An Analogue of Herglotz-Zagier-Novikov Function

M.Sc. Thesis

by

Pragya Singh

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Pragya Singh

(Roll No. **2203141006**)

Under the guidance of

Dr. Bibekananda Maji

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**An Analogue of Herglotz-Zagier-Novikov Function**" in the partial fulfilment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF MATHEMATICS**, **Indian Institute of Technology Indore**, is an authentic record of my work carried out during the period from July 2023 to May 2024 under the supervision of **Dr. Bibekananda Maji**, Assistant Professor, Department of Mathematics, IIT Indore.

The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

Pragya singh

Signature of the student with date

30/05/2024 (**Pragya Singh**)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Bibekananda Maji 30/05/2024

Signature of Thesis Supervisor with date

(**Dr. Bibekananda Maji**)

Pragya Singh has successfully given her M.Sc. Oral Examination held on 27th May, 2024.

Bibekananda Maji 30/05/2024

Signature of supervisor of M.Sc Thesis

Date:

V.K.Theri

Signature of Convener, DPGC

Date: 29/05/2024

Dedicated to my family and teachers

"It is better to live your own density imperfectly than to live an imitation of somebody else's life with perfection.."
–***Bhagavad Gita 3.35***

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Abstract

In mathematics, evaluating an integral in terms of well-known constants is always a fascinating and challenging task. Recently, Choie and Kumar [1] extensively studied the Herglotz-Zagier-Novikov function $\mathcal{F}(z;u,v)$. It is defined as the following integral:

$$\mathcal{F}(z;u,v) := \int_0^1 \frac{\log(1-ut^z)}{v^{-1}-t}dt, \quad \text{for} \quad \mathfrak{Re}(z) > 0, \tag{0.1}$$

where $u \in \mathbb{L}$ and $v \in \mathbb{L}'$. They obtained two-term, three-term and six-term functional equations for $\mathcal{F}(z;u,v)$ and also evaluated special values in terms of di-logarithmic functions. Motivated from their work, in this thesis, we study the following two integrals, for $\mathfrak{Re}(z) > 0$, and any natural number k,

$$\mathcal{F}(z;u,v,w) := \int_0^1 \frac{\log(1-ut^z)\log(1-wt^z)}{v^{-1}-t}dt, \tag{0.2}$$

$$\mathcal{F}_k(z;u,v) := \int_0^1 \frac{\log^k(1-ut^z)}{v^{-1}-t}\,dt, \tag{0.3}$$

where $u \in \mathbb{L}$ and $v \in \mathbb{L}'$. For $k=1$, the above integral (0.3) reduces to (0.1). This allows to recover the properties of $\mathcal{F}(z;u,v)$ by studying the properties of $\mathcal{F}_k(z;u,v)$. One of the main aims of this thesis is to evaluate special values of these two integrals in terms of poly-logarithmic functions.

Contents

List of Symbols **3**

1 Introduction **5**

2 Preliminaries from Number Theory **7**

2.0.1 Polylogarithm function 7

3 Well known results on $\mathcal{F}(z; u, v)$ **9**

3.0.1 Characteristics of $\mathcal{F}(z; u, v)$ 9

3.0.2 A particular case of $\mathcal{F}(z; u, v)$ 10

4 Main Results **13**

4.1 Generalization of Duplication Formula for HZN function 13

4.2 Extended domain for $J(z)$ 13

4.3 Analogue of H-Z-N function $\mathcal{F}(z; u, v, w)$ 14

4.3.1 Properties of $\mathcal{F}(z; u, v, w)$ 14

4.3.2 Special Evaluations of $\mathcal{F}(z; u, v, w)$ 15

5 Important Lemmas **17**

6 Proof of Main Results **19**

7 Generalization of H-Z-N function **23**

8 Proof of results for $\mathcal{F}_k(z; u, v)$ **27**

9 Concluding Thoughts **29**

List of Symbols

Symbol	Description
$\mathrm{Li}_s(z)$	Polylogarithm function
$\mathfrak{Re}(s)$	Real part of a complex number s
$\mathrm{Im}(s)$	Imaginary part of a complex number s
$\mathbb{E}$	$\{z \in \mathbb{C}\backslash\{0\} : \|z\| \leq 1\}$
$\mathbb{E}'$	$\mathbb{E}\backslash\{1\}$
$\mathbb{E}_1$	$\{z \in \mathbb{C} : \|z\| = 1\}$
$\mathbb{E}_1'$	$\mathbb{E}_1\backslash\{1\}$
$\mathbb{L}$	$\mathbb{C}\backslash\{(1,\infty) \cup \{0\}\}$
$\mathbb{L}'$	$\mathbb{L}\backslash\{1\}$

Chapter 1

Introduction

Zagier's [2] ground breaking exploration of the Kronecker limit formula for real quadratic fields has ignited considerable interest among number theorists. Novikov [3], building upon Zagier's work introduced a novel function within the Kronecker limit formula paradigm. Recently, Choie and Kumar [1] introduced the following function $\mathcal{F}(z;u,v)$, named as Herglotz-Zagier-Novikov function.(Throughout the thesis, we use H-Z-N function rather than Herglotz-Zagier-Novikov function)

Definition 1.1. *For $\mathfrak{Re}(z) > 0$, it is defined as*

$$\mathcal{F}(z;u,v) := \int_0^1 \frac{\log(1-ut^z)}{v^{-1}-t}dt, \tag{1.1}$$

where $u \in \mathbb{L}$ and $v \in \mathbb{L}'$.

Choie and Kumar studied properties inherent within $\mathcal{F}(z;u,v)$. This function serves as a unified framework encompassing three distinct functions extensively studied by Herglotz [4], Zagier [2], and Muzaffar Williams [5]. These functions are individually defined as follows:
For $\mathfrak{Re}(z) > 0$,

$$\mathcal{F}(z) := \int_0^1 \left(\frac{1}{1-y} + \frac{1}{logy}\right) \log(1-y^z)\, \frac{dy}{y},$$

$$J(z) := \int_0^1 \frac{\log(1+t^z)}{1+t}dt.$$

The function $F(x)$ is encountered in Herglotz's [4] work related to the Kronecker limit formula for real quadratic fields. Its analytical continuation can be achieved using the expression [7, equation(1.2)]:

$$F(x) = \sum_{n\geq 1} \frac{\psi(nx) - \log(nx)}{n} \quad x \in \mathbb{C} \setminus (-\infty, 0],$$

where $\psi(x) := \Gamma'(x)/\Gamma(x)$ is the digamma function.
Recently, Radchenko and Zagier [7] conducted an extensive study of the functions

$F(x)$ and $J(x)$. They uncovered connections to Stark's conjecture, Hecke operators, and the cohomology of the modular group $PSL_2(\mathbb{Z})$. Additionally, they identified a relationship between the functions $F(x)$ and $J(x)$ [7].

$$J(z) = F(2z) - 2F(z) + F\left(\frac{z}{2}\right) + \frac{\pi^2}{12z}.$$

In their paper, the function $F(x)$ is termed the Herglotz function, while Masri [8] refers to it as the Herglotz-Zagier function. Expanding on these connections, we introduce the function $\mathcal{F}(z; u, v)$ as the Herglotz-Zagier-Novikov function, as it emerges in Novikov's work and also converges to $F(x)$ as the limits of u and v approach 1. In addition to $F(z)$, the function $J(z)$ can also be derived from $\mathcal{F}(z; u, v)$ by putting $u = v = -1$.

