A BRIEF SURVEY ON MUCKENHOUPT WEIGHTS AND L^p-BOUNDEDNESS OF CERTAIN TRANSLATION-INVARIANT OPERATORS

M.Sc. Thesis

by

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DEPARTMENT OF MATHEMATICS

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A Brief Survey on Muckenhoupt Weights and L^p-Boundedness of Certain Translation-Invariant Operators

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"A brief survey on Muckenhoupt Weights and L^p-boundedness of certain translation-invariant operators"** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF MATHEMATICS**, **Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2023 to May 2024 under the supervision of **Dr. Ashisha Kumar**, Assistant Professor, Department of Mathematics, IIT Indore. The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

Uttam Kumar Dolai
30/05/2024

Signature of the student with date

(**Uttam Kumar Dolai**)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Ashisha Kumar 30/05/24

Signature of Thesis Supervisor with date

(**Dr. Ashisha Kumar**)

Uttam Kumar Dolai has successfully given his M.Sc. Oral Examination held on 27th May, 2024.

Ashisha Kumar	V.K. Tiwari
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Date: 30/05/24	Date: 30/05/2024

Dedicated to my family

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Abstract

In this survey, we try to understand boundedness of translation-invariant (linear) operators on Lebesgue spaces. Translation-invariant operators are an important part of Fourier Analysis. These operators enjoy "nice"-properties. For instance, it is known, due to Hörmander, that translation-invariant operators are "L^p-improving". Our main aim is to see the boundedness of such operators on L^p-spaces of the Euclidean space not only with the usual Lebesgue measure, but also with measures induced by positive (measurable) functions. Such functions are referred to as weights.

A natural question arises: Are all weights "good"? At the first glance, this question is ambiguous and does not merit an answer at all! How does one define "good" weights? A part of this thesis also describes some literature in this direction. Study of averages of functions on the real line was done nearly a century ago by Hardy and Littlewood, in the context of differentiability properties of integrable functions. Their study gave rise to a whole new area of *Maximal functions*. Muckenhoupt further developed theory on the weighted boundedness of maximal functions, and characterized all (positive) weights for which the Hardy-Littlewood

maximal functions are bounded on L^p. These classes of weights are now famously known as Muckenhoupt classes A_p. In our survey, we focus on these weights. Our aim, therefore, is twofold: One, to understand Muckenhoupt weights, their characterizations, possible generalization, and "nice" properties; and two, to use this knowledge in understanding the L^p-boundedness of translation-invariant operators. While, in generality, there are a variety of translation-invariant operators, we deal with those that are of convolution type. Particularly, in this thesis, we study the L^p-boundedness of (generalized) Calderón-Zygmund operators, and a few multipliers. For the latter, we require Littlewood-Payley theory, which is also dealt with. We see that the Muckenhoupt classes and the boundedness results for (Hardy-Littlewood) Maximal functions play an important role in this study. In the final chapter of the thesis, we give a few (and in no way exhaustive) directions that one can approach with this knowledge.

Contents

CHAPTER 1

Introduction

The study of averages is an important part of mathematics. The knowledge of average behaviour of a system is of interest to many fields, such as Dynamics, Ergodic Theory, and even Harmonic analysis. In this survey, we are interested in the Harmonic analysis point of view.

While the origin of the study of averages cannot be pinned to a particular event in mathematical history, we briefly describe the the view of Hardy and Littlewood for the study of averages of functions defined on the Euclidean space $\mathbb{R}^n$. The motivation for them goes back to the first fundamental theorem of calculus. We know that given a continuous function $f : [a, b] \to \mathbb{R}$, the "*area function*", $F : [a, b] \to \mathbb{R}$, defined by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t,$$

is differentiable, and in fact, $F'(x) = f(x)$, for every $x \in (a, b)$. Let us try to understand the derivative of F in detail. By definition, we have,

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}.$$

However, since the limit exists, we may very well write it as

$$F'(x) = \lim_{h\to 0} \frac{F(x) - F(x-h)}{h}.$$

By adding the two expressions, we get

$$f(x) = F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x-h)}{2h} = \lim_{h\to 0} \frac{1}{2h} \int\limits_{x-h}^{x+h} f(t)\,\mathrm{d}t.$$

If we notice carefully, the expression $\frac{1}{2h} \int\limits_{x-h}^{x+h} f(t)\,\mathrm{d}t$ gives the average of f over the interval $(x-h, x+h)$. Therefore, essentially, the first fundamental theorem of calculus says that for continuous functions, the averages over intervals centered at x converge to the function f as the size of the interval shrinks to zero.

