Zeros of Koshliakov Zeta Functions

M.Sc. Thesis

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Zeros of Koshliakov Zeta Functions

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by

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"Zeros of Koshliakov Zeta Functions"** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF MATHEMATICS**, **Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2023 to May 2024 under the supervision of **Dr. Bibekananda Maji**, Assistant Professor, Department of Mathematics, IIT Indore. The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

Yashovardhan Singh Gautam
30/05/2024

Signature of the student with date

(**Yashovardhan Singh Gautam**)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Bibekananda Maji 30/05/2024

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Yashovardhan Singh Gautam has successfully given his M.Sc. Oral Examination held on 27th May, 2024.

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Date: 30/05/2024

Dedicated to my
family and teachers

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Abstract

This thesis investigates the properties of zeros associated with the Riemann zeta function $\zeta(s)$ and the Hurwitz zeta function $\zeta(s, a)$. The thesis builds upon the work of N. S. Koshliakov [11, Chapter 1, 3], analyzing the properties of $\zeta_p(s)$ and $\eta_p(s)$ alongside their analytic continuations with the relationship between $\zeta_p(s)$ and $\eta_p(s)$. The research then delves into explicit formulas for the summation of specific infinite series. A key contribution of this thesis is the identification of a zero-free region for $\zeta_p(s)$ within the right half-plane.

Contents

CHAPTER 1

Introduction

The investigation of the zeros of the Riemann zeta function and the Hurwitz zeta function has been a fundamental topic in number theory and analytic number theory. The Riemann zeta function, $\zeta(s)$, is defined for $s \in \mathbb{C}$ with $\Re(s) > 1$ by the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^s}$, plays a crucial role in various mathematical disciplines, including complex analysis, number theory, and physics. The non-trivial zeros of the Riemann zeta function, $\zeta(s)$, are found within the critical strip $0 < \Re(s) < 1$. It is conjectured that these all zeros lie on the critical line $\Re(s) = \frac{1}{2}$, a hypothesis that holds great importance because of its profound connection to the distribution of prime numbers.

Furthermore, the infinite series $\sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$ for $s \in \mathbb{C}$ and $0 < a < 1$ with $\Re(s) > 1$, known as Hurwitz zeta function $\zeta(s,a)$, generalizes $\zeta(s)$. Spira [14] showed that all the non-trivial zeros of $\zeta(s,a)$ do not lie on half-line as well as he explored zero free region of $\zeta(s,a)$. Recently, Dixit and Gupta [3] explored the Koshliakov's manuscript [11]. The manuscript contains two generalizations of $\zeta(s)$ which are $\zeta_p(s)$ and $\eta_p(s)$. We are interested to study zero free regions for $\zeta_p(s)$ and $\eta_p(s)$.

CHAPTER 2

Riemann zeta function

The Riemann zeta function is the complex function defined on the half plane

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } \Re(s) > 1. \tag{2.1}$$

This series converges absolutely for $\Re(s) > 1$ and converges uniformly on any compact subset within this region and hence establishes an analytic function within this domain.

$\zeta(s)$ can be analytically extended to the entire complex plane, excluding a pole at $s = 1$ of order one. The functional equation for $\zeta(s)$ by Riemann [13] is given by:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{2.2}$$

Here, we use the reflection formula of the gamma function $\Gamma(s)$,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \quad s \notin \mathbb{Z}.$$

to rewrite the functional equation in the following asymmetric form

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\zeta(s)\cos\left(\frac{\pi s}{2}\right). \tag{2.3}$$

$\zeta(s)$ also has a product representation known as Euler's identity or Euler's product defined as:

$$\zeta(s) = \prod_{p:\ prime} \frac{1}{(1-p^{-s})}, \quad \text{for } \Re(s) > 1. \tag{2.4}$$

From Euler's formula one can say, $\zeta(s)$ is non-vanishing for $\Re(s) > 1$. According to the results established by Hadamard [4] and Vallée-Poussin [12] in the late 19th century, $\zeta(s)$ is non-vanishing for $\Re(s) = 1$.

The symmetric form (2.2) for the $\zeta(s)$ implies that $\zeta(s)$ remains non-zero for $\Re(s) \leq 0$, except for the trivial zeros occurring at negative even integers.

So the remaining region $0 < \Re(s) < 1$ is the critical region for $\zeta(s)$, in this region, all the non-trivial zeros of $\zeta(s)$ are found.

Riemann's conjecture, as proposed in his work [13], states that all non-trivial zeros of $\zeta(s)$ are located on the critical line $\Re(s) = \frac{1}{2}$, known as Riemann hypothesis.

CHAPTER 3

Hurwitz zeta function

The Hurwitz zeta function [9] for $0 < a \leq 1$ is defined as :

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad \text{for } \Re(s) > 1. \tag{3.1}$$

The series converges absolutely for $\Re(s) > 1$ and converges uniformly on any compact subset within this region and hence establishes an analytic function within this domain. The $\zeta(s,a)$ generalizes $\zeta(s)$ for $a = 1$ i.e. $\zeta(s,1) = \zeta(s)$.

The functional equation of $\zeta(s,a)$ [9] for $0 < a \leq 1$ and $\Re(s) > 1$ is given by,

$$\zeta(1-s,a) = 2\Gamma(s)\cos\left(\frac{\pi s}{2}\right)(2\pi)^{-s} \sum_{n=1}^{\infty} \frac{\tan\left(\frac{\pi s}{2}\right)\sin(2\pi n a) + \cos(2\pi n a)}{n^s}. \tag{3.2}$$

Spira [14], explored the zero-free region of the Hurwitz zeta function. Additionally, he analysed the pattern of non-trivial zeros for the specific cases at $a = \frac{1}{3}$ and $a = \frac{2}{3}$.

3.1 Zero free region in right half plane

In the following theorem, Spira [14] showed that the $\zeta(s,a)$ is non-vanishing for $\Re(s) \geq 1+a$.

Theorem 3.1. *Let $s = \sigma + it$. We have $\zeta(s,a) \neq 0$, if $\Re(s) \geq 1+a$.*

Proof. From (3.1), we have

$$\begin{aligned} |\zeta(s,a)| &= \left| \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \right| \\ &\geq a^{-\sigma} - \sum_{n=1}^{\infty} \frac{1}{(n+a)^\sigma} \\ &> a^{-\sigma} - (1+a)^{-\sigma} - \int_1^{\infty} \frac{1}{(n+a)^\sigma} \,\mathrm{d}x \\ &= a^{-\sigma} - (1+a)^{-\sigma} - \frac{(1+a)^{1-\sigma}}{\sigma-1}. \end{aligned}$$

For $a^{-\sigma} - (1+a)^{-\sigma} - \frac{(1+a)^{1-\sigma}}{\sigma-1} > 0$ we need $\left(1+\frac{1}{a}\right)^{\sigma} > 1 + \frac{a+1}{\sigma-1}$. By the binomial theorem we know $\left(1+\frac{1}{a}\right)^{\sigma} > 1 + \frac{\sigma}{a}$. So, this will give $1 + \frac{\sigma}{a} > 1 + \frac{a+1}{\sigma-1}$. Hence the condition follows if we have $\sigma \geq a+1$. □

3.2 Zero free region in left half plane

Unlike $\zeta(s)$, the functional equation of $\zeta(s,a)$ is not symmetric, hence we can not say that it is non-vanishing in $\Re(s) \leq -1-a$. In this context, Spira showed that for $\Re(s) \leq -1$ and $|\Im(s)| \geq 1$, the $\zeta(s,a)$ is non-vanishing.

Theorem 3.2. *If $|t| \geq 1$ and $\sigma \leq -1$, then $\zeta(s,a) \neq 0$.*

Proof. From the functional equation (3.2) of $\zeta(s,a)$, we have

$$\begin{aligned} \zeta(1-s,a) &= \frac{2\Gamma(s)}{(2\pi)^s} \left[\cos\left(\frac{\pi}{2}s\right) \sum_{m=1}^{\infty} \frac{\cos 2\pi ma}{m^s} + \sin\left(\frac{\pi}{2}s\right) \sum_{m=1}^{\infty} \frac{\sin 2\pi ma}{m^s} \right] \\ &= \frac{\Gamma(s)}{(2\pi)^s} \sum_{m=1}^{\infty} \frac{\cos\left(\frac{\pi}{2}s - 2\pi ma\right)}{m^s}. \end{aligned}$$

Now separating the first term, we get

$$\zeta(1-s,a)=\frac{\Gamma(s)\cos\left(\frac{\pi}{2}s-2\pi a\right)}{(2\pi)^s}\left[1+\sum_{m=2}^{\infty}\frac{\cos\left(\frac{\pi}{2}s-2\pi ma\right)}{\cos\left(\frac{\pi}{2}s-2\pi a\right)}\frac{1}{m^s}\right].$$

Now, for $|t|\geq 1$, one can have the following inequality:

$$\left|\frac{\cos\left(\frac{\pi}{2}s-2\pi ma\right)}{\cos\left(\frac{\pi}{2}s-2\pi a\right)}\right|\leqslant\frac{e^{\pi t}+1}{e^{\pi t}-1}\leqslant\frac{e^{\pi}+1}{e^{\pi}-1}<1.09.$$

Then,

$$\begin{aligned}|\zeta(1-s,a)|&=\left|\frac{\Gamma(s)\cos\left(\frac{\pi}{2}s-2\pi a\right)}{(2\pi)^s}\right|\left|1+\sum_{m=2}^{\infty}\frac{\cos\left(\frac{\pi}{2}s-2\pi ma\right)}{\cos\left(\frac{\pi}{2}s-2\pi a\right)}m^{-s}\right|\\&\geq\left|\frac{\Gamma(s)\cos\left(\frac{\pi}{2}s-2\pi a\right)}{(2\pi)^s}\right|\left\{1-\sum_{m=2}^{\infty}\left|\frac{\cos\left(\frac{\pi}{2}s-2\pi ma\right)}{\cos\left(\frac{\pi}{2}s-2\pi a\right)}\right|m^{-\sigma}\right\}\\&\geq\left|\frac{\Gamma(s)\cos\left(\frac{\pi}{2}s-2\pi a\right)}{(2\pi)^s}\right|\left\{1-1.09\sum_{m=2}^{\infty}m^{-\sigma}\right\}.\end{aligned}$$

Now, we need $1-1.09(\zeta(\sigma)-1)>0$ so, $\zeta(\sigma)<1.917$.

But we already have $\zeta(\sigma)<\zeta(2)(\approx 1.6449)$ for $\sigma\geq 2$. Also, $\Gamma(s),(2\pi)^{-s}$ and $\cos\left(\frac{\pi}{2}s-2\pi a\right)$ are all non-zero in $|t|\geqslant 1,\sigma\geqslant 2$, hence the theorem holds. □

3.3 The trivial zeros

In the next theorem, Spira [14] proved that if the real part $\Re(s)<(4a+1+2[1-2a])$, then only zeros are trivial zeros, with an imaginary part $\Im(s)\leq 1$.

Theorem 3.3. *If $\sigma<-(4a+1+2[1-2a])$ and $|t|\leqslant 1$, then $\zeta(s,a)\neq 0$ except for trivial zeros on the negative real axis, one in each interval*

$$[-2n-4a-1,-2n-4a+1],$$

for $n\geq 1-2a$.

