MONOTONE ITERATIVE METHODS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

Ph.D. Thesis

By

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MONOTONE ITERATIVE METHODS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Linia Anie Sunny

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **MONOTONE ITERATIVE METHODS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DISCIPLINE OF MATHEMATICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2013 to August 2017 under the supervision of Dr. V. Antony Vijesh, Associate Professor, IIT Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Signature of the student with date
(**LINIA ANIE SUNNY**)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Signature of Thesis Supervisor with date
(**DR. V. ANTONY VIJESH**)

LINIA ANIE SUNNY has successfully given his Ph.D. Oral Examination held on **October 30, 2018**.

Sign. of Chairperson(OEB) Date:	Sign. of External Examiner Date:	Sign. of Thesis Supervisor Date:
Sign. of PSPC Member #1 Date:	Sign. of PSPC Member #2 Date:	Sign. of Convener, DPGC Date:
Sign. of Head of Discipline Date:		

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ABSTRACT

A uniform approach is adopted throughout this thesis by appropriately approximating the solutions of nonlinear differential equations by sequences of linear ones that monotonically converge to the unique solution of the problem. The existence and uniqueness of the solutions of different nonlinear partial differential equations with initial and/or boundary conditions arising from mathematical models are obtained for both continuous and/or discretized domains. All the proposed methods supply lower and upper bounds for the solutions of the given nonlinear differential equations. The efficacy of the proposed iterative schemes in terms of their faster convergence and/or higher flexibility in choosing the initial guess are demonstrated through numerical simulations.

In **Chapter 1** provides an outline of the historic development of the method of monotone iterations as a powerful tool for nonlinear differential equations of various types. Few basic results and definitions that are relevant to the rest of the chapters are also given in this chapter.

Chapter 2 deals with an accelerated monotone iterative procedure for a coupled system of partial differential equations arising from a catalytic converter model. The monotone property as well as the convergence analysis and the error estimate of the proposed iterative schemes for continuous domain as well as discretized domain based on finite difference approximations are proved theoretically. The efficiency of the proposed scheme is illustrated by providing a comparative numerical study with the existing method.

In **Chapter 3**, an alternative approach to the one provided in Chapter 2 is proposed in which one has to evaluate the derivative only once throughout the procedure. The proposed scheme also accelerates the procedure studied in the literature. An interesting theoretical study on the monotone convergence as well as error estimate of the proposed

iterative procedure are provided for continuous as well as finite difference based discretized problems.

Chapter 4 proposes an accelerated iterative procedure for a nonlinear fourth order elliptic equation with nonlocal boundary conditions. Theoretically, the monotone property as well as the convergence analysis are proved for both the continuous and finite difference discretized cases. The proposed scheme not only accelerates the scheme in the literature but also provides a greater flexibility in choosing the initial guess. The efficacy of the proposed scheme is demonstrated through a comparative numerical study with the recent literature.

In **Chapter 5**, a finite difference method based monotone iterative technique is employed to solve an important class of Volterra type parabolic partial integro-differential equations. The monotone property, convergence analysis and an error estimate in terms of the stopping criteria are proved theoretically. The effectiveness of the proposed scheme is demonstrated by applying it to nonlinear integro-partial differential equations arising in population models and nuclear reactor models.

KEYWORDS: bending beams, catalytic converter, coupled system, finite difference, fourth order elliptic equations, monotone iterations, nonlocal boundary conditions, population models, quasilinearization, successive approximations, Volterra type integrals

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CHAPTER 1

INTRODUCTION

This chapter intends to provide a short literature survey on the method of monotone iterations and its evolution as a powerful tool to deal with nonlinear differential equations of various types. The historical developments are traced by the aid of some major references from [**24**] and [**72**]. Certain preliminary results and definitions along with a brief description of the works explained in the following chapters are also provided in this chapter.

1.1. A SHORT REVIEW ON MONOTONE ITERATIONS

Many of the real life problems can be mathematically modeled into differential equations. When these models are nonlinear, dealing with them is of great complexity. In particular, nonlinear partial differential equations with initial and boundary conditions occur frequently in varied fields such as Biology, Economics, Engineering and Physics. The qualitative and quantitative properties of such models and numerical techniques to approximate their solutions are of great importance to mathematicians.

There are various classical methods available in the literature that provide existence and uniqueness of solutions of nonlinear differential equations. Out of them, one of the systematic approaches that also provides iterative schemes to approximate solutions is the method of lower and upper solutions. During this process of iterations, it not only gives an existence-uniqueness theorem but also provides lower and upper bounds for the solution.

The method of lower and upper solutions marked its commencement during 1890s when E. Picard [**24**] investigated on the existence and uniqueness of the solution for the following nonlinear boundary value problem:

$$x^{''} + f(t,x) = 0,\ x(a) = 0,\ x(b) = 0. \tag{1.1}$$

Based on the assumptions that it has a trivial solution and the function f is increasing, sequence of successive approximation was developed from the iterative scheme

$$-\alpha_n^{''} = f(t, \alpha_{n-1}),\ \alpha_n(a) = 0,\ \alpha_n(b) = 0$$

that converge to the solution of (1.1). The case when f is decreasing was also handled by him. Later, he recursively generated two sequences α_n and β_n by solving the following sequences of linear problems:

$$\begin{aligned} \beta_n^{''} + f(t, \alpha_{n-1}) &= 0,\ \beta_n(a) = 0,\ \beta_n(b) = 0 \\ \alpha_n^{''} + f(t, \beta_{n-1}) &= 0,\ \alpha_n(a) = 0,\ \alpha_n(b) = 0, \end{aligned}$$

where α_n is increasing, β_n is decreasing and both converge to functions u and $v \geq u$ respectively that are bounds for the solution. By providing suitable examples, it was concluded that these bounds need not be equal in general. However, sufficient conditions were provided to ensure that these bounds are equal and thus a solution of (1.1). In the beginning of 1900s, independently using the comparison between solutions of differential inequalities, existence of solutions for the first order Cauchy problem

$$x^{'} + f(t, x) = 0,\ x(0) = x_0, \tag{1.2}$$

was studied by O. Perron [**24**] and its extension to systems by M. Muller [**24**]. A crucial advancement in this direction occurred due to G.S. Dragoni during 1930s while he considered a more general form of (1.1)

$$x^{''} = f(t, x, x^{'}),\ x(a) = A,\ x(b) = B. \tag{1.3}$$

The role of modern day lower and upper solutions were explicitly recognised for the first time through this work. More specifically, on the assumption of the existence of functions $\alpha, \beta \in C^2[a, b]$ satisfying

$$\begin{gathered} \alpha^{''}(t) + f(t, \alpha(t), y) \geq 0 \text{ if } t \in [a, b],\ y \leq \alpha^{'}(t)\ (y \geq \alpha^{'}(t)), \\ \alpha(a) \leq A,\ \alpha(b) \leq B \\ \beta^{''}(t) + f(t, \beta(t), y) \leq 0 \text{ if } t \in [a, b],\ y \leq \beta^{'}(t)\ (y \geq \beta^{'}(t)), \\ \beta(a) \geq A,\ \beta(b) \geq B \end{gathered} \tag{1.4}$$

with $\alpha \le \beta$ on $[a, b]$, the existence of a solution u of (1.3) was deduced between α and β. However, there were no explicit ways available to practically find these functions for a given problem. Later, this led to the study of constructions of lower and upper solutions in 1960s by K. Ako [**4**] followed by R.E. Gaines [**30**] in 1972. For details, see [**24**]. Later, by relaxing the smoothness as well as modifying the conditions of lower and upper solutions (1.4), M. Nagumo obtained at least one solution for (1.1) for certain Nagumo class of functions [**24**].

The class of lower and upper solutions was enlarged and quasi-subsolution and quasi-supersolution were defined by M. Nagumo [**61**] in 1954 while studying a nonlinear partial differential equation of elliptic type

$$\triangle u + f(x, u, \nabla u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where the lower solution α and upper solution β were supposed to be belonging to $C^2(\overline{\Omega})$, f is a Hölder continuous function satisfying

$$|f(x, u, v)| \le B\|v\|^2 + C$$

with the restriction

$$16MB < 1, \text{ where } M = \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}. \tag{1.5}$$

Many authors tried various extensions of this work and it was F. Tomi [**24**] in 1969 who extended this work by removing the restriction (1.5).

B.N. Babkin [**24**] in 1954 studied (1.1) by replacing the condition on f by the weaker one: $f + Ku$ is increasing in u for some $K > 0$. With suitable lower solution α_0 and upper solution $\beta_0 \ge \alpha_0$, the iterative scheme proposed was as follows:

$$\begin{aligned} -\alpha_n'' + K\alpha_n &= f(t, \alpha_{n-1}) + K\alpha_{n-1}, \ \alpha_n(a) = 0, \ \alpha_n(b) = 0 \\ -\beta_n'' + K\beta_n &= f(t, \beta_{n-1}) + K\beta_{n-1}, \ \beta_n(a) = 0, \ \beta_n(b) = 0. \end{aligned}$$

The unique solution of (1.1) was obtained as the limits of these sequences.

It was observed that the sequences generated by Picard's procedure and its modifications could achieve only a linear order of convergence. For handling (1.1), (1.2), (1.3) and nonlinear elliptic and parabolic problems, R.E. Kalaba [**41**] in 1959 replaced frequently used Picard's idea of linearizing the nonlinear problems by the idea of quasilinearization which was first used by R.E. Bellman [**8**] in 1955 in the context of dynamic programming. R.E. Kalaba also proved that the order of convergence of his iterative procedure is quadratic; one major enhancement in this direction. Originally, the quasilinearization technique gained its motivation from the well known Newton's method. One of the results due to Kalaba is given below:

Theorem 1.1.1. *The sequence of functions* $\{u_n(x)\}$ *given by*

$$\begin{aligned} u_0' &= f(v_0) + (u_0 - v_0)f'(v_0),\ u_0(0) = c \\ u_{n+1}' &= f(u_n) + (u_{n+1} - u_n)f'(u_n),\ u_{n+1}(0) = c, \end{aligned}$$

where $n = 0, 1, \ldots$ *is monotone increasing in the interval* $[0, b]$ *and converges to the unique solution* $u(x)$ *of*

$$u' = f(x, u),\ u(0) = c,\ 0 \le x \le b,$$

where f *is assumed to be continuous in* u *and* x *and strictly convex in* u *with a bounded derivative with respect to* u *for all* u *and* x. *Moreover, the sequence* $\{u_n(x)\}$ *is quadratically convergent on the interval* $[a, b]$.

For details on the contributions of R.E. Bellman and R.E. Kalaba, one can refer to [**9**].

Using the idea of quasi-subsolution and quasi-supersolution of M. Nagumo [**61**] and K. Ako [**3**] established the existence of solutions for a more general class of quasilinear elliptic differential equation of second order. In this direction, another approach in linearizing the nonlinear equations was suggested by G.V. Gendzhoyan [**24**] in 1964 to obtain the existence and uniqueness of the solution of

$$x'' + f(t, x, x') = 0,\ x(a) = 0,\ x(b) = 0.$$

Using lower and upper solutions α_0 and $\beta_0 \geq \alpha_0$, the sequences studied were

$$
\begin{aligned}
-\alpha_n'' + l(t)\alpha_n' + k(t)\alpha_n &= f(t, \alpha_{n-1}, \alpha_{n-1}') + l(t)\alpha_{n-1}' + k(t)\alpha_{n-1} \\
\alpha_n(a) &= 0,\ \alpha_n(b) = 0, \\
-\beta_n'' + l(t)\beta_n' + k(t)\beta_n &= f(t, \beta_{n-1}, \beta_{n-1}') + l(t)\beta_{n-1}' + k(t)\beta_{n-1} \\
\beta_n(a) &= 0,\ \beta_n(b) = 0,
\end{aligned}
$$

where $k(t)$ and $l(t)$ are functions related to the assumptions on f. A set of sufficient conditions were provided to ensure the convergence of the above iterative scheme to the unique solution of the problem along with its error estimate. It is worth mentioning that S.R. Bernfeld and J. Chandra adopted another approach to handle a second order boundary value problem $x'' + f(t, x, x') = 0$ with mixed boundary conditions in [**11**].

An interesting existence-uniqueness theorem for a class of mildly nonlinear elliptic boundary value problem where the nonlinearities might occur both in the boundary conditions and governing equation was obtained by H.B. Keller [**43**] using monotone iterations. The monotone iterative method gained wide attention due to Keller and few others in the late sixties and early seventies. In early 1970s, the works of H. Amann [**5**] for nonlinear elliptic boundary value problem and D.H. Sattinger [**92**] for parabolic boundary value problem provided the construction of monotone sequences using lower and upper solutions systematically. These ideas were extended by C.V. Pao during 1970s to a semilinear parabolic equation given by

$$
u_t + Lu = f(t, x, u),\ t \in [0, T],\ x \in \Omega, \tag{1.6}
$$

with

$$
L = \sum_{i,j=1}^{n} a_{i,j}(t, x) u_{x_i x_j} + \sum_{i=1}^{n} b_i(t, x) u_{x_i} - c(t, x) u
$$

under the initial condition $u(x, 0) = u_0(x)$ in $x \in \Omega$ and the nonlinear boundary conditions

$$
\beta \frac{\partial u}{\partial \nu} - g(t, x, u) = h(t, x),\ t \in (0, T],\ x \in \partial\Omega
$$

and $\lim_{|x| \to \infty} u(t, x) = 0,\ t \in (0, T]$ where f and g are nonlinear functions of u, ν is the outward unit normal vector on $\partial\Omega$, $\beta \geq 0$ is a constant and Ω is a bounded or unbounded domain in the n-dimensional Euclidean space $\mathbb{R}^n$ with boundary $\partial\Omega$. The existence and uniqueness of a positive solution for this problem was obtained through the construction of monotone iterations in [**65**]. The main theorem in this paper is given below.

Theorem 1.1.2. *Suppose*

(i) $f(t,x,0) \geq 0$ *and* $u_0(x) \geq 0$, $(t,x) \in D$,

(ii) $g(t,x,0) = 0$ *and* $h(t,x) \geq 0$, $(t,x) \in S$ *and*

(iii) *there exist constants* c_1, c_2 *such that*

$$f(t,x,\eta_1) - f(t,x,\eta_2) \quad \geq \quad -c_1(\eta_2 - \eta_1),\ 0 \leq \eta_1 \leq \eta_2 \leq \overline{\rho}.$$

Also let there exist an upper solution $\tilde{u}$. *Then the minimal sequence* $\{\underline{u}^{(k)}\}$ *converges from below to a minimal solution* $\underline{u}$ *of* (1.6) *and the maximal sequence* $\{\underline{u}^{(k)}\}$ *converges from above to a maximal solution* $\overline{u}$. *Furthermore, the convergence of these sequences are uniform on every bounded subdomain of* D *and*

$$0 \leq \underline{u}^{(1)} \leq \underline{u}^{(2)} \leq \cdots \leq \underline{u} \leq \overline{u} \leq \overline{u}^{(2)} \leq \overline{u}^{(1)} \leq \tilde{u}.$$

This idea was further extended for a variety of problems by X. Lu, C.V. Pao, Y.M. Wang and their collaborators in their subsequent works. By suitably coupling various linear approximations and monotone iterations, they handled a wide variety of partial differential equations with initial and boundary conditions such as semilinear parabolic and elliptic boundary value problems, coupled partial differential equations, fourth order elliptic equations with nonlocal boundary conditions, integro-differential equations of Fredhlom and Volterra type etc. For example, see [**66, 67, 72, 75, 81**].

In 1979, S. Bernfeld and V. Lakshmikantham [**12**] successively solved nonlinear boundary value problem in Banach space using monotone methods. An extension of the method of lower and upper solutions to a Volterra integral equation

$$x(t) = f(t) + \int_0^t K(t,s,x(s))\mathrm{d}s$$

where $x, f \in C[I,\mathbb{R}^n]$, $K \in C[I^2 \times \mathbb{R}^n, \mathbb{R}^n]$ and $I = [0,T]$ was done by G.S. Ladde et. al in [**45**].

The study of the method of upper and lower solutions for hyperbolic differential equation given by

$$
\begin{aligned}
& u_{xy} = f(x, y, u, u_x, u_y),\ (x, y) \in [0, a] \times [0, b] \\
& u(x, 0) = \sigma(x),\ x \in [0, a] \\
& u(0, y) = \tau(y),\ y \in [0, b] \\
& \sigma(0) = \tau(0) = u_0,
\end{aligned} \tag{1.7}
$$

where $f \in C([0, a] \times [0, b] \times \mathbb{R}^3)$, $\sigma \in C^1([0, a] \times \mathbb{R})$ and $\tau \in C^1([0, b] \times \mathbb{R})$ was initiated in 1985 by V. Lakshmikantham and S.G. Pandit in [**48**]. V. Lakshmikantham and his collaborators employed this technique widely to study different kinds of nonlinear differential equations with/without quasilinearization [**1, 46, 49, 50, 51**].

Many more interesting works can be found in the literature towards this direction among which the contributions of C.D. Coster [**24**], R. Courant and D. Hilbert [**25**], L. Kantorovich [**42**], E. Zeidler [**103**] and their references are worth noticing. The method of lower and upper solutions has proved its applicability in various other classifications of differential equations such as fractional [**57, 63**], fuzzy [**54**], hybrid [**94**], matrix [**60**] and stochastic [**26, 27**] differential equations. The investigations on utilization of this method for various equations are actively going on.

1.1.1. Monotone Finite Difference Methods

Monotone iterative methods are not only used for obtaining existence and uniqueness of the solution of nonlinear differential equations but also acts as an effective tool to obtain the numerical solutions using finite difference approximations. In 1965, S.V. Parter [**84**] initiated the study of numerical solutions of elliptic differential equations by aiding the partnership of monotone iterations and finite difference approximations. The iterative scheme proposed was given by

$$
\begin{aligned}
& \triangle_h Z^{n+1} - kZ^{n+1} = f(P, Z^n(P)) - kZ^n,\ P \in G(h) \\
& Z^{n+1}(P) = \hat{g}(P),\ P \in \Omega(h),
\end{aligned} \tag{1.8}
$$

where $\triangle_h$ was a finite dimensional linear operator approximating the Laplace operator, $G(h)$ denoted the set of interior points and $F(h)$ denoted the set of boundary points. A set of conditions was provided that ensured the convergence of the iterative procedure defined by (1.8). More specifically, when initial approximations were chosen as the lower and upper solutions respectively, increasing and decreasing sequences were generated that converge to the solution of the problem considered.

By adapting a suitable finite difference method, A.C. Lazer [**52**] in 1982 proposed a numerical method to solve a system of semilinear elliptic differential equations arising from prey-predator models numerically. The boundary value problem considered was

$$
\begin{aligned}
&u''(x) + u(x)[a - bu(x) - cv(x)] = 0, \ \alpha < x < \beta \\
&v''(x) + v(x)[e - fu(x) - gv(x)] = 0, \ \alpha < x < \beta \\
&u(\alpha) = u(\beta) = v(\alpha) = v(\beta) = 0,
\end{aligned}
\tag{1.9}
$$

where a, b, c, e, f, g are positive parameters. His approach provided two sequences, one increasing and the other decreasing, that converge to the unique solution of the nonlinear difference scheme.

This technique of lower and upper solutions was most extensively employed by C.V. Pao for numerically handling various problems. In 1985, C.V. Pao [**68**] obtained an existence-uniqueness theorem for a nonlinear finite difference scheme of a class of reaction diffusion equations via monotone iterations. In particular, for a bounded domain Ω in $\mathbb{R}^p$, $p = 1, 2, \ldots$, the following parabolic boundary value problem was considered:

$$
\begin{aligned}
&u_t - D\nabla^2 u = f(x, t, u), \ x \in \Omega, \ 0 < t \leq T \\
&\alpha(x_0)\tfrac{\partial u}{\partial \nu} + \beta(x_0)u = g(x_0, t), \ x_0 \in \partial\Omega, \ 0 < t \leq T \\
&u(0, x) = \psi(x), \ x \in \Omega,
\end{aligned}
\tag{1.10}
$$

where ∇^2 is the Laplacian operator, $\partial\Omega$ is the boundary of Ω and $\frac{\partial}{\partial \nu}$ is the outward normal derivative on $\partial\Omega$. It was assumed that the function $D \equiv D(x, t)$ is positive on $\overline{\Omega} \times [0, T]$, $\alpha(x_0) \geq 0$, $\beta(x_0) \geq 0$ with $\alpha(x_0) + \beta(x_0) > 0$ on $\partial\Omega$ and the nonlinear function f and the data g, ψ are known functions in their respective domains.

One of the interesting works where the quasilinearization technique was coupled with monotone iterations for numerically solving a reaction-diffusion-convection equation was given by C.V. Pao [**73**] in 1998. In this study, solution of the resultant nonlinear finite difference scheme was obtained from two monotone sequences generated by quasilinearization that converge quadratically. One of the main theorems in this article is given below.

Theorem 1.1.3. *Let $\tilde{U}_n, \hat{U}_n$ be a pair of ordered upper and lower solutions for the finite difference approximation*

$$(I + k_n A_n)U_n = U_{n-1} + k_n F(U_n),\ U_0 = \Psi, \tag{1.11}$$

where $U_n = (u_{1,n}, \dots, u_{N,n})$ with N denoting the total number of unknowns, A_n an $N \times N$ band matrix associated with the elliptic and boundary operators in the problem and $F(U_n)$ a vector in the form $F(U_n) = (f^*(u_{1,n}), f^*(u_{1,n}), \dots, f^*(u_{N,n})))$ with $f^*(u_{i,n}) = f(u_{i,n}) + g^*(u_{i,n})$. Let $A_n = (a_{i,j}^{(n)})$ be an irreducible matrix with $a_{i,j}^{(n)} \le 0$ for $i \neq j$, $a_{ii}^{(n)} \ge 0$ for all i and $\sum_{j=1}^{N} a_{i,j}^{(n)} \ge 0$ for all $i = 1, \dots, N$ and $n = 1, 2, \cdots$. Assume that $k_n(\sigma_n - \mu_n) < 1,\ n = 1, 2, \dots,$ where μ_n is the smallest eigenvalue of A_n. Then (1.11) has a unique solution $U_n^* \in < \tilde{U}_n, \hat{U}_n >$. Moreover, the sequences $\{\overline{U}_n^{(m)}\}$, $\{\underline{U}_n^{(m)}\}$ given by

$$P_n^{(m)} U_n^{(m+1)} = U_{n-1}^* + k_n[C_n^{(m)} U_n^{(m)} + F(U_n^{(m)})],\ U_0^{(m+1)} = \Psi,$$

where $m = 0, 1, 2, \dots,$ $U_n^{(0)}$ is either $\tilde{U}_n$ or $\hat{U}_n$, $P_n^{(m)} \equiv I + k_n A_n + k_n C_n^{(m)}$, $C_n^{(m)} \equiv diag(c_{1,n}^{(m)}, c_{2,n}^{(m)}, \dots, c_{N,n}^{(m)})$ and $c_{i,n}^{(m)} = \max\{-f_u^*(u_{i,n});\ \underline{u}_{i,n}^{(m)} \le u_{i,n}, \overline{u}_{i,n}^{(m)}\}$ converge monotonically from above and below, respectively, to the unique solution U_n^*.

In 2002 [**78**], this idea was further explored for a coupled system of reaction diffusion equations with nonlinear boundary conditions and time delays. The time dependent reaction diffusion system considered was of the form

$$\begin{aligned} &\tfrac{\partial u^{(l)}}{\partial t} - L^{(l)} u^{(l)} = f^{(l)}(x, t, \mathbf{u}, \mathbf{u}_\tau),\ x \in \Omega,\ t > 0 \\ &B^{(l)} u^{(l)} = g^{(l)}(x, t, \mathbf{u}, \mathbf{u}_{\tau'}),\ x \in \partial\Omega,\ t > 0 \\ &u^{(l)}(x, t) = \psi^{(l)}(x, t),\ x \in \Omega,\ -\overline{\tau}_t \le t \le 0, \end{aligned} \tag{1.12}$$

where for each $l = 1, 2, \dots, N,$ $L^{(l)}$ and $B^{(l)}$ are the respective diffusion-convection operator and boundary operator given by

$$\begin{aligned} &L^{(l)} u^{(l)} = D^{(l)} \nabla^2 u^{(l)} + \mathbf{v}^{(l)} \triangle u^{(l)} \\ &B^{(l)} u^{(l)} = \alpha^{(l)} \tfrac{\partial u}{\partial \nu} + \beta^{(l)} u^{(l)}. \end{aligned} \tag{1.13}$$

Here $\frac{\partial u}{\partial \nu}$ denotes the outward normal derivative of $u^{(l)}$ on $\partial\Omega$, $D^{(l)} \equiv D^{(l)}(x,t)$, $\beta^{(l)} \equiv \beta^{(l)}(x,t)$ and $\mathbf{v}^{(l)} \equiv (v_1^{(l)}, \dots, v_p^{(l)})$ with $v_\nu^{(l)} \equiv v_\nu^{(l)}(x,t)$, $\nu = 1, 2, \dots, p$ are continuous functions of (x,t).

In this direction, extensive contributions were made by C.V. Pao and his collaborators to this area. Using finite difference approximations, the method of lower and upper solutions was immensely utilized by X. Lu, C.V. Pao and Y.M. Wang and their collaborators to deal with diverse partial differential equations like reaction diffusion equations [**68**], semilinear parabolic equations [**69**], coupled systems of nonlinear boundary value problems [**70**], nonlinear parabolic boundary value problems [**71**], nonlinear parabolic equations with time delays [**74**], nonlinear integro-parabolic equations of Fredholm type [**76**], fourth-order nonlinear elliptic boundary value problems [**77, 80, 83**], reaction diffusion systems with coupled boundary conditions [**78**], nonlinear elliptic boundary value problems [**79**], coupled system of differential equations [**82**] etc.

In most of these recent studies, the convergence of the discretized solutions to their continuous counterparts as the mesh size tends to zero is based on the convergence of their continuous cases. In contrast to this, convergence analysis for discretized solutions that were completely independent of their continuous cases was studied by I. Bogalev, one of the active contributors in this area. He utilized the method of lower and upper solutions to a large extend especially for discretized problems obtained from differential equations. This approach was exclusively dependent on comparison results for discretized domain proposed by A. Samarskii in [**90**]. Using this technique, different aspects of finite difference based numerical solutions for nonlinear integro-partial differential equations were studied in his recent publications. [**13**] dealt with solving of nonlinear integro-parabolic problems using finite difference approximations based on the method of upper and lower solutions numerically. The same problem was further improved in [**14**] by adopting weighted average scheme for approximations. This study was then extended for coupled systems of nonlinear parabolic equations based on a nonlinear ADI scheme in [**15**] and a coupled system of two nonlinear integro-parabolic equations of Volterra type in [**16**]. The integro-parabolic equation that was considered in [**16**] is given by

$$\frac{\partial u_i}{\partial t} - L_i u_i + f_i(x,t,u) + \int_0^t g_i^*(x,t,s,u(x,s))\mathrm{d}s = 0,\ (x,t) \in \omega \times (0,T]$$
$$u_i(x,t) = \phi_i(x,t),\ (x,t) \in \partial\omega \times (0,T]; \quad u_i(x,0) = \psi_i(x),\ x \in \overline{\omega},\ i = 1,2,$$

where $u_i = (u_1, u_2)$, ω is a connected bounded domain in $\mathbb{R}^k$ $(k = 1, 2, \cdots)$ with boundary $\partial\omega$. The differential operators L_i are given by

$$L_i u_i = \sum_{\alpha=1} \kappa \frac{\partial}{\partial x_\alpha}\left(D_i(x,t)\frac{\partial u_i}{\partial x_\alpha}\right) + v_{i,\alpha}(x,t)\frac{\partial u_i}{\partial x_\alpha},$$

where the coefficients of the differential operators are smooth and $D_i,\ i = 1,2$ are positive in $\omega \times [0,T]$. It was also assumed that the functions $f_i,\ g_i^*,\ \phi_i$ and $\psi_i,\ i = 1,2$ are smooth in their respective domains. Based on the monotone iterative method for solving the nonlinear difference scheme, the existence and uniqueness of a discrete solution and error estimates of the iterative method were acquired in this article. In this approach, at each step of the iterative scheme one has to solve linear difference equations of the form

$$\begin{aligned} &(\mathcal{L}_i + \overline{c}_i)W_i(p,t_k) = \Psi_i(p,t_k),\ p \in \omega^h \\ &\overline{c}_i(p,t_k) \geq 0,\ W_i(p,t_k) = \phi_i(p,t_k),\ p \in \partial\omega^h,\ i = 1,2. \end{aligned} \tag{1.14}$$

The existence and uniqueness as well as the monotone property of proposed iterative scheme was obtained using the following maximum principle and error estimate by A. Samarskii [**90**].

Lemma 1.1.1. *Let the coefficients of the difference operator $\mathcal{L}_i^h,\ i = 1,2$ satisfy the assumptions on their coefficients and the mesh $\overline{\omega}^h$ be connected.*

(i) *If a mesh function $W_i(p,t_k),\ i = 1,2$ satisfy the conditions*

$$(\mathcal{L}_i + \overline{c}_i)W_i(p,t_k) \geq 0 (\leq 0), \quad p \in \omega^h$$
$$W_i(p,t_k) \geq 0 (\leq 0), \quad p \in \partial\omega^h,$$

then $W_i(p,t_k) \geq 0 (\leq 0)$ in $\overline{\omega}^h$.

(ii) *The following estimate to the solution to (1.14) hold true.*

$$\|W_i(.,t_k)\|_{\overline{\omega}^h} \leq \max\left\{\|g_i(.,t_k)\|_{\partial\omega^h}, \max_{p\in\omega^h} \frac{|\Phi_i(p,t_k)|}{\overline{c}_i(p,t_k) + \tau_k^{-1}}\right\},\ i = 1,2,$$

where $\|W_i(.,t_k)\|_{\overline{\omega}^h} = \max_{p\in\partial\omega^h} |W_i(p,t_k)|$ and $\|g_i(.,t_k)\|_{\partial\omega^h} = \max_{p\in\partial\omega^h} |g_i(p,t_k)|$.

1.1.2. Monotone Iterations in Abstract Space

It is interesting to note that solutions of differential and integro-differential equations can be seen as solutions of fixed point of nonlinear operators in the abstract space. In this regard, fixed point theorems in the setting of partially ordered abstract space can be utilized for proving existence and uniqueness theorems for these equations that also guarantee the existence of monotone sequences that will converge to the solutions. The literature for fixed point theorems is very vast and out of all those works, a few latest results are recalled in this subsection. For a detailed study, one can refer to [**29, 34, 36, 44, 103**] and the references therein. For example, the following fixed point theorem in the abstract space [**29**] can be used to obtain the monotone successive iterative schemes for the two-point boundary value problem $-\ddot{x}(t) = f(t, x(t)),\ t \in (0, 1),\ x(0) = x(1) = 0,$ where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a function continuous in the first variable and continuously differentiable in the second one.

Theorem 1.1.4. [**29**, pp. 396] *Let X be a real Banach space with a normal order cone and $T : X \to X$. Assume that u_0 and v_0 is a subsolution and a supersolution of the operator equation $u = T(u)$ respectively and $u_0 \leq v_0$. If T is a compact monotone increasing operator on the order interval $[u_0, v_0]$, then both the iterative sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ given by $u_{n+1} = T(u_n),\ v_{n+1} = T(v_n),\ n = 0, 1, 2, \ldots$ are defined, converge and $u = \lim\limits_{n\to\infty} u_n$ is the smallest fixed point and $v = \lim\limits_{n\to\infty} v_n$ is the largest fixed point respectively of T in $[u_0, v_0]$. Furthermore, we have the error estimates $u_n \leq u \leq v \leq v_n$ for all $n = 0, 1, \cdots$.*

Attempting to accelerate the monotone iterative procedure in partially ordered abstract spaces, fixed point theorems via quasilinearization scheme and its applications to differential equations were also investigated. In this direction, A. Buica and R. Precup [**17**], M.A. El-Gebeily et.al. [**31**], V. Lakshmikantham et.al. [**51**] and V.A. Vijesh and K.H. Kumar [**98**] are a few. [**31**] studied monotone quasilinearization method for nonlinear periodic boundary value problem

$$\begin{aligned} &u^{(n)}(t) = f(t, u(t)),\ t \in I = [0, T] \\ &u^{(i)}(0) - u^{(i)}(T) = c_i,\ i = 0, 1, \ldots, n-1 \end{aligned}$$

by proving monotone quasilinearization method for an operator equation in reflexive Banach space. One of the main theorems in this study is given below:

Theorem 1.1.5. [**31**] *Assume that the nonlinear problem $Au = f$, where A is a nonlinear operator has a lower solution α_0 and an upper solution β_0 such that $\alpha_0 \leq \beta_0$. Assume also that A satisfies strict positivity, differentiability and sup-positivity conditions. Then the two sequences of lower and upper solutions, $\{\alpha_n\}$ and $\{\beta_n\}$ quadratically converge to a solution of the nonlinear problem.*

An interesting generalized quasilinearization method for operator equation was studied in [**17**] and its applications to nonlinear elliptic problems were considered in [**18**]. Similar results were obtained by V. Lakshmikantham et. al [**51**] and existence-uniqueness theorems for initial value problem and semilinear parabolic initial boundary value problem via monotone quasilinearization were successfully deduced. One of the abstract fixed point results in [**51**] is as follows:

Theorem 1.1.6. [**51**] *Let E be an ordered Banach space with regular order cone E_+. Assume that $T : E \to E$ satisfies the following hypotheses:*

(i) *There exist $v_0, w_0 \in E$ such that $v_0 \leq Tv_0,\ Tw_0 \leq w_0$ and $v_0 \leq w_0$.*

(ii) *The Frechet derivative $T^{'}(u)$ exists for every $u \in [v_0, w_0]$ and $u \mapsto T^{'}uv$ is increasing on $[v_0, w_0]$ for all $v \in E_+$.*

(iii) *$[I - T^{'}(u)]^{-1}$ exists and is a bounded and positive operator for all $u \in [v_0, w_0]$. Then, for $n \in \mathbb{N}$, relations*

$$v_{n+1} = Tv_n + T^{'}(v_n)(v_{n+1} - v_n);\ v_{n+1} = Tv_n + T^{'}(v_n)(v_{n+1} - v_n)$$

define an increasing sequence $(v_n)_{n=0}^{\infty}$ and a decreasing sequence $(w_n)_{n=0}^{\infty}$ which converge to fixed points of T. These fixed points are equal if

(iv) *$Tu_1 - Tu_0 < u_1 - u_0$ whenever $v_0 \leq u_0 < u_1 \leq w_0$. Moreover, if (i) $-$ (iv) hold along with*

(v) *$\|T^{'}(u) - T^{'}(v)\| \leq L\|u - v\|$ for some $L > 0$ whenever $v_0 \leq v \leq u \leq w_0$.*

(vi) $\sup\{[I - T^{'}(u)]^{-1} :\ u \in [v_0, w_0]\} < \infty$, *then the sequences $(v_n)_{n=0}^{\infty}$ and $(w_n)_{n=0}^{\infty}$ converge quadratically to the same fixed point of T.*

The literature on the method of lower and upper solutions is enormous. This review is an outcome of a short literature survey done on this area and our references are mere representatives only.

1.2. PRELIMINARIES

This section provides certain preliminary definitions and results that are of use in the following chapters.

Definition 1.2.1. *A matrix A is said to be inverse positive if A^{-1} exists and is positive.*

Definition 1.2.2. *A matrix A is said to be monotone if $Ax \geq 0 \Rightarrow x \geq 0$.*

Definition 1.2.3. *An $n \times n$ real matrix $A = (a_{i,j})$ is said to be a $\mathbb{Z}$-matrix if $a_{i,j} \leq 0$ for all $i \neq j;\ 1 \leq i, j \leq n$ [**86**].*

Definition 1.2.4. *An $n \times n$ matrix A that can be expressed in the form $A = sI - B$ where $B = (b_{i,j})$ with $b_{i,j} \geq 0$ for all $1 \leq i, j \leq n$ and $s \geq \rho(B)$, the maximum of the moduli of the eigenvalues of B is called an $\mathbb{M}$-matrix [**86**].*

If A is an $n \times n$ real $\mathbb{Z}$-matrix, then the following statements are equivalent to A being a nonsingular $\mathbb{M}$-matrix.

- All the principal minors of A are positive.
- A is inverse positive.
- A is monotone.

Theorem 1.2.1. [**102**] *A matrix A of order N is irreducible if and only if $N = 1$ or given any two distinct integers i and j with $1 \leq i, j \leq N$, then $a_{i,j} \neq 0$ or there exist $i_1, i_2, \ldots, i_n$ such that*

$$a_{i,i_1} a_{i_1,i_2} \ldots a_{i_n j} \neq 0.$$

Let Ω be either a bounded or an unbounded open domain in $\mathbb{R}^n$. Then $C^m(\Omega)$ denotes the collection of all continuous functions whose partial derivatives up to the m^{th} order are continuous in Ω.

Definition 1.2.5. *A function $u \in C(\Omega)$ is said to be Hölder continuous of order $\alpha \in (0, 1)$ if*

$$H_\alpha \equiv \sup \left\{ \frac{|u(x) - u(\xi)|}{|x - \xi|^\alpha};\ x, \xi \in \Omega \text{ and } x \neq \xi \right\} < \infty.$$

$C^{m+\alpha}(\Omega)$ denotes the collection of all functions in $C^m(\Omega)$ that are Hölder continuous in Ω with exponent $\alpha \in (0, 1)$.

1.2.1. Approximating Derivatives using Finite Difference

In finite difference approximations, the following formula are used to approximate the derivatives of a function $u(x, t)$ of two variables. Note that h and k denote the step sizes in x and t directions respectively.

1. **Forward difference approximation** of $u_x(x, t)$

$$u_x(x, t) \approx \frac{u(x + h, t) - u(x, t)}{h}$$

2. **Forward difference approximation** of $u_t(x, t)$

$$u_t(x, t) \approx \frac{u(x, t + k) - u(x, t)}{k}$$

3. **Backward difference approximation** of $u_x(x, t)$

$$u_x(x, t) \approx \frac{u(x, t) - u(x - h, t)}{h}$$

4. **Backward difference approximation** of $u_t(x, t)$

$$u_t(x, t) \approx \frac{u(x, t) - u(x, t - k)}{k}$$

5. **Central difference approximation** of $u_x(x, t)$

$$u_x(x, t) \approx \frac{u(x + h, t) - u(x - h, t)}{2h}$$

6. **Central difference approximation** of $u_{xx}(x, t)$

$$u_{xx}(x, t) \approx \frac{u(x + h, t) - 2u(x, t) + u(x - h, t)}{h^2}$$

1.3. DESCRIPTION OF THE RESEARCH WORK

Chapter 2 deals with a mathematical model arising in interphase heat transfer for a catalytic converter where the vehicle and converter temperatures are governed by a coupled system of first order partial differential equations given by

$$
\begin{aligned}
&\tfrac{\partial u}{\partial t} + a\tfrac{\partial u}{\partial x} + cu = cv, \quad t > 0,\ 0 < x \leq l,\\
&\tfrac{\partial v}{\partial t} + bv = bu + \lambda \exp(v), \quad t > 0,\ 0 < x \leq l,\\
&u(0,t) = \eta,\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ t > 0,\ 0 < x \leq l.
\end{aligned}
\tag{1.15}
$$

An accelerated iterative procedure is proposed by a modification to the iterative scheme in [**21**] by combining successive iteration and quasilinearization together with monotone method. The first part of the chapter proves an existence and uniquenss result for (1.15) via the proposed accelerated iterative procedure. In the second part, based on this iterative procedure, a finite difference scheme is proposed to solve the coupled system numerically. Interestingly, the proposed iterative scheme not only accelerates but also preserves the monotone property. Moreover, a detailed error estimate is also derived. The following are the two major theorems given in this chapter.

Theorem 1.3.1. *Let (α, β) and $(\overline{\alpha}, \overline{\beta})$ be a pair of ordered lower and upper solutions of* (1.15)*. Then the minimal sequence $\{(\alpha^n, \beta^n)\}$ and the maximal sequence $\{(\overline{\alpha}^n, \overline{\beta}^n)\}$ converge monotonically to the unique solution (u^*, v^*) of* (1.15) *in S and the relation*

$$(\alpha, \beta) \leq (\alpha^n, \beta^n) \leq (\alpha^{n+1}, \beta^{n+1}) \leq (\alpha^*, \beta^*) \leq (\overline{\alpha}^*, \overline{\beta}^*) \leq (\overline{\alpha}^{n+1}, \overline{\beta}^{n+1}) \leq (\overline{\alpha}^n, \overline{\beta}^n) \leq (\overline{\alpha}, \overline{\beta})$$

holds for $n = 1, 2, \ldots,$ where $S = \{(u, v) \in C(\overline{Q}) \times C(\overline{Q}) : (\alpha, \beta) \leq (u, v) \leq (\overline{\alpha}, \overline{\beta})\}$. Moreover, $\|u^* - \alpha^{n+1}\| \leq \|u^* - \alpha^n\|$ and $\|v^* - \beta^{n+1}\| \leq C\left(\|u^* - \alpha^{n+1}\| + \|v^* - \beta^n\|^2\right)$ also hold for all $n \in \mathbb{N}$ with some positive constant C.

Let $h = \triangle x,\ k = \triangle k$ be the space and time increments and let $x_i = ih,\ t_j = jk$ be a mesh point in $[0, l] \times [0, T]$. The sets of mesh points (x_i, t_j) in $[0, l] \times [0, T]$ is denoted by $\bar{\Lambda}$. Define $u_{i,j} = u(x_i, t_j)$ and $v_{i,j} = v(x_i, t_j)$. Using the backward implicit approximation for first order differential equations, (1.15) is approximated by the finite difference system

$$
\begin{aligned}
&A^1 u_{i,j} = ckv_{i,j} + u_{i,j-1} + \tfrac{ak}{h} u_{i-1,j}, \\
&A^2 v_{i,j} = v_{i,j-1} + bku_{i,j} + k\lambda \exp(v_{i,j}), \\
&u_{0,j} = \eta_j,\ u_{i,0} = \psi_i,\ v_{i,0} = \phi_i;\ i = 1, 2, \ldots, M,\ j = 1, 2, \ldots, N
\end{aligned}
\tag{1.16}
$$

with $A^1 = 1 + \frac{ak}{h} + ck$ and $A^2 = 1 + bk$.

Theorem 1.3.2. *Let* $(\alpha_{i,j}, \beta_{i,j})$ *and* $(\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})$ *be a pair of ordered lower and upper solutions of* (1.16). *Then the minimal sequence* $\{(\alpha^n_{i,j}, \beta^n_{i,j})\}$ *and the maximal sequence* $\{(\overline{\alpha}^n_{i,j}, \overline{\beta}^n_{i,j})\}$ *are well defined and converge monotonically to the unique solution* $(u^*_{i,j}, v^*_{i,j})$ *of* (1.16) *in* $\bar{\Lambda}$ *and the relation*

$$
\begin{aligned}
(\alpha_{i,j}, \beta_{i,j}) \leq (\alpha^n_{i,j}, \beta^n_{i,j}) \leq (\alpha^{n+1}_{i,j}, \beta^{n+1}_{i,j}) \leq (\alpha^*_{i,j}, \beta^*_{i,j}) \leq (\overline{\alpha}^*_{i,j}, \overline{\beta}^*_{i,j}) \\
\leq (\overline{\alpha}^{n+1}_{i,j}, \overline{\beta}^{n+1}_{i,j}) \leq (\overline{\alpha}^n_{i,j}, \overline{\beta}^n_{i,j}) \leq (\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})
\end{aligned}
$$

holds for every $(i,j) \in \bar{\Lambda}$ *and* $n = 1, 2, \ldots$ *and if* $(u^*_{i,j}, v^*_{i,j})$ *for all* $(i,j) \in \bar{\Lambda}$ *is the solution of* (1.16), *then there exists a positive constant* C *such that*

$$
\left\| \begin{pmatrix} e^{n+1}_{i,j} \\ \underline{e}^{n+1}_{i,j} \end{pmatrix} \right\|_\infty \leq C \left[\left\| \begin{pmatrix} e^{n+1}_{i,j-1} + \frac{ak}{h} e^{n+1}_{i-1,j} \\ \underline{e}^{n+1}_{i,j-1} \end{pmatrix} \right\|_\infty + \left\| \begin{pmatrix} ck\underline{e}^n_{i,j} \\ k\lambda \exp(\xi^*) \left(\underline{e}^n_{i,j}\right)^2 \end{pmatrix} \right\|_\infty \right]
$$

where $e^{n+1}_{i,j} = u^*_{i,j} - \alpha^{n+1}_{i,j}$, $\underline{e}^{n+1}_{i,j} = v^*_{i,j} - \beta^{n+1}_{i,j}$ *and* $\xi^* = \max\{\overline{\beta}_{i,j} : (i,j) \in \bar{\Lambda}\}$.

Chapter 3 focuses on developing an alternative iterative procedure for solving (1.15). In the proposed procedure at each step of the iterative scheme, instead of solving two linear PDEs separately one has to solve a coupled linear PDE, a modification to the second chapter and [**82**]. The chapter renders the existence and uniqueness of (1.15). Based on the new procedure, a finite difference method is developed to solve the coupled system numerically and its convergence analysis is also provided. One of the main theorems in this chapter is

Theorem 1.3.3. *Let* $(\alpha_{i,j}, \beta_{i,j})$ *and* $(\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})$ *be a pair of ordered lower and upper solutions of* (1.16). *Then the minimal sequence* $\left\{(\alpha^n_{i,j}, \beta^n_{i,j})\right\}$ *and the maximal sequence* $\left\{(\overline{\alpha}^n_{i,j}, \overline{\beta}^n_{i,j})\right\}$ *converge monotonically to the unique solution* $(u^*_{i,j}, v^*_{i,j})$ *of* (1.16) *in*

$$
S^d = \{(u_{i,j}, v_{i,j}) \in \mathbb{R}^2 : (\alpha_{i,j}, \beta_{i,j}) \leq (u_{i,j}, v_{i,j}) \leq (\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})\}.
$$

Also, the relation

$$(\alpha_{i,j},\beta_{i,j})\le(\alpha^n_{i,j},\beta^n_{i,j})\le(\alpha^{n+1}_{i,j},\beta^{n+1}_{i,j})\le(u^*_{i,j},v^*_{i,j})\le(\overline{\alpha}^{n+1}_{i,j},\overline{\beta}^{n+1}_{i,j})\le(\overline{\alpha}^{n}_{i,j},\overline{\beta}^{n}_{i,j})\le(\overline{\alpha}_{i,j},\overline{\beta}_{i,j})$$

holds for every $(i,j)\in\bar{\Lambda}$ *and* $n=1,2,3,\cdots$. *Moreover, the following estimate holds*

$$\left\|\begin{pmatrix} e^{n+1}_{i,j} \\ \underline{e}^{n+1}_{i,j}\end{pmatrix}\right\|_\infty \le C\left\|\begin{pmatrix} e^{n+1}_{i,j-1}+\frac{ak}{h}e^{n+1}_{i-1,j} \\ \underline{e}^{n+1}_{i,j-1}+k\lambda\exp(\xi^*)\left(\frac{(\underline{e}^n_{i,j})^2+(\underline{e}_{i,j})^2)}{2}\right)\end{pmatrix}\right\|_\infty,$$

where $e^{n+1}_{i,j}=u^*_{i,j}-\alpha^{n+1}_{i,j}$, $\underline{e}^{n+1}_{i,j}=v^*_{i,j}-\beta^{n+1}_{i,j}$, $\underline{e}_{i,j}=v^*_{i,j}-\beta_{i,j}$ *and* $\xi^*=\max\{\overline{\beta}_{i,j}:(i,j)\in\bar{\Lambda}\}$.