Radchenko and Zagier [7] have derived special values of $F(x)$ at positive rational and quadratic units, which enable them to calculate value of $J(x)$. For instant :

$$J(4+\sqrt{17}) = -\frac{\pi^2}{6} + \frac{1}{2}\log^2(2) + \frac{1}{2}\log(2)\log\left(2\left(4+\sqrt{17}\right)\right),$$

$$J\left(\frac{2}{5}\right) = \frac{11\pi^2}{240} + \frac{3}{4}\log^2(2) - 2\log^2\left(\frac{\sqrt{5}+1}{2}\right).$$

Earlier Herglotz [4],and Muzaffar and Williams [5], had computed such integrals, but specifically for $J(n+\sqrt{n^2-1})$. For example, Herglotz showed:

$$J(4+\sqrt{15}) = -\frac{\pi^2}{12}\left(\sqrt{15}-2\right) + \log(2)\log\left(\sqrt{3}+\sqrt{5}\right) + \log\left(\frac{\sqrt{5}+1}{2}\right)\log(2+\sqrt{3}).$$

Chowla [15, p 372] remarked on the difficulty of direct evaluation of such integrals, noting their complexity and the necessity of methods from both analytic and algebraic number theory.

In this thesis, we first find a generalization of duplication formula (3.2), (3.3) for any natural number n (instead of 2) for the case of $\mathcal{F}(z; u, v)$. Furthermore, we extend the domain of $J(z)$ (an special case of $\mathcal{F}(z; u, v)$) to cover any $z \in \mathbb{C}$.

By multiplying another log term in the numerator, we defined an analogue of $\mathcal{F}(z; u, v)$, namely,

$$\mathcal{F}(z; u, v, w) := \int_0^1 \frac{\log(1-ut^z)\log(1-wt^z)}{v^{-1}-t}dt, \quad \mathfrak{Re}(z) > 0,$$

where $u, w \in \mathbb{L}$ and $v \in \mathbb{L}'$. We have studied the similar properties for our analogue function $\mathcal{F}(z; u, v, w)$ throughout the thesis.

Chapter 2

Preliminaries from Number Theory

First, we define an important function, namely, polylogarithm function $\mathrm{Li}_s(z)$, which will be useful in the later part of the thesis.

2.0.1 Polylogarithm function

Definition 2.1 (**Polylogarithm function**). *The polylogarithm function, denoted as* $\mathrm{Li}_s(z)$*, represents a fundamental mathematical concept with a dual representation: it can be expressed both as a power series in* z *and as a Dirichlet series in* s*. It is defined as*

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s}. \tag{2.1}$$

This definition holds true for every complex order s *and for any complex argument* z *where* $|z| < 1$*; analytic continuation allows it to be extended to* $|z| \geq 1$*.*

Another way to describe this function is as a repeated integral of itself.

$$\mathrm{Li}_s(z) := \int_0^z \frac{\mathrm{Li}_{s-1}(t)}{t} dt. \tag{2.2}$$

Properties and some special values

The polylogarithm function has numerous unique qualities and values, but in this thesis, we will focus on a few key characteristics and values.

1. The polylogarithm function's derivative can be expressed as follows:

$$\frac{\partial \mathrm{Li}_s(z)}{\partial z} = \frac{\mathrm{Li}_{s-1}(z)}{z}. \tag{2.3}$$

2. For $s = 1$, the polylogarithm function simplifies to the natural logarithm:

$$\mathrm{Li}_1(z) = -\log(1 - z). \tag{2.4}$$

3. When $z = 1$, the polylogarithm function simplifies to the Riemann zeta function $\zeta(s)$

$$\operatorname{Li}_s(1) = \zeta(s), \qquad \mathfrak{Re}(s) > 1. \tag{2.5}$$

Chapter 3

Well known results on $\mathcal{F}(z;u,v)$

Recall that, the definition of the H-Z-N function (1.1), that is,

$$\mathcal{F}(z;u,v) = \int_0^1 \frac{\log(1-ut^z)}{v^{-1}-t}dt, \qquad \mathfrak{Re}(z) > 0,$$

where $u \in \mathbb{L}$ and $v \in \mathbb{L}'$.

3.0.1 Characteristics of $\mathcal{F}(z;u,v)$

Here, we will note down some properties of $\mathcal{F}(z;u,v)$ which were proved by Choie and Kumar [1].

Theorem 3.1. *For $|u| < 1$ and $|v| < 1$, $\mathcal{F}(z;u,v)$ can be expressed as*

$$\mathcal{F}(z;u,v) := -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{u^m v^n}{m(mz+n)}, \quad z \neq -\frac{p}{q} \text{ where } p,q \in \mathbb{N}. \tag{3.1}$$

This gives analytic continuation of $\mathcal{F}(z;u,v)$ for any complex z except at negative rational number.

Theorem 3.2. *(Duplication formula). For $u \in \mathbb{E}$, $v, v^2 \in \mathbb{E}'$ and $\mathfrak{Re}(z) > 0$,*

$$\mathcal{F}(2z;u^2,v) = \mathcal{F}(z;u,v) + \mathcal{F}(z;-u,v), \tag{3.2}$$

$$\mathcal{F}\left(\frac{z}{2};u,v^2\right) = \mathcal{F}(z;u,v) + \mathcal{F}(z;u,-v). \tag{3.3}$$

Theorem 3.3. *$\mathcal{F}(z;u,v)$ fulfils two and three-term functional equations as shown below:*

1. *For $u, v \in \mathbb{E}'$, we have*

$$\mathcal{F}(z;u,v) + \mathcal{F}\left(\frac{1}{z};v,u\right) = -\log(1-u)\log(1-v). \tag{3.4}$$

2. For $u, v, uv \in \mathbb{E}'$, we have

$$\mathcal{F}(z; u, v) - \mathcal{F}(z+1; uv, v) - \mathcal{F}\left(\frac{z}{z+1}; u, uv\right) = \log(1-u)\log(1-uv) + \mathrm{Li}_2(u)$$
$$- \mathrm{Li}_2\left(\frac{v}{v-1}\right) + 2\mathrm{Li}_2\left(\frac{u}{u-1}\right) - \mathrm{Li}_2\left(\frac{u-v}{1-v}\right) - \left(\frac{1}{z+1} - \frac{1}{2}\right)\mathrm{Li}_2(uv)$$
$$- \sum_{j=1}^{2} \mathrm{Li}_2\left(\frac{u+\sqrt{uv}e^{\pi ij}}{u-1}\right) - \mathrm{Li}_2\left(\frac{v+\sqrt{uv}e^{\pi ij}}{v-1}\right). \tag{3.5}$$

Additionally, Choie and Kumar [1, Theorem 2.3] provided a six-term functional equation for $\mathcal{F}(z; u, v)$. They [1, Theorem 2.5] also provided an explicit expression for $z \in \mathbb{Q}$.