The next step is to generalize this concept to higher dimensions. Hardy and Littlewood in [14], study these averages through their corresponding maximal function, and derive a differentiation theorem, famously known as the Lebesgue differentiation theorem. When generalizing this idea to higher dimensions, the most natural way is to consider balls in place of intervals, and study the averages of the form $\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(t)\,\mathrm{d}t$, where $|\cdot|$ denotes the Lebesgue measure of a set in $\mathbb{R}^n$. This was done by Wiener in [26] in the context of Ergodic Theory. These averages are the main component of our study.

Speaking of averages, we easily notice that they are "smoothing" operators. Vaguely, taking averages of functions increase their regularity, at least in terms of differentiation. One might ask whether taking averages increase the integrability of functions? That is, given a p-integrable function on $\mathbb{R}$, is its average also p-integrable? Can we expect it to be q-integrable for $q \neq p$? These questions are the basic framework of our study.

Apart from the averages, another important operation in Harmonic analysis is the convolution. Formally, given two "nice" functions $f, g : \mathbb{R}^n \to \mathbb{C}$, we define their convolution by the function $f * g : \mathbb{R}^n \to \mathbb{C}$, as

$$f * g\,(x) = \int\limits_{\mathbb{R}^n} f\,(x - y)\, g\,(y)\, \mathrm{d}y.$$

A particular case of convolutions is the averaging operator. So, we may ask similar questions about integrability of convolution operators. Particularly, we fix a nice function K defined on $\mathbb{R}^n$, called the *kernel of convolution*, and define an operator T by

$$Tf = K * f.$$

Now, we ask whether Tf is q-integrable whenever f is p-integrable. In fact, we ask a stronger question: Is T bounded from L^p to L^q? Due to a famous result by Hörmander ([15]), it is known that a necessary condition is $q \geq p$. In fact, Hörmander proved the result translation-invariant operators. Formally, an operator T is translation-invariant, if for any $x \in \mathbb{R}^n$, we have $T \circ \tau_x = \tau_x \circ T$. Here, τ_x is an operator defined by

$$\tau_x f\,(y) = f\,(y - x)\,.$$

Translation-invariant operators form the other part of our study. It is known (see, for instance, [12]) that translation-invariant operators are of convolution type. More precisely, they can be written as convolutions for a "nice" class of functions. Owing to their importance in analysis, we study two types of translation-invariant operators in this thesis. First, we consider the operators that mimic convolutions in some sense. This leads us to the Calderón-Zygmund theory. Next, we study the (translation-invariant) operators whose Fourier transform is a multiplication by a bounded function. Such operators are called **multipliers**. While the study of multipliers is vast, and a lot of research is going on, we deal with three major multiplier theorems in this survey.

This thesis is organized as follows: Chapter 2 deals with preliminaries required later. Here, we begin with recalling basic concepts from measure theory and functional analysis. We do not give most of the proofs and details of the results mentioned in this chapter, but rather only their references. The main section of this chapter deals with the development of Bochner integral (also known as vector-valued or Banach-valued integral), which is not usually covered in a first course on Functional Analysis. Hence, we give all the required details. The next chapter (Chapter 3) deals with the study of averages and maximal functions, and their L^p-boundedness. In Chapter 4, we try and characterize all the weight functions w that make the Hardy-Littlewood maximal function bounded on the weighted Lebesgue spaces $L^p(w)$. This completes the first part of our thesis. The next part deals with the study of L^p-boundedness of translation-invariant operators. In Chapter 5, we start by a prototypical example of a convolution-type operator, and build our way to the Calderón-Zygmund theory. We see later that this study is also useful in our study of multipliers. The penultimate chapter (Chapter 6) first deals with Littlewood-Paley theory, which is further required to study multipliers. In our study of multiplier theory, we deal with three important results due to Hörmander, Marcinkiewicz and Bochner. Finally, in Chapter 7, we conclude the thesis and discuss a few directions of further study.