Proof. To prove this theorem, we will use Rouche's theorem. Let

$$u(s)=2\Gamma(s)(2\pi)^{-s}\cos\left(\frac{\pi}{2}s-2\pi a\right),$$

and

$$v(s) = u(s) \sum_{m=2}^{\infty} \frac{\cos\left(\frac{\pi}{2}s - 2\pi m a\right)}{\cos\left(\frac{\pi}{2}s - 2\pi a\right)} \frac{1}{m^s}.$$

Considering the rectangle with the four vertices $2n+1+4a \pm 1 \pm i$, and taking $2n+4a > 2$, on the horizontal boundary, one can get:

$$\left| \sum_{m=2}^{\infty} \frac{\cos\left(\frac{\pi}{2}s - 2\pi m a\right)}{\cos\left(\frac{\pi}{2}s - 2\pi a\right)} \frac{1}{m^s} \right| \leq \sum_{m=2}^{\infty} \left| \frac{\cos\left(\frac{\pi}{2}s - 2\pi m a\right)}{\cos\left(\frac{\pi}{2}s - 2\pi a\right)} \right| \frac{1}{m^\sigma}$$

$$\leq 1.09(\zeta(\sigma)) < 1.$$

On the vertical edges, one can get the following:

$$\left| \frac{\cos\left(\frac{\pi}{2}(2n+4a+it) - 2\pi m a\right)}{\cos\left(\frac{\pi}{2}(2n+4a+it) - 2\pi a\right)} \right|$$

$$= \left\{ \frac{\cos^2\left(\frac{\pi}{2}(2n+4a) - 2\pi m a\right) + \sinh^2 \frac{\pi}{2}t}{1 + \sinh^2 \frac{\pi}{2}t} \right\}^{1/2} \leqslant 1,$$

So, $|v(s)| \leqslant |u(s)|(\zeta(2) - 1) < |u(s)|$ for $\sigma \geqslant 2$. Given the symmetry properties, the single zero of $\zeta(s, a)$ within this rectangle resides on the real axis. □

3.4 The non-trivial zeros of $\zeta\left(s, \frac{1}{3}\right)$ and $\zeta\left(s, \frac{2}{3}\right)$

Spira [14] found the zeros of $\zeta\left(s, \frac{1}{3}\right)$ and $\zeta\left(s, \frac{2}{3}\right)$. He calculated the non-trivial zeros in the region $-1 \leq \sigma \leq 2,\ 0 \leq t \leq 100$. He used the regula falsi method to calculate the zeros.

One of the theorems by Davenport and Heilbronn in [2] tells that $\zeta\left(s, \frac{1}{3}\right)$ and $\zeta\left(s, \frac{2}{3}\right)$ will be zero for $\Re(s) > 1$ but Spira [14] did not get any zeros with $\Re(s) > 1$ in his calculation. The following two tables by Spira represent non-trivial zeros of Hurwitz zeta function at $a = \frac{1}{3},\ \frac{2}{3}$ with some error.

Table 1

Zeros Of $\zeta(s, 1/3)$

Re	**Im**	**Re**	**Im**
-3.356739	0.0	.382710	59.883303
-1.411510	0.0	.297271	61.558057
.431293	0.0	.087972	63.133469
-.159430	7.184833	.457258	65.195474
.342658	11.431373	.563711	66.783468
.241817	15.189346	.483299	69.515697
-.036837	17.768793	-.067285	70.180061
.591803	20.690440	.401279	72.270589
.193280	23.897873	.328136	74.292420
.127972	25.706324	.520017	75.643415
.334406	28.524914	.330148	77.920206
.429111	30.646264	.495597	79.533738
.462075	33.643477	-.147097	80.830920
-.147835	35.008686	.579068	82.764724
.506383	37.571524	.533603	84.515894
.472657	39.696042	.332095	86.302724
.151364	42.257863	.185455	88.479212
.343298	43.633735	.350208	88.904925
.093732	46.080690	.293221	91.542684
.622256	47.737933	.414854	92.638777
.442315	50.224064	.542324	94,468657
.159285	52.406133	.478417	96.639483
.140729	53.307053	-.016600	97.910995
.580453	56.035147	.316376	99.026723
.322000	57.568636		

Table 2

Zeros Of $\zeta(s, 2/3)$

Re	**Im**	**Re**	**Im**
-4.582225	0.0	.658788	60.192874
-2.629836	0.0	.136371	62.718934
-.534265	0.0	.694397	65.153529
.166871	10.821929	.145692	66.578130
.570050	16.605888	.510215	69.521528
.002611	20.525222	.799305	71.819557
.850931	24.340409	-.085295	73.824766
-.113795	28.078257	.459084	75.622482
.721490	30.792111	.745029	78.673253
.365790	34.136686	.430076	79.806836
.460172	37.583838	.163703	82.879125
.203952	39.160036	.288050	84.291484
.197874	43.008712	.766662	86.328553
.356658	45.347383	.533808	88.742453
.127852	47.671788	.014239	91.063946
.595766	50.633212	.718618	92.638399
.684428	52.731898	.348818	94.360457
.235421	55.856118	.359310	97.077760
.775805	57.447893	.742946	98.666525

CHAPTER 4

The manuscripts of N. S. Koshliakov

Recently, Dixit and Gupta [3] have uncovered a manuscript that was overlooked after World War II. The manuscript, authored by Nikolai Sergeevich Koshliakov, a distinguished Russian mathematician, contains significant contributions to differential equations and analytic number theory. The challenging circumstances under which Koshliakov wrote this manuscript and how it became known to the mathematical community are detailed in an article [1].

During the 1930s and even after World War II began, scholars in Leningrad faced repression. In 1942, during the blockade of Leningrad, Koshliakov and others were arrested on false charges and sentenced to 10 years of hard labor. After the verdict, Koshliakov was exiled to a camp in the Urals due to his health issues, including severe exhaustion and pellagra. Despite harsh conditions and a severe paper shortage, Koshliakov managed to write two lengthy memoirs "*Issledovanie nekotorykh voprosov analyticheskoi teorii rational'nogo i kvadratichnogo polya* (A study of some questions in the analytic theory of rational and quadratic fields)" and "*Issledovanie odnogo klassa transtsendentnykh funktsii, opredelyaemykh obobshchennym yravneniem Riemann* [11] " between 1943 and 1944. Unfortunately, the first memoir was lost while being transferred from jail to the mathematical community. However, it is believed that Koshliakov recreated its contents in three subsequent papers after his release from jail.

The second memoir, authored under the name N. S. Sergeev (Koshliakov's patronymic name), attracted the interest of renowned mathematicians I. M. Vinogradov, S. Bernstein, and Yu. V. Linnik, who recommended its publication.

A problem in heat conduction [10] serves as the basis for this analysis. Let us consider a sphere with radius R_2. Imagine heat radiating outward from its surface $r = R_2$ into a surrounding medium at zero temperature. The sphere starts with a uniform temperature of zero $u = 0$. Now, suppose there are internal heat sources distributed throughout a spherical region of radius R_1 (where $0 < R_1 < R_2$). The total rate of heat generation within the sphere is denoted by Q, then the problem involves determining the sphere's temperature at time $t > 0$. Assuming the material properties are characterized by thermal conductivity k, surface emissivity h, specific heat c, and density ρ, the relevant heat equation for this scenario is:

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial r^2}, \quad a = \sqrt{\frac{c}{k\rho}};$$
$$v|_{r=0} = 0, \ \frac{\partial v}{\partial r}|_{r=R_2} + \left(H - \frac{1}{R_2}\right) v|_{r=R_2} = 0, \ H = \frac{h}{k}.$$

Using the separation of variables method, one can easily confirm that the characteristic solution to the given system is:

$$\mu \cos \mu + p \sin \mu = 0, \quad \text{where } p = R_2 H - 1 \text{ and } \mu \text{ is its eigenvalue.} \tag{4.1}$$

Hamburger [5] obtained the following characterization of $\zeta(s)$.

Theorem 4.1. *Let*

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad \textit{and} \quad G(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s},$$

be two Dirichlet series absolutely convergent for large $\Re(s)$, *and suppose that* $F(s)$

can be analytically extended except for finitely many poles and obeying

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)F(s) = \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)G(1-s),$$

then $F(s) = G(s) = C\zeta(s)$*, where* C *is some constant.*

Hamburger's research [5], [6], [7], and [8] interests include the analytical behaviour of functions defined in the following equation.

For a Dirichlet series $F(s)$ that is valid in a certain right half-plane, the generalized functional equation links $F(s)$ with $G(s)$ as

$$F(1-s) = 2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)G(s), \tag{4.2}$$

where $G(s)$, in general, need not be a Dirichlet series.

Hamburger did not give any examples of such $G(s)$ and $F(s)$. It is important to consider Hurwitz's formula, valid for $\Re(s) > 1$ and $0 < a \leq 1$, given by

$$\zeta(1-s,a) = 2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\sum_{n=1}^{\infty}\frac{(\cos(2\pi na) + \tan\left(\frac{\pi s}{2}\right)\sin(2\pi na)}{n^s},$$

where $\zeta(s,a)$ is the Hurwitz zeta function.

The left-hand side represents a Dirichlet series, while the right-hand side does not. However, it is important to note that this relationship is not universally valid for all $s \in \mathbb{C}$. Koshliakov introduced such pairs of functions $F(s)$ and $G(s)$ exhibiting similar characteristics.

CHAPTER 5

Koshliakov's Generalizations of Riemann zeta function

Koshliakov [11, Chapter 1] studied the properties of $\zeta_p(s)$ and $\eta_p(s)$. In this chapter of thesis, I am presenting the the chapter 1 of Koshliakov manuscript [11].

In (4.1), replacing μ by $\pi\lambda$ and p by πp respectively, one gets the following equation:

$$p\sin(\pi\lambda) + \lambda\cos(\pi\lambda) = 0, \qquad p > 0. \tag{5.1}$$

Let $\lambda_1, \lambda_2, \lambda_3, \cdots$ represent the positive roots of the given transcendental equation (5.1) in increasing order.

Koshliakov considered the following Dirichlet series:

$$\zeta_p(s) := \sum_{j=1}^{\infty} \frac{p^2 + \lambda_j^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_j^2} \cdot \frac{1}{\lambda_j^s} \qquad \text{for } \Re(s) > 1.$$

The series $\zeta_p(s)$ converges absolutely for $\Re(s) > 1$ and converges uniformly on any compact subset within this region and hence establishes an analytic function within this domain.

Koshliakov explored an alternative generalization of the Riemann zeta function, namely,

$$\eta_p(s) := \sum_{k=1}^{\infty} \frac{(s, 2\pi pk)_k}{k^s} \quad \text{for} \quad \Re(s) > 1,$$

where

$$(s, \nu k)_k := \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-x} \left(\frac{k\nu - x}{k\nu + x} \right)^k x^{s-1} \mathrm{d}x.$$

The non-Dirichlet series, $\eta_p(s)$ converges absolutely for $\Re(s) > 1$ and converges uniformly on any compact subset within this region and hence establishes an analytic function within this domain.

In two limiting cases $p \to 0$ and $p \to \infty$, the roots λ_j of the equation (5.1) will be $j - \frac{1}{2}$ and j respectively ($j \in \mathbb{N}$). Hence, one can easily confirm,

$$\lim_{p\to\infty} \zeta_p(s) = \zeta(s), \quad \lim_{p\to\infty} \eta_p(s) = \zeta(s);$$

$$\lim_{p\to 0} \zeta_p(s) = (2^s - 1)\zeta(s), \quad \lim_{p\to 0} \eta_p(s) = (2^{1-s} - 1)\zeta(s).$$

To study the properties of $\zeta_p(s)$, Koshliakov [11] considered a complex valued function with the help of which he showed the analytic continuation of $\zeta_p(s)$. The function is given by,

$$\frac{1}{\sigma(z)e^{2\pi z} - 1}, \tag{5.2}$$

where

$$\sigma(z) = \frac{p+z}{p-z}.$$

The following corollary provides an alternate representation of equation (5.2) with which one can easily identify the poles of this equation.

