Stability analysis of a delayed predator-prey model with Cosner-type functional response

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by

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Stability analysis of a delayed predator-prey model with Cosner-type functional response

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by

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INDIAN INSTITUTE OF TECHNOLOGY INDORE CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "Stability analysis of a delayed predator-prey model with Cosner-type functional response" in the partial fulfillment of the requirements for the award of the degree of MASTER OF SCIENCE and submitted in the **DEPARTMENT OF MATHEMATICS**, Indian Institute of Technology Indore, is an authentic record of my work carried out during the period from July 2022 to June 2023 under the supervision of **Dr. Bapan Ghosh**, Assistant Professor, Department of Mathematics, IIT Indore. The matter presented in this thesis has not been submitted for the award of any other degree of this or any other institute.

Suman Mondal 30/05/2024

Signature of the student with date

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This is to certify that the above statement made by the candidate is Bapan Cheth 30.05.2024 correct to the best of my knowledge.

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Abstract

We consider a predator-prey model with a Cosner-type functional response. We incorporate time delay in the predation process. Delay played an important role in the population dynamics. The analysis of positive solution in the delayed model is challenging. We will prove the positivity and boundedness of solutions of both the models. Apart from trivial and boundary equilibrium under some conditions, the system consists of either no interior equilibrium, a unique interior equilibrium, or two distinct equilibria between boundary equilibria. We discuss different dynamic behaviors due to variation of time delay. Based on the parameter conditions, the stable co-existing equilibrium of the non-delayed model remains stable for increasing time delay. For some other parameter restrictions the stable equilibrium may experience instability through a Hopf bifurcation at a critical delay threshold.There does not exists any delay induced stability switching phenomena in this system.

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Chapter 1

Introduction

Delay differential equations (DDEs) are a class of differential equations in which the rate of change of a variable at any given time depends on its values at previous times. Unlike ordinary differential equations, which mainly depend on the current state, DDEs incorporate delays, making them particularly useful for modeling systems [9] where past states influence current dynamics. In ecology, DDEs are used to model population growth.

The exponential growth model [12] is a simple way to describe how populations change over time. It assumes that the population grows at a rate proportional to its current size. The differential equation for this model is:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = rR(t),\tag{1.1}$$

where R is the population, and r is the intrinsic growth rate of the population. The solution of the equation is $R(t) = R_0 e^{rt}$, where R_0 is the initial population size.

We consider the Lotka-Volterra model [1] as follows:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = rR - \alpha RC,$$

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \alpha \beta RC - mC,$$
(1.2)

where R and C are prey and predator biomass, respectively. Here r is the intrinsic growth rate of the prey, m denotes the specific death rate of the predator, α is the attack rate on the prey by the predator, and β is the

conversion coefficient.

After introducing logistic terms in the Lotka-Volterra model (1.2) we get,

$$\frac{\mathrm{d}R}{\mathrm{d}t} = rR\left(1 - \frac{R}{K}\right) - \alpha RC,$$

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \alpha\beta RC - mC,$$
(1.3)

where K is the environmental carrying capacity. This model is popularly known as logistic Lotka-Volterra model.

We now consider the Rosenzweig-MacArthur predator-prey model [10] read as:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = rR\left(1 - \frac{R}{K}\right) - \frac{\alpha R}{1 + \alpha hR}C,$$

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{\alpha\beta R}{1 + \alpha hR}C - mC,$$
(1.4)

where h represents the handling time. We are familiar with various types of responses, e.g., Holling type II [13], Holling type III [5], Beddington-DeAngelis [14]. The model with Beddington-DeAngelis type functional response [8] can taken in the form:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = rR\left(1 - \frac{R}{K}\right) - \frac{\alpha R}{1 + \beta_1 R + \beta_2 C}C,$$

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{e\alpha R}{1 + \beta_1 R + \beta_2 C}C - mC,$$
(1.5)

where β_1 and β_2 are parameters that measure the effect of prey and predator density on the functional response [4], respectively. We consider another type of functional response, known as the Cosner-type functional response. We consider a model with a Cosner-type functional response [11] and use delay in its predation process. We will discuss the stability analysis for both delayed and non-delayed model in the successive chapters.

Chapter 2

A delayed predator-prey model with Cosner-type functional response

2.1 Motivation

For the Lotka-Volterra model (1.2), the functional response part is in a very simple form. We can easily prove the positivity and the boundedness of that model. However, for a predator-prey model with a Cosner-type functional response [3], in the functional response contains prey and predator biomass in the denominator part. Thus, we face a lot of challenges to prove the positivity and the boundedness of the solutions. We consider a predator-prey model with a Cosner-type functional response. And introduce a delay in that model. Our main goal is to analyze the stability of delayed model. We will discuss the positivity and the boundedness of the solutions of that model.

Barman and Ghosh [2] have also provep that stability switching is never possible with respect to time delay. Therefore, for a predator-prey model with a Cosner-type functional response, we will examine whether stability switching is possible.

2.2 Main model

We consider a predator-prey model with the Cosner-type functional response [6] is as follows,

$$\frac{dX}{dT} = rX(T)\left(1 - \frac{X(T)}{K}\right) - \frac{se_0X(T)Y(T)}{1 + h_1se_0X(T)Y(T)}Y(T),
\frac{dY}{dT} = \frac{ase_0X(T)Y(T)}{1 + h_1se_0X(T)Y(T)}Y(T) - \mu Y(T),$$
(2.1)

with X(0) > 0, Y(0) > 0.

