}$ and $\tilde{\pi}_\sigma(\Lambda_1) \subset J(\sigma)$. Also, there is bijection $W \mapsto J$. We refer [3,12,20,24,29,61,62] for more details about shift-invariant spaces and associated range functions for the abelian and non-abelian setups.

Proposition 6.2.2. *The range function J associated with the $\Lambda_1\Lambda_0$-invariant space $W = \mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A})$ satisfies*

$$J(\sigma) = \overline{\operatorname{span}}\{\mathscr{F}(L_{\lambda_1}\varphi)(\sigma) : \varphi \in \mathscr{A}, \lambda_1 \in \Lambda_1\} \ \textit{a.e. } \sigma \in \Sigma. \tag{6.2.1}$$

Proof. From the intertwining property (6.1.1), we get $\mathscr{F}(L_{\lambda_1\lambda_0}\varphi)(\sigma) = e^{2\pi i\langle\sigma,\lambda_1\rangle}\tilde{\pi}(\lambda_1)\mathscr{F}\varphi(\sigma)$ for $\lambda_1\lambda_0 \in \Lambda_1\Lambda_0$, and a.e. $\sigma \in \Sigma$, and hence, $\mathscr{F}(\mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A}))$ is invariant under exponential and $\tilde{\pi}(\Lambda_1)\mathscr{F}(\mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A})) \subset \mathscr{F}(\mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A}))$. Therefore, we get the result by observing $\mathscr{F}(\mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A})) = M_J$, where the space M_J is defined by

$$M_J = \{f \in L^2(\Sigma, \ell^2({\Lambda_0}^{\perp}, \mathcal{HS}(L^2(\mathbb{R}^d)) : f(\sigma) \in J(\sigma) \text{ for a.e. } \sigma \in \Sigma\} \tag{6.2.2}$$

for the range function J given in (6.2.1). For this let us consider a function $g \in \mathscr{F}(\mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A}))$. Choose a sequence (g_i) converging to g such that $\mathscr{F}^{-1}g_i \in \operatorname{span}\{L_{\lambda_1\lambda_0}\varphi : \lambda_1\lambda_0 \in \Lambda_1\Lambda_0, \varphi \in \mathscr{A}\}$. Then we have $g_i(\sigma) \in J(\sigma)$ in view of (6.1.1), and hence $g(\sigma) \in J(\sigma)$ since $J(\sigma)$ is closed. Therefore, $g \in M_J$, i.e., $\mathscr{F}(\mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A})) \subset M_J$. For the equality $\mathscr{F}(\mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A})) = M_J$, we need to show $\mathscr{F}(\mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A}))^{\perp} \cap M_J = 0$. Choose $h \in \mathscr{F}(S^{\Lambda_1\Lambda_0}(\mathscr{A}))^{\perp} \cap M_J$. Then for any $f \in \operatorname{span}\{\mathscr{F}(L_{\lambda_1}\varphi) : \lambda_1 \in \Lambda_1, \varphi \in \mathscr{A}\}$ and $\lambda_0 \in \Lambda_0$, we have $e^{2\pi i\langle \cdot,\lambda_0\rangle} f(\cdot) \in \mathscr{F}(\mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A}))$, and then we obtain

$$0 = \int_{\Sigma} \langle e^{2\pi i\langle \cdot,\lambda_0\rangle} f(\sigma), h(\sigma)\rangle \, d\sigma = \int_{\Sigma} e^{2\pi i\langle \cdot,\lambda_0\rangle} \langle f(\sigma), h(\sigma)\rangle \, d\sigma.$$

Hence, all the Fourier coefficients of a scalar function given by $\sigma \mapsto \langle f(\sigma), h(\sigma)\rangle$ are zero. Thus, $\langle f(\sigma), h(\sigma)\rangle = 0$ a.e. $\sigma \in \Sigma$ and $f(\sigma) \in J(\sigma)$, i.e., $h(\sigma) \in J(\sigma)^{\perp}$ a.e. $\sigma \in \Sigma$. □

6.3. Reproducing formulas associated with continuous frames

Through out the section, we assume G to be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. The Haar measure on G can be realised as a Lebesgue measure on $\mathbb{R}^d$. Further, assume an arbitrary measurable subset

$$\Lambda_1 \subseteq \exp \mathbb{R}X_{r+1} \ldots \exp \mathbb{R}X_n \text{ (not necessarily discrete)}$$

and the integer lattice $\Lambda_0 = \exp \mathbb{Z}X_1 \ldots \exp \mathbb{Z}X_r$ in G. For the countable collection of functions $\mathscr{A} = \{\varphi_k : k \in I\}$ and $\mathscr{A}' = \{\psi_k : k \in I\}$ in $L^2(G)$, we recall the Λ-translation generated systems

$$\mathcal{E}^{\Lambda}(\mathscr{A}) = \{L_{\lambda}\varphi_k : \lambda \in \Lambda, k \in I\} \text{ and } \mathcal{E}^{\Lambda}(\mathscr{A}') = \{L_{\lambda}\psi_k : \lambda \in \Lambda, k \in I\},$$

where $\Lambda = \Lambda_1\Lambda_0 = \{\lambda_1\lambda_0 : \lambda_1 \in \Lambda_1, \lambda_0 \in \Lambda_0\}$. The set $\Lambda = \Lambda_1\Lambda_0$ is measurable.

We now define the translation generated dual and its types in $L^2(G)$:

Definition 6.3.1. Suppose $\mathscr{A} = \{\varphi_k\}_{k\in I}, \mathscr{A}' = \{\psi_k\}_{k\in I}$ are families of functions in $L^2(G)$ such that $\mathcal{E}^{\Lambda}(\mathscr{A})$ is a continuous frame for the span closure $\mathcal{S}^{\Lambda}(\mathscr{A})$, and $\mathcal{E}^{\Lambda}(\mathscr{A}')$ is Bessel. We recall the definitions of $\mathcal{E}^{\Lambda}(\mathscr{A}')$ to be an *alternate dual*, *oblique dual*, *type-I dual*, *type-II dual*, and *dual frame* for $\mathcal{E}^{\Lambda}(\mathscr{A})$ from Definition 4.1.1

First we proceed by defining the terms $G^k_{\lambda_1}(\alpha)$ and $H^k_{\lambda_1}(\alpha)$ for each $\lambda_1 \in \Lambda_1, k \in I$ and a.e. $\alpha \in \mathbb{T}^r$ as follows. For $f, g \in L^2(G)$,

(6.3.1) $$G^k_{\lambda_1}(\alpha) := \langle \mathscr{F}f(\alpha), \tilde{\pi}_{\alpha}(\lambda_1)\mathscr{F}\varphi_k(\alpha)\rangle \text{ and } H^k_{\lambda_1}(\alpha) := \langle \mathscr{F}g(\alpha), \tilde{\pi}_{\alpha}(\lambda_1)\mathscr{F}\psi_k(\alpha)\rangle.$$

Proposition 6.3.2. *For each $k \in I$ and $\lambda_1 \in \Lambda_1$, the functions $G^k_{\lambda_1}$ and $H^k_{\lambda_1}$ are in $L^1(\mathbb{T}^r)$ and their Fourier transforms $\widehat{G^k_{\lambda_1}}$ and $\widehat{H^k_{\lambda_1}}$ are the members of $\ell^2(\mathbb{Z}^r)$.*

Proof. Applying the Cauchy-Schwarz inequality and using the property of $\mathscr{F}$, we have

$$\int_{\mathbb{T}^r} |G^k_{\lambda_1}(\alpha)|\, d\alpha \leqslant \left(\int_{\mathbb{T}^r} \sum_{m\in\mathbb{Z}^r} \|\mathscr{F}f(\alpha)(m)\|^2\, d\alpha\right)^{1/2} \left(\int_{\mathbb{T}^r} \sum_{m\in\mathbb{Z}^r} \|\mathscr{F}L_{\lambda_1}\varphi_k(\alpha)(m)\|^2\, d\alpha\right)^{1/2}$$
$$= \|\mathscr{F}f\|\|\mathscr{F}L_{\lambda_1}\varphi_k\| = \|f\|\|\varphi_k\| < \infty,$$

since $\mathscr{F}(L_{\lambda_1}\varphi_k)(\alpha) = e^{2\pi i\langle\alpha,0\rangle_{\mathbb{T}^r}}\tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\varphi(\alpha) = \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\varphi(\alpha)$ and the left translation L_{λ_1} is an isometry. Hence $G^k_{\lambda_1} \in L^1(\mathbb{T}^r)$. Similarly, $H^k_{\lambda_1} \in L^1(\mathbb{T}^r)$. The Fourier transform of $G^k_{\lambda_1}$ and $H^k_{\lambda_1}$ at $\lambda_0 \in \Lambda_0$ is given by

$$\widehat{G^k_{\lambda_1}}(\lambda_0) = \int_{\mathbb{T}^r} G^k_{\lambda_1}(\alpha) e^{-2\pi i\langle\alpha,\lambda_0\rangle}\, d\alpha, \text{ and } \widehat{H^k_{\lambda_1}}(\lambda_0) = \int_{\mathbb{T}^r} H^k_{\lambda_1}(\alpha) e^{-2\pi i\langle\alpha,\lambda_0\rangle}\, d\alpha.$$

Then the sequence $\{\widehat{G^k_{\lambda_1}}(\lambda_0)\}_{\lambda_0\in\Lambda_0} \in \ell^2(\mathbb{Z}^r)$, follows by observing the Bessel property of $\mathcal{E}^\Lambda(\mathscr{A})$ and properties of $\mathscr{F}$ (Proposition 6.1.2) in the following calculations:

$$\infty > \sum_{k\in I}\int_\Lambda |\langle f, L_\lambda\varphi_k\rangle|^2\, d\lambda = \sum_{k\in I}\int_{\Lambda_1}\sum_{\lambda_0\in\Lambda_0}\left|\int_{\mathbb{T}^r}\langle \mathscr{F}f(\alpha), \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\varphi_k(\alpha)\rangle e^{-2\pi i\langle\alpha,\lambda_0\rangle}\, d\alpha\right|^2 d\lambda_1$$
$$= \sum_{k\in I}\int_{\Lambda_1}\sum_{\lambda_0\in\Lambda_0}|\widehat{G^k_{\lambda_1}}(\lambda_0)|^2\, d\lambda_1.$$

Similarly, we have $\{\widehat{H^k_{\lambda_1}}(\lambda_0)\}_{\lambda_0\in\Lambda_0} \in \ell^2(\mathbb{Z}^r)$. □

Proposition 6.3.3. *For all $f, g \in L^2(G)$, we have*

$$\sum_{k\in I}\int_\Lambda \langle f, L_\lambda\varphi_k\rangle\langle L_\lambda\psi_k, g\rangle\, d\lambda = \sum_{k\in I}\int_{\Lambda_1}\int_{\mathbb{T}^r} G^k_{\lambda_1}(\alpha)\overline{H^k_{\lambda_1}(\alpha)}\, d\alpha\, d\lambda_1,$$

where $\mathcal{E}^\Lambda(\mathscr{A})$ and $\mathcal{E}^\Lambda(\mathscr{A}')$ are Bessel systems.

Proof. Applying the map $\mathscr{F}$, we have

$$\sum_{k\in I}\int_\Lambda \langle f, L_\lambda\varphi_k\rangle\langle L_\lambda\psi_k, g\rangle\, d\lambda = \sum_{k\in I}\int_\Lambda \langle \mathscr{F}f, \mathscr{F}L_\lambda\varphi_k\rangle\langle \mathscr{F}L_\lambda\psi_k, \mathscr{F}g\rangle\, d\lambda$$
$$= \sum_{k\in I}\int_\Lambda \left(\int_{\mathbb{T}^r}\langle \mathscr{F}f(\alpha), \mathscr{F}L_\lambda\varphi_k(\alpha)\rangle\, d\alpha\right)\left(\int_{\mathbb{T}^r}\langle \mathscr{F}L_\lambda\psi_k(\alpha), \mathscr{F}g(\alpha)\rangle\, d\alpha\right) d\lambda$$
$$= \sum_{k\in I}\int_\Lambda\int_{\mathbb{T}^r}\langle \mathscr{F}f(\alpha), \tilde{\pi}_\alpha(\lambda)\mathscr{F}\varphi_k(\alpha)\rangle\, d\alpha\int_{\mathbb{T}^r}\langle \tilde{\pi}_\alpha(\lambda)\mathscr{F}\psi_k(\alpha), \mathscr{F}g(\alpha)\rangle\, d\alpha\, d\lambda.$$

Writing $\lambda = \lambda_1\lambda_0$, where $\lambda_1 \in \Lambda_1, \lambda_0 \in \Lambda_0$, we get $\tilde{\pi}_\alpha(\lambda_1\lambda_0) = e^{2\pi i\langle\alpha,\lambda_0\rangle}\tilde{\pi}_\alpha(\lambda_1)$, and hence the above expression can be written as:

$$\begin{aligned}
&\sum_{k\in I}\int_{\Lambda}\langle f, L_\lambda\varphi_k\rangle\langle L_\lambda\psi_k, g\rangle\, d\lambda \\
&\quad= \sum_{k\in I}\int_{\Lambda_1}\sum_{\lambda_0\in\Lambda_0}\left(\int_{\mathbb{T}^r}\langle \mathscr{F}f(\alpha), \tilde{\pi}_\alpha(\lambda_1\lambda_0)\mathscr{F}\varphi_k(\alpha)\rangle\, d\alpha\right)\times \\
&\qquad\qquad\qquad\left(\int_{\mathbb{T}^r}\langle \tilde{\pi}_\alpha(\lambda_1\lambda_0)\mathscr{F}\psi_k(\alpha), \mathscr{F}g(\alpha)\rangle\, d\alpha\right) d\lambda_1 \\
&\quad= \sum_{k\in I}\int_{\Lambda_1}\sum_{\lambda_0\in\Lambda_0}\left(\int_{\mathbb{T}^r} e^{-2\pi i\langle\alpha,\lambda_0\rangle}\langle \mathscr{F}f(\alpha), \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\varphi_k(\alpha)\rangle\, d\alpha\right)\times \\
&\qquad\qquad\qquad\left(\int_{\mathbb{T}^r} e^{2\pi i\langle\alpha,\lambda_0\rangle}\langle \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\psi_k(\alpha), \mathscr{F}g(\alpha)\rangle\, d\alpha\right) d\lambda_1 \\
&\quad= \sum_{k\in I}\int_{\Lambda_1}\sum_{\lambda_0\in\Lambda_0}\left(\int_{\mathbb{T}^r} e^{-2\pi i\langle\alpha,\lambda_0\rangle}G^k_{\lambda_1}(\alpha)\, d\alpha\right)\left(\int_{\mathbb{T}^r} e^{2\pi i\langle\alpha,\lambda_0\rangle}\overline{H^k_{\lambda_1}(\alpha)}\, d\alpha\right) d\lambda_1 \\
&\quad= \sum_{k\in I}\int_{\Lambda_1}\sum_{\lambda_0\in\Lambda_0}\widehat{G^k_{\lambda_1}}(\lambda_0)\overline{\widehat{H^k_{\lambda_1}}(\lambda_0)}\, d\lambda_1 = \sum_{k\in I}\int_{\Lambda_1}\langle\widehat{G^k_{\lambda_1}}, \widehat{H^k_{\lambda_1}}\rangle_{\ell^2(\mathbb{Z}^r)}\, d\lambda_1 \\
&\quad= \sum_{k\in I}\int_{\Lambda_1}\langle G^k_{\lambda_1}, H^k_{\lambda_1}\rangle\, d\lambda_1 = \sum_{k\in I}\int_{\Lambda_1}\int_{\mathbb{T}^r} G^k_{\lambda_1}(\alpha)\overline{H^k_{\lambda_1}(\alpha)}\, d\alpha\, d\lambda_1.
\end{aligned}$$

Hence the result follows. □

Now we state our main results of this section which characterizes alternate (oblique) duals and type-I (type-II) duals in the nilpotent Lie group setup. Our characterizations are based on the range function techniques for SI/Z Lie group associated with the representation $\tilde{\pi}_\alpha$.