CHAPTER 2

Preliminaries

In this chapter, we provide preliminary results required later, and make some notation precise. We refrain from giving detailed proofs of simple results and rather refer to the sources at most places. Two sections of this chapter are of importance to us: First, the section on interpolation results, and second, on the Bochner integral. In these sections we provide complete details of all results discussed.

2.1 L^p spaces

We start by recalling certain basic definitions and results from integration theory. Most of the material presented here can be found in [22] or [9]. Throughout this chapter, (X, μ) denotes a σ-finite measure space. If $1 \leq p < \infty$, the space $L^p(X, \mu)$ consists of all complex valued measurable functions on X such that

$$\int_X |f(x)|^p \, \mathrm{d}\mu(x) < \infty.$$

To simplify the notation, we write $L^p(X)$. If $f \in L^p(X)$, the L^p norm of f is defined as

$$||f||_{L^p(X)} = \left(\int\limits_X |f(x)|^p \, \mathrm{d}\mu(x) \right)^{1/p}.$$

Sometimes we abbreviate the norm as $\|f\|_{L^p}$ or $\|f\|_p$.

Similarly, we define $L^\infty(X)$ as the collection of all essentially bounded measurable functions. A function $f : X \to \mathbb{C}$ is essentially bounded if there is some $M > 0$ such that the $\mu(\{x \in X | \, |f(x)| > M\}) = 0$. The uniform norm of essentially bounded functions $\|f\|_\infty$ is defined to be the smallest $M > 0$ such that $\mu(\{x \in X | \, |f(x)| > M\}) = 0$.

Remark 2.1. Often, in statements of many results, we abuse notation and write $\|f\|_p = \left(\int\limits_X |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}$ for $p = +\infty$ as well. It is to be understood that in this case, the norm is taken as the uniform norm.

The following are some important examples of L^p spaces.

1. If $X = \mathbb{R}^n$ and μ equals Lebesgue measure then the L^p space is denoted by $L^p(\mathbb{R}^n)$. There, we write

$$||f||_p = \left(\int\limits_{\mathbb{R}^n} |f(x)| \, \mathrm{d}x \right)^{1/p}.$$

2. If we take $X = \mathbb{Z}$, μ equal to the counting measure, we get discrete L^p spaces. They are denoted by $\ell^p(\mathbb{Z})$. Measurable functions are simply sequences $f = (x_n)_{n \in \mathbb{Z}}$ of complex numbers, and

$$|| (x_n)_{n \in \mathbb{Z}} ||_{L^p} = \left(\sum_{n \in \mathbb{Z}} |a_n|^p \right)^{1/p}.$$

It is known that the space $L^p(X)$, is a Banach space with the norm defined above. The following inequalities are crucial to us. In what follows, given $1 \leq p \leq +\infty$, the conjugate exponent p' to p is given by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 2.1 (Hölder's inequality [9]). *Let f and g be two measurable functions on X. Let p and p' be conjugate exponents, $1 \leq p \leq \infty$. Then, we have*

$$\int_X |f(x)g(x)| \, \mathrm{d}\mu(x) \leq \left(\int_X |f(x)|^p \, \mathrm{d}\mu(x) \right)^{1/p} \left(\int_X |g(x)|^{p'} \, \mathrm{d}\mu(x) \right)^{1/p'},$$

where, the case $p = 1$ or $p = +\infty$ is understood accordingly.

Theorem 2.2 (Minkowski's inequality [17]). *If $1 \leq p \leq \infty$ and $f, g \in L^p(X)$, then $f + g \in L^p(X)$, and $||f + g||_{L^p} \leq ||f||_{L^p} + ||g||_{L^p}$.*

We also have the continuous version of Minkowski's inequality.

Theorem 2.3 (Minkowski's Integral Inequality [7]). *Let (X, μ) and (Y, ν) be measure spaces and $F : X \times Y \to \mathbb{C}$ be a measurable function. Then, we have for $1 \leq p < \infty$,*

$$\left[\int_X \left| \int_Y F(x, y) \, \mathrm{d}y \right|^p \mathrm{d}x \right]^{\frac{1}{p}} \leq \int_Y \left[\int_X |F(x, y) \, \mathrm{d}x| \right]^{\frac{1}{p}} \mathrm{d}y. \tag{2.1}$$

Remark 2.2. The spaces $L^p(X)$ can be defined for $0 < p < 1$ analogously. However, in these cases, the map $\| \cdot \|_p$ is not a norm, since it no longer satisfies the triangle inequality. Nonetheless, we call it a "norm", and use it at some places where the triangle inequality is not important.