Table 2.1: The variables and parameters are defined in Table:

Variables/parameters	Represents
Т	Time
X(T)	The population biomass of prey
Y(T)	The population biomass of predator
r	The specific birth rate of prey
8	The no. of prey caught by a predator during each encounter
h_1	The handling time per prey
<i>e</i> ₀	The overall encounter coefficient between the prey and predator
a	The effectiveness rate of the prey in predator

Including delay in a predator-prey model helps capture more realistic dynamics of ecosystems as biological processes do not happen instantly. In particular, digesting prey biomass by predator might takes more time. After introducing a delay [2] in the predation, the model (2.1) becomes,

$$\frac{\mathrm{d}X}{\mathrm{d}T} = rX(T)\left(1 - \frac{X(T)}{K}\right) - \frac{se_0X(T)Y(T)}{1 + h_1se_0X(T)Y(T)}Y(T),$$

$$\frac{\mathrm{d}Y}{\mathrm{d}T} = \frac{ase_0X(T - \sigma)Y(T - \sigma)}{1 + h_1se_0X(T - \sigma)Y(T - \sigma)}Y(T - \sigma) - \mu Y(T),$$
(2.2)

with $X(\phi) > 0$, $Y(\phi) > 0$, where $\phi \in [-\sigma, 0]$ and σ is gestation delay.

After applying the following scaling

$$R = \frac{N}{K}, \ C = Kh_1 se_0 Y, \ a_1 = \frac{1}{rse_0(h_1 K)^2}, \ t = rT, a_2 = \frac{a}{rh_1}, \ \gamma = \frac{\mu}{r}, \ \text{and} \ \sigma = r\tau,$$

the model (2.1) becomes,

$$\frac{dR}{dt} = R(t) (1 - R(t)) - \frac{a_1 R(t) C(t)}{1 + R(t) C(t)} C(t),$$

$$\frac{dC}{dt} = \frac{a_2 R(t) C(t)}{1 + R(t) C(t)} C(t) - \gamma C(t),$$
(2.3)

with R(0) > 0, C(0) > 0.

And the model (2.2) becomes,

$$\frac{dR}{dt} = R(t) (1 - R(t)) - \frac{a_1 R(t) C(t)}{1 + R(t) C(t)} C(t),
\frac{dC}{dt} = \frac{a_2 R(t - \tau) C(t - \tau)}{1 + R(t - \tau) C(t - \tau)} C(t - \tau) - \gamma C(t),$$
(2.4)

with the initial conditions $R(\theta) > 0$, $C(\theta) > 0$, where $\theta \in [-\tau, 0]$.

2.3 Positivity of solutions

We will establish the positivity and the boundedness of all solutions of both models (2.3) and (2.4).

Theorem 2.1. The solutions of the predator-prey model (2.3) remain positive for any time t > 0 with the initial conditions R(0) > 0, C(0) > 0.

Proof. Let, $R(0) = R_0 > 0$ and $C(0) = C_0 > 0$ are the initial conditions. If we consider any point $(0, C_0)$ on the *C*-axis as an initial condition, then $\frac{\mathrm{d}R}{\mathrm{d}t} = 0$. This implies that R(t) will remain zero for any initial condition $(0, C_0)$. Hence, R = 0 axis is an invariant manifold.

Similarly, if we consider any point $(R_0, 0)$ on the *R*-axis as an initial condition, then $\frac{dC}{dt} = 0$. This implies that C(t) will remain zero for any initial condition $(R_0, 0)$. Hence, C = 0 axis is an invariant manifold. Thus, (R(t), C(t)) remain non-negative for R(0) > 0, C(0) > 0.

Now We show that the solutions are positive. The fraction $\frac{R(t)C^2(t)}{1+R(t)C(t)}$

is always non-negative, this term does not have any singularity. Therefore,

$$R(t) = R(0) \exp\left(\int_{0}^{t} \left\{1 - R(s) - \frac{a_{1}C^{2}(s)}{1 + R(s)C(s)}\right\} ds\right),$$

$$C(t) = C(0) \exp\left(\int_{0}^{t} \left\{\frac{a_{2}R(s)C(s)}{1 + R(s)C(s)} - \gamma\right\} ds\right),$$
(2.5)

are solutions. Hence, R(t) and C(t) both are strictly positive for any time t > 0.

However, for the delayed model, we can not defined R and C axes as invariant manifold. Therefore, the proof of positivity is not straight forward.

Theorem 2.2. The solutions of the predator-prey model (2.4) remain positive for any time t > 0 with initial conditions $R(\theta) > 0$, $C(\theta) > 0$, where $\theta \in [-\tau, 0]$.

Proof. We have,

$$\dot{C}(t) = \frac{a_2 R(t-\tau) C^2(t-\tau)}{1 + R(t-\tau) C(t-\tau)} - \gamma C(t).$$
(2.6)

We consider two different cases.

Case-I: Let there exists a least $t = t_1 > 0$ such that R(t) > 0, C(t) > 0for $t < t_1$ and $R(t_1) = 0$, $C(t_1) \ge 0$. First integration of (2.5) is well defined. Hence $R(t_1) > 0$, which is a contradiction to $R(t_1) = 0$.