Theorem 6.3.4. *For a.e. $\alpha \in \mathbb{T}^r$, we consider the range function*

$$J_{\mathscr{A}}(\alpha) = \overline{\operatorname{span}}\{\tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\varphi(\alpha) : \varphi \in \mathscr{A}, \lambda_1 \in \Lambda_1\} \subseteq \ell^2(\mathbb{Z}^r, \mathcal{H}S(L^2(\mathbb{R}^d))), \tag{6.3.2}$$

associated with the representation $\tilde{\pi}_\alpha$. Then $\mathcal{E}^\Lambda(\mathscr{A}')$ is an alternate (oblique) dual for $\mathcal{E}^\Lambda(\mathscr{A})$ if and only if for a.e. $\alpha \in \mathbb{T}^r$, the system $\{\tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\psi(\alpha) : \psi \in \mathscr{A}', \lambda \in \Lambda_1\}$ is an alternate (oblique) dual for the continuous frame $\{\tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\varphi(\alpha) : \varphi \in \mathscr{A}, \lambda_1 \in \Lambda_1\}$ of $J_{\mathscr{A}}(\alpha)$, i.e., for all $h \in J_{\mathscr{A}}(\alpha)$,

$$h = \sum_{k\in I}\int_{\Lambda_1}\langle h, \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\psi_k(\alpha)\rangle\tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\varphi_k(\alpha)\, d\lambda_1.$$

Proof. We prove the result for alternate duals and proceeding by a similar manner we can conclude for oblique duals. For $f, g \in \mathcal{S}^{\Lambda}(\mathscr{A})$, we have

$$\sum_{k\in I}\int_{\Lambda}\langle f, L_\lambda\varphi_k\rangle\,\langle L_\lambda\psi_k, g\rangle\,d\lambda = \sum_{k\in I}\int_{\Lambda_1}\int_{\mathbb{T}^r} G^k_{\lambda_1}(\alpha)\overline{H^k_{\lambda_1}(\alpha)}\,d\alpha\,d\lambda_1$$

$$= \sum_{k\in I}\int_{\Lambda_1}\int_{\mathbb{T}^r}\langle \mathscr{F}f(\alpha), \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\varphi_k(\alpha)\rangle\,\langle \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\psi_k(\alpha), \mathscr{F}g(\alpha)\rangle\,d\alpha\,d\lambda_1 \tag{6.3.3}$$

due to the Proposition 6.3.3. Assume the system $\{\tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\psi_k(\alpha) : k \in I, \lambda_1 \in \Lambda_1\}$ is an alternate dual for the continuous frame $\{\tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\varphi_k(\alpha) : k \in I, \lambda_1 \in \Lambda_1\}$ for a.e. $\alpha \in \mathbb{T}^r$, i.e.,

$$\sum_{k\in I}\int_{\Lambda_1}\langle a, \tilde{\pi}_{\lambda_1}(\alpha)\mathscr{F}\varphi_k(\alpha)\rangle\,\langle \tilde{\pi}_{\lambda_1}(\alpha)\mathscr{F}\psi_k(\alpha), b\rangle\,d\lambda_1 = \langle a, b\rangle \text{ for all } a, b \in J_{\mathscr{A}}(\alpha).$$

Employing (6.3.3), we obtain $\sum_{k\in I}\int_{\Lambda}\langle f, L_\lambda\varphi_k\rangle\langle L_\lambda\psi_k, g\rangle\,d\lambda = \int_{\mathbb{T}^r}\langle \mathscr{F}f(\alpha), \mathscr{F}g(\alpha)\rangle\,d\alpha = \langle f, g\rangle$, since $f \in \mathcal{S}^{\Lambda}(\mathscr{A})$ implies $\mathscr{F}f(\alpha) \in J_{\mathscr{A}}(\alpha)$ for a.e. $\alpha \in \mathbb{T}^r$. Therefore, $\mathcal{E}^{\Lambda}(\mathscr{A}')$ is an alternate dual for $\mathcal{E}^{\Lambda}(\mathscr{A})$.

Conversely, assume that $\mathcal{E}^{\Lambda}(\mathscr{A}')$ is an alternate dual for $\mathcal{E}^{\Lambda}(\mathscr{A})$, i.e., $\sum_{k\in I}\int_{\Lambda}\langle f, L_\lambda\varphi_k\rangle\langle L_\lambda\psi_k, g\rangle\,d\lambda = \langle f, g\rangle$ for all $f, g \in \mathcal{S}^{\Lambda}(\mathscr{A})$. Now using (6.3.3), we obtain

$$\int_{\mathbb{T}^r}\langle \mathscr{F}f(\alpha), \mathscr{F}g(\alpha)\rangle\,d\alpha = < f, g > = \sum_{k\in I}\int_{\Lambda}\langle f, L_\lambda\varphi_k\rangle\langle L_\lambda\psi_k, g\rangle\,d\lambda$$

$$= \sum_{k\in I}\int_{\Lambda_1}\int_{\mathbb{T}^r}\langle \mathscr{F}f(\alpha), \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\varphi_k(\alpha)\rangle\,\langle \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\psi_k(\alpha), \mathscr{F}g(\alpha)\rangle\,d\alpha\,d\lambda_1. \tag{6.3.4}$$

Then, the expression $\sum_{k\in I}\int_{\Lambda_1}\langle \mathscr{F}f(\alpha), \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\varphi_k(\alpha)\rangle\,\langle \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\psi_k(\alpha), \mathscr{F}g(\alpha)\rangle\,d\lambda_1$ is equal to $\langle \mathscr{F}f(\alpha), \mathscr{F}g(\alpha)\rangle$ for a.e. $\alpha \in \mathbb{T}^r$. Suppose this does not hold. Then there exists a measurable set $\mathscr{D} \subseteq \mathbb{T}^r$ with positive measure such that it is not equal for a.e. $\alpha \in \mathscr{D}$.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a countable dense subset of $\ell^2(\mathbb{Z}^r, \mathcal{HS}(L^2(\mathbb{R}^d)))$ and let $P_{J_{\mathscr{A}}}(\alpha)$ be an orthogonal projection on $J_{\mathscr{A}}(\alpha)$. Clearly $\{P_{J_{\mathscr{A}}}(\alpha)x_n\}_{n\in\mathbb{N}}$ is dense in $J_{\mathscr{A}}(\alpha)$. For each $i, j \in \mathbb{N}$, we consider the set

$$S_{i,j} = \Big\{\alpha \in \mathbb{T}^r : \rho_{i,j}(\alpha) := \sum_{k\in I}\int_{\Lambda_1}\langle P_{J_{\mathscr{A}}}(\alpha)x_i, \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\varphi_k(\alpha)\rangle\,\langle \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\psi_k(\alpha), P_{J_{\mathscr{A}}}(\alpha)x_j\rangle\,d\lambda_1$$
$$-\langle P_{J_{\mathscr{A}}}(\alpha)x_i, P_{J_{\mathscr{A}}}(\alpha)x_j\rangle \neq 0\Big\}.$$

Then there exist $i_0, j_0 \in \mathbb{N}$ such that the set $E := S_{i_0,j_0} \cap \mathscr{D}$ is of positive measure, and hence one of the sets, viz., $E_1 = \{\alpha \in \mathbb{T}^r : \mathrm{Re}(\rho_{i_0,j_0}) > 0\}$, $E_2 = \{\alpha \in \mathbb{T}^r : \mathrm{Re}(\rho_{i_0,j_0}) < 0\}$, $E_3 = \{\alpha \in \mathbb{T}^r : \mathrm{Im}(\rho_{i_0,j_0}) > 0\}$ and $E_4 = \{\alpha \in \mathbb{T}^r : \mathrm{Im}(\rho_{i_0,j_0}) < 0\}$, must have positive measure, assume E_1. By choosing $\mathscr{F}f(\alpha) = \chi_{E_1} P_{J_{\mathscr{A}}}(\alpha)x_{j_0}$ and $\mathscr{F}g(\alpha) = \chi_{E_1} P_{J_{\mathscr{A}}}(\alpha)x_{i_0}$, we have $f, g \in \mathcal{S}^{\Lambda}(\mathscr{A})$ and in view of (6.3.4), we reach on a contradiction that the measure of $\mathscr{D}$ is positive. Similarly, we get contradictions with respect to the other sets E_2, E_3 and E_4. Hence the result follows for alternate duals. $\square$

Proposition 6.3.5. *For $k \in I$ and $\lambda_1 \in \Lambda_1$, let us assume a measurable $\mathbb{Z}^r$-periodic function $p^k_{\lambda_1}$ satisfying $\sum_{k\in I}\int_{\Lambda_1}\int_{\mathbb{T}^r} |p^k_{\lambda_1}(\alpha)|^2 \, d\alpha \, d\lambda_1 < \infty$. Then, for a Bessel system $\mathcal{E}^{\Lambda}(\mathscr{A})$, the following are equivalent:*

(i) $\sum_{k\in I}\int_{\Lambda_1}\int_{\mathbb{T}^r} G^k_{\lambda_1}(\alpha)p^k_{\lambda_1}(\alpha) \, d\alpha \, d\lambda_1 = 0$.

(ii) *For a.e. $\alpha \in \mathbb{T}^r$, $\sum_{k\in I}\int_{\Lambda_1} G^k_{\lambda_1}(\alpha)p^k_{\lambda_1}(\alpha) \ \ d\lambda_1 = 0$.*

Proof. Assume (i). If (ii) does not hold true, there exists a measurable set $\mathscr{D} \subseteq \mathbb{T}^r$ of positive measure such that $\sum_{k\in I}\int_{\Lambda_1} G^k_{\lambda_1}(\alpha)p^k_{\lambda_1}(\alpha) \, d\lambda_1 \neq 0$ for a.e. $\alpha \in \mathscr{D}$. Let $\{x_i\}_{i\in\mathbb{N}}$ be a countable dense subset of $\ell^2(\mathbb{Z}^r, \mathcal{HS}(L^2(\mathbb{R}^d)))$ and for a.e. $\alpha \in \mathbb{T}^r$, let $P_{J_{\mathscr{A}}}(\alpha)$ be an orthogonal projection on $J_{\mathscr{A}}(\alpha)$. Then $\{P_{J_{\mathscr{A}}}(\alpha)x_i\}_{i\in\mathbb{N}}$ is dense in $J_{\mathscr{A}}(\alpha)$ and there exists an $i_0 \in \mathbb{N}$ such that

$$h(\alpha) := \sum_{k\in I}\int_{\Lambda_1} \langle P_{J_{\mathscr{A}}}(\alpha)x_{i_0}, \tilde{\pi}_{\alpha}(\lambda_1)\mathscr{F}\varphi(\alpha)\rangle p^k_{\lambda_1}(\alpha) \, d\lambda_1 \neq 0$$

on some measurable set Y in $\mathscr{D}$ having positive measure. Now, the proof follows by considering the real and imaginary parts, and by choosing suitable function, the way we did for the proof of Theorem 6.3.4 . Conversely, we assume (ii). The part (i) follows easily by integrating (ii) with respect to the torus $\mathbb{T}^r$. $\square$

It can be noted further that the below characterization for type-I and type-II duals behaves similar to the Theorem 6.3.4 of the alternate (oblique) duals while type-I and type-II duals are the particular cases of the alternate (oblique) duals.

Theorem 6.3.6. *$\mathcal{E}^{\Lambda}(\mathscr{A}')$ is a type-I (type-II) dual for $\mathcal{E}^{\Lambda}(\mathscr{A})$ if and only if for a.e. $\alpha \in \mathbb{T}^r$, the system $\{\tilde{\pi}_{\alpha}(\lambda_1)\mathscr{F}\psi(\alpha) : \psi \in \mathscr{A}', \lambda \in \Lambda_1\}$ is a type-I (type-II) dual for the continuous frame $\{\tilde{\pi}_{\alpha}(\lambda_1)\mathscr{F}\varphi(\alpha) : \varphi \in \mathscr{A}, \lambda_1 \in \Lambda_1\}$ of $J_{\mathscr{A}}(\alpha)$, where the range function $J_{\mathscr{A}}(\alpha)$ is defined by (6.3.2) for a.e. $\alpha \in \mathbb{T}^r$.*

Proof. We first prove the result for type-I duals and then for type-II duals. Let $T_{\mathscr{A}(\alpha)}$ and $T_{\mathscr{A}'(\alpha)}$ be analysis operators associated with $\mathscr{F}\mathscr{A}(\alpha) := \{\tilde{\pi}_{\alpha}(\lambda_1)\mathscr{F}\varphi_k(\alpha) : k \in I, \lambda_1 \in \Lambda_1\}$

and $\mathscr{F}\mathscr{A}'(\alpha) := \{\tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\psi_k(\alpha) : k \in I, \lambda_1 \in \Lambda_1\}$, respectively. It's adjoint operators are $T^*_{\mathscr{A}(\alpha)}$ and $T^*_{\mathscr{A}'(\alpha)}$.

For Type-I duals: In view of Theorem 6.3.4, it suffices to show range $T^*_{\mathcal{E}^\Lambda(\mathscr{A}')} \subseteq$ range $T^*_{\mathcal{E}^\Lambda(\mathscr{A})}$ if and only if range $T^*_{\mathscr{A}'(\alpha)} \subseteq$ range $T^*_{\mathscr{A}(\alpha)}$. Equivalently, $\left[T^*_{\mathcal{E}^\Lambda(\mathscr{A}')}(L^2(I \times \Lambda))\right] \cap \mathcal{S}^\Lambda(\mathscr{A}') \subseteq \left[T^*_{\mathcal{E}^\Lambda(\mathscr{A})}(L^2(I \times \Lambda))\right] \cap S^\Lambda(\mathscr{A})$ if and only if for a.e. $\alpha \in \mathbb{T}^r$, $\left[T^*_{\mathscr{A}'(\alpha)}(L^2(I \times \Lambda_1))\right] \cap J_{A'}(\alpha) \subseteq \left[T^*_{\mathscr{A}(\alpha)}(L^2(I \times \Lambda_1))\right] \cap J_{\mathscr{A}}(\alpha)$. For this it is enough to verify on the generators φ_k, ψ_k. Then the result follows just by observing: $L_\lambda \psi_k \in \mathcal{S}^\Lambda(\mathscr{A})$ for $k \in I$ and $\lambda \in \Lambda$ if and only if for a.e. $\alpha \in \mathbb{T}^r$, $\mathscr{F} L_{\lambda_1}\psi_{k'}(\alpha) \in J_{\mathscr{A}}(\alpha)$ for $k' \in I, \lambda_1 \in \Lambda_1$.

For Type-II duals: In view of Theorem 6.3.4, it suffices to show range $T_{\mathcal{E}^\Lambda(\mathscr{A}')} \subseteq$ range $T_{\mathcal{E}^\Lambda(\mathscr{A})}$ if and only if for a.e. $\alpha \in \mathbb{T}^r$, range $T_{\mathscr{A}'}(\alpha) \subseteq$ range $T_{\mathscr{A}(\alpha)}$. First we assume range $T_{\mathscr{A}'(\alpha)} \subseteq$ range $T_{\mathscr{A}(\alpha)}$ a.e. $\alpha \in \mathbb{T}^r$. Then the family $\{a_{k,\lambda_1}\}_{k\in I, \lambda_1 \in \Lambda_1}$ in $L^2(I \times \Lambda_1)$ satisfies for a.e. $\alpha \in \mathbb{T}^r$,

(6.3.5)
$$\sum_{k\in I}\int_{\Lambda_1} \langle h, \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\varphi_k(\alpha)\rangle \overline{a_{k,\lambda_1}}\, d\lambda_1 = 0 \implies \sum_{k\in I}\int_{\Lambda_1} \langle h, \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\psi_k(\alpha)\rangle \overline{a_{k,\lambda_1}}\, d\lambda_1 = 0 \text{ for all } h \in J_{\mathscr{A}}(\alpha).$$

To prove range $T_{\mathcal{E}^\Lambda(\mathscr{A}')} \subseteq$ range $T_{\mathcal{E}^\Lambda(\mathscr{A})}$, we calculate the following for $\{c_{k,\lambda}\}_{k\in I, \lambda\in\Lambda} = \{c_{k,\lambda_0,\lambda_1}\}_{k\in I, \lambda_0\in\Lambda_0, \lambda_1\in\Lambda_1}$ in $L^2(I \times \Lambda)$ as follows:

$$\begin{aligned}
\sum_{k\in I}\int_{\Lambda} \langle f, L_\lambda \varphi_k\rangle \overline{c_{k,\lambda}}\, d\lambda &= \sum_{k\in I}\int_{\Lambda_1}\sum_{\lambda_0\in\Lambda_0}\int_{\mathbb{T}^r} \langle \mathscr{F} f(\alpha), \mathscr{F} L_{\lambda_1\lambda_0}\varphi_k(\alpha)\rangle \overline{c_{k,\lambda_1,\lambda_0}}\, d\alpha\, d\lambda_1 \\
&= \sum_{k\in I}\int_{\Lambda_1}\sum_{\lambda_0\in\Lambda_0}\int_{\mathbb{T}^r} \langle \mathscr{F} f(\alpha), \mathscr{F} L_{\lambda_1}\varphi_k(\alpha)\rangle e^{-2\pi i\langle \alpha,\lambda_0\rangle}\overline{c_{k,\lambda_1,\lambda_0}}\, d\alpha\, d\lambda_1 \\
&= \sum_{k\in I}\int_{\Lambda_1}\int_{\mathbb{T}^r} \langle \mathscr{F} f(\alpha), \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\varphi_k(\alpha)\rangle \overline{p^k_{\lambda_1}(\alpha)}\, d\alpha\, d\lambda_1,
\end{aligned}$$

where $p^k_{\lambda_1}(\alpha) = \sum_{\lambda_0\in\Lambda_0} c_{k,\lambda_1,\lambda_0} e^{2\pi i\langle\alpha,\lambda_0\rangle}$ satisfies

$$\sum_{k\in I}\int_{\Lambda_1}\int_{\mathbb{T}^r} |p^k_{\lambda_1}(\alpha)|^2\, d\alpha\, d\lambda_1 = \sum_{k\in I}\int_{\Lambda_1}\sum_{\lambda_0\in\Lambda_0} |c_{k,\lambda_1,\lambda_0}|^2\, d\lambda_1 < \infty.$$

Similarly, we can obtain

$$\sum_{k\in I}\int_{\Lambda} \langle f, L_\lambda \psi_k\rangle \overline{c_{k,\lambda}}\, d\lambda = \sum_{k\in I}\int_{\Lambda_1}\int_{\mathbb{T}^r} \langle \mathscr{F} f(\alpha), \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\psi_k)(\alpha)\rangle \overline{p^k_{\lambda_1}(\alpha)}\, d\alpha\, d\lambda_1.$$

By assuming $\sum_{k\in I}\int_{\Lambda}\langle f, L_\lambda \varphi_k\rangle \overline{c_{k,\lambda}}\, d\lambda = 0$ for all $f \in \mathcal{S}^{\Lambda}(\mathscr{A})$, and applying Proposition 6.3.5 we get for a.e. $\alpha \in \mathbb{T}^r$,

$$\sum_{k\in I}\int_{\Lambda_1}\langle \mathscr{F} f(\alpha), \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\varphi_k(\alpha)\rangle \overline{p^k_{\lambda_1}(\alpha)}\, d\lambda_1 = 0$$

$$\Longrightarrow \sum_{k\in I}\int_{\Lambda_1}\langle \mathscr{F} f(\alpha), \tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\psi_k(\alpha)\rangle \overline{p^k_{\lambda_1}(\alpha)}\, d\lambda_1 = 0 \tag{6.3.6}$$

from (6.3.5). Therefore, we get $\sum_{k\in I}\int_{\lambda\in\Lambda}\langle f, L_\lambda \psi_k\rangle \overline{c_{k,\lambda}}\, d\lambda = 0$ for $f \in \mathcal{S}^{\Lambda}(\mathscr{A})$. Thus, range $T_{\mathcal{E}^{\Lambda}(\mathscr{A}')} \subseteq$ range $T_{\mathcal{E}^{\Lambda}(\mathscr{A})}$. Conversely, we assume range $T_{\mathcal{E}^{\Lambda}(\mathscr{A}')} \subseteq$ range $T_{\mathcal{E}^{\Lambda}(\mathscr{A})}$. For range $T_{\mathscr{A}'(\alpha)} \subseteq$ range $T_{\mathscr{A}(\alpha)}$ a.e. $\alpha \in \mathbb{T}^r$, we can proceed with the help of (6.3.6) and choosing $p^k_{\lambda_1}(\alpha) = c_{k,\lambda_1}$ for all $\alpha \in \mathbb{T}^r$. □