Corollary 5.1.

$$\frac{1}{\sigma(z)e^{2\pi z} - 1} + \frac{1}{2} = \frac{1}{2}.\frac{p\cosh(\pi z) + z\sinh(\pi z)}{z\cosh(\pi z) + p\sinh(\pi z)}.$$

Proof. From (5.2), we have

$$\begin{aligned}\frac{1}{\sigma(z)e^{2\pi z}-1}+\frac{1}{2} &= \frac{p-z}{(p+z)e^{2\pi z}-(p-z)}+\frac{1}{2}\\ &= \frac{(p-z)e^{-\pi z}}{(p+z)e^{\pi z}-(p-z)e^{-\pi z}}+\frac{1}{2}\\ &= \frac{2pe^{-\pi z}-2ze^{-\pi z}+pe^{\pi z}+ze^{\pi z}-pe^{-\pi z}+ze^{-\pi z}}{2((p+z)e^{\pi z}-(p-z)e^{-\pi z})}\\ &= \frac{1}{2}.\frac{p\cosh(\pi z)+z\sinh(\pi z)}{z\cosh(\pi z)+p\sinh(\pi z)}.\end{aligned}$$

□

The function in (5.2) has infinite number of simple poles located at points $z=0$ and $z=\pm i\lambda_j$ with residues

$$R_0=\frac{1}{2\pi}\frac{1}{1+\frac{1}{\pi p}},\qquad R_{\pm i\lambda_j}=\frac{1}{2\pi}\frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2}.$$

Now by partial fraction decomposition by residue method, one gets the following equation

$$\frac{1}{\sigma(z)e^{2\pi z}-1}=-\frac{1}{2}+\frac{1}{2\pi}\frac{1}{1+\frac{1}{\pi p}}\frac{1}{z}+\frac{z}{\pi}\sum_{j=1}^{\infty}\frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2}\frac{1}{z^2+\lambda_j^2}. \tag{5.3}$$

In the two limiting cases $p\to 0$ and $p\to\infty$, (5.3) gives the following classical Euler's formulas respectively,

$$\frac{1}{e^{2\pi z}+1}=\frac{1}{2}-\frac{z}{\pi}\sum_{j=1}^{\infty}\frac{1}{z^2+\left(j-\frac{1}{2}\right)^2},$$

$$\frac{1}{e^{2\pi z}-1}=-\frac{1}{2}+\frac{1}{2\pi}\frac{1}{z}+\frac{z}{\pi}\sum_{j=1}^{\infty}\frac{1}{z^2+j^2}.$$

By using these formulae, one can have the following well-known relations:

$$\sum_{j=1}^{\infty}\frac{1}{\left(j-\frac{1}{2}\right)^{2k}}=\frac{(2\pi)^{2k}}{2(2k)!}(2^{2k}-1)B_k,\quad \sum_{j=1}^{\infty}\frac{1}{j^{2k}}=\frac{(2\pi)^{2k}}{2(2k)!}B_k,$$

where k is a positive integer and $B_1, B_2, B_3, \cdots$ are Bernoulli numbers.

Let,

$$\sigma_k^{(p)}=\sum_{j=1}^{\infty}\frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2}.\frac{1}{\lambda_j^{2k}},\quad k=1,2,3,4,\cdots$$

From (5.3) one gets

$$\frac{p-z}{(p+z)e^{2\pi z}-p+z} = -\frac{1}{2}+\frac{1}{2\pi z}\frac{p}{p+\frac{1}{\pi}}+\frac{z}{\pi}\sum_{j=1}^{\infty}\frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2} \times \frac{1}{1+\left(\frac{z}{\lambda_j}\right)^2}\frac{1}{\lambda_j^2}$$

$$\Rightarrow \frac{(p-z)2\pi z}{(p+z)e^{2\pi z}-p+z} = -\pi z+\frac{p}{p+\frac{1}{\pi}}+2z^2\sum_{j=1}^{\infty}\frac{p^2+\lambda_j^2}{p\left(p\frac{1}{\pi}\right)+\lambda_j^2} \times \left(\sum_{k=0}^{\infty}\frac{z^{2k}}{\lambda_j^{2k}}\frac{(-1)^k}{\lambda_j^2}\right)$$

$$\Rightarrow \frac{(p-z)2\pi z e^{-\pi z}}{(p+z)e^{\pi z}+(z-p)e^{-\pi z}}+\pi z = \frac{p}{p+\frac{1}{\pi}}+2z^2\sum_{j=1}^{\infty}\frac{p^2+\lambda_j^2}{p\left(p\frac{1}{\pi}\right)+\lambda_j^2} \times \left(\sum_{k=1}^{\infty}\frac{z^{2k-2}}{\lambda_j^{2k}}(-1)^{k-1}\right)$$

$$\Rightarrow \frac{(p-z)2\pi z e^{-\pi z}}{(p+z)e^{\pi z}+(z-p)e^{-\pi z}}+\pi z = \frac{p}{p+\frac{1}{\pi}}+2z^2\sum_{j=1}^{\infty}\frac{p^2+\lambda_j^2}{p\left(p\frac{1}{\pi}\right)+\lambda_j^2} \times \left(\sum_{k=1}^{\infty}\frac{z^{2k-2}}{\lambda_j^{2k}}(-1)^{k-1}\right)$$

$$\Rightarrow \frac{(p-z)2\pi z e^{-\pi z}}{(p+z)e^{\pi z}+(z-p)e^{-\pi z}}+\pi z = \frac{p}{p+\frac{1}{\pi}}+2\sum_{k=1}^{\infty}(-1)^{k-1}\sigma_k^{(p)}z^{2k}.$$

Thus, upon using the Maclaurian series of e^x in the left hand side, one gets the following equation,

$$\pi p z+(\pi^3 p+2\pi^2)\frac{z^3}{2!}+(\pi^5 p+4\pi^4)\frac{z^5}{4!}+\cdots = \left\{\frac{p}{p+\frac{1}{\pi}}+2\sum_{k=1}^{\infty}(-1)^{k-1}\sigma_k(p)z^{2k}\right\} \left\{(\pi p+1)z+(\pi^3 p+3\pi^2)\frac{z^3}{3!}+(\pi^5 p+5\pi^4)\frac{z^5}{5!}+\cdots\right\}.$$

Comparing coefficients of the same powers of z on both sides of the equation yields the expansion for $\sigma_1^{(p)}, \sigma_2^{(p)}, \ldots$, and so on. In particular,

$$\sigma_1^{(p)} = \frac{\pi^2}{6} . \frac{1 + \frac{3}{\pi p}\left(1 + \frac{1}{\pi p}\right)}{\left(1 + \frac{1}{\pi p}\right)^2},$$

$$\sigma_2^{(p)} = \frac{\pi^4}{90} . \frac{1 + \frac{6}{\pi p} + \frac{5}{\pi^2 p^2}\left(1 + \frac{1}{\pi p}\right)}{\left(1 + \frac{1}{\pi p}\right)^3}.$$

In general, we have

$$\sigma_k^{(p)} = \sum_{j=1}^{\infty} \frac{p^2 + \lambda_j^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_j^2} . \frac{1}{\lambda_j^{2k}} = \frac{(2\pi)^{2k}}{2(2k)!} B_k^{(p)}, \quad k = 1, 2, 3, 4, \cdots \tag{5.4}$$

where $B_k^{(p)}$ exhibits a kind of polynomial behaviour. The first two such polynomial values, $B_1^{(p)}$ and $B_2^{(p)}$, are given by

$$B_1^{(p)} = \frac{1}{6} \frac{1}{\left(1 + \frac{1}{\pi p}\right)^2} - \frac{1}{2} \frac{1}{1 + \frac{1}{\pi p}} + \frac{1}{2},$$

$$B_2^{(p)} = -\frac{1}{6} \frac{1}{\left(1 + \frac{1}{\pi p}\right)^3} + \frac{7}{10} \frac{1}{\left(1 + \frac{1}{\pi p}\right)^2} - \frac{1}{1 + \frac{1}{\pi p}} + \frac{1}{2}.$$

5.1 The analytic continuation of $\zeta_p(s)$

Koshliakov's work [11, Chapter 1, p. 17, Equation (16)] established the analytic continuation of $\zeta_p(s)$ across the complex plane, except a simple pole at $s = 1$.

Theorem 5.2. *The function $\zeta_p(s)$ can be extended analytically to the entire complex plane, with the exception of simple pole at $s = 1$. It can be shown that*

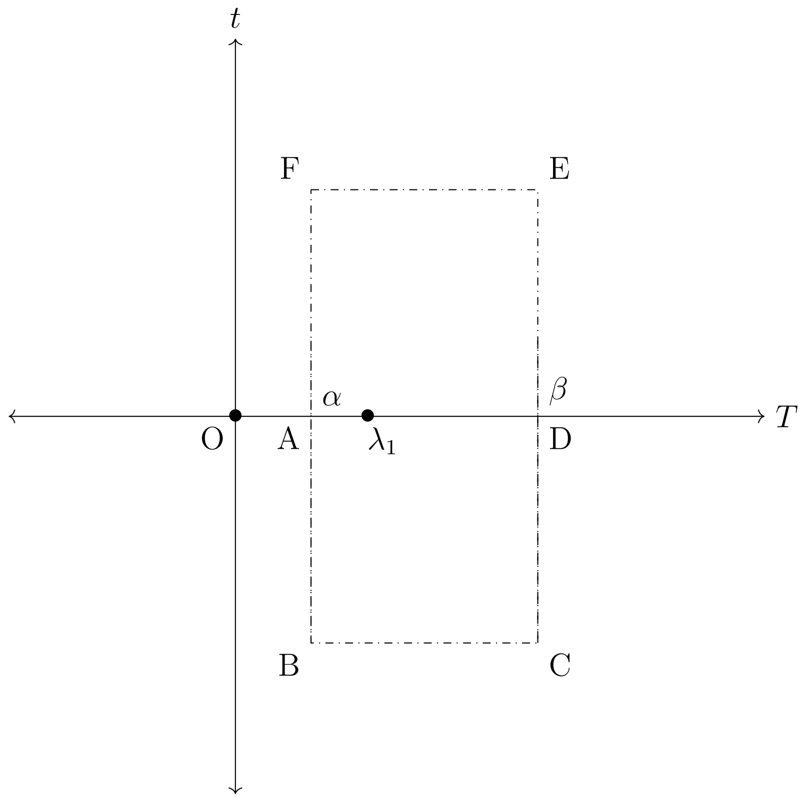
$$\zeta_p(s) = \frac{\alpha^{1-s}}{s-1} + \int_{\alpha}^{\alpha - i\infty} \frac{z^{-s}}{\sigma(iz)e^{2\pi iz} - 1} \mathrm{d}z + \int_{\alpha}^{\alpha + i\infty} \frac{z^{-s}}{\sigma(-iz)e^{-2\pi iz} - 1} \mathrm{d}z, \tag{5.5}$$

where

$$0 < \alpha < \lambda_1, \ \textit{and} \ \ \sigma(z) = \frac{p+z}{p-z}.$$

Proof. As a function of complex variable $z = T + it$, $\frac{1}{\sigma(iz)e^{2\pi iz} - 1}$ has the pole of order one at $z = 0$ and $z = \lambda_j$ with residues

$$\frac{1}{2\pi i} \frac{1}{\left(1 + \frac{1}{\pi p}\right)}, \quad \frac{1}{2\pi i} \frac{p^2 + \lambda_j^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_j^2},$$



respectively. Consider a closed loop in the complex plane. This loop has four sides and resembles a rectangle with height $2h$. The rectangle is centered on the real axis i.e. it is perfectly symmetrical with respect to the real axis. The sides of the rectangle intersect the real axis at points α and β.
Here, $\beta = n \in \mathbb{N}$ and $0 < \alpha < \lambda_1$. For all the roots of (5.1), one has to take $\beta \to \infty$.