Theorem 3.4. *Assume $p, q \in \mathbb{N}$ and $(u, v) \in \mathbb{E} \times \mathbb{E}'$. For $z = \frac{p}{q}$, the function $\mathcal{F}(z; u, v)$ can be calculated as*

$$\mathcal{F}\left(\frac{p}{q}; u, v\right) = \frac{q}{p}\mathrm{Li}_2(u) + \sum_{\alpha^p=1}\sum_{\beta^q=1} \mathrm{Li}_2\left(\frac{\beta v^{\frac{1}{q}}}{\beta v^{\frac{1}{q}} - 1}\right) - \mathrm{Li}_2\left(\frac{\alpha u^{\frac{1}{p}} - \beta v^{\frac{1}{q}}}{1 - \beta v^{\frac{1}{q}}}\right).$$

One can evaluate $\mathcal{F}(n; u, v)$ directly by entering $p = n$ and $q = 1$. Similarly, $\mathcal{F}\left(\frac{1}{n}; u, v\right)$ can be obtained directly by entering $p = 1$ and $q = n$.

3.0.2 A particular case of $\mathcal{F}(z; u, v)$

Herglotz in 1923 studied a function $J(z)$, which is interestingly turns out to be a particular instance of $\mathcal{F}(z; u, v)$. For $\mathfrak{Re}(z) > 0$, we have

$$\mathcal{F}(z; -1, -1) = -J(z).$$

Properties of the function $J(z)$

Choie and Kumar [1] studied properties of $J(z)$ as a particular case of their function and gave the following two term functional equation for $J(z)$.

Theorem 3.5. *Let $\mathfrak{Re}(z) > 0$, then we have*

$$J(z) + J\left(\frac{1}{z}\right) = \log^2(z). \tag{3.6}$$

Letting $u = v = -1$ in (3.4) gives (3.6).

They also evaluated $J(z)$ at $z = \frac{1}{m}$ and $z = m$ for $m \in \mathbb{N}$.

Theorem 3.6. *For any natural number m, one has*

$$J(m) = \frac{\pi^2}{12}\left(\frac{1}{m} - m\right) + \frac{m}{2}\log^2(2) + \sum_{j=1}^{m} \mathrm{Li}_2\left(\frac{1}{2}\left(1 + e^{\frac{\pi i}{m}(2j+1)}\right)\right). \tag{3.7}$$

and from equation (3.6), we have

$$J\left(\frac{1}{m}\right) = \frac{\pi^2}{12}\left(m - \frac{1}{m}\right) + \left(1 - \frac{m}{2}\right)\log^2(2) - \sum_{j=1}^{m} \mathrm{Li}_2\left(\frac{1}{2}\left(1 + e^{\frac{\pi i}{m}(2j+1)}\right)\right). \quad (3.8)$$

Chapter 4

Main Results

We highlight the major points of this thesis in this chapter. The equations (3.2), (3.3) illustrate the duplication formula for $\mathcal{F}(z; u, v)$ provided by Choie and Kumar. Inspired by this, we obtained a generalization of the duplication formula of $\mathcal{F}(z; u, v)$.

4.1 Generalization of Duplication Formula for HZN function

Theorem 4.1. *For any natural number n, we have*

$$\mathcal{F}(nz; u^n, v) = \sum_{\alpha^n=1} \mathcal{F}(z; u\alpha, v). \tag{4.1}$$

$$\mathcal{F}\left(\frac{z}{n}; u, v^n\right) = \sum_{\alpha^n=1} \mathcal{F}(z; u, v\alpha). \tag{4.2}$$

Remark 1. *Putting $n = 2$, we get the duplication formula* (3.2) *and* (3.3) *by Choie and Kumar [1].*

4.2 Extended domain for $J(z)$

Radchenko and Zagier defined Herglotz function $J(z)$ for $\mathfrak{Re}(z) > 0$, but we find that the integral in $J(z)$ is defined for any $z \in \mathbb{C}$. This is because whatever t we choose between 0 to 1, $\arg(1 + t^z) \neq -\pi$, hence

$$J(z) = \int_0^1 \frac{\log(1 + t^z)}{1 + t} dt,$$

is valid for any $z \in \mathbb{C}$.

Theorem 4.1. *For any $z \in \mathbb{C}$, we have*

$$J(-z) = J(z) + \frac{z\pi^2}{12}.$$

Using equations (3.7), (3.8) and the above theorem, one can directly evaluate $J(-m)$ and $J\left(-\frac{1}{m}\right)$.

4.3 Analogue of H-Z-N function $\mathcal{F}(z; u, v, w)$

Recall that our analogue function $\mathcal{F}(z; u, v, w)$ is given as

$$\mathcal{F}(z; u, v, w) = \int_0^1 \frac{\log(1 - ut^z)\log(1 - wt^z)}{v^{-1} - t} dt, \quad \mathfrak{Re}(z) > 0, \tag{4.1}$$

where $u, w \in \mathbb{L}$ and $v \in \mathbb{L}'$.

4.3.1 Properties of $\mathcal{F}(z; u, v, w)$

Note that the function $\mathcal{F}(z; u, v, w)$ is defined for $\mathfrak{Re}(z) > 0$. However, we extend its domain by analytical continuation.

Theorem 4.1. *(Analytic Continuation) For non-zero complex numbers $u, v,$ and w such that $|u| < 1$, $|v| < 1$, and $|w| < 1$,*

$$\mathcal{F}(z; u, v, w) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{u^m w^n v^k}{mn(mz + nz + k)}. \tag{4.2}$$

This holds for any complex z such that $pz + q \neq 0$ for any $p, q \in \mathbb{N}$.

We have also found the following multiplication relations for $\mathcal{F}(z; u, v, w)$ similar to the formula in Theorem 4.1 for $\mathcal{F}(z; u, v)$.

Theorem 4.2. *(Multiplication formula) Take $u, w \in \mathbb{E}$, $v, v^n \in \mathbb{E}'$ and $\mathfrak{Re}(z) > 0$ then for any given natural number n, we have*

$$\mathcal{F}(nz; u^n, v, w^n) = \sum_{\alpha^n=1} \sum_{\beta^n=1} \mathcal{F}(z; u\alpha, v, w\beta). \tag{4.3}$$

$$\mathcal{F}\left(\frac{z}{n}; u, v^n, w\right) = \sum_{\alpha^n=1} \mathcal{F}(z; u, v\alpha, w). \tag{4.4}$$

In the next section, we are going to explore a few particular values of $\mathcal{F}(z; u, v, w)$ for $z \in \mathbb{Q}$.

4.3.2 Special Evaluations of $\mathcal{F}(z; u, v, w)$

Theorem 4.3. *Let $u \in \mathbb{E}, v \in \mathbb{E}', w \in \mathbb{E}$ and $p, q \in \mathbb{N}$. Then $\mathcal{F}\left(\frac{p}{q}; u, v, u\right)$ can be given as*

$$\mathcal{F}\left(\frac{p}{q}; u, v, u\right) = \sum_{\alpha^q=1}\sum_{\beta^p=1}\sum_{\gamma^p=1} \mathcal{F}(1; u^{\frac{1}{p}}\beta, v^{\frac{1}{q}}\alpha, u^{\frac{1}{p}}\gamma). \tag{4.5}$$

We can evaluate $\mathcal{F}(x; u, v, w)$ on two different arguments using above theorem, one by putting $p = n, q = 1$ and another with $p = 1, q = n$.