Remark 6.3.7. Let G be a d-dimensional Heisenberg group denoted by $\mathbb{H}^d$. Consider $\Lambda_1 = \mathbb{R}^d \times \mathbb{R}^d \times \{0\}$ and $\Lambda_0 = \{(0,0)\} \times \mathbb{Z}$, then the set $\Lambda = \Lambda_1\Lambda_0$ can be identified with $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{Z}$. For $\mathscr{A} \subset L^2(\mathbb{H}^d)$, the Λ-generated system $\mathcal{E}^{\Lambda}(\mathscr{A})$ will be of the form

$$\mathcal{E}^{\Lambda}(\mathscr{A}) = \{L_{\lambda_1\lambda_0}\varphi : \varphi \in \mathscr{A}, \lambda_1 \in \mathbb{R}^d \times \mathbb{R}^d \times \{0\}, \lambda_0 \in \{(0,0)\} \times \mathbb{Z}\}.$$

For a.e. $\alpha \in \mathbb{T}$, we consider the range function

$$J_{\mathscr{A}}(\alpha) = \overline{\operatorname{span}}\{\tilde{\pi}_\alpha(\lambda_1)\mathscr{F}\varphi(\alpha) : \varphi \in \mathscr{A}, \lambda_1 \in \Lambda_1\} \subseteq \ell^2(\mathbb{Z}, \mathcal{HS}(L^2(\mathbb{R}^d)))$$

associated with the representation $\tilde{\pi}_\alpha$, which is defined by

$$\tilde{\pi}_\alpha(\lambda_1)z(m) := \pi_{\alpha+m}(\lambda_1)\circ z(m), \text{where } (z(m)) \in \ell^2(\mathbb{Z}, \mathcal{HS}(L^2(\mathbb{R}^d))).$$

Then we can state Theorems 6.3.4 and 6.3.6 for the continuous setup.
Similarly, the results can be developed for Λ_1 of the form $\Lambda_1 = \Gamma_1 \times \Gamma_2 \times \{0\}$ and $\Lambda_0 = \{(0,0) \times m\mathbb{Z}\}$, where Γ_1, Γ_2 are additive subgroups of $\mathbb{R}^d$ and $m \in \mathbb{N}$.

6.3.1. Super dual frame pair

Now, we will discuss the properties of dual frame for super-frames in orthogonal direct sum of Hilbert spaces, introduced by Han and Larson [43] and Balan [11]. This concept has been carried out by many authors in the context of translation-invariant system and Gabor systems including Lie and Lian [58], and Lopez and Han [59] in the super Hilbert spaces. We will address it for Lie group. By a *super Hilbert space* $L^2(G) \oplus \cdots \oplus L^2(G)$ (N-copies) or $\oplus^N L^2(G)$ (see (5.3.2)), we mean it is a collection of functions of the form $\{\oplus_{n=1}^N f^{(n)} := (f^{(1)}, f^{(2)}, \dots, f^{(N)}) : f^{(n)} \in L^2(G), 1 \leqslant n \leqslant N\}$ with the inner product

$\langle \oplus_{n=1}^N f^{(n)}, \oplus_{n=1}^N g^{(n)} \rangle = \sum_{n=1}^N \langle f^{(n)}, g^{(n)} \rangle$. Indeed, $\oplus^N L^2(G)$ is nothing but the Hilbert space $L^2(G \times \mathbb{Z}_N)$, where $\mathbb{Z}_N$ is an abelian group with modulo N. For each $\lambda \in \Lambda$, define the translation operator $\mathcal{L}_\lambda := \oplus^N L_\lambda$ which acts on an element $\oplus_{n=1}^N f^{(n)}$ by $\mathcal{L}_\lambda(\oplus_{n=1}^N f^{(n)}) = \oplus_{n=1}^N L_\lambda f^{(n)}$.

The following results characterize the (super) dual frame in the super Hilbert space $\oplus^N L^2(G)$. Recalling that, two Bessel families $\mathcal{X} = \{f_k\}_{k\in I}$ and $\mathcal{Y} = \{g_k\}_{k\in I}$ in $\mathcal{H}$, are said to be orthogonal if $\sum_{k\in I} \langle f, g_k \rangle f_k = 0$ for all $f \in \mathcal{H}$.

We will characterize two orthogonal Bessel pair using range function:

Theorem 6.3.8. *Let $\mathscr{A} = \{\varphi_k : k \in I\}$ and $\mathscr{A}' = \{\psi_k : k \in \mathcal{I}\}$ be two sequences of functions in $L^2(G)$ such that the Λ-translation generated systems $\mathcal{E}^\Lambda(\mathscr{A})$ and $\mathcal{E}^\Lambda(\mathscr{A}')$ are Bessel and they form an orthogonal pair if and only if for a.e. $\alpha \in \mathbb{T}^r$, the system $\mathscr{F}\mathscr{A}'(\alpha) = \{\mathscr{F} L_{\lambda_1}\psi(\alpha) : \psi \in \mathscr{A}', \lambda_1 \in \Lambda_1\}$ and $\mathscr{F}\mathscr{A}(\alpha) = \{\mathscr{F} L_{\lambda_1}\varphi(\alpha) : \varphi \in \mathscr{A}, \lambda_1 \in \Lambda_1\}$ are orthogonal Bessel pair in $\ell^2(\mathbb{Z}^r, \mathcal{HS}(L^2(\mathbb{R}^d)))$.*

Proof. Let $\mathcal{E}^\Lambda(\mathscr{A})$ and $\mathcal{E}^\Lambda(\mathscr{A}')$ are orthogonal, i.e., for $f, g \in L^2(G)$, first assume

$$\sum_{k\in I} \int_\Lambda \langle f, L_\lambda \psi_k \rangle \, \langle L_\lambda \varphi_k, g \rangle \, d\lambda = 0,$$

which is equivalent to $\sum_{k\in I} \int_{\mathbb{T}^r} \langle \mathscr{F} f(\alpha), \mathscr{F}\varphi_k(\alpha) \rangle \langle \mathscr{F}\psi_k(\alpha), \mathscr{F} g(\alpha) \rangle \, d\alpha \, d\lambda = 0$ from Proposition 6.3.3. For a.e. $\alpha \in \mathbb{T}^r$, we need to show $\sum_{k\in I} \langle \mathscr{F} f(\alpha), \mathscr{F}\varphi_k(\alpha) \rangle \langle \mathscr{F}\psi_k(\alpha), \mathscr{F} g(\alpha) \rangle = 0$. For this, let $(e_i)_{i\in\mathbb{Z}}$ be an orthonormal basis for $\ell^2(\mathbb{Z}^r, \mathcal{HS}(L^2(\mathbb{R}^d)))$ and $\mathscr{P}_{J_{\mathscr{A}}}(\alpha)$ is an orthogonal projection onto $J_{\mathscr{A}}(\alpha)$ for a.e. $\alpha \in \mathbb{T}^r$. Assume on the contrary, there exists $i_0 \in \mathbb{Z}$ such that $h(\alpha) = \sum_{k\in I} \langle \mathscr{P}(\alpha) e_{i_0}, \mathscr{F}\varphi_k(\alpha) \rangle \, \overline{\langle \mathscr{F}\psi_k(\alpha), \mathscr{F} g(\alpha) \rangle} \neq 0$ on a measurable set $\mathscr{D} \subseteq \mathbb{T}^r$ with $\mu(\mathscr{D}) > 0$. The rest part of the proof follows from the similar steps of Theorem 6.3.4 (i). The converse part follows easily from Proposition 6.3.3. □

Theorem 6.3.9. *For $1 \leqslant n \leqslant N$, $N \in \mathbb{N}$, let $\{\varphi_k^{(n)}\}_{k\in I}$ and $\{\psi_k^{(n)}\}_{k\in I}$ be two collections of functions in $L^2(G)$ such that $\{L_\lambda \varphi_k^{(n)}\}_{k\in I, \lambda\in\Lambda}$ and $\{L_\lambda \psi_k^{(n)}\}_{k\in I, \lambda\in\Lambda}$ are Bessel. Then $\{\mathcal{L}_\lambda(\oplus_{n=1}^N \varphi_k^{(n)})\}_{k\in I, \lambda\in\Lambda}$ and $\{\mathcal{L}_\lambda(\oplus_{n=1}^N \psi_k^{(n)})\}_{k\in I, \lambda\in\Lambda}$ are (super) dual frames in $\oplus^N L^2(G)$ if and only if :*

(i) *For a.e. $\alpha \in \mathbb{T}^r$ and $1 \leqslant n_1 \neq n_2 \leqslant N$, $\left\{\mathscr{F} L_{\lambda_1}\varphi_k^{(n)}(\alpha)\right\}_{\lambda_1\in\Lambda_1, k\in I}$ and $\left\{\mathscr{F} L_{\lambda_1}\psi_k^{(n)}(\alpha)\right\}_{\lambda_1\in\Lambda_1, k\in I}$ are orthogonal pair.*

(ii) *For a.e. $\alpha \in \mathbb{T}^r$ and $1 \leqslant n \leqslant N$, $\left\{\mathscr{F} L_{\lambda_1}\varphi_k^{(n)}(\alpha)\right\}_{\lambda_1\in\Lambda_1, k\in I}$ and $\{\mathscr{F} L_{\lambda_1}\psi_k(\alpha)\}_{k\in I, \lambda_1\in\Lambda_1}$ are dual frames in $\ell^2(\mathbb{Z}^r, \mathcal{HS}(L^2(\mathbb{R}^d)))$.*

Proof. Assume the systems $\{\mathcal{L}_\lambda(\oplus_{n=1}^N \varphi_k^{(n)})\}_{k\in I,\lambda\in\Lambda}$ and $\{\mathcal{L}_\lambda(\oplus_{n=1}^N \psi_k^{(n)})\}_{k\in I,\lambda\in\Lambda}$ are (super) dual frames in $\oplus^N L^2(G)$. For the part (i), let $1 \leqslant n_1 \neq n_2 \leqslant N$ and $h \in \oplus^N L^2(G)$. Then, we have $\mathscr{P}_{n_1}(\mathscr{P}_{n_2}h) = 0$, where $\mathscr{P}_{n_1}(\mathscr{P}_{n_2}h)$ is equal to

$$\sum_{k\in I}\int_\Lambda \langle \mathscr{P}_{n_2}h, \mathscr{P}_{n_2}^*\mathcal{L}_\lambda(\oplus_{n=1}^N \varphi_k^{(n)})\rangle \mathscr{P}_{n_1}(\mathcal{L}_\lambda \oplus_{n=1}^N \psi_k^{(n)})\, d\lambda = \sum_{k\in I}\int_\Lambda \langle \mathscr{P}_{n_2}h, L_\lambda\varphi_k^{(n_2)}\rangle L_\lambda \psi_k^{(n_1)}\, d\lambda$$

Hence, $\mathcal{E}^\Lambda(\{\varphi_k^{(n_2)}\}_{k\in I})$ and $\mathcal{E}^\Lambda(\{\psi_k^{(n_1)}\}_{k\in I})$ are orthogonal pair. Therefore (i) follows. Then for each $1 \leqslant n \leqslant N$, (ii) follows by just applying the orthogonal projection $\mathscr{P}_n$ on it. Conversely, let us assume (i) and (ii) hold. Both $\{\mathcal{L}_\lambda(\oplus_{n=1}^N \varphi_k^{(n)})\}_{k\in I,\lambda\in\Lambda}$ and $\{\mathcal{L}_\lambda(\oplus_{n=1}^N \psi_k^{(n)})\}_{k\in I,\lambda\in\Lambda}$ are Bessel families in $\oplus^N L^2(G)$, follows by the below calculations for $h \in L^2(G)^N$ and the Bessel property of $\{L_\lambda\varphi_k^{(n)}\}_{k\in I,\lambda\in\Lambda}$ with Bessel bound $B^{(n)}$:

$$\begin{aligned}\sum_{k\in I}\int_\Lambda |\langle h, \mathcal{L}_\lambda(\oplus_{n=1}^N \varphi_k^{(n)})\rangle|^2 d\lambda &= \sum_{k\in I}\int_\Lambda |\langle \oplus_{n=1}^N \mathscr{P}_n h, \mathcal{L}_\lambda(\oplus_{n=1}^N \varphi_k^{(n)})\rangle|^2\, d\lambda \\ &= \sum_{k\in I}\int_\Lambda \Big|\sum_{n=1}^N \langle \mathscr{P}_n h, L_\lambda \varphi_k^{(n)}\rangle\Big|^2\, d\lambda \leqslant C\|h\|^2 \sum_{n=1}^N B^{(n)}\end{aligned}$$

for some constant $C > 0$. Similarly for $\{L_\lambda\psi_k^{(n)}\}_{k\in I,\lambda\in\Lambda}$. Then the rest follows by observing the reproducing formula for $h \in \oplus^N L^2(G)$ and writing $h = \oplus_{n=1}^N \mathscr{P}_n h$ in the below calculations:

$$\begin{aligned}&\sum_{k\in I}\int_\Lambda \langle h, \mathcal{L}_\lambda(\oplus_{n=1}^N \psi_k^{(n)})\rangle \mathcal{L}_\lambda(\oplus_{n=1}^N \varphi_k^{(n)})\, d\lambda = \sum_{k\in I}\int_\Lambda \sum_{n=1}^N \langle \mathscr{P}_n h, L_\lambda \psi_k^{(n)}\rangle \mathcal{L}_\lambda(\oplus_{n=1}^N \varphi_k^{(n)})\, d\lambda \\ &= \sum_{k\in I}\int_\Lambda \sum_{n=1}^N \langle \mathscr{P}_n h, L_\lambda\psi_k^{(n)}\rangle L_\lambda \varphi_k^{(1)}\, d\lambda \oplus \cdots \oplus \sum_{k\in I}\int_\Lambda \sum_{n=1}^N \langle \mathscr{P}_n h, L_\lambda \psi_k^{(n)}\rangle L_\lambda\varphi_k^{(N)}\, d\lambda \\ &= \mathscr{P}_1 h \oplus \cdots \oplus \mathscr{P}_N h = h,\end{aligned}$$

in view of Theorem 6.3.8. □

6.4. Reproducing formulas by the action of discrete translations

In this section, we assume a discrete subset $\Lambda_1 \subseteq \exp\mathbb{R}X_{r+1}\ldots\exp\mathbb{R}X_n$ and the integer lattice $\Lambda_0 = \exp\mathbb{Z}X_1\ldots\exp\mathbb{Z}X_r$ in G. Our aim is to obtain results related to reproducing formula of a Bessel family in terms of the bracket map for a Λ-translation generated system having biorthogonal property, where $\Lambda = \Lambda_1\Lambda_0$.

In the sequel, we use the operator $[\cdot,\cdot] : L^2(G) \times L^2(G) \to L^1(\mathbb{T}^r)$, known as *bracket map*, defined as follows for a.e. $\alpha \in \mathbb{T}^r$ and $\varphi, \psi \in L^2(G)$:

$$[\varphi,\psi](\alpha) := \langle \mathscr{F}\varphi(\alpha), \mathscr{F}\psi(\alpha)\rangle_{\ell^2(\mathbb{Z}^r,\mathcal{HS}(L^2(\mathbb{R}^d)))} = \sum_{m\in\mathbb{Z}^r} \langle \mathcal{F}\varphi(\alpha+m), \mathcal{F}\psi(\alpha+m)\rangle |\mathbf{Pf}(\alpha+m)|,$$

to address the results related to the reproducing formulas [12]. Recall, the Λ-translation generated system $\mathcal{E}^\Lambda(\varphi) = \{L_\lambda\varphi : \lambda \in \Lambda\}$ and its associated Λ-translation invariant space $\mathcal{S}^\Lambda(\varphi) = \overline{\operatorname{span}}\mathcal{E}^\Lambda(\varphi)$ in $L^2(G)$, where $\varphi \in L^2(G)$. Then, for the reproducing formulas associated with the system $\mathcal{E}^\Lambda(\varphi)$, we proceed by considering biorthogonal systems generated by the discrete translations. $\mathcal{E}^\Lambda(\varphi)$ and $\mathcal{E}^\Lambda(\psi)$ are *biorthogonal* if $\langle\varphi, L_\lambda\psi\rangle = \delta_{\lambda,0}$ for all $\lambda \in \Lambda$. We obtain a necessary and sufficient condition for the biorthogonality and orthogonality of translation generated systems in terms of the bracket map.

Proposition 6.4.1. *Let $\varphi, \psi \in L^2(G)$ be non-zero functions. Then the following hold:*

(i) *$\mathcal{E}^\Lambda(\varphi)$ and $\mathcal{E}^\Lambda(\psi)$ are biorthogonal if and only if $[\varphi, L_{\lambda_1}\psi](\alpha) = \delta_{\lambda_1,0}$ for all $\lambda_1 \in \Lambda_1$, a.e. $\alpha \in \mathbb{T}^r$.*

(ii) *The subspace generated by $\mathcal{E}^\Lambda(\varphi)$ is orthogonal to $\mathcal{E}^\Lambda(\psi)$ if and only if*

$$[\varphi, L_{\lambda_1}\psi](\alpha) = 0 \text{ for all } \lambda_1 \in \Lambda_1, \quad \text{a.e. } \alpha \in \Omega_\varphi := \{\alpha \in \mathbb{T}^r : [\varphi,\varphi](\alpha) \neq 0\}.$$

In particular, $\mathcal{E}^\Lambda(\varphi)$ is an orthogonal system of functions if and only if the orthogonality condition

$$(\mathcal{O}_\varphi): \ [\varphi, L_{\lambda_1}\varphi](\alpha) = 0 \text{ for all } \lambda_1 \in \Lambda_1\backslash\{0\}, \quad \text{a.e. } \alpha \in \Omega_\varphi, \ \text{holds.} \tag{6.4.1}$$

The proof of Proposition 6.4.1 can be realized on the same techniques followed in [7, 12] for the Heisenberg group. In the wake of Proposition 6.4.1, we observe that the biorthogonality (or orthogonality) of Λ-translation generated systems is equivalent to the corresponding biorthogonality (or orthogonality) of Λ_0-translation generated systems.