In **Chapter 4**, a class of fourth order nonlocal elliptic boundary value problem of the form

$$\begin{aligned}
&\triangle^2u-b_0\triangle u+c_0u=f(x,u),\ x\in\Omega,\\
&u(x)=\textstyle\int_\Omega\gamma(x,\xi)u(\xi)\mathrm{d}\xi+g^{(1)}(x),\ x\in\partial\Omega,\\
&(\triangle u)(x)=\textstyle\int_\Omega\gamma(x,\xi)(\triangle u)(\xi)\mathrm{d}\xi-g^{(0)}(x),\ x\in\partial\Omega,
\end{aligned}\tag{1.17}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n=1,2,\ldots)$ with boundary $\partial\Omega$, $b_0\ge 0$ and c_0 are constants and $f(x,u)$, $\gamma(x',x)$ and $g^{(l)}(x')$ $(l=0,1)$ are continuous functions in their respective domains is handled. The major aim of this chapter is to accelerate the iterative scheme in [**83**] ensuring the monotone property without any additional assumptions. The proposed alternative iterative scheme is found to be much more efficient than the scheme in [**83**] as it exhibits an immense reduction in the number of iterations required and provides greater flexibility in choosing the initial guess during numerical experiments. The proposed iterative scheme also ensures the existence and uniqueness of the solution of (1.17). (1.17) can be rewritten as

$$\begin{aligned}
&-\triangle u+\mu u=v,\ -\triangle v+\mu^+v=F(x,u),\ x\in\Omega,\\
&u(x)=\textstyle\int_\Omega\gamma(x,\xi)u(\xi)\mathrm{d}\xi+g^{(1)}(x),\ x\in\partial\Omega,\\
&v(x)=\textstyle\int_\Omega\gamma(x,\xi)v(\xi)\mathrm{d}\xi+g^{(2)}(x),\ x\in\partial\Omega,
\end{aligned}\tag{1.18}$$

where $g^{(2)}=g^{(0)}+\mu g^{(1)}$, $\mu=\frac{b_0-\sqrt{b_0^2-4c^*}}{2}$, $\mu^+=\frac{b_0+\sqrt{b_0^2-4c^*}}{2}$, $c^*=c_0+\overline{c}\ge 0$, $b_0^2\ge 4c^*$, $\overline{c}\ge\max\left\{-\frac{\partial f}{\partial u}(x,u)\ :\widehat{\alpha}\le u\le\widetilde{\alpha},\ x\in\Omega\right\}$ and $F(x,u)=\overline{c}u+f(x,u)$ [**81**]. Discretizing (1.17) using central difference approximation and rewriting it as a coupled equation as in the continuous case, one can have

$$
\begin{aligned}
&-\triangle_h u_i + \mu u_i = v_i,\ i \in \Omega_h; \quad u_j = J[x_j', u] + g_j^{(1)},\ j \in \partial\Omega_h \\
&-\triangle_h v_i + \mu^+ v_i = F(x_i, u_i),\ i \in \Omega_h; \quad v_j = J[x_j', v] + g_j^{(2)},\ j \in \partial\Omega_h,
\end{aligned} \tag{1.19}
$$

where $F(x_i, u_i) = \bar{c}u_i + f(x_i, u_i)$ and $g^{(2)} = g^{(0)} + \mu g^{(1)}$. Let u_k represents the approximation of $u(x_k)$ for any mesh point x_k. The following is one of the main theorems in this chapter.

Theorem 1.3.4. *Let* $((\widehat{\alpha}, \widehat{\beta}), (\widetilde{\alpha}, \widetilde{\beta}))$, $((\widehat{\alpha}_k, \widehat{\beta}_k), (\widetilde{\alpha}_k, \widetilde{\beta}_k))$ *be ordered lower and upper solutions of* (1.18) *and* (1.19) *respectively. Then the minimal solution* $(\underline{\alpha}_k^*, \underline{\beta}_k^*)$ *and the maximal solution* $(\overline{\alpha}_k^*, \overline{\beta}_k^*)$ *of* (1.19) *converge respectively to the minimal solution* $(\underline{\alpha}^*(x_k), \underline{\beta}^*(x_k))$ *and the maximal solution* $(\overline{\alpha}^*(x_k), \overline{\beta}^*(x_k))$ *of* (1.18) *at every point as mesh size tends to zero.*

Chapter 5 deals with nonlinear parabolic integro-differential equations of the form

$$
\begin{aligned}
&\tfrac{\partial u}{\partial t} - \tfrac{1}{\theta}\tfrac{\partial^2 u}{\partial x^2} + f(u, v) = q(x, t),\ (x, t) \in \omega \times (0, T], \\
&u(x, t) = h(x, t),\ (x, t) \in \partial\omega \times (0, T]; \quad u(x, 0) = \psi(x),\ x \in \overline{\omega},
\end{aligned} \tag{1.20}
$$

where $\theta \in \mathbb{R}^+ \cup \{0\}$, ω is a connected bounded domain in $\mathbb{R}^n$ $(n = 1, 2, \cdots)$, $h,\ q,\ \psi$ are smooth functions in their domains where $v(x, t)$ stands for $\int_0^t \exp(\lambda s)\kappa(t - s)u(x, s)\mathrm{d}s$, λ being a positive constant arising in nuclear reactors and population models [**72**]. The monotone property, convergence analysis and an error estimate in terms of stopping criteria are derived for nonlinear integro-differential equation of Volterra type. This work also generalizes the recent work of I. Bogleav in [**13**]. In contrast to the previous chapters, in this chapter, monotone property and convergence analysis are obtained using the comparison theorem for discretized problems by S. Samarskii [**90**].

Let $\overline{\omega}^h$ and $\overline{\omega}^\tau$ be the corresponding meshes for the space and time domains respectively, and h and τ_k denote the respective step sizes in x and t directions with $t_0 = 0$. Applying backward and central difference approximations for time and space respectively in (1.20), one can get the following:

$$
(1.21) \qquad \begin{aligned}
&\mathcal{L}U(p,t_k) + f(p,t_k,U,V) - \tau_k^{-1}U(p,t_{k-1}) = Q(p,t_k),\ (p,t_k) \in \omega^h \times (\omega^\tau \setminus \{0\}) \\
&U(p,t_k) = h(p,t_k),\ (p,t_k) \in \partial\omega^h \times (\omega^\tau \setminus \{0\}); \quad U(p,0) = \psi(p),\ p \in \overline{\omega}^h,
\end{aligned}
$$

where $\partial\omega^h$ denotes the boundary of $\overline{\omega}^h$. The following theorem that guarantees the convergence of the solution of the nonlinear difference scheme (1.21) to the solution of (1.20) as the mesh sizes tend to zero is the main theorem in this chapter.

Theorem 1.3.5. *Let $\widehat{U}(p,t_k)$ and $\widetilde{U}(p,t_k)$ be a pair of coupled lower and upper solutions of* (1.21)*. Then the minimal sequence $\{U_{-1}^{n+1}(p,t_k)\}$ and the maximal sequence $\{U_1^{n+1}(p,t_k)\}$ converge monotonically to the unique solution of* (1.21) *in the sector $< \widehat{U}(p,t_k), \widetilde{U}(p,t_k) >$ and for $p \in \overline{\omega}^h$ and $n \in \mathbb{N}$, satisfy*

$$
\widehat{U}(p,t_k) \le U_{-1}^{n}(p,t_k) \le U_{-1}^{n+1}(p,t_k) \le U_1^{n+1}(p,t_k) \le U_1^{n}(p,t_k) \le \widetilde{U}(p,t_k).
$$

CHAPTER 2

A COUPLED SYSTEM OF DIFFERENTIAL EQUATIONS FOR A CATALYTIC CONVERTER

This chapter[1] deals with an accelerated iterative procedure for a coupled system of partial differential equations arising from a catalytic converter model.

2.1. Introduction

Catalytic converter is a reliable emissions control device that converts toxic pollutants in exhaust gas to less toxic pollutants and is located in the exhaust system of automobiles. The increasing concern about the atmospheric pollution caused due to the harmful emissions from the vehicles leads to the development of various mathematical models for the study of interphase heat-transfer problem in catalytic converter [**22, 37, 38, 40, 62, 85, 87, 91**]. One of such models is studied in [**53**] where the vehicle and converter temperatures are governed by a coupled system of a first order partial differential equation and an ordinary differential equation. After suitable simplifications [**20, 21, 82**], the problem reduces to the following system.

$$
\begin{aligned}
&\tfrac{\partial u}{\partial t} + a\tfrac{\partial u}{\partial x} + cu = cv, \quad t > 0,\ 0 < x \le l \\
&\tfrac{\partial v}{\partial t} + bv = bu + \lambda \exp(v), \quad t > 0,\ 0 < x \le l \\
&u(0,t) = \eta,\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), \quad t > 0,\ 0 < x \le l.
\end{aligned} \tag{2.1}
$$

The existence and uniqueness of the classical solution of the above coupled system has been proved using the contraction principle in [**20**]. Later by coupling successive iteration and monotone method, the existence and uniqueness as well as the blowup property of the solution have been discussed in [**21**]. Based on the main theorem in [**21**], a finite difference based iterative procedure has been developed in [**82**] to solve the coupled system numerically. The study in [**82**] has also proved that the finite difference scheme preserves

[1]This chapter forms the paper by L.A. Sunny, R. Roy and V. A. Vijesh in Jornal of Mathematical Analysis and Applications, 445(2017), 318–336.

the monotone property. It is important to note that for the numerical method in [**82**] based on the successive approximation discussed in [**21**], the performance of the numerical scheme is slow.

In this chapter, to accelerate the iterative procedure, a modification to the iterative scheme in [**21**] is proposed. More specifically, by combining successive iteration and quasilinearization together with monotone method, an accelerated iterative procedure is proposed. The first part of the chapter discusses about the convergence analysis, error estimate as well as the monotone property of the proposed accelerated iterative procedure for the continuous case. In the second part, based on this iterative procedure, a new iterative scheme based on finite difference method is proposed to solve the coupled system numerically. This part also proves the convergence and the monotone property of the discretized version of the iterative procedure. Moreover, a detailed error estimate is also derived.

In the proposed iterative scheme, at each step one has to solve a system with variable coefficients distinct from [**21**] and [**82**] where constant coefficients are only dealt with. Consequently in the discretized case, a new comparison theorem is developed to obtain the monotone property of the sequences.

This chapter is organised as follows. Section 2.2 provides certain basic results that are used in the following sections. In Section 2.3, the existence and uniqueness of the coupled system (2.1) is proved via the new accelerated iterative scheme. This section also provides the error estimate for the iterative procedure. Section 2.4 gives the convergence analysis as well as the error estimate for the proposed numerical scheme. The convergence of the finite difference solution to the continuous solution as the mesh sizes tend to zero is obtained in Section 2.5. Some numerical results are given in Section 2.6 to illustrate the efficiency of the proposed scheme. A comparative study is also provided in this section.

2.2. Preliminaries

In this section, some basic results are stated that will be used to obtain the results in the following sections. The existence and uniqueness theorem for (2.1) using contraction principle discussed in [**20**] can be stated as follows.

Theorem 2.2.1. *[Theorem 6; [**20**]] Suppose $u_0(x) = u(x, 0) \in C^1[0, l]$ and $v_0(x) = v(x, 0) \in C^1[0, l]$ with $u_0(0) = \eta$. There is a constant $t_{max} > 0$ such that $[0, t_{max})$ is the maximal time interval for the unique solution (u, v) of the differential equation (2.1) on the interval $[0, l] \times [0, t_{max})$.*

In the first part of the chapter, the following lemmas are used to obtain the monotone property of the sequences.

Lemma 2.2.1. *[Lemma 1; [**21**]] If $w \in C^1(\overline{Q})$ satisfies the inequalities*

$$\begin{aligned} &\tfrac{\partial w}{\partial t} + a\tfrac{\partial w}{\partial x} + bw \geq 0; \quad (x, t) \in Q \\ &w(0, t) \geq 0 \quad t \in [0, T]; \quad w(x, 0) \geq 0 \quad x \in [0, l], \end{aligned}$$

where $a \geq 0$ and $b > 0$ are constants, then $w \geq 0$ on $\overline{Q}$.

Lemma 2.2.2. *Let $v \in C(\overline{Q})$ be continuously differentiable with respect to t such that*

$$\frac{\partial v}{\partial t} - f(x, t)v \geq 0,$$

where $f(x, t)$ is a continuous function defined on $\overline{Q}$ with $v(x, 0) \geq 0$ for $0 < x \leq l$. Then $v(x, t) \geq 0$ on $\overline{Q}$.

2.3. Convergence Analysis for the Continuous Case

This section provides a modification to the iterative procedure discussed in [**21**] which deals with variable coefficients unlike that in [**21**]. It also proves the convergence, error and the monotone property of the new iterative scheme. Let Q denote $(0, l] \times (0, T]$ and $\overline{Q}$ denote $[0, l] \times [0, T]$ where l and T are arbitrary positive constants.

Definition 2.3.1. *A function* $(\overline{\alpha}, \overline{\beta}) \in C^1(\overline{Q}) \times C^1(\overline{Q})$ *is called an upper solution of* (2.1) *if it satisfies*

$$(2.2) \qquad \begin{array}{l} \frac{\partial \overline{\alpha}}{\partial t} + a\frac{\partial \overline{\alpha}}{\partial x} + c\overline{\alpha} \geq c\overline{\beta}, \quad t > 0, 0 < x \leq l \\ \frac{\partial \overline{\beta}}{\partial t} + b\overline{\beta} \geq b\overline{\alpha} + \lambda \exp(\overline{\beta}), \quad t > 0, 0 < x \leq l \\ \overline{\alpha}(0,t) \geq \eta, \ \overline{\alpha}(x,0) \geq u_0(x), \ \overline{\beta}(x,0) \geq v_0(x), \ t > 0, 0 < x \leq l. \end{array}$$

Similarly $(\alpha, \beta) \in C^1(\overline{Q}) \times C^1(\overline{Q})$ *is called a lower solution if it satisfies* (2.2) *with the inequalities reversed.*

For a given pair of ordered lower and upper solutions of (2.1), set

$$S = \{(u,v) \in C(\overline{Q}) \times C(\overline{Q}) : (\alpha, \beta) \leq (u,v) \leq (\overline{\alpha}, \overline{\beta})\}.$$

Using (α, β) and $(\overline{\alpha}, \overline{\beta})$ respectively as the initial iterations (u^0, v^0), two sequences can be computed. Applying the successive approximation method to the first equation and the quasilinearization technique to the second in (2.1), an iterative scheme can be obtained as follows.

$$(2.3) \qquad \begin{array}{l} \frac{\partial u^{n+1}}{\partial t} + a\frac{\partial u^{n+1}}{\partial x} + cu^{n+1} = cv^n, \quad t > 0, 0 < x \leq l \\ \frac{\partial v^{n+1}}{\partial t} + (b - \lambda \exp(v^n))v^{n+1} = bu^{n+1} + \lambda \exp(v^n)(1 - v^n), \quad t > 0, 0 < x \leq l \\ u^{n+1}(0,t) = \eta, \ u^{n+1}(x,0) = u_0(x), \ v^{n+1}(x,0) = v_0(x), \ t > 0, 0 < x \leq l, \end{array}$$

$n = 0, 1, 2, \cdots$. Denote the sequence generated from the lower solution by $\{(\alpha^{n+1}, \beta^{n+1})\}$ and the upper solution by $\{(\overline{\alpha}^{n+1}, \overline{\beta}^{n+1})\}$ and refer to them as minimal and maximal sequences respectively. The iterative schemes are given by

$$(2.4) \qquad \begin{array}{l} \frac{\partial \alpha^{n+1}}{\partial t} + a\frac{\partial \alpha^{n+1}}{\partial x} + c\alpha^{n+1} = c\beta^n, \quad t > 0, 0 < x \leq l \\ \frac{\partial \beta^{n+1}}{\partial t} + (b - \lambda \exp(\beta^n))\beta^{n+1} = b\alpha^{n+1} + \lambda \exp(\beta^n) \\ \qquad\qquad\qquad\qquad -\lambda \exp(\beta^n)\beta^n, \quad t > 0, 0 < x \leq l \\ \alpha^{n+1}(0,t) = \eta, \ \alpha^{n+1}(x,0) = u_0(x), \ \beta^{n+1}(x,0) = v_0(x), \ t > 0, 0 < x \leq l. \end{array}$$

and

$$
(2.5)\qquad \begin{aligned}
&\frac{\partial\overline{\alpha}^{n+1}}{\partial t}+a\frac{\partial\overline{\alpha}^{n+1}}{\partial x}+c\overline{\alpha}^{n+1}=c\overline{\beta}^{n}, \quad t>0, 0<x\le l\\
&\frac{\partial\overline{\beta}^{n+1}}{\partial t}+(b-\lambda\exp(\beta^{n}))\overline{\beta}^{n+1}=b\overline{\alpha}^{n+1}+\lambda\exp(\overline{\beta}^{n})\\
&\qquad\qquad\qquad\qquad\qquad\qquad -\lambda\exp(\beta^{n})\overline{\beta}^{n}, \quad t>0, 0<x\le l\\
&\overline{\alpha}^{n+1}(0,t)=\eta,\ \overline{\alpha}^{n+1}(x,0)=u_0(x),\ \overline{\beta}^{n+1}(x,0)=v_0(x),\ t>0, 0<x\le l.
\end{aligned}
$$

respectively.

Remark 2.3.1. *When the lower and upper solutions $(\alpha,\beta),(\overline{\alpha},\overline{\beta})\in C^1(\overline{Q})\times C^1(\overline{Q})$, the iterative procedures* (2.4) *and* (2.5) *are well defined. For more details, one can refer to* [**20**].

Define $L_1u=\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}+cu$ and $L_2v=\frac{\partial v}{\partial t}+bv$. Then (2.3) can be rewritten as

$$
\begin{aligned}
&L_1u^{n+1}=cv^{n}, \quad t>0, 0<x\le l\\
&L_2v^{n+1}-\lambda\exp(v^{n})v^{n+1}=bu^{n+1}+\lambda\exp(v^{n})(1-v^{n}), \quad t>0, 0<x\le l\\
&u^{n+1}(0,t)=\eta,\ u^{n+1}(x,0)=u_0(x),\ v^{n+1}(x,0)=v_0(x),\ t>0, 0<x\le l,
\end{aligned}
$$

where $n=0,1,2,\cdots$. The monotone property of both the sequences (2.4) and (2.5) and their convergence to the unique solution of (2.1) is given in the following theorem.

Theorem 2.3.1. *Let (α,β) and $(\overline{\alpha},\overline{\beta})$ be a pair of ordered lower and upper solutions of* (2.1). *Then*

(i) *the minimal sequence $\{(\alpha^n,\beta^n)\}$ and the maximal sequence $\{(\overline{\alpha}^n,\overline{\beta}^n)\}$ converge monotonically to the unique solution (u^*,v^*) of* (2.1) *in S.*

(ii) *the relation*

$$
(2.6)\qquad \begin{aligned}
(\alpha,\beta)\le(\alpha^n,\beta^n) \ &\le\ (\alpha^{n+1},\beta^{n+1})\le(\alpha^*,\beta^*)\le\\
&(\overline{\alpha}^*,\overline{\beta}^*)\le(\overline{\alpha}^{n+1},\overline{\beta}^{n+1})\le(\overline{\alpha}^{n},\overline{\beta}^{n})\le(\overline{\alpha},\overline{\beta})
\end{aligned}
$$

holds for $n=1,2,\cdots$.

Proof. The monotone property of both the minimal and maximal sequences is obtained first. Let $\overline{w}^0=\overline{\alpha}^0-\overline{\alpha}^1=\overline{\alpha}-\overline{\alpha}^1$ and $\overline{z}^0=\overline{\beta}^0-\overline{\beta}^1=\overline{\beta}-\overline{\beta}^1$.

$$
L_1\overline{w}^0=L_1\overline{\alpha}-L_1\overline{\alpha}^1\ge c\overline{\beta}-c\overline{\beta}=0
$$

$\overline{w}^0(0,t) = \overline{\alpha}(0,t) - \eta \geq 0$ and $\overline{w}^0(x,0) = \overline{\alpha}(x,0) - u_0(x) \geq 0$. By Lemma 2.2.1, $\overline{w}^0 \geq 0$ on $\overline{Q}$. Hence $\overline{\alpha} \geq \overline{\alpha}^1$. Also

$$\begin{aligned}
L_2\overline{z}^0 - \lambda\exp(\beta)\overline{z}^0 &= (L_2\overline{\beta} - \lambda\exp(\beta)\overline{\beta}) - (L_2\overline{\beta}^1 - \lambda\exp(\beta)\overline{\beta}^1) \\
&= \left(\frac{\partial\overline{\beta}}{\partial t} + b\overline{\beta} - \lambda\exp(\beta)\overline{\beta}\right) - \left(\frac{\partial\overline{\beta}^1}{\partial t} + b\overline{\beta}^1 - \lambda\exp(\beta)\overline{\beta}^1\right) \\
L_2\overline{z}^0 - \lambda\exp(\beta)\overline{z}^0 &\geq \big(b\overline{\alpha} + \lambda\exp(\overline{\beta}) - \lambda\exp(\beta)\overline{\beta}\big) - \big(b\overline{\alpha}^1 + \lambda\exp(\overline{\beta}) - \lambda\exp(\beta)\overline{\beta}\big) \\
L_2\overline{z}^0 - \lambda\exp(\beta)\overline{z}^0 &\geq b\overline{\alpha} - b\overline{\alpha}^1 \geq 0 \\
\overline{z}^0(x,0) &= \overline{\beta}(x,0) - v_0(x) \geq 0.
\end{aligned}$$

Hence $\overline{z}^0 \geq 0$ and thus $(\overline{\alpha}, \overline{\beta}) \geq (\overline{\alpha}^1, \overline{\beta}^1)$ on $\overline{Q}$. Similarly $(\alpha, \beta) \leq (\alpha^1, \beta^1)$. Now let $w^1 = \overline{\alpha}^1 - \alpha^1$ and $z^1 = \overline{\beta}^1 - \beta^1$.

$$L_1 w^1 = L_1\overline{\alpha}^1 - L_1\alpha^1 = c\overline{\beta} - c\beta \geq 0$$

and $w^1(0,t) = 0,\ w^1(x,0) = 0$. By Lemma 2.2.1, $w^1 \geq 0$ on $\overline{Q}$. Hence $\overline{\alpha}^1 \geq \alpha^1$. Also

$$\begin{aligned}
L_2 z^1 - \lambda\exp(\beta)z^1 &= (L_2\overline{\beta}^1 - \lambda\exp(\beta)\overline{\beta}^1) - (L_2\beta^1 - \lambda\exp(\beta)\beta^1) \\
&= \left(\frac{\partial\overline{\beta}^1}{\partial t} + b\overline{\beta}^1 - \lambda\exp(\beta)\overline{\beta}^1\right) - \left(\frac{\partial\beta^1}{\partial t} + b\beta^1 - \lambda\exp(\beta)\beta^1\right) \\
&= \big(b\overline{\alpha}^1 + \lambda\exp(\overline{\beta}) - \lambda\exp(\beta)\overline{\beta}\big) - \big(b\alpha^1 + \lambda\exp(\beta) - \lambda\exp(\beta)\beta\big) \\
&\geq \lambda\exp(\tilde{\beta})(\overline{\beta} - \beta) - \lambda\exp(\beta)(\overline{\beta} - \beta); \quad \beta \leq \tilde{\beta} \leq \overline{\beta} \\
L_2 z^1 - \lambda\exp(\beta)z^1 &\geq 0
\end{aligned}$$

together with $z^1(x,0) = 0$ conclude that $z^1 \geq 0$ and thus $(\alpha^1, \beta^1) \leq (\overline{\alpha}^1, \overline{\beta}^1)$ on $\overline{Q}$. The above conclusions show that

$$(\alpha, \beta) \leq (\alpha^1, \beta^1) \leq (\overline{\alpha}^1, \overline{\beta}^1) \leq (\overline{\alpha}, \overline{\beta}).$$

Assume that

$$(\alpha^{n-1}, \beta^{n-1}) \leq (\alpha^n, \beta^n) \leq (\overline{\alpha}^n, \overline{\beta}^n) \leq (\overline{\alpha}^{n-1}, \overline{\beta}^{n-1})$$

for some $n > 1$. Clearly $(\alpha^{n+1}, \beta^{n+1})$ and $(\overline{\alpha}^{n+1}, \overline{\beta}^{n+1})$ exist. Define $\overline{w}^n = \overline{\alpha}^n - \overline{\alpha}^{n+1}$ and $\overline{z}^n = \overline{\beta}^n - \overline{\beta}^{n+1}$.

$$L_1\overline{w}^n = L_1\overline{\alpha}^n - L_1\overline{\alpha}^{n+1} = c\overline{\beta}^{n-1} - c\overline{\beta}^n \geq 0$$

with $\overline{w}^n(0,t) = 0;\ \overline{w}^n(x,0) = 0$. By Lemma 2.2.1, $\overline{w}^n \geq 0$ on $\overline{Q}$. Hence $\overline{\alpha}^n \geq \overline{\alpha}^{n+1}$. Also

$$\begin{aligned}
L_2\overline{z}^n - \lambda\exp(\beta^n)\overline{z}^n &= (L_2\overline{\beta}^n - \lambda\exp(\beta^n)\overline{\beta}^n) - (L_2\overline{\beta}^{n+1} - \lambda\exp(\beta^n)\overline{\beta}^{n+1}) \\
&= \Big(b\overline{\alpha}^n + \lambda\exp(\overline{\beta}^{n-1}) + \lambda\exp(\beta^{n-1})(\overline{\beta}^n - \overline{\beta}^{n-1}) - \lambda\exp(\beta^n)\overline{\beta}^n\Big) \\
&\quad - \Big(b\overline{\alpha}^{n+1} + \lambda\exp(\overline{\beta}^n) - \lambda\exp(\beta^n)\overline{\beta}^n\Big) \\
&\geq \lambda\exp(\overline{\beta}^{n-1}) + \lambda\exp(\beta^{n-1})(\overline{\beta}^n - \overline{\beta}^{n-1}) - \lambda\exp(\overline{\beta}^n) \\
&\geq \lambda\exp(\hat{\beta})(\overline{\beta}^{n-1} - \overline{\beta}^n) - \lambda\exp(\beta^{n-1})(\overline{\beta}^{n-1} - \overline{\beta}^n);\ \overline{\beta}^n \leq \hat{\beta} \leq \overline{\beta}^{n-1} \\
L_2\overline{z}^n - \lambda\exp(\beta^n)\overline{z}^n &\geq 0
\end{aligned}$$

together with $\overline{z}^n(x,0) = 0$ conclude that $\overline{z}^n \geq 0$ and thus $(\overline{\alpha}^n, \overline{\beta}^n) \geq (\overline{\alpha}^{n+1}, \overline{\beta}^{n+1})$ on $\overline{Q}$. A similar reasoning using the property of lower solution gives $(\alpha^n, \beta^n) \leq (\alpha^{n+1}, \beta^{n+1})$. Now let $w^{n+1} = \overline{\alpha}^{n+1} - \alpha^{n+1}$ and $z^{n+1} = \overline{\beta}^{n+1} - \beta^{n+1}$.

$$L_1 w^{n+1} = L_1\overline{\alpha}^{n+1} - L_1\alpha^{n+1} = c\overline{\beta}^n - c\beta^n \geq 0$$

with $w^{n+1}(0,t) = 0,\ w^{n+1}(x,0) = 0$. By Lemma 2.2.1, $w^{n+1} \geq 0$ on $\overline{Q}$. Hence $\overline{\alpha}^{n+1} \geq \alpha^{n+1}$. Also

$$\begin{aligned}
L_2 z^{n+1} - \lambda\exp(\beta^n)z^{n+1} &= (L_2\overline{\beta}^{n+1} - \lambda\exp(\beta^n)\overline{\beta}^{n+1}) - (L_2\beta^{n+1} - \lambda\exp(\beta)^n\beta^{n+1}) \\
&= b(\overline{\alpha}^{n+1} - \alpha^{n+1}) + \lambda\exp(\overline{\beta}^n) - \lambda\exp(\beta^n) \\
&\qquad + \lambda\exp(\beta^n)\beta^n - \lambda\exp(\beta^n)\overline{\beta}^n \\
L_2 z^{n+1} - \lambda\exp(\beta^n)z^{n+1} &\geq 0
\end{aligned}$$

together with $z^{n+1}(x,0) = 0$ conclude that $z^{n+1} \geq 0$ and thus $(\alpha^{n+1}, \beta^{n+1}) \leq (\overline{\alpha}^{n+1}, \overline{\beta}^{n+1})$ on $\overline{Q}$. Thus

$$(\alpha, \beta) \leq (\alpha^n, \beta^n) \leq (\alpha^{n+1}, \beta^{n+1}) \leq (\overline{\alpha}^{n+1}, \overline{\beta}^{n+1}) \leq (\overline{\alpha}^n, \overline{\beta}^n) \leq (\overline{\alpha}, \overline{\beta})$$

for all n and this guarantees the existence of the limits

$$\lim_{n\to\infty}(\alpha^n, \beta^n) = (\alpha^*, \beta^*), \quad \lim_{n\to\infty}(\overline{\alpha}^n, \overline{\beta}^n) = (\overline{\alpha}^*, \overline{\beta}^*). \tag{2.7}$$

Moreover, both the limits are solutions of (2.1). The uniqueness of the solution $(\alpha^*, \beta^*) = (\overline{\alpha}^*, \overline{\beta}^*) = (u^*, v^*)$ follows from Theorem 6 of [**20**]. □

Theorem 2.3.2. *For all $n \in \mathbb{N}$, the following error estimates hold.*

$$\|u^* - \alpha^{n+1}\| \leq \|u^* - \alpha^n\| \tag{2.8}$$

$$\|v^* - \beta^{n+1}\| \leq C\left(\|u^* - \alpha^{n+1}\| + \|v^* - \beta^n\|^2\right) \tag{2.9}$$

for some positive constant C.

Proof. From the monotone property, (2.8) trivially holds. Now define $e^{n+1} = u^*(x,t) - \alpha^{n+1}(x,t)$ and $\underline{e}^{n+1} = v^*(x,t) - \beta^{n+1}(x,t)$.

$$\begin{aligned}
\underline{e}^{n+1} &= v^*(x,t) - v^{n+1}(x,t) \\
&= b\int_0^t \exp(b(\tau-t))e^{n+1}\mathrm{d}\tau + \lambda\int_0^t \exp(b(\tau-t))\left(\exp(v^*(x,\tau) - \exp(v^n(x,\tau))\right)\mathrm{d}\tau \\
&\quad -\lambda\int_0^t \exp(b(\tau-t))\exp(v^n(x,\tau))(v^{n+1}(x,\tau) - v^n(x,\tau))\mathrm{d}\tau \\
&= b\int_0^t \exp(b(\tau-t))e^{n+1}\mathrm{d}\tau + \lambda\int_0^t \exp(b(\tau-t))\exp(\hat{v}(x,\tau))\underline{e}^n\mathrm{d}\tau \\
&\quad +\lambda\int_0^t \exp(b(\tau-t))\exp(v^n(x,\tau))\underline{e}^{n+1}\mathrm{d}\tau \\
&\quad -\lambda\int_0^t \exp(b(\tau-t))\exp(v^n(x,\tau))\underline{e}^n\mathrm{d}\tau; \quad v^n \leq \hat{v} \leq v^* \\
&\leq b\int_0^t \exp(b(\tau-t))e^{n+1}\mathrm{d}\tau + \lambda\int_0^t \exp(b(\tau-t))\exp(v^n(x,\tau))\underline{e}^{n+1}\mathrm{d}\tau \\
&\quad +\lambda\int_0^t \exp(b(\tau-t))(\exp(v^*(x,\tau)) - \exp(v^n(x,\tau)))\underline{e}^n\mathrm{d}\tau \\
&\leq b\int_0^t \exp(b(\tau-t))e^{n+1}\mathrm{d}\tau + \lambda\int_0^t \exp(b(\tau-t))\exp(v^*(x,\tau))\underline{e}^{n+1}\mathrm{d}\tau \\
&\quad +\lambda\int_0^t \exp(b(\tau-t))\exp(v^*(x,\tau))(\underline{e}^n)^2\mathrm{d}\tau \\
&\leq b\int_0^T \exp(b(\tau-t))e^{n+1}\mathrm{d}\tau + \lambda\int_0^t \exp(b(\tau-t))\exp(v^*(x,\tau))\underline{e}^{n+1}\mathrm{d}\tau \\
&\quad +\lambda\int_0^T \exp(b(\tau-t))\exp(v^*(x,\tau))(\underline{e}^n)^2\mathrm{d}\tau \\
&\leq bK_1\|e^{n+1}\| + \lambda K_2\|\underline{e}^n\|^2 + \lambda\int_0^t |\exp(b(\tau-t))\exp(v^*(x,\tau))|\underline{e}^{n+1}\mathrm{d}\tau \\
\underline{e}^{n+1} &\leq K_3\left(\|e^{n+1}\| + \|\underline{e}^n\|^2\right) + \lambda\int_0^t |\exp(b(\tau-t))\exp(v^*(x,\tau))|\underline{e}^{n+1}\mathrm{d}\tau,
\end{aligned}$$

where $K_3 = \max\{bK_1, \lambda K_2\}$ and K_1 and K_2 are positive constants. Applying Gronwall's inequality,

$$\begin{aligned} \underline{e}^{n+1} &\leq K_3 \left(\|e^{n+1}\| + \|\underline{e}^n\|^2\right) \left\{1 + \lambda \int_0^t |\exp(b(\tau - t)) \exp(v^*(x,\tau))| \right. \\ & \left. \exp\left(\int_\tau^t \exp(b(s-t)) \exp(v^*(x,s)) \mathrm{d}s\right) \mathrm{d}\tau \right\} \\ \|\underline{e}^{n+1}\| &\leq K_3 \left(\|e^{n+1}\| + \|\underline{e}^n\|^2\right) (1 + K_4) = C \left(\|e^{n+1}\| + \|\underline{e}^n\|^2\right) \end{aligned}$$

where $C = K_3(1 + K_4)$ and K_4 is a positive constant. □

Remark 2.3.2. *Similar error estimate can be obtained in the case of maximal sequence given by* (2.5) *also.*

2.4. Convergence Analysis for the Discretized Case

In this section, a finite difference system is developed using Theorem 2.3.1 for solving the coupled equation (2.1) numerically. More specifically, the derivative terms in the iterative procedure are discretized using backward finite difference formula. This section discusses the monotone property as well as the convergence analysis of the proposed finite difference scheme. Unlike [**82**], variable coefficients are dealt with at each step and as a result, a new comparison lemma is obtained to prove the monotone property of the proposed scheme. The error estimate for the iterative procedure is also derived.

Let $h = \triangle x = \frac{l}{M}$, $k = \triangle k = \frac{T}{N}$ be the space and time increment and let $x_i = ih$, $t_j = jk$ be a mesh point in $[0, l] \times [0, T]$ where M and N are the total numbers of intervals in $[0, l]$ and $[0, T]$ respectively. The sets of mesh points (x_i, t_j) in $(0, l] \times (0, T]$ and $[0, l] \times [0, T]$ are denoted respectively by Λ and $\bar{\Lambda}$. Define $u_{i,j} = u(x_i, t_j)$ and $v_{i,j} = v(x_i, t_j)$. Using the backward implicit approximation for first order differential equations, (2.1) is approximated by the finite difference system

$$\begin{aligned} &\tfrac{u_{i,j} - u_{i,j-1}}{k} + a\tfrac{u_{i,j} - u_{i-1,j}}{h} + cu_{i,j} = cv_{i,j} \\ &\tfrac{v_{i,j} - v_{i,j-1}}{k} + bv_{i,j} = bu_{i,j} + \lambda \exp(v_{i,j}) \\ &u_{0,j} = \eta_j,\ u_{i,0} = \psi_i,\ v_{i,0} = \phi_i;\ i = 1, 2, \ldots, M,\ j = 1, 2, \ldots, N. \end{aligned}$$

This can be rewritten as

$$
\begin{aligned}
&A^1 u_{i,j} = ckv_{i,j} + u_{i,j-1} + \tfrac{ak}{h} u_{i-1,j} \\
(2.10) \qquad &A^2 v_{i,j} = v_{i,j-1} + bku_{i,j} + k\lambda \exp(v_{i,j}) \\
&u_{0,j} = \eta_j,\ u_{i,0} = \psi_i,\ v_{i,0} = \phi_i;\ i = 1, 2, \ldots, M,\ j = 1, 2, \ldots, N
\end{aligned}
$$

with $A^1 = 1 + \frac{ak}{h} + ck$ and $A^2 = 1 + bk$.

Definition 2.4.1. *A function* $(\alpha_{i,j}, \beta_{i,j})$ *defined on* $\overline{\Lambda}$ *is called a lower solution of* (2.10) *if it satisfies*

$$
\begin{aligned}
&A^1 \alpha_{i,j} \le ck\beta_{i,j} + \alpha_{i,j-1} + \tfrac{ak}{h} \alpha_{i-1,j} \\
(2.11) \qquad &A^2 \beta_{i,j} \le \beta_{i,j-1} + bk\alpha_{i,j} + k\lambda \exp(\beta_{i,j}) \\
&\alpha_{0,j} \le \eta_j,\ \alpha_{i,0} \le \psi_i,\ \beta_{i,0} \le \phi_i;\ i = 1, 2, \ldots, M,\ j = 1, 2, \ldots, N.
\end{aligned}
$$

Similarly $(\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})$ *is called an upper solution if it satisfies* (2.11) *with inequalities reversed.*

For a given pair of ordered lower and upper solutions $(\alpha_{i,j}, \beta_{i,j})$ and $(\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})$ of (2.10), set

$$S = \{(u_{i,j}, v_{i,j}) \in \mathbb{R}^2 : (\alpha_{i,j}, \beta_{i,j}) \le (u_{i,j}, v_{i,j}) \le (\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})\}.$$

As explained in Section 2.3, using $(\alpha_{i,j}, \beta_{i,j})$ and $(\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})$ respectively as the initial iterations $(u^0_{i,j}, v^0_{i,j})$, two sequences can be constructed by applying successive approximation and quasilinearization technique to the first and second equations of (2.10) respectively which yields

$$
\begin{aligned}
&\tfrac{u^{n+1}_{i,j} - u^{n+1}_{i,j-1}}{k} + a\tfrac{u^{n+1}_{i,j} - u^{n+1}_{i-1,j}}{h} + cu^{n+1}_{i,j} = cv^n_{i,j} \\
&\tfrac{v^{n+1}_{i,j} - v^{n+1}_{i,j-1}}{k} + bv^{n+1}_{i,j} = bu^{n+1}_{i,j} + \lambda \exp(v^n_{i,j}) + \lambda \exp(v^n_{i,j}) \left(v^{n+1}_{i,j} - v^n_{i,j}\right) \\
&u^{n+1}_{0,j} = \eta_j,\ u^{n+1}_{i,0} = \psi_i,\ v^{n+1}_{i,0} = \phi_i;\ i = 1, 2, \ldots, M,\ j = 1, 2, \ldots, N
\end{aligned}
$$

with $n = 0, 1, \ldots$ where $\eta_j = \eta(t_j), \psi_i = \psi(x_i)$ and $\phi_i = \phi(x_i)$. The above system can be written in the form

$$
\begin{aligned}
&A^1 u^{n+1}_{i,j} - ckv^n_{i,j} = u^{n+1}_{i,j-1} + \tfrac{ak}{h} u^{n+1}_{i-1,j} \\
(2.12) \qquad &-bku^{n+1}_{i,j} + B^n_{i,j} v^{n+1}_{i,j} = v^{n+1}_{i,j-1} + k\lambda \exp(v^n_{i,j}) - k\lambda \exp(v^n_{i,j}) v^n_{i,j} \\
&u^{n+1}_{0,j} = \eta_j,\ u^{n+1}_{i,0} = \psi_i,\ v^{n+1}_{i,0} = \phi_i;\ i = 1, 2, \ldots, M,\ j = 1, 2, \ldots, N
\end{aligned}
$$

with $n = 0, 1, \ldots$ and $B^n_{i,j} = \left(A^2 - k\lambda \exp(v^n_{i,j})\right)$. Equivalently,

$$(2.13) \qquad \begin{array}{l} \mathbb{A}^n \begin{pmatrix} u^{n+1}_{i,j} \\ v^{n+1}_{i,j} \end{pmatrix} = \begin{pmatrix} u^{n+1}_{i,j-1} + \frac{ak}{h} u^{n+1}_{i-1,j} + ckv^n_{i,j} \\ v^{n+1}_{i,j-1} + k\lambda \exp(v^n_{i,j}) - k\lambda \exp(v^n_{i,j}) v^n_{i,j} \end{pmatrix} \\ u^{n+1}_{0,j} = \eta_j, \ u^{n+1}_{i,0} = \psi_i, \ v^{n+1}_{i,0} = \phi_i; \ i = 1, 2, \ldots, M, \ j = 1, 2, \ldots, N \end{array}$$

with $n = 0, 1, \ldots$ and $\mathbb{A}^n = \begin{pmatrix} A^1 & 0 \\ -bk & B^n_{i,j} \end{pmatrix}$. Throughout this section, assume that the time step k satisfies the condition (3.7) in [**82**], i.e.;

$$(2.14) \qquad \frac{1}{k} > \max\{b - c - \frac{a}{h}, c - b + \lambda \exp(\xi^*)\},$$

where $\xi^* = \max\{\overline{\beta}_{i,j} : (i,j) \in \bar{\Lambda}\}$. Denote the sequence generated from the lower solution by $\{(\alpha^{n+1}_{i,j}, \beta^{n+1}_{i,j})\}$ and the upper solution by $\{(\overline{\alpha}^{n+1}_{i,j}, \overline{\beta}^{n+1}_{i,j})\}$ and refer to them as minimal and maximal sequences respectively. With $B^n_{i,j} = \left(A^2 - k\lambda \exp(\beta^n_{i,j})\right)$, the iterative schemes respectively are constructed by

$$(2.15) \qquad \begin{array}{l} A^1 \alpha^{n+1}_{i,j} = ck\beta^n_{i,j} + \alpha^{n+1}_{i,j-1} + \frac{ak}{h} \alpha^{n+1}_{i-1,j} \\ B^n_{i,j} \beta^{n+1}_{i,j} = bk\alpha^{n+1}_{i,j} + \beta^{n+1}_{i,j-1} + k\lambda \exp(\beta^n_{i,j}) - k\lambda \exp(\beta^n_{i,j}) \beta^n_{i,j} \\ \alpha^{n+1}_{0,j} = \eta_j, \ \alpha^{n+1}_{i,0} = \psi_i, \ \beta^{n+1}_{i,0} = \phi_i; \ i = 1, 2, \ldots, M, \ j = 1, 2, \ldots, N \end{array}$$

and

$$(2.16) \qquad \begin{array}{l} A^1 \overline{\alpha}^{n+1}_{i,j} = ck\overline{\beta}^n_{i,j} + \overline{\alpha}^{n+1}_{i,j-1} + \frac{ak}{h} \overline{\alpha}^{n+1}_{i-1,j} \\ B^n_{i,j} \overline{\beta}^{n+1}_{i,j} = bk\overline{\alpha}^{n+1}_{i,j} + \overline{\beta}^{n+1}_{i,j-1} + k\lambda \exp(\overline{\beta}^n_{i,j}) - k\lambda \exp(\beta^n_{i,j}) \overline{\beta}^n_{i,j} \\ \overline{\alpha}^{n+1}_{0,j} = \eta_j, \ \overline{\alpha}^{n+1}_{i,0} = \psi_i, \ \overline{\beta}^{n+1}_{i,0} = \phi_i; \ i = 1, 2, \ldots, M, \ j = 1, 2, \ldots, N. \end{array}$$

The following lemma is vital to prove the monotone property of the sequences.

Lemma 2.4.1. *Let $a_1, a_{i,j} > 0$ for all $(i,j) \in \bar{\Lambda}$ with b_1, b_2, c_1 and $c_2 \geq 0$. If $w_{i,j}$ and $z_{i,j}$ satisfy*

$$(2.17) \qquad a_1 w_{i,j} - b_1 w_{i-1,j} - c_1 w_{i,j-1} \ \geq \ 0, \quad (i,j) \in \Lambda$$

$$(2.18) \qquad a_{i,j} z_{i,j} - b_2 w_{i,j} - c_2 z_{i,j-1} \ \geq \ 0, \quad (i,j) \in \Lambda$$

with

$$(2.19) \qquad w_{0,j} \geq 0, w_{i,0} \geq 0, z_{i,0} \geq 0,$$

then $w_{i,j} \geq 0$ and $z_{i,j} \geq 0$ for all $(i,j) \in \bar{\Lambda}$.

Proof. The proof is by an induction process. (2.17) gives

$$a_1 w_{i,j} \geq b_1 w_{i-1,j} + c_1 w_{i,j-1}.$$

Let $i = 1$ and using (2.19),

$$\begin{aligned} a_1 w_{1,j} &\geq b_1 w_{0,j} + c_1 w_{1,j-1} \\ a_1 w_{1,j} &\geq c_1 w_{1,j-1}. \end{aligned} \tag{2.20}$$

Now for $i = 1$ and using (2.20) in (2.18),

$$a_{1,j} z_{1,j} \geq \frac{b_2 c_1}{a_1} w_{1,j-1} + c_2 z_{1,j-1}. \tag{2.21}$$

Hence for $j = 1$ and using (2.19), one can conclude that,

$$a_{1,1} z_{1,1} \geq \frac{b_2 c_1}{a_1} w_{1,0} + c_2 z_{1,0} \geq 0. \Rightarrow z_{1,1} \geq 0$$

Similarly for $j = 1$ in (2.20), $w_{1,1} \geq 0$. Assume that $w_{1,j} \geq 0$ and $z_{1,j} \geq 0$ for $j = n - 1$. For $j = n$, from (2.21)

$$a_{1,n} z_{1,n} \geq \frac{b_2 c_1}{a_1} w_{1,n-1} + c_2 z_{1,n-1} \geq 0 \Rightarrow z_{1,n} \geq 0.$$

Similarly for $j = n$ in (2.20), $w_{1,n} \geq 0$. Thus $w_{1,j} \geq 0$ and $z_{1,j} \geq 0$ for all j. Now assume that $w_{i,j} \geq 0$ and $z_{i,j} \geq 0$ for all j and $i = n - 1$. For $i = n$ in (2.17)

$$\begin{aligned} a_1 w_{n,j} &\geq b_1 w_{n-1,j} + c_1 w_{n,j-1} \\ a_1 w_{n,j} &\geq c_1 w_{n,j-1}. \end{aligned} \tag{2.22}$$

For $i = n$ in (2.18) and using (2.22),

$$a_{n,j} z_{n,j} \geq \frac{b_2 c_1}{a_1} w_{n,j-1} + c_2 z_{n,j-1}. \tag{2.23}$$

For $j = 1$ in (2.23) and by using (2.19), one can conclude that $z_{n,1} \geq 0$. Similarly, $w_{n,1} \geq 0$. Assume that $w_{n,j} \geq 0$ and $z_{n,j} \geq 0$ for $j = k - 1$. For $j = k$ from (2.23),

$$a_{n,k} z_{n,k} \geq \frac{b_2 c_1}{a_1} w_{n,k-1} + c_2 z_{n,k-1} \geq 0 \Rightarrow z_{n,k} \geq 0.$$

Similarly from (2.22),

$$a_1 w_{n,k} \geq c_1 w_{n,k-1} \Rightarrow w_{n,k} \geq 0.$$

Thus $w_{i,j} \geq 0$ and $z_{i,j} \geq 0$ for all $(i, j) \in \bar{\Lambda}$. $\square$

Theorem 2.4.1. *Let $(\alpha_{i,j}, \beta_{i,j})$ and $(\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})$ be a pair of ordered lower and upper solutions of* (2.10)*. Then the following statements hold:*

(i) *The iterative schemes* (2.15) *and* (2.16) *are well defined.*

(ii) *The minimal sequence $\{(\alpha^n_{i,j}, \beta^n_{i,j})\}$ and the maximal sequence $\{(\overline{\alpha}^n_{i,j}, \overline{\beta}^n_{i,j})\}$converge monotonically to the unique solution $(u^*_{i,j}, v^*_{i,j})$ of* (2.10) *in S.*

(iii) *The relation*

$$
\begin{aligned}
(\alpha_{i,j}, \beta_{i,j}) \;\; \leq \;\; & (\alpha^n_{i,j}, \beta^n_{i,j}) \leq (\alpha^{n+1}_{i,j}, \beta^{n+1}_{i,j}) \leq (\alpha^*_{i,j}, \beta^*_{i,j}) \leq \\
& (\overline{\alpha}^*_{i,j}, \overline{\beta}^*_{i,j}) \leq (\overline{\alpha}^{n+1}_{i,j}, \overline{\beta}^{n+1}_{i,j}) \leq (\overline{\alpha}^n_{i,j}, \overline{\beta}^n_{i,j}) \leq (\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})
\end{aligned} \tag{2.24}
$$

holds for every $(i, j) \in \bar{\Lambda}$ and $n = 1, 2, \cdots$.