2.2 Operators on Banach space

In this section, we recall a few definitions and results concerning linear operators on Banach spaces. A normed linear space consists of an underlying vetor space V over a field of scalars (the real or complex number), together with a norm $|| \cdot || : V \longrightarrow \mathbb{R}^+$ that satisfies:

1. $||v|| = 0$ if and only if $v = 0$.

2. $||\alpha v|| = |\alpha| ||v||$, where α is a scalar and $v \in V$.

3. $||v + w|| \leq ||v|| + ||w||$ for all $v, w \in V$.

The space V is said to be complete if every Cauchy sequence in V is convergent. A complete normed linear space is called a **Banach space**.

2.2.1 Bounded Linear Operator

Continuous linear operators between Banach spaces are of importance to us. Here, we recall a few related definitions and results from Functional Analysis.

Definition 2.1 (Bounded Linear Operator)**.** *Let X and Y be normed linear spaces, a linear operator $T : X \longrightarrow Y$ is bounded if there is a constant $C > 0$ such that for all $x \in X$,*

$$||T(x)|| \leq C||x||.$$

Operator norm of a bounded linear operator $T : X \to Y$ is

$$\|T\| := \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in X \\ \|x\|=1}} \|Tx\| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} \|Tx\|.$$

The following result is the link between bounded operators and continuous operators.

Theorem 2.4. *A linear operator from a normed space X to another normed space Y is bounded if and only if it is continuous.*

2.2.2 Linear functional and the dual of Banach space

Let $\mathcal{B}$ be a Banach space over a field $\mathbb{F}$. For our purpose, $\mathbb{F} = \mathbb{C}$. A linear functional is a map $\Lambda : \mathcal{B} \longrightarrow \mathbb{F}$ that satisfies

$$\Lambda(\alpha u + \beta v) = \alpha\Lambda(u) + \beta\Lambda(v),$$

for all $\alpha, \beta \in \mathbb{F}$, and $u, v \in \mathcal{B}$. We know that the set of all continuous linear functionals over $\mathcal{B}$ is a vector space over $\mathbb{F}$. It is denoted by $\mathcal{B}'$. Equipped with the operator norm, $\mathcal{B}'$ is a Banach space.

The following well-known duality is well-known.

Theorem 2.5. *For $1 \leq p < \infty$, the normed dual of $L^p(X)$ is $L^{p'}(X)$. Moreover, we have for measurable functions $f : X \to \mathbb{C}$,*

$$\|f\|_{L^p} = \sup \left\{ \left| \int\limits_X f(x)\, g(x)\, \mathrm{d}\mu(x) \right| : \|g\|_{L^{p'}(X)} \leq 1 \right\}.$$

2.2.3 Transpose of a linear operator

We also require the definition of transpose of a linear operator.

Definition 2.2 (Transpose)**.** *Let $T : X \longrightarrow Y$ be a linear operator where X and Y are normed spaces. Then the transpose operator $T^t : Y' \longrightarrow X'$ is defined by*

$$(T^t g)(x) = g(Tx),$$

The following result is well known.

Theorem 2.6. *If $T : X \to Y$ is a bounded linear operator, then $T^t : Y' \to X'$ is also a bounded linear operator. Moreover, we have $\|T\| = \|T^t\|$.*

2.3 Weak-type inequalities and Interpolation

In this section, we give important techniques useful to our study. Since the results discussed here might be new to some readers, we give complete details. We begin by defining weak boundedness of operators on L^p spaces.

Definition 2.3 (Weak type boundedness)**.** *Let (X, μ) and (Y, ν) be measure spaces, and let T be an operator from $L^p(X)$ into the space of measurable functions from Y to $\mathbb{C}$. We say that T is of weak type (p, q), for $q < \infty$, if*

$$\nu(\{y \in Y : |Tf(y)| > \lambda\}) \leq \left(\frac{C||f||_p}{\lambda} \right)^q,$$

and we say that T is weak (p, ∞) if it is bounded operator from $L^p(X)$ to $L^\infty(Y)$.