Case-II: Let there exists a least $t = t_1 > 0$ such that R(t) > 0, C(t) > 0for $t < t_1$ and $R(t_1) > 0$, $C(t_1) = 0$. Since $R(t - \tau)$ is strictly positive and $C(t - \tau)$ is non-negative, for $t \in (0, t_1]$, the first term in the right side of the given equation (2.6) is non-negative, and hence

$$\dot{C}(t) \geq -\gamma C(t), \ 0 < t \leq t_1.$$

Now, since C(t) is continuous on $[0, t_1]$, by the theory of differential inequality, we get,

$$C(t) \ge C(0) \exp\left(-\int_0^t \gamma \, \mathrm{d}s\right).$$

This implies $C(t_1) > 0$. Thus, our initial assumption $C(t_1) = 0$ is wrong. Hence, prey and predator populations of the predator-prey model (2.4) always remain positive.

Now, we will prove that the solutions are bounded for both the models (2.3) and (2.4).

2.4 Boundedness of solutions

Theorem 2.3. All solutions of the system (2.3) bounded for any time t > 0with R(0) > 0, C(0) > 0.

Proof. Choose
$$Z(t) = R(t) + \frac{a_1}{a_2}C(t)$$
. Thus,

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \frac{\mathrm{d}R}{\mathrm{d}t} + \frac{a_1}{a_2}\frac{\mathrm{d}C}{\mathrm{d}t},$$

$$= R\left(1-R\right) - \frac{a_1}{a_2}\gamma C$$

We have

$$\begin{aligned} \frac{\mathrm{d}Z}{\mathrm{d}t} &\leq -\gamma \left(R + \frac{a_1}{a_2}C \right) + 1 - \left(R - \frac{(\gamma+1)}{2} \right)^2, \\ &= -\gamma \left(R + \frac{a_1}{a_2}C \right) + 1 \end{aligned}$$

Therefore, $\frac{\mathrm{d}Z}{\mathrm{d}t} + \gamma Z \leq 1$. Now using comparison inequality, we obtain $0 < Z(t) < \frac{1 - e^{-\gamma t}}{\gamma} + Z(0)e^{-\gamma t}$. As $t \to \infty, 0 < Z < \frac{1}{\gamma}$. Thus, $Z(t) = R(t) + \frac{a_1}{a_2}C(t)$ implies that both R(t) and C(t) of the system (2.3) are bounded.

Since delay is included in the predation process for the delayed model, we need to use a different approach to prove the boundedness of the delayed model.

Theorem 2.4. All solutions of the system (2.4) bounded for any time t > 0with $R(\theta) > 0$, $C(\theta) > 0$, where $\theta \in [-\tau, 0]$.

Proof. Since R(t) > 0, $\forall t > 0$, we are able to write,

$$R(t) \le R(t)(1 - R(t)).$$

That implies

$$\lim_{t \to +\infty} \sup R(t) \le 1.$$

This implies, for any $\delta > 0$, there is a $t^* > 0$ such that $0 < R(t) \le 1 + \delta$ for $t \ge t^*$.

Thus for $t \ge t^*$, we obtain

$$a_{2}\dot{R}(t) + a_{1}\dot{C}(t+\tau) = a_{2}R(t)(1-R(t)) - \gamma a_{1}C(t+\tau)$$

$$\leq \frac{a_{2}}{4} - \gamma a_{1}C(t+\tau)$$

$$\left[\text{since } R(t)(1-R(t)) \leq \frac{1}{4}\right]$$

$$= \frac{a_{2}}{4} + \gamma a_{2}(1+\delta) - \gamma \left\{a_{2}R(t) + a_{1}C(t+\tau)\right\}.$$

For δ arbitrarily very small, $\lim_{t \to +\infty} \sup \{a_2 R(t) + a_1 C(t + \tau)\} \leq \frac{a_2}{4\gamma} + a_2 = L$. Thus, $\lim_{t \to +\infty} \sup C(t) \leq L/a_1$.

Chapter 3

Stability analysis of the non-delayed model

Analytic solutions are not easy to find for a nonlinear system. However, we can easily calculate equilibrium points for a nonlinear system. The equilibrium points are the constant solutions for a given system.

3.1 Equilibria

The equilibrium points for the system (2.3) are:

- (i) The trivial equilibrium (0,0) which always exists.
- (ii) The boundary equilibrium (1,0).
- (iii) The interior equilibrium (R^*, C^*) , where it satisfy:

$$(1 - R^*) - \frac{a_1 C^{*2}}{1 + R^* C^*} = 0$$
 and $\frac{a_2 C^{*2}}{1 + R^* C^*} - \gamma = 0$.

Solving above two equations, we obtain, $C^* = \frac{\gamma}{(a_2 - \gamma)R^*}$ and $HR^{*3} + GR^{*2} + L = 0$, where $L = a_1\gamma^2$, $G = -a_2^2 + a_2\gamma$, $H = -a_2\gamma + a_2^2$.

If $a_2 > \gamma$, then L > 0, G < 0, and H > 0. Since $\phi(R) = HR^3 + GR^2 + L = 0$ has two changes of sign, it has either two positive roots or no positive root. Again, $\phi(-R) = -HR^3 + GR^2 + L = 0$ has one change of sign. So it has at most one negative root. Now $\frac{d\phi}{dR} = 3HR^2 + 2GR = 0$ and $\frac{d^2\phi}{dR^2} = 6HR + 2G$. To find a local maximum or local minimum, $\frac{d\phi}{dR} = 0$. Thus, R(3R - 2) = 0. It is obtained that possible extreme points are R = 0, $\frac{2}{3}$. Now, $\frac{d^2(\phi)}{dR^2}|_{\phi=0} = 2G =$ $-2a_2^2 + 2a_2\gamma < 0$ and $\frac{d^2(\phi)}{dR^2}|_{\phi=\frac{2}{3}} = -2G = 2a_2^2 - 2a_2\gamma > 0$. Therefore, ϕ has a local maximum at 0 and the maximum value is $a_1\gamma^2$. Also, ϕ has a local minimum at $\frac{2}{3}$ and the minimum value is $a_1\gamma^2 - \frac{4}{27}(a_2^2 - a_2\gamma)$.