Proof. The proof is given by induction on n. For $n = 0$, $(\alpha_{i,j}, \beta_{i,j}) \in S$ and using (2.14) one can conclude that $\mathbb{A}^0$ is invertible. Hence $(\alpha^1_{i,j}, \beta^1_{i,j})$ and $(\overline{\alpha}^1_{i,j}, \overline{\beta}^1_{i,j})$ exist for all i and j. Note that since $(\alpha_{i,j}, \beta_{i,j}) \in S$, (2.14) ensures that $B^0_{i,j} = A^2 - k\lambda \exp(\beta_{i,j}) > A^2 - k\lambda \exp(\xi^*) > 0$ for all i and j. Let $w^0_{i,j} = \alpha^1_{i,j} - \alpha_{i,j}$ and $z^0_{i,j} = \beta^1_{i,j} - \beta_{i,j}$. Consider

$$
\begin{aligned}
A^1 w^0_{i,j} - \frac{ak}{h} w^0_{i-1,j} &= A^1 \alpha^1_{i,j} - A^1 \alpha_{i,j} - \frac{ak}{h}\alpha^1_{i-1,j} + \frac{ak}{h}\alpha_{i-1,j} \\
&= ck\beta_{i,j} + \alpha^1_{i,j-1} + \frac{ak}{h}\alpha^1_{i-1,j} - A^1 \alpha_{i,j} \\
&\qquad - \frac{ak}{h}\alpha^1_{i-1,j} + \frac{ak}{h}\alpha_{i-1,j}
\end{aligned}
$$

$$
A^1 w^0_{i,j} - \frac{ak}{h} w^0_{i-1,j} = ck\beta_{i,j} + \alpha^1_{i,j-1} - A^1 \alpha_{i,j} + \frac{ak}{h}\alpha_{i-1,j} \tag{2.25}
$$

and

$$
B^0_{i,j} z^0_{i,j} = bk\alpha^1_{i,j} + \beta^1_{i,j-1} + k\lambda \exp(\beta_{i,j})\,(1 - \beta_{i,j}) - B^0_{i,j}\beta_{i,j}. \tag{2.26}
$$

From (2.25) for $j = 1$,

$$
\begin{aligned}
A^1 w^0_{i,1} - \frac{ak}{h} w^0_{i-1,1} &= ck\beta_{i,1} + \alpha^1_{i,0} - A^1 \alpha_{i,1} + \frac{ak}{h}\alpha_{i-1,1} \\
&\geq ck\beta_{i,1} + \alpha_{i,0} - A^1 \alpha_{i,1} + \frac{ak}{h}\alpha_{i-1,1}
\end{aligned}
$$

$$
A^1 w^0_{i,1} - \frac{ak}{h} w^0_{i-1,1} \geq 0. \tag{2.27}
$$

From (2.26) for $j = 1$,

$$\begin{aligned}
B^0_{i,1} z^0_{i,1} &= bk\alpha^1_{i,1} + \beta^1_{i,0} + k\lambda \exp(\beta_{i,1})\,(1-\beta_{i,1}) - B^0_{i,1}\beta_{i,1} \\
B^0_{i,1} z^0_{i,1} &\geq bk\alpha^1_{i,1} + bk\alpha_{i,1} - bk\alpha_{i,1} + \beta_{i,0} + k\lambda \exp(\beta_{i,1})\,(1-\beta_{i,1}) - B^0\beta_{i,1} \\
B^0_{i,1} z^0_{i,1} &\geq bkw^0_{i,1}. \qquad (2.28)
\end{aligned}$$

From the boundary and initial conditions, $w^0_{0,1} = \alpha^1_{0,1} - \alpha_{0,1} = \eta_1 - \alpha_{0,1} \geq 0$, $z^0_{i,0} = \beta^1_{i,0} - \beta_{i,0} = \phi_i - \beta_{i,0} \geq 0$ and $w^0_{i,0} = \alpha^1_{i,0} - \alpha_{i,0} = \psi_i - \alpha_{i,0} \geq 0$. For $i = 1$, (2.27) and (2.28) give

$$\begin{aligned}
A^1 w^0_{1,1} &\geq \frac{ak}{h} w^0_{0,1} \geq 0 \Rightarrow w^0_{1,1} \geq 0. \\
B^0_{1,1} z^0_{1,1} &\geq bkw^0_{1,1} \Rightarrow z^0_{1,1} \geq 0.
\end{aligned}$$

For $i = 2$, (2.27) and (2.28) give

$$\begin{aligned}
A^1 w^0_{2,1} &\geq \frac{ak}{h} w^0_{1,1} \geq 0 \Rightarrow w^0_{2,1} \geq 0. \\
B^0_{2,1} z^0_{2,1} &\geq bkw^0_{2,1} \Rightarrow z^0_{2,1} \geq 0.
\end{aligned}$$

Proceeding like this, one can prove that $w^0_{i,1} \geq 0$ and $z^0_{i,1} \geq 0$ for all i. i.e., $\alpha_{i,1} \leq \alpha^1_{i,1}$ and $\beta_{i,1} \leq \beta^1_{i,1}$ for all i. From (2.25) for $j = 2$,

$$\begin{aligned}
A^1 w^0_{i,2} - \frac{ak}{h} w^0_{i-1,2} &= ck\beta_{i,2} + \alpha^1_{i,1} - A^1\alpha_{i,2} + \frac{ak}{h}\alpha_{i-1,2} \\
&\geq ck\beta_{i,2} + \alpha_{i,1} - A^1\alpha_{i,2} + \frac{ak}{h}\alpha_{i-1,2} \\
A^1 w^0_{i,2} - \frac{ak}{h} w^0_{i-1,2} &\geq 0. \qquad (2.29)
\end{aligned}$$

From (2.26) for $j = 2$,

$$\begin{aligned}
B^0_{i,2} z^0_{i,2} &= bk\alpha^1_{i,2} + \beta^1_{i,1} + k\lambda \exp(\beta_{i,2})\,(1-\beta_{i,2}) - B^0_{i,2}\beta_{i,2} \\
&\geq bk\alpha^1_{i,2} + bk\alpha_{i,2} - bk\alpha_{i,2} + \beta_{i,1} + k\lambda \exp(\beta_{i,2})\,(1-\beta_{i,2}) - B^0_{i,2}\beta_{i,2} \\
B^0_{i,2} z^0_{i,2} &\geq bkw^0_{i,2}. \qquad (2.30)
\end{aligned}$$

From the boundary and initial conditions, $w^0_{0,2} \geq 0$, $z^0_{i,0} \geq 0$ and $w^0_{i,0} \geq 0$. For $i = 1$, (2.29) and (2.30) give

$$\begin{aligned} A^1 w^0_{1,2} &\geq \frac{ak}{h} w^0_{0,2} \geq 0 \Rightarrow w^0_{1,2} \geq 0. \\ B^0_{1,2} z^0_{1,2} &\geq bkw^0_{1,2} \Rightarrow z^0_{1,2} \geq 0. \end{aligned}$$

For $i = 2$, (2.29) and (2.30) give

$$\begin{aligned} A^1 w^0_{2,2} &\geq \frac{ak}{h} w^0_{1,2} \geq 0 \Rightarrow w^0_{2,2} \geq 0. \\ B^0_{2,2} z^0_{2,2} &\geq bkw^0_{2,2} \Rightarrow z^0_{2,2} \geq 0. \end{aligned}$$

Proceeding like this, one can prove that $w^0_{i,2} \geq 0$ and $z^0_{i,2} \geq 0$ for all i. i.e., $\alpha_{i,2} \leq \alpha^1_{i,2}$ and $\beta_{i,2} \leq \beta^1_{i,2}$ for all i. Repeating the similar argument for $j = 3, \ldots, N$ leads to $\alpha_{i,j} \leq \alpha^1_{i,j}$ and $\beta_{i,j} \leq \beta^1_{i,j}$ for all $(i,j) \in \bar{\Lambda}$. Similarly, $\overline{\alpha}_{i,j} \geq \overline{\alpha}^1_{i,j}$ and $\overline{\beta}_{i,j} \geq \overline{\beta}^1_{i,j}$ for every i and j. Now let $w^1_{i,j} = \overline{\alpha}^1_{i,j} - \alpha^1_{i,j}$ and $z^1_{i,j} = \overline{\beta}^1_{i,j} - \beta^1_{i,j}$.

$$A^1 w^1_{i,j} = ck\overline{\beta}_{i,j} + \overline{\alpha}^1_{i,j-1} + \frac{ak}{h}\overline{\alpha}^1_{i-1,j} - ck\beta_{i,j} - \alpha^1_{i,j-1} - \frac{ak}{h}\alpha^1_{i-1,j}.$$

Thus $A^1 w^1_{i,j} - w^1_{i,j-1} - \frac{ak}{h} w^1_{i-1,j} \geq 0$. Now

$$\begin{aligned} B^0_{i,j} z^1_{i,j} &= bkw^1_{i,j} + z^1_{i,j-1} + k\lambda \exp(\overline{\beta}_{i,j}) - k\lambda \exp(\beta_{i,j}) - k\lambda \exp(\beta_{i,j}) \left(\overline{\beta}_{i,j} - \beta_{i,j}\right) \\ B^0_{i,j} z^1_{i,j} &= bkw^1_{i,j} + z^1_{i,j-1} + k\lambda \exp(\tilde{v}_{i,j}) \left(\overline{\beta}_{i,j} - \beta_{i,j}\right) - k\lambda \exp(\beta_{i,j}) \left(\overline{\beta}_{i,j} - \beta_{i,j}\right). \\ & \quad [\beta_{i,j} \leq \tilde{v}_{i,j} \leq \overline{\beta}_{i,j}] \end{aligned}$$

Thus $B^0_{i,j} z^1_{i,j} - bkw^1_{i,j} - z^1_{i,j-1} \geq 0$. Note that $w^1_{0,j} = 0$, $z^1_{i,0} = 0$, $w^1_{i,0} = 0$. Hence by Lemma 2.4.1, $\alpha^1_{i,j} \leq \overline{\alpha}^1_{i,j}$ and $\beta^1_{i,j} \leq \overline{\beta}^1_{i,j}$ for all $(i,j) \in \bar{\Lambda}$. The above conclusions show that

$$(\alpha_{i,j}, \beta_{i,j}) \leq (\alpha^1_{i,j}, \beta^1_{i,j}) \leq (\overline{\alpha}^1_{i,j}, \overline{\beta}^1_{i,j}) \leq (\overline{\alpha}_{i,j}, \overline{\beta}_{i,j}). \tag{2.31}$$

Assume by induction that $(\alpha^n_{i,j}, \beta^n_{i,j})$ and $(\overline{\alpha}^n_{i,j}, \overline{\beta}^n_{i,j})$ exist and

$$(\alpha_{i,j}, \beta_{i,j}) \leq (\alpha^{n-1}_{i,j}, \beta^{n-1}_{i,j}) \leq (\alpha^n_{i,j}, \beta^n_{i,j}) \leq (\overline{\alpha}^n_{i,j}, \overline{\beta}^n_{i,j}) \leq (\overline{\alpha}^{n-1}_{i,j}, \overline{\beta}^{n-1}_{i,j}) \leq (\overline{\alpha}_{i,j}, \overline{\beta}_{i,j}) \tag{2.32}$$

for some $n > 1$. From induction hypothesis and using (2.14), one can conclude that $\mathbb{A}^n$ is invertible. Consequently, $(\alpha^{n+1}_{i,j}, \beta^{n+1}_{i,j})$ and $(\overline{\alpha}^{n+1}_{i,j}, \overline{\beta}^{n+1}_{i,j})$ exist for all i and j. The induction hypothesis and (2.14) also confirm that $B^n_{i,j} > 0$ for all i and j. Let $w^n_{i,j} = \alpha^{n+1}_{i,j} - \alpha^n_{i,j}$ and $z^n_{i,j} = \beta^{n+1}_{i,j} - \beta^n_{i,j}$. Consider

$$\begin{aligned}
A^1 w^n_{i,j} - \frac{ak}{h} w^n_{i-1,j} &= A^1\alpha^{n+1}_{i,j} - A^1\alpha^n_{i,j} - \frac{ak}{h}\alpha^{n+1}_{i-1,j} + \frac{ak}{h}\alpha^n_{i-1,j} \\
&= ck\beta^n_{i,j} + \alpha^{n+1}_{i,j-1} - ck\beta^{n-1}_{i,j} - \alpha^n_{i,j-1} \\
A^1 w^n_{i,j} - \frac{ak}{h} w^n_{i-1,j} &\geq w^n_{i,j-1}.
\end{aligned}$$

Thus $A^1 w^n_{i,j} - \frac{ak}{h} w^n_{i-1,j} - w^n_{i,j-1} \geq 0$ for all $(i,j) \in \bar{\Lambda}$. Moreover,

$$\begin{aligned}
A^2 z^n_{i,j} &= bk\alpha^{n+1}_{i,j} + \beta^{n+1}_{i,j-1} + k\lambda\exp(\beta^n_{i,j})\left(1 + \beta^{n+1}_{i,j} - \beta^n_{i,j}\right) \\
&\quad - bk\alpha^n_{i,j} - \beta^n_{i,j-1} - k\lambda\exp(\beta^{n-1}_{i,j})\left(1 + \beta^n_{i,j} - \beta^{n-1}_{i,j}\right) \\
&= bkw^n_{i,j} + z^n_{i,j-1} + k\lambda\exp(\beta^n_{i,j}) z^n_{i,j} + k\lambda\exp(\beta^n_{i,j}) - k\lambda\exp(\beta^{n-1}_{i,j}) \\
&\quad - k\lambda\exp(\beta^{n-1}_{i,j})\left(\beta^n_{i,j} - \beta^{n-1}_{i,j}\right) \\
A^2 z^n_{i,j} &= bkw^n_{i,j} + z^n_{i,j-1} + k\lambda\exp(\beta^n_{i,j}) z^n_{i,j} + k\lambda\exp(\tilde{v}_{i,j})\left(\beta^n_{i,j} - \beta^{n-1}_{i,j}\right) \\
&\quad - k\lambda\exp(\beta^{n-1}_{i,j})\left(\beta^n_{i,j} - \beta^{n-1}_{i,j}\right); \;\; \beta^{n-1}_{i,j} \leq \tilde{v}_{i,j} \leq \beta^n_{i,j} \\
B^n_{i,j} z^n_{i,j} &\geq bkw^n_{i,j} + z^n_{i,j-1}.
\end{aligned}$$

Thus $B^n_{i,j} z^n_{i,j} - bkw^n_{i,j} - z^n_{i,j-1} \geq 0$ for all $(i,j) \in \bar{\Lambda}$. Moreover, $w^n_{0,j} = 0,\ z^n_{i,0} = 0,\ w^n_{i,0} = 0$. Hence by Lemma 2.4.1, $w^n_{i,j} \geq 0$ and $z^n_{i,j} \geq 0$ or equivalently $\alpha^n_{i,j} \leq \alpha^{n+1}_{i,j}$ and $\beta^n_{i,j} \leq \beta^{n+1}_{i,j}$ for all $(i,j) \in \bar{\Lambda}$. A similar argument gives $(\overline{\alpha}^{n+1}_{i,j}, \overline{\beta}^{n+1}_{i,j}) \leq (\overline{\alpha}^n_{i,j}, \overline{\beta}^n_{i,j})$ and $(\alpha^{n+1}_{i,j}, \beta^{n+1}_{i,j}) \leq (\overline{\alpha}^{n+1}_{i,j}, \overline{\beta}^{n+1}_{i,j})$. Thus

$$(\alpha_{i,j}, \beta_{i,j}) \leq (\alpha^n_{i,j}, \beta^n_{i,j}) \leq (\alpha^{n+1}_{i,j}, \beta^{n+1}_{i,j}) \leq (\overline{\alpha}^{n+1}_{i,j}, \overline{\beta}^{n+1}_{i,j}) \leq (\overline{\alpha}^n_{i,j}, \overline{\beta}^n_{i,j}) \leq (\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})$$

guarantees the existence of the limits

$$\lim_{m\to\infty}(\alpha^n_{i,j}, \beta^n_{i,j}) = (\alpha^*_{i,j}, \beta^*_{i,j}); \quad \lim_{m\to\infty}(\overline{\alpha}^n_{i,j}, \overline{\beta}^n_{i,j}) = (\overline{\alpha}^*_{i,j}, \overline{\beta}^*_{i,j}). \tag{2.33}$$

Hence (2.24) holds and in the limiting case, both the limits are solutions of (2.1). The proof for the uniqueness of the solution follows from Theorem 3.1 in [**82**]. □

The following remark is similar to Remark 3.1(b) in [**82**].

Remark 2.4.1. *Theorem 2.4.1 holds true for the more general system*

$$\begin{aligned}
&\tfrac{\partial u}{\partial t} + a\tfrac{\partial u}{\partial x} + cu = f(x,t,u,v), \quad t>0,\ 0<x\le l\\
&\tfrac{\partial v}{\partial t} + bv = g(x,t,u,v) \quad t>0,\ 0<x\le l\\
&u(0,t)=\eta(t),\ u(x,0)=u_0(x),\ v(x,0)=v_0(x),\ t>0,\ 0<x\le l,
\end{aligned}$$

where $f(x,t,u,v)$ and $g(x,t,u,v)$ are continuous and C^1-functions of (u,v) such that $f_u(x,t,u,v), f_v(x,t,u,v),\ g_u(x,t,u,v)$ are positive and $g_v(x,t,u,v)$ is nondecreasing and Lipschitz with respect to u and v for $\alpha \le u \le \overline{\alpha}$ and $\beta \le v \le \overline{\beta}$. The monotone iteration process is given by

$$\begin{aligned}
&\tfrac{\partial u^{n+1}}{\partial t} + a\tfrac{\partial u^{n+1}}{\partial x} + cu^{n+1} = f(x,t,u^n,v^n), \quad t>0,\ 0<x\le l\\
&\tfrac{\partial v^{n+1}}{\partial t} + bv^{n+1} = g(x,t,u^{n+1},v^n) + g_v(x,t,u^{n+1},v^n)(v^{n+1}-v^n), \quad t>0,\ 0<x\le l\\
&u^{n+1}(0,t)=\eta,\ u^{n+1}(x,0)=u_0(x),\ v^{n+1}(x,0)=v_0(x),\ t>0,\ 0<x\le l.
\end{aligned}$$

Theorem 2.4.2. *If $(u^*_{i,j}, v^*_{i,j})$ for all $(i,j)\in\bar{\Lambda}$ is the solution of* (2.10)*, then there exists a positive constant C such that*

$$\left\| \begin{pmatrix} e^{n+1}_{i,j} \\ \underline{e}^{n+1}_{i,j} \end{pmatrix} \right\|_\infty \le C \left[\left\| \begin{pmatrix} e^{n+1}_{i,j-1} + \frac{ak}{h} e^{n+1}_{i-1,j} \\ \underline{e}^{n+1}_{i,j-1} \end{pmatrix} \right\|_\infty + \left\| \begin{pmatrix} ck\underline{e}^{n}_{i,j} \\ k\lambda \exp(\xi^*)\left(\underline{e}^{n}_{i,j}\right)^2 \end{pmatrix} \right\|_\infty \right],$$

where $e^{n+1}_{i,j} = u^*_{i,j} - \alpha^{n+1}_{i,j}$, $\underline{e}^{n+1}_{i,j} = v^*_{i,j} - \beta^{n+1}_{i,j}$ and $\xi^* = \max\{\overline{\beta}_{i,j} : (i,j)\in\bar{\Lambda}\}$.

Proof. From (2.15),

$$\begin{aligned}
&(2.34)\quad A^1\alpha^{n+1}_{i,j} = ck\beta^n_{i,j} + \alpha^{n+1}_{i,j-1} + \frac{ak}{h}\alpha^{n+1}_{i-1,j}\\
&(2.35)\quad A^2\beta^{n+1}_{i,j} = bk\alpha^{n+1}_{i,j} + \beta^{n+1}_{i,j-1} + k\lambda\exp(\beta^n_{i,j}) + k\lambda\exp(\beta^n_{i,j})\left(\beta^{n+1}_{i,j} - \beta^n_{i,j}\right).
\end{aligned}$$

If $(u^*_{i,j}, v^*_{i,j})$ is the solution of (2.10) in S, then

$$A^1u^*_{i,j} = ckv^*_{i,j} + u^*_{i,j-1} + \frac{ak}{h}u^*_{i-1,j} \qquad (2.36)$$

$$A^2v^*_{i,j} = bku^*_{i,j} + v^*_{i,j-1} + k\lambda\exp(v^*_{i,j}). \qquad (2.37)$$

Hence,

$$A^1e^{n+1}_{i,j} - e^{n+1}_{i,j-1} - \frac{ak}{h}e^{n+1}_{i-1,j} = ck\underline{e}^n_{i,j}.$$

Similarly,

$$\begin{aligned}
A^2\underline{e}_{i,j}^{n+1} &= bke_{i,j}^{n+1} + \underline{e}_{i,j-1}^{n+1} + k\lambda\left(\exp(v_{i,j}^*) - \exp(\beta_{i,j}^n)\right) - k\lambda\exp(\beta_{i,j}^n)\left(\beta_{i,j}^{n+1} - \beta_{i,j}^n\right)\\
A^2\underline{e}_{i,j}^{n+1} &= bke_{i,j}^{n+1} + \underline{e}_{i,j-1}^{n+1} + k\lambda\exp(\hat{\beta}_{i,j})\left(v_{i,j}^* - \beta_{i,j}^n\right)\\
&\quad -k\lambda\exp(\beta_{i,j}^n)\left(\beta_{i,j}^{n+1} - v_{i,j}^* + v_{i,j}^* - \beta_{i,j}^n\right);\ \beta_{i,j}^n \le \hat{\beta}_{i,j} \le v_{i,j}^*\\
B^n\underline{e}_{i,j}^{n+1} &= bke_{i,j}^{n+1} + \underline{e}_{i,j-1}^{n+1} + k\lambda\exp(\hat{\beta}_{i,j})\underline{e}_{i,j}^n - k\lambda\exp(\beta_{i,j}^n)\underline{e}_{i,j}^n\\
&\le bke_{i,j}^{n+1} + \underline{e}_{i,j-1}^{n+1} + k\lambda\underline{e}_{i,j}^n\left(\exp(v_{i,j}^*) - \exp(\beta_{i,j}^n)\right)\\
B^n\underline{e}_{i,j}^{n+1} &\le bke_{i,j}^{n+1} + \underline{e}_{i,j-1}^{n+1} + k\lambda\exp(\xi^*)\left(\underline{e}_{i,j}^n\right)^2.
\end{aligned}$$

Thus,

$$\mathbb{A}^n\begin{pmatrix} e_{i,j}^{n+1}\\ \underline{e}_{i,j}^{n+1}\end{pmatrix} - \begin{pmatrix} e_{i,j-1}^{n+1} + \frac{ak}{h}e_{i-1,j}^{n+1}\\ \underline{e}_{i,j-1}^{n+1}\end{pmatrix} \le \begin{pmatrix} ck\underline{e}_{i,j}^n\\ k\lambda\exp(\xi^*)\left(\underline{e}_{i,j}^n\right)^2\end{pmatrix}. \tag{2.38}$$

Note that for each $n \in \mathbb{N}$, $\mathbb{A}^n$ is a nonsingular $\mathbb{M}$-matrix. Hence (2.38) can be written as

$$\begin{pmatrix} e_{i,j}^{n+1}\\ \underline{e}_{i,j}^{n+1}\end{pmatrix} \le (\mathbb{A}^n)^{-1}\left[\begin{pmatrix} e_{i,j-1}^{n+1} + \frac{ak}{h}e_{i-1,j}^{n+1}\\ \underline{e}_{i,j-1}^{n+1}\end{pmatrix} + \begin{pmatrix} ck\underline{e}_{i,j}^n\\ k\lambda\exp(\xi^*)\left(\underline{e}_{i,j}^n\right)^2\end{pmatrix}\right].$$

Consequently,

$$\left\|\begin{pmatrix} e_{i,j}^{n+1}\\ \underline{e}_{i,j}^{n+1}\end{pmatrix}\right\|_\infty \le C\left[\left\|\begin{pmatrix} e_{i,j-1}^{n+1} + \frac{ak}{h}e_{i-1,j}^{n+1}\\ \underline{e}_{i,j-1}^{n+1}\end{pmatrix}\right\|_\infty + \left\|\begin{pmatrix} ck\underline{e}_{i,j}^n\\ k\lambda\exp(\xi^*)\left(\underline{e}_{i,j}^n\right)^2\end{pmatrix}\right\|_\infty\right],$$

where $C = \max\left\{1, \frac{1}{1+bk-k\lambda e^{\xi^*}} + \frac{b}{c}\right\}$. □

Remark 2.4.2. *Similar error estimate can be obtained in the case of maximal sequence given by* (2.16) *also.*

2.5. Convergence of Finite Difference Solutions

In this section, the convergence of $(u_{i,j}^*, v_{i,j}^*)$ to the continuous solution $(u^*(x_i,t_j), v^*(x_i,t_j))$ as the mesh size tends to zero is obtained. The following theorem is similar to Theorem 5.1 in [**82**].

Theorem 2.5.1. *Let $(u^*(x,t), v^*(x,t))$ and $(u^*_{i,j}, v^*_{i,j})$ be the respective solutions of* (2.1) *and* (2.10) *respectively and let $\bar{\Lambda}$ be a given partition of $\bar{Q} = [0,l] \times [0,T]$. Then*

$$(u^*_{i,j}, v^*_{i,j}) \to (u^*(x_i,t_j), v^*(x_i,t_j)) \text{ as } h+k \to 0$$

at every mesh point (x_i, t_j) in $\bar{\Lambda}$.

Proof. To prove this theorem, for given any $\epsilon > 0$ it has to be shown that there exists $\delta > 0$ such that

$$|u^*(x_i,t_j) - u^*_{i,j}| + |v^*(x_i,t_j) - v^*_{i,j}| < \epsilon \text{ when } h+k < \delta. \tag{2.39}$$

Let $(\alpha^0, \beta^0) = (\alpha^0_{i.j}, \beta^0_{i,j}) = (0,0)$ for both the minimal sequences given by (2.3) and (2.12) respectively. By Theorem 2.3.1 and Theorem 2.4.1, there exists an integer $n = n^*(\epsilon)$ such that

$$\begin{aligned} |u^* - \alpha^{n+1}| &+ |v^* - \beta^{n+1}| < \frac{\epsilon}{3} \\ |u^*_{i,j} - \alpha^{n+1}_{i,j}| &+ |v^*_{i,j} - \beta^{n+1}_{i,j}| < \frac{\epsilon}{3}, \end{aligned}$$

where $(i,j) \in \bar{\Lambda}$ for all $n \geq n^*$. Note that

$$\begin{aligned} |u^*(x_i,t_j) - u^*_{i,j}| &\leq |u^*(x_i,t_j) - \alpha^{n^*}(x_i,t_j)| + |\alpha^{n^*}(x_i,t_j) - \alpha^{n^*}_{i,j}| + |\alpha^{n^*}_{i,j} - u^*_{i,j}| \\ |v^*(x_i,t_j) - v^*_{i,j}| &\leq |v^*(x_i,t_j) - \beta^{n^*}(x_i,t_j)| + |\beta^{n^*}(x_i,t_j) - \beta^{n^*}_{i,j}| + |\beta^{n^*}_{i,j} - v^*_{i,j}|. \end{aligned}$$

Hence the proof is complete if one can prove that

$$|\alpha^{n^*}(x_i,t_j) - \alpha^{n^*}_{i,j}| + |\beta^{n^*}(x_i,t_j) - \beta^{n^*}_{i,j}| < \frac{\epsilon}{3};\ (i,j) \in \bar{\Lambda}. \tag{2.40}$$

From (2.3) and (2.12), it can be seen that $(\alpha^{n+1}(x_i,t_j), \beta^{n+1}(x_i,t_j))$ satisfies the equations

$$\begin{aligned} &A^1\alpha^{n+1}(x_i,t_j) - ck\beta^n(x_i,t_j) = \alpha^{n+1}(x_i,t_{j-1}) + \tfrac{ak}{h}\alpha^{n+1}(x_{i-1},t_j) + o(h,k) \\ &B^n\beta^{n+1}(x_i,t_j) - bk\alpha^{n+1}(x_i,t_j) = \beta^{n+1}(x_i,t_{j-1}) + \\ &\quad k\lambda\exp(\beta^n(x_i,t_j))\,(1 - \beta^n(x_i,t_j)) + o(h,k) \\ &\alpha^{n+1}(0,t_j) = \eta_j, \alpha^{n+1}(x_i,0) = \psi(x_i), \beta^{n+1}(x_i,0) = \phi(x_i), \end{aligned} \tag{2.41}$$

where $o(h,k) \to 0$ as $h+k \to 0$. Let

$$e^{n+1}_{i,j} = \alpha^{n+1}(x_i,t_j) - \alpha^{n+1}_{i,j};\ \underline{e}^{n+1}_{i,j} = \beta^{n+1}(x_i,t_j) - \beta^{n+1}_{i,j}. \tag{2.42}$$

Subtracting (2.12) from (2.41) and using mean value theorem,

$$
\begin{aligned}
&A^1 e_{i,j}^{n+1} - ck\underline{e}_{i,j}^{n} = e_{i,j-1}^{n+1} + \tfrac{ak}{h} e_{i-1,j}^{n+1} + o(h,k) \\
&B^n \underline{e}_{i,j}^{n+1} - bk e_{i,j}^{n+1} = \underline{e}_{i,j-1}^{n+1} - k\lambda \hat{\beta}_{i,j}^{n} \exp(\hat{\beta}_{i,j}^{n}) \underline{e}_{i,j}^{n} + o(h,k) \\
&e_{0,j}^{n+1} = 0,\ e_{i,0}^{n+1} = 0,\ \underline{e}_{i,0}^{n+1} = 0,
\end{aligned}
\tag{2.43}
$$

where $\hat{\beta}_{i,j}^{n}$ is an intermediate value between $\beta^n(x_i, t_j)$ and $\beta_{i,j}^{n}$. Define column vectors E_j^{n+1} and $\underline{E}_j^{n+1}$ respectively by

$$
\begin{aligned}
E_j^{n+1} &= \left(e_{1,j}^{n+1}, e_{2,j}^{n+1}, \ldots, e_{M,j}^{n+1}\right)^T \\
\underline{E}_j^{n+1} &= \left(\underline{e}_{1,j}^{n+1}, \underline{e}_{2,j}^{n+1}, \ldots, \underline{e}_{M,j}^{n+1}\right)^T,
\end{aligned}
\tag{2.44}
$$

where $(\cdot)^T$ denotes the transpose of a row vector. Let $\mathbb{A}$ be an $M \times M$ bidiagonal matrix and D_j^n a diagonal matrix given by

$$
\mathbb{A} = (a_{jk}) = \begin{pmatrix}
A^1 & & & & \\
-\frac{ak}{h} & A^1 & & & \\
& -\frac{ak}{h} & A^1 & & \\
& & \ddots & \ddots & \\
& & & -\frac{ak}{h} & A^1
\end{pmatrix}
$$

$$
D_j^n = diag\left(-\hat{\beta}_{1,j}^{n} \exp(\hat{\beta}_{1,j}^{n}), -\hat{\beta}_{2,j}^{n} \exp(\hat{\beta}_{2,j}^{n}), \ldots, -\hat{\beta}_{M,j}^{n} \exp(\hat{\beta}_{M,j}^{n})\right)
$$

Using the same argument in [**82**], one can conclude that $\mathbb{A}^{-1}$ exists and is a nonnegative matrix. Also, $\mathbb{A}$ has the positive smallest eigenvalue $\mu_0 = A^1$. (2.43) can be written as

$$
\begin{aligned}
&\mathbb{A} E_j^{n+1} = E_{j-1}^{n+1} + ck\underline{E}_j^{n} + O(h,k) \\
&B^n \underline{E}_j^{n+1} = \underline{E}_{j-1}^{n+1} + bkE_j^{n+1} + k\lambda D_j^n \underline{E}_j^{n} + O(h,k) \\
&E_0^{n+1} = 0,\ \underline{E}_0^{n+1} = 0,
\end{aligned}
$$

where $\|O(h,k)\| \to 0$ for any suitable norm $\|\cdot\|$ in $\mathbb{R}^M$ as $h+k \to 0$. Define $\sigma_1 = \xi^* \exp(\xi^*)$, $\sigma_0 = k\lambda\sigma_1$ and $\tilde{B} = 1 + bk - k\lambda \exp(\xi^*)$, where $\xi^* = \max\{\overline{\beta}_{i,j} : (i,j) \in \bar{\Lambda}\}$. Proceeding as in Theorem 5.1 in [**82**], one will end up with

$$
\begin{aligned}
&\|E_j^{n+1}\| \le \tfrac{2}{\mu_0}\left[\|E_{j-1}^{n+1}\| + ck\|\underline{E}_j^{n}\| + \|O(h,k)\|\right] \\
&\|\underline{E}_j^{n+1}\| \le \tfrac{1}{\tilde{B}}\left[\|\underline{E}_{j-1}^{n+1}\| + bk\|E_j^{n+1}\| + \sigma_0\|\underline{E}_j^{n}\| + \|O(h,k)\|\right] \\
&\|E_0^{n+1}\| = \|\underline{E}_0^{n+1}\| = 0.
\end{aligned}
\tag{2.45}
$$

Define $S_j^{n+1} = \|E_j^{n+1}\| + \|\underline{E}_j^{n+1}\|$, $\overline{\gamma_1} = \max\{\frac{2}{\mu_0}, \frac{1}{\bar{B}}\}$, $\overline{\gamma_2} = \frac{bk}{\bar{B}}$ and $\overline{\gamma_3} = \max\{\frac{2ck}{\mu_0}, \frac{\sigma_0}{\bar{B}}\}$. Since $h+k \to 0$, one can choose k such that $k\lambda \exp(\xi^*) < 1$. Consequently, $\overline{\gamma_2} < 1$. Note that $S_0^{n+1} = S_j^0 = 0$ for all n and j. Proceeding similar to Theorem 5.1 in [**82**], for given any $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that

$$
\begin{aligned}
&(1-\overline{\gamma_2})\, S_j^{n+1} \le \overline{\gamma_1} S_{j-1}^{n+1} + \overline{\gamma_3} S_j^n + \epsilon_1 \text{ when } h+k < \delta_1 \\
&S_0^{n+1} = S_j^0 = 0;\ j = 1, 2, \ldots, N,\ n = 0, 1, \ldots, n^*.
\end{aligned} \tag{2.46}
$$

Define $\beta^* = 1 + \frac{\overline{\gamma_3}}{1-\overline{\gamma_2}} + (\frac{\overline{\gamma_3}}{1-\overline{\gamma_2}})^2 + \ldots + (\frac{\overline{\gamma_3}}{1-\overline{\gamma_2}})^{n^*-1}$. An induction argument in j leads to

$$
S_j^{n+1} \le \frac{\beta^*}{(1-\overline{\gamma_2})} \left[\left(\frac{\overline{\gamma_1}\beta^*}{1-\overline{\gamma_2}} \right)^{j-1} + \left(\frac{\overline{\gamma_1}\beta^*}{1-\overline{\gamma_2}} \right)^{j-2} + \cdots + \frac{\overline{\gamma_1}\beta^*}{1-\overline{\gamma_2}} + 1 \right] \epsilon_1
$$

for all $n \le n^*$ and $j \le N$. Thus $S_j^{n^*} \le K\epsilon_1$ for all $j = 1, 2, \ldots, N$, where $K = \frac{\beta^*}{(1-\overline{\gamma_2})} \left[\left(\frac{\overline{\gamma_1}\beta^*}{1-\overline{\gamma_2}} \right)^{N-1} + \left(\frac{\overline{\gamma_1}\beta^*}{1-\overline{\gamma_2}} \right)^{N-2} + \cdots + \frac{\overline{\gamma_1}\beta^*}{1-\overline{\gamma_2}} + 1 \right]$. For the choice of $\epsilon_1 < \frac{\epsilon}{3K}$, there exists $\delta > 0$ such that $S_j^{n^*} = \|E_j^{n^*}\| + \|\underline{E}_j^{n^*}\| < \frac{\epsilon}{3}$ when $h+k < \delta$. This leads to (2.40). Thus (2.39) holds and hence the theorem. □

2.6. Numerical Examples

In this section, the accelerated iterative technique is illustrated by applying to different examples. The existence and uniqueness of the solution and the convergence of the proposed examples are followed by Theorem 2.4.1 and Theorem 2.5.1 respectively. The iterative schemes taken for the numerical solution of the examples are same as the iterative scheme discussed in Remark 2.4.1. Throughout this section, n denotes the number of iterations required for the stopping criteria

$$
\max_{(i,j)} \left[|u_{i,j}^{n+1} - u_{i,j}^n| + |v_{i,j}^{n+1} - v_{i,j}^n| \right] \le \epsilon.
$$

Example 2.6.1.

Consider the following differential system discussed in [**82**].

$$
\begin{aligned}
&\tfrac{\partial u}{\partial t} + \tfrac{\partial u}{\partial x} + u = v + q_1(x,t), \quad 0 < x \le 1,\ 0 < t \le T \\
&\tfrac{\partial v}{\partial t} + v = u + \lambda \exp(v) + q_2(x,t), \quad 0 < x \le 1,\ 0 < t \le T \\
&u(0,t) = 2 - \exp(-t), \quad 0 < t \le T \\
&u(x,0) = 1 - x^2,\ v(x,0) = 1 - x^2, \quad 0 < x \le 1,
\end{aligned} \tag{2.47}
$$

where $\lambda > 0$ is considered as a parameter with $q_1(x,t) = (1-x)^2 \exp(-t)$ and $q_2(x,t) = (1+x^2)\exp(-t) - \lambda \exp(2-(1+x^2)\exp(-t))$. The solution of (2.47) is given by $u(x,t) = v(x,t) = 2-(1+x^2)\exp(-t)$. Numerical results for the minimal solution and the exact solution are given in Table 2.1 and Table 2.2. From Table 2.1 and Table 2.2, one can conclude that the proposed scheme performs faster than the scheme in [**82**]. Here $T = 1,\ \lambda = 0.05,\ h = k = 10^{-3}$ and $\epsilon = 2 \times 10^{-5}$.

Example 2.6.2.

Consider the following differential system discussed in [**82**].

$$\begin{aligned}
&\tfrac{\partial u}{\partial t} + \tfrac{\partial u}{\partial x} + u = \lambda_1 v^{p_1} + q_1(x,t), \quad 0 < x \le 1,\ 0 < t \le T \\
&\tfrac{\partial v}{\partial t} + v = u + \lambda_2 v^{p_2} + q_2(x,t), \quad 0 < x \le 1,\ 0 < t \le T \\
&u(0,t) = 2 - \exp(-t), \quad 0 < t \le T \\
&u(x,0) = 1-x,\ v(x,0) = 1-x, \quad 0 < x \le 1,
\end{aligned} \tag{2.48}$$

where $\lambda_i, p_i > 0$ for $i = 1,2$ with $q_1(x,t) = (2-\exp(-t)) - \lambda_1[2-(1+x)\exp(-t)]^{p_1}$ and $q_2(x,t) = (1+x)\exp(-t) - \lambda_2[2-(1+x)\exp(-t)]^{p_2}$. The solution of (2.48) is given by $u(x,t) = v(x,t) = 2-(1+x)\exp(-t)$. Numerical results for the minimal solution and the exact solution are given in Table 2.3 and Table 2.4 for the choice of $T = 1,\ h = k = 10^{-3},\ p_1 = 2,\ p_2 = 3,\ \lambda_1 = \frac{1}{4},\ \lambda_2 = \frac{1}{8e}$ and $\epsilon = 2 \times 10^{-5}$. From Table 2.3 and Table 2.4, one can conclude that the proposed scheme outperforms the scheme in [**82**].

2.7. Conclusion

The existence and uniqueness of the solution for a coupled system of partial differential equations is established through an accelerated iterative scheme. The monotonicity, convergence and the error estimate of the sequences obtained from both continuous and discrete cases are also obtained. The efficiency of the proposed scheme is proved by comparing with the existing scheme available in the literature.

TABLE 2.1. Numerical solution of $u(x,t)$ for Example 2.6.1.

Grid Point (x_i, t_j)	Exact Solution	Successive [82] $n = 10$	Proposed Scheme $n = 6$
(0.4,0.2)	1.050272326	1.050008303	1.050008298
(0.8,0.2)	0.657281565	0.656981110	0.656981108
(0.4,0.4)	1.222428747	1.221989786	1.221989780
(0.8,0.4)	0.900675125	0.900170790	0.900170784
(0.4,0.6)	1.363378502	1.362970024	1.362970010
(0.8,0.6)	1.099948917	1.099303519	1.099303500
(0.4,0.8)	1.478778402	1.478408732	1.478408701
(0.8,0.8)	1.263100499	1.262364658	1.262364626
(0.4,1)	1.573259848	1.572925710	1.572925661
(0.8,1)	1.396677716	1.395978881	1.395978804

TABLE 2.2. Numerical solution of $v(x,t)$ for Example 2.6.1.

Grid Point (x_i, t_j)	Exact Solution	Successive [82] $n = 10$	Proposed Scheme $n = 6$
(0.4,0.2)	1.050272326	1.050149607	1.050149601
(0.8,0.2)	0.657281565	0.657115969	0.657115962
(0.4,0.4)	1.222428747	1.222179075	1.222179049
(0.8,0.4)	0.900675125	0.900349866	0.900349838
(0.4,0.6)	1.363378502	1.363022778	1.363022712
(0.8,0.6)	1.099948917	1.099477231	1.099477158
(0.4,0.8)	1.478778402	1.478349819	1.478349717
(0.8,0.8)	1.263100499	1.262497886	1.262497772
(0.4,1)	1.573259848	1.572783575	1.572783541
(0.8,1)	1.396677716	1.395970568	1.395970586

TABLE 2.3. Numerical solution of $u(x,t)$ for Example 2.6.2.

Grid Point	Exact Solution	Successive [**82**]	Proposed Scheme
(x_i, t_j)		$n = 10$	$n = 5$
(0.4,0.2)	0.853776946	0.853665705	0.853665703
(0.8,0.2)	0.526284644	0.526141887	0.526141886
(0.4,0.4)	1.061551936	1.061374624	1.061374625
(0.8,0.4)	0.793423917	0.793190963	0.793190960
(0.4,0.6)	1.231663710	1.231492264	1.231492262
(0.8,0.6)	1.012139055	1.011846944	1.011846935
(0.4,0.8)	1.370939450	1.370774925	1.370774903
(0.8,0.8)	1.191207865	1.190878991	1.190878980
(0.4,1)	1.484968782	1.484809943	1.484809874
(0.8,1)	1.337817006	1.337493421	1.337493371

TABLE 2.4. Numerical solution of $v(x,t)$ for Example 2.6.2.

Grid Point	Exact Solution	Successive [**82**]	Proposed Scheme
(x_i, t_j)		$n = 10$	$n = 5$
(0.4,0.2)	0.853776946	0.853650111	0.853650112
(0.8,0.2)	0.526284644	0.526122512	0.526122511
(0.4,0.4)	1.061551936	1.061322831	1.061322829
(0.8,0.4)	0.793423917	0.793132909	0.793132909
(0.4,0.6)	1.231663710	1.231358465	1.231358458
(0.8,0.6)	1.012139055	1.011746640	1.011746639
(0.4,0.8)	1.370939450	1.370581962	1.370581955
(0.8,0.8)	1.191207865	1.190735411	1.190735413
(0.4,1)	1.484968782	1.484575963	1.484575964
(0.8,1)	1.337817006	1.337284272	1.337284280

CHAPTER 3

A COUPLED SYSTEM OF DIFFERENTIAL EQUATIONS FOR A CATALYTIC CONVERTER - AN ALTERNATIVE TECHNIQUE

This chapter[1] deals with an alternative iterative procedure for the coupled system of partial differential equations discussed in Chapter 2.

3.1. Introduction

The interphase heat transfer problems in catalytic converters draw much attention now a days due to its increasing relevance in the automobile emission control [**22, 37, 38, 40, 53, 62, 85, 87, 91**]. In this direction, [**20, 21, 82**], and [**95**] dealt with one of such problems where the vehicle temperature and converter temperature are given by a coupled partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + cu = cv, \ t > 0, 0 < x \leq l \\ \frac{\partial v}{\partial t} + bv = bu + \lambda \exp(v), \ t > 0, 0 < x \leq l \\ u(0,t) = \eta, \ t \geq 0; \ u(x,0) = u_0(x), \ v(x,0) = v_0(x), \ 0 \leq x \leq l. \end{cases} \tag{3.1}$$

where a, b, c and λ are positive constants, $u_0, v_0 \in C^1[0,l]$ and $u_0(0) = \eta$. Note that the iterative procedures available in the literature for the continuous problem [**20, 21, 95**] as well as the discretized problem [**82, 95**] are based on the successive approach. Consequently, at each step of the iterative procedure, one has to solve two linear partial differential equations seperately.

This chapter focuses on developing an alternative iterative procedure for solving (3.1). In the proposed procedure at each step of the iterative scheme, instead of solving two linear PDEs separately one has to solve a coupled linear PDE, a modification to [**82, 95**].

[1]This chapter forms the paper by L.A. Sunny, R. Roy and V. A. Vijesh in Applicable Analysis (2018), https://doi.org/10.1080/00036811.2018.1478077.

The chapter renders the convergence analysis and the monotone property of the proposed scheme for the continuous case. Based on the new procedure, a finite difference method is developed to solve the coupled system numerically. The convergence and montone property of the discretized version of the iterative procedure along with an error estimate is also provided. A new comparison lemma different from [**21, 95**] is proved to handle the coupled equation. In the discretized version, the monotone property is obtained by using the properties of the coefficient matrix.

This chapter is organised as follows. In Section 3.2, certain basic results are given that are used in the following sections. Section 3.3 provides the existence and uniqueness of the coupled system (3.1) using the proposed iterative scheme. The convergence analysis as well as the error estimate for the proposed numerical scheme is given in Section 3.4. Some numerical results are given in Section 3.5 to illustrate the efficiency of the proposed scheme. A comparative study is also provided in this section.

3.2. Preliminaries

To make this chapter self contained, this section provides basic results that will be used to prove the main results in the following sections.

Definition 3.2.1. [**86**] *An $n \times n$ real matrix $A = (a_{i,j})$ is said to be a $\mathbb{Z}$-matrix if $a_{i,j} \leq 0$ for all $i \neq j$; $1 \leq i, j \leq n$. An $n \times n$ matrix A that can be expressed in the form $A = sI - B$ where $B = (b_{i,j})$ with $b_{i,j} \geq 0$ for all $1 \leq i, j \leq n$ and $s \geq \rho(B)$, the maximum of the moduli of the eigenvalues of B is called an $\mathbb{M}$-matrix.*

For $\mathbb{M}$-matrix, [**86**] is a good reference.

Definition 3.2.2. *A matrix A is said to be inverse positive if A is invertible and $Ax \geq 0 \Rightarrow x \geq 0$.*

Throughout this chapter, Q and $\overline{Q}$ denote the sets $(0, l] \times (0, T]$ and $[0, l] \times [0, T]$ respectively.

Lemma 3.2.1. *[Lemma 1; [**21**]] If $w \in C^1(\overline{Q})$ satisfies the inequalities*

$$\begin{cases} \frac{\partial w}{\partial t} + a\frac{\partial w}{\partial x} + bw \geq 0; \quad (x,t) \in Q, \\ w(0,t) \geq 0 \quad t \in [0,T], \\ w(x,0) \geq 0 \quad x \in [0,l], \end{cases}$$

where $a \geq 0$ and $b > 0$ are constants, then $w \geq 0$ on $\overline{Q}$.