One might wonder about the notion of "strong" boundedness. The definition is not different from that of bounded operators (see Definition 2.1).

Definition 2.4 (Strong boundedness)**.** *Let T be an operator from $L^p(X)$ into the space of measurable functions from Y to $\mathbb{C}$. We say that T is of strong type (p,q) if it is bounded from $L^p(X)$ to $L^q(Y)$, i.e., there is a constant $C > 0$ such that for all $f \in L^p(X)$, we have*

$$||Tf||_q \leq C||f||_p.$$

It is easy to see that if an operator is strong (p,q) then it is weak (p,q). Indeed, let $E_\lambda = \{y \in Y : |Tf(y)| > \lambda\}$. Then,

$$\nu(E_\lambda) = \int_Y \chi_{E_\lambda} \mathrm{d}\nu(y) \leq \int_Y \left|\frac{Tf(y)}{\lambda}\right|^q \mathrm{d}\nu(y) \leq \frac{||Tf||_q^q}{\lambda^q} \leq \left(\frac{C||f||_p}{\lambda}\right)^q.$$

Remark 2.3. *When $(X,\mu) = (Y,\nu)$ and T is the identity operator, the weak (p,p) inequality is the classical Chebyshev (or Markov's) inequality (see for example, [9]).*

The relationship between weak (p,q) inequalities and almost everywhere convergence is given by the following result. Here, we assume that (X,μ)=(Y,ν).

Theorem 2.7. *Let $\{T_t\}_{t\in I}$ be a family of linear operators on $L^p(X)$ and define a maximal function associated to $\{T_t\}_{t\in I}$ by*

$$T^* f(x) = \sup_{t\in I} |T_t f(x)|.$$

Here, the index set I is a topological subspace of $\mathbb{R}$ with a limit point t_0. If T^* is weak (p,q) then the set

$$\left\{ f \in L^p(X) \;\middle|\; \lim_{t\longrightarrow t_0} T_t f(x) = f(x) \text{ a.e. } x \in X \right\}$$

is closed in $L^p(X)$.

Proof. Let us consider a sequence of functions $(f_n)_{n\in\mathbb{N}}$ in $L^p(X)$ which converges to the function $f \in L^p(X)$ in the L^p norm and such that $\forall n \in \mathbb{N}$, $T_t f_n$ converges to f_n almost everywhere. Now,

$$\begin{aligned} |T_t f(x) - f(x)| &= |T_t(f - f_n)(x) - (f(x) - f_n(x)) + T_t f_n(x) - f_n(x)| \\ &\leq |T_t(f - f_n)(x) - (f(x) - f_n(x))| + |T_t f_n(x) - f_n(x)|. \end{aligned}$$

Therefore, $\limsup_{t \longrightarrow t_0} |T_t f(x) - f(x)| \leq \limsup_{t \longrightarrow t_0} |T_t(f - f_n)(x) - (f(x) - f_n(x))| + \limsup_{t \longrightarrow t_0} |T_t f_n(x) - f_n(x)|$. As a result,

$$\begin{aligned}
&\left\{ x \in X \middle| \limsup_{t \longrightarrow t_0} |T_t f(x) - f(x)| > 2\lambda \right\} \subseteq \\
&\left\{ x \in X \middle| \limsup_{t \longrightarrow t_0} |T_t(f - f_n)(x) - (f(x) - f_n(x))| > \lambda \right\} \\
&\bigcup \left\{ x \in X : \limsup_{t \longrightarrow t_0} |T_t f_n(x) - f_n(x)| > \lambda \right\}.
\end{aligned}$$

Since $T_t f_n$ converges pointwise to f_n almost everywhere, we have

$$\limsup_{t \longrightarrow t_0} |T_t f_n(x) - f_n(x)| = 0,$$

for almost every $x \in X$. Therefore,

$$\mu\left(\left\{ x \in X : \limsup_{t \longrightarrow t_0} |T_t f_n(x) - f_n(x)| > \lambda \right\} \right) = 0.$$