We first note Proposition 6.4.1 motivates to decompose the principle translation-invariant space $\mathcal{S}^\Lambda(\varphi)$ into the orthogonal direct sum of Λ_1-translates.

Proposition 6.4.2. *Let $\varphi \in L^2(G)$ be such that it satisfies the orthogonality condition $(\mathcal{O}_\varphi)$ mentioned in (6.4.1). For $\lambda_1 \neq \lambda_1' \in \Lambda_1$, the subspace generated by $\mathcal{E}^{\Lambda_0}(L_{\lambda_1}\varphi)$ is orthogonal to $\mathcal{E}^{\Lambda_0}(L_{\lambda_1'}\varphi)$ and $\mathcal{S}^\Lambda(\varphi) = \bigoplus_{\lambda_1\in\Lambda_1} \mathcal{S}^{\Lambda_0}(L_{\lambda_1}\varphi)$.*

Moreover, $f \in S^{\Lambda}(\varphi)$ if and only if

$$(\mathscr{F} f)(\alpha)(m) = \sum_{\lambda_1 \in \Lambda_1} \mathfrak{p}_{\lambda_1}(\alpha) \mathscr{F} L_{\lambda_1}\varphi(\alpha)(m) \text{ for a.e. } \alpha \in \mathbb{T}^r,\ m \in \mathbb{Z}^r,$$

where $\mathfrak{p} = \{\mathfrak{p}_{\lambda_1}\}_{\lambda_1 \in \Lambda_1}$ is a member of the weighted space $L^2(\mathbb{T}^r, [\varphi, \varphi])$. Further, there exist unique $\varphi_{\lambda_1}, \psi_{\lambda_1}$ in $\mathcal{S}^{\Lambda_0}(L_{\lambda_1}\varphi)$ such that

$$f = \sum_{\lambda_1 \in \Lambda_1} \varphi_{\lambda_1},\ g = \sum_{\lambda_1 \in \Lambda_1} \psi_{\lambda_1},\ \text{and } \langle f, g\rangle = \sum_{\lambda_1 \in \Lambda_1} \langle \varphi_{\lambda_1}, \psi_{\lambda_1}\rangle \text{ for } f, g \in \mathcal{S}^{\Lambda}(\varphi).$$

Proof. At first we will show the condition $(\mathcal{O}_\varphi)$ is equivalent to

$$\langle \varphi, L_{\lambda_1\lambda_0}\varphi\rangle = \delta_{\lambda_1,0}\langle \varphi, L_{\lambda_0}\varphi\rangle \text{ for all } \lambda_1\lambda_0 \in \Lambda_1\Lambda_0.$$

First, assume the condition that is, for all $\lambda_1 \in \Lambda_1\backslash\{0\}$, $[\varphi, L_{\lambda_1}\varphi](\alpha) = 0$ for a.e. $\alpha \in \Omega_\varphi = \{\alpha \in \mathbb{T}^r : [\varphi, \varphi](\alpha) \neq 0\}$. If $\alpha \notin \Omega_\varphi$ then $[\varphi, L_\lambda\varphi](\alpha) = 0$ for all $\lambda \in \Lambda$, as for $\varphi, \psi \in L^2(G)$

$$\begin{aligned}|[\varphi, \psi](\alpha)| &\leqslant \sum_{j \in \mathbb{Z}^r} \left\| \mathscr{F}\varphi(\alpha)(j)\overline{\mathscr{F}\psi(\alpha)(j)} \right\|_{\mathcal{HS}} \leqslant \left(\sum_{j \in \mathbb{Z}^r} \|\mathscr{F}\varphi(\alpha)(j)\|^2_{\mathcal{HS}} \right)^{1/2} \left(\sum_{j \in \mathbb{Z}^r} \|\mathscr{F}\psi(\alpha)(j)\|^2_{\mathcal{HS}} \right)^{1/2} \\ &= ([\varphi, \varphi](\alpha))^{1/2}([\psi, \psi](\alpha))^{1/2},\end{aligned}$$

easily follows from Cauchy-Schwarz's inequality. Hence the condition $(\mathcal{O}_\varphi)$ equivalent to mention for a.e. $\alpha \in \mathbb{T}^r$. So for $\lambda_1 \neq 0$, $[\varphi, L_{\lambda_1}\varphi](\alpha) = 0$ a.e. $\alpha \in \mathbb{T}^r$. Now calculating

$$\begin{aligned}\langle \varphi, L_{\lambda_1\lambda_0}\varphi\rangle &= \int_{\mathbb{T}^r} \langle \mathscr{F}\varphi(\alpha), \mathscr{F}L_{\lambda_1\lambda_0}\varphi(\alpha)\rangle \, d\alpha = \int_{\mathbb{T}^r} \langle \mathscr{F}\varphi(\alpha), \mathscr{F}L_{\lambda_1}\varphi(\alpha)\rangle e^{-2\pi i\langle \alpha, \lambda_0\rangle} \, d\alpha \\ &= \int_{\mathbb{T}^r} [\varphi, L_{\lambda_1}\varphi](\alpha) e^{-2\pi i\langle \alpha, \lambda_0\rangle} \, d\alpha = \delta_{\lambda_1,0} \int_{\mathbb{T}^r} e^{-2\pi i\langle \alpha, \lambda_0\rangle} \, d\alpha \\ &= \delta_{\lambda_1,0} \int_{\mathbb{T}^r} [\varphi, \varphi](\alpha) e^{-2\pi i\langle \alpha, \lambda_0\rangle} \, d\alpha = \delta_{\lambda_1,0}\langle \varphi, L_{\lambda_0}\varphi\rangle.\end{aligned}$$

Conversely, assume $\langle \varphi, L_{\lambda_1\lambda_0}\varphi\rangle = \delta_{\lambda_1,0}\langle \varphi, L_{\lambda_0}\varphi\rangle$, then $\langle \varphi, L_\lambda\psi\rangle = \int_{\mathbb{T}^r} [\varphi, L_{\lambda_1}\psi] e^{-2\pi i\langle \alpha, \lambda_0\rangle} d\alpha$ and from the uniqueness of Fourier coefficients, we have $[\varphi, L_{\lambda_1}\varphi](\alpha) = 0$ for all $\alpha \in \mathbb{T}^r$.

Let $0 \neq f \in S^{\Lambda_0}(L_{\lambda_1}\varphi)$, $0 \neq g \in S^{\Lambda_0}(L_{\lambda_1'}\varphi)$ then $f = \sum_{\lambda_0 \in \Lambda_0} c_{\lambda_0} L_{\lambda_0}(L_{\lambda_1}\varphi)$ and $g = \sum_{\lambda_0 \in \Lambda_0} d_{\lambda_0} L_{\lambda_0}(L_{\lambda_1'}\varphi)$ for some non-zero c_{λ_0} and d_{λ_0}. Now,

$$\begin{aligned}\langle f, g\rangle &= \left\langle \sum_{\lambda_0 \in \Lambda_0} c_{\lambda_0} L_{\lambda_0}(L_{\lambda_1}\varphi), \sum_{\lambda_0 \in \Lambda_0} d_{\lambda_0} L_{\lambda_0}(L_{\lambda_1'}\varphi) \right\rangle = \sum_{\lambda_0 \in \Lambda_0} \left\langle c_{\lambda_0} L_{\lambda_0} L_{\lambda_1}\varphi, d_{\lambda_0} L_{\lambda_0} L_{\lambda_1'}\varphi \right\rangle \\ &= \sum_{\lambda_0 \in \Lambda_0} \left\langle c_{\lambda_0}\varphi, d_{\lambda_0} L_{\lambda_1^{-1}\lambda_1'}\varphi \right\rangle = \sum_{\lambda_0 \in \Lambda_0} c_{\lambda_0} d_{\lambda_0}^{-1} \langle \varphi, L_{\lambda_1{}^{-1}\lambda_1'}\varphi\rangle \\ &= \sum_{\lambda_0 \in \Lambda_0} c_{\lambda_0} d_{\lambda_0}^{-1} \langle \varphi, L_{\eta_1}\varphi\rangle = \delta_{\lambda_1^{-1}\lambda_1',0} \sum_{\lambda_0 \in \Lambda_0} c_{\lambda_0} d_{\lambda_0}^{-1} = 0 \text{ (assume } \eta_1 = \lambda_1^{-1}\lambda_1').\end{aligned}$$

Hence $f \perp g$, i.e., $\mathcal{S}^{\Lambda_0}(L_{\lambda_1}\varphi) \perp \mathcal{S}^{\Lambda_0}(L_{\lambda_1'}\varphi)$.

Let $f \in S^{\Lambda}(\varphi)$, then $f = \sum_{\lambda\in\Lambda} c_\lambda L_\lambda \varphi = \sum_{\lambda_1\lambda_0\in\Lambda_1\Lambda_0} c_{\lambda_1\lambda_0} L_{\lambda_1\lambda_0}\varphi = \sum_{\lambda_1\lambda_0\in\Lambda_1\Lambda_0} c_{\lambda_1\lambda_0} L_{\lambda_1} L_{\lambda_0}\varphi = \sum_{\lambda_1\in\Lambda_1} d_{\lambda_1} L_{\lambda_1}(L_{\lambda_0}\varphi)$ where $d_{\lambda_1} = c_{\lambda_1\lambda_0}$, which implies that $f \in S^{\Lambda_0}(L_{\lambda_1}\varphi)$.

The next part follows easily.

□

We now provide a characterization of Bessel family under the orthogonality condition. Recall, the sequence $\mathcal{E}^{\Lambda}(\varphi)$ is called Bessel in $\mathcal{S}^{\Lambda}(\varphi)$ if $\sum_{\lambda\in\Lambda} |\langle f, L_\lambda\varphi\rangle|^2 \leqslant B\|f\|^2$ for all $f \in \mathcal{S}^{\Lambda}(\varphi)$.

Proposition 6.4.3. *Let $\varphi \in L^2(G)$ be such that it satisfies the orthogonality condition $(\mathcal{O}_\varphi)$ mentioned in (6.4.1). Then $\mathcal{E}^{\Lambda}(\varphi)$ is Bessel sequence in $\mathcal{S}^{\Lambda}(\varphi)$. Also, $\mathcal{E}^{\Lambda}(\varphi)$ is a Bessel sequence in $\mathcal{S}^{\Lambda}(\varphi)$ is equivalent to $\mathcal{E}^{\Lambda_0}(\varphi)$ is a Bessel sequence in $\mathcal{S}^{\Lambda_0}(\varphi)$.*

Proof. The Bessel condition of $\mathcal{E}^{\Lambda}(\varphi)$ is followed from the Parseval equality of the orthonormal system $\mathcal{E}^{\Lambda}(\varphi)$. To show the equivalence, let $\mathcal{E}^{\Lambda_0}(\varphi)$ be a Bessel sequence in $\mathcal{S}^{\Lambda_0}(\varphi)$ with bound B. For $f \in \mathcal{S}^{\Lambda}(\varphi)$, there exists φ_{λ_1} in $\mathcal{S}^{\Lambda_0}(L_{\lambda_1}\varphi)$ such that $f = \sum_{\lambda_1\in\Lambda_1} \varphi_{\lambda_1}$, and hence by using Proposition 6.4.2, we obtain

$$\sum_{\lambda\in\Lambda} |\langle f, L_\lambda\varphi\rangle|^2 = \sum_{\eta_1\in\Lambda_1}\sum_{\eta_0\in\Lambda_0} |\langle \sum_{\lambda_1\in\Lambda_1} \varphi_{\lambda_1}, L_{\eta_1}L_{\eta_0}\varphi\rangle|^2 = \sum_{\eta_1\in\Lambda_1}\sum_{\eta_0\in\Lambda_0} |\langle L_{\eta_1{}^{-1}}\varphi_{\eta_1}, L_{\eta_0}\varphi\rangle|^2$$
$$\leqslant B \sum_{\eta_1\in\Lambda_1} \|L_{\eta_1{}^{-1}}\varphi_{\eta_1}\|^2 = B\sum_{\eta_1\in\Lambda_1} \|\varphi_{\eta_1}\|^2 = B\|f\|^2,$$

since $L_{\eta_1^{-1}}\varphi_{\eta_1} \in \mathcal{S}^{\Lambda_0}(\varphi)$ and $L_{\eta_1{}^{-1}}$ is an isometry. Hence $\mathcal{E}^{\Lambda}(\varphi)$ is also Bessel sequence in $\mathcal{S}^{\Lambda}(\varphi)$.

Conversely, assume $\mathcal{E}^{\Lambda}(\varphi)$ is a Bessel sequence in $\mathcal{S}^{\Lambda}(\varphi)$. Using Proposition 6.4.2, we have $\langle\varphi, L_{\lambda_1}L_{\lambda_0}\varphi\rangle = 0$ for $\lambda_1 \neq 0$ as $\varphi \in \mathcal{S}^{\Lambda_0}(\varphi)$, and hence the result follows by noting

$$\sum_{\lambda_0\in\Lambda_0} |\langle\varphi, L_{\lambda_0}\varphi\rangle|^2 = \sum_{\lambda_1\in\Lambda_1}\sum_{\lambda_0\in\Lambda_0} |\langle\varphi, L_{\lambda_1}L_{\lambda_0}\varphi\rangle|^2 \leqslant B\|\varphi\|^2.$$

□

Next, we observe that the orthogonality condition (6.4.1) transfers the nature of the reproducing formula of Λ-translation generated systems to the Λ_0-translation generated systems.

Proposition 6.4.4. *Let $\varphi, \psi \in L^2(G)$ be two functions such that φ and ψ satisfy the orthogonality conditions $(\mathcal{O}_\varphi)$ and $(\mathcal{O}_\psi)$ mentioned in (6.4.1), respectively, then the following are equivalent:*

(i) $\langle f, g\rangle = \sum_{\lambda\in\Lambda}\langle f, L_\lambda\psi\rangle\,\langle L_\lambda\varphi, g\rangle$ *for all* $f, g \in \mathcal{S}^\Lambda(\varphi)$.

(ii) $\langle f, g\rangle = \sum_{\lambda_0\in\Lambda_0}\langle f, L_{\lambda_0}\psi\rangle\,\langle L_{\lambda_0}\varphi, g\rangle$ *for all* $f, g \in \mathcal{S}^{\Lambda_0}(\varphi)$.

Proof. First we note that the summations used in (i) and (ii) are well defined in view of Proposition 6.4.3, since $\mathcal{E}^\Lambda(\varphi)$ and $\mathcal{E}^\Lambda(\psi)$ are Bessel, and φ and ψ satisfy the orthogonality conditions $(\mathcal{O}_\varphi)$ and $(\mathcal{O}_\psi)$, respectively.

Now assume (ii) holds. By choosing $\varphi_{\lambda_1}, \psi_{\lambda_1} \in \mathcal{S}^{\Lambda_0}(L_{\lambda_1}\varphi)$ such that $f = \sum_{\lambda_1\in\Lambda_1}\varphi_{\lambda_1}, g = \sum_{\lambda_1\in\Lambda_1}\psi_{\lambda_1}$ and using Proposition (6.4.2), we have

$$\begin{aligned}\sum_{\lambda\in\Lambda}\langle f, L_\lambda\psi\rangle\langle L_\lambda\varphi, g\rangle &= \sum_{\eta_1\in\Lambda_1}\sum_{\eta_0\in\Lambda_0}\left\langle \sum_{\lambda_1\in\Lambda_1}\varphi_{\lambda_1}, L_{\eta_1}L_{\eta_0}\psi\right\rangle\left\langle L_{\eta_1}L_{\eta_0}\varphi, \sum_{\lambda_1\in\Lambda_1}\psi_{\lambda_1}\right\rangle\\ &= \sum_{\lambda_1\in\Lambda_1}\sum_{\eta_0\in\Lambda_0}\left\langle L_{\lambda_1^{-1}}\varphi_{\lambda_1}, L_{\eta_0}\psi\right\rangle\left\langle L_{\eta_0}\varphi, L_{\lambda_1^{-1}}\psi_{\lambda_1}\right\rangle.\end{aligned}$$

Since $L_{\lambda_1^{-1}}\varphi_{\lambda_1},\ L_{\lambda_1^{-1}}\psi_{\lambda_1} \in \mathcal{S}^{\Lambda_0}(\varphi)$, we obtain the following in view of the assumption (ii):

$$\sum_{\lambda\in\Lambda}\langle f, L_\lambda\psi\rangle\langle L_\lambda\varphi, g\rangle = \sum_{\lambda_1\in\Lambda_1}\left\langle L_{\lambda_1^{-1}}\varphi_{\lambda_1}, L_{\lambda_1^{-1}}\psi_{\lambda_1}\right\rangle = \sum_{\lambda_1\in\Lambda_1}\langle\varphi_{\lambda_1}, \psi_{\lambda_1}\rangle = \langle f, g\rangle.$$

Thus, (i) holds. Conversely, assume (i) holds. For $f, g \in \mathcal{S}^{\Lambda_0}(\varphi)$, we have $\langle f, L_{\lambda_1}L_{\lambda_0}\varphi\rangle = 0$ and $\langle g, L_{\lambda_1}L_{\lambda_0}\varphi\rangle = 0$ for all $\lambda_1 \neq 0 \in \Lambda_1$, by Proposition 6.4.2. Then the result follows immediately. □

Now we state our main result for a Λ-translation generated system in $L^2(G)$ to form reproducing formula. Unlike the case of the Euclidean setup, we observe that a necessary condition is involved related to the orthogonality of Λ-translation generated system of functions.