The following coupled linear partial differential equation plays a key role in the proof of the main theorem in Section 3.

(3.2)
$$\begin{cases} \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + cu = cv, \ t > 0, 0 < x \leq l \\ \frac{\partial v}{\partial t} + b(v-u) - \lambda \exp(f_1(x,t))v = \lambda \exp(f_2(x,t))(1 - f_2(x,t)), \ t > 0, 0 < x \leq l \\ u(0,t) = \eta, \ t \geq 0; \ u(x,0) = u_0(x), \ v(x,0) = v_0(x), \ 0 \leq x \leq l. \end{cases}$$

where $f_1, f_2 \in C^1(\overline{Q})$. In this section, $\lambda \exp(f_2(x,t))(1 - f_2(x,t))$ is denoted by $h(x,t)$.

Definition 3.2.3. *A function $(\overline{u}, \overline{v}) \in C^1(\overline{Q}) \times C^1(\overline{Q})$ is called an upper solution of (3.2) if it satisfies*

$$\begin{cases} \frac{\partial \overline{u}}{\partial t} + a\frac{\partial \overline{u}}{\partial x} + c\overline{u} \geq c\overline{v}, \ t > 0, 0 < x \leq l \\ \frac{\partial \overline{v}}{\partial t} + b(\overline{v} - \overline{u}) - \lambda \exp(f_1(x,t))\overline{v} \geq h(x,t), \ t > 0, 0 < x \leq l \\ \overline{u}(0,t) \geq \eta, \ t \geq 0; \ \overline{u}(x,0) \geq u_0(x), \ \overline{v}(x,0) \geq v_0(x), 0 \leq x \leq l. \end{cases}$$

Similarly $(\underline{u}, \underline{v}) \in C^1(\overline{Q}) \times C^1(\overline{Q})$ is called a lower solution if it satisfies (3.2) with the inequalities reversed.

The following theorem ensures the well defined property of the proposed iterative scheme.

Theorem 3.2.1. *Let $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ in $C^1(\overline{Q}) \times C^1(\overline{Q})$ be a pair of ordered lower and upper solutions of (3.2). Then (3.2) has a unique solution $(u^*, v^*) \in C^1(\overline{Q}) \times C^1(\overline{Q})$ such that $(\underline{u}, \underline{v}) \leq (u^*, v^*) \leq (\overline{u}, \overline{v})$.*

Proof. The solution of the linear coupled partial differential equations (3.2) is obtained as the limit of the following successive iterative procedure.

$$(3.3)\qquad \begin{cases} L_1u^{n+1} = cv^n,\ t > 0, 0 < x \leq l \\ L_2v^{n+1} - \lambda \exp(f_1(x,t))v^{n+1} = bu^{n+1} + h(x,t),\ t > 0, 0 < x \leq l \\ u^{n+1}(0,t) = \eta,\ t \geq 0,\ u^{n+1}(x,0) = u_0(x),\ v^{n+1}(x,0) = v_0(x), 0 \leq x \leq l. \end{cases}$$

where $L_1u = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + cu,\ L_2v = \frac{\partial v}{\partial t} + bv$ and $n = 0, 1, 2, \cdots$ with $(u^0, v^0) = (\underline{u}, \underline{v})$. The iterative scheme (3.3) is equivalent to the following iterative procedure [**20, 21**]. For $(x,t) \in \overline{Q}$,

$$u^{n+1}(x,t) \ = \ \begin{cases} \exp(-ct)u^{n+1}(x-at,0) + c\int_0^t \exp(c(\tau - t))v^n(x + a\tau - at, \tau)\mathrm{d}\tau \\ \ 0 \leq t < xa^{-1} \\ \exp(-cxa^{-1})\eta + c\int_0^{xa^{-1}} \exp(c(\tau - xa^{-1}))v^n(a\tau, t - xa^{-1} + \tau)\mathrm{d}\tau \\ \ xa^{-1} \leq t < T. \end{cases}$$

$$v^{n+1}(x,t) \ = \ e^{-g(x,t)}v^{n+1}(x,0) + e^{-g(x,t)}\left[b\int_0^t u^{n+1}(x,\tau)e^{g(x,\tau)}\mathrm{d}\tau + \lambda\int_0^t h(x,t)e^{g(x,\tau)}\mathrm{d}\tau\right]$$

where $g(x,t) = \int_0^t (b - \lambda\exp(f_1(x,s))\mathrm{d}s$. Clearly, the iterative scheme (3.3) is well defined. i.e.; at each step, u^{n+1} and v^{n+1} exist. Let $w^0 = u^1 - \underline{u}$ and $z^0 = v^1 - \underline{v}$.

$$L_1w^0 \ = \ L_1u^1 - L_1\underline{u} \geq c\underline{v} - c\underline{v} = 0$$

with $w^0(0,t) = \eta - \underline{u}(0,t) \geq 0$ and $w^0(x,0) = u_0(x) - \underline{u}(x,0) \geq 0$. By Lemma 3.2.1, $w^0 \geq 0$ on $\overline{Q}$. Hence $\underline{u} \leq u^1$. Also

$$\begin{aligned} L_2z^0 - \lambda\exp(f_1(x,t))z^0 \ &= \ (L_2v^1 - \lambda\exp(f_1(x,t))v^1) - (L_2\underline{v} - \lambda\exp(f_1(x,t))\underline{v}) \\ &\geq \ bu^1 + h(x,t) - b\underline{u} - h(x,t) \\ L_2z^0 - \lambda\exp(f_1(x,t))z^0 \ &\geq \ bu^1 - b\underline{u} \geq 0 \end{aligned}$$

with $z^0(x,0) = v_0(x) - \underline{v}(x,0) \geq 0$. Hence $z^0 \geq 0$ and thus $(\underline{u}, \underline{v}) \leq (u^1, v^1)$ on $\overline{Q}$. Now let $w = \overline{u} - u^1$ and $z = \overline{v} - v^1$.

$$L_1w = L_1\overline{u} - L_1u^1 \geq c\overline{v} - c\underline{v} \geq 0$$

with $w(0,t) \geq 0;\ w(x,0) \geq 0$. By Lemma 3.2.1, $w \geq 0$ on $\overline{Q}$. Hence $u^1 \leq \overline{u}$. Also

$$L_2z - \lambda\exp(f_1(x,t))z \ = \ b\overline{u} - bu^1 \geq 0$$

together with $z(x,0) \geq 0$, conclude that $z \geq 0$ and thus $(\underline{u}, \underline{v}) \leq (u^1, v^1) \leq (\overline{u}, \overline{v})$ on $\overline{Q}$. Assume that $(\underline{u}, \underline{v}) \leq (u^{n-1}, v^{n-1}) \leq (u^n, v^n) \leq (\overline{u}, \overline{v})$ for some $n > 1$. Clearly (u^{n+1}, v^{n+1}) exists. Define $w^n = u^{n+1} - u^n$ and $z^n = v^{n+1} - v^n$.

$$L_1 w^n = L_1 u^{n+1} - L_1 u^n = cv^n - cv^{n-1} \geq 0$$

with $w^n(0,t) = 0;\ w^n(x,0) = 0$. By Lemma 3.2.1, $w^n \geq 0$ on $\overline{Q}$. Hence $u^n \leq u^{n+1}$. Also

$$L_2 z^n - \lambda \exp(f_1(x,t)) z^n \quad = \quad bu^{n+1} - bu^n \geq 0$$

together with $z^n(x,0) = 0$ conclude that $z^n \geq 0$ and thus $(u^n, v^n) \leq (u^{n+1}, v^{n+1})$ on $\overline{Q}$. Now let $w = \overline{u} - u^{n+1}$ and $z = \overline{v} - v^{n+1}$.

$$L_1 w = L_1 \overline{u} - L_1 u^{n+1} \geq c\overline{v} - cv^n \geq 0$$

with $w(0,t) \geq 0;\ w(x,0) \geq 0$. By Lemma 3.2.1, $w \geq 0$ on $\overline{Q}$. Hence $u^{n+1} \leq \overline{u}$. Also

$$L_2 z - \lambda \exp(f_1(x,t)) z \quad = \quad b\overline{u} - bu^{n+1} \geq 0$$

together with $z(x,0) \geq 0$, conclude that $z \geq 0$ and thus

$$(\underline{u}, \underline{v}) \leq (u^1, v^1) \leq \ldots \leq (u^n, v^n) \leq (\overline{u}, \overline{v})$$

for all n. This guarantees the existence of the limit $\lim_{n\to\infty}(u^n, v^n) = (u^*, v^*)$. Moreover the limit is a solution of (3.2) satisfying $(\underline{u}, \underline{v}) \leq (u^*, v^*) \leq (\overline{u}, \overline{v})$. Let (u_1, v_1) and (u_2, v_2) be two solutions of (3.2) in $C^1(\overline{Q}) \times C^1(\overline{Q})$. Put $U = u_1 - u_2$ and $V = v_1 - v_2$. Then U and V satisfy

$$(3.4) \qquad \begin{cases} \frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} + c(U - V) = 0, \\ \frac{\partial V}{\partial t} + b(V - U) - \lambda \exp(f_1(x,t))V = 0, \\ U(0,t) = 0,\ U(x,0) = 0,\ V(x,0) = 0,\ t > 0, 0 < x \leq l. \end{cases}$$

For $(x,t) \in \overline{Q}$, the corresponding integral representation of (3.4) is given by

$$(3.5)\, U(x,t) \quad = \quad \begin{cases} c\int_0^t \exp(c(\tau - t))V(x + a\tau - at, \tau)\mathrm{d}\tau;\ 0 \leq t < xa^{-1} \\ c\int_0^{xa^{-1}} \exp(c(\tau - xa^{-1}))V(a\tau, t - xa^{-1} + \tau)\mathrm{d}\tau;\ xa^{-1} \leq t \leq T. \end{cases}$$

$$V(x,t) \quad = \quad be^{-g(x,t)} \int_0^t U(x,\delta)e^{g(x,\delta)}\mathrm{d}\delta$$

From (3.5), one can conclude that V satisfies the following integral equation.

$$(3.6)\quad V(x,t)=\begin{cases} cbe^{-g(x,t)}\int_0^t\int_0^{\delta}\exp(c(\tau-\delta))V(x+a\tau-a\delta,\tau)e^{g(x,\delta)}\mathrm{d}\tau\mathrm{d}\delta \\ 0\le t<xa^{-1} \\ cbe^{-g(x,t)}\int_0^t\int_0^{xa^{-1}}\exp(c(\tau-xa^{-1}))V(a\tau,\delta-xa^{-1}+\tau)e^{g(x,\delta)}\mathrm{d}\tau\mathrm{d}\delta \\ xa^{-1}\le t\le T. \end{cases}$$

Note that $V=0$ is a solution for the integral equation (3.6). Now one can show that the above integral equation has a unique solution in $C(\overline{Q})$ using contraction principle. Define $T: C(\overline{Q})\to C(\overline{Q})$ by

$$TV(x,t)\quad=\quad\begin{cases} cbe^{-g(x,t)}\int_0^t\int_0^{\delta}\exp(c(\tau-\delta))V(x+a\tau-a\delta,\tau)e^{g(x,\delta)}\mathrm{d}\tau\mathrm{d}\delta \\ \quad 0\le t<xa^{-1} \\ cbe^{-g(x,t)}\int_0^t\int_0^{xa^{-1}}\exp(c(\tau-xa^{-1}))V(a\tau,\delta-xa^{-1}+\tau)e^{g(x,\delta)}\mathrm{d}\tau\mathrm{d}\delta \\ \quad xa^{-1}\le t\le T. \end{cases}$$

One can show that $\|T^n(V_1-V_2)\|\le\frac{2(KT^2)^n}{(2n)!}\|V_1-V_2\|$. Hence for sufficiently large $n\in\mathbb{N}$, T^n is a contraction. Hence $Tx=x$ has a unique solution. Consequently, $V\equiv 0$ is the only solution of (3.6). This also leads to $U\equiv 0$. Hence the uniqueness. $\square$

3.3. Convergence Analysis for the Continuous Case

In this section, the iterative scheme based on successive and quasilinearization discussed in [**95**] is altered. Note that the procedure discussed in [**95**] is based on quasilinearition. Consequently, one has to update the derivative at each step. In the proposed scheme, the evaluation of the derivative at each step is avoided carefully by evaluating the derivative only once at a suitable initial guess. The monotone property as well as the convergence of the sequences produced by the proposed iterative new procedure is proved in this section.

Definition 3.3.1. *A function* $(\overline{\alpha}, \overline{\beta}) \in C^1(\overline{Q}) \times C^1(\overline{Q})$ *is called an upper solution of* (3.1) *if it satisfies*

$$(3.7) \qquad \begin{cases} \frac{\partial \overline{\alpha}}{\partial t} + a\frac{\partial \overline{\alpha}}{\partial x} + c\overline{\alpha} \geq c\overline{\beta}, \quad t > 0, 0 < x \leq l \\ \frac{\partial \overline{\beta}}{\partial t} + b\overline{\beta} \geq b\overline{\alpha} + \lambda \exp(\overline{\beta}), \quad t > 0, 0 < x \leq l \\ \overline{\alpha}(0,t) \geq \eta,\ \overline{\alpha}(x,0) \geq u_0(x),\ \overline{\beta}(x,0) \geq v_0(x),\ t > 0, 0 < x \leq l. \end{cases}$$

Similarly $(\alpha, \beta) \in C^1(\overline{Q}) \times C^1(\overline{Q})$ *is called a lower solution if it satisfies* (3.7) *with the inequalities reversed.*

Denote the set $\{(u,v) \in C(\overline{Q}) \times C(\overline{Q}) : (\alpha, \beta) \leq (u,v) \leq (\overline{\alpha}, \overline{\beta})\}$ by S for a given pair of lower and upper solutions. In the quasilinearization iterative procedure for the coupled equation (3.1), at each step if the derivative is evaluated only at the initial guess (u^0, v^0), then the iterative procedure becomes

$$(3.8) \quad \begin{cases} \frac{\partial u^{n+1}}{\partial t} + a\frac{\partial u^{n+1}}{\partial x} + c\left(u^{n+1} - cv^{n+1}\right) = 0, \quad t > 0, 0 < x \leq l \\ \frac{\partial v^{n+1}}{\partial t} + (b - \lambda \exp(v^0))v^{n+1} - bu^{n+1} = \lambda \exp(v^n) - \lambda \exp(v^0)v^n, \quad t > 0, 0 < x \leq l \\ u^{n+1}(0,t) = \eta,\ t \geq 0;\ u^{n+1}(x,0) = u_0(x),\ v^{n+1}(x,0) = v_0(x), 0 \leq x \leq l. \end{cases}$$

where $n = 0, 1, 2, \cdots$. Using (α, β) and $(\overline{\alpha}, \overline{\beta})$ respectively as the initial iterations (u^0, v^0), two sequences can be constructed. The minimal sequence $\{(\alpha^{n+1}, \beta^{n+1})\}$ is defined by

$$(3.9) \quad \begin{cases} \frac{\partial \alpha^{n+1}}{\partial t} + a\frac{\partial \alpha^{n+1}}{\partial x} + c\left(\alpha^{n+1} - \beta^{n+1}\right) = 0, \quad t > 0, 0 < x \leq l \\ \frac{\partial \beta^{n+1}}{\partial t} + (b - \lambda \exp(\beta))\beta^{n+1} - b\alpha^{n+1} = \lambda \exp(\beta^n) - \lambda \exp(\beta)\beta^n, \quad t > 0, 0 < x \leq l \\ \alpha^{n+1}(0,t) = \eta,\ t \geq 0;\ \alpha^{n+1}(x,0) = u_0(x),\ \beta^{n+1}(x,0) = v_0(x), 0 \leq x \leq l. \end{cases}$$

with $(u^0, v^0) = (\alpha, \beta)$. Similarly denote the maximal sequence $\{(\overline{\alpha}^{n+1}, \overline{\beta}^{n+1})\}$ is defined by

(3.10)
$$\begin{cases} \frac{\partial \overline{\alpha}^{n+1}}{\partial t} + a\frac{\partial \overline{\alpha}^{n+1}}{\partial x} + c(\overline{\alpha}^{n+1} - \overline{\beta}^{n+1}) = 0, \quad t > 0, 0 < x \leq l \\ \frac{\partial \overline{\beta}^{n+1}}{\partial t} + (b - \lambda \exp(\beta))\overline{\beta}^{n+1} - b\overline{\alpha}^{n+1} = \lambda \exp(\overline{\beta}^{n}) - \lambda \exp(\beta)\overline{\beta}^{n}, \quad t > 0, 0 < x \leq l \\ \overline{\alpha}^{n+1}(0,t) = \eta, \ t \geq 0; \ \overline{\alpha}^{n+1}(x,0) = u_0(x), \ \overline{\beta}^{n+1}(x,0) = v_0(x), 0 \leq x \leq l. \end{cases}$$

with $(u^0, v^0) = (\overline{\alpha}, \overline{\beta})$. The comparison lemma, Lemma 3.2.1 proved in [**21**] is applicable only for a single equation. Hence a new comparison lemma is proved to handle the coupled equation.

Lemma 3.3.1. *If $(u, v) \in C^1(\overline{Q}) \times C^1(\overline{Q})$ is such that*

$$\begin{cases} \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + c(u - v) > 0, \\ \frac{\partial v}{\partial t} + b(v - u) + f(x,t)v \geq 0, \end{cases} \tag{3.11}$$

where $f(x,t) \in C(\overline{Q})$ with $u(0,t) \geq 0,\ 0 \leq t \leq T$ and $u(x,0) \geq 0,\ v(x,0) \geq 0,\ 0 \leq x \leq l$, then $(u,v) \geq 0$ on $\overline{Q}$.

Proof. Assume on contrary that u has a negative minimum in Q. Then there exists some $(x_0, t_0) \in Q$ such that $u(x_0, t_0) < 0$. Then (3.11) gives $c(u(x_0,t_0) - v(x_0,t_0)) > 0$. i.e., $u(x_0,t_0) > v(x_0,t_0)$. Thus one can find a neighborhood of (x_0,t_0), say $Q_\delta(x_0,t_0) \subset Q$ such that $u(x,t) > v(x,t)$ for all $(x,t) \in Q_\delta(x_0,t_0)$. Let $(x,t) \in Q_\delta(x_0,t_0)$. Then from (3.11),

$$\begin{aligned} 0 &< \frac{\partial v}{\partial t} + f(x,t)v \\ 0 &< \exp\left(\int_{t_0}^{t} f(x,s)ds\right)\frac{\partial v}{\partial t} + \exp\left(\int_{t_0}^{t} f(x,s)ds\right) f(x,t)v \\ 0 &< \frac{d}{dt}\left(\exp\left(\int_{t_0}^{t} f(x,s)ds\right)v\right) \\ 0 &< \exp\left(\int_{t_0}^{t} f(x,s)ds\right)v(x,t) - v(x,t_0) \\ 0 &< 0 \quad \text{for the choice of } t = t_0 \end{aligned}$$

which is a contradiction. Hence u cannot be negative. Consequently for $(x,t) \in Q_\delta(x_0,t_0)$, (3.11) becomes

$$
\begin{aligned}
0 &\le \frac{\partial v}{\partial t} + (b+f(x,t))v - bu \le \frac{\partial v}{\partial t} + (b+f(x,t))v \\
0 &\le \exp\left(\int_0^t (b+f(x,s))ds\right)\frac{\partial v}{\partial t} + \exp\left(\int_0^t (b+f(x,s))ds\right)(b+f(x,t))v \\
0 &\le \frac{d}{dt}\left(\exp\left(\int_0^t (b+f(x,s))ds\right)v\right) \\
0 &\le \exp\left(\int_0^t (b+f(x,s))ds\right)v(x,t) - v(x,0) \\
0 &\le \exp\left(\int_0^t (b+f(x,s))ds\right)v(x,t) \\
0 &\le v(x,t)
\end{aligned}
$$

Thus v is also nonnegative. Hence $(u,v) \ge (0,0)$ on $\overline{Q}$. □

The following theorem renders the monotone property of both the sequences from (3.9) and (3.10) and their convergence to the unique solution of (3.1).

Theorem 3.3.1. *Let (α,β) and $(\overline{\alpha},\overline{\beta})$ be a pair of ordered lower and upper solutions of* (3.1)*. Then the minimal sequence $\{(\alpha^n,\beta^n)\}$ and the maximal sequence $\{(\overline{\alpha}^n,\overline{\beta}^n)\}$ converge monotonically to the unique solution (u^*,v^*) of* (3.1) *in S. Also, the relation*

$$
\begin{aligned}
(\alpha,\beta) \le (\alpha^n,\beta^n) &\le (\alpha^{n+1},\beta^{n+1}) \le (u^*,v^*) \\
&\le (\overline{\alpha}^{n+1},\overline{\beta}^{n+1}) \le (\overline{\alpha}^n,\overline{\beta}^n) \le (\overline{\alpha},\overline{\beta})
\end{aligned}
\tag{3.12}
$$

holds for $n = 1,2,3,\cdots$.

Proof. Since (α,β) is a lower solution of (3.1), from Definition 3.3.1,

$$
\begin{aligned}
\frac{\partial \beta}{\partial t} + b\beta &\le b\alpha + \lambda\exp(\beta) \\
\frac{\partial \beta}{\partial t} + (b - \lambda\exp(\beta))\beta &\le b\alpha + \lambda\exp(\beta) - \lambda\exp(\beta)\beta.
\end{aligned}
$$

Hence (α, β) is a lower solution for (3.9) with $n = 0$. Similarly since $(\overline{\alpha}, \overline{\beta})$ is an upper solution for (3.1), using Definition 3.3.1,

$$\begin{aligned}
\frac{\partial \overline{\beta}}{\partial t} + b\overline{\beta} &\geq b\overline{\alpha} + \lambda \exp(\overline{\beta}) \\
&= b\overline{\alpha} + \lambda \exp(\overline{\beta}) - \lambda \exp(\beta) + \lambda \exp(\beta) \\
&\geq b\overline{\alpha} + \lambda \exp(\beta)(\overline{\beta} - \beta) + \lambda \exp(\beta) \\
\frac{\partial \overline{\beta}}{\partial t} + (b - \lambda \exp(\beta))\overline{\beta} - b\overline{\alpha} &\geq \lambda \exp(\beta) - \lambda \exp(\beta)\beta.
\end{aligned}$$

Thus $(\overline{\alpha}, \overline{\beta})$ is an upper solution for (3.9) with $n = 0$ and by Theorem 3.2.1, (α^1, β^1) exists and satisfies $(\alpha, \beta) \leq (\alpha^1, \beta^1) \leq (\overline{\alpha}, \overline{\beta})$ on $\overline{Q}$. Now to show that (α^1, β^1) and $(\overline{\alpha}, \overline{\beta})$ are ordered lower and upper solutions of (3.10) with $n = 0$ given by

$$(3.13) \qquad \begin{array}{l}
\frac{\partial \overline{\alpha}^1}{\partial t} + a\frac{\partial \overline{\alpha}^1}{\partial x} + c(\overline{\alpha}^1 - \overline{\beta}^1) = 0, \quad t > 0, 0 < x \leq l \\
\frac{\partial \overline{\beta}^1}{\partial t} + (b - \lambda \exp(\beta))\overline{\beta}^1 - b\overline{\alpha}^1 = \lambda \exp(\overline{\beta}) - \lambda \exp(\beta)\overline{\beta}, \quad t > 0, 0 < x \leq l \\
\overline{\alpha}^1(0, t) = \eta,\ t \geq 0;\ \overline{\alpha}^1(x, 0) = u_0(x),\ \overline{\beta}^1(x, 0) = v_0(x), 0 \leq x \leq l.
\end{array}$$

Consider (3.9) with $n = 0$.

$$\begin{aligned}
\frac{\partial \alpha^1}{\partial t} + a\frac{\partial \alpha^1}{\partial x} + c\left(\alpha^1 - \beta^1\right) &= 0 \\
\frac{\partial \beta^1}{\partial t} + (b - \lambda \exp(\beta))\beta^1 - b\alpha^1 &= \lambda \exp(\beta) - \lambda \exp(\beta)\beta \\
&= \lambda \exp(\beta) - \lambda \exp(\beta)\beta - \lambda \exp(\overline{\beta}) + \lambda \exp(\overline{\beta}) \\
&\leq \lambda \exp(\beta)(\beta - \overline{\beta}) - \lambda \exp(\beta)\beta + \lambda \exp(\overline{\beta}) \\
\frac{\partial \alpha^1}{\partial t} + a\frac{\partial \alpha^1}{\partial x} + c\left(\alpha^1 - \beta^1\right) &\leq \lambda \exp(\overline{\beta}) - \lambda \exp(\beta)\overline{\beta}.
\end{aligned}$$

Also from Definition 3.3.1, one can have

$$\begin{aligned}
\frac{\partial \overline{\alpha}}{\partial t} + a\frac{\partial \overline{\alpha}}{\partial x} + c\overline{\alpha} &\geq c\overline{\beta} \\
\frac{\partial \overline{\beta}}{\partial t} + b\overline{\beta} &\geq b\overline{\alpha} + \lambda \exp(\overline{\beta}) \\
\frac{\partial \overline{\beta}}{\partial t} + (b - \lambda \exp(\beta))\overline{\beta} &\geq b\overline{\alpha} + \lambda \exp(\overline{\beta}) - \lambda \exp(\beta)\overline{\beta}.
\end{aligned}$$

Hence by Theorem 3.2.1, $(\overline{\alpha}^1, \overline{\beta}^1)$ exists and satisfies $(\alpha^1, \beta^1) \leq (\overline{\alpha}^1, \overline{\beta}^1) \leq (\overline{\alpha}, \overline{\beta})$ on $\overline{Q}$ and thus

$$(\alpha, \beta) \leq (\alpha^1, \beta^1) \leq (\overline{\alpha}^1, \overline{\beta}^1) \leq (\overline{\alpha}, \overline{\beta}).$$

Assume that

$$(\alpha, \beta) \leq (\alpha^{n-1}, \beta^{n-1}) \leq (\alpha^n, \beta^n) \leq (\overline{\alpha}^n, \overline{\beta}^n) \leq (\overline{\alpha}^{n-1}, \overline{\beta}^{n-1}) \leq (\overline{\alpha}, \overline{\beta})$$

for some $n > 1$. Now one has to prove that (α^n, β^n) and $(\overline{\alpha}^n, \overline{\beta}^n)$ are ordered lower and upper solutions of (3.9). From (3.9),

$$\begin{aligned}
\frac{\partial \alpha^n}{\partial t} + a\frac{\partial \alpha^n}{\partial x} + c\left(\alpha^n - \beta^n\right) &= 0 \\
\frac{\partial \beta^n}{\partial t} + (b - \lambda \exp(\beta))\beta^n - b\alpha^n &= \lambda \exp(\beta^{n-1}) - \lambda \exp(\beta)\beta^{n-1} \\
&= \lambda \exp(\beta^{n-1}) - \lambda \exp(\beta)\beta^{n-1} - \lambda \exp(\beta^n) + \lambda \exp(\beta^n) \\
&\leq \lambda \exp(\beta)(\beta^{n-1} - \beta^n) - \lambda \exp(\beta)(\beta^{n-1}) + \lambda \exp(\beta^n) \\
\frac{\partial \beta^n}{\partial t} + (b - \lambda \exp(\beta))\beta^n - b\alpha^n &\leq \lambda \exp(\beta^n) - \lambda \exp(\beta)\beta^n.
\end{aligned}$$

Also from (3.10),

$$\begin{aligned}
\frac{\partial \overline{\alpha}^n}{\partial t} + a\frac{\partial \overline{\alpha}^n}{\partial x} + c\left(\overline{\alpha}^n - \overline{\beta}^n\right) &= 0 \\
\frac{\partial \overline{\beta}^n}{\partial t} + (b - \lambda \exp(\beta))\overline{\beta}^n - b\overline{\alpha}^n &= \lambda \exp(\overline{\beta}^{n-1}) - \lambda \exp(\beta)\overline{\beta}^{n-1} \\
&= \lambda \exp(\overline{\beta}^{n-1}) - \lambda \exp(\beta)\overline{\beta}^{n-1} - \lambda \exp(\beta^n) + \lambda \exp(\beta^n) \\
&\geq \lambda \exp(\beta)(\overline{\beta}^{n-1} - \beta^n) - \lambda \exp(\beta)(\overline{\beta}^{n-1}) + \lambda \exp(\beta^n) \\
\frac{\partial \overline{\beta}^n}{\partial t} + (b - \lambda \exp(\beta))\overline{\beta}^n - b\overline{\alpha}^n &\geq \lambda \exp(\beta^n) - \lambda \exp(\beta)\beta^n.
\end{aligned}$$

Hence by Theorem 3.2.1, $(\alpha^{n+1}, \beta^{n+1})$ exists and satisfies $(\alpha^n, \beta^n) \leq (\alpha^{n+1}, \beta^{n+1}) \leq (\overline{\alpha}^n, \overline{\beta}^n)$ on $\overline{Q}$. To obtain (3.12), it has to be proved that $(\alpha^{n+1}, \beta^{n+1})$ and $(\overline{\alpha}^n, \overline{\beta}^n)$

are ordered lower and upper solutions of (3.10). From (3.9),

$$\begin{aligned}
\frac{\partial \alpha^{n+1}}{\partial t}+a\frac{\partial \alpha^{n+1}}{\partial x}+c\left(\alpha^{n+1}-\beta^{n+1}\right) &= 0\\
\frac{\partial \beta^{n+1}}{\partial t}+(b-\lambda\exp(\beta))\beta^{n+1}-b\alpha^{n+1} &= \lambda\exp(\beta^{n})-\lambda\exp(\beta)\beta^{n}\\
&= \lambda\exp(\beta^{n})-\lambda\exp(\beta)\beta^{n}-\lambda\exp(\overline{\beta}^{n})+\lambda\exp(\overline{\beta}^{n})\\
&\leq \lambda\exp(\beta)(\beta^{n}-\overline{\beta}^{n})-\lambda\exp(\beta)(\beta^{n})+\lambda\exp(\overline{\beta}^{n})\\
\frac{\partial \beta^{n+1}}{\partial t}+(b-\lambda\exp(\beta))\beta^{n+1}-b\alpha^{n+1} &\leq \lambda\exp(\overline{\beta}^{n})-\lambda\exp(\beta)\overline{\beta}^{n}.
\end{aligned}$$

Also from (3.10),

$$\begin{aligned}
\frac{\partial \overline{\alpha}^{n}}{\partial t}+a\frac{\partial \overline{\alpha}^{n}}{\partial x}+c\left(\overline{\alpha}^{n}-\overline{\beta}^{n}\right) &= 0\\
\frac{\partial \overline{\beta}^{n}}{\partial t}+(b-\lambda\exp(\beta))\overline{\beta}^{n}-b\overline{\alpha}^{n} &= \lambda\exp(\overline{\beta}^{n-1})-\lambda\exp(\beta)\overline{\beta}^{n-1}\\
&= \lambda\exp(\overline{\beta}^{n-1})-\lambda\exp(\beta)\overline{\beta}^{n-1}-\lambda\exp(\overline{\beta}^{n})+\lambda\exp(\overline{\beta}^{n})\\
&\geq \lambda\exp(\beta)(\overline{\beta}^{n-1}-\overline{\beta}^{n})-\lambda\exp(\beta)(\overline{\beta}^{n-1})+\lambda\exp(\overline{\beta}^{n})\\
\frac{\partial \overline{\beta}^{n}}{\partial t}+(b-\lambda\exp(\beta))\overline{\beta}^{n}-b\overline{\alpha}^{n} &\geq \lambda\exp(\overline{\beta}^{n})-\lambda\exp(\beta)\overline{\beta}^{n}.
\end{aligned}$$

Hence by Theorem 3.2.1, $(\overline{\alpha}^{n+1},\overline{\beta}^{n+1})$ exists and satisfies $(\alpha^{n+1},\beta^{n+1})\leq(\overline{\alpha}^{n+1},\overline{\beta}^{n+1})\leq(\overline{\alpha}^{n},\overline{\beta}^{n})$ on $\overline{Q}$. Thus

$$(\alpha,\beta)\leq(\alpha^{n},\beta^{n})\leq(\alpha^{n+1},\beta^{n+1})\leq(\overline{\alpha}^{n+1},\overline{\beta}^{n+1})\leq(\overline{\alpha}^{n},\overline{\beta}^{n})\leq(\overline{\alpha},\overline{\beta})$$

for all n and this guarantees the existence of the limits

$$\lim_{n\to\infty}(\alpha^{n},\beta^{n})=(\alpha^{*},\beta^{*});\quad \lim_{n\to\infty}(\overline{\alpha}^{n},\overline{\beta}^{n})=(\overline{\alpha}^{*},\overline{\beta}^{*})$$

Moreover both the limits are solutions of (3.1). The uniqueness of the solution of (3.2) can be obtained using similar argument in Theorem 3.2.1. Let (u_1,v_1) and (u_2,v_2) be two solutions of (3.1) in $C^1(\overline{Q})\times C^1(\overline{Q})$. Put $U=u_1-u_2$ and $V=v_1-v_2$. Then U and V satisfy

$$\begin{cases}
\frac{\partial U}{\partial t}+a\frac{\partial U}{\partial x}+c(U-V)=0,\\
\frac{\partial V}{\partial t}+b(V-U)=\lambda(\exp(v_1)-\exp(v_2)),\\
U(0,t)=0,\ t\geq 0;\ U(x,0)=0,\ V(x,0)=0, 0\leq x\leq l.
\end{cases}\tag{3.14}$$

For $(x,t) \in \overline{Q}$, the corresponding integral representation of (3.4) is given by

$$
(3.15)\quad \begin{aligned}
U(x,t) &= \begin{cases} c\int_0^t \exp(c(\tau-t))V(x+a\tau-at,\tau)\mathrm{d}\tau;\ 0 \le t < xa^{-1} \\ c\int_0^{xa^{-1}} \exp(c(\tau-xa^{-1}))V(a\tau, t-xa^{-1}+\tau)\mathrm{d}\tau;\ xa^{-1} \le t \le T. \end{cases} \\
V(x,t) &= b\int_0^t e^{\delta-t}U(x,\delta)\mathrm{d}\delta + \lambda\int_0^t e^{\delta-t}(\exp(v_1((x,\delta)) - \exp(v_2((x,\delta)))\mathrm{d}\delta
\end{aligned}
$$

From (3.15), one can conclude that V satisfies the following integral equation.

$$
V(x,t) = \begin{cases} bc\int_0^t e^{\delta-t}\int_0^\delta \exp(c(\tau-\delta))V(x+a\tau-a\delta,\tau)\mathrm{d}\tau\mathrm{d}\delta \\ +\lambda\int_0^t e^{\delta-t}(\exp(v_1((x,\delta)) - \exp(v_2((x,\delta)))\mathrm{d}\delta, \quad 0 \le t < xa^{-1} \\ bc\int_0^t e^{\delta-t}\int_0^{xa^{-1}} \exp(c(\tau-xa^{-1}))V(a\tau,\delta-xa^{-1}+\tau)\mathrm{d}\tau\mathrm{d}\delta \\ +\lambda\int_0^t e^{\delta-t}(\exp(v_1((x,\delta)) - \exp(v_2((x,\delta)))\mathrm{d}\delta, \quad xa^{-1} \le t \le T. \end{cases}
$$

Note that $V \equiv 0$ is a solution for the integral equation. Similar to that of the linear case in Theorem 3.2.1, one can show that the above integral equation has a unique solution in $C(\overline{Q})$ using contraction principle. □

Remark 3.3.1. *Applying the classical quasilinearization technique to the coupled equation* (3.1)*, the following iterative procedure can be obtained.*

$$
(3.16)\quad \begin{cases} \frac{\partial u^{n+1}}{\partial t} + a\frac{\partial u^{n+1}}{\partial x} + c\left(u^{n+1} - v^{n+1}\right) = 0, \quad t>0, 0<x\le l \\ \frac{\partial v^{n+1}}{\partial t} + (b-\lambda\exp(v^n))v^{n+1} - bu^{n+1} = \lambda\exp(v^n) - \lambda\exp(v^n)v^n, \quad t>0, 0<x\le l \\ u^{n+1}(0,t) = \eta,\ t\ge 0;\ u^{n+1}(x,0) = u_0(x),\ v^{n+1}(x,0) = v_0(x), 0\le x\le l. \end{cases}
$$

where $n = 0,1,2,\cdots$. *Using* (α,β) *and* $(\overline{\alpha},\overline{\beta})$ *respectively as the initial iterations* (u^0,v^0)*, two sequences can be constructed. The minimal sequence is defined by*

$$
(3.17)\quad \begin{cases} \frac{\partial \alpha^{n+1}}{\partial t} + a\frac{\partial \alpha^{n+1}}{\partial x} + c\left(\alpha^{n+1} - \beta^{n+1}\right) = 0, \quad t>0, 0<x\le l \\ \frac{\partial \beta^{n+1}}{\partial t} + (b-\lambda\exp(\beta^n))\beta^{n+1} - b\alpha^{n+1} = \lambda\exp(\beta^n) - \lambda\exp(\beta^n)\beta^n, \quad t>0, 0<x\le l \\ \alpha^{n+1}(0,t) = \eta,\ t\ge 0;\ \alpha^{n+1}(x,0) = u_0(x),\ \beta^{n+1}(x,0) = v_0(x), 0\le x\le l. \end{cases}
$$

where $(u^0, v^0) = (\alpha, \beta)$ *and the maximal sequence is defined by*

(3.18)
$$\begin{cases} \frac{\partial \overline{\alpha}^{n+1}}{\partial t} + a\frac{\partial \overline{\alpha}^{n+1}}{\partial x} + c(\overline{\alpha}^{n+1} - \overline{\beta}^{n+1}) = 0, \quad t > 0, 0 < x \leq l \\ \frac{\partial \overline{\beta}^{n+1}}{\partial t} + (b - \lambda \exp(\beta^n))\overline{\beta}^{n+1} - b\overline{\alpha}^{n+1} = \lambda \exp(\overline{\beta}^n) - \lambda \exp(\beta^n)\overline{\beta}^n, \quad t > 0, 0 < x \leq l \\ \overline{\alpha}^{n+1}(0,t) = \eta, \ t \geq 0; \ \overline{\alpha}^{n+1}(x,0) = u_0(x), \ \overline{\beta}^{n+1}(x,0) = v_0(x), 0 \leq x \leq l. \end{cases}$$

where $(u^0, v^0) = (\overline{\alpha}, \overline{\beta})$.

The following theorem ensures the monotone convergence of the iterative schemes (3.17) and (3.18).

Theorem 3.3.2. *Let* (α, β) *and* $(\overline{\alpha}, \overline{\beta})$ *be a pair of ordered lower and upper solutions of* (3.1)*. Then the minimal sequence* $\{(\alpha^n, \beta^n)\}$ *and the maximal sequence* $\{(\overline{\alpha}^n, \overline{\beta}^n)\}$ *given by* (3.17) *and* (3.18) *respectively converge monotonically to the unique solution* (u^*, v^*) *of* (3.1) *in* S*. Also, the relation*

$$\begin{aligned} (\alpha, \beta) \leq (\alpha^n, \beta^n) \ &\leq \ (\alpha^{n+1}, \beta^{n+1}) \leq (u^*, v^*) \\ &\leq (\overline{\alpha}^{n+1}, \overline{\beta}^{n+1}) \leq (\overline{\alpha}^n, \overline{\beta}^n) \leq (\overline{\alpha}, \overline{\beta}) \end{aligned}$$

holds for $n = 1, 2, 3, \cdots$.

Proof. The proof of this theorem is similar to that of Theorem 3.3.1. □

3.4. Convergence Analysis for the Discretized Case

Employing the main theorem from Section 3, a finite difference scheme is acquired to solve (3.1) numerically. The convergence, error estimate and the monotonicity of the sequences are obtained just by utilizing the properties of the coefficient matrix in the iterative scheme. The iterative scheme discussed in [**95**] improves the algorithm in [**82**] by quasilinearizing the second equation in the coupled system (3.1). Similar to the algorithm in [**95**], the proposed algorithm also accelerates the algorithm in [**82**]. The major difference between the proposed scheme and [**95**] is that in [**95**], the evaluation of derivative at each iteration is required whereas for the proposed scheme, the derivative needs to be evaluated only once. Let $h = \triangle x = \frac{l}{M}$, $k = \triangle t = \frac{T}{N}$ be the step sizes in x and t directions respectively. Λ and $\bar{\Lambda}$ respectively denote the sets of mesh points (x_i, t_j) in $(0, l] \times (0, T]$ and $[0, l] \times [0, T]$. Define $u_{i,j} = u(x_i, t_j)$ and $v_{i,j} = v(x_i, t_j)$. Throghout this section, assume that the time step k satisfies the same condition in [**82**] and [**95**], i.e.;

$$\frac{1}{k} > \max\{b - c - \frac{a}{h}, c - b + \lambda \exp(\xi^*)\} \tag{3.19}$$

where $\xi^* = \max\{\overline{\beta}_{i,j} : (i, j) \in \bar{\Lambda}\}$. The convergence analysis of the proposed finite difference iterative scheme is based on this assumption. Once again, the backward finite difference approximation is used for discretizing (3.1). Consequently, the discretized version of (3.1) can be represented by

$$\begin{cases} \begin{pmatrix} \mu_1 & -ck \\ -bk & \mu_2 \end{pmatrix} \begin{pmatrix} u_{i,j} \\ v_{i,j} \end{pmatrix} = \begin{pmatrix} u_{i,j-1} + \frac{ak}{h} u_{i-1,j} \\ v_{i,j-1} + k\lambda \exp(v_{i,j}) \end{pmatrix} \\ u_{0,j} = \eta_j, \ u_{i,0} = \psi_i, \ v_{i,0} = \phi_i; \ i = 1, 2, \ldots, M, \ j = 1, 2, \ldots, N. \end{cases} \tag{3.20}$$

where $\mu_1 = 1 + \frac{ak}{h} + ck, \mu_2 = 1 + bk$.

Definition 3.4.1. *A function $(\alpha_{i,j}, \beta_{i,j})$ defined on $\overline{\Lambda}$ is called a lower solution of* (3.20) *if it satisfies*

$$\begin{cases} \begin{pmatrix} \mu_1 & -ck \\ -bk & \mu_2 \end{pmatrix} \begin{pmatrix} \alpha_{i,j} \\ \beta_{i,j} \end{pmatrix} \leq \begin{pmatrix} \alpha_{i,j-1} + \frac{ak}{h} \alpha_{i-1,j} \\ \beta_{i,j-1} + k\lambda \exp(\beta_{i,j}) \end{pmatrix} \\ \alpha_{0,j} \leq \eta_j, \ \alpha_{i,0} \leq \psi_i, \ \beta_{i,0} \leq \phi_i; \ i = 1, 2, \ldots, M, \ j = 1, 2, \ldots, N. \end{cases} \tag{3.21}$$

Similarly $(\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})$ is called an upper solution if it satisfies (3.21) *with inequalities reversed.*

For a given pair of ordered lower and upper solutions $(\alpha_{i,j}, \beta_{i,j})$ and $(\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})$ of (3.20), set

$$S^d = \{(u_{i,j}, v_{i,j}) \in \mathbb{R}^2 : (\alpha_{i,j}, \beta_{i,j}) \leq (u_{i,j}, v_{i,j}) \leq (\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})\}.$$

Applying modified quasilinearization technique to both the equations of (3.20) simultaneously yields

$$\text{(3.22)} \qquad \begin{cases} A\begin{pmatrix} u_{i,j}^{n+1} \\ v_{i,j}^{n+1} \end{pmatrix} = \begin{pmatrix} u_{i,j-1}^{n+1} + \frac{ak}{h} u_{i-1,j}^{n+1} \\ v_{i,j-1}^{n+1} + k\lambda \exp(v_{i,j}^{n}) - k\lambda \exp(v_{i,j}^{0}) v_{i,j}^{n} \end{pmatrix} \\ u_{0,j}^{n+1} = \eta_j, \ u_{i,0}^{n+1} = \psi_i, \ v_{i,0}^{n+1} = \phi_i; \ i = 1, 2, \ldots, M, \ j = 1, 2, \ldots, N. \end{cases}$$

for $n = 0, 1, \cdots$ where $\eta_j = \eta(t_j), \psi_i = \psi(x_i)$, $\phi_i = \phi(x_i)$, $\eta_1 = (\mu_2 - k\lambda \exp(v_{i,j}^0))$ and $A = \begin{pmatrix} \mu_1 & -ck \\ -bk & \eta_1 \end{pmatrix}$. Due to (3.19), the matrix A is an $\mathbb{M}$-matrix and is inverse positive. This assures the existence of $(u_{i,j}^n, v_{i,j}^n)$ for all i, j and n. Consequently (3.22) becomes

$$\begin{pmatrix} u_{i,j}^{n+1} \\ v_{i,j}^{n+1} \end{pmatrix} = A^{-1} \begin{pmatrix} u_{i,j-1}^{n+1} + \frac{ak}{h} u_{i-1,j}^{n+1} \\ v_{i,j-1}^{n+1} + k\lambda \exp(v_{i,j}^{n}) - k\lambda \exp(v_{i,j}^{0}) v_{i,j}^{n} \end{pmatrix}$$

Using $(\alpha_{i,j}, \beta_{i,j})$ and $(\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})$ respectively as the initial iterations $(u_{i,j}^0, v_{i,j}^0)$, two sequences can be constructed by using (3.22). Denote the minimal sequence by $\left\{(\alpha_{i,j}^n, \beta_{i,j}^n)\right\}$ defined by

$$\text{(3.23)} \qquad \begin{cases} A\begin{pmatrix} \alpha_{i,j}^{n+1} \\ \beta_{i,j}^{n+1} \end{pmatrix} = \begin{pmatrix} \alpha_{i,j-1}^{n+1} + \frac{ak}{h} \alpha_{i-1,j}^{n+1} \\ \beta_{i,j-1}^{n+1} + k\lambda \exp(\beta_{i,j}^{n}) - k\lambda \exp(\beta_{i,j}) \beta_{i,j}^{n} \end{pmatrix} \\ \alpha_{0,j}^{n+1} = \eta_j, \ \alpha_{i,0}^{n+1} = \psi_i, \ \beta_{i,0}^{n+1} = \phi_i; \ i = 1, 2, \ldots, M, \ j = 1, 2, \ldots, N. \end{cases}$$

with $(u_{i,j}^0, v_{i,j}^0) = (\alpha_{i,j}, \beta_{i,j})$. Similarly denote the maximal sequence by $\left\{(\overline{\alpha}_{i,j}^n, \overline{\beta}_{i,j}^n)\right\}$ defined by

$$\text{(3.24)} \qquad \begin{cases} A\begin{pmatrix} \overline{\alpha}_{i,j}^{n+1} \\ \overline{\beta}_{i,j}^{n+1} \end{pmatrix} = \begin{pmatrix} \overline{\alpha}_{i,j-1}^{n+1} + \frac{ak}{h} \overline{\alpha}_{i-1,j}^{n+1} \\ \overline{\beta}_{i,j-1}^{n+1} + k\lambda \exp(\overline{\beta}_{i,j}^{n}) - k\lambda \exp(\beta_{i,j}) \overline{\beta}_{i,j}^{n}, \end{pmatrix} \\ \overline{\alpha}_{0,j}^{n+1} = \eta_j, \ \overline{\alpha}_{i,0}^{n+1} = \psi_i, \ \overline{\beta}_{i,0}^{n+1} = \phi_i; \ i = 1, 2, \ldots, M, \ j = 1, 2, \ldots, N. \end{cases}$$

with $(u_{i,j}^0, v_{i,j}^0) = (\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})$.