So, we have the following:

$$\begin{aligned}
&\mu\left(\left\{ x \in X : \limsup_{t \longrightarrow t_0} |T_t f(x) - f(x)| > \lambda \right\} \right) \\
&\leq \mu\left(\left\{ x \in X : \limsup_{t \longrightarrow t_0} |T_t(f - f_n)(x) - (f(x) - f_n(x))| > \lambda \right\} \right) \\
&\leq \mu\left(\left\{ x \in X \ |T^*(f - f_n\right) (x) > \frac{\lambda}{2} \right\}) + \mu\left(\left\{ x \in X \ ||(f - f_n)(x)| > \frac{\lambda}{2} \right\} \right) \\
&\leq \left(\frac{2C}{\lambda} ||f - f_n||_p \right)^q + \left(\frac{2}{\lambda} ||f - f_n||_p \right)^p.
\end{aligned}$$

The last inequality follows from the fact that T^* is weak (p, q), and the Chebyshev's inequality. Since $(f_n)_{n \in \mathbb{N}}$ converges to the function f in $L^p(X)$ norm, the right hand side of above inequality tends to 0 as $n \longrightarrow \infty$. So, for a given $\lambda > 0$ we have

$$\mu\left(\left\{ x \in X : \limsup_{t \longrightarrow t_0} |T_t f(x) - f(x)| > \lambda \right\} \right) = 0.$$

Now, we have

$$\begin{aligned}
&\left\{ x \in X \middle| \limsup_{t \longrightarrow t_0} |T_t f(x) - f(x)| > 0 \right\} \\
&= \bigcup_{k=1}^{\infty} \left\{ x \in X \middle| \limsup_{t \longrightarrow t_0} |T_t f(x) - f(x)| > \frac{1}{k} \right\}.
\end{aligned}$$

Hence,

$$\mu\left(\left\{x \in X : \limsup_{t \longrightarrow t_0} |T_t f(x) - f(x)| > 0\right\}\right) = 0.$$

$\square$

Next, we define the distribution function of a given measurable function. It plays a role in proving the interpolation results of this section. We keep the notation as in [12], and refer to reader to this source for further details on this topic.

Definition 2.5 (Distribution function)**.** *Let (X, μ) be a measure space and let $f : X \longrightarrow \mathbb{C}$ be a measurable function. We call the function $d_f : (0, \infty) \to \mathbb{R}$, defined by*

$$d_f(\lambda) = \mu(\{x \in X : |f(x)| > \lambda\}),$$

the distribution function of f associated with μ.

In the next result we see the relation between the L^p norm of a function f and its distribution function, d_f.

Lemma 2.8. *Let $f \in L^p(X)$, with $0 < p < \infty$. Then, $||f||_p^p = p \int_0^\infty \alpha^{p-1} d_f(\alpha) \mathrm{d}\alpha$.*

Proof. We prove this result as follows.

$$\begin{aligned}
p\int_0^\infty \alpha^{p-1} d_f(\alpha)\mathrm{d}\alpha &= p\int_0^\infty \alpha^{p-1} \int_X \chi_{\{x:|f(x)|>\alpha\}}(x)\ \mathrm{d}\mu(x)\ \mathrm{d}\alpha \\
&= p\int_0^\infty \int_X \alpha^{p-1}\chi_{(0,|f(x)|)}(\alpha)\ \mathrm{d}\mu(x)\ \mathrm{d}\alpha \\
&= p\int_X \left(\int_0^\infty \alpha^{p-1}\chi_{(0,|f(x)|)}(\alpha)\ \mathrm{d}\alpha\right) \mathrm{d}\mu(x) \\
&= p\int_X \left[\frac{\alpha^p}{p}\right]_0^{|f(x)|} \mathrm{d}\mu(x) \\
&= \int_X |f(x)|^p\ \mathrm{d}\mu(x) \\
&= ||f||_p^p.
\end{aligned}$$

□

With the help of the Lemma 2.8, we now prove an important result, known as the Marcinkiewicz Interpolation theorem. This theorem says that if we know that an operator is of weak type (p_0, p_0) and weak type (p_1, p_1), then it is of strong type (p, p), for all $p_0 < p < p_1$. We use this theorem frequently in the sequel. There is a general version of the Marcinkiewicz interpolation, and can be found in [2]. Since we only deal with L^p-L^p boundedness of operators, we do not require it in the full generality. We state and prove the result in a manner convenient to us.