Lemma 3.1.

- (i) If $a_1\gamma^2 = \frac{4}{27}(a_2^2 a_2\gamma)$, we can conclude that (2.3) has exactly one interior equilibria $\left(\frac{2}{3}, \frac{3\gamma}{2(a_2-\gamma)}\right)$.
- (ii) If $\gamma^2 > \frac{4}{27}(a_2^2 a_2\gamma)$, we can conclude that (2.3) has no interior equilibria.
- (iii) If $a_1\gamma^2 < \frac{4}{27}(a_2^2 a_2\gamma)$, we can get that (2.3) has exactly two distinct interior equilibria.

Proof. Clearly $\frac{d^2\phi(R)}{dR^2} = 6HR + 2G$. We obtain $\frac{d^2\phi(0)}{dR^2} = -2(a_2^2 - a_2\gamma) < 0$ and $\frac{d^2\phi\left(\frac{2}{3}\right)}{dR^2} = 2(a_2^2 - a_2\gamma) > 0$. We obtain $\phi(R)$ has local maximum $a_1\gamma^2$ at R = 0 and local minimum $a_1\gamma^2 - \frac{4}{27}(a_2^2 - a_2\gamma)$ at $R = \frac{2}{3}$.

- (i) If $\gamma^2 = \frac{4}{27}(a_2^2 a_2\gamma)$, then $\phi(R)$ touches *R*-axis one at $R = \frac{2}{3}$ as $g(\frac{2}{3}) = 0$. That implies the given system has a unique interior equilibrium point $\left(\frac{2}{3}, \frac{3\gamma}{2(a_2-\gamma)}\right)$.
- (ii) If $\gamma^2 > \frac{4}{27}(a_2^2 a_2\gamma)$, the graph of $\phi(R)$ never crosses *R*-axis for R > 0. This implies that the given system (2.3) has no interior equilibrium.

(iii) If $\gamma^2 < \frac{4}{27}(a_2^2 - a_2\gamma)$, then $\phi(R)$ exceeds *R*-axis at exactly two points for 0 < R < 1. That implies the given system has exactly two distinct interior equilibria.

Now we will discuss the stability analysis of the non-delayed model.

3.2 Linearization process of the non-delayed model

The Jacobian matrices at the interior equilibrium (R^*, C^*) of the delayed model (2.4) are:

$$J_0 = \begin{bmatrix} 1 - 2R^* - \frac{a_1 C^{*2}}{(1 + R^* C^*)^2} & -\frac{a_1 R^* (2C^* + R^* C^{*2})}{(1 + R^* C^*)^2} \\ 0 & -\gamma \end{bmatrix}$$

and

$$J_{\tau} = \begin{bmatrix} 0 & 0 \\ \frac{a_2 C^{*2}}{(1+R^*C^*)^2} & \frac{a_2 R^* (2C^*+R^*C^{*2})}{(1+R^*C^*)^2} \end{bmatrix}.$$

The characteristic equation of the model (2.4) is:

$$\det \left(J_0 + J_\tau e^{-\lambda \tau} - \lambda I \right) = 0,$$

i.e.,

$$\begin{vmatrix} 1 - 2R^* - \frac{a_1 C^{*2}}{(1 + R^* C^*)^2} - \lambda & -\frac{a_1 R^* (2C^* + R^* C^{*2})}{(1 + R^* C^*)^2} \\ \frac{a_2 C^{*2}}{(1 + R^* C^*)^2} e^{-\lambda \tau} & -\gamma + \frac{a_2 R^* (2C^* + R^* C^{*2})}{(1 + R^* C^*)^2} e^{-\lambda \tau} - \lambda \end{vmatrix} = 0.$$

The characteristic equation can be represented as

$$\lambda^2 + \Omega_1 \lambda + \Omega_2 \lambda e^{-\lambda \tau} + \Omega_3 e^{-\lambda \tau} + \Omega_4 = 0, \qquad (3.1)$$

where

$$\begin{split} \Omega_1 &= \gamma - 1 + 2R^* + \frac{a_1 C^{*2}}{(1 + R^* C^*)^2}, \\ \Omega_2 &= -\frac{a_2 R^* (2C^* + R^* C^{*2})}{(1 + R^* C^*)^2}, \\ \Omega_3 &= \left(\frac{a_1 C^{*2}}{(1 + R^* C^*)} + 1 - 2R^* - \frac{a_1 C^{*2}}{(1 + R^* C^*)^2}\right) \left(\frac{a_2 R^* (2C^* + R^* C^{*2})}{(1 + R^* C^*)^2}\right), \\ \Omega_4 &= -\gamma \left(1 - 2R^* - \frac{a_1 C^{*2}}{(1 + R^* C^*)^2}\right). \\ \text{For } \tau = 0, \text{ the characteristic equation (3.1) becomes} \end{split}$$

$$\lambda^2 + (\Omega_1 + \Omega_2)\lambda + (\Omega_3 + \Omega_4) = 0.$$
(3.2)

From the characteristic equation, it is clear that when $\tau = 0$, interior equilibrium $E = (R^*, C^*)$ has either a stable, unstable, or saddle point under some parameter conditions.