Theorem 3.4.1. *Let $(\alpha_{i,j}, \beta_{i,j})$ and $(\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})$ be a pair of ordered lower and upper solutions of* (3.20). *Then the minimal sequence $\left\{(\alpha_{i,j}^n, \beta_{i,j}^n)\right\}$ and the maximal sequence $\left\{(\overline{\alpha}_{i,j}^n, \overline{\beta}_{i,j}^n)\right\}$ converge monotonically to the unique solution $(u_{i,j}^*, v_{i,j}^*)$ of* (3.20) *in S^d.*

Also, the relation

$$(\alpha_{i,j},\beta_{i,j}) \le (\alpha^n_{i,j},\beta^n_{i,j}) \le (\alpha^{n+1}_{i,j},\beta^{n+1}_{i,j}) \le (u^*_{i,j},v^*_{i,j}) \le (\overline{\alpha}^{n+1}_{i,j},\overline{\beta}^{n+1}_{i,j}) \le (\overline{\alpha}^{n}_{i,j},\overline{\beta}^{n}_{i,j}) \le (\overline{\alpha}_{i,j},\overline{\beta}_{i,j}) \tag{3.25}$$

holds for every $(i,j) \in \bar{\Lambda}$ *and* $n = 1, 2, 3, \cdots$.

Proof. The monotone property of the discretized sequences is obtained by using the inverse positivity of A. The proof is by an induction on n. From (3.23) for $n = 0$

$$\begin{pmatrix} \mu_1 & -ck \\ -bk & \mu_2 \end{pmatrix} \begin{pmatrix} \alpha^1_{i,j} \\ \beta^1_{i,j} \end{pmatrix} = \begin{pmatrix} \alpha^1_{i,j-1} + \frac{ak}{h}\alpha^1_{i-1,j} \\ \{\beta^1_{i,j-1} + k\lambda\exp(\beta_{i,j}) \\ +k\lambda\exp(\beta_{i,j})\left(\beta^1_{i,j} - \beta_{i,j}\right)\} \end{pmatrix} \tag{3.26}$$

Subtracting (3.21) from (3.26) one can get

$$\begin{pmatrix} \mu_1 & -ck \\ -bk & \mu_2 \end{pmatrix} \begin{pmatrix} \alpha^1_{i,j} - \alpha_{i,j} \\ \beta^1_{i,j} - \beta_{i,j} \end{pmatrix} \ge \begin{pmatrix} \alpha^1_{i,j-1} - \alpha_{i,j-1} + \frac{ak}{h}\left(\alpha^1_{i-1,j} - \alpha_{i-1,j}\right) \\ \beta^1_{i,j-1} - \beta_{i,j-1} + k\lambda\exp(\beta_{i,j})\left(\beta^1_{i,j} - \beta_{i,j}\right) \end{pmatrix}.$$

Consequently

$$A\begin{pmatrix} \alpha^1_{i,j} - \alpha_{i,j} \\ \beta^1_{i,j} - \beta_{i,j} \end{pmatrix} \ge \begin{pmatrix} \alpha^1_{i,j-1} - \alpha_{i,j-1} + \frac{ak}{h}\left(\alpha^1_{i-1,j} - \alpha_{i-1,j}\right) \\ \beta^1_{i,j-1} - \beta_{i,j-1} \end{pmatrix} \tag{3.27}$$

Putting $j = 1$, (3.27) becomes

$$A\begin{pmatrix} \alpha^1_{i,1} - \alpha_{i,1} \\ \beta^1_{i,1} - \beta_{i,1} \end{pmatrix} \ge \begin{pmatrix} \alpha^1_{i,0} - \alpha_{i,0} + \frac{ak}{h}\left(\alpha^1_{i-1,1} - \alpha_{i-1,1}\right) \\ \beta^1_{i,0} - \beta_{i,0} \end{pmatrix} \tag{3.28}$$

Clearly $\alpha^1_{i,0} - \alpha_{i,0} \ge 0$ and $\beta^1_{i,0} - \beta_{i,0} \ge 0$ for all i. Hence,

$$A\begin{pmatrix} \alpha^1_{i,1} - \alpha_{i,1} \\ \beta^1_{i,1} - \beta_{i,1} \end{pmatrix} \ge \begin{pmatrix} \frac{ak}{h}\left(\alpha^1_{i-1,1} - \alpha_{i-1,1}\right) \\ 0 \end{pmatrix} \tag{3.29}$$

For $i = 1$, from the boundary conditions one can obtain that $\alpha^1_{0,1} - \alpha_{0,1} \ge 0$. The inverse positivity of A together with (3.29) implies $\begin{pmatrix} \alpha^1_{1,1} - \alpha_{1,1} \\ \beta^1_{1,1} - \beta_{1,1} \end{pmatrix} \ge \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Using the relation between $(\alpha^1_{1,1}, \beta^1_{1,1})$ and $(\alpha_{1,1}, \beta_{1,1})$ in (3.29) for $i = 2$, one can conclude that

$\begin{pmatrix} \alpha^1_{2,1} - \alpha_{2,1} \\ \beta^1_{2,1} - \beta_{2,1} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Repeating the above argument for $i = 3, \ldots, M$, one can obtain

$$\begin{pmatrix} \alpha^1_{i,1} - \alpha_{i,1} \\ \beta^1_{i,1} - \beta_{i,1} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \forall i = 1, 2, \ldots, M. \tag{3.30}$$

Now for $j = 2$, (3.27) becomes

$$A \begin{pmatrix} \alpha^1_{i,2} - \alpha_{i,2} \\ \beta^1_{i,2} - \beta_{i,2} \end{pmatrix} \geq \begin{pmatrix} \alpha^1_{i,1} - \alpha_{i,1} + \frac{ak}{h}\left(\alpha^1_{i-1,2} - \alpha_{i-1,2}\right) \\ \beta^1_{i,1} - \beta_{i,1} \end{pmatrix} \tag{3.31}$$

Using (3.30) in (3.31) leads to

$$A \begin{pmatrix} \alpha^1_{i,2} - \alpha_{i,2} \\ \beta^1_{i,2} - \beta_{i,2} \end{pmatrix} \geq \begin{pmatrix} \frac{ak}{h}\left(\alpha^1_{i-1,2} - \alpha_{i-1,2}\right) \\ 0 \end{pmatrix}$$

Now using similar argument for $i = 1, 2, \ldots, M$, one can conclude that $\begin{pmatrix} \alpha^1_{i,2} - \alpha_{i,2} \\ \beta^1_{i,2} - \beta_{i,2} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all i. Repeating the above process for $j = 3, \ldots, N$, one can obtain $\begin{pmatrix} \alpha^1_{i,j} - \alpha_{i,j} \\ \beta^1_{i,j} - \beta_{i,j} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for every i and j. Similarly $\begin{pmatrix} \overline{\alpha}_{i,j} - \overline{\alpha}^1_{i,j} \\ \overline{\beta}_{i,j} - \overline{\beta}^1_{i,j} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for every i and j. From (3.24) for $n = 0$ gives

$$A \begin{pmatrix} \overline{\alpha}^1_{i,j} \\ \overline{\beta}^1_{i,j} \end{pmatrix} = \begin{pmatrix} \overline{\alpha}^1_{i,j-1} + \frac{ak}{h}\overline{\alpha}^1_{i-1,j} \\ \overline{\beta}^1_{i,j-1} + k\lambda\exp(\overline{\beta}_{i,j}) - k\lambda\exp(\beta_{i,j})\overline{\beta}_{i,j} \end{pmatrix} \tag{3.32}$$

Subtracting (3.26) from (3.32) and by using mean value theorem

$$A \begin{pmatrix} \overline{\alpha}^1_{i,j} - \alpha^1_{i,j} \\ \overline{\beta}^1_{i,j} - \beta^1_{i,j} \end{pmatrix} = \begin{pmatrix} \overline{\alpha}^1_{i,j-1} - \alpha^1_{i,j-1} + \frac{ak}{h}\left(\overline{\alpha}^1_{i-1,j} - \alpha^1_{i-1,j}\right) \\ \Big\{\overline{\beta}^1_{i,j-1} - \beta^1_{i,j-1} + k\lambda\left(\exp(\overline{\beta}_{i,j}) - \exp(\beta_{i,j})\right) \\ -k\lambda\exp(\beta_{i,j})\left(\overline{\beta}_{i,j} - \beta_{i,j}\right)\Big\} \end{pmatrix}$$

$$A \begin{pmatrix} \overline{\alpha}^1_{i,j} - \alpha^1_{i,j} \\ \overline{\beta}^1_{i,j} - \beta^1_{i,j} \end{pmatrix} \geq \begin{pmatrix} \overline{\alpha}^1_{i,j-1} - \alpha^1_{i,j-1} + \frac{ak}{h}\left(\overline{\alpha}^1_{i-1,j} - \alpha^1_{i-1,j}\right) \\ \overline{\beta}^1_{i,j-1} - \beta^1_{i,j-1} \end{pmatrix}. \tag{3.33}$$

For $j = 1$ in (3.33),

$$A \begin{pmatrix} \overline{\alpha}_{i,1}^1 - \alpha_{i,1}^1 \\ \overline{\beta}_{i,1}^1 - \beta_{i,1}^1 \end{pmatrix} \geq \begin{pmatrix} \overline{\alpha}_{i,0}^1 - \alpha_{i,0}^1 + \frac{ak}{h}\left(\overline{\alpha}_{i-1,1}^1 - \alpha_{i-1,1}^1\right) \\ \overline{\beta}_{i,0}^1 - \beta_{i,0}^1 \end{pmatrix} \tag{3.34}$$

Clearly $\overline{\alpha}_{i,0}^1 - \alpha_{i,0}^1 \geq 0$ and $\overline{\beta}_{i,0}^1 - \beta_{i,0}^1 \geq 0$ for all i. Consequently,

$$A \begin{pmatrix} \overline{\alpha}_{i,1}^1 - \alpha_{i,1}^1 \\ \overline{\beta}_{i,1}^1 - \beta_{i,1}^1 \end{pmatrix} \geq \begin{pmatrix} \frac{ak}{h}\left(\overline{\alpha}_{i-1,1}^1 - \alpha_{i-1,1}^1\right) \\ 0 \end{pmatrix} \tag{3.35}$$

For $i = 1$, from the boundary conditions one can obtain $\overline{\alpha}_{0,1}^1 - \alpha_{0,1}^1 \geq 0$. The inverse positivity of A together with (3.35) leads to $\begin{pmatrix} \overline{\alpha}_{1,1}^1 - \alpha_{1,1}^1 \\ \overline{\beta}_{1,1}^1 - \beta_{1,1}^1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Using the relation between $(\overline{\alpha}_{1,1}^1, \overline{\beta}_{1,1}^1)$and $(\alpha_{1,1}^1, \beta_{1,1}^1)$ in (3.35) for $i = 2$, one can conclude that $\begin{pmatrix} \overline{\alpha}_{2,1}^1 - \alpha_{2,1}^1 \\ \overline{\beta}_{2,1}^1 - \beta_{2,1}^1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. By repeating the above argument for $i = 3, \ldots, M$, one can obtain that

$$\begin{pmatrix} \overline{\alpha}_{i,1}^1 - \alpha_{i,1}^1 \\ \overline{\beta}_{i,1}^1 - \beta_{i,1}^1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \forall i = 1, 2, \ldots, M. \tag{3.36}$$

Now for $j = 2$, (3.33) becomes

$$A \begin{pmatrix} \overline{\alpha}_{i,2}^1 - \alpha_{i,2}^1 \\ \overline{\beta}_{i,2}^1 - \beta_{i,2}^1 \end{pmatrix} \geq \begin{pmatrix} \overline{\alpha}_{i,1}^1 - \alpha_{i,1}^1 + \frac{ak}{h}\left(\overline{\alpha}_{i-1,2}^1 - \alpha_{i-1,2}^1\right) \\ \overline{\beta}_{i,1}^1 - \beta_{i,1}^1 \end{pmatrix} \tag{3.37}$$

Using (3.36) in (3.37) leads to

$$A \begin{pmatrix} \overline{\alpha}_{i,2}^1 - \alpha_{i,2}^1 \\ \overline{\beta}_{i,2}^1 - \beta_{i,2}^1 \end{pmatrix} \geq \begin{pmatrix} \frac{ak}{h}\left(\overline{\alpha}_{i-1,2}^1 - \alpha_{i-1,2}^1\right) \\ 0 \end{pmatrix}.$$

Using the inverse positivity of A and above recurrence relation for $i = 1, 2, \ldots, M$, one can conclude that $\begin{pmatrix} \overline{\alpha}_{i,2}^1 - \alpha_{i,2}^1 \\ \overline{\beta}_{i,2}^1 - \beta_{i,2}^1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all i. Repeating the above process for $j = 3, \ldots, N$, one can obtain $\begin{pmatrix} \overline{\alpha}_{i,j}^1 - \alpha_{i,j}^1 \\ \overline{\beta}_{i,j}^1 - \beta_{i,j}^1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for every i and j. Hence

$$(\alpha_{i,j}, \beta_{i,j}) \leq (\alpha_{i,j}^1, \beta_{i,j}^1) \leq (\overline{\alpha}_{i,j}^1, \overline{\beta}_{i,j}^1) \leq (\overline{\alpha}_{i,j}, \overline{\beta}_{i,j}). \tag{3.38}$$

Assume that the result is true for some $n > 1$.

$$(\alpha_{i,j}^{n-1}, \beta_{i,j}^{n-1}) \leq (\alpha_{i,j}^{n}, \beta_{i,j}^{n}) \leq (\overline{\alpha}_{i,j}^{n}, \overline{\beta}_{i,j}^{n}) \leq (\overline{\alpha}_{i,j}^{n-1}, \overline{\beta}_{i,j}^{n-1}) \tag{3.39}$$

From (3.23), one can have

$$A\begin{pmatrix} \alpha_{i,j}^{n+1} - \alpha_{i,j}^{n} \\ \beta_{i,j}^{n+1} - \beta_{i,j}^{n} \end{pmatrix} = \begin{pmatrix} \alpha_{i,j-1}^{n+1} - \alpha_{i,j-1}^{n} + \frac{ak}{h}\left(\alpha_{i-1,j}^{n+1} - \alpha_{i-1,j}^{n}\right) \\ \left\{\beta_{i,j-1}^{n+1} - \beta_{i,j-1}^{n} + k\lambda\left(\exp(\beta_{i,j}^{n}) - \exp(\beta_{i,j}^{n-1})\right)\right. \\ \left. -k\lambda\exp(\beta_{i,j})\left(\beta_{i,j}^{n} - \beta_{i,j}^{n-1}\right)\right\} \end{pmatrix}$$

$$A\begin{pmatrix} \alpha_{i,j}^{n+1} - \alpha_{i,j}^{n} \\ \beta_{i,j}^{n+1} - \beta_{i,j}^{n} \end{pmatrix} \geq \begin{pmatrix} \alpha_{i,j-1}^{n+1} - \alpha_{i,j-1}^{n} + \frac{ak}{h}\left(\alpha_{i-1,j}^{n+1} - \alpha_{i-1,j}^{n}\right) \\ \beta_{i,j-1}^{n+1} - \beta_{i,j-1}^{n} \end{pmatrix} \tag{3.40}$$

For $j = 1$, (3.40) becomes

$$A\begin{pmatrix} \alpha_{i,1}^{n+1} - \alpha_{i,1}^{n} \\ \beta_{i,1}^{n+1} - \beta_{i,1}^{n} \end{pmatrix} \geq \begin{pmatrix} \alpha_{i,0}^{n+1} - \alpha_{i,0}^{n} + \frac{ak}{h}\left(\alpha_{i-1,1}^{n+1} - \alpha_{i-1,1}^{n}\right) \\ \beta_{i,0}^{n+1} - \beta_{i,0}^{n} \end{pmatrix}$$

Using the boundary conditions, the above inequality reduces to

$$A\begin{pmatrix} \alpha_{i,1}^{n+1} - \alpha_{i,1}^{n} \\ \beta_{i,1}^{n+1} - \beta_{i,1}^{n} \end{pmatrix} \geq \begin{pmatrix} \frac{ak}{h}\left(\alpha_{i-1,1}^{n+1} - \alpha_{i-1,1}^{n}\right) \\ 0 \end{pmatrix} \tag{3.41}$$

From the boundary conditions and the inverse positivity of A, one can obtain that $\begin{pmatrix} \alpha_{1,1}^{n+1} - \alpha_{1,1}^{n} \\ \beta_{1,1}^{n+1} - \beta_{1,1}^{n} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Further using this in (3.41) for $i = 2$, one can conclude that $\begin{pmatrix} \alpha_{2,1}^{n+1} - \alpha_{2,1}^{n} \\ \beta_{2,1}^{n+1} - \beta_{2,1}^{n} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Hence from (3.41),

$$\begin{pmatrix} \alpha_{i,1}^{n+1} - \alpha_{i,1}^{n} \\ \beta_{i,1}^{n+1} - \beta_{i,1}^{n} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \forall i = 1, 2, \ldots, M. \tag{3.42}$$

Now for $j = 2$, (3.40) becomes

$$A\begin{pmatrix} \alpha_{i,2}^{n+1} - \alpha_{i,2}^{n} \\ \beta_{i,2}^{n+1} - \beta_{i,2}^{n} \end{pmatrix} \geq \begin{pmatrix} \alpha_{i,1}^{1} - \alpha_{i,1} + \frac{ak}{h}\left(\alpha_{i-1,2}^{n+1} - \alpha_{i-1,2}^{n}\right) \\ \beta_{i,1}^{n+1} - \beta_{i,1}^{n} \end{pmatrix} \tag{3.43}$$

Using (3.42) in (3.43) leads to

$$A\begin{pmatrix} \alpha_{i,2}^{n+1} - \alpha_{i,2}^{n} \\ \beta_{i,2}^{n+1} - \beta_{i,2}^{n} \end{pmatrix} \geq \begin{pmatrix} \frac{ak}{h}\left(\alpha_{i-1,2}^{n+1} - \alpha_{i-1,2}^{n}\right) \\ 0 \end{pmatrix}$$

Now using similar argument for $i = 1, 2, \ldots, M$, one can conclude that $\begin{pmatrix} \alpha_{i,2}^{n+1} - \alpha_{i,2}^{n} \\ \beta_{i,2}^{n+1} - \beta_{i,2}^{n} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all i. Repeating the above process for $j = 3, \ldots, N$, one can obtain $\begin{pmatrix} \alpha_{i,j}^{n+1} - \alpha_{i,j}^{n} \\ \beta_{i,j}^{n+1} - \beta_{i,j}^{n} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for every i, j and n. A similar argument gives $\begin{pmatrix} \overline{\alpha}_{i,j}^{n} - \overline{\alpha}_{i,j}^{n+1} \\ \overline{\beta}_{i,j}^{n} - \overline{\beta}_{i,j}^{n+1} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \overline{\alpha}_{i,j}^{n+1} - \alpha_{i,j}^{n+1} \\ \overline{\beta}_{i,j}^{n+1} - \beta_{i,j}^{n+1} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for every i and j. Hence for all $n \in \mathbb{N}$,

$$(\alpha_{i,j}, \beta_{i,j}) \leq (\alpha_{i,j}^{n}, \beta_{i,j}^{n}) \leq (\alpha_{i,j}^{n+1}, \beta_{i,j}^{n+1}) \leq (\overline{\alpha}_{i,j}^{n+1}, \overline{\beta}_{i,j}^{n+1}) \leq (\overline{\alpha}_{i,j}^{n}, \overline{\beta}_{i,j}^{n}) \leq (\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})$$

This guarantees the existence of the limits

$$\lim_{m\to\infty} (\alpha_{i,j}^{n}, \beta_{i,j}^{n}) = (\alpha_{i,j}^{*}, \beta_{i,j}^{*}); \quad \lim_{m\to\infty} (\overline{\alpha}_{i,j}^{n}, \overline{\beta}_{i,j}^{n}) = (\overline{\alpha}_{i,j}^{*}, \overline{\beta}_{i,j}^{*}) \tag{3.44}$$

and the limits are solutions of the discretized equation (3.20). To complete the proof one has to show that $(\alpha_{i,j}^{*}, \beta_{i,j}^{*}) = (\overline{\alpha}_{i,j}^{*}, \overline{\beta}_{i,j}^{*})$. From (3.20)

$$\begin{pmatrix} \mu_1 & -ck \\ -bk & \mu_2 \end{pmatrix} \begin{pmatrix} \overline{\alpha}_{i,j}^{*} - \alpha_{i,j}^{*} \\ \overline{\beta}_{i,j}^{*} - \beta_{i,j}^{*} \end{pmatrix} = \begin{pmatrix} \overline{\alpha}_{i,j-1}^{*} - \alpha_{i,j-1}^{*} + \frac{ak}{h}(\overline{\alpha}_{i-1,j}^{*} - \alpha_{i-1,j}^{*}) \\ \overline{\beta}_{i,j-1}^{*} - \beta_{i,j-1}^{*} + k\lambda \left(\exp(\overline{\beta}_{i,j}^{*}) - \exp(\beta_{i,j}^{*})\right) \end{pmatrix}$$

$$\begin{pmatrix} \mu_1 & -ck \\ -bk & \mu_2 \end{pmatrix} \begin{pmatrix} \overline{\alpha}_{i,j}^{*} - \alpha_{i,j}^{*} \\ \overline{\beta}_{i,j}^{*} - \beta_{i,j}^{*} \end{pmatrix} = \begin{pmatrix} \overline{\alpha}_{i,j-1}^{*} - \alpha_{i,j-1}^{*} + \frac{ak}{h}(\overline{\alpha}_{i-1,j}^{*} - \alpha_{i-1,j}^{*}) \\ \overline{\beta}_{i,j-1}^{*} - \beta_{i,j-1}^{*} + k\lambda \exp(\tilde{\beta}_{i,j})(\overline{\beta}_{i,j}^{*} - \beta_{i,j}^{*}) \end{pmatrix} \tag{3.45}$$

where $\beta_{i,j}^{*} \leq \tilde{\beta}_{i,j} \leq \overline{\beta}_{i,j}^{*}$. Define $M_{i,j} = \begin{pmatrix} \mu_1 & -ck \\ -bk & (\mu_2 - k\lambda \exp(\tilde{\beta}_{i,j})) \end{pmatrix}$. Then (3.45) can be written as

$$M_{i,j} \begin{pmatrix} \overline{\alpha}_{i,j}^{*} - \alpha_{i,j}^{*} \\ \overline{\beta}_{i,j}^{*} - \beta_{i,j}^{*} \end{pmatrix} = \begin{pmatrix} \overline{\alpha}_{i,j-1}^{*} - \alpha_{i,j-1}^{*} + \frac{ak}{h}(\overline{\alpha}_{i-1,j}^{*} - \alpha_{i-1,j}^{*}) \\ \overline{\beta}_{i,j-1}^{*} - \beta_{i,j-1}^{*} \end{pmatrix} \tag{3.46}$$

Clearly $M_{i,j}$ is an invertible matrix for all i and j. For $j = 1$, (3.46) becomes

$$M_{i,1} \begin{pmatrix} \overline{\alpha}_{i,1}^{*} - \alpha_{i,1}^{*} \\ \overline{\beta}_{i,1}^{*} - \beta_{i,1}^{*} \end{pmatrix} = \begin{pmatrix} \overline{\alpha}_{i,0}^{*} - \alpha_{i,0}^{*} + \frac{ak}{h}(\overline{\alpha}_{i-1,1}^{*} - \alpha_{i-1,1}^{*}) \\ \overline{\beta}_{i,0}^{*} - \beta_{i,0}^{*} \end{pmatrix} \tag{3.47}$$

It is clear that $\overline{\alpha}^*_{i,0} - \alpha^*_{i,0} = 0$ and $\overline{\beta}^*_{i,0} - \beta^*_{i,0} = 0$ for all i. Consequently (3.47) becomes

$$M_{i,1} \begin{pmatrix} \overline{\alpha}^*_{i,1} - \alpha^*_{i,1} \\ \overline{\beta}^*_{i,1} - \beta^*_{i,1} \end{pmatrix} = \begin{pmatrix} \frac{ak}{h}(\overline{\alpha}^*_{i-1,1} - \alpha^*_{i-1,1}) \\ 0 \end{pmatrix} \tag{3.48}$$

For the choice of $i = 1$, (3.48) becomes $\begin{pmatrix} \overline{\alpha}^*_{1,1} - \alpha^*_{1,1} \\ \overline{\beta}^*_{1,1} - \beta^*_{1,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Hence using this relation in (3.48) for $i = 2$, one can conclude that $\begin{pmatrix} \overline{\alpha}^*_{2,1} - \alpha^*_{2,1} \\ \overline{\beta}^*_{2,1} - \beta^*_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. By repeating the above argument fro $i = 3, \ldots, M$, one can obtain that

$$\begin{pmatrix} \overline{\alpha}^*_{i,1} - \alpha^*_{i,1} \\ \overline{\beta}^*_{i,1} - \beta^*_{i,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \forall i = 1, 2, \ldots, M. \tag{3.49}$$

Now for $j = 2$ in (3.46),

$$M_{i,2} \begin{pmatrix} \overline{\alpha}^*_{i,2} - \alpha^*_{i,2} \\ \overline{\beta}^*_{i,2} - \beta^*_{i,2} \end{pmatrix} = \begin{pmatrix} \overline{\alpha}^*_{i,1} - \alpha^*_{i,1} + \frac{ak}{h}\left(\overline{\alpha}^*_{i-1,2} - \alpha^*_{i-1,2}\right) \\ \overline{\beta}^*_{i,1} - \beta^*_{i,1} \end{pmatrix} \tag{3.50}$$

Using (3.49) in (3.50) leads to

$$M_{i,2} \begin{pmatrix} \overline{\alpha}^*_{i,2} - \alpha^*_{i,2} \\ \overline{\beta}^*_{i,2} - \beta^*_{i,2} \end{pmatrix} = \begin{pmatrix} \frac{ak}{h}\left(\overline{\alpha}^*_{i-1,2} - \alpha^*_{i-1,2}\right) \\ 0 \end{pmatrix}.$$

Using similar argument for $i = 1, 2, \ldots, M$, one can conclude that $\begin{pmatrix} \overline{\alpha}^*_{i,2} - \alpha^*_{i,2} \\ \overline{\beta}^*_{i,2} - \beta^*_{i,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all i. Repeating the above process for $j = 3, \ldots, N$, one can obtain $\begin{pmatrix} \overline{\alpha}^*_{i,j} - \alpha^*_{i,j} \\ \overline{\beta}^*_{i,j} - \beta^*_{i,j} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for every i and j. Hence $(\alpha^*_{i,j}, \beta^*_{i,j}) = (\overline{\alpha}^*_{i,j}, \overline{\beta}^*_{i,j})$ for all $(i,j) \in \bar{\Lambda}$. □

Define $C = \max\left\{ \frac{1+\frac{ak}{h}+ck+bk}{(1+\frac{ak}{h}+ck)(1+bk-k\lambda\exp(\xi^*))-bck^2}, \frac{1+bk-k\lambda+ck}{(1+\frac{ak}{h}+ck)(1+bk-k\lambda\exp(\xi^*))-bck^2} \right\}$. The following theorem gives the error estimate for the iterative scheme given by (3.22). The proof is similar to the proof of Theorem 4.2 in [**95**].

Theorem 3.4.2. *If* $(u^*_{i,j}, v^*_{i,j})$ *for all* $(i,j) \in \bar{\Lambda}$ *is the solution of* (3.20), *then*

$$\left\| \begin{pmatrix} e^{n+1}_{i,j} \\ \underline{e}^{n+1}_{i,j} \end{pmatrix} \right\|_\infty \le C \left\| \begin{pmatrix} e^{n+1}_{i,j-1} + \frac{ak}{h} e^{n+1}_{i-1,j} \\ \underline{e}^{n+1}_{i,j-1} + k\lambda \exp(\xi^*) \left(\frac{(\underline{e}^n_{i,j})^2 + (\underline{e}_{i,j})^2)}{2} \right) \end{pmatrix} \right\|_\infty$$

where $e^{n+1}_{i,j} = u^*_{i,j} - \alpha^{n+1}_{i,j}$; $\underline{e}^{n+1}_{i,j} = v^*_{i,j} - \beta^{n+1}_{i,j}$; $\underline{e}_{i,j} = v^*_{i,j} - \beta_{i,j}$ *and* $\xi^* = \max\{\overline{\beta}_{i,j} : (i,j) \in \bar{\Lambda}\}$

Remark 3.4.1. *Similar error estimate can be obtained in the case of minimal sequence derived from* (3.22) *also.*

Remark 3.4.2. *Applying quasilinearization technique to both the equations of* (3.20) *simultaneously leads to*

$$\text{(3.51)} \qquad \begin{cases} A^n \begin{pmatrix} u^{n+1}_{i,j} \\ v^{n+1}_{i,j} \end{pmatrix} = \begin{pmatrix} u^{n+1}_{i,j-1} + \frac{ak}{h} u^{n+1}_{i-1,j} \\ v^{n+1}_{i,j-1} + k\lambda \exp(v^n_{i,j})(1 - v^n_{i,j}) \end{pmatrix} \\ u^{n+1}_{0,j} = \eta_j, \ u^{n+1}_{i,0} = \psi_i, \ v^{n+1}_{i,0} = \phi_i; \ i = 1, 2, \ldots, M, \ j = 1, 2, \ldots, N. \end{cases}$$

for $n = 0, 1, \cdots$ *where* $\eta_j = \eta(t_j), \psi_i = \psi(x_i)$, $\phi_i = \phi(x_i)$, $\mu_1 = 1 + \frac{ak}{h} + ck$, $\mu_2 = 1 + bk$, $\eta^n = (\mu_2 - k\lambda \exp(v^n_{i,j}))$ *and* $A^n = \begin{pmatrix} \mu_1 & -ck \\ -bk & \eta^n \end{pmatrix}$. *The condition* (3.19) *assures that* (3.51) *is well defined. Using* $(\alpha_{i,j}, \beta_{i,j})$ *and* $(\overline{\alpha}_{i,j}, \overline{\beta}_{i,j})$ *as the initial iteration* $(u^0_{i,j}, v^0_{i,j})$, *minimal and maximal sequences respectively can be constructed by using* (3.51). *Under the same hypotheses of Theorem 3.4.1 without any additional assumptions, the conclusions of Theorem 3.4.1 hold true for the minimal and maximal sequences obtained from* (3.51). *Also if* $(u^*_{i,j}, v^*_{i,j})$ *for all* $(i,j) \in \bar{\Lambda}$ *is the solution of* (3.20), *then*

$$\left\| \begin{pmatrix} e^{n+1}_{i,j} \\ \underline{e}^{n+1}_{i,j} \end{pmatrix} \right\|_\infty \le C \left\| \begin{pmatrix} e^{n+1}_{i,j-1} + \frac{ak}{h} e^{n+1}_{i-1,j} \\ \underline{e}^{n+1}_{i,j-1} + k\lambda \exp(\xi^*) \left(\underline{e}^n_{i,j} \right)^2 \end{pmatrix} \right\|_\infty$$

where $e^{n+1}_{i,j} = u^*_{i,j} - \alpha^{n+1}_{i,j}$; $\underline{e}^{n+1}_{i,j} = v^*_{i,j} - \beta^{n+1}_{i,j}$ *and* $\xi^* = \max\{\overline{\beta}_{i,j} : (i,j) \in \bar{\Lambda}\}$.

The following theorem guarantees the convergence of $(u^*_{i,j}, v^*_{i,j})$ to the continuous solution $(u^*(x_i, t_j), v^*(x_i, t_j))$ as the mesh size tends to zero. The proof is similar to the proof of Theorem 5.1 in [**82, 95**].

Theorem 3.4.3. *Let* $(u^*(x,t), v^*(x,t))$ *and* $(u^*_{i,j}, v^*_{i,j})$ *be the respective solutions of* (3.1) *and* (3.20) *respectively and let* $\bar{\Lambda}$ *be a given partition of* $\bar{Q} = [0,l] \times [0,T]$*. Then*

$$(u^*_{i,j}, v^*_{i,j}) \to (u^*(x_i,t_j), v^*(x_i,t_j)) \text{ as } h + k \to 0$$

at every mesh point (x_i, t_j) *in* $\bar{\Lambda}$*.*

Let Scheme 1 denote the iterative procedure (3.22) and Scheme 2 denote the quasi-linearization method discussed in (3.51).

Remark 3.4.3. *The schemes discussed in* [**82, 95**] *and the proposed scheme can be expressed as follows.*
Scheme in [**82**]*:*

$$\begin{cases} \begin{pmatrix} u^{n+1}_{i,j} \\ v^{n+1}_{i,j} \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}^{-1} \begin{pmatrix} u^{n+1}_{i,j-1} + \frac{ak}{h} u^{n+1}_{i-1,j} + ckv^n_{i,j} \\ v^{n+1}_{i,j-1} + bku^n_{i,j} + k\lambda \exp(v^n_{i,j}) \end{pmatrix} \\ u^{n+1}_{0,j} = \eta_j,\ u^{n+1}_{i,0} = \psi_i,\ v^{n+1}_{i,0} = \phi_i;\ i = 1,2,\ldots,M,\ j = 1,2,\ldots,N. \end{cases}$$

Scheme in [**95**]*:*

$$\begin{cases} \begin{pmatrix} u^{n+1}_{i,j} \\ v^{n+1}_{i,j} \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ -bk & B^n_{i,j} \end{pmatrix}^{-1} \begin{pmatrix} u^{n+1}_{i,j-1} + \frac{ak}{h} u^{n+1}_{i-1,j} + ckv^n_{i,j} \\ v^{n+1}_{i,j-1} + k\lambda \exp(v^n_{i,j}) - k\lambda \exp(v^n_{i,j}) v^n_{i,j} \end{pmatrix} \\ u^{n+1}_{0,j} = \eta_j,\ u^{n+1}_{i,0} = \psi_i,\ v^{n+1}_{i,0} = \phi_i;\ i = 1,2,\ldots,M,\ j = 1,2,\ldots,N. \end{cases}$$

where $B^n_{i,j} = (\mu_2 - k\lambda \exp(v^n_{i,j}))$*.*
Proposed Scheme 1:

$$\begin{cases} \begin{pmatrix} u^{n+1}_{i,j} \\ v^{n+1}_{i,j} \end{pmatrix} = \begin{pmatrix} \mu_1 & -ck \\ -bk & \eta_1 \end{pmatrix}^{-1} \begin{pmatrix} u^{n+1}_{i,j-1} + \frac{ak}{h} u^{n+1}_{i-1,j} \\ v^{n+1}_{i,j-1} + k\lambda \exp(v^n_{i,j}) - k\lambda \exp(v^0_{i,j}) v^n_{i,j} \end{pmatrix} \\ u^{n+1}_{0,j} = \eta_j,\ u^{n+1}_{i,0} = \psi_i,\ v^{n+1}_{i,0} = \phi_i;\ i = 1,2,\ldots,M,\ j = 1,2,\ldots,N. \end{cases}$$

where $\eta_1 = (\mu_2 - k\lambda \exp(v^0_{i,j}))$*. Note that to accelerate the iterative procedure in* [**82**]*, the scheme in* [**95**] *uses a new matrix at each step whereas the proposed scheme requires only one constant matrix throughout to accelerate* [**82**]*.*

3.5. Numerical Examples

In this section, the proposed schemes are applied to the same example illustrated in [**82**] and [**95**] to do the comparitiive study. Theorem 3.4.1, Remark 3.4.2 and Theorem 3.4.3 guaranty the existence and uniqueness of the solution as well as the convergence of the proposed iterative method for the following example. To stop the iterative procedure, the following stopping criteria is used as in [**82**] and [**95**].

$$\max_{(i,j)} \left[|u_{i,j}^{n+1} - u_{i,j}^{n}| + |v_{i,j}^{n+1} - v_{i,j}^{n}| \right] \leq \epsilon = 2 \times 10^{-5}.$$

Example 3.5.1.

Consider the following differential system discussed in [**95**, Example 6.1].

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u = v + q_1(x,t), \quad 0 < x \leq 1, 0 < t \leq T \\ \frac{\partial v}{\partial t} + v = u + \lambda \exp(v) + q_2(x,t), \quad 0 < x \leq 1, 0 < t \leq T \\ u(0,t) = 2 - \exp(-t), \quad 0 < t \leq T \\ u(x,0) = 1 - x^2, \ v(x,0) = 1 - x^2, \quad 0 < x \leq 1, \end{cases} \tag{3.52}$$

where $\lambda > 0$ is considered as a parameter with $q_1(x,t) = (1-x)^2 \exp(-t)$ and $q_2(x,t) = (1+x^2)\exp(-t) - \lambda \exp(2 - (1+x^2)\exp(-t))$. The solution of (3.52) is given by $u(x,t) = v(x,t) = 2 - (1+x^2)\exp(-t)$. It can be verified that $(0,0)$ is a lower solution of (3.52) which is taken as the initial guess. Here $T = 1$, $\lambda = 0.05$ and $h = k = 10^{-3}$.

Remark 3.5.1. *From Table 1 and 2 one can conclude that similar to the scheme in* [**95**]*, the proposed schemes Scheme 1 and Scheme 2 also accelerate the scheme discussed in* [**82**]*. Hence the proposed schemes are efficient alternatives to* [**95**].

3.6. Conclusion

In this study, an alternative iterative procedure for solving a coupled system of PDE in interphase heat transfer is proposed. Though the proposed iterative procedure (3.22) uses only a fixed constant matrix at every iterative steps, it produces similar results as that in [**95**] where the matrix is updated to a new one at each iterative step.

TABLE 3.1. Numerical solution of $u(x,t)$ for Example 3.5.1.

Grid Point (x_i, t_j)	Exact	Scheme in [82] $m = 10$	Scheme in [95] $m = 6$	Scheme 1 $m = 5$	Scheme 2 $m = 4$
(0.4,0.2)	1.050272326	1.050008303	1.050008298	1.050008298	1.050008298
(0.8,0.2)	0.657281565	0.656981110	0.656981108	0.656981108	0.656981108
(0.4,0.4)	1.222428747	1.221989786	1.221989780	1.221989780	1.221989780
(0.8,0.4)	0.900675125	0.900170790	0.900170784	0.900170784	0.900170784
(0.4,0.6)	1.363378502	1.362970024	1.362970010	1.362970010	1.362970010
(0.8,0.6)	1.099948917	1.099303519	1.099303500	1.099303500	1.099303500
(0.4,0.8)	1.478778402	1.478408732	1.478408701	1.478408699	1.478408702
(0.8,0.8)	1.263100499	1.262364658	1.262364626	1.262364627	1.262364627
(0.4,1)	1.573259848	1.572925710	1.572925661	1.572925654	1.572925667
(0.8,1)	1.396677716	1.395978881	1.395978804	1.395978817	1.395978820

TABLE 3.2. Numerical solution of $v(x,t)$ for Example 3.5.1.

Grid Point (x_i, t_j)	Exact	Scheme in [82] $m = 10$	Scheme in [95] $m = 6$	Scheme 1 $m = 5$	Scheme 2 $m = 4$
(0.4,0.2)	1.050272326	1.050149607	1.050149601	1.050149601	1.050149601
(0.8,0.2)	0.657281565	0.657115969	0.657115962	0.657115962	0.657115962
(0.4,0.4)	1.222428747	1.222179075	1.222179049	1.222179049	1.222179049
(0.8,0.4)	0.900675125	0.900349866	0.900349838	0.900349838	0.900349838
(0.4,0.6)	1.363378502	1.363022778	1.363022712	1.363022709	1.363022712
(0.8,0.6)	1.099948917	1.099477231	1.099477158	1.099477158	1.099477158
(0.4,0.8)	1.478778402	1.478349819	1.478349717	1.478349700	1.478349717
(0.8,0.8)	1.263100499	1.262497886	1.262497772	1.262497771	1.262497772
(0.4,1)	1.573259848	1.572783575	1.572783541	1.572783462	1.572783542
(0.8,1)	1.396677716	1.395970568	1.395970586	1.395970578	1.395970587

CHAPTER 4

FOURTH ORDER ELLIPTIC EQUATION WITH NONLOCAL BOUNDARY CONDITIONS

This chapter[1] provides an accelerated iterative procedure for a nonlinear fourth order elliptic equation with nonlocal boundary conditions.

4.1. Introduction

Nonlinear elliptic equations of fourth order have been receiving significant attention in recent literature. The bending of an elastic beam with simply-supported ends can be mathematically formulated using such equations especially in the area of two point boundary value problems [**7, 104**]. The existence and uniqueness theorem for fourth order elliptic equations with Dirichlet boundary conditions [**2, 23, 32, 55, 56, 75, 101**], mixed boundary conditions [**7, 33, 59, 100**] and multi point boundary conditions [**104**] are obtained using various classical fixed point theorems and monotone iterative techniques. Though there are few discussions done for fourth order elliptic equations with nonlocal two point boundary conditions [**58**], hardly any work can be found in the literature with nonlocal boundary conditions in higher dimensional spatial domains except for [**81**]. In [**81**], using monotone iterations, the existence and uniqueness of the solution for a class of fourth order nonlocal elliptic boundary value problem of the form

$$\begin{aligned}
&\triangle^2 u - b_0\triangle u + c_0 u = f(x,u),\ x\in\Omega\\
&u(x) = \textstyle\int_\Omega \gamma(x,\xi)u(\xi)\mathrm{d}\xi + g^{(1)}(x),\ x\in\partial\Omega \qquad (4.1)\\
&(\triangle u)(x) = \textstyle\int_\Omega \gamma(x,\xi)(\triangle u)(\xi)\mathrm{d}\xi - g^{(0)}(x),\ x\in\partial\Omega
\end{aligned}$$

is proved where Ω is a bounded domain in $\mathbb{R}^n$ $(n = 1, 2, \ldots)$ with boundary $\partial\Omega$, $b_0 \geq 0$ and c_0 are constants and $f(x,u)$, $\gamma(x',x)$ and $g^{(l)}(x')$ $(l = 0, 1)$ are continuous functions in their respective domains. Based on [**81**], an interesting finite difference iterative scheme

[1]This chapter forms the paper by L.A. Sunny and V. A. Vijesh in Journal of Scientific Computing, 76(2018), 275-298.

is developed in [**83**] recently, to obtain the numerical solution of (4.1). Similar to the iterative scheme in [**81**], [**83**] also ensures the monotone property of the finite difference iterative scheme. The numerical solutions for fourth order elliptic equations with Dirichlet and mixed boundary conditions are obtained using finite difference based monotone iterative methods in [**80, 79, 77, 100**].

The major aim of this chapter is to accelerate the iterative scheme in [**83**] ensuring the monotone property without any additional assumptions. In [**83**], the solution of the fourth order elliptic equation with nonlocal boundary conditions is approximated using a coupled second order elliptic equation with Dirichlet boundary conditions. Interestingly, in this proposed work, a better approximation is done using two second order elliptic equations with nonlocal boundary conditions. The proposed iterative scheme is found to be much more efficient than the scheme in [**83**] as it exhibits an immense reduction in the number of iterations required.

This chapter is organised as follows. Some basic notations, assumptions and formulations are provided in Section 4.2. Section 4.3 supplies the monotone property as well as the convergence analysis of the proposed scheme for the continuous case. Based on this, the convergence analysis and the monotonicity of the proposed numerical scheme is rendered in Section 4.4. In Section 4.5, three computational algorithms namely Picard, Gauss-Seidel and Jacobi are presented along with their monotone properties and comparison relations. The convergence of the finite difference solution to the corresponding continuous solution as mesh size tends to zero is guaranteed in Section 4.6. Section 4.7 provides the algorithms used in the numerical implementations of the schemes in comparison for Picard's iterations. The numerical implementation is done in Section 4.8 to illustrate the efficacy of the proposed numerical scheme. A comparative numerical study with the recent literature is also done in this section.

4.2. Preliminaries

This section presents basic definitions, notations and preliminary results that are required in the following sections. The lower and upper solutions for the fourth order partial differential equation (4.1) are defined as follows.

Definition 4.2.1. *A function $\widetilde{\alpha} \in C^4(\Omega) \cap C^2(\overline{\Omega})$ is called an upper solution of* (4.1) *if*

$$
\begin{aligned}
&\triangle^2 \widetilde{\alpha} - b_0 \triangle \widetilde{\alpha} + c_0 \widetilde{\alpha} \geq f(x, \widetilde{\alpha}), \ x \in \Omega \\
&\widetilde{\alpha}(x) \geq \textstyle\int_\Omega \gamma(x,\xi) \widetilde{\alpha}(\xi) \mathrm{d}\xi + g^{(1)}(x), \ x \in \partial\Omega \\
&(\triangle \widetilde{\alpha})(x) \leq \textstyle\int_\Omega \gamma(x,\xi)(\triangle \widetilde{\alpha})(\xi) \mathrm{d}\xi - g^{(0)}(x), \ x \in \partial\Omega.
\end{aligned} \tag{4.2}
$$

Similarly $\widehat{\alpha}$ is called a lower solution if it satisfies (4.2) *with the inequalities reversed.*

If the lower and upper solutions $\widehat{\alpha}$ and $\widetilde{\alpha}$ satisfy $\widehat{\alpha} \leq \widetilde{\alpha}$ and $\triangle \widehat{\alpha} \geq \triangle \widetilde{\alpha}$, then they are said to be ordered. For a given pair of ordered lower and upper solutions $\widehat{\alpha}, \widetilde{\alpha}$, let

$$
< \widehat{\alpha}, \widetilde{\alpha} >= \{u \in C(\overline{\Omega}); \ \widehat{\alpha} \leq u \leq \widetilde{\alpha}\}
$$

Throughout this chapter, the following hypotheses in [**81**] hold true:

($\mathbf{H}_1$) $\gamma(x,\xi)$ is non-negative in $\partial\Omega \times \Omega$, piecewise continuous in ξ for $x \in \partial\Omega$ and are in $C^{2+\alpha}$ in x for $\xi \in \Omega$.

($\mathbf{H}_2$) $f(x,u)$ is Hölder continuous in x and is continuously differentiable in u for $\widehat{\alpha} \leq u \leq \widetilde{\alpha}$.

($\mathbf{H}_3$) $\int_\Omega \gamma(x,\xi)\mathrm{d}\xi < 1, \ x \in \partial\Omega$ and $\frac{\partial f}{\partial u}(x,u) \leq c_0$ for $\widehat{\alpha} \leq u \leq \widetilde{\alpha}$, $x \in \overline{\Omega}$.

($\mathbf{H}_4$) Let $\overline{c}$ be any non-negative constant satisfying

$$
\overline{c} \geq \max\left\{ -\frac{\partial f}{\partial u}(x,u) \ : \widehat{\alpha} \leq u \leq \widetilde{\alpha}, \ x \in \Omega \right\},
$$

$c^* = c_0 + \overline{c} \geq 0$ and $b_0^2 \geq 4c^*$.

Define $F(x,u) = \overline{c}u + f(x,u)$. It is clear that $F(x,u)$ is monotone non-decreasing in u for all $u \in < \widehat{\alpha}, \widetilde{\alpha} >$. Define

$$
\mu = \frac{b_0 - \sqrt{b_0^2 - 4c^*}}{2}, \quad \mu^+ = \frac{b_0 + \sqrt{b_0^2 - 4c^*}}{2}.
$$

Thus (4.1) can be rewritten as

$$
\begin{aligned}
&-\triangle u + \mu u = v, \ -\triangle v + \mu^+ v = F(x,u), \ x \in \Omega \\
&u(x) = \textstyle\int_\Omega \gamma(x,\xi) u(\xi) \mathrm{d}\xi + g^{(1)}(x), \ x \in \partial\Omega \\
&v(x) = \textstyle\int_\Omega \gamma(x,\xi) v(\xi) \mathrm{d}\xi + g^{(2)}(x), \ x \in \partial\Omega,
\end{aligned} \tag{4.3}
$$

where $g^{(2)} = g^{(0)} + \mu g^{(1)}$ [**81**]. Hence the fourth order partial differential equation becomes a coupled second order problem. The existence and uniqueness of the solution of (4.3) is obtained using monotone iterations for which lower and upper solutions of (4.3) are defined as follows.