Theorem 2.9 (Marcinkiewicz Interpolation). *Let (X, μ) and (Y, ν) be measure spaces, $1 \leq p_0 < p_1 \leq \infty$, and let T be sublinear operator from $L^{p_0}(X) + L^{p_1}(X)$ to the space of measurable functions on Y that is of weak type (p_0, p_0) and weak type (p_1, p_1). Then T is of strong type (p, p) for any $p_0 < p < p_1$.*

Proof. Given $f \in L^p(X)$, for each $\lambda > 0$ we decompose $f = f_0 + f_1$, where,

$$f_0 = f\chi_{\{x \in X : |f(x)| > c\lambda\}},$$

$$f_1 = f\chi_{\{x \in X : |f(x)| \leq c\lambda\}}.$$

The constant $c > 0$ can be fixed later in the proof. Let $r = p - p_0$ and consider the set $A_\lambda := \{x \in X : |f(x)| > c\lambda\}$. Then

$$\begin{aligned}
\int_X |f_0(x)|^{p_0} \mathrm{d}x &= \int_{A_\lambda} |f(x)|^{p_0} \mathrm{d}x \\
&= \int_{A_\lambda} |f(x)|^p |f(x)|^{-r} \mathrm{d}x \\
&\leq (c\lambda)^{-r} \int_{A_\lambda} |f(x)|^p \mathrm{d}x \\
&\leq (c\lambda)^{-r} ||f(x)||_p^p < \infty.
\end{aligned}$$

That is, $f_0 \in L^{p_0}(X)$. Now, let $s = p_1 - p$ and consider the set $B_\lambda := \{x \in X : |f(x)| \leq c\lambda\}$. Then,

$$\int_X |f_1(x)|^{p_1} \mathrm{d}x = \int_{B_\lambda} |f(x)|^{p_1} \mathrm{d}x$$

$$= \int\limits_{B_\lambda} |f(x)|^p |f(x)|^s \mathrm{d}x$$
$$\leq (c\lambda)^s \int\limits_{B_\lambda} |f(x)|^p$$
$$\leq (c\lambda)^s ||f(x)||_p^p < \infty.$$

That is, $f_1 \in L^{p_1}(X)$. Now if y is such that $|Tf(y)| > \lambda$, then from the sublinearity of T, $\lambda < |Tf(y)| \leq |Tf_0(y)| + |Tf_1(y)|$. Therefore $\{y \in Y : |Tf(y)| > \lambda\} \subseteq \{y \in Y : |Tf_0(y)| > \frac{\lambda}{2}\} \bigcup \{y \in Y : |Tf_1(y)| > \frac{\lambda}{2}\}$. So we have $\nu(\{y \in Y : |Tf(y)| > \lambda\}) \leq \nu(\{y \in Y : |Tf_0(y)| > \frac{\lambda}{2}\}) + \nu(\{y \in Y : |Tf_1(y)| > \frac{\lambda}{2}\})$. That is $d_{Tf}(\lambda) \leq d_{Tf_0}(\frac{\lambda}{2}) + d_{Tf_1}(\frac{\lambda}{2})$. Let us consider the following cases.

Case 1 : $p_1 = \infty$. Choose $c = \frac{1}{2A_1}$, where A_1 is such that $||Tg||_\infty \leq A_1||g||_\infty$. Therefore, $d_{Tf_1}(\frac{\lambda}{2}) = \nu(\{y \in Y : |Tf_1(y)| > \frac{\lambda}{2}\}) = 0$. By the weak (p_0, p_0) inequality, there exists a constant A_0 such that

$$d_{Tf_0}\left(\frac{\lambda}{2}\right) \leq \left(\frac{2A_0}{\lambda}||f_0||_{p_0}\right)^{p_0}.$$