3.3 Stability analysis of the non-delayed model

The Jacobian matrix of the non-delayed model (2.3) is:

$$J = \begin{bmatrix} 1 - 2R - \frac{a_1 C^2}{(1 + RC)^2} & -\frac{a_1 R (2C + RC^2)}{(1 + RC)^2} \\ \frac{a_2 C^2}{(1 + RC)^2} & \frac{a_2 R (2C + RC^2)}{(1 + RC)^2} - \gamma \end{bmatrix}.$$

Now we will discuss the stability nature of each equilibrium.

Trivial equilibrium: For the trivial equilibrium (0,0), the eigenvalues of J are 1 and $-\gamma$. Hence, the trivial equilibrium (0,0) is always a saddle point. **Boundary equilibrium:** For the boundary equilibrium (1,0), the eigenvalues of J are -1 and $-\gamma$. Therefore, the boundary equilibrium (1,0) is always locally asymptotically stable.

Interior equilibrium: We plot the diagram (3.1) of R^* for two interior equilibria for varying γ for the parameters $a_1 = 0.8$, $a_2 = 0.7$.



Figure 3.1: The diagram illustrates R^* variation with changing γ for the parameters: $a_1 = 0.8$, $a_2 = 0.7$.

We will discuss the stability analysis of interior equilibrium for the nondelayed model. For the parameters: $a_1 = 0.8$, $a_2 = 0.7$, $\gamma = 0.2$, we obtain two interior equilibria are $E_1 = (0.882, 0.453)$ and $E_2 = (0.385, 1.036)$.

(i) For interior equilibrium $E_1 = (0.882, 0.453)$, the determinant of J

is -0.092. Therefore, the interior equilibrium $E_1 = (0.882, 0.453)$ is a saddle point for the non-delayed model.

(ii) For interior equilibrium $E_2 = (0.385, 1.036)$, the determinant of J is 0.121, which is positive. Also, the trace of J is -0.067. Thus, eigenvalues are complex with negative real parts. Therefore, the interior equilibrium $E_2 = (0.385, 1.036)$ is a stable focus for the non-delayed model. The phase portrait (3.2) gives E_2 is a stable focus.



Figure 3.2: The phase portrait illustrates E_2 is a stable focus for the parameters: $a_1 = 0.8$, $a_2 = 0.7$, with initial condition (0.4, 1.04).

Chapter 4

Stability analysis of the delayed model

The equilibriums of delayed and non-delayed models are equal. We now present delay induced stability for the equilibria. In order to proceed further, we present the following theorem:

Theorem 4.1. If an equilibrium is a saddle point when $\tau = 0$, then the equilibrium cannot be stable for varying delay. [Ref. Theorem 4.7 [7]]

Trivial equilibrium: The trivial equilibrium (0,0) is a saddle point in the absence of delay. Using the above Theorem 4.1 we can say that it is always a saddle point for any time delay τ .

Boundary equilibrium: The boundary equilibrium (1,0) is always stable for the non-delayed case. Since J_{τ} is independent of delay, we obtain the characteristic equation delay-free. Therefore, we can state that both the eigenvalues remain negative. Hence it is always locally stable for any time delay.

Theorem 4.2. The boundary equilibrium (1,0) is globally stable whenever $a_2 < \gamma$.

Proof. We have established the positivity and the boundedness of the solu-

tions. Clearly,

$$\lim_{t \to +\infty} \sup R(t) \le 1.$$

Thus, for any positive ϕ , there exists $U = U(\phi)$ such that $R(t) \leq 1 + \phi$, whenever t > U. We understand the concept from that for $t > U + \tau$,

$$\dot{C}(t) \le a_2 \frac{R(t-\tau)C^2(t-\tau)}{R(t-\tau)C(t-\tau)} - \gamma C(t)$$
$$a_2 \le C(t-\tau) - \gamma C(t)$$

Therefore, $\lim_{t \to +\infty} C(t) = 0$ provided $a_2 < \gamma$. Thus, for any given $\phi_1 > 0$, there exists a $U(\phi_1) > 0$ such that for $t > U(\phi_1)$, we have

$$\dot{R}(t) \ge R(t)(1 - R(t) - a_1\phi_1).$$

Thus $\lim_{t \to +\infty} \inf R(t) \ge 1$. Now $\lim_{t \to +\infty} \sup R(t) \le 1$ and $\lim_{t \to +\infty} \inf R(t) \ge 1$ implies, $\lim_{t \to +\infty} R(t) = 1$. Finally, we find $\lim_{t \to +\infty} C(t) = 0$ and $\lim_{t \to +\infty} R(t) = 1$. Therefore, (1,0) is globally stable whenever $a_2 < \gamma$.

Interior equilibrium: We will explore the stability changes in the system for time delay τ . Since equilibriums are independent of time delay τ , bifurcations such as saddle-node, and transcritical are not possible while considering τ as the bifurcation parameter. For the existence of Hopf bifurcation there must exist a pair of eigenvalues $\lambda = \pm i\omega$ ($\omega > 0$) of the characteristic function.

Without loss of generality, putting $\lambda = i\omega$ in the characteristic equation and separating the real and imaginary parts, we obtain,

$$-\omega^2 + \Omega_2 \omega \sin \omega \tau + \Omega_3 \cos \omega \tau + \Omega_4 = 0, \qquad (4.1)$$

$$\Omega_1 \omega + \Omega_2 \omega \cos \omega \tau - \Omega_3 \sin \omega \tau = 0. \tag{4.2}$$

After eliminating τ from equations (4.1) and (4.2), we can write,

$$(\omega^2 - \Omega_4)^2 + \Omega_1^2 \omega^2 = \Omega_2^2 \omega^2 + \Omega_3^2.$$

After simplifying we obtain,

$$\omega^4 + (\Omega_1^2 - \Omega_2^2 - 2\Omega_4)\omega^2 + \Omega_4^2 - \Omega_3^2 = 0.$$
(4.3)

The roots of the equation (4.3) are easy to calculate. We examine the stability analysis of the system derived from the roots of the equation (4.3).