By the Cauchy residue theorem, one gets the following equation for $\Re(s) > 1$,

$$\lim_{\beta \to \infty} \int_{ABCDEFA} \frac{z^{-s}}{\sigma(iz)e^{2\pi iz} - 1} \mathrm{d}z = \sum_{j=1}^{\infty} \frac{p^2 + \lambda_j^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_j^2} \frac{1}{\lambda_j^s},$$

since

$$\frac{1}{\sigma(iz)e^{2\pi iz} - 1} = -1 - \frac{1}{\sigma(-iz)e^{-2\pi iz} - 1}.$$

If we subdivide the original contour, ABCDEFA, into two smaller contours, ABCD and DEFA, we obtain the following equation,

$$\begin{aligned}\zeta_p(s) &= \lim_{\beta\to\infty}\int_{ABCD}\frac{z^{-s}}{\sigma(iz)e^{2\pi iz}-1}\mathrm{d}z + \lim_{\beta\to\infty}\int_{DEFA}\frac{z^{-s}}{\sigma(iz)e^{2\pi iz}-1}\mathrm{d}z\\ &= \lim_{\beta\to\infty}\int_{ABCD}\frac{z^{-s}}{\sigma(iz)e^{2\pi iz}-1}\mathrm{d}z + \lim_{\beta\to\infty}\int_{DEFA}\left(-1-\frac{1}{\sigma(-iz)e^{-2\pi iz}-1}\right)\frac{\mathrm{d}z}{z^s}\\ \Rightarrow \zeta_p(s) &= \lim_{\beta\to\infty}\int_{AFED}\frac{\mathrm{d}z}{z^{-s}} + \lim_{\beta\to\infty}\int_{ABCD}\frac{z^{-s}}{\sigma(iz)e^{2\pi iz}-1}\mathrm{d}z\\ &\qquad + \lim_{\beta\to\infty}\int_{AFED}\frac{z^{s}}{\sigma(-iz)e^{-2\pi iz}-1}\mathrm{d}z.\end{aligned}$$

The integrals along the segments BC and FE are examined by

$$\left|\int_{BC}\frac{z^{-s}}{\sigma(iz)e^{2\pi iz}-1}\mathrm{d}z\right| < Me^{-2\pi h}\int_{\alpha}^{\beta}\frac{\mathrm{d}x}{|x-ih|^{\sigma}}.$$

Here, M is a finite value that depends on h and β. Because of this, the right-hand side of the equation can be made arbitrarily small for any value of β as long as h is sufficiently large. As h approaches positive infinity, the integral along the segment BC necessarily tends towards zero for any value of β. The same logic can be applied to show that the integral along the segment FE also tends to zero as h approaches infinity for any value of β.

For sections CD and DE, we can easily derive the following inequality,

$$\left|\int_{CD}\frac{z^{-s}}{\sigma(iz)e^{2\pi iz}-1}\mathrm{d}z\right| < N\int_{0}^{\infty}\frac{e^{-2\pi t}}{(\beta^2+t^2)^{\sigma}}\mathrm{d}t.$$

Here, N represents a finite value independent of β. As β approaches positive infinity, the right-hand side of the inequality tends to zero. This behaviour can also be verified for the integral along path DE. Consequently, we arrive at the following equation

$$\zeta_p(s) = \int_{AFED}\frac{\mathrm{d}z}{z^s} + \int_{\alpha}^{\alpha-i\infty}\frac{z^{-s}}{\sigma(iz)e^{2\pi iz}-1}\mathrm{d}z + \int_{\alpha}^{\alpha+i\infty}\frac{z^{s}}{\sigma(iz)e^{2\pi iz}-1}\mathrm{d}z. \quad (5.6)$$

For the integral $\int_{AFED}\frac{\mathrm{d}z}{z^s}$ take the quadrilateral AFEDA and apply the Cauchy residue theorem over z^{-s}. One can get the following equality

$$\int_{AFED} \frac{\mathrm{d}z}{z^s} + \int_{DA} \frac{\mathrm{d}z}{z^s} = 0$$

$$\int_{AFED} \frac{\mathrm{d}z}{z^s} = \int_{AD} \frac{\mathrm{d}z}{z^s}$$

$$\Rightarrow \lim_{\beta\to\infty} \int_{AFED} \frac{\mathrm{d}z}{z^s} = \int_{\alpha}^{\infty} \frac{\mathrm{d}z}{z^s} = \frac{\alpha^{1-s}}{s-1}.$$

Thus from (5.6), one can get

$$\zeta_p(s) = \frac{\alpha^{1-s}}{s-1} + \int_{\alpha}^{\alpha-i\infty} \frac{z^{-s}}{\sigma(iz)e^{2\pi iz}-1}\,\mathrm{d}z + \int_{\alpha}^{\alpha+i\infty} \frac{z^{-s}}{\sigma(-iz)e^{-2\pi iz}-1}\mathrm{d}z.$$

which proves (5.5) and gives analytic continuation of $\zeta_p(s)$ in the whole complex plane except for a pole of order one at $s = 1$. □

5.2 The relation between $\zeta_p(s)$ and $\eta_p(s)$

Koshliakov [11, Chapter 1, p. 20, Equation (30)] showed that $\zeta_p(s)$ and $\eta_p(s)$ follow the Hamburger's functional equation (4.2). If we take α approaching to zero, then we have to restrict ourselves to the region $\Re(s) < 0$. We obtain the equation from (5.5) given by,

$$\zeta_p(s) = 2\sin\left(\frac{\pi s}{2}\right)\int_0^{\infty} \frac{x^{-s}}{\sigma(x)e^{2\pi x}-1}\mathrm{d}x. \tag{5.7}$$

Thus (5.7) implies that $\zeta_p(s)$ vanishes at the points $s = -2, -4, -6, \cdots$.

From (5.7) one gets the following equality:

$$\zeta_p(1-s) = 2\cos\left(\frac{\pi s}{2}\right)\int_0^{\infty} \frac{x^{s-1}}{\sigma(x)e^{2\pi x}-1}\mathrm{d}x, \quad \Re(s) > 1. \tag{5.8}$$

For $x > 0$, we know

$$\frac{p-x}{p+x} < 1.$$

Thus one can write

$$\frac{1}{\sigma(x)e^{2\pi x}-1} = \frac{\sigma(-x)e^{-2\pi x}}{1-\sigma(-x)e^{-2\pi x}} = \sum_{k=1}^{\infty}\left(\frac{p-x}{p+x}\right)^k e^{-2\pi kx},$$

and hence one gets the following representation of (5.8),

$$\zeta_p(1-s) = \frac{2\cos\left(\frac{\pi s}{2}\right)}{(2\pi)^s}\sum_{k=1}^{\infty}\frac{1}{k^s}\int_0^{\infty} e^{-x}\left(\frac{2k\pi p - x}{2k\pi p + x}\right)^k x^{s-1}\mathrm{d}x. \tag{5.9}$$

Now, let us take the integral from the right hand side of (5.9)

$$I_s(\lambda) = \int_0^\infty e^{-x} \left(\frac{\lambda - x}{\lambda + x}\right)^k x^{s-1} \mathrm{d}x, \quad \Re(x) < 0, \tag{5.10}$$

where $k \in \mathbb{N} \cup \{0\}$.

The given integral representation is the gamma function

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} \mathrm{d}x, \quad \Re(s) > 0.$$

Consider the integral of the form

$$e^{\frac{-\lambda}{k}\xi} \frac{\Gamma(s)}{\xi^s} = \int_0^\infty e^{-\left(x + \frac{\lambda}{k}\right)\xi} x^{s-1} \mathrm{d}x, \quad \xi > 0.$$

We now proceed by integrating each term on the right-hand side of the equation successively k times with respect to ξ over the interval (ξ, ∞). Next, we multiply the resulting expression by $e^{\frac{2\lambda}{k}\xi}$ and differentiate it with respect to ξ a total of k times. This sequence of operations lead us to the final result,

$$\Gamma(s) D_\xi^{(k)} \left\{ e^{2\frac{\lambda}{k}\xi} \int_\xi^\infty \int_{\xi_{k-1}}^\infty \cdots \int_{\xi_1}^\infty e^{-\frac{\lambda}{k}t} t^{-s} \mathrm{d}t \mathrm{d}\xi \cdots \mathrm{d}\xi_{k-1} \right\}$$
$$= \int_0^\infty e^{\left(\frac{\lambda}{k} - x\right)\xi} \left(\frac{\lambda - kx}{\lambda + kx}\right)^k x^{s-1} \mathrm{d}x.$$

Putting $\xi = k$ to get

$$\Gamma(s) D_\xi^{(k)} \left\{ e^{2\frac{\lambda}{k}\xi} \int_\xi^\infty \int_{\xi_{k-1}}^\infty \cdots \int_{\xi_1}^\infty e^{-\frac{\lambda}{k}t} t^{-s} \mathrm{d}t \mathrm{d}\xi \cdots \mathrm{d}\xi_{k-1} \right\}_{\xi = k}$$
$$= \int_0^\infty e^{\left(\frac{\lambda}{k} - x\right)k} \left(\frac{\lambda - kx}{\lambda + kx}\right)^k x^{s-1} \mathrm{d}x.$$

Hence by (5.10), we obtain the following equation

$$I_s(\lambda) = \Gamma(s) k^s e^{-\lambda} D_\xi^{(k)} \left\{ e^{2\frac{\lambda}{k}\xi} \int_\xi^\infty \int_{\xi_{k-1}}^\infty \cdots \int_{\xi_1}^\infty e^{-\frac{\lambda}{k}t} t^{-s} \mathrm{d}t \mathrm{d}\xi \cdots \mathrm{d}\xi_{k-1} \right\}_{\xi = k}.$$

The multiple integral within the curly brackets can be transformed into a single integral using the formula,

$$\int_\xi^\infty \int_{\xi_{k-1}}^\infty \cdots \int_{\xi_1}^\infty e^{-\frac{\lambda}{k}t} t^{-s} \mathrm{d}t \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{k-1} = \int_\xi^\infty \frac{(t - \xi)^{k-1}}{(k-1)!} e^{\frac{-\lambda}{k}t} t^{-s} \mathrm{d}t.$$

Thus, one gets the following representation of $I_s(\lambda)$

$$I_s(\lambda) = \Gamma(s) k^s e^{-\lambda} D_\xi^{(k)} \left\{ e^{\frac{2\lambda\xi}{k}} \int_\xi^\infty \frac{(t - \xi)^{k-1}}{(k-1)!} e^{\frac{-\lambda}{k}t} t^{-s} dt \right\}_{\xi = k}.$$

Koshliakov introduced a new function in his analysis $(s, \lambda)_k$. This function depends on two complex variables, s and λ, and is defined by

$$(s,\lambda)_k = k^s e^{-\lambda} D_\xi^{(k)} \left\{ e^{\frac{2\lambda\xi}{k}} \int_\xi^\infty e^{-\frac{\lambda}{k}t} \frac{(t-\xi)^{k-1}}{(k-1)!} \frac{\mathrm{d}t}{t^s} \right\}_{\xi=k}.$$

Thus, we obtain

$$I_s(\lambda) = \Gamma(s)(s,\lambda)_k.$$

We can now express equation (5.9) in the following form:

$$\zeta_p(1-s) = \frac{2\cos\left(\frac{\pi s}{2}\right)\Gamma(s)}{(2\pi)^s} \sum_{k=1}^\infty \frac{(s,2\pi pk)_k}{k^s}. \tag{5.11}$$

We know that,

$$\frac{(s,\lambda k)_k}{k^s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-kx} \left(\frac{\lambda - x}{\lambda + x}\right)^k x^{s-1}\mathrm{d}x. \tag{5.12}$$

Consider Koshliakov's second generalized Riemann function [11, chapter 1, p. 20, Equation (29)], symbolized by $\eta_p(s)$. This function is defined by an infinite series:

$$\eta_p(s) = \sum_{k=1}^\infty \frac{(s,2\pi pk)_k}{k^s}, \quad \Re(s) > 1. \tag{5.13}$$

The function $\eta_p(s)$ is absolutely and uniformly convergent in $\Re(s) > 1$. Based on (5.11), we derive the following equality:

$$\zeta_p(1-s) = \frac{2\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\eta_p(s)}{(2\pi)^s}. \tag{5.14}$$

The restriction requiring $\Re(s) > 1$ in the proof can actually be relaxed. This is because both sides of equation (5.14) demonstrate analytic behaviour with respect to the variables, except for a few isolated points. These isolated points are the singularities occurring at s = 0.