Definition 4.2.2. *A function* $(\widehat{\alpha}, \widehat{\beta})$ *is called a lower solution of* (4.3) *if it satisfies*

$$(4.4) \qquad \begin{aligned} &-\triangle\widehat{\alpha} + \mu\widehat{\alpha} \leq \widehat{\beta},\ x \in \Omega; \quad \widehat{\alpha}(x) \leq \textstyle\int_\Omega \gamma(x,\xi)\widehat{\alpha}(\xi)\mathrm{d}\xi + g^{(1)}(x),\ x \in \partial\Omega \\ &-\triangle\widehat{\beta} + \mu^+\widehat{\beta} \leq F(x,\widehat{\alpha}),\ x \in \Omega; \quad \widehat{\beta}(x) \leq \textstyle\int_\Omega \gamma(x,\xi)\widehat{\beta}(\xi)\mathrm{d}\xi + g^{(2)}(x),\ x \in \partial\Omega. \end{aligned}$$

Similarly $(\widetilde{\alpha}, \widetilde{\beta})$ *is called an upper solution of* (4.3) *if it satisfies* (4.4) *with inequalities reversed.*

The lower and upper solutions $(\widehat{\alpha}, \widehat{\beta})$ and $(\widetilde{\alpha}, \widetilde{\beta})$ are said to be ordered if $\widehat{\alpha} \leq \widetilde{\alpha}$ and $\widehat{\beta} \leq \widetilde{\beta}$.

Remark 4.2.1. *It can be easily verified that if* $\widehat{\alpha}$ *and* $\widetilde{\alpha}$ *are ordered lower and upper solutions of* (4.1)*, then the pair* $(\widehat{\alpha}, -\triangle\widehat{\alpha} + \mu\widehat{\alpha})$ *and* $(\widetilde{\alpha}, -\triangle\widetilde{\alpha} + \mu\widetilde{\alpha})$ *are ordered lower and upper solutions of* (4.3)*.*

For ordered lower and upper solutions $(\widehat{\alpha}, \widehat{\beta})$ and $(\widetilde{\alpha}, \widetilde{\beta})$, define the set $S = \{(u,v) \in C(\overline{\Omega});\ (\widehat{\alpha}, \widehat{\beta}) \leq (u,v) \leq (\widetilde{\alpha}, \widetilde{\beta})\}$.

4.3. Convergence Analysis for the Continuous Case

In the setting of function space, convergence and monotone property of the proposed iterative scheme is proved in this section. In the proposed iterative scheme, at each step (4.3) is approximated using two second order linear elliptic equations with nonlocal boundary conditions. In this regard, the existence and uniqueness of the following equation play a vital role in the proof of the main theorem in this section.

$$(4.5) \qquad \begin{aligned} &-\triangle w + aw = h(x), \quad x \in \Omega \\ &w(x) = \textstyle\int_\Omega \gamma(x,\xi)w(\xi)\mathrm{d}\xi + g(x), \quad x \in \partial\Omega, \end{aligned}$$

where $h(x)$ is Hölder continuous in x and a is any non-negative constant. To obtain the existence and uniqueness of (4.5), successive approximation coupled with monotone iterations is employed. The lower and upper solutions for (4.5) are defined as follows.

Definition 4.3.1. *A function* $\underline{w}^0$ *is called a lower solution of* (4.5) *if* $\underline{w}^0 \in C^2(\Omega) \cap C(\overline{\Omega})$ *it satisfies*

$$\begin{aligned} &-\triangle \underline{w}^0 + a\underline{w}^0 \leq h(x), \quad x \in \Omega \\ &\underline{w}^0(x) \leq \textstyle\int_\Omega \gamma(x,\xi)\underline{w}^0(\xi)\mathrm{d}\xi + g(x), \quad x \in \partial\Omega. \end{aligned} \tag{4.6}$$

Similarly $\overline{w}^0$ *is called an upper solution of* (4.5) *if satisfies* (4.6) *with inequalities reversed.*

Lemma 4.3.1. *Let* $\underline{w}^0$ *and* $\overline{w}^0$ *be an ordered lower and upper solutions of* (4.5). *Then* (4.5) *has a unique solution* w^* *such that* $\underline{w}^0 \leq w^* \leq \overline{w}^0$.

Proof. The solution of (4.5) is obtained as the limit of the following successive iterative scheme.

$$\begin{aligned} &-\triangle w^{n+1} + aw^{n+1} = h(x), \quad x \in \Omega \\ &w^{n+1}(x) = \textstyle\int_\Omega \gamma(x,\xi)w^n(\xi)\mathrm{d}\xi + g(x), \quad x \in \partial\Omega. \end{aligned} \tag{4.7}$$

Choosing the initial iteration w^0 as $\underline{w}^0$ and $\overline{w}^0$ respectively, two sequences $\{\underline{w}^{n+1}\}$ and $\{\overline{w}^{n+1}\}$ can be constructed. Clearly, the iterative scheme (4.7) is well defined [**72**]. The proof is done through an induction argument. Let $z = \underline{w}^1 - \underline{w}^0$.

$$\begin{aligned} -\triangle z + az &= (-\triangle \underline{w}^1 + a\underline{w}^1) - (-\triangle \underline{w}^0 + a\underline{w}^0) \geq h(x) - h(x) = 0 \\ z(x) &= \underline{w}^1(x) - \underline{w}^0(x) = \int_\Omega \gamma(x,\xi)\underline{w}^0(\xi)\mathrm{d}\xi + g(x) - \underline{w}^0 \geq 0, \quad x \in \partial\Omega. \end{aligned}$$

The maximum principle for second order elliptic equations implies that $z \geq 0$ and thus $\underline{w}^0 \leq \underline{w}^1$ in $\overline{\Omega}$. Similarly $\overline{w}^1 \leq \overline{w}^0$ in $\overline{\Omega}$. Now let $z = \overline{w}^1 - \underline{w}^1$. Clearly for all $x \in \Omega$, $-\triangle w + aw = 0$. Moreover,

$$\begin{aligned} z(x) &= \overline{w}^1(x) - \underline{w}^1(x) \\ &= \int_\Omega \gamma(x,\xi)\overline{w}^0(\xi)\mathrm{d}\xi - \int_\Omega \gamma(x,\xi)\underline{w}^0(\xi)\mathrm{d}\xi \geq 0, \quad x \in \partial\Omega. \end{aligned}$$

Hence $\underline{w}^0 \leq \underline{w}^1 \leq \overline{w}^1 \leq \overline{w}^0$ in $\overline{\Omega}$. Assume that $\underline{w}^0 \leq \underline{w}^{n-1} \leq \underline{w}^n \leq \overline{w}^n \leq \overline{w}^{n-1} \leq \overline{w}^0$ for some $n > 1$. Let $z = \underline{w}^{n+1} - \underline{w}^n$. Note that for all $x \in \Omega$, one have $-\triangle z + az = 0$ and for all $x \in \partial\Omega$,

$$\begin{aligned} z(x) &= \underline{w}^{n+1}(x) - \underline{w}^{n}(x) \\ &= \int_\Omega \gamma(x,\xi)\underline{w}^{n}(\xi)\mathrm{d}\xi - \int_\Omega \gamma(x,\xi)\underline{w}^{n-1}(\xi)\mathrm{d}\xi \geq 0. \end{aligned}$$

Thus $\underline{w}^n \leq \underline{w}^{n+1}$ in $\overline{\Omega}$. Similarly $\overline{w}^{n+1} \leq \overline{w}^n$ in $\overline{\Omega}$. Let $z = \overline{w}^{n+1} - \underline{w}^{n+1}$. Consequently, $-\triangle z + az = 0$ for all $x \in \Omega$. Also,

$$\begin{aligned} z(x) &= \overline{w}^{n+1}(x) - \underline{w}^{n+1}(x) \\ &= \int_\Omega \gamma(x,\xi)\overline{w}^{n}(\xi)\mathrm{d}\xi - \int_\Omega \gamma(x,\xi)\underline{w}^{n}(\xi)\mathrm{d}\xi \geq 0, \quad x \in \partial\Omega. \end{aligned}$$

Hence $\underline{w}^n \leq \underline{w}^{n+1} \leq \overline{w}^{n+1} \leq \overline{w}^n$ for all n in $\overline{\Omega}$. This guarantees the existence of the limit $\lim\limits_{n\to\infty} \underline{w}^n = \underline{w}$ and $\lim\limits_{n\to\infty} \overline{w}^n = \overline{w}$. Moreover, the limits are the solutions of (4.5) satisfying $\underline{w} \leq w^* \leq \overline{w}$. Put $W = \underline{w} - \overline{w}$. Then $W \leq 0$ satisfies

$$\begin{aligned} &-\triangle W + aW = 0, \quad x \in \Omega \\ &W(x) = \textstyle\int_\Omega \gamma(x,\xi)W(\xi)\mathrm{d}\xi, \ x \in \partial\Omega \end{aligned} \tag{4.8}$$

To show that $W \geq 0$, assume by contradiction that it is not true. Then there exists some $x_0 \in \overline{\Omega}$ such that $W(x_0)$ is negative and it is the minimum of $W(x)$ on $\overline{\Omega}$. Then

$$W(x_0) = \int_\Omega \gamma(x_0,\xi)W(\xi)\mathrm{d}\xi \geq W(x_0)\int_\Omega \gamma(x_0,\xi)\mathrm{d}\xi$$

which is possible only if $\int_\Omega \gamma(x_0,\xi)\mathrm{d}\xi \geq 1$, a contradiction to $(\mathbf{H}_3)$. Hence $W = 0$ and the uniqueness. □

To accelerate the iterative procedure in [**81**], the following iterative scheme is proposed to solve the coupled equation (4.3). With initial (u^0, v^0) and $n \in \mathbb{N}$,

$$\begin{aligned} &-\triangle u^{n+1} + \mu u^{n+1} = v^n, \quad x \in \Omega \\ &u^{n+1}(x) = \textstyle\int_\Omega \gamma(x,\xi)u^{n+1}(\xi)\mathrm{d}\xi + g^{(1)}(x), \quad x \in \partial\Omega \\ &-\triangle v^{n+1} + \mu^+ v^{n+1} = F(x, u^{n+1}), \quad x \in \Omega \\ &v^{n+1}(x) = \textstyle\int_\Omega \gamma(x,\xi)v^{n+1}(\xi)\mathrm{d}\xi + g^{(2)}(x), \quad x \in \partial\Omega. \end{aligned} \tag{4.9}$$

Using $(\widehat{\alpha}, \widehat{\beta})$ and $(\widetilde{\alpha}, \widetilde{\beta})$ respectively as the initial iterations, two sequences namely minimal and maximal sequences can be generated from the proposed iterative procedure. The minimal sequence $\{(\alpha^n, \beta^n)\},\ n \in \mathbb{N}$ is given by

$$\text{(4.10a)} \qquad \begin{aligned} &-\triangle\alpha^{n+1} + \mu\alpha^{n+1} = \beta^n, \quad x \in \Omega \\ &\alpha^{n+1}(x) = \int_\Omega \gamma(x,\xi)\alpha^{n+1}(\xi)\mathrm{d}\xi + g^{(1)}(x), \quad x \in \partial\Omega. \end{aligned}$$

$$\text{(4.10b)} \qquad \begin{aligned} &-\triangle\beta^{n+1} + \mu^+\beta^{n+1} = F(x, \alpha^{n+1}), \quad x \in \Omega \\ &\beta^{n+1}(x) = \int_\Omega \gamma(x,\xi)\beta^{n+1}(\xi)\mathrm{d}\xi + g^{(2)}(x), \quad x \in \partial\Omega. \end{aligned}$$

and the maximal sequence $\{(\overline{\alpha}^n, \overline{\beta}^n)\},\ n \in \mathbb{N}$ is given by

$$\text{(4.11a)} \qquad \begin{aligned} &-\triangle\overline{\alpha}^{n+1} + \mu\overline{\alpha}^{n+1} = \overline{\beta}^n, \quad x \in \Omega \\ &\overline{\alpha}^{n+1}(x) = \int_\Omega \gamma(x,\xi)\overline{\alpha}^{n+1}(\xi)\mathrm{d}\xi + g^{(1)}(x), \quad x \in \partial\Omega \end{aligned}$$

$$\text{(4.11b)} \qquad \begin{aligned} &-\triangle\overline{\beta}^{n+1} + \mu^+\overline{\beta}^{n+1} = F(x, \overline{\alpha}^{n+1}), \quad x \in \Omega \\ &\overline{\beta}^{n+1}(x) = \int_\Omega \gamma(x,\xi)\overline{\beta}^{n+1}(\xi)\mathrm{d}\xi + g^{(2)}(x), \quad x \in \partial\Omega. \end{aligned}$$

The following theorem provides the monotone and well defined properties of the minimal and maximal sequences of (4.10) and (4.11) and their convergence to the unique solution of (4.3).

Theorem 4.3.1. *Let $(\widehat{\alpha}, \widehat{\beta})$ and $(\widetilde{\alpha}, \widetilde{\beta})$ be a pair of ordered lower and upper solutions of* (4.3)*. Then the minimal sequence $\{(\alpha^n, \beta^n)\}$ and the maximal sequence $\{(\overline{\alpha}^n, \overline{\beta}^n)\}$ are well defined and converge monotonically to the unique solution (u^*, v^*) of* (4.3) *in S. Moreover, the following relation holds for $n \in \mathbb{N}$.*

$$\text{(4.12)} \quad (\widehat{\alpha}, \widehat{\beta}) \leq (\alpha^n, \beta^n) \leq (\alpha^{n+1}, \beta^{n+1}) \leq (u^*, v^*) \leq (\overline{\alpha}^{n+1}, \overline{\beta}^{n+1}) \leq (\overline{\alpha}^n, \overline{\beta}^n) \leq (\widetilde{\alpha}, \widetilde{\beta}).$$

Proof. The proof is done by an induction on n. First the following inequality is proved by using Lemma 4.3.1. To prove $(\widehat{\alpha}, \widehat{\beta}) \leq (\alpha^1, \beta^1) \leq (\widetilde{\alpha}, \widetilde{\beta})$, it is enough to show that $(\widehat{\alpha}, \widehat{\beta})$ and $(\widetilde{\alpha}, \widetilde{\beta})$ are lower and upper solutions respectively of (4.10) for $n = 0$. From Definition 4.2.2, one can have

$$\begin{aligned} &-\triangle\widehat{\alpha} + \mu\widehat{\alpha} \leq \widehat{\beta}, \quad x \in \Omega \\ &\widehat{\alpha}(x) \leq \int_\Omega \gamma(x,\xi)\widehat{\alpha}(\xi)\mathrm{d}\xi + g^{(1)}(x), \quad x \in \partial\Omega \end{aligned}$$

and

$$-\triangle\widetilde{\alpha}+\mu\widetilde{\alpha}\geq\widetilde{\beta}\geq\widehat{\beta},\quad x\in\Omega$$
$$\widetilde{\alpha}(x)\geq\int_{\Omega}\gamma(x,\xi)\widetilde{\alpha}(\xi)\mathrm{d}\xi+g^{(1)}(x),\quad x\in\partial\Omega.$$

Thus by Lemma 4.3.1, (4.10a) with $n=0$ has a unique solution α^1 between $\widehat{\alpha}$ and $\widetilde{\alpha}$. Thus $\widehat{\alpha}\leq\alpha^1\leq\overline{\alpha}^1$. Similarly from Definition 4.2.2,

$$-\triangle\widehat{\beta}+\mu^{+}\widehat{\beta}\leq F(x,\widehat{\alpha})\leq F(x,\alpha^1),\quad x\in\Omega$$
$$\widehat{\beta}(x)\leq\int_{\Omega}\gamma(x,\xi)\widehat{\beta}(\xi)\mathrm{d}\xi+g^{(2)}(x),\quad x\in\partial\Omega$$

and

$$-\triangle\widetilde{\beta}+\mu^{+}\widetilde{\beta}\geq F(x,\widetilde{\alpha})\geq F(x,\alpha^1),\quad x\in\Omega$$
$$\widetilde{\beta}(x)\geq\int_{\Omega}\gamma(x,\xi)\widetilde{\beta}(\xi)\mathrm{d}\xi+g^{(2)}(x),\quad x\in\partial\Omega.$$

Thus by Lemma 4.3.1, (4.10b) with $n=0$ has a unique solution β^1 between $\widehat{\beta}$ and $\widetilde{\beta}$. Hence $(\widehat{\alpha},\widehat{\beta})\leq(\alpha^1,\beta^1)\leq(\overline{\alpha}^1,\overline{\beta}^1)$. To prove $(\alpha^1,\beta^1)\leq(\overline{\alpha}^1,\overline{\beta}^1)\leq(\widetilde{\alpha},\widetilde{\beta})$, it is enough to show that (α^1,β^1) and $(\widetilde{\alpha},\widetilde{\beta})$ are lower and upper solutions respectively of (4.11) with $n=0$. From (4.10a) for $n=0$,

$$-\triangle\alpha^1+\mu\alpha^1=\widehat{\beta}\leq\widetilde{\beta},\quad x\in\Omega$$
$$\alpha^1(x)=\int_{\Omega}\gamma(x,\xi)\alpha^1(\xi)\mathrm{d}\xi+g^{(1)}(x),\quad x\in\partial\Omega.$$

From Definition 4.2.2

$$-\triangle\widetilde{\alpha}+\mu\widetilde{\alpha}\geq\widetilde{\beta},\quad x\in\Omega$$
$$\widetilde{\alpha}(x)\geq\int_{\Omega}\gamma(x,\xi)\widetilde{\alpha}(\xi)\mathrm{d}\xi+g^{(1)}(x),\quad x\in\partial\Omega.$$

Thus by Lemma 4.3.1, (4.11a) with $n=0$ has a unique solution $\overline{\alpha}^1$ between α^1 and $\widetilde{\alpha}$. Thus $\alpha^1\leq\overline{\alpha}^1\leq\widetilde{\alpha}$. Similarly from (4.10b) for $n=0$,

$$-\triangle\beta^1+\mu^{+}\beta^1=F(x,\alpha^1)\leq F(x,\overline{\alpha}^1),\quad x\in\Omega$$
$$\beta^1(x)=\int_{\Omega}\gamma(x,\xi)\beta^1(\xi)\mathrm{d}\xi+g^{(2)}(x),\quad x\in\partial\Omega.$$

From Definition 4.2.2,

$$-\triangle\widetilde{\beta}+\mu^{+}\widetilde{\beta}\geq F(x,\widetilde{\alpha})\geq F(x,\overline{\alpha}^1),\quad x\in\Omega$$
$$\widetilde{\beta}(x)\geq\int_{\Omega}\gamma(x,\xi)\widetilde{\beta}(\xi)\mathrm{d}\xi+g^{(2)}(x),\quad x\in\partial\Omega.$$

Thus by Lemma 4.3.1, (4.11b) with $n=0$ has a unique solution $\overline{\beta}^1$ between β^1 and $\widetilde{\beta}$. Thus $(\alpha^1,\beta^1)\leq(\overline{\alpha}^1,\overline{\beta}^1)\leq(\widetilde{\alpha},\widetilde{\beta})$. Hence $(\widehat{\alpha},\widehat{\beta})\leq(\alpha^1,\beta^1)\leq(\overline{\alpha}^1,\overline{\beta}^1)\leq(\widetilde{\alpha},\widetilde{\beta})$. Assume that

$$(\widehat{\alpha}, \widehat{\beta}) \le (\alpha^{n-1}, \beta^{n-1}) \le (\alpha^n, \beta^n) \le (\overline{\alpha}^n, \overline{\beta}^n) \le (\overline{\alpha}^{n-1}, \overline{\beta}^{n-1}) \le (\widetilde{\alpha}, \widetilde{\beta})$$

for some $n > 1$. To complete the proof for the monotone property, one has to show that $(\alpha^n, \beta^n) \le (\alpha^{n+1}, \beta^{n+1}) \le (\overline{\alpha}^{n+1}, \overline{\beta}^{n+1}) \le (\overline{\alpha}^n, \overline{\beta}^n)$. To do this, the following inequality is proved first; $(\alpha^n, \beta^n) \le (\alpha^{n+1}, \beta^{n+1}) \le (\overline{\alpha}^n, \overline{\beta}^n)$. It is enough to show that (α^n, β^n) and $(\overline{\alpha}^n, \overline{\beta}^n)$ are lower and upper solutions respectively of (4.10). From (4.10a),

$$\begin{aligned}
&-\triangle \alpha^n + \mu \alpha^n = \beta^{n-1} \le \beta^n, \quad x \in \Omega \\
&\alpha^n(x) = \textstyle\int_\Omega \gamma(x,\xi)\alpha^n(\xi)\mathrm{d}\xi + g^{(1)}(x), \quad x \in \partial\Omega
\end{aligned}$$

and from (4.11a),

$$\begin{aligned}
&-\triangle \overline{\alpha}^n + \mu \overline{\alpha}^n = \overline{\beta}^{n-1} \ge \beta^n, \quad x \in \Omega \\
&\overline{\alpha}^n(x) = \textstyle\int_\Omega \gamma(x,\xi)\overline{\alpha}^n(\xi)\mathrm{d}\xi + g^{(1)}(x), \quad x \in \partial\Omega.
\end{aligned}$$

Thus by Lemma 4.3.1, (4.10a) has a unique solution α^{n+1} between α^n and $\overline{\alpha}^n$. Thus $\alpha^n \le \alpha^{n+1} \le \overline{\alpha}^n$. Similarly from (4.10b),

$$\begin{aligned}
&-\triangle \beta^n + \mu^+ \beta^n = F(x, \alpha^n) \le F(x, \alpha^{n+1}), \quad x \in \Omega \\
&\beta^n(x) = \textstyle\int_\Omega \gamma(x,\xi)\beta^n(\xi)\mathrm{d}\xi + g^{(2)}(x), \quad x \in \partial\Omega
\end{aligned}$$

and from (4.11b),

$$\begin{aligned}
&-\triangle \overline{\beta}^n + \mu^+ \overline{\beta}^n = F(x, \overline{\alpha}^n) \ge F(x, \alpha^{n+1}), \quad x \in \Omega \\
&\overline{\beta}^n(x) = \textstyle\int_\Omega \gamma(x,\xi)\overline{\beta}^n(\xi)\mathrm{d}\xi + g^{(2)}(x), \quad x \in \partial\Omega.
\end{aligned}$$

Thus by Lemma 4.3.1, (4.10b) has a unique solution β^{n+1} between β^n and $\overline{\beta}^n$. Hence $(\alpha^n, \beta^n) \le (\alpha^{n+1}, \beta^{n+1}) \le (\overline{\alpha}^n, \overline{\beta}^n)$. To prove $(\alpha^{n+1}, \beta^{n+1}) \le (\overline{\alpha}^{n+1}, \overline{\beta}^{n+1}) \le (\overline{\alpha}^n, \overline{\beta}^n)$ it is enough to show that $(\alpha^{n+1}, \beta^{n+1})$ and $(\overline{\alpha}^n, \overline{\beta}^n)$ are lower and upper solutions respectively of (4.11). From (4.10a),

$$\begin{aligned}
&-\triangle \alpha^{n+1} + \mu \alpha^{n+1} = \beta^n \le \overline{\beta}^n, \quad x \in \Omega \\
&\alpha^{n+1}(x) = \textstyle\int_\Omega \gamma(x,\xi)\alpha^{n+1}(\xi)\mathrm{d}\xi + g^{(1)}(x), \quad x \in \partial\Omega
\end{aligned}$$

and from (4.11a),

$$\begin{aligned}
&-\triangle \overline{\alpha}^n + \mu \overline{\alpha}^n = \overline{\beta}^{n-1} \ge \overline{\beta}^n, \quad x \in \Omega \\
&\overline{\alpha}^n(x) = \textstyle\int_\Omega \gamma(x,\xi)\overline{\alpha}^n(\xi)\mathrm{d}\xi + g^{(1)}(x), \quad x \in \partial\Omega.
\end{aligned}$$

Thus by Lemma 4.3.1, (4.11a) has a unique solution $\overline{\alpha}^{n+1}$ between α^{n+1} and $\overline{\alpha}^{n}$. Thus $\alpha^{n+1} \leq \overline{\alpha}^{n+1} \leq \overline{\alpha}^{n}$. Similarly from (4.10b),

$$\begin{aligned}
&-\triangle\beta^{n+1} + \mu^{+}\beta^{n+1} = F(x, \alpha^{n+1}) \leq F(x, \overline{\alpha}^{n+1}), \quad x \in \Omega \\
&\beta^{n+1}(x) = \textstyle\int_{\Omega} \gamma(x,\xi)\beta^{n+1}(\xi)\mathrm{d}\xi + g^{(2)}(x), \quad x \in \partial\Omega
\end{aligned}$$

and from (4.11b),

$$\begin{aligned}
&-\triangle\overline{\beta}^{n} + \mu^{+}\overline{\beta}^{n} = F(x, \overline{\alpha}^{n}) \geq F(x, \overline{\alpha}^{n+1}), \quad x \in \Omega \\
&\overline{\beta}^{n}(x) = \textstyle\int_{\Omega} \gamma(x,\xi)\overline{\beta}^{n}(\xi)\mathrm{d}\xi + g^{(2)}(x), \quad x \in \partial\Omega.
\end{aligned}$$

Thus by Lemma 4.3.1, (4.11b) has a unique solution $\overline{\beta}^{n+1}$ between β^{n+1} and $\overline{\beta}^{n}$. Hence $(\alpha^{n+1}, \beta^{n+1}) \leq (\overline{\alpha}^{n+1}, \overline{\beta}^{n+1}) \leq (\overline{\alpha}^{n}, \overline{\beta}^{n})$. Thus $(\alpha^{n}, \beta^{n}) \leq (\alpha^{n+1}, \beta^{n+1}) \leq (\overline{\alpha}^{n+1}, \overline{\beta}^{n+1}) \leq (\overline{\alpha}^{n}, \overline{\beta}^{n})$. Hence (4.12) holds true for $n \in \mathbb{N}$. This guarantees the existence of the limits $\lim\limits_{n\to\infty}(\alpha^{n}, \beta^{n}) = (\underline{\alpha}, \underline{\beta})$ and $\lim\limits_{n\to\infty}(\overline{\alpha}^{n}, \overline{\beta}^{n}) = (\overline{\alpha}, \overline{\beta})$ where $(\underline{\alpha}, \underline{\beta}) \leq (\overline{\alpha}, \overline{\beta})$ and they are the solutions of (4.3). The proof for the uniqueness is similar to the proof of Theorem 3.1 in [**81**] and hence omitted. □

Remark 4.3.1. *Under the given conditions, Theorem 4.3.1 also holds good for* (4.1) *with the boundary conditions replaced with*

$$\begin{aligned}
&u(x) = \textstyle\int_{\Omega} \gamma(x,\xi)u(\xi)\mathrm{d}\xi + g^{(1)}(x), \ x \in \partial\Omega \\
&\triangle u(x) = -g^{(0)}(x), \ x \in \partial\Omega.
\end{aligned}$$

4.4. Convergence Analysis for the Discretized Case

Keeping the same finite difference approximation as in [**83**], an accelerated iterative procedure is proposed in this section to solve the problem numerically. The convergence to the unique solution and the monotone property of the proposed iterative scheme are also shown in this section. Let h_ν be the spatial increment in the x_ν direction. Let $i = (i_1, i_2, \ldots, i_n)$ be a multiple index with $i_\nu = 1, 2, \ldots, M_\nu$ for $\nu = 1, 2, \ldots, n$ and let $x_i = (x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ be an interior point in Ω, $x_j^{'} = (x_{j_1}^{'}, x_{j_2}^{'}, \ldots, x_{j_n}^{'})$ a boundary point on $\partial\Omega$ and $x_k = (x_{k_1}, x_{k_2}, \ldots, x_{k_n})$ denotes any point in $\overline{\Omega}$. The set of mesh points in Ω, $\overline{\Omega}$ and $\partial\Omega$ are denoted by Ω_h, $\overline{\Omega}_h$ and $\partial\Omega_h$ respectively. Write $i \in \Omega_h$, $j \in \partial\Omega_h$ and $k \in \overline{\Omega}_h$ respectively for $x_i \in \Omega_h$, $x_j^{'} \in \partial\Omega_h$ and $x_k \in \overline{\Omega}_h$, when there is no confusion. The boundary conditions are approximated by

$$J[x_j,u]=\sum_{i_1=1}^{M_1}\sum_{i_2=1}^{M_2}\cdots\sum_{i_n=1}^{M_n} w_{i_1}w_{i_2}\cdots w_{i_n}\gamma(x_j,\xi_{i_1},\xi_{i_2},\cdots,\xi_{i_n})u(\xi_{i_1},\xi_{i_2},\cdots,\xi_{i_n}),$$

where $\xi_i=(\xi_{i_1},\xi_{i_2},\cdots,\xi_{i_n})$ denotes an arbitrary mesh point in Ω_h and $\{w_{i_1},w_{i_2},\cdots,w_{i_n}\}$ is any set of quadrature weights such that $0<w_{i_\nu}\leq 1$. Let u_k represents the approximation of $u(x_k)$ for any mesh point x_k. Then by the central difference approximation

$$\triangle_h u_i=\sum_{\nu=1}^{n}\triangle_h^{(\nu)}u_i=\sum_{\nu=1}^{n}h_\nu^{-2}[u_{i+e_\nu}-2u_i+u_{i-e_\nu}],$$

where e_ν is the unit vector in $\mathbb{R}^n$ with ν^{th} component one and zero elsewhere. Discretizing (4.1) using central difference approximation leads to

$$\begin{aligned}
&\triangle_h(\triangle_h u_i)-b_0\triangle_h u_i+c_0u_i=f(x_i,u_i), \quad i\in\Omega_h\\
&u_j=J[x_j',u]+g_j^{(1)}, \quad j\in\partial\Omega_h\\
&\triangle_h u_j=J[x_j',\triangle_h u]-g_j^{(0)}, \quad j\in\partial\Omega_h.
\end{aligned}\tag{4.13}$$

The lower and upper solutions for the discretized fourth order elliptic partial differential equation (4.13) are defined as follows.

Definition 4.4.1. *A function $\widetilde{\alpha}_k$ is called an upper solution of* (4.13) *if*

$$\begin{aligned}
&\triangle_h(\triangle_h\widetilde{\alpha}_i)-b_0\triangle_h\widetilde{\alpha}_i+c_0\widetilde{\alpha}_i\geq f(x_i,\widetilde{\alpha}_i), \quad i\in\Omega_h\\
&\widetilde{\alpha}_j\geq J[x_j',\widetilde{\alpha}_i]+g_j^{(1)}, \quad j\in\partial\Omega_h\\
&\triangle_h\widetilde{\alpha}_j\leq J[x_j',\triangle_h\widetilde{\alpha}_i]-g_j^{(0)}, \quad j\in\partial\Omega_h.
\end{aligned}\tag{4.14}$$

Similarly $\widehat{\alpha}_k$ is called a lower solution if it satisfies (4.14) *with the inequalities reversed.*

The lower and upper solutions $\widehat{\alpha}_k$ and $\widetilde{\alpha}_k$ are said to be ordered if $\widehat{\alpha}_k\leq\widetilde{\alpha}_k$ and $\triangle\widehat{\alpha}_k\geq\triangle\widetilde{\alpha}_k$. For a given pair of ordered lower and upper solutions $\widehat{\alpha}_k,\widetilde{\alpha}_k$, let

$$<\widehat{\alpha}_k,\widetilde{\alpha}_k>=\{u_k\in C(\overline{\Omega}_h);\ \widehat{\alpha}_k\leq u_k\leq\widetilde{\alpha}_k\}.$$

For the rest of the discussion, the following hypotheses in [**83**] hold true:

($\mathbf{H}_5$) $J[x_j',1]<1$ for $j\in\partial\Omega_h$ and $\frac{\partial f}{\partial u}(x_k,u_k)\leq c_0$ for $\widehat{\alpha}_k\leq u_k\leq\widetilde{\alpha}_k$.

($\mathbf{H}_6$) Let $\overline{c}$ be any constant satisfying $\overline{c}\geq\max\left\{-\frac{\partial f}{\partial u}(x_k,u_k)\ :\widehat{\alpha}_k\leq u_k\leq\widetilde{\alpha}_k,\ x_k\in\overline{\Omega}_h\right\}$, $c^*=c_0+\overline{c}\geq 0$ and $b_0^2\geq 4c^*$.

As in Section 4.3, (4.13) can be rewritten as a coupled equation as follows.

$$
\begin{aligned}
&-\triangle_h u_i + \mu u_i = v_i,\ i \in \Omega_h; \quad u_j = J[x_j', u] + g_j^{(1)},\ j \in \partial\Omega_h \\
&-\triangle_h v_i + \mu^+ v_i = F(x_i, u_i),\ i \in \Omega_h; \quad v_j = J[x_j', v] + g_j^{(2)},\ j \in \partial\Omega_h
\end{aligned} \tag{4.15}
$$

where $F(x_i, u_i) = \bar{c}u_i + f(x_i, u_i)$ and $g^{(2)} = g^{(0)} + \mu g^{(1)}$. For the above coupled equation (4.15), the lower and upper solutions are defined as follows.

Definition 4.4.2. *A function* $(\widehat{\alpha}_k, \widehat{\beta}_k)$ *is called a lower solution of* (4.15) *if it satisfies*

$$
\begin{aligned}
&-\triangle_h \widehat{\alpha}_i + \mu \widehat{\alpha}_i \leq \widehat{\beta}_i,\ i \in \Omega_h; \quad \widehat{\alpha}_j \leq J[x_j', \widehat{\alpha}] + g_j^{(1)};\ j \in \partial\Omega_h \\
&-\triangle_h \widehat{\beta}_i + \mu^+ \widehat{\beta}_i \leq F(x_i, \widehat{\alpha}_i),\ i \in \Omega_h; \quad \widehat{\beta}_j \leq J[x_j', \widehat{\beta}] g_j^{(2)},\ j \in \partial\Omega_h.
\end{aligned} \tag{4.16}
$$

Similarly $(\widetilde{\alpha}_k, \widetilde{\beta}_k)$ *is called an upper solution of* (4.15) *if it satisfies* (4.16) *with inequalities reversed.*

The lower and upper solutions $(\widehat{\alpha}_i, \widehat{\beta}_i)$ and $(\widetilde{\alpha}_i, \widetilde{\beta}_i)$ are said to be ordered if $\widehat{\alpha}_i \leq \widetilde{\alpha}_i$ and $\widehat{\beta}_i \leq \widetilde{\beta}_i$.

Remark 4.4.1. *It can be easily verified that if* $\widehat{\alpha}_i$ *and* $\widetilde{\alpha}_i$ *are ordered lower and upper solutions of* (4.13)*, then the pair* $(\widehat{\alpha}_i, -\triangle_h \widehat{\alpha}_i + \mu \widehat{\alpha}_i)$ *and* $(\widetilde{\alpha}_i, -\triangle_h \widetilde{\alpha}_i + \mu \widetilde{\alpha}_i)$ *are ordered lower and upper solutions of* (4.15)*.*

For ordered lower and upper solutions $(\widehat{\alpha}_i, \widehat{\beta}_i)$ and $(\widetilde{\alpha}_i, \widetilde{\beta}_i)$, define the set $S^d = \{(u_k, v_k) : (\widehat{\alpha}_k, \widehat{\beta}_k) \leq (u_k, v_k) \leq (\widetilde{\alpha}_k, \widetilde{\beta}_k)\}$. As in Section 4.3, at first the numerical solution for a second order linear elliptic equation with nonlocal boundary condition is obtained using central difference approximation and monotone iterations. More specifically, a numerical technique is developed to solve (4.5) numerically. The proposed algorithm for solving (4.5) also ensures the existence of the unique solution of the corresponding discretized problem given by

$$
-\triangle_h w_i + a w_i = h(x_i), \quad i \in \Omega_h; \quad w_j = J[x_j', w] + g_j,\ j \in \partial\Omega_h. \tag{4.17}
$$

Since the proposed iterative scheme for (4.17) is based on monotone iterations, one requires the following lower and upper solutions.

Definition 4.4.3. *A function $\widehat{w}_k$ is called a lower solution of* (4.17) *if it satisfies*

$$-\triangle_h \underline{w}_i^0 + a\underline{w}_i^0 \leq h(x_i), \quad i \in \Omega_h; \quad \underline{w}_j^0 \leq J[x_j^{'}, \underline{w}^0] + g_j, \quad j \in \partial\Omega_h. \tag{4.18}$$

Similarly $\overline{w}_k^0$ is called an upper solution of (4.17) *if it satisfies* (4.18) *with inequalities reversed.*

The following lemma guarantees the existence of the unique solution of (4.17).

Lemma 4.4.1. *Let $\underline{w}_k^0$ and $\overline{w}_k^0$ be ordered lower and upper solutions of* (4.17)*. Then* (4.17) *has a unique solution w_k^* such that $\underline{w}_k^0 \leq w_k^* \leq \overline{w}_k^0$.*

Proof. The solution of (4.17) is obtained as the limit of the following successive iterative scheme.

$$\begin{aligned} -\triangle_h w_i^{n+1} + a w_i^{n+1} = h(x_i), \quad i \in \Omega_h \\ w_j^{n+1} = J[x_j^{'}, w^n] + g_j, \quad j \in \partial\Omega_h. \end{aligned} \tag{4.19}$$

Choosing the initial iteration w_k^0 as $\underline{w}_k^0$ and $\overline{w}_k^0$ respectively, two sequences $\{\underline{w}_k^{n+1}\}$ and $\{\overline{w}_k^{n+1}\}$ can be constructed. Clearly the iterative scheme (4.19) is well defined. The proof is done through an induction argument. Let $z_k = \underline{w}_k^1 - \underline{w}_k^0$.

$$\begin{aligned} -\triangle_h z_i + a z_i &= (-\triangle_h \underline{w}_i^1 + a\underline{w}_i^1) - (-\triangle_h \underline{w}_i^0 + a\underline{w}_i^0) \\ -\triangle_h z_i + a z_i &\geq h(x_i) - h(x_i) = 0, \quad i \in \Omega_h \\ z_j &= \underline{w}_j^1 - \underline{w}_j^0 = J[x_j^{'}, \underline{w}^0] + g_j - \underline{w}_j^0 \geq 0, \quad j \in \partial\Omega_h. \end{aligned}$$

The positivity lemma for second order finite difference elliptic boundary value problems [**70**] implies that $z_k \geq 0$ and thus $\underline{w}_k^0 \leq \underline{w}_k^1$ in $\overline{\Omega}_h$. Similarly $\overline{w}_k^1 \leq \overline{w}_k^0$ in $\overline{\Omega}_h$. Now let $z_k = \overline{w}_k^1 - \underline{w}_k^1$. Clearly for $i \in \Omega_h$, $-\triangle_h w_i + a w_i = 0$. Moreover,

$$z_j = \overline{w}_j^1 - \underline{w}_j^1 = J[x_j^{'}, \overline{w}^0] - J[x_j^{'}, \underline{w}^0] \geq 0, \quad j \in \partial\Omega_h.$$

Hence $\underline{w}_k^0 \leq \underline{w}_k^1 \leq \overline{w}_k^1 \leq \overline{w}_k^0$ in $\overline{\Omega}_h$. Assume that $\underline{w}_k^0 \leq \underline{w}_k^{n-1} \leq \underline{w}_k^n \leq \overline{w}_k^n \leq \overline{w}_k^{n-1} \leq \overline{w}_k^0$ for some $n > 1$. Let $z_k = \underline{w}_k^{n+1} - \underline{w}_k^n$. Note that for all $i \in \Omega_h$, one has $-\triangle_h z_i + a z_i = 0$. For all $j \in \partial\Omega_h$,

$$z_j = \underline{w}_j^{n+1} - \underline{w}_j^n = J[x_j^{'}, \underline{w}^n] - J[x_j^{'}, \underline{w}^{n-1}] \geq 0.$$

Thus $\underline{w}_k^n \le \underline{w}_k^{n+1}$ in $\overline{\Omega}_h$. Similarly $\overline{w}_k^{n+1} \le \overline{w}_k^n$ in $\overline{\Omega}_h$. Let $z_k = \overline{w}_k^{n+1} - \underline{w}_k^{n+1}$. Consequently, $-\triangle_h z_i + a z_i = 0$ for all $i \in \Omega_h$. Also

$$z_j \;=\; \overline{w}_j^{n+1} - \underline{w}_j^{n+1} = J[x_j^{'}, \overline{w}^n] - J[x_j^{'}, \underline{w}^n] \ge 0, \quad j \in \partial\Omega_h.$$

Hence $\underline{w}_k^n \le \underline{w}_k^{n+1} \le \overline{w}_k^{n+1} \le \overline{w}_k^n$ for all n in $\overline{\Omega}_h$. This guarantees the existence of the limit $\lim\limits_{n\to\infty} \underline{w}_k^n = \underline{w}_k$ and $\lim\limits_{n\to\infty} \overline{w}_k^n = \overline{w}_k$. Moreover the limits are the solutions of (4.17) satisfying $\underline{w}_k \le w_k^* \le \overline{w}_k$. Let $W_k = \underline{w}_k - \overline{w}_k$. Then $W_k \le 0$ satisfies

$$\begin{aligned} &-\triangle_h W_i + a W_i = 0, \quad i \in \Omega_h \\ &W_j = J[x_j^{'}, W], \;\; j \in \partial\Omega_h. \end{aligned}$$

To show that $W_k \ge 0$, assume by contradiction that it is not true. Then there exists some $j_0 \in \partial\Omega_h$ such that W_{j_0} is negative and it is the minimum on $\overline{\Omega}_h$. Then $W_{j_0} = J[x_{j_0}^{'}, W] \ge J[x_{j_0}^{'}, 1] W_{j_0}$, which is possible only if $J[x_{j_0}^{'}, 1] \ge 1$, a contradiction to $(\mathbf{H}_5)$. Hence $W_k = 0$ and the uniqueness. □

Based on iterative scheme (4.9) discussed in Section 4.3, the following discretized iterative scheme is proposed to solve (4.15). With initial guess (u^0, v^0) and $n \in \mathbb{N}$,

$$\begin{aligned} &-\triangle_h u_i^{n+1} + \mu u_i^{n+1} = v_i^n, \quad i \in \Omega_h \\ &u_j^{n+1} = J[x_j^{'}, u^{n+1}] + g_j^{(1)}, \quad j \in \partial\Omega_h \\ &-\triangle_h v_i^{n+1} + \mu^+ v_i^{n+1} = F(x_i, u_i^{n+1}), \quad i \in \Omega_h \\ &v_j^{n+1} = J[x_j^{'}, v^{n+1}] + g_j^{(2)}, \quad j \in \partial\Omega_h. \end{aligned} \tag{4.20}$$

Using $(\widehat{\alpha}_k, \widehat{\beta}_k)$ and $(\widetilde{\alpha}_k, \widetilde{\beta}_k)$ respectively as the initial iterations, two sequences namely minimal and maximal sequences can be generated from the proposed iterative procedure. The minimal sequence $\{(\alpha_k^n, \beta_k^n)\}$, $n \in \mathbb{N}$ is given by

$$\begin{aligned} &-\triangle_h \alpha_i^{n+1} + \mu \alpha_i^{n+1} = \beta_i^n, \quad i \in \Omega_h \\ &\alpha_j^{n+1} = J[x_j^{'}, \alpha^{n+1}] + g_j^{(1)}, \quad j \in \partial\Omega \end{aligned} \tag{4.21a}$$

and

$$\begin{aligned} &-\triangle_h \beta_i^{n+1} + \mu^+ \beta_i^{n+1} = F(x_i, \alpha_i^{n+1}), \quad i \in \Omega_h \\ &\beta_j^{n+1} = J[x_j^{'}, \beta^{n+1}] + g_j^{(2)}, \quad j \in \partial\Omega_h \end{aligned} \tag{4.21b}$$

and the maximal sequence $\{(\overline{\alpha}_k^n, \overline{\beta}_k^n)\}$, $n \in \mathbb{N}$ is given by

(4.22a)
$$\begin{aligned} -\triangle_h \overline{\alpha}_i^{n+1} + \mu \overline{\alpha}_i^{n+1} &= \overline{\beta}_i^n, \quad i \in \Omega_h \\ \overline{\alpha}_j^{n+1} &= J[x_j', \overline{\alpha}] + g_j^{(1)}, \quad j \in \partial\Omega_h. \end{aligned}$$

(4.22b)
$$\begin{aligned} -\triangle_h \overline{\beta}_i^{n+1} + \mu^+ \overline{\beta}_i^{n+1} &= F(x_i, \overline{\alpha}_i^{n+1}), \quad i \in \Omega_h \\ \overline{\beta}_j^{n+1} &= J[x_j', \overline{\beta}^{n+1}] + g_j^{(2)}, \quad j \in \partial\Omega_h. \end{aligned}$$

The following theorem provides the monotone and well defined properties of the minimal and maximal sequences of (4.21) and (4.22) and their convergence to the unique solution of (4.15).