Corollary 4.4. *For any natural number n and $u, v, \in \mathbb{E}$ we have*

$$\mathcal{F}(n; u, v, u) = \sum_{\alpha^n=1}\sum_{\beta^n=1} \mathcal{F}(1; u^{\frac{1}{n}}\alpha, v, u^{\frac{1}{n}}\beta).$$

Corollary 4.5. *For any $u, v \in \mathbb{E}$ such that $u \neq v, u \neq 1$ and $v \neq 1$,*

$$\mathcal{F}\left(\frac{1}{n}; u, v^n, u\right) = \sum_{\alpha^n=1} -\log^2(1-u)\log\left(\frac{u(1-\alpha v)}{u-\alpha v}\right) - 2\log(1-u)$$
$$\mathrm{Li}_2\left(\frac{\alpha(-1+u)v}{u-\alpha v}\right) - 2\mathrm{Li}_3\left(\frac{\alpha v}{\alpha v - u}\right) + 2\mathrm{Li}_3\left(\frac{\alpha(-1+u)v}{u-\alpha v}\right). \tag{4.6}$$

Letting u tends to 1 and $v \in \mathbb{E} \setminus \{1\}$, we have

$$\mathcal{F}\left(\frac{1}{n}; 1, v^n, 1\right) = -\sum_{\alpha^n=1} 2\mathrm{Li}_3\left(\frac{\alpha v}{\alpha v - 1}\right). \tag{4.7}$$

When n is any even natural number. Letting u tend to 1 and $v \in \mathbb{E} \setminus \{-1, 1\}$ in equation 4.7, we get

$$\mathcal{F}\left(\frac{1}{n}; 1, v^n, 1\right) = -\sum_{\alpha^n=1} 2\mathrm{Li}_3\left(\frac{\alpha v}{\alpha v - 1}\right).$$

Again when n is any odd natural number. Letting u tend to 1 and $v \in \mathbb{E} \setminus \{1\}$ in equation 4.7 , we get

$$\mathcal{F}\left(\frac{1}{n}; 1, v^n, 1\right) = -\sum_{\alpha^n=1} 2\mathrm{Li}_3\left(\frac{\alpha v}{\alpha v - 1}\right).$$

On putting $v = -1$ in above equation, we get

$$\mathcal{F}\left(\frac{1}{n}; 1, (-1)^n, 1\right) = -\sum_{\alpha^n=1} 2\mathrm{Li}_3\left(\frac{\alpha}{\alpha + 1}\right). \tag{4.8}$$

One can evaluate value of $\mathcal{F}\left(\frac{1}{n}; 1, -1, 1\right)$ for any odd natural number directly using equation 4.8. For $n = 1$, one gets

$$\mathcal{F}(1; 1, -1, 1) = 2\,\mathrm{Li}_3\left(\frac{1}{2}\right).$$

Theorem 4.6. *For any $u \in \mathbb{E}$ such that $u \neq 1$,*

$$\mathcal{F}(1; u, u, u) = -\frac{1}{3}\log^3(1-u). \tag{4.9}$$

Our next corollary illustrates that a certain combination of $\mathcal{F}(z;u,v,w)$ can also be represented in terms of the tri-logarithmic function only.

Corollary 4.7. *For $n \in \mathbb{N}$ and $v \in \mathbb{E}'$ we have*

1.
$$\mathcal{F}\left(\frac{1}{n};1,v,1\right)+\mathcal{F}\left(\frac{1}{n};1,\frac{v}{1-v},1\right)$$
$$=-2\sum_{\alpha^n=1}\mathrm{Li}_3\left(\frac{\alpha v^{\frac{1}{n}}}{-1+\alpha v^{\frac{1}{n}}}\right)-2\sum_{\alpha^n=1}\mathrm{Li}_3\left(\frac{\alpha\left(\frac{v}{v-1}\right)^{\frac{1}{n}}}{-1+\alpha\left(\frac{v}{v-1}\right)^{\frac{1}{n}}}\right).$$

2. *For $0<\mathfrak{Re}(v)<1$, one has*

(a)
$$\mathcal{F}\left(\frac{1}{n};1,v,1\right)+\mathcal{F}\left(\frac{1}{n};1,1-v,1\right)$$
$$=-2\sum_{\alpha^n=1}\mathrm{Li}_3\left(\frac{\alpha v^{\frac{1}{n}}}{-1+\alpha v^{\frac{1}{n}}}\right)-2\sum_{\alpha^n=1}\mathrm{Li}_3\left(\frac{\alpha\left(1-v\right)^{\frac{1}{n}}}{-1+\alpha\left(1-v\right)^{\frac{1}{n}}}\right).$$

(b)
$$\mathcal{F}\left(\frac{1}{n};1,\frac{v}{1-v},1\right)+\mathcal{F}\left(\frac{1}{n};1,1-v,1\right)$$
$$=-2\sum_{\alpha^n=1}\mathrm{Li}_3\left(\frac{\alpha(1-v)^{\frac{1}{n}}}{-1+\alpha(1-v)^{\frac{1}{n}}}\right)-2\sum_{\alpha^n=1}\mathrm{Li}_3\left(\frac{\alpha\left(\frac{v}{v-1}\right)^{\frac{1}{n}}}{-1+\alpha\left(\frac{v}{v-1}\right)^{\frac{1}{n}}}\right).$$

3. *For $\mathfrak{Re}(v)<0$, we get*

(a)
$$\mathcal{F}\left(\frac{1}{n};1,v,1\right)+\mathcal{F}\left(\frac{1}{n};1,\frac{1}{1-v},1\right)$$
$$=-2\sum_{\alpha^n=1}\mathrm{Li}_3\left(\frac{\alpha v^{\frac{1}{n}}}{-1+\alpha v^{\frac{1}{n}}}\right)-2\sum_{\alpha^n=1}\mathrm{Li}_3\left(\frac{\alpha\left(\frac{1}{1-v}\right)^{\frac{1}{n}}}{-1+\alpha\left(\frac{1}{1-v}\right)^{\frac{1}{n}}}\right).$$

(b)
$$\mathcal{F}\left(\frac{1}{n};1,1-v,1\right)+\mathcal{F}\left(\frac{1}{n};1,\frac{1}{1-v},1\right)$$
$$=-2\sum_{\alpha^n=1}\mathrm{Li}_3\left(\frac{\alpha(1-v)^{\frac{1}{n}}}{-1+\alpha(1-v)^{\frac{1}{n}}}\right)-2\sum_{\alpha^n=1}\mathrm{Li}_3\left(\frac{\alpha\left(\frac{1}{1-v}\right)^{\frac{1}{n}}}{-1+\alpha\left(\frac{1}{1-v}\right)^{\frac{1}{n}}}\right).$$

Chapter 5

Important Lemmas

In this chapter, we will look at a few important lemmas that will help us to prove our main results.