To analyze the singularities of the function $\eta_p(s)$, let us consider (5.12). We arrive at the following equality:

$$\begin{aligned} \eta_p(s) =& \frac{1}{\Gamma(s)} \int_0^\infty \sum_{k=1}^\infty e^{-kx} \left(\frac{p - \frac{x}{2\pi}}{p + \frac{x}{2\pi}}\right)^k x^{s-1}\mathrm{d}x \\ =& \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{\sigma\left(\frac{x}{2\pi}\right)e^x - 1}\mathrm{d}x \qquad \Re(s) > 1. \end{aligned}$$

Now consider the Hankel integral under the assumption that $\Re(s) > 1$,

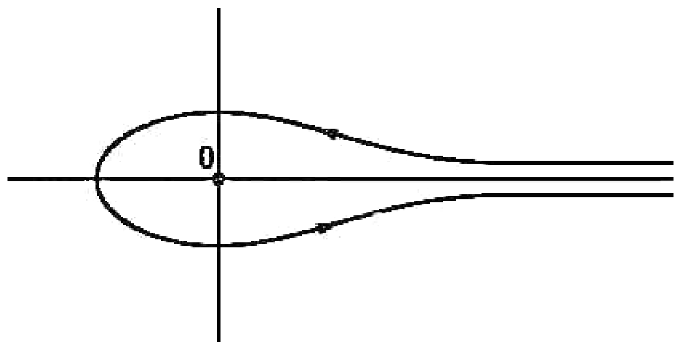
$$\int_\infty^{(0+)} \frac{(-z)^{s-1}\mathrm{d}z}{\sigma\left(\frac{z}{2\pi}\right)e^z - 1}, -\pi < \arg(-z) < \pi. \tag{5.15}$$

The integration contour avoids all the poles of the integrand function. These poles are located at $\pm 2\pi i\lambda_j, (j = 1, 2, \ldots)$, and hold for $\Re(s) > 1$,

$$\int_{\infty}^{(+0)} \frac{(-z)^{s-1}}{\sigma\left(\frac{z}{2\pi}\right)e^z - 1}\mathrm{d}z = \left\{e^{\pi i(s-1)} - e^{-\pi i(s-1)}\right\} \int_0^{\infty} \frac{x^{s-1}\mathrm{d}x}{\sigma\left(\frac{x}{2\pi}\right)e^x - 1}.$$

From Euler's reflection formula, one gets the the following integral representation of the function $\eta_p(s)$:

$$\eta_p(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{\sigma\left(\frac{z}{2\pi}\right)e^z - 1}\mathrm{d}z. \tag{5.16}$$



Consider the integration contour C, it starts at infinity, circles around $z = 0$ in a positive direction, and then returns to its starting point. Because integral (5.16) defines a single-valued analytic function of the variable s for all its values, the singularities of $\eta_p(s)$ must coincide with the singularities of $\Gamma(1-s)$, which occur at $s = 1, 2, \cdots$. However, we previously presented that $\eta_p(s)$ is also an analytic function of s for $\Re(s) > 1$. Therefore, the only remaining singular point of $\eta_p(s)$ within this region is $s = 1$. From (5.16), it follows that

$$\lim_{s\to 1} \frac{\eta_p(s)}{\Gamma(1-s)} = -\frac{1}{2\pi i} \int_C \frac{\mathrm{d}z}{\sigma\left(\frac{z}{2\pi}\right)e^z - 1}.$$

The expression on the right-hand side represents the residue of the integrand at the pole $z = 0$. Using (5.3), Koshliakov determined that this residue is equal to $\frac{1}{1+\frac{1}{\pi p}}$. Consequently, the function $\eta_p(s)$ possesses a single singular point at $s = 1$. This singularity is a pole of order one with a residue of $\frac{1}{1+\frac{1}{\pi p}}$. Now, we can claim that the functional equation (5.14), previously established for $\Re(s) > 1$, remains valid for all values of s except for the isolated points mentioned earlier. Some proven properties of $\zeta_p(s)$ and $\eta_p(s)$ are given by:

$$\zeta_p(0) = -\frac{1}{2}.\frac{1}{1+\frac{1}{\pi p}}, \quad \eta_p(0) = -\frac{1}{2};$$

$$\lim_{s\to 1}\left\{\zeta_p(s) - \frac{1}{s-1}\right\} = c + \log 2\pi + 2\eta_p'(0);$$

$$\lim_{s\to 1}\left\{\eta_p(s) - \frac{1}{1+\frac{1}{\pi p}}\frac{1}{s-1}\right\} = C + \log 2\pi + 2\zeta_p'(0);$$

$$\zeta_p(-2k) = 0; \quad \eta_p(-2k) = 0 \quad , k = 1, 2, \ldots;$$

$$\zeta_p(2k) = \frac{(2\pi)^{2k}}{2\cdot(2k)!}B_k^{(p)}, \ \eta_p(-(2k-1)) = \frac{(-1)^k B_k^{(p)}}{2k}.$$

In particular,

$$\zeta_p(2) = \frac{\pi^2}{6}\cdot\frac{1}{\left(1+\frac{1}{\pi p}\right)^2} - \frac{\pi^2}{2}\cdot\frac{1}{1+\frac{1}{\pi p}} + \frac{\pi^2}{2},$$

$$\eta_p(-1) = -\frac{1}{12}\cdot\frac{1}{\left(1+\frac{1}{\pi p}\right)^2} + \frac{1}{4}\cdot\frac{1}{1+\frac{1}{\pi p}} - \frac{1}{4}.$$

5.3 Asymptotic estimations of Koshliakov zeta functions

Koshliakov [11, Chapter 1, p. 24, Equation (43)] gave an asymptotic estimation of functions $\zeta_p(s)$ and $\eta_p(s)$ for $s = \sigma + it$ and for large values of $|t|$.

To arrive at this estimate, we analyze two separate cases.

1) $\sigma > 0$ and

2) $\sigma < 0$

For $\Re(s) > 1$, the following equality holds

$$\zeta_p(s) = \zeta(s) + \sum_{j=1}^{\infty}\left\{\frac{1}{\lambda_j^s} - \frac{1}{j^s}\right\} - \frac{p}{\pi}\sum_{j=1}^{\infty}\frac{1}{p\left(p+\frac{1}{\pi}\right)+\lambda_j{}^2}\frac{1}{\lambda_j^s}. \tag{5.17}$$

Let us take $\zeta(s)$ in the following form of series

$$\zeta(s) = \sum_{n=0}^{N}\frac{1}{(n+1)^s} + \frac{(N+1)^{1-s}}{s-1} + \sum_{n=N}^{\infty} f_n(s), \tag{5.18}$$

where

$$f_n(s) = \frac{1}{s-1}\left\{\frac{1}{(n+2)^{s-1}} - \frac{1}{(n+1)^{s-1}}\right\} + \frac{1}{(n+2)^s} = s\int_{n+1}^{n} \frac{u-n}{(u+1)^{s+1}}\mathrm{d}u.$$

Equation (5.18) is known to provide an analytic continuation of $\zeta(s)$, for values of $\sigma > 0$. This is because the infinite series $\sum_{n=0}^{\infty} f_n(s)$ converges absolutely and uniformly in the region $\sigma > \delta$, where $(\delta > 0)$, and its terms are analytic functions in that region. Consequently, for $\sigma > 0$, equality (5.17) can be adopted as the definition of a new function, denoted by $\zeta_p(s)$. To prove it, one can use the following inequality

$$\left| s\sum_{j=m}^{\infty}\left(\frac{1}{\lambda_j^s} - \frac{1}{j^s}\right)\right| < |s|\sum_{j=m}^{\infty}\int_{\lambda_j}^{s}\frac{\mathrm{d}u}{u^{\sigma+1}},$$

where $j - \frac{1}{2} < \lambda_j < j$. Since,

$$\int_{\lambda_j}\frac{du}{u^{\sigma+1}} < \int_{\lambda_j}^{j}\frac{du}{\lambda_j^{\sigma+1}} = \frac{j-\lambda_j}{\lambda_j^{\sigma+1}} < \frac{1}{2}\frac{1}{\lambda_j^{\sigma+1}},$$

it is clear that the series,

$$\sum_{j=1}^{\infty}\left\{\frac{1}{\lambda_j^s} - \frac{1}{j^s}\right\},$$

will be uniformly convergent for $\sigma > \delta; (\delta > 0)$. Similarly, it follows the same for the series

$$\sum_{j=1}^{\infty}\frac{1}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2}.\frac{1}{\lambda_j^s},$$

which proves the statement that the series (5.16) is valid in $\Re(s) > 0$.

As the absolute value of $|t|$ increases, estimating the value of $\zeta_p(s)$ becomes increasingly similar to estimating the value of the Riemann zeta function. This makes sense because both the functions share the same singularity at $s = 1$, as demonstrated in equation (5.18).

For the second case, when $\sigma < 0$, we need to use the functional equation

$$\zeta_p(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\eta_p(1-s),$$

where

$$\eta_p(1-s) = \sum_{k=1}^{\infty}\frac{(1-s, 2k\pi p)_k}{k^{1-s}}.$$

The series will converge absolutely and uniformly for $\sigma < -\delta$; $(\delta > 0)$. This property implies that

$$\zeta_p(s) = O\left(|t|^{\frac{1}{2}-\sigma}\right), \quad \sigma < 0. \tag{5.19}$$

Thus the desired asymptotic estimation of function $\zeta_p(s)$ turns out to be the same as for the ordinary Riemann function,

$$\zeta_p(s) = O\left(|t|^{\lambda} \log |t|\right), \qquad \text{at } t \to \infty, \tag{5.20}$$

where

$$\lambda = \frac{1}{2} - \sigma,\ (\sigma \leq 0); \lambda = \frac{1}{2}, \quad (0 \leq \lambda \leq 1/2);$$
$$\lambda = 1 - \sigma, \quad (\frac{1}{2} \leq \sigma \leq 1);\ \lambda = 0, \quad (1 \leq \sigma).$$

5.4 Series representation of $(s, \lambda)_k$

Koshliakov [11, Chapter 1, p. 25, Equation (50)] gave the series representation by the help of which he studied some properties of $(s, \lambda)_k$. The function $\eta_p(s)$ as an infinite series (5.13), involves the function $(s, \lambda k)_k$ at $\lambda = 2\pi p > 0$. We can compute these functions either using the following formula

$$(s, \lambda k)_k = k^s e^{-\lambda k} D_\xi^{(k)} \left\{ e^{2\lambda\xi} \int_\xi^\infty e^{-\lambda t} \frac{(t-\xi)^{k-1}}{(k-1)!} \frac{\mathrm{d}t}{t^s} \right\}_{\xi = k},$$

or by formula

$$(s, \lambda k)_k = k^s e^{-\lambda k} D_\xi^{(k)} \left\{ e^{2\lambda\xi} \int_{\xi_\varphi}^\infty \int_{\xi_{k-1}}^\infty \cdots \int_{\xi_1}^\infty \frac{e^{-\lambda t}}{t^s} \mathrm{d}t \mathrm{d}\xi, \cdots \mathrm{d}\xi_{k-1} \right\}. \tag{5.21}$$

Consider a k-dimensional integral on the right-hand side of the equation. If we differentiate this integral using Leibniz rule, we obtain

$$D_\xi^{(k)}[u.v] = \sum_{r=0}^{k} \frac{k!}{r!(k-r)!} u^{(r)} v^{(k-r)},$$

Substituting our functions into u and v:

$$u = e^{2\lambda\xi},\ v = \int_\xi^\infty \int_{\xi_{k-1}}^\infty \cdots \int_{\xi_1}^\infty \frac{e^{-\lambda t}}{t^s} \mathrm{d}t \mathrm{d}\xi_1 \mathrm{d}\xi_{k-1}.$$

Then by Leibniz rule, the given equation holds

$$(s, \lambda k)_k = (-1)^k + e^{\lambda k} \sum_{r=1}^{k} \frac{(-1)^{k+r} k! (2\lambda)^r}{r!(k-r)!} \int_k^\infty \int_{\xi_{r-1}}^\infty \cdots \int_{\xi_1}^\infty e^{-\lambda t} \left(\frac{k}{t}\right)^s \mathrm{d}t d\xi_1 \cdots d\xi_{k-1}. \quad (5.22)$$

After evaluating the multiple integrals on the right-hand side, we arrive at the following result

$$(s, \lambda k)_k = (-1)^k + e^{\lambda k} \sum_{r=1}^{k} \frac{(-1)^{k+r} k! (2\lambda)^r}{r!(k-r)!} \int_{-k}^\infty e^{-\lambda t} \frac{(t-k)^{r-1}}{(r-1)!} \left(\frac{k}{t}\right)^s \mathrm{d}t.$$

By substituting appropriate values, we can rewrite the integral on the right-hand side in terms of incomplete gamma function,

$$Q_\mu(s) = \int_\mu^\infty e^{-t} t^{s-1} \mathrm{d}t, \quad \mu > 0.$$

Hence, the following equation holds,

$$(s, \lambda k)_k = e^{\lambda k} \sum_{r=1}^{k} \sum_{q=0}^{r-1} \frac{(-1)^{k+r+q} k! 2^r (\lambda k)^{q+s}}{r!(k-r)!(r-q-1)!} Q_{\lambda k}(r-q-s) + (-1)^k.$$