Theorem 6.4.5. *Let $\varphi, \psi \in L^2(G)$ be two functions such that they satisfy the orthogonality conditions $(\mathcal{O}_\varphi)$ and $(\mathcal{O}_\psi)$ mentioned in (6.4.1). If $\mathcal{E}^\Lambda(\psi)$ is biorthogonal to $\mathcal{E}^\Lambda(\varphi)$, the following reproducing formula holds true:*

$$f = \sum_{\lambda\in\Lambda}\langle f, L_\lambda\psi\rangle L_\lambda\varphi \text{ for all } f \in \mathcal{S}^\Lambda(\varphi).$$

Proof. Since $\mathcal{E}^{\Lambda}(\varphi)$ and $\mathcal{E}^{\Lambda}(\psi)$ are biorthogonal, we have $\langle \varphi, L_{\lambda_0}\psi\rangle = \delta_{\lambda_0,0}$ for $\lambda_0 \in \Lambda_0$, and hence for $f \in \operatorname{span} \mathcal{E}^{\Lambda_0}(\varphi)$, we write $f = \sum_{\lambda_0 \in \Lambda_0'} \langle f, L_{\lambda_0}\psi\rangle L_{\lambda_0}\varphi$ for some finite subset Λ_0' of Λ_0. Since the function $f \mapsto \sum_{\lambda_0\in\Lambda_0}\langle f, L_{\lambda_0}\psi\rangle L_{\lambda_0}\varphi$ is continuous, the expansion of f holds for all $f \in \mathcal{S}^{\Lambda_0}(\varphi)$. Thus the result follows by Proposition 6.4.4. □

Next, we discuss reproducing formula for a Λ-translation generated system and find an easily verifiable condition to satisfy the reproducing formula.

Theorem 6.4.6. *Let $\varphi, \psi \in L^2(G)$ be such that φ and ψ satisfy the orthogonality conditions $(\mathcal{O}_\varphi)$ and $(\mathcal{O}_\psi)$ mentioned in (6.4.1), respectively, then the following are equivalent:*

(i) $f = \sum_{\lambda\in\Lambda}\langle f, L_\lambda\psi\rangle L_\lambda\varphi$ *for all* $f \in \mathcal{S}^{\Lambda}(\varphi)$.

(ii) $[\varphi,\psi](\alpha) = 1$ *a.e.* $\alpha \in \Omega_\varphi$.

Proof. Let us assume (i). Equivalently, $f = \sum_{\lambda_0\in\Lambda_0}\langle f, L_{\lambda_0}\psi\rangle L_{\lambda_0}\varphi$ for all $f \in \mathcal{S}^{\Lambda_0}(\varphi)$ by Proposition 6.4.4. Applying the map $\mathscr{F}$ on the both sides for a.e. $\alpha \in \mathbb{T}^r$ and using the relation $(\mathscr{F}L_{\lambda_0}\varphi)(\alpha) = e^{-2\pi i\langle\alpha,\lambda_0\rangle}\mathscr{F}\varphi(\alpha)$, we have

$$\begin{aligned}\mathscr{F}\left(\sum_{\lambda_0\in\Lambda_0}\langle f, L_{\lambda_0}\psi\rangle L_{\lambda_0}\varphi\right)(\alpha) &= \sum_{\lambda_0\in\Lambda_0}\langle f, L_{\lambda_0}\psi\rangle(\mathscr{F}L_{\lambda_0}\varphi)(\alpha) = (\mathscr{F}\varphi)(\alpha)\sum_{\lambda_0\in\Lambda_0}\langle f, L_{\lambda_0}\psi\rangle e^{-2\pi i\langle\alpha,\lambda_0\rangle}\\ &= (\mathscr{F}\varphi)(\alpha)\sum_{\lambda_0\in\Lambda_0}\int_{\mathbb{T}^r}\langle\mathscr{F}f(\alpha), \mathscr{F}L_{\lambda_0}\psi(\alpha)\rangle\, d\alpha\; e^{-2\pi i\langle\alpha,\lambda_0\rangle}\\ &= (\mathscr{F}\varphi)(\alpha)\sum_{\lambda_0\in\lambda_0}\int_{\mathbb{T}^r} e^{2\pi i\langle\alpha,\lambda_0\rangle}[f,\psi](\alpha)\, d\alpha\; e^{-2\pi i\langle\alpha,\lambda_0\rangle}\\ &= [f,\psi](\alpha)\;(\mathscr{F}\varphi)(\alpha) \text{ for a.e. } \alpha\in\mathbb{T}^r,\end{aligned}$$

and hence we get $\mathscr{F}f(\alpha) = \mathscr{F}\left(\sum_{\lambda_0\in\Lambda_0}\langle f, L_{\lambda_0}\psi\rangle L_{\lambda_0}\varphi\right)(\alpha) = (\mathscr{F}\varphi)(\alpha)[f,\psi](\alpha)$ for all $f \in \mathcal{S}^{\Lambda_0}(\varphi)$. By choosing $f = \varphi$, we have $(\mathscr{F}\varphi)(\alpha)\,(1 - [\varphi,\psi](\alpha)) = 0$ for a.e. $\alpha \in \mathbb{T}^r$. Therefore, we get $[\varphi,\psi](\alpha) = 1$ for a.e. $\alpha \in \Omega_\varphi$.

Conversely, assume (ii), i.e., $[\varphi,\psi](\alpha) = 1$ a.e. $\alpha \in \Omega_\varphi$. Then it is enough to show $f = \sum_{\lambda_0\in\Lambda_0}\langle f, L_{\lambda_0}\psi\rangle L_{\lambda_0}\varphi$ for all $f \in \mathcal{S}^{\Lambda_0}(\varphi)$ in view of the Proposition 6.4.4. Since the function $f \mapsto \sum_{\lambda_0\in\Lambda_0}\langle f, L_{\lambda_0}\psi\rangle L_{\lambda_0}\varphi$ from $\mathcal{S}^{\Lambda_0}(\varphi)$ to $L^2(G)$ is continuous, it suffices to show the result for $f = L_\eta\varphi$, $\eta \in \Lambda_0$. Therefore the result follows in view of the calculations

$$\begin{aligned}\mathscr{F}\left(\sum_{\lambda\in\Lambda}\langle L_\eta\varphi, L_\lambda\psi\rangle L_\lambda\varphi\right)(\alpha) &= (\mathscr{F}\varphi)(\alpha)[(L_\eta\varphi),\psi](\alpha) = e^{2\pi i\langle\alpha,\eta\rangle}(\mathscr{F}\varphi)(\alpha)[\varphi,\psi](\alpha)\\ &= \mathscr{F}(L_\eta\varphi)(\alpha)[\varphi,\psi](\alpha) = \mathscr{F}(L_\eta\varphi)(\alpha).\end{aligned}$$

□

6.5. Gabor system and Heisenberg group

We now discuss a reproducing formula for the Heisenberg group $\mathbb{H}^d$ associated with the orthonormal Gabor systems of $L^2(\mathbb{R}^d)$ using Theorem 6.4.6. For $y \in \mathbb{R}^*$, we define functions v_y and w_y from [12], such that $v_y(x) = |y|^{d/2}v(yx)$ and $w_y(x) = |y|^{d/2}w(yx)$, $x \in \mathbb{R}^d$, where $v, w \in L^2(\mathbb{R}^d)$ with $\|v\| = 1, \|w\| = 1$. Corresponding to v_y and w_y, we consider the rank one projection operators $\mathcal{P}_y$ and $\mathcal{Q}_y$ defined as follows:

$$\mathcal{P}_y = v_y \otimes v_y : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \text{ by } \quad f \to \langle f, v_y\rangle v_y,$$

$$\text{and } \mathcal{Q}_y = w_y \otimes w_y : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \text{ by } \quad f \to \langle f, w_y\rangle w_y.$$

Next, for all $t \in (0,1)$ we define

$$\mathscr{H}_t(y) = \begin{cases} \mathcal{P}_y & \text{for } y \in (t,1], \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \mathscr{G}_t(y) = \begin{cases} \mathcal{Q}_y & \text{for } y \in (t,1], \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathscr{H}_t, \mathscr{G}_t \in L^2(\mathbb{R}^*, \mathcal{HS}(L^2(\mathbb{R}^d))|\lambda|^d d\lambda)$ since

$$\|\mathscr{H}_t(y)\|_{\mathcal{HS}} = \begin{cases} 1 & \text{for } y \in (t,1], \\ 0, & \text{otherwise,} \end{cases} = \|\mathscr{G}_t(y)\|_{\mathcal{HS}},$$

and hence for each $t \in (0,1)$, $\varphi_t, \psi_t \in L^2(\mathbb{H}^d)$, where $\varphi_t = \mathcal{F}^{-1}\mathscr{H}_t$ and $\psi_t = \mathcal{F}^{-1}\mathscr{G}_t$.

Theorem 6.5.1. *Let $A, B \in GL(d,\mathbb{R})$ such that $AB^t \in \mathbb{Z}$. For $0 < \alpha \leqslant 1$, let the Gabor systems*

$$\{\pi_\alpha(m,n)v_\alpha : (m,n) \in A\mathbb{Z}^d \times B\mathbb{Z}^d\}, \text{ and } \{\pi_\alpha(m,n)w_\alpha : (m,n) \in A\mathbb{Z}^d \times B\mathbb{Z}^d\}$$

be orthonormal in $L^2(\mathbb{R}^d)$. Then for each $0 < t < \alpha \leqslant 1$, the systems $\mathcal{E}^\Lambda(\varphi_t)$ and $\mathcal{E}^\Lambda(\psi_t)$ satisfy the reproducing formula

$$f = \sum_{\lambda\in\Lambda}\langle f, L_\lambda\psi_t\rangle L_\lambda\varphi_t \text{ for all } f \in \mathcal{S}^\Lambda(\varphi_t) \text{ if and only if } |\langle v_\alpha, w_\alpha\rangle_{L^2(\mathbb{R}^d)}| = \frac{1}{\alpha^{d/2}},$$

where $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d \times \mathbb{Z}$.

Proof. Observe that for each $0 < \alpha \leqslant 1$, $0 < t < \alpha \leqslant 1$ and $\lambda_1 \in \Lambda_1 = A\mathbb{Z}^d \times B\mathbb{Z}^d$, we have

$$\begin{aligned}
[L_{\lambda_1}\varphi_t, \varphi_t](\alpha) &= \sum_{m\in\mathbb{Z}} \langle \mathcal{F}L_{\lambda_1}\varphi_t(\alpha+m), \mathcal{F}\varphi_t(\alpha+m)\rangle_{\mathcal{HS}} |\alpha+m|^d \\
&= \sum_{m\in\mathbb{Z}} \langle \pi_{\alpha+m}(\lambda_1)\mathscr{H}_t(\alpha+m), \mathscr{H}_t(\alpha+m)\rangle_{\mathcal{HS}} |\alpha+m|^d \\
&= \langle \pi_\alpha(\lambda_1)\mathscr{H}_t(\alpha), \mathscr{H}_t(\alpha)\rangle_{\mathcal{HS}} |\alpha|^d = \langle \pi_\alpha(\lambda_1)v_\alpha, v_\alpha\rangle_{L^2(\mathbb{R}^d)} |\alpha|^d
\end{aligned}$$

since $\pi_\alpha(\lambda_1)\mathcal{P}_\alpha = (\pi_\alpha(\lambda_1)\mathcal{P}_\alpha v_\alpha) \otimes v_\alpha$ and

$$\langle \pi_\alpha(\lambda_1)\mathscr{H}_t(\alpha), \mathscr{H}_t(\alpha)\rangle_{\mathcal{HS}} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \overline{v_\alpha(s)}(\pi_\alpha(\lambda_1)v_\alpha(w))\overline{v_\alpha(w)}v_\alpha(s)\, ds\, dw = \langle \pi_\alpha(\lambda_1)v_\alpha, v_\alpha\rangle_{L^2(\mathbb{R}^d)}.$$

Similarly, we can calculate $[L_{\lambda_1}\psi_t, \psi_t](\alpha) = \langle \pi_\alpha(\lambda_1)w_\alpha, w_\alpha\rangle_{L^2(\mathbb{R}^d)}|\alpha|^d$. Therefore for each $0 < t < \alpha \leqslant 1$, the functions φ_t and ψ_t satisfy orthogonality conditions $(\mathcal{O}_{\varphi_t})$ and $(\mathcal{O}_{\psi_t})$ (mentioned in (6.4.1)) due to the orthonormal Gabor systems $\{\pi_\alpha(\lambda_1)v_\alpha : \lambda_1 \in \Lambda_1\}$ and $\{\pi_\alpha(\lambda_1)w_\alpha : \lambda_1 \in \Lambda_1\}$.

Further, observe that $\mathcal{E}^\Lambda(\varphi_t)$ is Bessel since $[\varphi_t, \varphi_t](\alpha)$ is bounded above follows by noting

$$\begin{aligned}
[\varphi_t, \varphi_t](\alpha) &= \sum_{m\in\mathbb{Z}} \|\mathcal{F}\varphi_t(\alpha+m)\|^2_{\mathcal{HS}} |\alpha+m|^d \\
&= \sum_{m\in\mathbb{Z}} \|\mathscr{H}_t(\alpha+m)\|^2_{\mathcal{HS}} |\alpha+m|^d \\
&= \|\mathscr{H}_t(\alpha)\|^2 |\alpha|^d \\
&= \begin{cases} \alpha^d, & t < \alpha \leqslant 1, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

In a similar way, we can obtain $\mathcal{E}^\Lambda(\psi_t)$ to be a Bessel system. Next we calculate $[\varphi_t, \psi_t](\alpha)$ as follows:

$$\begin{aligned}
[\varphi_t, \psi_t](\alpha) &= \sum_{m\in\mathbb{Z}} \langle \mathcal{F}\varphi_t(\alpha+m), \mathcal{F}\psi_t(\alpha+m)\rangle |\alpha+m|^d = \sum_{m\in\mathbb{Z}} \langle \mathscr{H}_t(\alpha+m), \mathscr{G}_t(\alpha+m)\rangle |\alpha+m|^d \\
&= \langle \mathscr{H}_t(\alpha), \mathscr{G}_t(\alpha)\rangle_{\mathcal{HS}} |\alpha|^d = |\alpha|^d \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \overline{v_\alpha(s)}v_\alpha(u)\overline{w_\alpha(u)}w_\alpha(s)\, ds\, du \\
&= |\langle v_\alpha, w_\alpha\rangle|^2 |\alpha|^d.
\end{aligned}$$

Hence the result follows from Theorem 6.4.6 provided $[\varphi_t, \psi_t](\alpha) = 1$ for $0 < \alpha \leqslant 1$ and $0 < t < \alpha \leqslant 1$. □

The properties of the TI spaces are very important in establishing the characteristics of dual frames but the researchers are also interested in looking for extra-invariant spaces. Next, we aim to characterize all $\Lambda_1\Lambda_0$-invariant space W to become $\Lambda_1\Theta$-invariant, where Θ is a closed subgroup of Z and $\Lambda_0 \subset \Theta$. This is known as the *extra-invariance* of a translation-invariant space W. In the context of a connected, simply connected nilpotent Lie group, whose representations are square-integrable modulo the center, we find characterization results of extra-invariant spaces under the left translations associated with the range functions [67]. Consequently, the theory is valid for the Heisenberg group $\mathbb{H}^d$, a 2-step nilpotent Lie group.

6.6. Extra invariances on Lie group

Translation-invariant spaces have enormous applications in sampling, approximation, wavelets, etc. Bownik in [16] characterized all $\mathbb{Z}^d$-invariant subspaces in $L^2(\mathbb{R}^d)$ followed by the works of Ron and Shen in [61]. For the locally compact abelian group setup, the theory of TI spaces were studied in [20, 22, 38]. Moving towards the non-abelian group setup, Currey et al. in [29] provided a characterization of all $\Lambda_1\Lambda_0$-invariant spaces using the range function for the SI/Z nilpotent Lie group.

Next we define an invariance set in the center Z of the SI/Z nilpotent Lie group G.

Definition 6.6.1. For a given $\Lambda_1\Lambda_0$-invariant subspace W, an *invariance set* Θ is defined by

$$\Theta = \{\lambda_0 \in Z : L_{\lambda_1\lambda_0} f \in W \text{ for all } \lambda_1 \in \Lambda_1 \text{ and } f \in W\}. \tag{6.6.1}$$

The set Θ is a closed subgroup of Z containing Λ_0. For this, let us consider a net $(\lambda_{0,\alpha})$ in Θ such that $\lim_\alpha \lambda_{0,\alpha} = \lambda_0$, say. Then we have $\lim_\alpha \|L_{\lambda_1\lambda_{0,\alpha}} f - L_{\lambda_1\lambda_{0,\alpha}} f\| = 0$ for $f \in W$ and each $\lambda_1 \in \Lambda_1$, and hence $\lambda_0 \in \Theta$ since W is a closed subspace. Therefore, Θ is a closed set. Since Θ is a semigroup of Z and the image of quotient map from Z to Z/Λ_0 on Θ is closed in Z/Λ_0 and hence compact, therefore, the group property of Θ follows from the fact that a compact semigroup of Z/Λ_0 is a group.

The main aim is to characterize all $\Lambda_1\Lambda_0$-invariant spaces W to become $\Lambda_1\Theta$-invariant, where Θ is a closed subgroup of Z and $\Lambda_0 \subset \Theta$. This is known as the *extra-invariance* of a translation-invariant space W. The current study of extra-invariance encompasses the non-abelian setup for a nilpotent Lie group which is considered a high degree of non-abelian structure. Consequently, the theory is valid for the d-dimensional Heisenberg group $\mathbb{H}^d$, a 2-step nilpotent Lie group. Shift-invariant spaces that are $\frac{1}{n}\mathbb{Z}$-invariant in $L^2(\mathbb{R})$ were completely characterized by Aldroubi et al. in [2] for the one-dimensional Euclidean case, and Anastasio et al. in [5,6] for higher dimensions and locally compact abelian groups. In this chain of researches we continue with various necessary and sufficient conditions under which a $\Lambda_1\Lambda_0$-invariant space becomes $\Lambda_1\Theta$-invariant in the context of nilpotent Lie group G whose representations are SI/Z type. The characterization results below are based on the Plancherel transform. Unlike the Euclidean and LCA group setup, the Plancherel transform of a function is operator-valued so that the techniques used in the Euclidean and LCA groups is restrained. We now state the main result.