Now we obtain the possible ω as follows:

(i) If $\Omega_2^2 - \Omega_1^2 + 2\Omega_4 < 0$ and $\Omega_4^2 - \Omega_3^2 > 0$ or $(\Omega_2^2 - \Omega_1^2 + 2\Omega_4)^2 < 4(\Omega_4^2 - \Omega_3^2)$, then we have no positive ω .

(ii) If $\Omega_4^2 - \Omega_3^2$ is strictly less than zero, then we obtain exactly one positive ω represented by ω_+ .

(iii) For $\Omega_2^2 - \Omega_1^2 + 2\Omega_4 > 0$, $(\Omega_2^2 - \Omega_1^2 + 2\Omega_4)^2 = 4(\Omega_4^2 - \Omega_3^2)$, then we obtain a two-fold or two equal positive ω .

(iv) For $\Omega_4^2 - \Omega_3^2 > 0$, $\Omega_2^2 - \Omega_1^2 + 2\Omega_4 > 0$ and $(\Omega_2^2 - \Omega_1^2 + 2\Omega_4)^2 > 4(\Omega_4^2 - \Omega_3^2)$, then we obtain two unequal positive values of ω , and we denote terms as ω_+ and ω_- .

Lemma 4.3. If $\Omega_2^2 - \Omega_1^2 + 2\Omega_4 > 0$, then $\Omega_4^2 - \Omega_3^2 < 0$.

Proof. Let us consider $L_1 = 1 - 2R^* - \frac{a_1 C^{*2}}{(1 + R^* C^*)^2}$ and $L_2 = \frac{a_2 R^* (2C^* + R^* C^{*2})}{(1 + R^* C^*)^2}$. Therefore we get,

$$\Omega_{1} = \gamma - L_{1},$$

$$\Omega_{2} = -L_{2},$$

$$\Omega_{3} = \left(\frac{a_{1}C^{*2}}{(1 + R^{*}C^{*})}L_{2} + L_{1}L_{2}\right),$$

$$\Omega_{4} = -\gamma L_{1}.$$

Now,

$$\begin{split} \Omega_2^2 - \Omega_1^2 + 2\Omega_4 &= (-L_2)^2 - (\gamma - L_1)^2 + 2(-\gamma L_1) \\ &= L_2^2 - \gamma^2 - L_1^2 > 0 \\ \text{We get, } L_2^2 > \gamma^2 + L_1^2. \end{split}$$

Also,

$$\Omega_4^2 - \Omega_3^2 = (-\gamma L_1)^2 - \left(\frac{a_1 C^{*2}}{(1+R^*C^*)}L_2 + L_1 L_2\right)^2$$
$$= L_1^2 \gamma^2 - L_2^2 \left(\frac{a_1 C^{*2}}{(1+R^*C^*)} + L_1\right)^2$$
$$= L_1^2 (\gamma^2 - L_2^2) - L_2^2 (P^2 + 2PL_1)$$
$$\left[\text{Here, } \frac{a_1 C^{*2}}{(1+R^*C^*)} = P\right]$$
$$< -L_1^4 - L_2^2 (P^2 + 2PL_1) < 0$$

Thus, if $\Omega_2^2 - \Omega_1^2 + 2\Omega_4$ is strictly positive, then $\Omega_4^2 - \Omega_3^2$ is strictly negative. So, there is no case, in which we can get two positive ω .

The implication of the above lemma is that the two positive ω 's are impossible.

Now we will find the critical τ values corresponding to the positive $\omega = \omega_+$. After simplifying the equations (4.1) and (4.2), we obtain,

$$\cos\omega\tau = \frac{(\omega^2 - \Omega_4)\Omega_3 - \Omega_1\Omega_2\omega^2}{\Omega_2^2\omega^2 + \Omega_3^2},\tag{4.4}$$

$$\sin \omega \tau = \frac{\Omega_2 \omega (\omega^2 - \Omega_4) + \Omega_1 \Omega_3 \omega}{\Omega_2^2 \omega^2 + \Omega_3^2}.$$
(4.5)

Now we can easily calculate the values of $\tau = \tau_j^+$.

The transversality condition in the context of Hopf-bifurcation refers to the condition that ensures that a pair of eigenvalues crosses the imaginary axis. If $\left[\frac{d\text{Re}(\lambda)}{d\tau}\right]_{\lambda=\pm i\omega} \neq 0$, then the eigenvalues $\pm i\omega$ cross the imaginary axis. Now we verify the transversality condition for $\tau = \tau_j^+$.