In particular,

$$(s, \lambda)_1 = 2e^\lambda \lambda^s Q_\lambda(1-s) - 1,$$

$$(s, \lambda)_2 = 4e^{2\lambda} 2\lambda^s Q_{2\lambda}(2-s) - 4e^{2\lambda} 2\lambda^s Q_{2\lambda}(1-s) + 1.$$

5.5 Properties of $(s, \lambda)_k$

Koshliakov gave some necessary properties of the function $(s, \lambda k)_k$, given by:

Property 1: For $\lambda > 0$

$$(0, \lambda k)_k = 1 \quad (5.23)$$

A straightforward approach to proving this statement involves utilizing the formula (5.21). Applying this formula, we obtain the following equation

$$\begin{aligned}(0, \lambda k)_k &= e^{-\lambda k} D_\xi^{(k)} \left\{ e^{2\pi\xi} \int_\xi^\infty \int_{\xi_{k-1}}^\infty \cdots \int_{\xi_1}^\infty e^{-\lambda t} \mathrm{d}t \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{k-1} \right\} \\ &= e^{\lambda k} \frac{1}{\lambda^k} \lambda^k e^{-\lambda k} = 1.\end{aligned}$$

Property 2**:** For $\lambda > 0$, we have

$$(-1, \lambda k)_k = 1 + \frac{2}{\lambda}. \tag{5.24}$$

Furthermore, based on the same formula (5.21) and the equality,

$$e^{2\lambda\xi} \int_{\xi}^{\infty} \int_{\xi_{k-1}}^{\infty} \cdots \int_{\xi_1}^{\infty} e^{-\lambda t} t \mathrm{d}t \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{k-1} = \frac{e^{\lambda}(\lambda\xi + k)}{\lambda^{k+1}},$$

we have $(-1, \lambda k)_k = \frac{e^{-\lambda k}}{k} D_{\xi}^{(k)} \left\{ \frac{e^{\lambda\xi}(\lambda\xi+k)}{\lambda^{k+1}} \right\}_{\xi=k} = 1 + \frac{2}{\lambda}$.

Property 3**:** At $\lambda > 0$

$$(s, -\lambda)_{-k} = (s, \lambda)_k. \tag{5.25}$$

We obtain this property by examining the following representation of the function $(s, \lambda)_k$,

$$(s, \lambda)_k = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-x} \left(\frac{\lambda - x}{\lambda + x} \right)^k x^{s-1} \mathrm{d}x = (s, -\lambda)_{-k}. \tag{5.26}$$

Property 4**:** At $\lambda > 0$,

$$\lim_{k \to \infty} (s, \lambda k)_k = \frac{1}{\left(1 + \frac{2}{\lambda}\right)^s}. \tag{5.27}$$

We rewrite formula (5.26) as follows:

$$(s, \lambda k)_k = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-x} \left(1 - \frac{x}{\lambda k}\right)^k \left(1 + \frac{x}{\lambda k}\right)^{-k} x^{s-1} \mathrm{d}x,$$

and because,

$$\lim_{k \to +\infty} \left(1 - \frac{x}{\lambda k}\right)^k = e^{\frac{-x}{\lambda}}; \qquad \lim_{k \to +\infty} \left(1 + \frac{x}{\lambda k}\right)^{-k} = e^{\frac{-x}{\lambda}},$$

hence, we get,

$$\begin{aligned} \lim_{k \to \infty} (s, \lambda k)_k =& \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-x} e^{-\frac{x}{\lambda}} e^{-\frac{x}{\lambda}} x^{s-1} \mathrm{d}x \\ =& \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-x\left(1+\frac{2}{\lambda}\right)} x^{s-1} \mathrm{d}x \\ =& \frac{1}{\left(1 + \frac{2}{\lambda}\right)^s}. \end{aligned}$$

Hence the property follows.

Property 5**:** The following equality holds, when $-1 < x < 1$,

$$\sum_{n=0}^{\infty} (n + 1, \lambda k)_k x^n = \frac{(1, \lambda k(1 - \lambda))_k}{(1 - x)}. \tag{5.28}$$

From the integral representation of $(s,\lambda)_k$, one gets the following

$$(n+1,\lambda k)_k = \frac{k^{n+1}}{n!}\int_0^\infty e^{-kx}\left(\frac{\lambda-x}{\lambda+x}\right)^k x^n \mathrm{d}x.$$

Then we have

$$\begin{aligned}\sum_{n=0}^{\infty}(n+1,\lambda k)_k x^n &= k\int_0^\infty e^{-k(1-x)t}\left(\frac{\lambda-t}{\lambda+t}\right)^k \mathrm{d}t \\ &= \frac{1}{1-x}\int_0^\infty e^{-u}\left(\frac{\lambda k(1-x)-u}{\lambda k(1-x)+u}\right)^k \mathrm{d}u = \frac{(1,\lambda k(1-\lambda))_k}{(1-x)}.\end{aligned}$$

Therefore, we get the formula (5.28).

Property 6**:** For $\lambda > 0$ the given relation holds,

$$\sum_{n=0}^{\infty}\frac{(-1)^r(-r,\lambda)_1}{r!(n-r)!}x^{n-r} = \frac{2}{\lambda^n}\sum_{q=0}^{n}\frac{(-1)^q(\lambda(x-1))^{n-q}}{(n-q)!} - \frac{(x-1)^n}{n!}. \tag{5.29}$$

Koshliakov demonstrated this formula by the properties of $(s,\lambda)_1$ and utilizing a specific relation,

$$Q_\lambda(r+1) = e^{-\lambda}\left\{\lambda^r + r\lambda^{r+1} + r(r-1)\lambda^{r-2}\cdots r(r-1)\cdots 2\cdot 1\right\}.$$

After evaluating the sum on the left-hand side of formula (5.29), we obtain the following summation

$$\begin{aligned}&2\sum_{r=0}^{n}\sum_{q=0}^{r}\frac{(r-1)^r x^{n-\mu}}{(r-q)!(n-r)!}\frac{1}{\lambda^q} - \frac{(x-1)^n}{n!} \\ &= 2\sum_{q=0}^{n}\sum_{r=q}^{n}\frac{(-1)^r x^{n-r}}{(r-q)!(n-r)!}\frac{1}{\lambda^q} - \frac{(x-1)^n}{n!}.\end{aligned}$$

On the right-hand side, we substitute the variable r with a new variable s, defined as $s = r - q$. This substitution gives the following equation

$$\sum_{s=0}^{n-q}\frac{(-1)^{q+s}x^{n-q-s}}{s!(n-q-s)!} = (-1)^q\frac{(x-1)^{n-q}}{(n-q)!}.$$

This result leads directly to the formula (5.29).

Property 7**:** For $x > 0$ the following formula turns out to be valid,

$$D_x^{(m-1)}\left\{\frac{(s,\lambda(x+k)_k)}{(x+k)^s}\right\} = (-1)^{m-1}\frac{\Gamma(m+s-1)}{\Gamma(s)}\frac{(m+s-1,\lambda(x+k))_k}{(x+k)^{m+s-1}}. \tag{5.30}$$

To demonstrate this, we differentiate the expression $(m-1)$ times with respect to the variable x

$$\frac{(s,\lambda(x+k))_k}{(x+k)^s}=\frac{1}{\Gamma(s)}\int_0^\infty e^{-(x+k)t}\left(\frac{\lambda-t}{\lambda+t}\right)^k t^{s-1}\mathrm{d}t.$$

Then one gets,

$$\left\{\frac{(s,\lambda(x+k))_k}{(x+k)^s}\right\}=\frac{(-1)^{m-1}}{\Gamma(s)(x+k)^m}\int_0^\infty e^{-u}\left(\frac{\lambda(x+k)-u}{\lambda(x+k)+u}\right)^k u^{m+s-2}\mathrm{du}.$$

Hence, (5.30) follows directly.

In particular, we obtain,

$$\frac{1}{(m-1)!}D_x^{(m-1)}\left\{\frac{(1,\lambda(x+k))_k}{x+k}\right\}=\frac{(-1)^{m-1}(m,\lambda(x+k))_k}{(x+k)^m}.$$

Property 8**:** This equality holds for $\Re(s)>1$,

$$\int_x^\infty \frac{(s,\lambda(x+k))_k}{(x+k)^s}\mathrm{dx}=-\frac{1}{s-1}D_x^{(1)}\left\{\frac{(s-1,\lambda(x+k))_k}{(x+k)^{s-1}}\right\}. \tag{5.31}$$

To establish this formula, we use formula (5.30) by taking $m=2$ and substituting s with $s-1$. Hence, the following equation holds,

$$\frac{(s,\lambda(x+k))k}{(x+k)^s}=\frac{-1}{s-1}D_x^{(1)}\left\{\frac{(s-1,\lambda(x+k))_k}{(x+k)^{s-1}}\right\}.$$

Integrating both sides of this equation, we arrive at the equation (5.31).

Property 9**:** The following equality holds for $x>0$,

$$\int_0^x \frac{(1,\lambda(x+k))_k}{x+k}\mathrm{d}x=\log\left(1+\frac{x}{k}\right)+D_s^{(1)}\left\{(s,\lambda k)_k-(s,\lambda(x+k))_k\right\}_{s=0}. \tag{5.32}$$

We demonstrate this statement by utilizing the formula (5.26). This formula gives

$$\begin{aligned}\int_0^x \frac{(1,\lambda(x+k))k}{(x+k)}dx &= \int_0^x e^{-kt}\left(\frac{\lambda-t}{\lambda+t}\right)^k\frac{1-e^{-xt}}{t}\mathrm{d}t\\ &=\lim_{s\to 0}\Gamma(s)\left\{\frac{(s,\lambda k)_k}{k^s}-\frac{(s,\lambda(x+k))_k}{(x+k)^s}\right\},\end{aligned}$$

where,

$$\begin{aligned}(s,\lambda k)_k &= (0,\lambda k)_k+\left\{D_s^{(1)}(s,\lambda k)_k\right\}_{s=0}s+\cdots\\ &=1+\left\{D_s^{(1)}(s,\lambda k)_k\right\}_{s=0}s+\cdots,\\ (s,\lambda(x+k))_k &= (0,\lambda(x+k))_k+\left\{D_s^{(1)}(s,\lambda(x+k))_k\right\}_{s=0}s+\cdots\\ &=1+\left\{D_s^{(1)}(s,\lambda(x+k))_k\right\}_{s=0}s+\cdots\end{aligned}$$

and

$$k^{-s} = 1 - \log ks + \cdots, \quad (x+k)^{-s} = 1 - \log(x+k)s + \cdots.$$

Therefore,

$$\int_0^x \frac{(1, \lambda(x+k))_k}{x+k} \mathrm{d}x = \lim_{s\to 0}\Gamma(s+1)\left[\left\{D_s^{(1)}(s, \lambda k)_k\right.\right.$$
$$\left.\left. D_s^{(\prime)}(s, \lambda(x+k))_k\right\}_{s=0} + \log\frac{x+k}{k}\right].$$

Based on the above steps, we obtain relation (5.32).