Theorem 4.4.1. *Let $(\widehat{\alpha}_k, \widehat{\beta}_k)$ and $(\widetilde{\alpha}_k, \widetilde{\beta}_k)$ be a pair of ordered lower and upper solutions of* (4.15). *Then the minimal sequence $\{(\alpha_k^n, \beta_k^n)\}$ and the maximal sequence $\{(\overline{\alpha}_k^n, \overline{\beta}_k^n)\}$ are well defined and converge monotonically to the unique solution (u_k^*, v_k^*) of* (4.15) *in S^d. Moreover, the following relation holds for $n \in \mathbb{N}$.*

(4.23)
$$(\widehat{\alpha}_k, \widehat{\beta}_k) \le (\alpha_k^n, \beta_k^n) \le (\alpha_k^{n+1}, \beta_k^{n+1}) \le (u_k^*, v_k^*) \le (\overline{\alpha}_k^{n+1}, \overline{\beta}_k^{n+1}) \le (\overline{\alpha}_k^n, \overline{\beta}_k^n) \le (\widetilde{\alpha}_k, \widetilde{\beta}_k).$$

Proof. The proof is done by an induction on n. Using Theorem 4.4.1 the following inequality is proved first. To prove $(\widehat{\alpha}_k, \widehat{\beta}_k) \le (\alpha_k^1, \beta_k^1) \le (\widetilde{\alpha}_k, \widetilde{\beta}_k)$, it is enough to show that $(\widehat{\alpha}_k, \widehat{\beta}_k)$ and $(\widetilde{\alpha}_k, \widetilde{\beta}_k)$ are lower and upper solutions respectively of (4.21) for $n = 0$. From Definition 4.4.2, one can have

$$\begin{aligned} -\triangle_h \widehat{\alpha}_i + \mu \widehat{\alpha}_i &\le \widehat{\beta}_i, \quad i \in \Omega_h \\ \widehat{\alpha}_j &\le J[x_j' \widehat{\alpha}] + g_j^{(1)}, \quad j \in \partial\Omega_h \end{aligned}$$

and

$$\begin{aligned} -\triangle_h \widetilde{\alpha}_i + \mu \widetilde{\alpha}_i &\ge \widetilde{\beta}_i \ge \widehat{\beta}_i, \quad i \in \Omega_h \\ \widetilde{\alpha}_j &\ge J[x_j', \widetilde{\alpha}] + g_j^{(1)}, \quad j \in \partial\Omega_h. \end{aligned}$$

Thus by Theorem 4.4.1, (4.21a) with $n = 0$ has a unique solution α_k^1 between $\widehat{\alpha}_k$ and $\widetilde{\alpha}_k$. Thus $\widehat{\alpha}_k \le \alpha_k^1 \le \overline{\alpha}_k^1$. Similarly from Definition 4.4.2,

$$\begin{aligned} -\triangle_h \widehat{\beta}_i + \mu^+ \widehat{\beta}_i &\le F(x_i, \widehat{\alpha}_i) \le F(x_i, \alpha_i^1), \quad i \in \Omega_h \\ \widehat{\beta}_j &\le J[x_j' \widehat{\beta}] + g_j^{(2)}, \quad j \in \partial\Omega_h \end{aligned}$$

and

$$\begin{aligned}
&-\triangle_h\widetilde{\beta}_i+\mu^+\widetilde{\beta}_i\geq F(x_i,\widetilde{\alpha}_i)\geq F(x_i,\alpha_i^1), \quad i\in\Omega_h\\
&\widetilde{\beta}_j\geq J[x_j',\widetilde{\beta}]+g_j^{(2)}, \quad j\in\partial\Omega_h.
\end{aligned}$$

Thus by Lemma 4.4.1, (4.21b) with $n=0$ has a unique solution β_k^1 between $\widehat{\beta}_k$ and $\widetilde{\beta}_k$. Hence $(\widehat{\alpha}_k,\widehat{\beta}_k)\leq(\alpha_k^1,\beta_k^1)\leq(\overline{\alpha}_k^1,\overline{\beta}_k^1)$. To prove $(\alpha_k^1,\beta_k^1)\leq(\overline{\alpha}_k^1,\overline{\beta}_k^1)\leq(\widetilde{\alpha}_k,\widetilde{\beta}_k)$ it is enough to show that (α_k^1,β_k^1) and $(\widetilde{\alpha}_k,\widetilde{\beta}_k)$ are lower and upper solutions respectively of (4.22) with $n=0$. From (4.21a) for $n=0$,

$$\begin{aligned}
&-\triangle_h\alpha_i^1+\mu\alpha_i^1=\widehat{\beta}_i\leq\widetilde{\beta}_i, \quad i\in\Omega_h\\
&\alpha_j^1=J[x_j',\alpha^1]+g_j^{(1)}, \quad j\in\partial\Omega_h
\end{aligned}$$

and from Definition 4.4.2,

$$\begin{aligned}
&-\triangle_h\widetilde{\alpha}_i+\mu\widetilde{\alpha}_i\geq\widetilde{\beta}_i, \quad i\in\Omega_h\\
&\widetilde{\alpha}_j\geq J[x_j',\widetilde{\alpha}]+g_j^{(1)}, \quad j\in\partial\Omega_h.
\end{aligned}$$

Thus by Lemma 4.4.1, (4.22a) with $n=0$ has a unique solution $\overline{\alpha}_k^1$ between α_k^1 and $\widetilde{\alpha}_k$.Thus $\alpha_k^1\leq\overline{\alpha}_k^1\leq\widetilde{\alpha}_k$. Similarly from (4.21b) for $n=0$,

$$\begin{aligned}
&-\triangle_h\beta_i^1+\mu^+\beta_i^1=F(x_i,\alpha_i^1)\leq F(x_i,\overline{\alpha}_i^1), \quad i\in\Omega_h\\
&\beta_j^1=J[x_j',\beta^1]+g_j^{(2)}, \quad j\in\partial\Omega_h
\end{aligned}$$

and from Definition 4.4.2,

$$\begin{aligned}
&-\triangle_h\widetilde{\beta}_i+\mu^+\widetilde{\beta}_i\geq F(x_i,\widetilde{\alpha}_i)\geq F(x_i,\overline{\alpha}_i^1), \quad i\in\Omega_h\\
&\widetilde{\beta}_j\geq J[x_j',\widetilde{\beta}]+g_j^{(2)}, \quad j\in\partial\Omega_h.
\end{aligned}$$

Thus by Lemma 4.4.1, (4.22b) with $n=0$ has a unique solution $\overline{\beta}_k^1$ between β_k^1 and $\widetilde{\beta}_k$. Thus $(\alpha_k^1,\beta_k^1)\leq(\overline{\alpha}_k^1,\overline{\beta}_k^1)\leq(\widetilde{\alpha}_k,\widetilde{\beta}_k)$. Hence $(\widehat{\alpha}_k,\widehat{\beta}_k)\leq(\alpha_k^1,\beta_k^1)\leq(\overline{\alpha}_k^1,\overline{\beta}_k^1)\leq(\widetilde{\alpha_k},\widetilde{\beta}_k)$. Assume that

$$(\widehat{\alpha}_k,\widehat{\beta}_k)\leq(\alpha_k^{n-1},\beta_k^{n-1})\leq(\alpha_k^n,\beta_k^n)\leq(\overline{\alpha}_k^n,\overline{\beta}_k^n)\leq(\overline{\alpha}_k^{n-1},\overline{\beta}_k^{n-1})\leq(\widetilde{\alpha}_k,\widetilde{\beta}_k)$$

for some $n>1$. To complete the proof for the monotone property, one has to show that $(\alpha_k^n,\beta_k^n)\leq(\alpha_k^{n+1},\beta_k^{n+1})\leq(\overline{\alpha}_k^{n+1},\overline{\beta}_k^{n+1})\leq(\overline{\alpha}_k^n,\overline{\beta}_k^n)$. To do this, the following inequality is proved first; $(\alpha_k^n, \beta_k^n) \le (\alpha_k^{n+1}, \beta_k^{n+1}) \le (\overline{\alpha}_k^n, \overline{\beta}_k^n)$. It is enough to show that (α_k^n, β_k^n) and $(\overline{\alpha}_k^n, \overline{\beta}_k^n)$ are lower and upper solutions respectively of (4.21). From (4.21a),

$$\begin{aligned}
&-\triangle_h \alpha_i^n + \mu \alpha_i^n = \beta_i^{n-1} \le \beta_i^n, \quad i \in \Omega_h \\
&\alpha_j^n = J[x_j', \alpha^n] + g_j^{(1)}, \quad j \in \partial\Omega_h
\end{aligned}$$

and from (4.22a),

$$\begin{aligned}
&-\triangle_h \overline{\alpha}_i^n + \mu \overline{\alpha}_i^n = \overline{\beta}_i^{n-1} \ge \beta_i^n, \quad i \in \Omega_h \\
&\overline{\alpha}_j^n = J[x_j', \overline{\alpha}^n] + g_j^{(1)}, \quad j \in \partial\Omega_h.
\end{aligned}$$

Thus by Lemma 4.4.1, (4.21a) has a unique solution α_k^{n+1} between α_k^n and $\overline{\alpha}_k^n$. Thus $\alpha_k^n \le \alpha_k^{n+1} \le \overline{\alpha}_k^n$. Similarly from (4.21b),

$$\begin{aligned}
&-\triangle_h \beta_i^n + \mu^+ \beta_i^n = F(x_i, \alpha_i^n) \le F(x_i, \alpha_i^{n+1}), \quad i \in \Omega_h \\
&\beta_j^n = J[x_j', \beta^n] + g_j^{(2)}, \quad j \in \partial\Omega_h
\end{aligned}$$

and from (4.22b),

$$\begin{aligned}
&-\triangle_h \overline{\beta}_i^n + \mu^+ \overline{\beta}_i^n = F(x_i, \overline{\alpha}_i^n) \ge F(x_i, \alpha_i^{n+1}), \quad i \in \Omega_h \\
&\overline{\beta}_j^n = J[x_j', \overline{\beta}^n] + g_j^{(2)}, \quad j \in \partial\Omega_h.
\end{aligned}$$

Thus by Lemma 4.4.1, (4.21b) has a unique solution β_k^{n+1} between β_k^n and $\overline{\beta}_k^n$. Hence $(\alpha_k^n, \beta_k^n) \le (\alpha_k^{n+1}, \beta_k^{n+1}) \le (\overline{\alpha}_k^n, \overline{\beta}_k^n)$. To prove $(\alpha_k^{n+1}, \beta_k^{n+1}) \le (\overline{\alpha}_k^{n+1}, \overline{\beta}_k^{n+1}) \le (\overline{\alpha}_k^n, \overline{\beta}_k^n)$, it is enough to show that $(\alpha_k^{n+1}, \beta_k^{n+1})$ and $(\overline{\alpha}_k^n, \overline{\beta}_k^n)$ are lower and upper solutions respectively of (4.22). From (4.21a),

$$\begin{aligned}
&-\triangle_h \alpha_i^{n+1} + \mu \alpha_i^{n+1} = \beta_i^n \le \overline{\beta}_i^n, \quad i \in \Omega_h \\
&\alpha_j^{n+1} = J[x_j', \alpha^{n+1}] + g_j^{(1)}, \quad j \in \partial\Omega_h
\end{aligned}$$

and from (4.22a),

$$\begin{aligned}
&-\triangle_h \overline{\alpha}_i^n + \mu \overline{\alpha}_i^n = \overline{\beta}_i^{n-1} \ge \overline{\beta}_i^n, \quad i \in \Omega_h \\
&\overline{\alpha}_j^n = J[x_j', \overline{\alpha}^n] + g_j^{(1)}, \quad j \in \partial\Omega_h.
\end{aligned}$$

Thus by Lemma 4.4.1, (4.22a) has a unique solution $\overline{\alpha}_k^{n+1}$ between α_k^{n+1} and $\overline{\alpha}_k^n$. Thus $\alpha_k^{n+1} \le \overline{\alpha}_k^{n+1} \le \overline{\alpha}_k^n$. Similarly from (4.21b),

$$\begin{aligned}
&-\triangle_h \beta_i^{n+1} + \mu^+ \beta_i^{n+1} = F(x_i, \alpha_i^{n+1}) \le F(x_i, \overline{\alpha}_i^{n+1}), \quad i \in \Omega_h \\
&\beta_j^{n+1} = J[x_j', \beta^{n+1}] + g_j^{(2)}, \quad j \in \partial\Omega_h
\end{aligned}$$

and from (4.22b),

$$
\begin{aligned}
-\triangle_h \overline{\beta}_i^n + \mu^+ \overline{\beta}_i^n &= F(x_i, \overline{\alpha}_i^n) \geq F(x_i, \overline{\alpha}_i^{n+1}), \quad i \in \Omega_h \\
\overline{\beta}_j^n &= J[x_j', \overline{\beta}^n] + g_j^{(2)}, \quad j \in \partial\Omega_h.
\end{aligned}
$$

Thus by Lemma 4.4.1, (4.22b) has a unique solution $\overline{\beta}_k^{n+1}$ between β_k^{n+1} and $\overline{\beta}_k^n$. Hence $(\alpha_k^{n+1}, \beta_k^{n+1}) \leq (\overline{\alpha}_k^{n+1}, \overline{\beta}_k^{n+1}) \leq (\overline{\alpha}_k^n, \overline{\beta}_k^n)$. Thus $(\alpha_k^n, \beta_k^n) \leq (\alpha_k^{n+1}, \beta_k^{n+1}) \leq (\overline{\alpha}_k^{n+1}, \overline{\beta}_k^{n+1}) \leq (\overline{\alpha}_k^n, \overline{\beta}_k^n)$. Hence (4.23) holds true for $n \in \mathbb{N}$. This guarantees the existence of the limits $\lim\limits_{n\to\infty} (\alpha_k^n, \beta_k^n) = (\underline{\alpha}_k, \underline{\beta}_k)$ and $\lim\limits_{n\to\infty} (\overline{\alpha}_k^n, \overline{\beta}_k^n) = (\overline{\alpha}_k, \overline{\beta}_k)$ where $(\underline{\alpha}_k, \underline{\beta}_k) \leq (\overline{\alpha}_k, \overline{\beta}_k)$ and they are the solutions of (4.15). The proof for the uniqueness is similar to the proof of Theorem 2.2 in [**83**] and hence omitted. □

Remark 4.4.2. *Under the given conditions, Theorem 4.4.1 also holds good for* (4.13) *with the boundary conditions replaced with*

$$
\begin{aligned}
u_j &= J[x_j', u] + g_j^{(1)}, \quad j \in \partial\Omega_h \\
\triangle_h u_j &= -g_j^{(0)}, \quad j \in \partial\Omega_h.
\end{aligned}
$$

4.5. Gauss-Seidel and Jacobi Iterations

To overcome the computational complexity while dealing with higher dimensions, Gauss-Seidel and Jacobi methods are used to develop iterative schemes similar to Section 3 in [**83**]. Let $M = M_1 M_2 \cdots M_n$ be the total number of mesh points in Ω_h and let $U = (u_1, \ldots, u_M)^T$ and $V = (v_1, \ldots, v_M)^T$ be the vector representations of the solutions u_k and v_k for all $k \in \overline{\Omega}_h$ arranged suitably. Then (4.15) can be written using vector representation as follows.

$$
\begin{aligned}
(A + B + \mu J)U &= V_1 + G^{(1)} \\
(A + B + \mu^+ J)V &= F(U) + G^{(2)}.
\end{aligned}
\tag{4.24}
$$

Here A is an $M \times M$ diagonally dominant matrix associated with the difference operator $-\triangle_h$ and boundary conditions and J is the $M \times M$ diagonal matrix with diagonal entries corresponding to the boundary mesh points zero and one elsewhere. B is also an $M \times M$ matrix associated with the boundary conditions whose entries are zero except for the negative entries in the rows corresponding to the boundary mesh points. $F(U) = (F(x_i, u_i))$ is defined to be a column vector of size M in which the entries corresponding to the boundary mesh points will be zero. Similarly the vector V_1 in the first equation of (4.24) is obtained from V by assigning zeros to the entries corresponding to the boundary mesh points. The vectors $G^{(1)}$ and $G^{(2)}$ corresponding to the boundary conditions will have non-zero entries only at the positions corresponding to the boundary mesh points. Denote the matrix $A+B$ by $\mathcal{M}$. Note that dissimilar to the vector representations given in [**83**], both interior and boundary mesh points are included in the vector representation (4.24).

Definition 4.5.1. *A pair of vectors* $(\widehat{U},\widehat{V})\in\mathbb{R}^M\times\mathbb{R}^M$ *is said to be a lower solution of* (4.24)*if it satisfies*

$$(4.25)\qquad \begin{aligned}(\mathcal{M}+\mu J)\widehat{U} &\leq \widehat{V}_1+G^{(1)}\\ (\mathcal{M}+\mu^+ J)\widehat{V} &\leq F(\widehat{U})+G^{(2)}.\end{aligned}$$

Similarly $(\widetilde{U},\widetilde{V})\in\mathbb{R}^M\times\mathbb{R}^M$ *is said to be an upper solution of* (4.24) *if it satisfies* (4.25) *with the inequalities reversed.*

For ordered lower and upper solutions $(\widehat{U},\widehat{V})$ and $(\widetilde{U},\widetilde{V})$, define the set $S^*=\{(U,V)\in\mathbb{R}^M\times\mathbb{R}^M\ :\ (\widehat{U},\widehat{V})\leq(U,V)\leq(\widetilde{U},\widetilde{V})\}$. Note that the matrix $\mathcal{M}$ is irreducible with $\mathcal{M}_{k,k}>0,\ \mathcal{M}_{k,l}\leq 0$ for $k\neq l$. Moreover, $\mathcal{M}$ is a strictly diagonally dominant matrix. Consequently, $\mathcal{M}$ is an M-matrix and is inverse positive [**10, 97, 102**]. $\mathcal{M}$ can be written as $\mathcal{M}=\mathcal{D}-\mathcal{L}-\mathcal{U}$ where $\mathcal{D}$, $-\mathcal{L}$ and $-\mathcal{U}$ are diagonal, lower triangular and upper triangular sub-matrices of $\mathcal{M}$ respectively. Define $\mathcal{P}^{(1)}=\mathcal{M}+\mu J$, $\mathcal{P}^{(2)}=\mathcal{M}+\mu^+J$, $\mathcal{G}^{(1)}=\mathcal{D}+\mu J-\mathcal{L}$, $\mathcal{G}^{(2)}=\mathcal{D}+\mu^+J-\mathcal{L}$, $\mathcal{J}^{(1)}=\mathcal{D}+\mu J$ and $\mathcal{J}^{(2)}=\mathcal{D}+\mu^+J$. Clearly, $\mathcal{G}^{(1)}$, $\mathcal{G}^{(2)}$, $\mathcal{J}^{(1)}$ and $\mathcal{J}^{(2)}$ are inverse positive matrices. Using $(\widehat{U},\widehat{V})$ and $(\widetilde{U},\widetilde{V})$ respectively as the initial iterations, minimal and maximal sequences $\{(\underline{U}^n,\underline{V}^n)\}$ and $\{(\overline{U}^n,\overline{V}^n)\}$ can be constructed from the following iterative procedures for $n\in\mathbb{N}$.

(a) Picard iteration:

$$(4.26)\qquad \begin{aligned}\mathcal{P}^{(1)}U^{n+1} &= V_1^n+G^{(1)}\\ \mathcal{P}^{(2)}V^{n+1} &= F(U^{n+1})+G^{(2)}.\end{aligned}$$

(b) Gauss-Seidel iteration:

$$\begin{aligned} \mathcal{G}^{(1)}U^{n+1} &= \mathcal{U}U^n + V_1^n + G^{(1)} \\ \mathcal{G}^{(2)}V^{n+1} &= \mathcal{U}V^n + F(U^{n+1}) + G^{(2)}. \end{aligned} \tag{4.27}$$

(c) Jacobi iteration:

$$\begin{aligned} \mathcal{J}^{(1)}U^{n+1} &= (\mathcal{U}+\mathcal{L})U^n + V_1^n + G^{(1)} \\ \mathcal{J}^{(2)}V^{n+1} &= (\mathcal{U}+\mathcal{L})V^n + F(U^{n+1}) + G^{(2)}. \end{aligned} \tag{4.28}$$

Clearly all the iterative schemes above are well defined. The following theorem provides the monotone convergence of the three iterative schemes.

Theorem 4.5.1. *Let $(\widehat{U},\widehat{V})$ and $(\widetilde{U},\widetilde{V})$ be a pair of ordered lower and upper solutions of* (4.24)*. Then the following holds good.*

(i) *The minimal sequence $\{(\underline{U}^n,\underline{V}^n)\}$ and the maximal sequence $\{(\overline{U}^n,\overline{V}^n)\}$ converge monotonically to the minimal solution $(\underline{U},\underline{V})$ and the maximal solution $(\overline{U},\overline{V})$ respectively of* (4.24) *in S^* and the following relation holds for $n\in\mathbb{N}$.*

$$\begin{aligned} (\widehat{U},\widehat{V}) &\le (\underline{U}^n,\underline{V}^n) \le (\underline{U}^{n+1},\underline{V}^{n+1}) \le (\underline{U},\underline{V}) \\ &\le (\overline{U},\overline{V}) \le (\overline{U}^{n+1},\overline{V}^{n+1}) \le (\overline{U}^n,\overline{V}^n) \le (\widetilde{U},\widetilde{V}). \end{aligned} \tag{4.29}$$

(ii) *If $(\underline{U},\underline{V}) = (\overline{U},\overline{V})(\equiv (U^*,V^*))$, then (U^*,V^*) is the unique solution of* (4.24) *in S^*.*

Proof. (a) Picard iteration: This is a direct implication of Theorem 4.4.1 as (4.26) is the vector representation of (4.20).
(b) Gauss-Seidel Iteration:

$$\begin{aligned} \mathcal{G}^{(1)}(U^1-\widehat{U}) &= \mathcal{U}\widehat{U} + \widehat{V}_1 + G^{(1)} - (\mathcal{D}+\mu J-\mathcal{L})\widehat{U} \\ &= \widehat{V}_1 + G^{(1)} - (\mathcal{M}+\mu J)\widehat{U} \ge 0 \end{aligned}$$

Due to the non-negative property of $(\mathcal{G}^{(1)})^{-1}$, $\widehat{U}\le U^1$. Also

$$\begin{aligned} \mathcal{G}^{(2)}(V^1-\widehat{V}) &= \mathcal{U}\widehat{V} + F(U^1) + G^{(2)} - (\mathcal{D}+\mu J-\mathcal{L})\widehat{V} \\ &= F(U^1) + G^{(2)} - (\mathcal{M}+\mu J)\widehat{V} \\ &\ge F(\widehat{U}) + G^{(2)} - (\mathcal{M}+\mu J)\widehat{V} \ge 0. \end{aligned}$$

Thus $\widehat{V} \leq V^1$ since $(\mathcal{G}^{(2)})^{-1}$ is non-negative. Similarly one can obtain $(\overline{U}^1, \overline{V}^1) \leq (\widetilde{U}, \widetilde{V})$. Now consider

$$\mathcal{G}^{(1)}(\overline{U}^1 - U^1) \;=\; (\mathcal{U}\widetilde{U} + \widetilde{V}_1 + G^{(1)}) - (\mathcal{U}\widehat{U} + \widehat{V}_1 + G^{(1)}) \geq 0.$$

Thus $U^1 \leq \overline{U}^1$. Also

$$\mathcal{G}^{(2)}(\overline{V}^1 - V^1) \;=\; (\mathcal{U}\widetilde{V} + F(\overline{U}^1) + G^{(2)}) - (\mathcal{U}\widehat{V} + F(U^1) + G^{(2)}) \geq 0.$$

Hence $V^1 \leq \overline{V}^1$ and therefore $(\widehat{U}, \widehat{V}) \leq (U^1, V^1) \leq (\overline{U}^1, \overline{V}^1) \leq (\widetilde{U}, \widetilde{V})$. Through an induction argument on n, one can show that $(\widehat{U}, \widehat{V}) \leq (U^n, V^n) \leq (U^{n+1}, V^{n+1}) \leq (\overline{U}^{n+1}, \overline{V}^{n+1}) \leq (\overline{U}^n, \overline{V}^n) \leq (\widetilde{U}, \widetilde{V})$ for all $n \in \mathbb{N}$. This assures that the limits $\lim\limits_{n\to\infty}(U^n, V^n) = (\underline{U}, \underline{V})$ and $\lim\limits_{n\to\infty}(\overline{U}^{n+1}, \overline{V}^{n+1}) = (\overline{U}, \overline{V})$ exists and they are the solutions of (4.24). This shows that (i) holds. Also, $(\underline{U}, \underline{V}) \leq (U^*, V^*) \leq (\overline{U}, \overline{V})$ and hence (ii) also holds true.
(c) Jacobi Iteration: The proof is similar to that of Gauss-Jacobi iteration and therefore omitted. □

Denote the minimal and maximal sequences of the three iterations respectively by $\{(U_P^n, V_P^n)\}$, $\{(\overline{U}_P^n, \overline{V}_P^n)\}$, $\{(U_G^n, V_G^n)\}$, $\{(\overline{U}_G^n, \overline{V}_G^n)\}$, $\{(U_J^n, V_J^n)\}$ and $\{(\overline{U}_J^n, \overline{V}_J^n)\}$. The following theorem supplies a comparison relation between the three iterative procedures (4.26), (4.27) and (4.28).

Theorem 4.5.2. *Let the conditions in Theorem 4.5.1 holds. Then for $n = 1, 2, \ldots,$*

$$\begin{aligned}(\overline{U}_P^n, \overline{V}_P^n) &\leq (\overline{U}_G^n, \overline{V}_G^n) \leq (\overline{U}_J^n, \overline{V}_J^n)\\ (U_P^n, V_P^n) &\geq (U_G^n, V_G^n) \geq (U_J^n, V_J^n)\end{aligned} \tag{4.30}$$

Proof. From (4.26), (4.27) and (4.29), one can have

$$\begin{aligned}\mathcal{P}^{(1)}\overline{U}_G^n &= (\mathcal{G}^{(1)} - \mathcal{U})\overline{U}_G^n = \mathcal{U}(\overline{U}_G^{n-1} - \overline{U}_G^n) + \overline{V}_{1_G}^{n-1} + G^{(1)} \geq \overline{V}_{1_G}^{n-1} + G^{(1)}\\ \mathcal{P}^{(2)}\overline{V}_G^n &= (\mathcal{G}^{(2)} - \mathcal{U})\overline{V}_G^n = \mathcal{U}(\overline{V}_G^{n-1} - \overline{V}_G^n) + F(\overline{U}_G^n) + G^{(2)} \geq F(\overline{U}_G^n) + G^{(2)}.\end{aligned} \tag{4.31}$$

Subtracting (4.26) from (4.31) yields

$$\begin{aligned}\mathcal{P}^{(1)}(\overline{U}_G^n - \overline{U}_P^n) &\geq \overline{V}_{1_G}^{n-1} - \overline{V}_{1_P}^{n-1}, \quad \forall\, n \in \mathbb{N}\\ \mathcal{P}^{(2)}(\overline{V}_G^n - \overline{V}_P^n) &\geq F(\overline{U}_G^n) - F(\overline{U}_P^n), \quad \forall\, n \in \mathbb{N}.\end{aligned} \tag{4.32}$$

For $n=0$, $(\overline{U}^0_G,\overline{V}^0_G)=(\overline{U}^0_P,\overline{V}^0_P)=(\widetilde{U},\widetilde{V})$. Then $\mathcal{P}^{(1)}(\overline{U}^1_G-\overline{U}^1_P)\geq 0$. Thus $\overline{U}^1_P\leq\overline{U}^1_G$. Consequently, $\mathcal{P}^{(2)}(\overline{V}^1_G-\overline{V}^1_P)\geq 0$ and thus $\overline{V}^1_P\leq\overline{V}^1_G$. Assume by induction that $(\overline{U}^{n-1}_P,\overline{V}^{n-1}_P)\leq(\overline{U}^{n-1}_G,\overline{V}^{n-1}_G)$ for some $n>1$. Then by using (4.32), one can obtain $\mathcal{P}^{(1)}(\overline{U}^n_G-\overline{U}^n_P)\geq 0$ and hence $\overline{U}^n_P\leq\overline{U}^n_G$. Similarly $\mathcal{P}^{(2)}(\overline{V}^n_G-\overline{V}^n_P)\geq 0$ and as a result $\overline{V}^n_P\leq\overline{V}^n_G$. Thus $(\overline{U}^n_P,\overline{V}^n_P)\leq(\overline{U}^n_G,\overline{V}^n_G)$ for all $n\in\mathbb{N}$. To show $(\overline{U}^n_G,\overline{V}^n_G)\leq(\overline{U}^n_J,\overline{V}^n_J)$, consider (4.27), (4.28) and (4.29).

$$\begin{aligned}
\mathcal{G}^{(1)}\overline{U}^n_J &= (\mathcal{J}^{(1)}-\mathcal{L})\overline{U}^n_J = \mathcal{U}\overline{U}^{n-1}_J+\mathcal{L}(\overline{U}^{n-1}_J-\overline{U}^n_J)+\overline{V}^{n-1}_{1_J}+G^{(1)}\\
&\geq \mathcal{U}\overline{U}^{n-1}_J+\overline{V}^{n-1}_{1_J}+G^{(1)}\\
\mathcal{G}^{(2)}\overline{V}^n_J &= (\mathcal{J}^{(2)}-\mathcal{L})\overline{V}^n_J = \mathcal{U}\overline{V}^{n-1}_J+\mathcal{L}(\overline{V}^{n-1}_J-\overline{V}^n_J)+F(\overline{U}^n_J)+G^{(2)}\\
&\geq \mathcal{U}\overline{V}^{n-1}_J+F(\overline{U}^n_J)+G^{(2)}.
\end{aligned}\tag{4.33}$$

Subtracting (4.27) from (4.33) yields

$$\begin{aligned}
&\mathcal{G}^{(1)}(\overline{U}^n_J-\overline{U}^n_G)\geq\mathcal{U}(\overline{U}^{n-1}_J-\overline{U}^{n-1}_G)+\overline{V}^{n-1}_{1_J}-\overline{V}^{n-1}_{1_G}, \quad \forall\, n\in\mathbb{N}\\
&\mathcal{G}^{(2)}(\overline{V}^n_J-\overline{V}^n_G)\geq\mathcal{U}(\overline{V}^{n-1}_J-\overline{V}^{n-1}_G)+F(\overline{U}^n_J)-F(\overline{U}^n_G), \quad \forall\, n\in\mathbb{N}.
\end{aligned}\tag{4.34}$$

For $n=0$, $(\overline{U}^0_J,\overline{V}^0_J)=(\overline{U}^0_G,\overline{V}^0_G)=(\widetilde{U},\widetilde{V})$. Then $\mathcal{G}^{(1)}(\overline{U}^1_J-\overline{U}^1_G)\geq 0$. Thus $\overline{U}^1_G\leq\overline{U}^1_J$. Consequently, $\mathcal{G}^{(2)}(\overline{V}^1_J-\overline{V}^1_G)\geq 0$ and thus $\overline{V}^1_G\leq\overline{V}^1_J$. Assume by induction that $(\overline{U}^{n-1}_G,\overline{V}^{n-1}_G)\leq(\overline{U}^{n-1}_J,\overline{V}^{n-1}_J)$ for some $n>1$. Then by using (4.34), one can obtain $\mathcal{G}^{(1)}(\overline{U}^n_J-\overline{U}^n_G)\geq 0$ and hence $\overline{U}^n_G\leq\overline{U}^n_J$. Similarly $\mathcal{G}^{(2)}(\overline{V}^n_J-\overline{V}^n_G)\geq 0$ and as a result $\overline{V}^n_G\leq\overline{V}^n_J$. Thus $(\overline{U}^n_G,\overline{V}^n_G)\leq(\overline{U}^n_J,\overline{V}^n_J)$ for all $n\in\mathbb{N}$. This proves the first inequality in (4.30). The proof for the second inequality is omitted as it can be proved along similar lines. □

Remark 4.5.1. *From Theorem 4.5.2 it is evident that both the minimal and maximal sequences of Picard iteration converge faster than that of Gauss-Seidel iteration which in turn converge faster than Jacobi iteration for the same initial iteration. The numerical results in Section 4.7 confirm this and reveal that the number of iterations by the three iterations differ extremely.*

4.6. Convergence of Finite Difference Solutions

The following theorem ensures the convergence of the discretized solutions to the exact solutions as the mesh size tends to zero. For simplicity, the uniform mesh size case is only considered here. The proof is similar to Theorem 4.1 in [**83**] and hence it is outlined.

Theorem 4.6.1. *Let* $((\widehat{\alpha},\widehat{\beta}),(\widetilde{\alpha},\widetilde{\beta}))$, $((\widehat{\alpha}_k,\widehat{\beta}_k),(\widetilde{\alpha}_k,\widetilde{\beta}_k))$ *be ordered lower and upper solutions of* (4.3) *and* (4.15) *respectively. Then the minimal solution* $(\underline{\alpha}_k^*,\underline{\beta}_k^*)$ *and the maximal solution* $(\overline{\alpha}_k^*,\overline{\beta}_k^*)$ *of* (4.15) *converge respectively to the minimal solution* $(\underline{\alpha}^*(x_k),\underline{\beta}^*(x_k))$ *and the maximal solution* $(\overline{\alpha}^*(x_k),\overline{\beta}^*(x_k))$ *of* (4.3) *at every point as mesh size tends to zero.*

Proof. Assume that for any $\epsilon>0$, there exists $\delta=\delta(\epsilon)>0$ such that whenever $|h|<\delta$,

$$|\widehat{\alpha}(x_k)-\widehat{\alpha}_k|+|\widehat{\beta}(x_k)-\widehat{\beta}_k|<\epsilon,\quad |\widetilde{\alpha}(x_k)-\widetilde{\alpha}_k|+|\widetilde{\beta}(x_k)-\widetilde{\beta}_k|<\epsilon \tag{4.35}$$

for all $x_k\in\overline{\Omega}_h$. The convergence of the maximal solution $(\overline{\alpha}_k^*,\overline{\beta}_k^*)$ to the corresponding maximal solution $(\overline{\alpha}^*(x_k),\overline{\beta}^*(x_k))$ as $|h|\to 0$ for every point $x_k\in\overline{\Omega}_h$ is proved here. For given any $\epsilon>0$, it has to be shown that there exists $\delta=\delta(\epsilon)>0$ such that

$$|\overline{\alpha}_k^*-\overline{\alpha}^*(x_k)|+|\overline{\beta}_k^*-\overline{\beta}^*(x_k)|<\epsilon \text{ when } |h|<\delta. \tag{4.36}$$

Let $\{(\overline{\alpha}_k^n,\overline{\beta}_k^n)\}$ and $\{(\overline{\alpha}^n(x_k),\overline{\beta}^n(x_k))\}$ be the respective maximal sequences of (4.22) and (4.11). Due to the convergence of the maximal sequences to the respective maximal solutions, there exists an integer $n^*=n^*(\epsilon)$ such that for all $n\geq n^*$,

$$|\overline{\alpha}_k^*-\overline{\alpha}_k^n|+|\overline{\alpha}^*(x_k)-\overline{\alpha}^n(x_k)|<\frac{\epsilon}{3};\quad |\overline{\beta}_k^*-\overline{\beta}_k^n|+|\overline{\beta}^*(x_k)-\overline{\beta}^n(x_k)|<\frac{\epsilon}{3}$$

for all $x_k\in\overline{\Omega}_h^*$. Note that for all $n\in\mathbb{N}$,

$$\begin{aligned}
|\overline{\alpha}_k^*-\overline{\alpha}^*(x_k)| &\leq |\overline{\alpha}_k^*-\overline{\alpha}_k^n|+|\overline{\alpha}_k^n-\overline{\alpha}^n(x_k)|+|\overline{\alpha}^n(x_k)-\overline{\alpha}^*(x_k)|\\
|\overline{\beta}_k^*-\overline{\beta}^*(x_k)| &\leq |\overline{\beta}_k^*-\overline{\beta}_k^n|+|\overline{\beta}_k^n-\overline{\beta}^n(x_k)|+|\overline{\beta}^n(x_k)-\overline{\beta}^*(x_k)|.
\end{aligned}$$

Hence the proof is complete if one can prove that there exists an $n_0\geq n^*$ and $\delta=\delta(\epsilon)>0$ such that

$$|\overline{\alpha}_k^{(n_0)}-\overline{\alpha}^{(n_0)}(x_k)|+|\overline{\beta}_k^{(n_0)}-\overline{\beta}^{(n_0)}(x_k)|<\frac{\epsilon}{3} \text{ when } |h|<\delta \tag{4.37}$$

for all $x_k \in \overline{\Omega}_h^*$. From (4.11) and (4.22), it can be seen that $(\overline{\alpha}^{n+1}(x_k), \overline{\beta}^{n+1}(x_k))$ satisfies the equations

$$
\begin{aligned}
&-\triangle_h \overline{\alpha}^{n+1}(x_i) + \mu \overline{\alpha}^{n+1}(x_i) = \overline{\beta}^n(x_i) + O^{n+1}(|h|^2), \quad i \in \Omega_h \\
&\overline{\alpha}^{n+1}(x_j^{'}) = J[x_j^{'}, \overline{\alpha}^{n+1}] + g_j^{(1)} + O^{n+1}(|h|), \quad j \in \partial\Omega_h \\
&-\triangle_h \overline{\beta}^{n+1}(x_i) + \mu^+ \overline{\beta}^{n+1}(x_i) = F(x_i, \overline{\alpha}^{n+1}(x_i)) + O^{n+1}(|h|^2), \quad i \in \Omega_h \\
&\overline{\beta}^{n+1}(x_j^{'}) = J[x_j^{'}, \overline{\beta}^{n+1}] + g_j^{(2)} + O^{n+1}(|h|), \quad j \in \partial\Omega_h,
\end{aligned}
\tag{4.38}
$$

where $O^n(|h|), O^n(|h|^2) \to 0$ as $|h| \to 0$. Let $\overline{w}_k^n = \overline{\alpha}^n(x_k) - \overline{\alpha}_k^n$ and $\overline{z}_k^n = \overline{\beta}^n(x_k) - \overline{\beta}_k^n$. Subtracting (4.22) from (4.38) and using mean value theorem,

$$
\begin{aligned}
&-\triangle_h \overline{w}_i^{n+1} + \mu \overline{w}_i^{n+1} = \overline{z}_i^n + O^{n+1}n(|h|^2), \quad i \in \Omega_h \\
&\overline{w}^{n+1}(x_j^{'}) = J[x_j^{'}, \overline{w}^{n+1}] + O^{n+1}(|h|), \quad j \in \partial\Omega_h \\
&-\triangle_h \overline{z}_i^{n+1} + \mu^+ \overline{z}_i^{n+1} = F_u(x_i, \xi_i^{n+1}) \overline{w}_i^{n+1} + O^{n+1}(|h|^2), \quad i \in \Omega_h \\
&\overline{z}^{n+1}(x_j^{'}) = J[x_j^{'}, \overline{z}^{n+1}] + O^{n+1}(|h|), \quad j \in \partial\Omega_h,
\end{aligned}
\tag{4.39}
$$

where ξ_i^{n+1} is an intermediate value between $\overline{\alpha}^{n+1}(x_i)$ and $\overline{\alpha}_i^{n+1}$. On account of (4.24), (4.39) can be rewritten in the vector form as

$$
\begin{aligned}
(\mathcal{M} + \mu J)\overline{W}^{n+1} &= \overline{Z}_1^n + \mathcal{O}^{n+1}(|h|) \\
(\mathcal{M} + \mu^+ J)\overline{Z}^{n+1} &= F_u(\xi)\overline{W}^{n+1} + \mathcal{O}^{n+1}(|h|),
\end{aligned}
\tag{4.40}
$$

where the matrices $\mathcal{M}, J$ and the vectors U, V, V_1 are as defined in Section 5. $F_u(\xi)$ is a diagonal matrix with the diagonal entries corresponding to the interior mesh points are $F_u(x_i, \xi_i^{n+1})$ and zero elsewhere. Since $\mathcal{M} + \mu J$ and $\mathcal{M} + \mu^+ J$ are inverse positive matrices there exists a positive constant K_1 such that

$$
\begin{aligned}
&\|\overline{W}^{n+1}\|_\infty \leq K_1 \left(\|\overline{Z}^n\|_\infty + \|\mathcal{O}^{n+1}(|h|)\|_\infty\right) \\
&\|\overline{Z}^{n+1}\|_\infty \leq K_1 \left(K_2 \|\overline{W}^{n+1}\|_\infty + \|\mathcal{O}^{n+1}(|h|)|\|_\infty\right),
\end{aligned}
\tag{4.41}
$$

where K_2 is the maximum of the elements $|F_u(x_i, \xi_i^{n+1})|$. Let $K_0 = K_1 \max\{1, K_2\}$. Then one can have

$$
\begin{aligned}
\|\overline{W}^{n+1}\|_\infty &\leq K_0^{3n+1} \|\widetilde{Z}\|_\infty + \mathcal{R}_1^{n+1}(|h|) \\
\|\overline{Z}^{n+1}\|_\infty &\leq K_0^{3n+3} \|\widetilde{Z}\|_\infty + \mathcal{R}_2^{n+1}(|h|),
\end{aligned}
\tag{4.42}
$$

where $R_1^{(n+1)}(|h|) = (K_0^{3n} + K_0^{3n-1})\|\mathcal{O}^1(|h|)\|_\infty + (K_0^{3n-2} + K_0^{3n-4})\|\mathcal{O}^2(|h|)\|_\infty + \cdots + (K_0^4 + K_0^2)\|\mathcal{O}^n(|h|)\|_\infty + K_0 \|\mathcal{O}^{n+1}(|h|)\|_\infty$ and $R_2^{(n+1)}(|h|) = (K_0^{3n+2} + K_0^{3n+1})\|\mathcal{O}^1(|h|)\|_\infty + (K_0^{3n} +$

$K_0^{3n-2})\|\mathcal{O}^2(|h|)\|_\infty + \cdots + (K_0^6 + K_0^4)\|\mathcal{O}^n(|h|)\|_\infty + (K_0^3 + K_0)\|\mathcal{O}^{n+1}(|h|)\|_\infty$. Let $n_0 \geq n^*$ be fixed. Since $R_1^{(n_0)}(|h|) \to 0$ and $R_2^{(n_0)}(|h|) \to 0$ as $|h| \to 0$, there exists a $\delta_1 > 0$ such that $R_i^{(n_0)}(|h|) \leq \frac{\epsilon}{12}$ whenever $|h| < \delta_1$ and $i = 1, 2$. Similarly using (4.35), one can get $\|\widetilde{W}\|_\infty + \|\widetilde{Z}\|_\infty \leq \frac{\epsilon}{6K^*}$ whenever $|h| < \delta_2$. Thus

$$\|\overline{W}^{n_0}\|_\infty + \|\overline{Z}^{n_0}\|_\infty \leq \frac{\epsilon}{3} \text{ whenever } |h| < \delta = \min\{\delta_1, \delta_2\}.$$

This leads to (4.37). Thus (4.36) holds and hence the theorem. The convergence of the minimal solution can also be obtained similarly. □

4.7. Algorithms

This section provides the algorithms used in the numerical implementations of the proposed scheme and the scheme in [**83**] for Picard's iterations. For a given matrix A, A_k stands for the matrix A whose entries corresponding to all the mesh points are included. When the entries corresponding to the boundary mesh points take their corresponding values and zero elsewhere, the matrix A is notated by A_j. Similarly, A_i stands for the matrix A in which the entries corresponding to the interior mesh points take their values and zero otherwise. The following algorithms are given in regard to the vector representations (4.24) given in Section 4.5 and that in Section 3 of [**83**] with $h_i = h$ for all i.

(i) **Proposed Scheme**:

Input: $\widehat{\alpha}_k, \widetilde{\alpha}_k, \widehat{\beta}_k, \widetilde{\beta}_k, \mathcal{M}_k + \mu J_i, \mathcal{M}_k + \mu^+ J_i$

while $\epsilon > 10^{-8}$ (say)

$\alpha_k^1 \leftarrow (\mathcal{M}_k + \mu J_i)^{-1}(\widehat{\beta}_i^0 + G_j^{(1)});$

$\overline{\alpha}_k^1 \leftarrow (\mathcal{M}_k + \mu J_i)^{-1}(\widetilde{\beta}_i^0 + G_j^{(1)});$

$\beta_k^1 \leftarrow (\mathcal{M}_k + \mu^+ J_i)^{-1}(F(\alpha_i^1) + G_j^{(2)});$

$\overline{\beta}_k^1 \leftarrow (\mathcal{M}_k + \mu^+ J_i)^{-1}(F(\overline{\alpha}_i^1) + G_j^{(2)});$

$\epsilon = \max(|\overline{\alpha}_k^1 - \alpha_k^1| + |\overline{\beta}_k^1 - \beta_k^1|);$

$\widehat{\alpha}_k \leftarrow \alpha_k^1;\ \widetilde{\alpha}_k \leftarrow \overline{\alpha}_k^1;$

$\widehat{\beta}_k \leftarrow \beta_k^1;\ \widetilde{\beta}_k \leftarrow \overline{\beta}_k^1;$

end

(ii) **Scheme in [83]:**

Input: $\widehat{\alpha}_k, \widetilde{\alpha}_k, \widehat{\beta}_k, \widetilde{\beta}_k, A_i + \mu I_i, A_i + \mu^+ I_i$

while $\epsilon > 10^{-8}$ (say)

Obtain $\alpha_j^1, \overline{\alpha}_j^1, \beta_j^1, \overline{\beta}_j^1$ using Composite Simpson's rule.

$\alpha_i^1 \leftarrow (A_i + \mu I_i)^{-1}(\frac{1}{h^2}\widehat{\alpha}_j^0 + \beta_i^0);$

$\overline{\alpha}_i^1 \leftarrow (A_i + \mu I_i)^{-1}(\frac{1}{h^2}\widetilde{\alpha}_j^0 + \widetilde{\beta}_i^0);$

$\beta_i^1 \leftarrow (A_i + \mu^+ I_i)^{-1}(\frac{1}{h^2}\widehat{\beta}_j^0 + F(\widehat{\alpha}_i));$

$\overline{\beta}_i^1 \leftarrow (A_i + \mu^+ I_i)^{-1}(\frac{1}{h^2}\widetilde{\beta}_j^0 + F(\widetilde{\alpha}_i));$

$\alpha_k^1 \leftarrow (\alpha_i^1; \alpha_j^1)$; $\overline{\alpha}_k^1 \leftarrow (\overline{\alpha}_i^1; \overline{\alpha}_j^1)$;

$\beta_k^1 \leftarrow (\beta_i^1; \beta_j^1)$; $\overline{\beta}_k^1 \leftarrow (\overline{\beta}_i^1; \overline{\beta}_j^1)$;

$\epsilon = \max(|\overline{\alpha}_k^1 - \alpha_k^1| + |\overline{\beta}_k^1 - \beta_k^1|);$

$\widehat{\alpha}_k \leftarrow \alpha_k^1$; $\widetilde{\alpha}_k \leftarrow \overline{\alpha}_k^1$;

$\widehat{\beta}_k \leftarrow \beta_k^1$; $\widetilde{\beta}_k \leftarrow \overline{\beta}_k^1$;

end

4.8. Numerical Examples

In this section, an efficient numerical illustration is done in comparison with all the examples discussed in [**83**]. The composite Simpson's rule is used to approximate the integral terms in the equations. All the numerical examples are performed with MATLAB R2010b. This section demonstrates the efficiency of the proposed iterative scheme in comparison with the iterative scheme in [**83**] by varying the initial guess, mesh size and stopping criterion. Throughout this section, N denotes that number of partitions and the stopping criterion chosen is $|\overline{\alpha}_k^n - \alpha_k^n| + |\overline{\beta}_k^n - \beta_k^n| \leq \epsilon$.

Example 4.8.1.

Consider the following one-dimensional fourth order nonlinear elliptic problem

$$
\begin{aligned}
& u'''' - 3u' + 2u = \tfrac{u}{1+u} + q(x), \quad 0 < x < 1, \\
& u(0) = \int_0^1 x^2 u(x)\mathrm{d}x + g^{(1)}(0), \quad u(1) = \int_0^1 x^2 u(x)\mathrm{d}x + g^{(1)}(1) \\
& u''(0) = \int_0^1 x^2 u''(x)\mathrm{d}x - g^{(0)}(0) \quad u''(1) = \int_0^1 x^2 u''(x)\mathrm{d}x - g^{(0)}(1),
\end{aligned} \tag{4.43}
$$

where $q(x) = 10 + 2x(1-x) - \frac{2+x(1-x)}{3+x(1-x)}$, $g^{(1)}(x^{'}) = \frac{77}{60}$ and $g^{(0)}(x^{'}) = \frac{77}{60} + \frac{4}{3}$ for $x^{'} = 0, 1$. Clearly $\widehat{\alpha}_i = 0$ and $\widetilde{\alpha}_i = 6$ are lower and upper solutions of (4.43). Hence (4.43) has a unique solution by Theorem 4.3.1. The corresponding discretized coupled system for (4.43) is given by

$$
\begin{aligned}
&-\triangle_h u_i + u_i = v_i,\ -\triangle_h v_i + 2v_i = \tfrac{u_1}{1+u_1} + q_i, \quad 0 < x_i < 1 \\
&u_j = J[x_j^{'}, u] + g_j^{(1)}, \quad v_j = J[x_j^{'}, v] + g_j^{(2)}, \quad x_j^{'} = 0, 1,
\end{aligned} \tag{4.44}
$$

where $g_j^{(2)} = g_j^{(0)} + g_j^{(1)}$. Clearly, $(\widehat{\alpha}_i, \widehat{\beta}_i) = (0, 0)$ and $(\widetilde{\alpha}_i, \widetilde{\beta}_i) = (6, 6)$ serve as the lower and upper solutions of (4.44). Hence by Theorem 4.4.1, $\{(\alpha_k^n, \beta_k^n)\}$ and $\{(\overline{\alpha}_k^n, \overline{\beta}_k^n)\}$ converge to the unique solution $(u_k^*, v_k^*) = (2 + x_k(1-x_k), 4 + x_k(1-x_k))$. Figure 4.1 and Figure 4.2 illustrate the monotone convergence of both the minimal and maximal sequences to the unique solution at $x_i = 0.5$. Table 4.1 provides the numerical error, order and number of iterations for the stopping criterion $\epsilon = 10^{-8}$ where $\text{order}(h) = \log_2\left(\frac{\text{error}(h)}{\text{error}(\frac{h}{2})}\right)$. For a fixed number of partitions $N = 100$, Table 4.2 provides the number of iterations for the proposed and the scheme in [**83**] for various stopping criteria ϵ.

Example 4.8.2.