Theorem 6.6.2. *Let Λ_0 be a uniform lattice in the center Z of G and Λ_1 be a discrete set lying outside the center Z containing the identity element e such that $\mathcal{J}$ and Σ are the Borel sections of ${\Lambda_0}^{\perp}/\Theta^{\perp}$ and $\widehat{Z}/{\Lambda_0}^{\perp}$, respectively, where Θ is a closed subgroup of Z containing Λ_0. If W is a $\Lambda_1\Lambda_0$-invariant subspace of $L^2(G)$, then it is $\Lambda_1\Theta$-invariant if and only if for each $j \in \mathcal{J}$, W contains V_j^{Θ}, where*

$$V_j^{\Theta} = \{f \in L^2(G) : \widehat{f} = \chi_{\mathcal{H}_j^{\Theta}}\widehat{g} \text{ with } g \in W\}, \text{ and } \mathcal{H}_j^{\Theta} := \Sigma + j + \Theta^{\perp}.$$

In this case, the space W can be decomposed as the orthogonal direct sum of V_j^{Θ}'s, i.e.,

$$W = \bigoplus_{j\in\mathcal{J}} V_j^{\Theta}.$$

6.6.1. Proof of the main result

Recall the $\Lambda_1\Lambda_0$-invariant subspace W from Definition 6.2.1, $\Lambda_1\Lambda_0$-invariant space $\mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A})$ from (1.3.1) generated by $\mathscr{A}$ and invariance set Θ from (6.6.1). Firstly, we concentrate for the properties of Θ. We observe the tilling property of Σ with respect to $\Theta^{\perp}$ in the following result.

Proposition 6.6.3. *For any section $\mathcal{J}$ of $\Lambda_0^{\perp}/\Theta^{\perp}$, the set $\Theta^{\perp} + \mathcal{J}$ is a tiling partner (Definition 6.1.1) of Σ for $\widehat{Z}$. That means, the collection $\{\mathcal{H}_j^{\Theta}\}_{j\in\mathcal{J}}$ is tiling of $\widehat{Z}$, where*

$$H_j^{\Theta} := \Sigma + j + \Theta^{\perp}. \tag{6.6.2}$$

Proof. The set Σ is a tiling set of $\widehat{Z}$, means, the collection $\{\Sigma + \lambda^* : \lambda^* \in \Lambda_0^\perp\}$ is a measurable partition of $\widehat{Z}$. Since $\widehat{Z/\Lambda_0} \cong \Lambda_0^\perp$ and Z/Λ_0 is compact, therefore $\Lambda_0^\perp$ is discrete and countable, and hence $\Theta^\perp$ is also discrete and countable follows from $\Theta^\perp \subset \Lambda_0^\perp$. Hence the collection $\{\Theta^\perp + j : j \in \mathcal{J}\}$ is a tiling of $\Lambda_0^\perp$ by considering a Borel section $\mathcal{J}$ of $\Lambda_0^\perp/\Theta^\perp$. Thus the result follows by employing the fact $\Lambda_0^\perp$ is tiling partner of Σ for $\widehat{Z}$ and $\mathcal{J}$ is a tiling partner of $\Theta^\perp$ for $\Lambda_0^\perp$. □

Example 6.6.4. For the Heisenberg group $\mathbb{H}^d$, the uniform lattice Λ_0 in the center $\mathbb{R}$ is the set of all integers $\mathbb{Z}$. Since the only proper closed additive subgroups of $\mathbb{R}$ containing $\mathbb{Z}$ are $\frac{1}{N}\mathbb{Z}$ for some natural number N, we consider the extra-invariance set $\Theta = \frac{1}{N}\mathbb{Z}$. Then the annihilators of Λ_0 and Θ are $\Lambda_0^\perp = \mathbb{Z}$ and $\Theta^\perp = N\mathbb{Z}$, respectively. Note that the set $\mathbb{R}$ can be tiled by a Borel section $\Sigma = [0,1)$ with the tiling partner $\Lambda_0^\perp = \mathbb{Z}$. By assuming the Borel section $\mathbb{Z}_N := \{0, 1, \ldots, N-1\}$ of $\Lambda_0^\perp/\Theta^\perp = \mathbb{Z}/N\mathbb{Z}$, the set $n\mathbb{Z} + \mathbb{Z}_N$ is a tiling partner of $[0,1)$ for $\mathbb{R}$, that means, the collection $\{\mathcal{H}_n^{\frac{1}{N}\mathbb{Z}}\}_{n=0}^{N-1}$ is tiling of $\mathbb{R}$, where

$$\mathcal{H}_n^{\frac{1}{N}\mathbb{Z}} = [0,1) + n + N\mathbb{Z} = \bigcup_{k\in\mathbb{Z}} [n, n+1) + Nk.$$

For the case of Heisenberg group $\mathbb{H}^d$, we consider the set $\Lambda_1 = A\mathbb{Z}^d \times B\mathbb{Z}^d$ from outside of the center of $\mathbb{H}^d$, where $A, B \in GL(d, \mathbb{R})$ such that $AB^t \in \mathbb{Z}$.

Employing the Proposition 6.2.2, we characterize a member of $\mathcal{S}^{\Lambda_1\Lambda_0}(\varphi)$ with the help of Plancherel transform.

Proposition 6.6.5. *For $f \in \mathcal{S}^{\Lambda_1\Lambda_0}(\varphi)$, the Plancherel transform of f is given by*

$$\mathcal{F}f(\omega) = \sum_{\lambda_1\in\Lambda_1} \beta_{\lambda_1}(\omega)\mathcal{F}(L_{\lambda_1}\varphi)(\omega) \text{ a.e. } \omega \in \mathfrak{z}^*, \tag{6.6.3}$$

where β_{λ_1} is a ${\Lambda_0}^\perp$–periodic function. Conversely if β_{λ_1} is an ${\Lambda_0}^\perp$-periodic function such that

$$\sum_{\lambda_1\in\Lambda_1} \beta_{\lambda_1}(\cdot)\mathcal{F}(L_{\lambda_1}\varphi)(\cdot) \in L^2(\mathfrak{z}^*; \mathcal{HS}(L^2(\mathbb{R}^d))),$$

then the function f defined by (6.6.3) is a member of $\mathcal{S}^{\Lambda_1\Lambda_0}(\varphi)$.

Proof. Applying the Plancherel transform followed by periodization $\mathscr{F}$ on a function $f \in \mathcal{S}^{\Lambda_1\Lambda_0}(\varphi)$, we get

$$\begin{aligned} \mathscr{F} f(\sigma) &= \mathscr{F}(Pf)(\sigma) = P(\sigma)\mathscr{F} f(\sigma) \\ &= \sum_{\lambda_1\in\Lambda_1} \frac{\langle \mathscr{F} f(\sigma), \mathscr{F}(L_{\lambda_1}\varphi)(\sigma)\rangle}{\|\mathscr{F}(L_{\lambda_1}\varphi)(\sigma)\|^2}\mathscr{F}(L_{\lambda_1}\varphi)(\sigma) \text{ a.e. } \sigma \in \Sigma, \end{aligned} \tag{6.6.4}$$

in view of Proposition 6.2.2 and commutativity of P and $\mathscr{F}$, where P and $P(\sigma)$ are orthogonal projections on $\mathcal{S}^{\Lambda_1\Lambda_0}(\varphi)$ and $J(\sigma)$, respectively. The above expression (6.6.4) can be written as $\mathscr{F} f(\sigma) = \sum_{\lambda_1\in\Lambda_1} \beta_\lambda(\sigma)\mathscr{F} L_{\lambda_1}\varphi(\sigma)$ for a.e. $\sigma \in \Sigma$, where the $\Lambda_0^\perp$–periodic function β_{λ_1} is defined by

$$\beta_{\lambda_1}(\sigma) = \begin{cases} \frac{\langle \mathscr{F} f(\sigma), \mathscr{F}(L_{\lambda_1}\varphi)(\sigma)\rangle}{\|\mathscr{F}(L_{\lambda_1}\varphi)(\sigma)\|^2} = \sum_{\lambda^*\in\Lambda_0^\perp} \frac{\langle \mathscr{F} f(\sigma)(\lambda^*), \mathscr{F}(L_{\lambda_1}\varphi)(\sigma)(\lambda^*)\rangle}{\|\mathscr{F}(L_{\lambda_1}\varphi)(\sigma)\|^2}, & \sigma \in \Sigma_\varphi(\sigma) = \\ & \left\{\sigma \in \Sigma : \|\mathscr{F}(L_{\lambda_1}\varphi)(\sigma)\|^2 \neq 0\right\}, \\ 0, & \text{otherwise.} \end{cases}$$

The function β_{λ_1} can be extended periodically on $\widehat{Z}$ since $\Lambda_0^\perp$ is a tiling partner of Σ for $\widehat{Z}$. Also observe that for any $w \in \widehat{Z}$, there exists unique $\sigma \in \Sigma$, $\lambda^* \in \Lambda_0^\perp$ such that $w = \sigma + \lambda^*$, and hence from (6.6.4), we obtain

$$\begin{aligned} \mathcal{F} f(w)|\mathbf{Pf}(\omega)| = \mathcal{F} f(\sigma + \lambda^*)|\mathbf{Pf}(\sigma + \lambda^*)| &= \mathscr{F} f(\sigma)(\lambda^*) = \sum_{\lambda_1\in\Lambda_1} \beta_{\lambda_1}(\sigma)\mathscr{F}(L_{\lambda_1}\varphi)(\sigma)(\lambda^*) \\ &= \sum_{\lambda_1\in\Lambda_1} \beta_{\lambda_1}(\sigma + \lambda^*)\mathcal{F}(L_{\lambda_1}\varphi)(\sigma)(\lambda^*)|\mathbf{Pf}(\sigma + \lambda^*)| \\ &= \sum_{\lambda_1\in\Lambda_1} \beta_{\lambda_1}(w)\mathcal{F}(L_{\lambda_1}\varphi)(w)|\mathbf{Pf}(\omega)|. \end{aligned}$$

The converse part follows from the above calculations by writing $\mathcal{F} f(\cdot) = \sum_{\lambda_1\in\Lambda_1} \beta_{\lambda_1}(\cdot)\mathcal{F}(L_{\lambda_1}\varphi)(\cdot)$ in the form $\mathscr{F} f(\sigma) = \sum_{\lambda_1\in\Lambda_1} \beta_{\lambda_1}(\sigma)\mathscr{F}(L_{\lambda_1}\varphi)(\sigma)$, and noting $\mathscr{F} f(\sigma) \in J(\sigma)$ gives $f \in \mathcal{S}^{\Lambda_1\Lambda_0}(\varphi)$ from Proposition 6.2.2. □

In the present section, our first goal is to prove Theorem 6.6.2 which characterizes invariant subspaces of $L^2(G)$ with the action of Θ in the center Z containing the uniform lattice Λ_0. The following lemma plays a crucial role to establish the Theorem 6.6.2.

Lemma 6.6.6. *Let W be a $\Lambda_1\Lambda_0$-invariant subspace of $L^2(G)$ and let $\mathcal{J}$ be a Borel section of $\Lambda_0^\perp/\Theta^\perp$. For each $j \in \mathcal{J}$, consider the space V_j^Θ given by*

$$V_j^\Theta = \{f \in L^2(G) : \widehat{f} = \chi_{\mathcal{H}_j^\Theta}\widehat{g} \text{ for some } g \in W\}, \tag{6.6.5}$$

where $\mathcal{H}_j^\Theta$ is defined in (6.6.2). If $V_j^\Theta \subset W$, it is a $\Lambda_1\Theta$-invariant (and hence $\Lambda_1\Lambda_0$-invariant) subspace of $L^2(G)$.

Proof. For $j \in \mathcal{J}$, let us consider the space $V_j^\Theta = \{f \in L^2(G) : \widehat{f} = \chi_{\mathcal{H}_j^\Theta}\widehat{g} \text{ for some } g \in W\}$, where $\mathcal{H}_j^\Theta = \Sigma + j + \Theta^\perp$. To prove it as a $\Lambda_1\Theta$-invariant subspace of $L^2(G)$, we first assume a sequence (φ_k) in V_j^Θ converging to $\varphi \in L^2(G)$. Then $\varphi \in W$ since $V_j^\Theta \subset W$ and W is closed, and hence $\varphi \in V_j^\Theta$. This follows by writing $\widehat{\varphi} = \chi_{\mathcal{H}_j^\Theta}\widehat{\varphi}$ since $\|\varphi_k - \varphi\| \to 0$ implies $\widehat{\varphi}\chi_{(\mathcal{H}_j^\Theta)^c} = 0$ from

$$\|\varphi_k - \varphi\|^2 \geqslant \|(\widehat{\varphi}_k - \widehat{\varphi})\chi_{(\mathcal{H}_j^\Theta)^c}\|^2 \geqslant \|\widehat{\varphi}\chi_{(\mathcal{H}_j^\Theta)^c}\|^2, \text{ where c denotes complement of the set.}$$

Therefore, V_j^Θ is closed. Further we observe that if $f \in V_j^\Theta$, then $\widehat{f} = \chi_{\mathcal{H}_j^\Theta}\widehat{g}$ for some $g \in W$, and hence for $\theta \in \Theta$ and $\lambda_1 \in \Lambda_1$, we can write $e^{2\pi i\langle\omega,\theta\rangle}\mathcal{F}(L_{\lambda_1}f)(\omega) = \chi_{\mathcal{H}_j^\Theta}(\omega)e^{2\pi i\langle\omega,\theta\rangle}\mathcal{F}(L_{\lambda_1}g)(\omega)$ for $\omega \in \mathfrak{z}^*$ since $\mathcal{F}(L_{\lambda_1}g)(\omega) = \pi_\omega(\lambda_1)\widehat{g}(\omega)$. For the $\Lambda_1\Theta$-invariant it suffices to show $e^{2\pi i\langle\omega,\theta\rangle}\mathcal{F}(L_{\lambda_1}g)(\omega) \in \mathcal{F}(W)$ that gives $e^{2\pi i\langle\omega,\theta\rangle}\mathcal{F}(L_{\lambda_1}f)(\omega) \in \mathcal{F}(V_j^\Theta)$. Observe that $e^{2\pi i\langle\omega,\theta\rangle}\mathcal{F}(L_{\lambda_1}g)(\omega) \in \mathcal{F}(\mathcal{S}^{\Lambda_1\Lambda_0}(g)) \subset \mathcal{F}(W)$ due to the converse part of Proposition 6.6.5, provided $t_\theta(\omega) := e^{2\pi i\langle\omega,\theta\rangle}$ a. e. $\omega \in \mathcal{H}_j^\Theta$, is a $\Lambda_0^\perp$-periodic function. Since $e^{2\pi i\langle\cdot,\theta\rangle}$ is $\Theta^\perp$-periodic we have $e^{2\pi i\langle\sigma+j,\theta\rangle} = e^{2\pi i\langle\sigma+j+\theta^*,\theta\rangle}$ for a.e $\sigma \in \Sigma$, $j \in \mathcal{J}$, and for every $\theta^* \in \Theta^\perp$, and then for each $\lambda^* \in \Lambda_0^\perp$ we define $t_\theta(\sigma + \lambda^*) = e^{2\pi i\langle\sigma+j,\theta\rangle}$ a.e. $\sigma \in \Sigma$. Thus the function t_θ is $\Lambda_0^\perp$-periodic on Σ, can be extended to $\mathfrak{z}*$ since $\Lambda_0^\perp$ is tiling partner of Σ for $\mathfrak{z}^* \cong \widehat{Z}$. □

Proof of Theorem 6.6.2. For each $j \in \mathcal{J}$, if $V_j^\Theta \subset W$, the space V_j^Θ is $\Lambda_1\Theta$-invariant from Lemma 6.6.6, and hence the space $\bigoplus_{j\in\mathcal{J}} V_j^\Theta \subset W$ is so. Since $\{\mathcal{H}_j^\Theta\}_{j\in\mathcal{J}}$ is a tiling of $\widehat{Z} \cong \mathfrak{z}^*$, therefore any element $f \in W$ can be written as $\widehat{f}(\omega) = \sum_{j\in\mathcal{J}}\widehat{g}_j(\omega)$ a.e. $\omega \in \mathfrak{z}^*$, where $\widehat{g}_j = \widehat{f}\chi_{\mathcal{H}_j^\Theta}$. By the definition of V_j^Θ, $g_j \in V_j^\Theta$ for every $j \in \mathcal{J}$ and hence $f \in \bigoplus_{j\in\mathcal{J}} V_j^\Theta$. Therefore, W is $\Lambda_1\Theta$-invariant.