Differentiating characteristic equation (3.1) w.r.t. τ , we obtain,

$$\left(2\lambda + \Omega_1 - \Omega_2\lambda e^{-\lambda\tau}\tau + \Omega_2 e^{-\lambda\tau} - \Omega_3 e^{-\lambda\tau}\tau\right)\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} - \left(\Omega_2\lambda e^{-\lambda\tau} + \Omega_3 e^{-\lambda\tau}\right)\lambda = 0.$$
(4.6)

Thus, after some simplification we obtain,

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = \frac{2\lambda + \Omega_1 + \Omega_2 e^{-\lambda\tau}}{(\Omega_2\lambda + \Omega_3)\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}.$$
(4.7)

From the equation (3.1) we get,

$$e^{-\lambda\tau} = \frac{-\lambda^2 - \Omega_1 \lambda - \Omega_4}{\Omega_2 \lambda + \Omega_3}$$

Therefore,

$$\left[\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} \right)^{-1} \right]_{\tau=\tau_j^+} = \left[\frac{-2\lambda - \Omega_1}{\lambda(\lambda^2 + \Omega_1\lambda + \Omega_4)} + \frac{\Omega_2}{\lambda(\Omega_2\lambda + \Omega_3)} - \frac{\tau}{\lambda} \right]_{\lambda=i\omega_{\pm}}$$

$$= \frac{\Omega_1^2 - 2(\Omega_4 - \omega^2)}{\Omega_1^2\omega^2 + (\omega^2 - \Omega_4)^2} - \frac{\Omega_2^2}{\Omega_2^2\omega^2 + \Omega_3^2} + \text{Some imag. part of} \left[\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} \right)^{-1} \right]_{\tau=\tau_j^+}.$$

We know the result:

$$\operatorname{sign}\left\{\left[\frac{\mathrm{d}(\operatorname{Re}(y))}{\mathrm{d}t}\right]\right\} = \operatorname{sign}\left\{\operatorname{Re}\left[\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^{-1}\right]\right\}$$

Using the above result we get, at
$$\tau = \tau_j^{\pm}$$
,

$$\operatorname{sign}\left\{ \left[\frac{\mathrm{d}(\mathrm{Re}\lambda)}{\mathrm{d}\tau} \right] \right\} = \operatorname{sign}\left\{ \frac{\Omega_1^2 - 2(\Omega_4 - \omega^2)}{\Omega_1^2 \omega^2 + (\omega^2 - \Omega_4)^2} - \frac{\Omega_2^2}{\Omega_2^2 \omega^2 + \Omega_3^2} \right.$$

$$= \operatorname{sign}\left\{ \Omega_1^2 - \Omega_2^2 - 2\Omega_4 + 2\omega^2 \right\}$$

$$\left[\operatorname{As} \Omega_1^2 \omega^2 + (\omega^2 - \Omega_4)^2 = \Omega_2^2 \omega^2 + \Omega_3^2 \right]$$

$$= \operatorname{sign}\left\{ \pm \sqrt{D} \right\},$$

where $D = (\Omega_2^2 - \Omega_1^2 + 2\Omega_4)^2 - 4(\Omega_4^2 - \Omega_3^2)$. Hence, we obtain the transversality conditions $\left[\frac{\mathrm{d}(\mathrm{Re}\lambda)}{\mathrm{d}\tau}\right]_{\tau=\tau_j^+}$ is strictly positive provided $\Delta \neq 0$.

We have already theoretically proven that two positive ω 's are impossible. We will provide example whether no or one positive ω exists.

Example 4.1. We consider $a_1 = 0.25$, $a_2 = 0.99$, $\gamma = 0.2$, then two interior equilibria are $E_1 = (0.121, 2.099)$ and $E_2 = (0.986, 0.256)$. The equilibrium E_1 is locally stable and E_2 is a saddle point when $\tau = 0$. At the interior equilibrium $E_1, \ the \ obtain \ \Omega_1 + \Omega_2 = 0.156 > 0 \ and \ \Omega_3 + \Omega_4 = 0.2 > 0. \ Now, \ (\Omega_2^2 - \Omega_1^2 + 2\Omega_4)^2 - \Omega_2^2 + \Omega_2^2 +$ $4(\Omega_4^2 - \Omega_3^2) = -1.328 < 0$. Therefore, in this case, ω does not exist. Thus, at all time delays, E_1 remains locally stable.

We already mentioned that the saddle equilibrium does not changes stability. However we examine if there exists any positive ω . At E_2 , $\Omega_1 + \Omega_2 =$ 0.707 > 0 and $\Omega_3 + \Omega_4 = 0.266 > 0$. Thus, E_2 is unstable or it has a saddle point when $\tau = 0$. There is no alteration in stability despite the variation in delay. As $\Omega_4^2 - \Omega_3^2 = -0.175 < 0$ at E_2 . Therefore, a positive $\omega_+ = 0.638$ exists. At τ_0^+ = 3.816, two eigenvalues shift from the negative complex plane to the positive complex plane, both with negative real parts. Thus, we observed that the instability remains unchanged. Thus, delay promotes instability for this saddle point.

Example 4.2. We consider another parametr set as $a_1 = 0.8$, $a_2 = 0.7$, $\gamma = 0.2$. Two interior equilibria are $E_1 = (0.882, 0.453)$ and $E_2 = (0.385, 1.036)$. For the equilibrium E_1 , the value of $\Omega_1 + \Omega_2 = 1.873$ and $\Omega_3 + \Omega_4 = -0.685$. We obtain, that the equilibrium E_1 is saddle point when τ is zero. Therefore, the time delay can not changed the stability.

The equilibrium E_2 is stable when $\tau = 0$. For the equilibrium E_2 , the value of $\Omega_1 + \Omega_2 = 1.273$ and $\Omega_3 + \Omega_4 = 0.375$. We check it manually and obtain $\tau_0^+ = 1.12$. Due to the transversality condition one pair of eigenvalues cross \mathbb{C}^0 (imag. axis) from \mathbb{C}^- (left- half complex plane) to \mathbb{C}^+ (right-half complex plane), when τ increases through τ_0^+ . The evaluation of eigenvalues are computed in the Figure (4.1).