Lemma 5.1. *For any $u, v \in \mathbb{C}$ such that $u \neq v, u \neq 1$ and $v \neq 1$, we have*

$$\begin{aligned}\mathcal{F}(1; u, v, u) &= \int_0^1 \frac{\log^2(1-ut)}{(v^{-1}-t)} dt, \\ &= -\log^2(1-u)\log\left(\frac{u-uv}{u-v}\right) - 2\log(1-u)\mathrm{Li}_2\left(\frac{uv-v}{u-v}\right) \\ &+ 2\mathrm{Li}_3\left(\frac{v-vu}{v-u}\right) - 2\mathrm{Li}_3\left(\frac{v}{v-u}\right).\end{aligned} \tag{5.1}$$

Proof. We know that

$$\mathcal{F}(1; u, v, u) = \int_0^1 \frac{\log^2(1-ut)}{(v^{-1}-t)} dt.$$

On integrating, we have

$$\mathcal{F}(1; u, v, u) = -\log^2(1-u)\log\left(\frac{u-uv}{u-v}\right) - 2\int_0^1 \frac{u}{1-ut}\log(1-ut)\log\left(\frac{u-uvt}{u-v}\right) dt.$$

Using the relation

$$\frac{d}{dz}\mathrm{Li}_{s+1}(z) = \frac{\mathrm{Li}_s(z)}{z}, \tag{5.2}$$

we have

$$\begin{aligned}\mathcal{F}(1; u, v, u) &= -\log^2(1-u)\log\left(\frac{u-uv}{u-v}\right) - 2\log(1-u)\mathrm{Li}_2\left(\frac{uv-v}{u-v}\right) \\ &- 2\int_0^1 \frac{u}{1-ut}\mathrm{Li}_2\left(\frac{uvt-v}{u-v}\right) dt,\end{aligned}$$

$$\mathcal{F}(1;u,v,u) = -\log^2(1-u)\log\left(\frac{u-uv}{u-v}\right) - 2\log(1-u)\mathrm{Li}_2\left(\frac{uv-v}{u-v}\right)$$
$$+ 2\mathrm{Li}_3\left(\frac{v-vu}{v-u}\right) - 2\mathrm{Li}_3\left(\frac{v}{v-u}\right).$$

□

Lemma 5.2. *For any $u, v \in \mathbb{C}$ such that $u \neq v, u \neq 1$ and $v \neq 1$,*

$$\mathcal{F}_k(1;u,v) := \int_0^1 \frac{\log^k(1-ut)}{v^{-1}-t}dt$$
$$= \sum_{j=1}^{k+1}(-1)^{j-1}\log^{k+1-j}(1-u)\,\mathrm{Li}_j\left(\frac{vu-v}{u-v}\right)\frac{k!}{(k+1-j)!} + (-1)^{k+1}k!$$
$$\mathrm{Li}_{k+1}\left(\frac{v}{v-u}\right). \tag{5.3}$$

Proof. We have

$$\mathcal{F}_k(1;u,v) = \int_0^1 \frac{\log^k(1-ut)}{v^{-1}-t}dt$$

On integrating, we get

$$\mathcal{F}_k(1;u,v) = -\log^k(1-u)\log\left(\frac{u(1-v)}{u-v}\right) - k\int_0^1 \log^{k-1}(1-ut)\frac{u}{1-ut}\log\left(\frac{u-uvt}{u-v}\right)dt.$$

Using the relation

$$\frac{d}{dz}\mathrm{Li}_{s+1}(z) = \frac{\mathrm{Li}_s(z)}{z}, \tag{5.4}$$

we have

$$\mathcal{F}_k(1;u,v) = -\log^k(1-u)\log\left(\frac{u-uv}{u-v}\right) - k\log^{k-1}(1-u)\mathrm{Li}_2\left(\frac{vu-v}{u-v}\right)$$
$$+ k(k-1)\int_0^1 \log^{k-2}(1-ut)\frac{u}{1-ut}\mathrm{Li}_2\left(\frac{v(ut-1)}{u-v}\right)dt,$$

$$\mathcal{F}_k(1;u,v) = -\log^k(1-u)\log\left(\frac{u-uv)}{u-v}\right) - k\log^{k-1}(1-u)\mathrm{Li}_2\left(\frac{vu-v}{u-v}\right) + k(k-1)$$
$$\log^{k-2}(1-u)\mathrm{Li}_3\left(\frac{vu-v}{u-v}\right) - \frac{k!}{(k-3)!}\int_0^1 \frac{u\log^{k-3}(1-ut)}{1-ut}\mathrm{Li}_3\left(\frac{v(ut-1)}{u-v}\right)dt,$$
$$= -\log^k(1-u)\log\left(\frac{u-uv}{u-v}\right) - k\log^{k-1}(1-u)\mathrm{Li}_2\left(\frac{v(-1+u)}{u-v}\right) + b\cdots$$
$$+ (-1)^k!\,\mathrm{Li}_{k+1}\left(\frac{v(u-1)}{u-v}\right) + (-1)^{k+1}k!\,\mathrm{Li}_{k+1}\left(\frac{-v}{u-v}\right),$$
$$= \sum_{j=1}^{k+1}(-1)^{j-1}\log^{k+1-j}(1-u)\mathrm{Li}_j\left(\frac{vu-v}{u-v}\right)\frac{k!}{(k+1-j)!} + (-1)^{k+1}k!\,\mathrm{Li}_{k+1}\left(\frac{v}{v-u}\right).$$

□

Chapter 6

Proof of Main Results

This chapter is dedicated to presenting the proofs of the main results outlined in this thesis.

Proof of Theorem 4.1. From equation (1.1), one has

$$\mathcal{F}(nz;u^n,v) = \int_0^1 \frac{\log(1-u^n t^{nz})}{v^{-1}-t}dt.$$

We know that

$$1 - z^n = (1-z)(1-z\alpha)(1-z\alpha^2)....(1-z\alpha^{n-1}), \tag{6.1}$$

where α is a primitive n^{th} root of unity. Hence

$$\begin{aligned}\mathcal{F}(nz;u^n,v) &= \int_0^1 \frac{\log(1-ut^z)(1-\alpha ut^z)...(1-\alpha^{n-1}ut^z)}{v^{-1}-t}dt,\\ &= \int_0^1 \frac{\sum_{\alpha^n=1}\log(1-\alpha ut^z)}{v^{-1}-t}dt.\end{aligned}$$

On swapping the order of summation and integration, one can directly get the result.