Consider the following two-dimensional problem defined on the rectangular domain $\Omega = \{(x, y) :\ 0 < x < 1,\ 0 < y < 2\}$.

$$
\begin{aligned}
&\triangle^2 u - 10\triangle u + u = q(x, y) - u^4, \quad (x, y) \in \Omega \\
&u(x^{'}, y^{'}) = \tfrac{1}{2}\int_0^1\int_0^2 xyu(x, y)\mathrm{d}x\mathrm{d}y + g^{(1)}(x^{'}, y^{'}), \quad (x^{'}, y^{'}) \in \partial\Omega \\
&(\triangle u)(x^{'}, y^{'}) = \tfrac{1}{2}\int_0^1\int_0^2 xy(\triangle u)(x, y)\mathrm{d}x\mathrm{d}y - g^{(0)}(x^{'}, y^{'}), \quad (x^{'}, y^{'}) \in \partial\Omega,
\end{aligned} \tag{4.45}
$$

where $\lambda_0 = \frac{5}{4}\pi^2$, $\phi(x, y) = \sin(\pi x)\sin(\frac{\pi y}{2})$, $\alpha = (\lambda_0^2 + 10\lambda_0 + 1)^{-1}$, $q(x, y) = (1 - \alpha\phi(x, y))^4 + 1 - \phi(x, y)$, $g^{(1)}(x^{'}, y^{'}) = \frac{1}{2} + 2\frac{\alpha}{\pi^2}$ and $g^{(0)}(x^{'}, y^{'}) = \frac{5}{2}\alpha$. Clearly $\widehat{\alpha}_{i,j} = 0$ and $\widetilde{\alpha}_{i,j} = 2$ are lower and upper solutions for (4.45) and thus by Theorem 4.3.1, (4.45) has a unique solution. The corresponding coupled system for (4.45) is given by

$$
\begin{aligned}
&-\triangle_h u_{i,j} + u_{i,j} = v_{i,j},\ -\triangle_h v_{i,j} + 9v_{i,j} = 8u_{i,j} - u_{i,j}^4 + q_{i,j}, \quad (i, j) \in \Omega_h, \\
&u_{i,j} = J[x_i^{'}, y_j^{'}, u] + g_{i,j}^{(1)}, \quad v_{i,j} = J[x_i^{'}, y_j^{'}, v] + g_{i,j}^{(2)}, (x_i^{'}, y_j^{'}) \in \partial\Omega_h,
\end{aligned} \tag{4.46}
$$

where $g_{i,j}^{(2)} = g_{i,j}^{(0)} + g_{i,j}^{(1)}$. Clearly, $(\widehat{\alpha}_{i,j}, \widehat{\beta}_{i,j}) = (0, 0)$ and $(\widetilde{\alpha}_{i,j}, \widetilde{\beta}_{i,j}) = (2, 2)$ serve as the lower and upper solutions of (4.46). Thus $\{(\alpha_k^n, \beta_k^n)\}$ and $\{(\overline{\alpha}_k^n, \overline{\beta}_k^n)\}$ converge to the unique solution $(u^*_{i,j}, v^*_{i,j}) = (1 - \alpha\phi(x_i, y_j), 1 - \alpha(1+\lambda_0)\phi(x_i, y_j))$ due to Theorem 4.4.1. For the stopping criterion $\epsilon = 10^{-8}$, Table 4.3 provides the numerical error, order and number of iterations for the proposed scheme with various mesh sizes. For a fixed number of partitions $N = 100$, the number of iterations for the proposed scheme and the scheme in [**83**] is given in Table 4.4. The basic computation algorithms (Picard, Gauss-Seidel and Jacobi) studied in [**83**] are compared with the algorithms proposed in Section 4.5. This comparison study for the stopping criterion $\epsilon = 10^{-4}$ and $g^{(1)}_{i,j} = g^{(2)}_{i,j} = 0$ are provided in Table 4.5 for various mesh sizes. For $\epsilon = 10^{-8}$, Table 4.6 presents a comparison study between the proposed scheme and the scheme in [**83**] for various initial guesses. For $N = 50$, it can be observed that the scheme in [**83**] fails to converge when the initial guess is chosen away from the exact solution whereas the proposed scheme converges even for an initial guess chosen comparatively much away from the exact solution. Hence this benchmark problem guarantees that the proposed iterative scheme not only accelerates the scheme in [**83**] but also provides a greater flexibility in choosing the initial guess.

Example 4.8.3.

Consider the following two-dimensional problem defined on the rectangular domain $\Omega = \{(x, y) : \ 0 < x < 1, \ 0 < y < 2\}$.

$$\begin{aligned} &\triangle^2 u - 5\triangle u + 2u = (1-u)e^{-u} + q(x,y), \quad (x,y) \in \Omega \\ &u(x', y') = \int_0^1 \int_0^2 x^2 y u(x,y) \mathrm{d}x\mathrm{d}y, \ (\triangle u)(x', y') = -g^{(0)}(x', y'), \ (x', y') \in \partial\Omega, \end{aligned} \tag{4.47}$$

where $q(x, y) = 8 + 10x(1-x) + 10y(2-y) + 2w - (1-w)e^{-w}$, $g^{(0)}(x', y') = 2x'(1 - x') + 2y'(2-y')$ and $w(x, y) = \frac{1}{5} + xy(1-x)(2-y)$. Clearly, $\widehat{\alpha}_{i,j} = 0$ and $\widetilde{\alpha}_{i,j} = 8$ are lower and upper solution for (4.47) and Theorem 4.3.1 assures the existence of a unique solutions of (4.47). The corresponding coupled system for (4.45) is given by

$$\begin{aligned} &-\triangle_h u_{i,j} + u_{i,j} = v_{i,j}, \ -\triangle_h v_{i,j} + 4v_{i,j} = 2u_{i,j} + (1-u_{i,j})e^{-u_{i,j}} + q_{i,j}; \ (i,j) \in \Omega_h, \\ &u_{i,j} = J[x'_i, y'_j, u], \quad v_{i,j} = J[x'_i, y'_j, u] + g^{(0)}_{i,j}; \ (x'_i, y'_j) \in \partial\Omega_h. \end{aligned} \tag{4.48}$$

Clearly, $(\widehat{\alpha}_{i,j}, \widehat{\beta}_{i,j}) = (0, 0)$ and $(\widetilde{\alpha}_{i,j}, \widetilde{\beta}_{i,j}) = (8, 8)$ serve as the lower and upper solutions of (4.48). Hence by Theorem 4.4.1, $\{(\alpha^n_k, \beta^n_k)\}$ and $\{(\overline{\alpha}^n_k, \overline{\beta}^n_k)\}$ converge to the unique solution $(u^*_{i,j}, v^*_{i,j}) = (\frac{1}{5} + x_i y_j (1-x_i)(2-y_j), \frac{1}{5} + x_i y_j (1-x_i)(2-y_j) + 2x_i(1-x_i) + 2y_j(2-y_j))$.

Table 4.7 provides the numerical error, order and number of iterations for the stopping criterion $\epsilon = 10^{-8}$. The number of iterations for the proposed scheme and the scheme in [**83**] are given in Table 4.8 for a fixed number of partitions $N = 100$.

4.9. Conclusion

An efficient accelerated iterative scheme is obtained to solve a class of fourth order elliptic equations with nonlocal boundary conditions. The monotone property as well as the convergence of the proposed iterative scheme are obtained for both the continuous and discretized cases. The efficiency of the proposed numerical scheme is illustrated through the numerical implementation. The higher laxity in choosing the initial guess is also a significant contribution of the proposed iterative scheme.

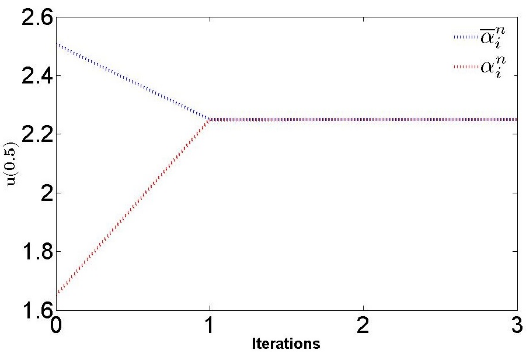

FIGURE 4.1. Monotone convergence of $\{\alpha_i^n, \overline{\alpha}_i^n\}$

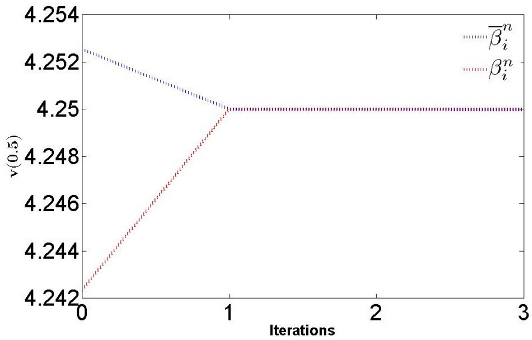

FIGURE 4.2. Monotone convergence of $\{\beta_i^n, \overline{\beta}_i^n\}$

TABLE 4.1. Numerical results for Example 4.8.1

N	8	16	32	64	128
Error	4.8498×10^{-5}	3.0315×10^{-6}	1.8979×10^{-7}	1.3081×10^{-8}	2.2385×10^{-9}
Order	3.9998	3.9976	3.8589	2.5469	—
Iterations	4	4	4	4	4

Table 4.2. Comparison of number of iterations for Example 4.8.1

ϵ	10^{-4}		10^{-8}		10^{-12}	
Scheme	Scheme in [**83**]	Proposed Scheme	Scheme in [**83**]	Proposed Scheme	Scheme in [**83**]	Proposed Scheme
Iterations	11	3	19	4	27	6

Table 4.3. Numerical results for Example 4.8.2

N	8	16	32	64	128
Error	1.0788×10^{-4}	2.6666×10^{-5}	6.6508×10^{-6}	1.6646×10^{-6}	4.1913×10^{-7}
Order	2.0164	2.0034	1.9984	1.9897	–
Iterations	6	6	6	6	6

Table 4.4. Comparison of number of iterations for Example 4.8.2

ϵ	10^{-5}		10^{-7}		10^{-9}	
Scheme	Scheme in [**83**]	Proposed Scheme	Scheme in [**83**]	Proposed Scheme	Scheme in [**83**]	Proposed Scheme
Iterations	20	4	27	6	34	7

Table 4.5. Comparison of different computational algorithms for Example 4.8.2

N	10		20		40	
Scheme	Scheme in [**83**]	Proposed Scheme	Scheme in [**83**]	Proposed Scheme	Scheme in [**83**]	Proposed Scheme
Picard	18	4	18	4	18	4
Gauss-Seidel	185	127	726	512	2897	2063
Jacobi	361	219	1449	882	5800	3538

TABLE 4.6. Comparison of number of iterations for Example 4.8.2

$(\widetilde{\alpha}_{i,j}, \widetilde{\beta}_{i,j})$	$(6, 6)$		$(7, 7)$		$(35, 35)$		$(36, 36)$	
Scheme	Scheme in [**83**]	Proposed Scheme	Scheme in [**83**]	Proposed Scheme	Scheme in [**83**]	Proposed Scheme	Scheme in [**83**]	Proposed Scheme
Iterations	35	6	–	7	–	10	–	–

TABLE 4.7. Numerical results for Example 4.8.3

N	8	16	32	64	128
Error	1.2798×10^{-4}	7.9986×10^{-6}	5.0009×10^{-7}	3.1453×10^{-8}	2.3055×10^{-9}
Order	4.0000	3.9995	3.9909	3.7700	–
Iterations	9	9	9	9	9

TABLE 4.8. Comparison of number of iterations for Example 4.8.3

ϵ	10^{-5}		10^{-7}		10^{-9}	
Scheme	Scheme in [**83**]	Proposed Scheme	Scheme in [**83**]	Proposed Scheme	Scheme in [**83**]	Proposed Scheme
Iterations	37	6	49	8	60	10

CHAPTER 5

A CLASS OF NONLINEAR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS OF VOLTERRA TYPE

This chapter[1] supplies a finite difference method based monotone iterative technique to solve an important class of Volterra type parabolic partial integro-differential equations.

5.1. Introduction

A wide variety of physical and biological problems arising from nuclear reactor models [**6, 72, 66, 64, 88, 89**] and population models [**39**] are dealt by using nonlinear parabolic integro-differential equations of the form

$$\begin{aligned}
&\theta\tfrac{\partial z}{\partial t}-\tfrac{\partial^2 z}{\partial x^2}-az+bz^2+czz_1=q_1(x,t),\ (x,t)\in\omega\times(0,T]\\
&z(x,t)=h_1(x,t),\ (x,t)\in\partial\omega\times(0,T]\\
&z(x,0)=\psi_1(x),\ x\in\overline{\omega},
\end{aligned}\tag{5.1}$$

where $\theta, a\in\mathbb{R}^+,\ b,c\in\mathbb{R}^+\cup\{0\},\ z_1$ stands for $\int_0^t\kappa(t-s)z(x,s)\mathrm{d}s,\ \omega$ is a connected bounded domain in $\mathbb{R}^n$ $(n=1,2,\cdots)$, $h_1,\ \psi_1$ are smooth functions and κ is a non-negative continuous function in their domains.

Recently in [**13**] and [**76**],finite difference method based on monotone iterations are studied for a class of nonlinear parabolic integro-differential equation of Volterra and Fredholm types respectively. More specifically in [**13**], existence, uniqueness, convergence analysis in terms of stopping criteria together with the monotone property of the iterative scheme for the discretized problem are obtained for the governing equation u_t-Lu+

[1]This chapter forms the paper by L.A. Sunny and V. A. Vijesh which is under preparation.

$f(x,t,u) + \int_0^t g_0(x,t,s,u(x,s))\mathrm{d}s = 0$ with Dirichlet initial and boundary conditions and Lu being $\sum_{\alpha=1} \kappa \frac{\partial}{\partial x_\alpha}\left(D(x,t)\frac{\partial u}{\partial x_\alpha}\right) + v_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}$.

By utilizing the change of variable $u(x,t) = \exp(\lambda t)z(x,t)$ where λ is a constant, (5.1) can be rewritten as follows for a broader class of functions:

$$
\begin{aligned}
&\tfrac{\partial u}{\partial t} - \tfrac{1}{\theta}\tfrac{\partial^2 u}{\partial x^2} + f(u,v) = q(x,t),\ (x,t) \in \omega \times (0,T] \\
&u(x,t) = h(x,t),\ (x,t) \in \partial\omega \times (0,T] \\
&u(x,0) = \psi(x),\ x \in \overline{\omega},
\end{aligned} \tag{5.2}
$$

where $v(x,t)$ stands for $\int_0^t \exp(\lambda s)\kappa(t-s)u(x,s)\mathrm{d}s$, $f(u,v) = a\frac{\theta\lambda-1}{\theta}u(x,t) + \frac{b}{\theta}\exp(\lambda t)u^2(x,t) + \frac{c}{\theta}u(x,t)v(x,t)$, $h(x,t) = \exp(-\lambda t)h_1(x,t)$, $\psi(x) = \psi_1(x)$ and $q(x,t) = \exp(-\lambda t)q_1(x,t)$. Note that one can choose λ so large that all the first and second derivatives of f with respect to u and v are non-negative. The monotone property, convergence analysis an error estimate in terms of stopping criteria are derived for the more general nonlinear integro-differential equation of Volterra type.

This chapter is organised as follows. In Section 5.2, the basic definitions and results that are used in the subsequent sections are provided. A detailed convergence analysis of the nonlinear difference scheme is rendered in Section 5.3. The numerical implementation is done in Section 5.4 to illustrate the efficacy of the proposed numerical scheme.

5.2. Construction of the Proposed Scheme

The discretization as well as the construction of minimal and maximal sequences are presented in this section for the following Volterra type partial integro-differential equation:

$$
\begin{aligned}
&\tfrac{\partial u}{\partial t} - \tfrac{1}{\theta}\tfrac{\partial^2 u}{\partial x^2} + f(u,v) = q(x,t),\ (x,t) \in \omega \times (0,T] \\
&u(x,t) = h(x,t),\ (x,t) \in \partial\omega \times (0,T]; \quad u(x,0) = \psi(x),\ x \in \overline{\omega},
\end{aligned}
$$

where $v(x,t) = \int_0^t \kappa(t,s)u(x,s)\mathrm{d}s$ and the functions f, q, h, ψ are smooth and κ is non-negative continuous in their respective domains. Let $\overline{\omega}^h$ and $\overline{\omega}^\tau$ be the corresponding meshes for the space and time domains respectively and h and τ_k denote the step sizes in x and t directions respectively with $t_0 = 0$. Applying backward and central difference approximations for time and space respectively in (5.2), one can get the following:

$$(5.3)\qquad \begin{aligned}&\mathcal{L}U(p,t_k)+f(p,t_k,U,V)-\tau_k^{-1}U(p,t_{k-1})=Q(p,t_k),\ (p,t_k)\in\omega^h\times(\omega^\tau\setminus\{0\})\\ &U(p,t_k)=h(p,t_k),\ (p,t_k)\in\partial\omega^h\times(\omega^\tau\setminus\{0\})\\ &U(p,0)=\psi(p),\ p\in\overline{\omega}^h,\end{aligned}$$

where $\partial\omega^h$ denotes the boundary of $\overline{\omega}^h$

$$\begin{aligned}\mathcal{L}U(p,t_k) &= \frac{1}{\theta}\mathcal{L}^hU(p,t_k)+\tau_k^{-1}U(p,t_k),\\ \mathcal{L}^hU(p,t_k) &= d(p,t_k)U(p,t_k)-\sum_{l'\in\sigma'(l)}a(l',t_k)U(l',t_k),\end{aligned}$$

where $\sigma'(l)=\sigma(l)\setminus\{l\},\sigma(l)$ is a stencil of the scheme at an interior mesh point $l\in\omega^h$. Assume that $\overline{\omega}^h$ is connected. Throughout this chapter, the coefficients of $\mathcal{L}^h$ satisfy the following conditions similar to (3) in [**13**]:

$$(5.4)\qquad \begin{aligned}&d(p,t_k)\geq 0,\ a(l',t_k)\geq 0,\ l'\in\sigma'(l);\\ &d(p,t_k)-\textstyle\sum_{l'\in\sigma'(l)}a(l',t_k)\geq 0,\ (p,t_k)\in\omega^h\times(\omega^\tau\setminus\{0\}).\end{aligned}$$

The integral in (5.2) can be approximated as $V(p,t_k)\approx\sum_{i=1}^k\tau_i\kappa(t_k,t_i)U(p,t_i)$ using Riemann sum. At each iteration, one has to solve a linear problem of the following type:

$$(5.5)\qquad \begin{aligned}&(\mathcal{L}+c)W(p,t_k)=\Psi(p,t_k,U),\quad p\in\omega^h,\\ &c(p,t_k)\geq 0,\quad W(p,t_k)=h(p,t_k),\quad p\in\partial\omega^h.\end{aligned}$$

The following lemma is a useful tool to prove the monotone property of the proposed iterative scheme.

Lemma 5.2.1. [**90**, P.261] *Let the coefficients of the difference operator $\mathcal{L}^h$ satisfy* (5.4) *and the mesh $\overline{\omega}^h$ be connected.*

(i) *If a mesh function $W(p,t_k)$ satisfies the conditions*

$$\begin{aligned}&(\mathcal{L}+c)W(p,t_k)\geq 0(\leq 0),\ p\in\omega^h\\ &W(p,t_k)\geq 0(\leq 0),\ p\in\partial\omega^h,\end{aligned}$$

then $W(p,t_k)\geq 0(\leq 0)$ in $\overline{\omega}^h$.

(ii) *The following estimate to the solution to* (5.5) *hold true.*

$$\|W(.,t_k)\|_{\overline{\omega}^h} \leq \max\left\{\|h(.,t_k)\|_{\partial\omega^h}, \max_{p\in\omega^h}\frac{|\Psi(p,t_k)|}{c(p,t_k)+\tau_k^{-1}}\right\}, \tag{5.6}$$

where $\|W(.,t_k)\|_{\overline{\omega}^h} = \max\limits_{p\in\overline{\omega}^h}|W(p,t_k)|$ *and* $\|h(.,t_k)\|_{\partial\omega^h} = \max\limits_{p\in\partial\omega^h}|h(p,t_k)|$.

Definition 5.2.1. *The mesh functions* $\widehat{U}(p,t_k)$ *and* $\widetilde{U}(p,t_k)$ *are called coupled lower and upper solutions of* (5.3) *if they satisfy* $\widehat{U}(p,t_k) \leq \widetilde{U}(p,t_k)$, $(p,t_k) \in \overline{\omega}^h \times \overline{\omega}^\tau$,

$$\begin{cases} \mathcal{L}\widehat{U}(p,t_k) + f(p,t_k,\widehat{U},\widetilde{V}) - \tau_k^{-1}\widehat{U}(p,t_{k-1}) \leq Q(p,t_k), \ (p,t_k) \in \omega^h \times (\omega^\tau \setminus \{0\}) \\ \widehat{U}(p,t_k) \leq h(p,t_k), \ p \in \partial\omega^h; \quad \widehat{U}(p,0) \leq \psi(p), \ p \in \overline{\omega}^h \end{cases}$$

and

$$\begin{cases} \mathcal{L}\widetilde{U}(p,t_k) + f(p,t_k,\widetilde{U},\widehat{V}) - \tau_k^{-1}\widetilde{U}(p,t_{k-1}) \geq Q(p,t_k), \ (p,t_k) \in \omega^h \times (\omega^\tau \setminus \{0\}) \\ \widetilde{U}(p,t_k) \geq h(p,t_k), \ p \in \partial\omega^h; \quad \widetilde{U}(p,0) \geq \psi(p), \ p \in \overline{\omega}^h. \end{cases}$$

For a fixed time step t_k and a given pair of ordered lower and upper solutions of (5.3), the sector is defined as

$$< \widehat{U}(p,t_k), \widetilde{U}(p,t_k) >= \{U(p,t_k) : \widehat{U}(p,t_k) \leq U(p,t_k) \leq \widetilde{U}(p,t_k), \ p \in \overline{\omega}^h\}.$$

Throughout this chapter, assume that f and the time step τ_k satisfy the following hypotheses:

($\mathbf{H}_1$) All the first and second derivatives of f with respect to u and v exist as a nonnegative continuous function in $\overline{\omega} \times [0,T]$.

($\mathbf{H}_2$) The k^{th} time step satisfies

$$\begin{aligned} &\tau_k < \sqrt{\tfrac{1}{\gamma M}}, \quad \gamma = \max_{(t,s)\in[0,T]\times[0,T]} \kappa(t,s) \\ &0 \leq \tfrac{\partial f}{\partial u}(p,t_k,U,V) \leq \mu, \quad \text{on} < \widehat{U}(p,t_k), \widetilde{U}(p,t_k) > \\ &0 \leq \tfrac{\partial f}{\partial v}(p,t_k,U,V) \leq M \quad \text{on} < \widehat{U}(p,t_k), \widetilde{U}(p,t_k) >, \end{aligned}$$

where μ and M are positive constants. Using $\widehat{U}(p,t_k)$ and $\widetilde{U}(p,t_k)$ as the initial iterations, motivated from [**76**], two sequences namely minimal sequence $\{U_{-1}^{n+1}(p,t_k)\}$ and maximal sequence $\{U_1^{n+1}(p,t_k)\}$ respectively can be generated as follows:

$$\begin{aligned}(\mathcal{L}+\mu)\,U_{-1}^{n+1}(p,t_k) &= \mu U_{-1}^{n}(p,t_k) - f(p,t_k,U_{-1}^{n},V_1^{n}) + \tau_k^{-1}U_{-1}(p,t_{k-1})\\ &\qquad +Q(p,t_k), \quad (p,t_k)\in\omega^h\times(\omega^\tau\setminus\{0\})\\ U_{-1}^{n+1}(p,t_k) &= h(p,t_k), \quad p\in\partial\omega^h, \quad U_{-1}(p,t_0)=\psi(p), \quad p\in\overline{\omega}^h\end{aligned} \tag{5.7}$$

$$\begin{aligned}(\mathcal{L}+\mu)\,U_{1}^{n+1}(p,t_k) &= \mu U_{1}^{n}(p,t_k) - f(p,t_k,U_{1}^{n},V_{-1}^{n}) + \tau_k^{-1}U_{1}(p,t_{k-1})\\ &\qquad +Q(p,t_k), \quad (p,t_k)\in\omega^h\times(\omega^\tau\setminus\{0\})\\ U_{1}^{n+1}(p,t_k) &= h(p,t_k), \quad p\in\partial\omega^h, \quad U_{1}(p,t_0)=\psi(p), \quad p\in\overline{\omega}^h,\end{aligned} \tag{5.8}$$

where $U_{-1}(p,t_k) = U_{-1}^{\underline{n}_k}(p,t_k)$ and $U_1(p,t_k) = U_1^{\overline{n}_k}(p,t_k)$ are the approximations of the exact solution on time step t_k and $n = 0,1,\cdots$. Clearly the proposed sequences in (5.7) and (5.8) are well defined.

5.3. Convergence Analysis

The monotone property of the sequences defined by (5.7) and (5.8) along with the existence and uniqueness of the solution for (5.3) are deduced in this section. The following theorem renders the monotone property of the proposed iterative schemes and their convergence to the unique solution of (5.3).

Theorem 5.3.1. *Let $\widehat{U}(p,t_k)$ and $\widetilde{U}(p,t_k)$ be a pair of ordered lower and upper solutions of* (5.3) *respectively. Then the minimal sequence $\{U_{-1}^{n+1}(p,t_k)\}$ and the maximal sequence $\{U_1^{n+1}(p,t_k)\}$ converge monotonically to the unique solution of* (5.3) *in $<\widehat{U}(p,t_k),\widetilde{U}(p,t_k)>$ and satisfy*

$$\widehat{U}(p,t_k)\le U_{-1}^{n}(p,t_k)\le U_{-1}^{n+1}(p,t_k)\le U_1^{n+1}(p,t_k)\le U_1^{n}(p,t_k)\le \widetilde{U}(p,t_k) \tag{5.9}$$

for $p\in\overline{\omega}^h$ and $n\in\mathbb{N}$.

Proof. First the monotone property of (5.7) and (5.8) at the time step t_1 is proved. For $n=0$ and $k=1$ in (5.7), one can get

$$(\mathcal{L}+\mu)\,(U_{-1}^{1}-\widehat{U})(p,t_1)\ge -\mathcal{L}\widehat{U}(p,t_1) - f(p,t_1,\widehat{U},\widetilde{V}) + \tau_1^{-1}\widehat{U}(p,t_0) + Q(p,t_1)\ge 0,$$

$p \in \omega^h$ with $(U_{-1}^1 - \widehat{U})(p,t_1) \geq 0,\ p \in \partial\omega^h$. Thus Lemma 5.2.1 leads to $\widehat{U}(p,t_1) \leq U_{-1}^1(p,t_1),\ p \in \overline{\omega}^h$. Similarly from (5.2.1) and (5.8) it can be concluded that $U_1^1(p,t_1) \leq \widetilde{U}(p,t_1),\ p \in \overline{\omega}^h$. Let $W = U_1^1 - U_{-1}^1$. Now

$$\begin{aligned}
(\mathcal{L}+\mu)\, W(p,t_1) &= \mu(\widetilde{U} - \widehat{U})(p,t_1) + f(p,t_1,\widehat{U},\widetilde{V}) - f(p,t_1,\widetilde{U},\widehat{V}) \\
&= \left(\mu - \tfrac{\partial f}{\partial u}(p,t_1,\hat{u},\hat{v})\right)(\widetilde{U} - \widehat{U})(p,t_1) + \tfrac{\partial f}{\partial v}(p,t_1,\hat{u},\hat{v})(\widetilde{V} - \widehat{V})(p,t_1) \\
&\qquad [(\widehat{U},\widehat{V}) \leq (\hat{u},\hat{v}) \leq (\widetilde{U},\widetilde{V})] \\
(\mathcal{L}+\mu)\, W(p,t_1) &\geq \tau_1\kappa(t_1,t_1)\tfrac{\partial f}{\partial v}(p,t_1,\hat{u},\hat{v})(\widetilde{U} - \widehat{U})(p,t_1).
\end{aligned}$$

Thus$(\mathcal{L}+\mu)\, W(p,t_1) \geq 0;\ p \in \omega^h;\ W(p,t_1) = h(p,t_1) - h(p,t_1) = 0,\ p \in \partial\omega^h$ and by Lemma 5.2.1, $W(p,t_1) \geq 0$ in $\overline{\omega}^h$. Consequently $U_{-1}^1(p,t_1) \leq U_1^1(p,t_1)$. Hence

$$\widehat{U}(p,t_1) \leq U_{-1}^1(p,t_1) \leq U_1^1(p,t_1) \leq \widetilde{U}(p,t_1), \quad p \in \overline{w}^h. \tag{5.10}$$

Assume that for some $n \in \mathbb{N}$,

$$\widehat{U}(p,t_1) \leq U_{-1}^{n-1}(p,t_1) \leq U_{-1}^n(p,t_1) \leq U_1^n(p,t_1) \leq U_1^{n-1}(p,t_1) \leq \widetilde{U}(p,t_1),\ p \in \overline{w}^h.$$

Let $W = U_{-1}^{n+1} - U_{-1}^n$. Now

$$\begin{aligned}
(\mathcal{L}+\mu)\, W(p,t_1) &= -\mathcal{L}U_{-1}^n(p,t_1) - f(p,t_1,U_{-1}^n,V_1^n) + \tau_1^{-1}U_{-1}(p,t_0) + Q(p,t_1) \\
&= \mu(U_{-1}^n - U_{-1}^{n-1})(p,t_1) + f(p,t_1,U_{-1}^{n-1},V_1^{n-1}) - f(p,t_1,U_{-1}^n,V_1^n) \\
&= \left(\mu - \tfrac{\partial f}{\partial u}(p,t_1,\hat{u},\hat{v})\right)(U_{-1}^n - U_{-1}^{n-1})(p,t_1) + \tfrac{\partial f}{\partial v}(p,t_1,\hat{u},\hat{v})(V_1^{n-1} - V_1^n)(p,t_1) \\
&\qquad [(U_{-1}^{n-1},V_1^n) \leq (\hat{u},\hat{v}) \leq (U_{-1}^n,V_1^{n-1})] \\
(\mathcal{L}+\mu)\, W(p,t_1) &\geq \tau_1\kappa(t_1,t_1)\tfrac{\partial f}{\partial v}(p,t_1,\hat{u},\hat{v})(U_1^{n-1} - U_1^n)(p,t_1).
\end{aligned}$$

Thus $(\mathcal{L}+\mu)\, W(p,t_1) \geq 0;\ p \in \omega^h;\ W(p,t_1) = 0,\ p \in \partial\omega^h$ and by Lemma 5.2.1 one can obtain $U_{-1}^n(p,t_1) \leq U_{-1}^{n+1}(p,t_1),\ p \in \overline{\omega}^h$. Similarly from (5.8) it can be concluded that $U_1^{n+1}(p,t_1) \leq U_1^n(p,t_1),\ p \in \overline{\omega}^h$. Now let $W = U_1^{n+1} - U_{-1}^{n+1}$.

$$\begin{aligned}
(\mathcal{L}+\mu)\, W(p,t_1) &= \mu(U_1^n - U_{-1}^n)(p,t_1) + f(p,t_1,U_{-1}^n,V_1^n) - f(p,t_1,U_1^n,V_{-1}^n) \\
&= \left(\mu - \tfrac{\partial f}{\partial u}(p,t_1,\hat{u},\hat{v})\right)(U_1^n - U_{-1}^n)(p,t_1) + \tfrac{\partial f}{\partial v}(p,t_1,\hat{u},\hat{v})(V_1^n - V_{-1}^n)(p,t_1) \\
&\qquad [(U_{-1}^n,V_{-1}^n) \leq (\hat{u},\hat{v}) \leq (U_1^n,V_1^n)] \\
(\mathcal{L}+\mu)\, W(p,t_1) &\geq \tau_1\kappa(t_1,t_1)\tfrac{\partial f}{\partial v}(p,t_1,\hat{u},\hat{v})(U_1^n - U_{-1}^n)(p,t_1).
\end{aligned}$$

Thus $(\mathcal{L}+\mu)\,W(p,t_1) \geq 0$; $p \in \omega^h$; $W(p,t_1) = 0$, $p \in \partial\omega^h$ and by Lemma 5.2.1, $W(p,t_1) \geq 0$ in $\overline{\omega}^h$. Hence $U_{-1}^{n+1}(p,t_1) \leq U_1^{n+1}(p,t_1)$. Thus

$$U_{-1}^{n}(p,t_1) \leq U_{-1}^{n+1}(p,t_1) \leq U_1^{n+1}(p,t_1) \leq U_1^{n}(p,t_1), \quad p \in \overline{\omega}^h.$$

Consequently, the maximal sequence $\{U_1^n(p,t_1)\}$ and the minimal sequence $\{U_{-1}^n(p,t_1)\}$ are monotonically decreasing and increasing sequences respectively that satisfy (5.9). Let $\overline{n}_k$ and $\underline{n}_k$ respectively denote the number of iterative steps applied for the time step t_k for the maximal and minimal sequences. Thus from (5.9) for $k=1$,

$$\widehat{U}(p,t_1) \leq U_{-1}^{\underline{n}_1}(p,t_1) \leq U_1^{\overline{n}_1}(p,t_1) \leq \widetilde{U}(p,t_1), \quad p \in \overline{\omega}^h. \tag{5.11}$$

Due to (5.11) and Definition 5.2.1, one can have

$$\begin{aligned} &\mathcal{L}\widehat{U}(p,t_2) + f(p,t_2,\widehat{U},\widetilde{V}) - \tau_2^{-1}U_{-1}^{\underline{n}_1}(p,t_1) \leq Q(p,t_2), \quad p \in \omega^h \\ &\mathcal{L}\widetilde{U}(p,t_2) + f(p,t_2,\widetilde{U},\widehat{V}) - \tau_2^{-1}U_1^{\overline{n}_1}(p,t_1) \geq Q(p,t_2), \quad p \in \omega^h. \end{aligned} \tag{5.12}$$

Hence $\widehat{U}(p,t_2)$ and $\widetilde{U}(p,t_2)$ are coupled lower and upper solutions with respect to $U_{-1}^{\underline{n}_1}(p,t_1)$ and $U_1^{\overline{n}_1}(p,t_1)$. Similarly by induction on k, $k \geq 2$, it can proved that the minimal sequence $\{U_{-1}^n(p,t_k)\}$ and the maximal sequence $\{U_1^n(p,t_k)\}$ are monotonically increasing and decreasing sequences respectively that satisfy (5.9). $\square$

Theorem 5.3.2. *Under the hypotheses of Theorem 5.3.1, the nonlinear difference scheme* (5.3) *has a unique solution.*

Proof. From the above theorem it can be observed that for each time step t_k, the limits $\lim_{n\to\infty} U_1^n(p,t_k) = U_1^*(p,t_k)$ and $\lim_{n\to\infty} U_{-1}^n(p,t_k) = U_{-1}^*(p,t_k)$ exist for $p \in \overline{\omega}^h$. Define $W(p,t_k) = U_1^*(p,t_k) - U_{-1}^*(p,t_k)$, $p \in \overline{\omega}^h, k > 0$. Now for $k=1$, one have

$$(\mathcal{L}+\mu)\,(U_1^{n+1} - U_1^n)(p,t_1) = -\mathcal{L}U_1^n(p,t_1) - f(p,t_1,U_1^n,V_{-1}^n) + \tau_1^{-1}U_1(p,t_0) + Q(p,t_1).$$

As $n \to \infty$,

$$\mathcal{L}U_1^*(p,t_1) + f(p,t_1,U_1^*,V_{-1}^*) - \tau_1^{-1}U_1^*(p,t_0) = Q(p,t_1).$$

Similarly

$$\mathcal{L}U_{-1}^*(p,t_1) + f(p,t_1,U_{-1}^*,V_1^*) - \tau_1^{-1}U_{-1}^*(p,t_0) = Q(p,t_1).$$

Thus

$$\mathcal{L}W(p,t_1) + f(p,t_1,U_1^*,V_{-1}^*) - f(p,t_1,U_{-1}^*,V_1^*) = 0, \quad p \in \omega^h$$
$$W(p,t_1) = 0,\ p \in \partial\omega^h.$$

Consequently for $p \in \omega^h$,

$$\begin{aligned}(\mathcal{L} + \tfrac{\partial f}{\partial u}(p,t_1,\hat{u},\hat{v}))W(p,t_1) &= \tfrac{\partial f}{\partial v}(p,t_1,\hat{u},\hat{v})(V_1^* - V_{-1}^*)(p,t_1),\\ &= \tfrac{\partial f}{\partial v}(p,t_1,\hat{u},\hat{v})\tau_1\kappa(t_1,t_1)W(p,t_1).\\ &[(U_{-1}^*,V_{-1}^*) \le (\hat{u},\hat{v}) \le (U_1^*,V_1^*)]\end{aligned}$$

Hence by Lemma 5.2.1, one can obtain

$$w(t_1) \le \frac{\tau_1\gamma M}{\tau_1^{-1}} w(t_1),$$

where $w(t_1) = \|W(.,t_k)\|_{\overline{\omega}^h}$. Since $w(t_1) \ge 0$ and due to $(\mathbf{H}_2)$, $w(t_1) = 0$. Hence $U_1^*(p,t_1) = U_{-1}^*(p,t_1) = U^*(p,t_1)$ and

$$\mathcal{L}U^*(p,t_1) + f(p,t_1,U^*,V^*) - \tau_1^{-1}U^*(p,t_0) = Q(p,t_1),\ p \in \omega^h.$$

Hence $U^*(p,t_1)$ is the unique solution of (5.3) at t_1. Similar to the above argument at $k = 2$, one can have

$$\begin{aligned}(\mathcal{L} + \tfrac{\partial f}{\partial u}(p,t_2,\hat{u},\hat{v}))W(p,t_2) &= \tfrac{\partial f}{\partial v}(p,t_2,\hat{u},\hat{v})(V_1^* - V_{-1}^*)(p,t_2),\\ &= \tfrac{\partial f}{\partial v}(p,t_2,\hat{u},\hat{v})[\tau_1\kappa(t_1,t_2)W(p,t_1) + \tau_2\kappa(t_2,t_2)W(p,t_2)]\\ &\qquad [(U_{-1}^*,V_{-1}^*) \le (\hat{u},\hat{v}) \le (U_1^*,V_1^*)]\\ &= \tfrac{\partial f}{\partial v}(p,t_2,\hat{u},\hat{v})\tau_2\kappa(t_2,t_2)W(p,t_2),\end{aligned}$$

$p \in \omega^h$ with $W(p,t_2) = 0,\ p \in \partial\omega^h$. Hence by Lemma 5.2.1, one can obtain

$$w(t_2) \le \frac{\tau_2\gamma M}{\tau_2^{-1}} w(t_2).$$

Since $w(t_2) \ge 0$ and due to $(\mathbf{H}_2)$, $w(t_2) = 0$. Hence $U_1^*(p,t_2) = U_{-1}^*(p,t_2) = U^*(p,t_2)$ and $\mathcal{L}U^*(p,t_2) + f(p,t_2,U^*,V^*) - \tau_2^{-1}U^*(p,t_1) = Q(p,t_2),\ p \in \omega^h$. Hence $U^*(p,t_2)$ is the unique solution of (5.3) at t_2. Thus by induction on k, one can prove that $w(t_k) = 0$ for $k \ge 1$. □

For the rest of the discussion, the following hypothesis also holds.

($\mathbf{H}_3$) The k^{th} time step satisfies

$$\tau_k < \min(\sqrt{\tfrac{1}{\gamma M}}, \tfrac{\nu}{\gamma M}),\ \ \nu \le \tfrac{\partial f}{\partial u}(p, t_k, U, V) \le \mu \text{ on } < \widehat{U}(p, t_k), \widetilde{U}(p, t_k) >$$

where ν is a positive constant.

The following theorem provides the error estimate in terms of the stopping criteria

$$\begin{aligned} &\| \, (\mathcal{L} + \mu) \, (U_1^{\overline{n}_k+1}(p, t_k) - U_1^{\overline{n}_k}(p, t_k))\|_{\omega^h} \le \epsilon \text{ and} \\ &\| \, (\mathcal{L} + \mu) \, (U_{-1}^{\underline{n}_k+1}(p, t_k) - U_{-1}^{\underline{n}_k}(p, t_k))\|_{\omega^h} \le \epsilon, \end{aligned} \tag{5.13}$$

where $\epsilon > 0$.

Theorem 5.3.3. *Under the hypotheses of Theorem 5.3.1, the following estimate holds.*

$$\max_{t_k \in \overline{\omega}^\tau} \left(\|U_1^n(p, t_k) - U^*(p, t_k)\|_{\overline{\omega}^h} + \|U_{-1}^n(p, t_k) - U^*(p, t_k)\|_{\overline{\omega}^h} \right) \le 2T\epsilon,$$

where $U^*(p, t_k)$ is the unique solution of (5.3).

Proof. From (5.8) for $U_1(p, t_k) = U_1^{\overline{n}_k}(p, t_k),\ k \ge 1,$

$$\begin{aligned} (\mathcal{L} + \mu) \, (U_1^{\overline{n}_k} - U_1^{\overline{n}_k+1})(p, t_k) &= \mathcal{L}U_1(p, t_k) + f(p, t_k, U_1, V_{-1}) \\ &\quad -\tau_k^{-1} U_1(p, t_{k-1}) - Q(p, t_k), \quad (p, t_k) \in \omega^h \times (\omega^\tau \setminus \{0\}) \\ U_1(p, t_k) = h(p, t_k), \quad p \in \partial\omega^h, &\quad U_1(p, t_0) = \psi(p), \quad p \in \overline{\omega}^h. \end{aligned}$$

Define $W_1(p, t_k) = U_1(p, t_k) - U^*(p, t_k)$ and $W_{-1}(p, t_k) = U_{-1}(p, t_k) - U^*(p, t_k)$ for $p \in \overline{\omega}^h$.

$$\begin{aligned} \mathcal{L}W_1(p, t_k) + f(p, t_k, U_1, V_{-1}) - f(p, t_k, U^*, V^*) &= \tau_k^{-1} W_1(p, t_{k-1}) \\ &\quad + (\mathcal{L} + \mu) \, (U_1^{\overline{n}_k} - U_1^{\overline{n}_k+1})(p, t_k) \\ (\mathcal{L} + \tfrac{\partial f}{\partial u}(p, t_k, \hat{u}, \hat{v}))W_1(p, t_k) &= (\mathcal{L} + \mu) \, (U_1^{\overline{n}_k} - U_1^{\overline{n}_k+1})(p, t_k) \\ &\quad +\tau_k^{-1} W_1(p, t_{k-1}) - \tfrac{\partial f}{\partial v}(p, t_k, \hat{u}, \hat{v})\tau_k \kappa(t_k, t_k) W_{-1}(p, t_k), \end{aligned}$$

$p \in \omega^h;\ W_1(p, t_k) = 0,\ p \in \partial\omega^h$. Hence using Lemma 5.2.1 and ($\mathbf{H}_2$), one can have

$$w_1(t_k) \le \frac{1}{\tau_k^{-1} + \nu} \left(\epsilon + \tau_k^{-1} w_1(t_{k-1}) + \tau_k \gamma M w_{-1}(t_k) \right). \tag{5.14}$$

Similarly from (5.7), one can obtain

$$w_{-1}(t_k) \le \frac{1}{\tau_k^{-1} + \nu} \left(\epsilon + \tau_k^{-1} w_{-1}(t_{k-1}) + \tau_k \gamma M w_1(t_k) \right). \tag{5.15}$$

Let $W(p,t_k) = W_1(p,t_k) + W_{-1}(p,t_k)$. Adding (5.14) and (5.15),

$$w(t_k) \le \frac{1}{\tau_k^{-1} + \nu}\left(2\epsilon + \tau_k^{-1} w(t_{k-1}) + \tau_k \gamma M w(t_k)\right).$$

Due to $(\mathbf{H}_3)$, it reduces to $w(t_k) \le 2\epsilon\tau_k + w(t_{k-1})$. Since $w_1(t_0) = 0$, by an induction argument on k, one can conclude that $w(t_k) \le 2\epsilon \sum_{l=1}^{k} \tau_l \le 2T\epsilon$. This proves the theorem. □

The following theorem guarantees the convergence of the solution of the nonlinear difference scheme (5.3) to the solution of (5.2) as the mesh sizes tend to zero.

Theorem 5.3.4. *Let $U^*(p,t_k)$ and $u^*(p,t_k)$ be the unique solutions of* (5.3) *and* (5.2) *respectively and $\Xi(p,t_k)$ be the local truncation error of $u^*(x,t)$ on* (5.3). *The error in the nonlinear difference scheme* (5.3) *satisfies the following inequality.*

$$e(t_k) \le C(T)\xi, \quad \xi = \max_{k\ge 1} \xi(t_k), \tag{5.16}$$

where $E(p,t_k) = U^*(p,t_k) - u^*(p,t_k)$, $e(t_k) = \|E(p,t_k)\|_{\overline{\omega}^h}$, $\xi(t,k) = \|\Xi(p,t_k)\|_{\overline{\omega}^h}$ and $C(T) = T\exp(\frac{MT^2}{2})$.

Proof. The proof is similar to the proof of Theorem 2 in [**13**] and hence is omitted. □

5.4. Numerical Examples

In this section, the proposed iterative scheme is illustrated by applying to different partial integro-differential equations arising from mathematical models. Throughout this section, N denotes the number of partitions and n denotes the number of iterations required for the stopping criteria (5.13) with $\epsilon = 10^{-5}$. Moreover, a uniform mesh size is used to demonstrate the algorithm.

Example 5.4.1.

Consider the partial integro-differential equation (5.1) with $a = 1,\ b = 0,\ c = 1,\ \theta = 1,\ \kappa = 1,\ \overline{w} = [0,1],\ T = 1$, $q_1(x,t) = \exp(2t)(x + xt + \frac{x^2 t}{2}(\exp(2t)(t - \frac{1}{2}) + \frac{1}{2}))$ and initial and boundary conditions $z(x,0)=0,\ 0\leq x\leq 1;\ z(0,t)=0,\ z(1,t)=\exp(2t)t,\ 0\leq t\leq 1$. After applying change of variable $z(x,t)=\exp(2t)u(x,t)$, it leads to

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+u+u\int_0^t \exp(2s)u(x,s)\mathrm{d}s=q(x,t),\ 0<x<1,\ 0<t\leq T \tag{5.17}$$

with the initial and boundary conditions $u(x,0)=0,\ 0\leq x\leq 1;\quad u(0,t)=0,\ u(1,t)=t,\ 0\leq t\leq 1$. Consequently, all the conditions hold true for (5.17) with $\widehat{U}=0$ and $\widetilde{U}=6$. Clearly $u(x,t)=xt$ is the solution of (5.17). Table 5.1 provides the numerical error at $T=1$, its order and the number of iterations where $\mathrm{error}(h)=\|U_1(.,T)-u(.,T)\|_{\overline{\omega}^h}$, $u(x,t)$ is the exact solution and $\mathrm{order}(h)=\log_2\left(\frac{\mathrm{error}(h)}{\mathrm{error}(\frac{h}{2})}\right)$.

Example 5.4.2.

Consider the partial integro-differential equation (5.1) with $a=1,b=1,c=1,\theta=1,\kappa=1,\overline{w}=[0,1]\times[0,1],T=1$, $\psi(x)=x_1(x_1-1)x_2(x_2-1)$, $\phi(x)=x_1(x_1-1)+x_2(x_2-1)$, $q_1(x,t)=\exp(t)\left(\psi(x)-2t\phi(x)-t\exp(t)\psi^2(x)+t\psi^2(x)\right)$ and initial and boundary conditions $z(x,t)=0,\ (x,t)\in\partial\omega\times(0,T],\quad z(x,0)=0,\ x\in\overline{\omega}$. After applying change of variable $z(x,t)=\exp(t)u(x,t)$, it leads to

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x_1^2}-\frac{\partial^2 u}{\partial x_2^2}+(\lambda-1)u+\exp(\lambda t)u^2+u\int_0^t \exp(\lambda s)u(x,s)\mathrm{d}s=q(x,t), \tag{5.18}$$

$(x,t)\in\omega\times(0,T]$ with the initial and boundary conditions $u(x,t)=0,\ (x,t)\in\partial\omega\times(0,T], u(x,0)=0,\ x\in\overline{\omega}$. Consequently, all the conditions hold true for (5.18) with $\widehat{U}=0$ and $\widetilde{U}=3$. Clearly $u(x,t)=t\psi(x)$ is the solution of (5.18). Table 5.2 provides the numerical error at $T=1$, its order and the number of iterations.

5.5. Conclusion

In this chapter, monotone iterations based on finite difference is proposed for an important class of parabolic partial integro-differential equations of Volterra type. This result also extends the recent work of [**13**] to problems of higher nonlinearity. The monotone property as well as the convergence of the new iterative scheme are obtained. The efficiency of the proposed numerical scheme is illustrated through its numerical implementation.

TABLE 5.1. Numerical results for Example 5.4.1

N	128	256	512	1024
Error	7.1285×10^{-4}	3.5561×10^{-4}	1.7760×10^{-4}	8.8746×10^{-5}
Order	1.0033	1.0017	1.0009	–
No. of iterations	6	6	5	5

TABLE 5.2. Numerical results for Example 5.4.2

N	16	32	64	128
Error	1.2473×10^{-5}	6.1388×10^{-6}	3.0467×10^{-6}	1.5183×10^{-6}
Order	1.0228	1.0107	1.0048	–
No. of iterations	6	6	5	5

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