Figure 4.1: The plot illustrates how the real parts of eigenvalues changes with respect to τ .



Figure 4.2: (4.2a) and (4.2b) are time series and phase potrait for $\tau = 0.9$. (4.2c) and (4.2d) are time series and phase potrait for $\tau = 1.3$.

A time series for a predator-prey model with Cosner-type functional responses involves simulating the dynamics of prey and predator biomass. A time series plot (4.2) showing prey population dynamics over time in a predator-prey model serves several important purposes: understanding population dynamics, evaluating stability, predicting future trends, etc. Time series plots of prey and predator population dynamics provide valuable insights into the ecological processes driving predator-prey interactions and ecosystem dynamics.



Figure 4.3: The bifurcation diagrams illustrates in (4.3a) how prey populations vary with changing τ and in (4.3b) how predator populations vary with changing τ .

Bifurcation diagrams are used to visualize the dynamics of the system as particular parametric changes. In the context of prey versus the parameter τ , the bifurcation diagram (4.3a) would show nature of the prey biomass for different τ values. In the other context of predator versus the parameter τ , the bifurcation diagram (4.3b) would show nature of the predator biomass for different τ values. It can reveal stable and unstable equilibrium points and periodic behavior in the predator-prey dynamics. This bifurcation diagram helps in understanding how changes in the parameter τ influence the stability and behavior of the prey population.

Chapter 5

Conclusion and future plan

In this study, we proposed a predator-prey model with Cosner-type functional response. Time delay is incorporated in predation process. Proving the positivity for the delayed model is not straight forward. We have use some sophisticated technique to proof the positivity of the delayed model. We have also proved the boundedness of solutions for both the delayed and non-delayed model.

We observed that there are either two positive equilibrium or no positive equilibrium. Trivial equilibrium is a saddle point and boundary equilibrium is locally asymptotically stable for both non-delayed and delayed models. It is established the boundary equilibrium is globally stable whenever $a_2 < \gamma$.

We have provided numerical examples of no change in stability of the coexisting equilibrium. We have already theoretically proved that no switching is possible for the delayed model.

While the Cosner-type functional response has been our focus, other nonlinear functional responses (e.g., Holling type III, Beddington-DeAngelis) could be explored. Comparing these responses might provide a better understanding of predator-prey interactions and their implications for ecosystem management. Using numerical simulations to explore the system's behavior under various initial conditions and parameter settings could reveal complex dynamics not captured by analytical methods.

Bibliography

- Nicolas Bacaër. Lotka, Volterra and the predator-prey system (1920– 1926), pages 71–76. Springer London, London, 2011.
- [2] Binandita Barman and Bapan Ghosh. Role of time delay and harvesting in some predator-prey communities with different functional responses and intra-species competition. *International Journal of Modelling and Simulation*, 42(6):883–901, 2022.
- [3] Chris Cosner. Effects of spatial heterogeneity on the population dynamics of diffusive predators and prey. In *Spatial Ecology: The Role of Space in Population Dynamics and Interspecific Interactions*, pages 274–296. Princeton University Press, 2003.
- [4] Chris Cosner, Donald L. DeAngelis, Jerald S. Ault, and Donald B. Olson. Effects of spatial grouping on the functional response of predators. *Theoretical Population Biology*, 56(1):65–75, 1999.
- [5] J.H.P. Dawes and M.O. Souza. A derivation of holling's type i, ii and iii functional responses in predator-prey systems. *Journal of Theoretical Biology*, **327**:11–22, 2013.
- [6] William F Fagan and Chris Cosner. Do habitat corridors really enhance population viability? *Ecology*, 75(3):667–681, 1994.
- [7] Bapan Ghosh, Binandita Barman, and Manideepa Saha. Multiple dynamics in a delayed predator-prey model with asymmetric functional

and numerical responses. *Mathematical Methods in the Applied Sciences*, **46**(5):5187–5207, 2023.

- [8] Mainul Haque. A detailed study of the beddington-deangelis predator-prey model. *Mathematical Biosciences*, 234(1):1–16, 2011.
- [9] Mainul Hossain, N.C. Pati, Saheb Pal, Sourav Rana, Nikhil Pal, and G.C. Layek. Bifurcations and multistability in a food chain model with nanoparticles. *Mathematics and Computers in Simulation*, **190**:808–825, 2021.
- [10] Min Lu, Chuang Xiang, Jicai Huang, and Shigui Ruan. Dynamics of the generalized rosenzweig-macarthur model in a changing and patchy environment. *Physica D: Nonlinear Phenomena*, 465:134197, 2024.
- [11] Biruk Tafesse Mulugeta, Jingli Ren, Qigang Yuan, and Liping Yu. Bifurcation analysis of predator-prey model with cosner type functional response and combined harvesting. *Mathematical Methods in the Applied Sciences*, 47(6):5286–5306, 2024.
- [12] Hal L Smith and Paul Waltman. The theory of the chemostat: dynamics of microbial competition. Cambridge University Press, 1995.
- [13] N. Stollenwerk, M. Aguiar, and B.W. Kooi. Modelling holling type ii functional response in deterministic and stochastic food chain models with mass conservation. *Ecological Complexity*, 49:100982, 2022.
- [14] Ruizhi Yang, Qiannan Song, and Yong An. Spatiotemporal dynamics in a predator-prey model with functional response increasing in both predator and prey densities. *Mathematics*, **10**(1), 2022.