In particular, we have:

$$\int_0^x \frac{(1, \lambda(x+1))_1}{(x+1)} \mathrm{d}x = \log(1+x) + 2\left\{e^{\lambda(x+1)}Q_{\lambda(1+x)}(0) - e^{\lambda}Q_{\lambda}^{(0)}\right\}.$$

From the formula of $(s, \lambda)_1$, it follows that

$$D_s^{(1)}\left\{(s,\lambda)_1 - (s, \lambda(x+1))_1\right\}_{s=0}$$
$$= 2e^{\lambda}\left\{\frac{Q_\lambda(1-s) - (1+x)^s e^{\lambda x} Q_{\lambda(1+x)}(1-s)}{s}\right\}_{s=0}.$$

But,

$$Q_\lambda(1-s) = Q_\lambda(1) - Q'(1)s + \cdots;$$
$$(1+x)^s = 1 + \log(1+x)s + \cdots;$$

therefore,

$$Q_\lambda(1-s) - (1+x)^s e^{\lambda x} Q_{\lambda(1+x)}(1-s)$$
$$= -2e^{\lambda(x+1)}\left\{Q'_\lambda(1) + e^{-\lambda x}\log(1+x) - Q'_{\lambda(1+x)}(1)\right\}s + \cdots$$

where, $Q'_\lambda(1) = \log \lambda e^{-\lambda} + Q_\lambda(0)$,

Let us make sure that

$$D_s^{(1)}\left\{(s,\lambda)_1 - (s,\lambda(x+1))_1\right\}_{s=0} = 2\left\{e^{\lambda(1+x)}Q_{\lambda(1+x)}(0) - e^{\lambda}Q_\lambda(0)\right\}.$$

Property 10**:** The following equality holds for $\alpha > 0$ and $n \geq 1$,

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{(s,\lambda k)k}{s(s+1)\dots(s+n)}\left(\frac{x}{k}\right)^{s+n}\mathrm{d}s$$
$$=\begin{cases}0,\ \text{at } k\geqslant x;\\ \frac{(-1)^k(x-k)^n}{k^n n!}\\ +\frac{e^{\lambda k}}{k^n}\sum_{r=1}^{k}\frac{(-1)^{k+r}k!(2\lambda)^r}{r!(k-r)!}\int_k^x e^{-\lambda t}\frac{(x-t)^n}{n!}\frac{(t-k)^{r-1}}{(r-1)!}\mathrm{d}t, \qquad \text{at } k\leqslant x.\end{cases} \tag{5.33}$$

To demonstrate this formula, let us consider the equality (5.22). Applying this, we will arrive at

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{(s,\lambda k)k}{s(s+1)\dots(s+n)}\left(\frac{x}{k}\right)^{s+n}\mathrm{d}s$$
$$=(-1)^k\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\left(\frac{x}{k}\right)^{s+n}}{s(s+1)\dots(s+n)}\mathrm{d}s+e^{\lambda k}\sum_{r=1}^{k}(-1)^{k+r}\frac{k!(2\lambda)^r}{r!(k-r)!}$$
$$\times\int_k^\infty e^{-\lambda\cdot t}\left(\frac{t}{k}\right)^n\frac{(t-k)^{r-1}}{(r-1)!}\frac{x}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\left(\frac{x}{t}\right)^{s+n}}{s(s+1)\dots(s+n)}\mathrm{d}s\mathrm{d}t. \tag{5.34}$$

To proceed, we examine each of the two cases individually

1) $x < k$ and

2) $x > k$.

In this first case, when we evaluate the double integral on the right-hand side of equality (5.34) with respect to the variable t, we find that the integration range is limited to $t > k$. Consequently, for this specific case, we have $t > x$. We can then utilize the following well-known property:

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\left(\frac{x}{t}\right)^{s+n}}{s(s+1)\cdots(s+n)}\mathrm{d}s=0 \tag{5.35}$$

from which we will get the following equality:

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{(s,\lambda k)_k}{s(s+1)(s+2)\cdots(s+n)}\left(\frac{x}{k}\right)^{s+n}2)s=0, \quad k>x.$$

Now, we consider the case where $x > k$. We analyse the double integral (5.34) by splitting the integration interval into two parts over t: (k, x) and (x, ∞). Within the first interval (k, x), we find that:

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\left(\frac{x}{t}\right)^{s+n}}{s(s+1)\cdots(s+n)}\mathrm{d}s=\frac{\left(\frac{x}{t}-1\right)}{n!}. \tag{5.36}$$

For the second interval, we have $t > x$. In this case, equality (70) holds true. This leads to:

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{(s,\lambda k)_k}{s(s+1)\cdots(s+n)}\left(\frac{x}{k}\right)^{s+n}\mathrm{d}s$$

$$=\frac{(-1)^k(x-k)^n}{k^n n!}+\frac{e^{\lambda k}}{k^n}\sum_{m+1}^{n}\frac{(-1)^{k+r}k!(2\lambda)^r}{r!(k-r)!}\int_k^x e^{-\lambda t}\frac{(x-t)^n}{n!}\frac{(t-k)^{n-1}}{(r-1)!}\mathrm{d}t,\ k<x.$$

Thus formula (5.33) is proved.

The first chapter of Koshliakov's manuscript [11] introduced the results we discussed previously. The subsequent chapter explores "Generalized Bernoulli polynomials of the first kind". In the following chapter of my thesis, I will present a series formula from the third chapter of his manuscript, titled "About the summation formulas for sums of the form $\sum_{j=1}^{\infty}\frac{p^2+1j^2}{p\left(p+\frac{1}{\pi}\right)+1_j^2}f\left(\lambda_j\right)$".

CHAPTER 6

Sums of type $\sum_{j=1}^{\infty} \frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2} f\left(\lambda_j\right)$

The functions $\zeta_p(s)$ and $\sigma_p(x)$, are some particular cases of sums of the form:

$$\sum_{j=1}^{\infty} \frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2} f\left(\lambda_j\right). \tag{6.1}$$

In Koshliakov manuscript [11, Chapter 3], he studied functions that are special cases of (6.1). He was interested to find an explicit formula of (6.1). Given a predetermined function $f(x)$, Koshliakov aimed to transform (6.1) into a definite integral. This transformation led to the derivation of several summation formulas, which are highly valuable in the context of Koshliakov's manuscript. Koshliakov derived these formulas using two methods, contour integration and Mellin transform. Here, we present the proof using contour integration.

Consider a holomorphic function $F(s) = F(\sigma + it)$, where s is a complex variable. Koshliakov analyzed this function within a domain defined by the following inequalities:

$$-\alpha < \sigma < \beta + 1, \quad 0 < \alpha < 1, \quad \beta > 0. \tag{6.2}$$

Suppose that the $F(s)$ function has a pole at $s = 0$ which is simple. Additionally, for all sufficiently large values of $|\Im(s)(=t)|$, the function follows certain asymptotic behaviour:

$$F(s) = O\left(e^{-\frac{\pi}{2}|t|}\ |t|^{-\lambda}\right). \tag{6.3}$$

Here, parameter λ must satisfy the condition $\lambda > 1$ for all values of σ within the set defined by formula (6.2). Let us consider a function, $f(x)$. Koshliakov defined this function for all positive x values using the following integral:

$$f(x) = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \frac{F(s)}{x^s}\mathrm{d}s, \quad 0<\tau<\beta+1. \tag{6.4}$$

The following infinite series converges absolutely, which can easily verified,

$$\sum_{j=1}^{\infty} \frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2}\ \frac{1}{\lambda_j^s}, \tag{6.5}$$

on the straight line with abscissa $\beta+1$, and the convergence of the integral $\int_{-\infty}^{\infty}|F(\beta+1+it)|\mathrm{d}t$ also follows by estimate (6.3).

Koshliakov wanted an integral representation of (6.1). Hence by (6.1) and (6.4) we get the following equality:

$$\sum_{j=1}^{\infty} \frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2} f\left(\lambda_j\right) = \frac{1}{2\pi i}\sum_{j=1}^{\infty} \frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2}\int_{\tau-i\infty}^{\tau+i\infty} \frac{F(s)}{\lambda_j^s}\mathrm{d}s.$$

The interchange of summation and integration is justified because (6.5) is absolutely convergent for $\Re(s)=\beta+1$, so we get

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2} f\left(\lambda_j\right) &= \frac{1}{2\pi i}\int_{\beta+1-i\infty}^{\beta+1+i\infty} F(s)\left(\sum_{j=1}^{\infty} \frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2}\frac{1}{\lambda_j^s}\right)\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{\beta+1-i\infty}^{\beta+1+i\infty} F(s)\zeta_p(s)\mathrm{d}s. \qquad (6.6)\end{aligned}$$

We now evaluate the integral of $F(s)\zeta_p(s)$ along the closed contour of a quadrilateral. The quadrilateral vertices are:

$$A(\beta+1-iT), B(\beta+1+iT), C(-\alpha+iT), D(-\alpha-iT).$$

The function $F(s)\zeta_p(s)$ has poles at $s=0,\ 1$ both of which lie inside the contour. If R_0 and R_1 denote the residue of integrand $F(s)\zeta_p(s)$ relative to its poles at $s=0$ and $s=1$, then by the Cauchy residue theorem, we have

$$\frac{1}{2\pi i}\left(\int_{\beta+1-iT}^{\beta+1+iT} + \int_{\beta+1+iT}^{-\alpha+iT} + \int_{-\alpha+iT}^{-\alpha-iT} + \int_{-\alpha-iT}^{\beta+1-iT}\right) F(s)\zeta_p(s)\mathrm{d}s = R_0+R_1,$$

because estimate (6.3) guarantees that the integrals along segments BC and DA vanish as $T \longrightarrow +\infty$. We now obtain the following equality

$$\frac{1}{2\pi i}\left(\int_{-\alpha+i\infty}^{\alpha-i\infty} F(s)\zeta_p(s)\mathrm{d}s + \frac{1}{2\pi i}\int_{\beta+1-i\infty}^{\beta+1+i\infty} F(s)\zeta_p(s)\mathrm{d}s\right) = R_0 + R_1.$$

Utilizing (6.6), we have

$$\sum_{j=1}^{\infty} \frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2} f\left(\lambda_j\right) = R_0 + R_1 + \frac{1}{2\pi i}\int_{-\alpha-i\infty}^{\alpha+i\infty} F(s)\zeta_p(s)\mathrm{d}s, \tag{6.7}$$

where R_0 and R_1 denote the residue of the function $F(s)\zeta_p(s)$ relative to its poles at $s=0$ and $s=1$. To determine the residue R_0, we consider the integral of the function $\frac{F(s)}{x^s}$ along the contour indicated in the above quadrilateral ABCD. As T approaches positive infinity, by estimate (6.3), we arrive at the following equality,

$$f(x) = R + O(x^{\alpha}), \quad 0 < \alpha < 1,$$

where R is the residue of the function $F(s)$ relative to its pole at $s=0$.

It follows that $R = f(0)$ and thus,

$$R_0 = \zeta_p(0)f(0) = -\frac{1}{2}\,\frac{1}{1+\frac{1}{\pi p}} f(0). \tag{6.8}$$

To find the residue R_1, we use equation (6.6) and Mellin's transformation, to obtain

$$F(s) = \int_0^{\infty} f(x)x^{s-1}\mathrm{d}x,$$

We want to transform the integral term in equality (6.7). One can easily verify that, due to (6.3) the definition of function $f(x)$ can also be defined for complex values of x with positive real part. Indeed for any $x \in \mathbb{C}$, we have

$$x = |x|e^{i\theta}, \quad -\frac{\pi}{2} \le \theta < +\frac{\pi}{2}. \tag{6.9}$$

One get the following asymptotic inequality

$$\left|\int_{-\alpha+i\infty}^{-\alpha+i\infty} \frac{F(s)}{x^s}\mathrm{d}s\right| < |x|\int_{-\infty}^{+\infty} e^{\frac{\pi}{2}|t|}|F(-\alpha+it)|\mathrm{d}t.$$

Hence, it is clear that the integral $\int_{-\alpha-i\infty}^{-\alpha+i\infty} \frac{F(s)}{x^s}\mathrm{d}s$ will be convergent, and therefore, $f(x)$ will be regular for all values of θ lying in the region (6.2), as a result of which one can write the following equation from (6.4):

$$\frac{f(ix)-f(-ix)}{2i} = -\frac{1}{2\pi i}\int_{\beta+1-i\infty}^{\beta+1+i\infty} F(s)\sin\left(\frac{\pi s}{2}\right)\frac{\mathrm{d}s}{x^s}, \quad x>0$$
$$= -\frac{1}{2\pi i}\int_{-\alpha-i\infty}^{-\alpha+i\infty} F(s)\sin\left(\frac{\pi s}{2}\right)\frac{\mathrm{d}s}{x^s}, \quad x>0. \tag{6.10}$$

Multiply both sides of the equation by $\frac{1}{\sigma(x)e^{2\pi x}-1}$ and integrate over the interval from $x=0$ to $x=+\infty$. Then rearranging the orders of integration based on Jordan's theorem and by the functional equation of $\zeta_p(s)$, one has

$$\zeta_p(s) = 2\sin\left(\frac{\pi s}{2}\right)\int_0^\infty \frac{x^{-s}\mathrm{d}x}{\sigma(x)e^{2\pi x}-1}, \quad \Re(s)<0.$$

By the above functional equation, one get the following equation

$$\frac{1}{2\pi i}\int_{-\alpha-i\infty}^{-\alpha+i\infty} F(s)\zeta_p(s)\mathrm{d}s = -2\int_0^\infty \frac{f(ix)-f(-ix)}{2i}\frac{\mathrm{d}x}{\sigma(x)e^{2\pi x}-1}. \tag{6.11}$$

Therefore, from (6.7) and the above equation one obtains the following relation

$$\sum_{j=1}^{\infty}\frac{p^2+\lambda_j^2}{p\left(p_{+}\frac{1}{\pi}\right)+\lambda_j^2}f\left(\lambda_j\right) = -\frac{1}{2}\ \frac{1}{1+\frac{1}{\pi p}}f(0)+\int_0^\infty f(x)\mathrm{d}x$$
$$-2\int_0^\infty \frac{f(ix)-f(-ix)}{2i}\frac{\mathrm{d}x}{\sigma(x)e^{2\pi x}-1}.$$

Koshliakov [11, Chapter 3, p. 54-56] utilized Mellin transforms to gave alternative prove for the formula we discussed earlier. He also explored specific examples of functions, $f(x)$, that can be employed within the infinite series. In the next chapter of my thesis, I will take a different direction, focusing on analysing zero-free regions for the function $\zeta_p(s)$ within the left half-plane.