Again from equation (1.1), we have

$$\mathcal{F}\left(\frac{z}{n};u,v^n\right) = \int_0^1 \frac{\log\left(1-u(t)^{\frac{z}{n}}\right)}{v^{-n}-t}\,dt.$$

Making the change of variable $t = e^{-t}$ in above equation, one obtains

$$\mathcal{F}\left(\frac{z}{n};u,v^n\right) = \int_0^\infty \frac{\log\left(1-ue^{\frac{-tz}{n}}\right)}{e^t v^{-n}-1}\,dt.$$

Now substituting $t = ny$ and using the relation

$$\frac{q}{1-y^q} = \sum_{\beta^q=1}\frac{1}{1-\beta y}, \tag{6.2}$$

we obtain,

$$\mathcal{F}\left(\frac{z}{n};u,v^n\right) = \sum_{\beta^n=1}\mathcal{F}(z;u,\beta v).$$

This proves the theorem. □

Proof of Theorem 4.1. Substituting $-z$ in place of z in $J(z)$ we get,

$$\begin{aligned} J(-z) &= \int_0^1 \frac{\log(1+t^{-z})}{1+t} dt, \\ &= \int_0^1 \frac{\log(1+t^{z})}{1+t} dt - \int_0^1 \frac{\log(t^{z})}{1+t} dt, \\ &= \int_0^1 \frac{\log(1+t^{z})}{1+t} dt - z \int_0^1 \frac{\log(t)}{1+t} dt. \end{aligned}$$

We know that

$$\int_0^1 \frac{\log(t)}{1+t} dt = \mathrm{Li}_2(-1).$$

Hence, we have

$$J(-z) = J(z) - \mathrm{Li}_2(-1)z,$$

$$J(-z) = J(z) + \frac{\pi^2 z}{12}.$$

This completes the proof of theorem. □

Proof of Theorem 4.1. Using the series expansion of logarithm and $\frac{1}{1-vt}$ around $t = 0$, we reach

$$\begin{aligned} \mathcal{F}(z; u, v, w) &= \int_0^1 \sum_{m=1}^{\infty} \frac{(ut^z)^m}{m} \sum_{n=1}^{\infty} \frac{(wt^z)^n}{n} \sum_{k=0}^{\infty} v^{(k+1)} t^k \, dt, \\ &= \sum_{m=1}^{\infty} \frac{u^m}{m} \sum_{n=1}^{\infty} \frac{w^n}{n} \sum_{k=0}^{\infty} (v^{k+1}) \frac{1}{(mz + zn + k + 1)}, \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{u^m w^n v^k}{mn(z(m+n)+k)}; \quad z \neq -\frac{p}{q} \text{ for any } p, q \in \mathbb{N}. \end{aligned}$$

This series represents an analytic function in u, v, w. It decays for the specified values of u, v, and w, suggesting that the series is uniformly convergent for any complex z other than a negative rational number. As a result, the function $\mathcal{F}(z; u, v, w)$ is analytic.

□

Proof of Theorem 4.2. From equation (4.1), we get

$$\mathcal{F}(nz, u^n, v, w^n) = \int_0^1 \frac{\log(1 - u^n t^{nz}) \log(1 - w^n t^{nz})}{v^{-1} - t} dt.$$

Using equation (6.1) one obtain,

$$\mathcal{F}(nz, u^n, v, w^n) = \int_0^1 \frac{\sum_{\alpha^n=1} \log(1 - u\alpha t^z) \sum_{\beta^n=1} \log(1 - w\beta t^z)}{v^{-1} - t} \, dt.$$

On interchanging sum and integration,

$$\mathcal{F}\left(nz, u^n, v, w^n\right) = \sum_{\alpha^n=1}\sum_{\beta^n=1}\int_0^1 \frac{\log(1-u\alpha t^z)\log(1-w\beta t^z)}{v^{-1}-t}\,dt,$$

$$= \sum_{\alpha^n=1}\sum_{\beta^n=1}\mathcal{F}\left(z; u\alpha, v, w\beta\right).$$

Again from equation (4.1), we have

$$\mathcal{F}\left(\frac{z}{n}; u, v^n, w\right) = \int_0^1 \frac{\log\left(1-ut^{\frac{z}{n}}\right)\log\left(1-wt^{\frac{z}{n}}\right)}{v^{-1}-t}dt. \tag{6.3}$$

Making the change of variable $t = e^{-t}$ in equation (6.3), one obtains

$$\mathcal{F}\left(\frac{z}{n}; u, v^n, w\right) = \int_0^\infty \frac{\log\left(1-ue^{\frac{-tz}{n}}\right)\log\left(1-we^{\frac{-tz}{n}}\right)}{e^t v^{-n}-1}\,dt.$$

Substituting $t = ny$ and using relation (6.2), we gets

$$\mathcal{F}\left(\frac{z}{n}; u, v^n, w\right) = \sum_{\beta^n=1}\mathcal{F}(z; u, \beta v, w).$$

Hence the proof. $\square$

Proof of Theorem 4.3 . From equation (4.4), we have

$$\mathcal{F}\left(\frac{p}{q}; u, v, u\right) = \sum_{\alpha^q=1}\mathcal{F}(p; u, v^{\frac{1}{q}}\alpha, u).$$

Again using equation (4.3) for $\mathcal{F}(p; u, v^{\frac{1}{q}}\alpha, u)$, we get

$$\mathcal{F}\left(\frac{p}{q}; u, v, u\right) = \sum_{\alpha^q=1}\sum_{\beta^p=1}\sum_{\gamma^p=1}\mathcal{F}(1; u^{\frac{1}{p}}\beta, v^{\frac{1}{q}}\alpha, u^{\frac{1}{p}}\gamma).$$

$\mathcal{F}(1; \zeta x, y, \delta x)$ is well defined and value of this integral can be found using integration by parts. $\square$

By putting $q = 1$ and $p = n$, one can directly get Corollary 4.4.

Proof of Corollary 4.5. Using duplication formula (4.4) for $\mathcal{F}\left(\frac{1}{n}; u, v^n, u\right)$, we get

$$\mathcal{F}\left(\frac{1}{n}; u, v^n, u\right) = \sum_{\alpha^n=1}\mathcal{F}(1; u, \alpha v, u).$$

Use equation (5.1) for $v = v\alpha$ to get result. $\square$

Proof of Theorem 4.7. Using duplication formula (4.4) for $\mathcal{F}\left(\frac{1}{n}; 1, v^n, 1\right)$, we get

$$\mathcal{F}\left(\frac{1}{n}; 1, v^n, 1\right) = \sum_{\alpha^n=1}\mathcal{F}(1; 1, \alpha v, 1),$$

Using equation (4.7) for αv in place of v one can directly get result. $\square$

Proof of Theorem 4.6. We know that

$$\mathcal{F}\left(1;u,u,u\right)=\int_0^1\frac{\log^2\left(1+ut\right)}{\frac{1}{u}-t}dt.$$

On integrating, one gets

$$\begin{aligned}\mathcal{F}\left(1;u,u,u\right)&=-\log^3\left(1+u\right)-2\mathcal{F}\left(1;u,u,u\right)\\ \mathcal{F}\left(1;u,u,u\right)&=-\frac{1}{3}\log^3\left(1-u\right).\end{aligned}$$

Hence the proof. □

Proof of Corollary 4.7. The result follows directly from (4.7). □

Chapter 7

Generalization of H-Z-N function

We define another function which is a generalization of the H-Z-N function as follows.

$$\mathcal{F}_k(z;u,v) := \int_0^1 \frac{\log^k(1-ut^z)}{v^{-1}-t}\,dt, \quad \text{for any } k\in\mathbb{N} \text{ and } \mathfrak{Re}(z)>0, \tag{7.1}$$

where $u \in \mathbb{L}$ and $v \in \mathbb{L}'$.