Conversely, let us assume that the $\Lambda_1\Lambda_0$-invariant space W is $\Lambda_1\Theta$-invariant. For $V_j^\Theta \subset W$, we choose $f \in V_j^\Theta$. Then, we have $\widehat{f} = \chi_{\mathcal{H}_j^\Theta}\widehat{g}$ for some $g \in W$. Employing the Plancherel transform followed by periodization (Proposition 6.1.2) $\mathscr{F}^\Theta$ from $L^2(G)$ to $L^2(\mathcal{D}, \ell^2(\Theta^\perp, \mathcal{HS}(L^2(\mathbb{R}^d)))$ given by

$$\mathscr{F}^\Theta f(\delta)(\theta^*) = \mathcal{F}f(\delta + \theta^*)|\mathbf{Pf}(\delta + \theta^*)|^{1/2}, \; f \in L^2(G), \theta^* \in \Theta^\perp \text{ and a.e. } \delta \in \mathcal{D},$$

where $\mathcal{D}$ is the Borel section of $\widehat{Z}/\Theta^{\perp}$, we obtain

$$\mathscr{F}^{\Theta} f(\delta)(\theta^*) = \chi_{\mathcal{H}_j^{\Theta}}(\delta)\mathscr{F}^{\Theta} g(\delta)(\theta^*),$$

since $\chi_{\mathcal{H}_j^{\Theta}}$ is $\Theta^{\perp}$-periodic due to the definition of $\mathcal{H}_j^{\Theta}$ in (6.6.2). Then for a.e. $\delta \in \mathcal{D}$, we have $\mathscr{F}^{\Theta}(L_e g)(\delta) \in J^{\Theta}(\delta)$, where $J^{\Theta}(\delta) = \overline{\text{span}}\{\mathscr{F}^{\Theta}(L_{\lambda_1} g)(\delta) : \lambda_1 \in \Lambda_1\}$, and hence $\mathscr{F}^{\Theta} f(\delta) \in J^{\Theta}(\delta)$ for a.e. $\delta \in \mathcal{D}$. Thus $f \in \mathcal{S}^{\Lambda_1\Theta}(g) \subset W$ due to Proposition 6.2.2. □

As a consequence, we find the below characterization result for $\mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A})$ to become $\Lambda_1\Theta$-invariant using the associated range function.

Theorem 6.6.7. *In addition to the hypotheses of Theorem 6.6.2, let $\mathscr{A}$ be a sequence of functions in $L^2(G)$. Then $\mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A})$ is a $\Lambda_1\Theta$-invariant if and only if the Plancherel transform followed by periodization $\mathscr{F}$ satisfies*

$$\mathscr{F}(L_{\lambda_1}\varphi^j)(\sigma) \in J(\sigma) \text{ a.e. } \sigma \in \Sigma, \text{ for all } j \in \mathcal{J} \text{ and } \lambda_1 \in \Lambda_1,$$

where the associated range function $J(\sigma) = \overline{\text{span}}\{\mathscr{F}(L_{\lambda_1}\varphi)(\sigma) : \varphi \in \mathscr{A}, \lambda_1 \in \Lambda_1\}$, and $\widehat{\varphi}^j = \widehat{\varphi}\chi_{\mathcal{H}_j^{\Theta}}$.

Proof. For $j \in \mathcal{J}$ assume $V_j^{\Theta} = \{f \in L^2(G) : \widehat{f} = \chi_{\mathcal{H}_j^{\Theta}}\widehat{g} \text{ for some } g \in W\}$, and $W_j = \{f \in L^2(G) : \text{supp}(\widehat{f}) \subset \mathcal{H}_j^{\Theta}\}$, where $\mathcal{H}_j^{\Theta} = \Sigma + j + \Theta^{\perp}$. Let P_j be the orthogonal projection on W_j. Then

$$P_j(\mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A})) = \{f^j : \widehat{f^j} = \widehat{f}\chi_{\mathcal{H}_j^{\Theta}}, f \in \mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A})\} = V_j^{\Theta},$$

whose associated range function is $J_{V_j^{\Theta}}(\sigma) = \overline{\text{span}}\{\mathscr{F}(L_{\lambda_1}\varphi^j)(\sigma) : \varphi \in \mathscr{A}, \lambda_1 \in \Lambda_1, \widehat{\varphi}^j = \widehat{\varphi}\chi_{\mathcal{H}_j^{\Theta}}\}$ for a.e. $\sigma \in \Sigma$. Therefore from Theorem 6.6.2, $\mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A})$ is a $\Lambda_1\Theta$-invariant if and only if $V_j^{\Theta} \subset \mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A})$ for each $j \in \mathcal{J}$. Further it is equivalent to $J_{V_j^{\Theta}}(\sigma) \subset J(\sigma)$ for a.e. $\sigma \in \Sigma$, for all $j \in \mathcal{J}$, where $J(\sigma)$ is the range function associated with $\mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A})$ [see Proposition 6.2.2 and [20]]. Thus the result follows. □

We further characterize this extra invariance using the dimension function. Given any $\Lambda_1\Lambda_0$-invariant subspace W of $L^2(G)$, we define the *dimension function* as

$$\dim_W : \Sigma \to \mathbb{N}_0 \bigcup \{\infty\} \text{ by } \dim_W(\sigma) := \dim(J_W(\sigma)) \text{ for a.e. } \sigma \in \Sigma,$$

where $J_W(\sigma)$ is the range function associated with W.

Theorem 6.6.8. *Under the standing hypotheses of Theorem 6.6.2, the $\Lambda_1\Lambda_0$-invariant space W is $\Lambda_1\Theta$-invariant if and only if the dimension function satisfies the following relation*

$$\dim_W(\sigma) = \sum_{j\in\mathcal{J}} \dim_{V_j^\Theta}(\sigma) \ \textit{a.e. } \sigma \in \Sigma.$$

Proof. From Theorem 6.6.2, we have $W = \bigoplus_{j\in\mathcal{J}} V_j^\Theta$ if W is $\Lambda_1\Theta$-invariant. Then for a.e. $\sigma \in \Sigma$, the range function satisfies $J_W(\sigma) = \bigoplus_{j\in\mathcal{J}} J_{V_j^\Theta}(\sigma)$, follows by observing the orthogonality of $J_{V_j^\Theta}(\sigma)$ and $J_{V_{j'}^\Theta}(\sigma)$ for $j \neq j'$, since $\{\mathcal{H}_j^\Theta\}_{j\in\mathcal{J}}$ is a tiling of $\widehat{Z} \cong \mathfrak{z}^*$. Hence $\dim_W(\sigma) = \sum_{j\in\mathcal{J}} \dim_{V_j^\Theta}(\sigma)$ a.e. $\sigma \in \Sigma$.

For the converse part, first observe that the $\Lambda_1\Lambda_0$-invariant space W is contained in $\bigoplus_{j\in\mathcal{J}} V_j^\Theta$. This follows by writing $f \in W$ as $\widehat{f}(\omega) = \sum_{j\in\mathcal{J}} \widehat{g}_j(\omega)$ a.e. $\omega \in \mathfrak{z}^*$, where $\widehat{g}_j = \widehat{f}\chi_{\mathcal{H}_j^\Theta}$ since $\{\mathcal{H}_j^\Theta\}_{j\in\mathcal{J}}$ is a tiling of $\widehat{Z} \cong \mathfrak{z}^*$. Then the range function satisfies $J_W(\sigma) \subset \bigoplus_{j\in\mathcal{J}} J_{V_j^\Theta}(\sigma)$, and hence we have $J_W(\sigma) = \bigoplus_{j\in\mathcal{J}} J_{V_j^\Theta}(\sigma)$ for a.e. $\sigma \in \Sigma$ due to the condition $\dim_W(\sigma) = \sum_{j\in\mathcal{J}} \dim_{V_j^\Theta}(\sigma)$ a.e. $\sigma \in \Sigma$. Therefore we get $J_{V_j^\Theta}(\sigma) \subset J_W(\sigma)$ for each $j \in \mathcal{J}$, i.e, $V_j^\Theta \subset W$ for all j. Thus W is $\Lambda_1\Theta$-invariant follows from Theorem 6.6.2. □

The following result can be established easily for the d-dimensional Heisenberg group $\mathbb{H}^d$, a 2-step nilpotent Lie group, using Theorems 6.6.2, 6.6.7 and 6.6.8. In this case, the uniform lattice is $\Lambda_0 = \mathbb{Z}$, Λ_1 is a discrete set of the form $A\mathbb{Z}^d \times B\mathbb{Z}^d$, and the extra invariance set Θ is of the form $\frac{1}{N}\mathbb{Z}$ where $A, B \in GL(d, \mathbb{R})$ with $AB^t \in \mathbb{Z}$, and $N \in \mathbb{N}$.

Theorem 6.6.9. *Let $A, B \in GL(d, \mathbb{R})$ such that $AB^t \in \mathbb{Z}$ and let $N \in \mathbb{N}$. If W is an $A\mathbb{Z}^d \times B\mathbb{Z}^d \times \mathbb{Z}$-invariant subspace of $L^2(\mathbb{H}^d)$, then it is $A\mathbb{Z}^d \times B\mathbb{Z}^d \times \frac{1}{N}\mathbb{Z}$-invariant if and only if for each $n \in \mathbb{Z}_N := \{0, 1, 2, \ldots, N-1\}$, W contains $V_n^{\frac{1}{N}\mathbb{Z}}$, where*

$$(6.6.6) \qquad V_n^{\frac{1}{N}\mathbb{Z}} = \{f \in L^2(G) : \widehat{f} = \chi_{\mathcal{H}_n^{\frac{1}{N}\mathbb{Z}}}\widehat{g} \textit{ with } g \in W\}, \textit{ and } \mathcal{H}_n^{\frac{1}{N}\mathbb{Z}} = [n, n+1) + N\mathbb{Z}.$$

In this case, the space $W = \bigoplus_{n\in\mathbb{Z}_N} V_n^{\frac{1}{N}\mathbb{Z}}$, and $\dim_W(\xi) = \sum_{n\in\mathbb{Z}_N} \dim_{V_n^{\frac{1}{N}\mathbb{Z}}}(\xi)$ a.e. $\xi \in [0, 1)$.

As an application of the above results, the following consequence provides an estimate to measure the support of the Plancherel transform of a generator of $\mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A})$.

Theorem 6.6.10. *In addition to the hypotheses of Theorem 6.6.2, let $\mathscr{A} = \{\varphi_i\}_{i=1}^n \subset L^2(G)$ and Λ_1 be a finite set having cardinality k, i.e., $|\Lambda_1| = k$. If $\mathcal{S}^{\Lambda_1\Lambda_0}(\mathscr{A})$ is $\Lambda_1\Theta$-invariant, then the following inequality holds:*

$$\mu(\{\delta \in \mathcal{D} : \widehat{\varphi}_i(\delta) \neq 0\}) \leqslant \sum_{m=0}^{nk} m\ \mu(\Sigma_m) \leqslant nk \text{ for all } i \in \{1, 2, \ldots, n\}, \tag{6.6.7}$$

where $\mathcal{D}$ is the Borel section of $\widehat{Z}/\Theta^{\perp}$, $\Sigma_m = \{\sigma \in \Sigma : \dim_W(\sigma) = m\}$ and 0 is in the sense of the zero operator.

Proof. For $\theta^* \in \Theta^{\perp}$ and $\varphi \in \mathscr{A}$, we first estimate the measure of following set:

$$\begin{aligned}
&\mu(\{(\sigma, j) \in \Sigma \times \mathcal{J} : \widehat{\varphi}(\sigma + j + \theta^*) \neq 0\}) \\
&\quad= \mu\left(\{(\sigma, j) \in \Sigma \times \mathcal{J} : \tilde{\pi}_{\sigma+j+\theta^*}(\lambda_1)\widehat{\varphi}(\sigma + j + \theta^*) \neq 0\}\right) \text{for any } \lambda_1 \in \Lambda_1 \\
&\quad= \mu\left(\{(\sigma, j) \in \Sigma \times \mathcal{J} : \mathcal{F}(L_{\lambda_1}\varphi)(\sigma + j + \theta^*) \neq 0\}\right) \\
&\quad= \int_{\Sigma} \left|S_\sigma^{\mathcal{J}}\right|\ d\sigma,
\end{aligned}$$

where the set $S_\sigma^{\mathcal{J}} := \{j \in \mathcal{J} : \mathcal{F}(L_{\lambda_1}\varphi)(\sigma + j + \theta^*) \neq 0\}$ and $|S_\sigma^{\mathcal{J}}|$ denotes the cardinality of $S_\sigma^{\mathcal{J}}$. For a.e. $\sigma \in \Sigma$, the set $S_\sigma^{\mathcal{J}}$ is contained in the set $\{j \in \mathcal{J} : \dim_{V_j^\Theta}(\sigma) \neq 0\}$ since $\dim_{V_j^\Theta}(\sigma) = \dim J_{V_j^\Theta}(\sigma)$, where $J_{V_j^\Theta}(\sigma) = \overline{\operatorname{span}}\{\mathscr{F}(L_{\lambda_1}\varphi^j)(\sigma) : \varphi \in \mathscr{A}, \lambda_1 \in \Lambda_1, \widehat{\varphi}^j = \widehat{\varphi}\chi_{\mathcal{H}_j^\Theta}\}$. Then, we have

$$|S_\sigma^{\mathcal{J}}| \leqslant |\{j \in \mathcal{J} : \dim_{V_j^\Theta}(\sigma) \neq 0\}| \leqslant \sum_{j \in \mathcal{J}} \dim_{V_j^\Theta}(\sigma) = \dim_W(\sigma) \text{ a. e. } \sigma \in \Sigma.$$

Since the set $\{\Sigma + j + \theta^*\}_{j \in \mathcal{J}, \theta^* \in \Theta^\perp}$ is a tiling set for $\widehat{Z}$, therefore for a fixed $\sigma \in \Sigma$ and $j \in \mathcal{J}$ there is a unique $\theta^*_{\sigma,j} \in \Theta^\perp$ such that $\sigma + j + \theta^*_{\sigma,j} \in \mathcal{D}$, and hence we have

$$\begin{aligned}
\mu\left(\{\delta \in \mathcal{D} : \widehat{\varphi}(\delta) \neq 0\}\right) &= \sum_{j \in \mathcal{J}} \mu\left(\left\{\sigma \in \Sigma : \widehat{\varphi}(\sigma + j + \theta^*_{\sigma,j}) \neq 0\right\}\right) \\
&= \mu\left(\left\{(\sigma, j) \in \Sigma \times \mathcal{J} : \widehat{\varphi}(\sigma + j + \theta^*_{\sigma,j}) \neq 0\right\}\right) \\
&= \int_\Sigma \left|S_\sigma^{\mathcal{J}}\right|\ d\sigma \\
&\leqslant \int_\Sigma \sum_{j \in \mathcal{J}} \dim_{V_j^\Theta}(\sigma)\ d\sigma = \int_\Sigma \dim_W(\sigma)\ d\sigma \\
&= \sum_{m=0}^{nk} m\ \mu(\Sigma_m) \leqslant nk,
\end{aligned}$$

where $|\Lambda_1| = k$ and $\Sigma_m = \{\sigma \in \Sigma : \dim_W(\sigma) = m\}$. Thus the result follows. □

We have a immediate consequence for the singly generated system.

Corollary 6.6.11. *Let* Σ *and* $\mathcal{J}$ *be the Borel sections of* $\widehat{Z}/\Lambda_0^{\perp}$ *and* $\Lambda_0^{\perp}/\Theta^{\perp}$*, respectively, such that the cardinality* $|\mathcal{J}|$ *of* $\mathcal{J}$ *and measure* $\mu(\Sigma)$ *of* Σ *satisfies the relation*

$$(|\mathcal{J}|\mu(\Sigma) - k) > 0, \text{ where } k \text{ is the cardinality of a nonempty set } \Lambda_1.$$

When the space $\mathcal{S}^{\Lambda_1\Lambda_0}(\varphi)$ *becomes* $\Lambda_1\Theta$*-invariant, then the Plancherel transform* $\widehat{\varphi}$ *of* φ *satisfies the following relation:*

$$\mu(\{\omega \in \mathfrak{z}^* : \widehat{\varphi}(\omega) = 0\}) \geqslant |\Theta^{\perp}|(|\mathcal{J}|\mu(\Sigma) - k) > 0.$$

Proof. Considering a Borel section $\mathcal{D}$ of $\widehat{Z}/\Theta^{\perp}$ and noting $\widehat{Z} \cong \mathfrak{z}^*$, we have the following from Theorem 6.6.10:

$$\begin{aligned}
\mu(\{\omega \in \mathfrak{z}^* : \widehat{\varphi}(\omega) = 0\}) &= \sum_{\theta^* \in \Theta^{\perp}} \mu(\{\delta \in \mathcal{D} + \theta^* : \widehat{\varphi}(\delta) = 0\}) \\
&= \sum_{\theta^* \in \Theta^{\perp}} \mu[(\mathcal{D} + \theta^*) \backslash (\{\delta \in \mathcal{D} + \theta^* : \widehat{\varphi}(\delta) \neq 0\})] \\
&= \sum_{\theta^* \in \Theta^{\perp}} \mu(\mathcal{D}) - \sum_{\theta^* \in \Theta^{\perp}} \mu(\{\delta \in \mathcal{D} : \widehat{\varphi}(\delta) \neq 0\}) \\
&= \sum_{\theta^* \in \Theta^{\perp}} \sum_{j \in \mathcal{J}} \mu(\Sigma + j) - \sum_{\theta^* \in \Theta^{\perp}} \mu(\{y \in \mathcal{D} : \widehat{\varphi}(y) \neq 0\}) \\
&\geqslant |\Theta^{\perp}||\mathcal{J}|\mu(\Sigma) - k|\Theta^{\perp}| = |\Theta^{\perp}|[|\mathcal{J}|\mu(\Sigma) - k] > 0.
\end{aligned}$$

Thus the result follows. □

Remark 6.6.12. For the Heisenberg group $\mathbb{H}^d$, the center $Z = \mathbb{R}$, the uniform lattice $\Lambda_0 = \mathbb{Z}$ and the extra-invariance set $\Theta = \frac{1}{N}\mathbb{Z}$. Then the annihilators of Λ_0 and Θ are $\Lambda_0^{\perp} = \mathbb{Z}$ and $\Theta^{\perp} = N\mathbb{Z}$, respectively. Consider $\Sigma = [0, 1)$ and $\mathcal{J} = \{0, 1, 2, \ldots, N-1\}$ be the Borel sections of $\widehat{Z}/\Lambda_0^{\perp} = \mathbb{R}/\mathbb{Z}$ and $\Lambda_0^{\perp}/\Theta^{\perp} = \mathbb{Z}/N\mathbb{Z}$, and choose $\Lambda_1 = \{0\}$. Then $\mathcal{D} = [0, N)$ is the Borel section of $\widehat{Z}/\Theta^{\perp}$ and the estimate (6.6.7) mentioned in Theorem 6.6.10 can be written as $\mu(\{\xi \in [0, N) : \widehat{\varphi}_i(\xi) \neq 0\}) \leqslant n$ for all $i \in \{1, 2, \ldots, n\}$, since the cardinality of Λ_1 is $k = 1$. For $N > 1$, when the space $\mathcal{S}^{\Lambda_1\Lambda_0}(\varphi)$ becomes $\Lambda_1\Theta$-invariant, the measure of the set $\{\xi \in \mathbb{R} : \widehat{\varphi}(\xi) = 0\}$ is infinite from Corollary 6.6.11.