CHAPTER 7

Zero-free region for $\zeta_p(s)$

In his work, Koshliakov did not address the zero-free regions for the functions $\zeta_p(s)$ and $\eta_p(s)$. Inspired by Spira's paper [14], we aim to identify the zero-free region and the critical region for these functions. Like the Hurwitz zeta function, where some non-trivial zeros lie outside the half-line and even lie outside of $0 < \Re(s) < 1$, we are curious to see if the same holds for the Koshliakov zeta function. The next theorem scrutinizes the area within the right half-plane where $\zeta_p(s)$ does not have any zeros, referred to as its zero-free region.

Theorem 7.1. *If* $\sigma > 1 + \frac{p\left(p+\frac{1}{\pi}\right)+1}{p^2+\frac{1}{4}}$*, then* $\zeta_p(s) \neq 0$.

Proof. From definition, we know for $\Re(s) > 1$,

$$\zeta_p(s) = \sum_{j=1}^{\infty} \frac{p^2 + \lambda_j^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_j^2} . \frac{1}{\lambda_j^s}.$$

By taking absolute value on both sides, we have

$$|\zeta_p(s)| = \left| \sum_{j=1}^{\infty} \frac{p^2 + \lambda_j^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_j^2} . \frac{1}{\lambda_j^s} \right|$$

$$\Rightarrow |\zeta_p(s)| = \left| \frac{p^2+\lambda_1^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_1^2} . \frac{1}{\lambda_1^s} + \sum_{j=2}^{\infty} \frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2} . \frac{1}{\lambda_j^s} \right| .$$

Use triangle inequality to see that

$$|\zeta_p(s)| \geqslant \left| \frac{p^2+\lambda_1^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_1^2} . \frac{1}{\lambda_1^s} \right| - \left| \sum_{j=2}^{\infty} \frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2} . \frac{1}{\lambda_j^s} \right|$$

$$|\zeta_p(s)| \geqslant \frac{p^2+\lambda_1^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_1^2} . \frac{1}{\lambda_1^\sigma} - \sum_{j=2}^{\infty} \left| \frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2} \right| . \frac{1}{\lambda_j^\sigma} . \tag{7.1}$$

Since $p^2+\lambda_j^2 < p\left(p+\frac{1}{\pi}\right)+\lambda_j^2 \implies \frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2} < 1$, then by equation (7.1) we get

$$\begin{aligned} |\zeta_p(s)| &> \frac{p^2+\lambda_1^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_1^2} . \frac{1}{\lambda_1^\sigma} - \sum_{j=2}^{\infty} \frac{1}{\lambda_j^\sigma} \\ &> \frac{p^2+\frac{1}{4}}{p\left(p+\frac{1}{\pi}\right)+1} - \int_1^\infty \frac{1}{x^\sigma}\,\mathrm{d}x \\ &= \frac{p^2+\frac{1}{4}}{p\left(p+\frac{1}{\pi}\right)+1} - \left(0 - \frac{1}{1-\sigma}\right) \\ &= \frac{p^2+\frac{1}{4}}{p\left(p+\frac{1}{\pi}\right)+1} + \frac{1}{1-\sigma} . \end{aligned}$$

We have to find p such that

$$\begin{aligned} &\frac{p^2+\frac{1}{4}}{p\left(p+\frac{1}{\pi}\right)+1} + \frac{1}{1-\sigma} > 0 \\ &\Rightarrow \frac{p^2+\frac{1}{4}}{p\left(p+\frac{1}{\pi}\right)+1} > \frac{1}{\sigma-1} \\ &\Rightarrow (\sigma-1) > \frac{p\left(p+\frac{1}{\pi}\right)+1}{p^2+\frac{1}{4}} \\ &\Rightarrow \sigma > 1 + \frac{p\left(p+\frac{1}{\pi}\right)+1}{p^2+\frac{1}{4}} . \end{aligned}$$

So, when $\sigma > 1+\frac{p\left(p+\frac{1}{\pi}\right)+1}{p^2+\frac{1}{4}}$, we have $\zeta_p(s) > 0$ i.e., $\zeta_p(s) \neq 0$. Hence the theorem is proved. □

7.0.1 Non-trivial zeros of $\zeta_p(s)$

Now we mention a few non-trivial zeros of the Koshliakov zeta function $\zeta_p(s)$. We used Newton-Raphson numerical method to find these non-trivial zeros of $\zeta_p(s)$.

We utilized Theorem 5.2 in Mathematica software to obtain the below table for different values of p and $0 < \alpha < \frac{1}{2}$.

For $p = 2$ **and** $\alpha = 0.4$

Re	**Im**
0.500606	$\pm$ 20.3221
0.584657	$\pm$ 13.7578

For $p = 2$ **and** $\alpha = 0.15$

Re	**Im**
0.500606	$\pm$ 20.3221
0.584657	$\pm$ 13.7578

For $p = 4$ **and** $\alpha = 0.4$

Re	**Im**
0.526852	$\pm$ 14.0575
0.505677	$\pm$ 20.8458

For $p = 5$ **and** $\alpha = 0.2$

Re	**Im**
0.516232	$\pm$14.0916
0.504739	$\pm$ 20.9185

For $p = 7$ **and** $\alpha = 0.25$

Re	**Im**
0.507009	$\pm$ 14.1175
0.502791	$\pm$ 20.9784

For $p = 13$ **and** $\alpha = 0.12$

Re	**Im**
0.501282	$\pm$ 14.3182
0.500664	$\pm$ 21.0142

For $p = 100$ **and** $\alpha = 0.2$

Re	**Im**
0.500003	$\pm$ 14.1347
0.500002	$\pm$ 21.022

For $p = 200$ **and** $\alpha = 0.2$

Re	**Im**
0.5000003	$\pm$ 14.1347242
0.5	$\pm$ 21.0220396

For $p = 300$ **and** $\alpha = 0.2$

Re	**Im**
0.5	$\pm$ 14.1347
0.500002	$\pm$ 21.0220396

For $p = 400$ **and** $\alpha = 0.2$

Re	**Im**
0.50000039	$\pm$ 14.1347242
0.5	$\pm$ 21.0220396

Here we would like to mention that the first two non-trivial zeros of $\zeta(s)$ are $0.5 + i\ 14.1347$ and $0.5 + i\ 21.022$. From the above table, one can clearly observe that the first two non-trivial zeros of $\zeta_p(s)$ are converging to the non-trivial zeros of $\zeta(s)$ as we know $\zeta_p(s)$ becomes $\zeta(s)$ when $p \to \infty$.

CHAPTER 8

Conclusion

Based on the above numerical evidences of the non-trivial zeros of the Koshliakov zeta function $\zeta_p(s)$, we can clearly see that $\zeta_p(s)$ does not obey the Riemann hypothesis, that is, there are non-trivial zeros that are not lying on the line $\Re(s) = 1/2$. In our investigation, we attempted to locate non-trial zeros in the left half-plane $\Re(s) < 0$. While unsuccessful in finding zero-free region in the left half-plane, this work led us to formulate conjecture about the location of non-trivial zeros of $\zeta_p(s)$.

- **Conjecture 1**: The Koshliakov zeta function $\zeta_p(s)$ possesses only trivial zeros in the left half-plane $\Re(s) < 0$, which occur at negative even integers.

- **Conjecture 2**: In Theorem 7.1, we have shown that $\zeta_p(s)$ has no zero in the right half plane $\Re(s) > 1 + \frac{p\left(p+\frac{1}{\pi}\right)+1}{p^2+\frac{1}{4}}$. Moreover, we feel that all the non-trivial zeros of the Koshliakov zeta functions $\zeta_p(s)$ will lie in the region $0 < \Re(s) < 1$.

Bibliography

[1] N. N. Bogolyubov, L. D. Faddeev, A. Yu. Ishlinskii, V. N. Koshlyakov and Yu. A. Mitropol'skii, *Nikolai Sergeevich Koshlyakov (on the centenary on his birth)*, Uspekhi Mat. Nauk **45** (1990), No. 4, 173-176; Enlish transl. in Russian Math. Surveys **45** (1990), No. 4,197-202.

[2] H. DAVENPORT & H. HHEILBRONN, "On the zeros of certain Dirichlet series," *J. London Math. Soc.*, **11**(1936), 181-185.

[3] A. Dixit and R. Gupta, *Koshliakov zeta functions I: modular relations* , Adv. Math. **393**, 108093 (2021).

[4] J. Hadamard. *Sur la distribution des zéros de la fonction* $\zeta(s)$ *et ses conséquences arithmétiques*. Bull. Soc. Math. France, **24**(1896), 199-–220.

[5] H. Hamburger,*Über die Riemannsche Funktionalgleichung der* ζ*-Funktion I, II*, Math. Z. **10** (1921), 240–254, Math. Z. **11** (1921), 224–245.

[6] H. Hamburger, *Über einige Beziehungen,die mit der Funktionalgleichung der Riemannschen* ξ*-Funktion äquivalent sind*, Math. Ann. **85** (1922) No. 1, 129–140.

[7] H. Hamburger, *Über eine Riemannsche Formel aus der Theorie der Dirichletschen Reihen*, Math. Z. **6**(1920), no. 1–2, 1–10.

[8] H. Hamburger, Über die Riemannsche Funktionalgleichung der ζ-Funktion, Dritte Mitteilung: Die Funktionalgleichung der L-Reihen, Math. Z.**13** No. 1 (1922), 283–311.

[9] A. Hurwitz, *Einige Eigenschaften der Dirichlet'schen Funktionen $F(s) = \sum(\frac{D}{n})\frac{1}{n^s}$, die bei der Bestimmung der Klassenzahlen binarer quadratischer Formen auftreten*, Zeitschr. für Math. u. Phys, **27** (1882) 86–101.

[10] N. S. Koshliakov, *An extension of Bernoulli's polynomials*, Mat. Sb. **42** no. 4 (1935), 425–434.

[11] N. S. Koshliakov, (under the name N. S. Sergeev) *Issledovanie odnogo klassa transtsendentnykh funktsii, opredelyaemykh obobshchennym yravneniem Riemann*, Trudy Mat. Inst. Steklov, Moscow, 1949, available online at https://dds.crl.edu/crldelivery/14052.

[12] C. J. de la Vallé Poussin. *Recherches analytiques la theorie des nombres premiers.* Ann. Soc. scient. Bruxelles, **20**(1896), 183–256.

[13] Bernhard Riemann, Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse, *Monatsberichte der Berliner Akademie,* Nov 1859.

[14] R. Spira, *Zeros of Hurwitz zeta functions*, Mathematics of computation, **30**(1976), 863-–866.