For $k = 1$, it is same as H-Z-N function $\mathcal{F}(z;u,v)$. Below we show the analytic continuation of $\mathcal{F}_k(z;u,v)$ to $\mathbb{C}$ except at negative rationals.

Analytic Continuation for $\mathcal{F}_k(z;u,v)$

Theorem 7.1. *For $|u| < 1$ and $|v| < 1$, we have*

$$\mathcal{F}_k(z;u,v) = \prod_{i=1}^{k} \sum_{m_i,l=1}^{\infty} \frac{u^{m_i} v^l}{m_i(z(m_1+m_2+\cdots+m_k)+l)}, \quad z \neq -\frac{p}{q} \quad \textit{where } p,q \in \mathbb{N}.$$

This result gives analytic continuation of $\mathcal{F}_k(z;u,v)$.

Multiplication Formula for $\mathcal{F}_k(z;u,v)$

Theorem 7.2. *Take $u, w \in \mathbb{E}$, $v, v^n \in \mathbb{E}'$, and $\mathfrak{Re}(z) > 0$, then for any given natural number n, we have*

$$\mathcal{F}_k\left(\frac{z}{n};u,v^n\right) = \sum_{\alpha^n=1} \mathcal{F}_k(z;u,v\alpha^{-1}). \tag{7.2}$$

Special Values for $\mathcal{F}_k(z;u,v)$

In the next theorem, we give the explicit evaluation of $\mathcal{F}_k\left(\frac{1}{n};u,v\right)$ for any $n \in \mathbb{N}$.

Theorem 7.3. *For any $u, v \in \mathbb{E}$ such that $u \neq v$ and $u \neq 1$,*

$$\mathcal{F}_k\left(\frac{1}{n}; u, v\right) = \sum_{\alpha^n=1}\sum_{j=1}^{k+1}(-1)^{j-1}\log^{k+1-j}(1-u)\operatorname{Li}_j\left(\frac{\alpha^{-1}v^{\frac{1}{n}}(u-1)}{u-\alpha^{-1}v^{\frac{1}{n}}}\right)\frac{k!}{(k+1-j)!}$$
$$+(-1)^{k+1}k!\operatorname{Li}_{k+1}\left(\frac{\alpha^{-1}v^{\frac{1}{n}}}{\alpha^{-1}v^{\frac{1}{n}}-u}\right). \tag{7.3}$$

We take limit u tends to 1 in the above Theorem 7.3 to have one more special case.

$$\mathcal{F}_k\left(\frac{1}{n}; 1, v\right) = \sum_{\alpha^n=1}(-1)^{k+1}\,k!\,\operatorname{Li}_{k+1}\left(\frac{\alpha^{-1}v^{\frac{1}{n}}}{\alpha^{-1}v^{\frac{1}{n}}-1}\right). \tag{7.4}$$

Next theorem consider $u = v$ to give the following result.

Theorem 7.4. *For any $u \in \mathbb{E}$ such that $u \neq 1$*

$$\mathcal{F}_k(1; u, u) = -\frac{1}{1+k}\log^{1+k}(1-u). \tag{7.5}$$

The function $\mathcal{F}_k(z; u, v)$ satisfies the following functional equation similar to that of $\mathcal{F}(z; u, v, w)$ given in Corollary 4.7.

Corollary 7.5. *For any natural number n and $v \in \mathbb{E}'$, we have*

1. $$\mathcal{F}_k\left(\frac{1}{n}; 1, v\right) + \mathcal{F}_k\left(\frac{1}{n}; 1, \frac{v}{v-1}\right)$$
$$= \sum_{\alpha^n=1}(-1)^{k+1}\,k!\left(\operatorname{Li}_{k+1}\left(\frac{\alpha^{-1}v^{\frac{1}{n}}}{-1+\alpha^{-1}v^{\frac{1}{n}}}\right)+\operatorname{Li}_{k+1}\left(\frac{\alpha^{-1}\left(\frac{v}{v-1}\right)^{\frac{1}{n}}}{-1+\alpha^{-1}\left(\frac{v}{v-1}\right)^{\frac{1}{n}}}\right)\right).$$

2. *Let $Re(v) \in (0, 1)$. Then*

   (a) $$\mathcal{F}_k\left(\frac{1}{n}; 1, v\right) + \mathcal{F}_k\left(\frac{1}{n}; 1, 1-v\right)$$
$$= \sum_{\alpha^n=1}(-1)^{k+1}\,k!\left(\operatorname{Li}_{k+1}\left(\frac{\alpha^{-1}v^{\frac{1}{n}}}{-1+\alpha^{-1}v^{\frac{1}{n}}}\right)+\operatorname{Li}_{k+1}\left(\frac{\alpha^{-1}(1-v)^{\frac{1}{n}}}{-1+\alpha^{-1}(1-v)^{\frac{1}{n}}}\right)\right).$$

   (b) $$\mathcal{F}_k\left(\frac{1}{n}; 1, 1-v\right) + \mathcal{F}_k\left(\frac{1}{n}; 1, \frac{v}{v-1}\right)$$
$$= \sum_{\alpha^n=1}(-1)^{k+1}\,k!\left(\operatorname{Li}_{k+1}\left(\frac{\alpha^{-1}(1-v)^{\frac{1}{n}}}{-1+\alpha^{-1}(1-v)^{\frac{1}{n}}}\right)+\operatorname{Li}_{k+1}\left(\frac{\alpha^{-1}\left(\frac{v}{v-1}\right)^{\frac{1}{n}}}{-1+\alpha^{-1}\left(\frac{v}{v-1}\right)^{\frac{1}{n}}}\right)\right).$$

3. *For $Re(v) < 0$,*

(a)

$$\mathcal{F}_k\left(\frac{1}{n};1,\frac{1}{1-v}\right)+\mathcal{F}_k\left(\frac{1}{n};1,v\right)$$
$$=\sum_{\alpha^n=1}(-1)^{k+1}\,k!\,\left(\mathrm{Li}_{k+1}\left(\frac{\alpha^{-1}v^{\frac{1}{n}}}{-1+\alpha^{-1}v^{\frac{1}{n}}}\right)+\mathrm{Li}_{k+1}\left(\frac{\alpha^{-1}\left(\frac{1}{1-v}\right)^{\frac{1}{n}}}{-1+\alpha^{-1}\left(\frac{1}{1-v}\right)^{\frac{1}{n}}}\right)\right).$$

(b)

$$\mathcal{F}_k\left(\frac{1}{n};1,1-v\right)+\mathcal{F}_k\left(\frac{1}{n};1,\frac{1}{1-v}\right)$$
$$=\sum_{\alpha^n=1}(-1)^{k+1}\,k!\,\left(\mathrm{Li}_{k+1}\left(\frac{\alpha^{-1}(1-v)^{\frac{1}{n}}}{-1+\alpha^{-1}(1-v)^{\frac{1}{n}}}\right)+\mathrm{Li}_{k+1}\left(\frac{\alpha^{-1}\left(\frac{1}{1-v}\right)^{\frac{1}{n}}}{-1+\alpha^{-1}\left(\frac{1}{1-v}\right)^{\frac{1}{n}}}\right)\right).$$

Table containing special values of $\mathcal{F}_k(z;u,v)$

As Herglotz, Radchenko and Zagier have discovered evaluations for the functions $J(x)$, Muzaffar and Williams also provided evaluations of $J(x)$. Choie and Kumar provide special value of $J(x)$. In the tables below, we present some special values of function $\mathcal{F}_k(z;u,v)$ using our Theorem 7.3 for different value of k.