CHAPTER 7

SUMMARY AND FUTURE DIRECTIONS

Chapter 1 gives an introduction to the research area and available literatures including preliminaries for the upcoming chapters. In Chapter 2, we characterize alternate (oblique) duals, and duals of type-I and type-II of a frame for an MI space on $L^2(X;\mathcal{H})$ corresponding to the pointwise conditions in $\mathcal{H}$. Besides we characterize these duals' uniqueness using the Gramian/dual Gramian operators, which become a discrete frame/Riesz basis for the associated range spaces. In Chapter 3, we discuss the construction of dual frames and their uniqueness for the multiplication generated frames on $L^2(X;\mathcal{H})$ using infimum cosine angle. Emplyoing the techniques of Zak transform for the pair $(\mathscr{G},\Gamma)$, in Chapter 4, we obtain characterizations of alternate (oblique) Γ-TG duals and Γ-TG duals of type-I, type-II, dual frames. When $\mathscr{G}$ becomes an abelian group $\mathcal{G}$, the fiberization map is used to characterize these duals by the action of its closed subgroup Λ.

Further in Chapter 5, we study $\mathcal{S}^\Gamma(\mathscr{A})$-subspace orthogonal and duals to a Bessel family/frame $\mathcal{E}^\Gamma(\mathscr{A})$ and obtain characterization results in terms of the Zak transform and Gramian operator. The Chapter 6 starts with a brief discussion about the Plancherel transform for the connected, simply connected nilpotent Lie group of SI/Z type. Employing the Plancherel transform followed by periodization, we discuss the reproducing formula for translation-invariant spces by an action of a non-abelian subgroup. Finally, Chapter 7 deals with concluding remarks and provides some directions for future studies.

It would be interesting to study the above problems for dual frames in K-translation generated systems in $L^2(\mathscr{G})$, where K is compact. This type of study will be promising since its beautiful interplay between representation theory and frames. From a geometric point of view, the theory of dual frames can be further studied for $L^2(M)$, where M is a smooth connected Riemannian manifold.

Looking from the perspective of quantum field/modern physics, the theory of dual frames may be discussed for unitary irreducible representations on solvable Lie groups.

The method of characterizing dual frames arises from the action of irreducible representations of some solvable Lie groups can also be studied.

BIBLIOGRAPHY

[1] A. Aldroubi, P. Abry, M. Unser, *Construction of biorthogonal wavelets starting from any two multiresolutions*, IEEE Trans. Signal Process. **46** (1998), no. 4, 1130–1133.

[2] A. Aldroubi, C. Cabrelli, C. Heil, K. Kornelson, U. Molter, *Invariance of a shift-invariant space*, J. Fourier Anal. Appl. **16** (2010), no. 1, 60–75.

[3] S. Adhikari and R. Radha, *A study of oblique dual of a system of left translates on the Heisenberg group*, Results Math. **78** (2023), no. 2, Paper No. 65, 21.

[4] S. T. Ali, J.-P. Antoine, J.-P. Gazeau, *Continuous frames in Hilbert space*, Ann. Physics **222** (1993), no. 1, 1–37.

[5] M. Anastasio, C. Cabrelli, V. Paternostro, *Invariance of a shift-invariant space in several variables*, Complex Anal. Oper. Theory **5** (2011), no. 4, 1031–1050.

[6] M. Anastasio, C. Cabrelli, V. Paternostro, *Extra invariance of shift-invariant spaces on LCA groups*, J. Math. Anal. Appl. **370** (2010), no. 2, 530–537.

[7] S. Arati, R. Radha, *Orthonormality of wavelet system on the Heisenberg group*, J. Math. Pures Appl. **131** (2019), no. 9, 171–192.

[8] A. A. Arefijamaal, *The continuous Zak transform and generalized Gabor frames*, Mediterr. J. Math. **10** (2013), no. 1, 353–365.

[9] D. Barbieri, E. Hernández , J. Parcet, *Riesz and frame systems generated by unitary actions of discrete groups*, Appl. Comput. Harmon. Anal. **39** (2015), no. 3, 369–399.

[10] D. Barbieri, E. Hernández, V. Paternostro, *The Zak transform and the structure of spaces invariant by the action of an LCA group*, J. Funct. Anal. **269** (2015), no. 5, 1327–1358.

[11] R. V. Balan, *Multiplexing of signals using superframes*, Wavelet Applications in Signal and Image Processing VIII, vol. 4119, International Society for Optics and Photonics, 2000, pp. 118–129.

[12] D. Barbieri, E. Hernández, A. Mayeli, *Bracket map for the Heisenberg group and the characterization of cyclic subspaces*, Appl. Comput. Harmon. Anal. **37** (2014), no. 2, 218–234.

[13] B. Behera, Q. Jahan, *Wavelet analysis on local fields of positive characteristic*, Indian Statistical Institute Series (2021), xvii+333.

[14] G. Bhatt, B. D. Johnson, E. Weber, *Orthogonal wavelet frames and vector-valued wavelet transforms*, Appl. Comput. Harm. Anal. **23** (2007), no. 2, 215–234.

[15] S. Bishop, C. Heil, Y. K. Yoo, J. K. Lim, *Invariances of frame sequences under perturbations*, Linear Algebra Appl. **432** (2010), no. 6, 1501–1514.

[16] M. Bownik, *The structure of shift-invariant subspaces of* $L^2(\mathbb{R}^n)$, J. Funct. Anal. **177** (2000), no. 2, 282–309.

[17] M. Bownik, *The structure of shift-modulation invariant spaces: The rational case*, J. Funct. Anal. **244** (2007), no. 1, 172-219.

[18] M. Bownik, G. Garrigós, *Biorthogonal wavelets, MRA's and shift-invariant spaces*, Studia Math. **160** (2004), 231–248.

[19] M. Bownik, J. W. Iverson, *Multiplication-invariant operators and the classification of LCA group frames*, J. Funct. Anal. **280** (2021), no. 2, Paper No. 108780, 59 pages.

[20] M. Bownik, K. A. Ross, *The structure of translation-invariant spaces on locally compact abelian groups*, J. Fourier Anal. Appl. **21** (2015), no. 4, 849–884.

[21] H. Bölcskei, F. Hlawatsch, H. G. Feichtinger, *Frame theoretic analysis of oversampled filter banks*, IEEE Trans. Signal Process. **46** (1998), no. 12, 3256–3268.

[22] C. Cabrelli, V. Paternostro, *Shift-invariant spaces on LCA groups*, J. Funct. Anal. **258** (2010), no. 6, 2034–2059.

[23] C. Cabrelli, V. Paternostro, *Shift-modulation invariant spaces on LCA groups*, Studia Math. **211** (2012), no. 1, 1–19.

[24] O. Christensen, *An introduction to frames and Riesz bases*, Springer, 2016.

[25] O. Christensen, Y. C. Eldar, *Generalized shift-invariant systems and frames for subspaces*, J. Fourier Anal. Appl. **11** (2005), no. 3, 299–313.

[26] O. Christensen, Y.C. Eldar, *Oblique dual frames and shift-invariant spaces*, Appl. Comput. Harmon. Anal. **17** (2004), no. 1, 48–68.

[27] J. B. Conway, *A course in functional analysis*, Graduate Texts in Mathematics, 96, Springer-Verlag, New York, 1990.

[28] L. Corwin, F. P. Greenleaf, *Representations of nilpotent Lie groups and their applications: Volume 1, part 1, Basic theory and examples*, vol. 18, Cambridge university press, 2004.

[29] B. Currey, A. Mayeli, V. Oussa, *Characterization of shift-invariant spaces on a class of nilpotent Lie groups with applications*, J. Fourier. Anal. Appl. **20** (2014), no. 2, 384–400.

[30] I. Daubechies, B. Han, *Pairs of dual wavelet frames from any two refinable functions*, Constr. Approx. **20** (2004), no. 3, 325–352.

[31] C. Deboor, R. A. DeVore, A. Ron, *The structure of finitely generated shift-invariant spaces in* $L_2(\mathbb{R}^d)$, J. Funct. Anal. **119** (1994), no. 1, 37–78.

[32] R. J. Duffin, A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), no. 1, 341–366.

[33] Y. C. Eldar, O. Christensen, *Characterization of oblique dual frame pairs*, EURASIP J. Adv. Signal Process. **16** (2006), no. 1, 1–11.

[34] G. B. Folland, *A course in abstract harmonic analysis. Textbooks in Mathematics*, 2016.

[35] D. Freeman, D. Speegle, *The discretization problem for continuous frames*, Adv. Math. **345** (2019), no. 2, 784–813.

[36] J.-P. Gabardo, D. Han, *Balian-Low phenomenon for subspace Gabor frames*, J. Math. Phys. Anal. **45** (2004), no. 8, 3362-3378.

[37] J.-P. Gabardo, A. A. Hemmat, *Properties of oblique dual frames in shift-invariant systems*, J. Math. Anal. Appl. **356** (2009), no. 1, 346–354.

[38] R. A. Kamyabi Gol, R. Raisi Tousi, *The structure of shift invariant spaces on a locally compact abelian group*, J. Math. Anal. Appl. **340** (2008), no. 1, 219–225.

[39] A. Gumber, N. K. Shukla, *Orthogonality of a pair of frames over locally compact abelian groups*, J. Math. Anal. Appl. **458** (2018), no. 2, 1344–1360.

[40] A. Gumber, N. K. Shukla, *Pairwise orthogonal frames generated by regular representations of LCA groups*, Bull. Sci. Math. **152** (2019), 40–60.

[41] B. Han, *Pairs of frequency-based nonhomogeneous dual wavelet frames in the distribution space*, Appl. Comput. Harmon. Anal. **29** (2010), no. 3, 330–353.

[42] D. Han, J.-P. Gabardo, *The uniqueness of the dual of Weyl-Heisenberg subspace frames*, Appl. Comput. Harmon. Anal. **17** (2004), no. 2, 226–240.

[43] D. Han, D. R. Larson, *Frames, bases and group representations*, vol. 697, Amer. Math. Soc., 2000.

[44] C. Heil, Y. Y. Koo, J. K. Lim, *Duals of frame sequences*, Acta Appl. Math. **107** (2009), no. 1-3, 75–90.

[45] H. Helson, *Lectures on invariant subspaces*, Academic press, New York-London, 1964, xi+130 pp.

[46] A. A. Hemmat, J.-P. Gabardo, *The uniqueness of shift-generated duals for frames in shift-invariant subspaces*, J. Fourier Anal. Appl. **13** (2007), no. 5, 589–606.

[47] E. Hernández, H. Šikic, G. Weiss, E. Wilson, *Cyclic subspaces for unitary representations of LCA groups; generalized Zak transform*, Colloq. Math, 2010, 313–332.

[48] E. Hewitt, K. A. Ross, *Abstract harmonic analysis. Vol. I: Structure of topological groups, Integration theory, group representations*, Die Grundlehren der mathematischen Wissenschaften, Bd. 115 Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963, viii+519 pp.

[49] J. W. Iverson, *Subspaces of $L^2(G)$ invariant under translation by an abelian subgroup*, J. Funct. Anal. **269** (2015), no. 3, 865–913.

[50] J. W. Iverson, *Frames generated by compact group actions*, Trans. Amer. Math. Soc. **370** (2018), no. 1, 509-551.

[51] M. S. Jakobsen, J. Lemvig, *Reproducing formulas for generalized translation invariant systems on locally compact abelian groups*, Trans. Amer. Math. Soc. **368** (2016), no. 12, 8447–8480.

[52] M. S. Jakobsen, J. Lemvig, *Co-compact Gabor systems on locally compact abelian groups*, J. Fourier Anal. Appl. **22** (2016), no. 1, 36–70.

[53] G. Kaiser, *A friendly guide to wavelets*, Birkhäuser Boston, Inc., Boston, MA, 1994, xviii+300 pp.

[54] H. O. Kim, R. Y. Kim, J. K. Lim, Z. Shen, *A pair of orthogonal frames*, J. Approx. Theory **147** (2007), no. 2, 196–204.

[55] H. O. Kim, R. Y. Kim, J. K. Lim, *Characterizations of biorthogonal wavelets which are associated with biorthogonal multiresolution analyses*, Appl. Comput. Harmon. Anal. **11** (2001), no. 2, 263–272.

[56] H. O. Kim, R. Y. Kim, J. K. Lim, *Quasi-biorthogonal frame multiresolution analyses and wavelets*, Adv. Comput. Math. **18** (2003), no. 2, 269–296.

[57] H. O. Kim, R. Y. Kim, J. K. Lim, *The infimum cosine angle between two finitely generated shift-invariant spaces and its applications*, Appl. Comput. Harmon. Anal. **19** (2005), no. 2, 253–281.

[58] Y. Z. Li, Q. F. Lian, *Super Gabor frames on discrete periodic sets*, Adv. Comput. Math. **38** (2013), no. 4, 763–799.

[59] J. Lopez, D. Han, *Discrete Gabor frames in $\ell^2(\mathbb{Z}^d)$*, Proc. Amer. Math. Soc. **141** (2013), no. 11, 3839–3851.

[60] G. W. Mackey, *Induced representations of locally compact groups I*, Ann. Math. **55** (1952), no. 1, 101–139.

[61] A. Ron, Z. Shen, *Frames and stable bases for shift-invariant subspaces of $L^2(\mathbb{R}^d)$*, Canad. J. Math. **47** (1995), no. 5, 1051–1094.

[62] R. Radha and S. Adhikari, *Shift-invariant spaces with countably many mutually orthogonal generators on the Heisenberg group*, Houston J. Math. **46** (2020), no. 2, 435–463.

[63] S. Sarkar, N. K. Shukla, *Translation generated oblique dual frames on locally compact groups*, Linear Multilinear Algebra, (2023), `doi:10.1080/03081087.2023.2173718`, 32 pages.

[64] S. Sarkar, N. K. Shukla, *Subspace dual and orthogonal frames by action of an abelian group*, preprint.

[65] S. Sarkar, N. K. Shukla, *Reproducing formula associated to translation generated systems on nilpotent Lie groups*, `arXiv:2301.03152`.

[66] S. Sarkar, N. K. Shukla, *A characterization of MG dual frames using infimum cosine angle*, `arXiv:2301.07448`.

[67] S. Sarkar, N. K. Shukla, *Characterizations of extra-invariant spaces under the left translations on a Lie group*, Advances in Operator theory, (2023), `https://doi.org/10.1007/s43036-023-00273-x`.

[68] S. Sarkar, S. Kalra, N. K. Shukla, *An application of the supremum cosine angle between multiplication invariant spaces in $L^2(X;\mathcal{H})$*, `arXiv:2211.15238`.

[69] N. K. Shukla, S. C. Maury *Super-wavelets on local fields of positive characteristic*, Math. Nachr. **291** (2018), no. 4, 704–719.

[70] N. K. Shukla, S. C. Maury *Semi-orthogonal Parseval wavelets associated with GM-RAs on local fields of positive characteristic*, Mediterr. J. Math. **16** (2019), no. 5, 20 pages.

[71] T. P. Srinivasan, *Doubly invariant subspaces*, Pacific J. Math. **14** (1964), no. 2, 701–707.

[72] W. S. Tang, *Oblique projections, biorthogonal Riesz bases and multiwavelets in Hilbert spaces*, Proc. Amer. Math. Soc. **128** (2000), no. 12, 463–473.

[73] E. Weber, *Orthogonal frames of translates*, Appl. Comput. Harmon. Anal. **17** (2004), no. 1, 69–90.

[74] D. F. Walnut, *An introduction to wavelet analysis*, Springer Science & Business Media, 2013.

INDEX

ρ-invariant space, 60

adjoint action, 102
alternate dual, 4
analysis operator, 3

B-spline, 98
Bessel system, 2
bracket map, 60, 117
biorthogonal system 87, 117
biorthogonal dual, 70

cannocial dual, 3
coadjoint action, 102
continuous frame, 2
complete frame, 68

determining set, 14
dimension function, 129
discrete-Fourier transform, 71
dual frame, 4
dual Gramian, 29, 36
dual integrable representation, 60

extra-invariance, 124

fiberization, 49
frame operator, 3

Gabor system, 58, 97
Gramian operator, 3, 36

Heisenberg group, 106

$\mathcal{K}$-subspace dual, 68
$\mathcal{K}$-subspace frame, 67
$\mathcal{K}$-subspace orthogonal, 68

left regular representation, 69
left translation operator, 7
linearly independent, 87

matrix element, 70
mixed Gramian operator, 36, 68
mixed Gramian operator, 36, 76
modulation operator, 58, 97
multiplication generated system, 13
multiplication invariant (MI) space, 14
multiplication operator, 13

oblique dual, 4
orthogonal direct sum, 35
orthogonal frame, 68
orthonormal Parseval determining set, 25

Parseval frame, 2
p-adic group, 55
Parseval determining set, 15
periodic funciton, 82
Pfaffian, 103
Plancherel transform, 103
pre-Gramian operator, 76

range function, 14, 50, 107
Riesz sequence, 28

Schrödinger representation, 106

semidirect product, 99

shift-invariant space, 5

square integrable modulo the center, 9

subspace Riesz basis, 90

super dual frame, 95, 114

supremum cosine angle, 43

synthesis operator, 3

tiling partner, 104

tiling, 104

translation-invariant space, 7, 48, 69, 107

type-I dual, 4

type-II dual, 4

unitary representation, 60

Zak transform, 49