Table 7.1: Special values of $\mathcal{F}_k(x;u,v)$

k	(z,u,v)	value of $\mathcal{F}_k(z;u,v)$
1	(1, 2, -1)	$-\log 2\log 3+\mathrm{Li}_2\left(\frac{2}{3}\right)-\mathrm{Li}_2\left(\frac{4}{3}\right)$
1	(1, 3, -1)	$-\log 2\log 4+\mathrm{Li}_2\left(\frac{3}{2}\right)-\mathrm{Li}_2\left(\frac{3}{4}\right)$
1	(1, 4, -1)	$-\log 2\log 5+\mathrm{Li}_2\left(\frac{4}{5}\right)-\mathrm{Li}_2\left(\frac{8}{5}\right)$
1	$\left(\frac{1}{2},1,-3\right)$	-$\mathrm{Li}_2\left(\frac{\sqrt{3}}{\sqrt{3}-i}\right)-\mathrm{Li}_2\left(\frac{\sqrt{3}}{\sqrt{3}+i}\right)$
1	$\left(\frac{1}{2},1,-2\right)$	-$\mathrm{Li}_2\left(\frac{\sqrt{2}}{\sqrt{2}-i}\right)-\mathrm{Li}_2\left(\frac{\sqrt{2}}{\sqrt{2}+i}\right)$
1	$\left(\frac{1}{2},1,-1\right)$	$\frac{-5\pi^2}{48}+\frac{\log^2 2}{4}$
2	(1, 1, -1)	$2\,\mathrm{Li}_3\left(\frac{1}{2}\right)$
2	(1, -2, -1)	$-\log 4\log^2 3-2\log 3\mathrm{Li}_2(-3)-2\mathrm{Li}_3(-1)+2\mathrm{Li}_3(-3)$
2	(1,-1,-1)	$\frac{\log^3(2)}{3}$
2	(1,2,-1)	$\pi^2\log\left(\frac{4}{3}\right)-2\pi i\mathrm{Li}_2\left(\frac{-1}{3}\right)-2\mathrm{Li}_2\left(\frac{1}{3}\right)+2\mathrm{Li}_3\left(\frac{-1}{3}\right)$
2	$\left(\frac{1}{2},1,2\right)$	$-2\mathrm{Li}_3\left(\frac{1}{1+\frac{i}{\sqrt{2}}}\right)-2\mathrm{Li}_3\left(\frac{2}{3}+\frac{i\sqrt{2}}{3}\right)$

Chapter 8

Proof of results for $\mathcal{F}_k(z;u,v)$

We are going to present the proofs of the results for $\mathcal{F}_k(z;u,v)$ in this chapter.

Proof of Theorem 7.1. Using $\frac{1}{1-vt}$ as the series expansion around $t=0$ and expanding the logarithm as series, we get

$$\mathcal{F}_k(z;u,v) = \int_0^1 \left(\sum_{m=1}^{\infty} \frac{(ut^z)^m}{m}\right)^k \sum_{l=0}^{\infty} v^{l+1} t^l \, dt$$
$$= \int_0^1 \sum_{m_1=1}^{\infty} \frac{(ut^z)^{m_1}}{m_1} \sum_{m_2=1}^{\infty} \frac{(ut^z)^{m_2}}{m_2} \cdots \sum_{m_k=1}^{\infty} \frac{(ut^z)^{m_k}}{m_k} \sum_{l=0}^{\infty} v^{l+1} t^l.$$

On switching the integration and summation order, it gives

$$\mathcal{F}_k(z;u,v) = \prod_{i=1}^{k} \sum_{m_i=1}^{\infty} \sum_{l=0}^{\infty} \frac{u^{m_i}}{m_i} v^{l+1} \int_0^1 t^{zm_1+zm_2+\cdots+zm_k+l} dt,$$
$$= \prod_{i=1}^{k} \sum_{m_i=1}^{\infty} \sum_{l=0}^{\infty} \frac{u^{m_i}}{m_i} \frac{v^{l+1}}{(z(m_1+m_2+m_3+\cdots+m_k)+l+1)},$$
$$= \prod_{i=1}^{k} \sum_{m_i=1}^{\infty} \sum_{l=1}^{\infty} \frac{u^{m_i}}{m_i} \frac{v^{l}}{(z(m_1+m_2+m_3+\cdots+m_k)+l)}.$$

□

Proof of Theorem 7.2. From equation (7.1), we have

$$\mathcal{F}_k\left(\frac{z}{n};u,v^n\right) = \int_0^1 \frac{\log^k\left(1-ut^{\frac{z}{n}}\right)}{v^{-n}-t} dt. \tag{8.1}$$

Changing the variable $t=e^{-t}$ in the above equation yields,

$$\mathcal{F}_k\left(\frac{z}{n};u,v^n\right) = \int_0^\infty \frac{\log^k\left(1-ue^{\frac{-tz}{n}}\right)}{e^t v^{-n}-1} \, dt.$$

Making the change of variable $t = ny$ and using the relation

$$\frac{q}{1-y^q} = \sum_{\alpha^q=1} \frac{1}{1-\alpha y}$$

in above equation, one obtains

$$\mathcal{F}_k\left(\frac{z}{n}; u, v^n\right) = \sum_{\beta^n=1} \mathcal{F}_k(z; u, \alpha^{n-1}v).$$

□

Proof of Theorem 7.3. From multiplication formula (7.2), we have

$$\mathcal{F}_k\left(\frac{1}{n}; u, v\right) = \sum_{\alpha^n=1} \mathcal{F}_k(1; u, \alpha^{-1}v^{\frac{1}{n}}).$$

Substituting value of $\mathcal{F}_k(1; u, \alpha^{-1}v^{\frac{1}{n}})$ from equation (5.2), one can directly obtained the result. □

Proof of Theorem 7.4. We know that

$$\mathcal{F}_k(1; u, u) = \int_0^1 \frac{\log^k(1-ut)}{\frac{1}{u}-t} dt.$$

On integrating, one gets

$$\mathcal{F}_k(1; u, u) = -\log^{k+1}(1-u) - k\mathcal{F}(1; u, u)$$

$$\mathcal{F}_k(1; u, u) = -\frac{1}{k+1}\log^{k+1}(1-u).$$

□

Proof of Corollary 7.5. The result follows directly from (7.4). □

Chapter 9

Concluding Thoughts

Throughout this thesis, our primary focus has been to study the properties of the H-Z-N function and to prove some of these properties for our newly defined analogue function. Additionally, we have discovered a generalization of the H-Z-N function. While it is quite challenging to identify all similar properties for our generalized function, we have successfully proven several of them. Furthermore, when $u = w$ in our analogue function, it reduces to a special case of our generalized function for $n = 2$. Investigating other properties of this generalized function would be a compelling direction for future research. Additionally, we plan to prove a three-term functional equation for $J(z)$.

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