Now, by Lemma 2.8 we have,

$$\begin{aligned}
||Tf||_p^p &\leq p \int\limits_0^\infty \lambda^{p-1} d_{Tf_0}(\frac{\lambda}{2}) \mathrm{d}\lambda \\
&\leq p \int\limits_0^\infty \lambda^{p-1} \left(\frac{2A_0}{\lambda}||f_0||_{p_0}\right)^{p_0} \mathrm{d}\lambda \\
&= (2A_0)^{p_0} p \int\limits_0^\infty \lambda^{p-p_0-1} ||f_0||_{p_0}^{p_0} \mathrm{d}\lambda \\
&= (2A_0)^{p_0} p \int\limits_0^\infty \lambda^{p-p_0-1} \int\limits_X |f(x)|^{p_0} |\chi_{A_\lambda}(x)|^{p_0} \mathrm{d}\mu(x) \mathrm{d}\lambda \\
&= (2A_0)^{p_0} p \int\limits_0^\infty \int\limits_X \lambda^{p-p_0-1} |f(x)|^{p_0} \chi_{\left(0, \frac{|f(x)|}{c}\right)}(\lambda) \mathrm{d}\mu(x) \mathrm{d}\lambda \\
&= (2A_0)^{p_0} p \int\limits_X |f(x)|^{p_0} \left(\int\limits_0^\infty \lambda^{p-p_0-1} \chi_{\left(0, \frac{|f(x)|}{c}\right)}(\lambda) \mathrm{d}\lambda\right) \mathrm{d}\mu(x)
\end{aligned}$$

$$= (2A_0)^{p_0} p \int_X |f(x)|^{p_0} \left[\frac{\lambda^{p-p_0}}{p-p_0}\right]_0^{\frac{|f(x)|}{c}} \mathrm{d}\mu(x)$$

$$= (2A_0)^{p_0} p \int_X |f(x)|^{p_0} \frac{|f(x)|^{(p-p_0)}}{(p-p_0)c^{(p-p_0)}} \mathrm{d}\mu(x)$$

$$= \frac{(2A_0)^{p_0} p}{(p-p_0)c^{(p-p_0)}} \int_X |f(x)|^p \mathrm{d}\mu(x)$$

$$= \frac{(2A_0)^{p_0} p (2A_1)^{p-p_0}}{(p-p_0)} ||f||_p^p$$

$$= \frac{2^p p A_0^{p_0} A_1^{p-p_0}}{(p-p_0)} ||f||_p^p.$$

Hence, $||Tf||_p^p \leq \dfrac{2^p p A_0^{p_0} A_1^{p-p_0}}{(p-p_0)} ||f||_p^p$, and we have the strong (p,p) inequality.

Case 2: $p_1 < \infty$.

We know that $f_0 \in L^{p_0}(\mu)$ and $f_1 \in L^{p_1}(\mu)$. Therefore, we have the following pair of inequalities

$$d_{Tf_1}\left(\frac{\lambda}{2}\right) \leq \left(\frac{2A_1}{\lambda}||f_1||_{p_1}\right)^{p_1}, \tag{2.2}$$

$$d_{Tf_0}\left(\frac{\lambda}{2}\right) \leq \left(\frac{2A_0}{\lambda}||f_0||_{p_0}\right)^{p_0}. \tag{2.3}$$

By Lemma 2.8, we also have

$$||Tf||_p^p = p\int_0^\infty \lambda^{p-1} d_{Tf}(\lambda)\mathrm{d}\lambda.$$

As $d_{Tf}(\lambda) \leq d_{Tf_0}(\frac{\lambda}{2}) + d_{Tf_1}(\frac{\lambda}{2})$, we have the following

$$||Tf||_p^p \leq p\int_0^\infty \lambda^{p-1} d_{Tf_0}\left(\frac{\lambda}{2}\right)\mathrm{d}\lambda + p\int_0^\infty \lambda^{p-1} d_{Tf_1}\left(\frac{\lambda}{2}\right)\mathrm{d}\lambda. \tag{2.4}$$

Now for the first integral, we have,

$$p\int_0^\infty \lambda^{p-1} d_{Tf_0}\left(\frac{\lambda}{2}\right)\mathrm{d}\lambda \leq p\int_0^\infty \lambda^{p-1}\left(\frac{2A_0}{\lambda}||f_0||_{p_0}\right)^{p_0}\mathrm{d}\lambda$$

$$= (2A_0)^{p_0} p \int_0^\infty \lambda^{p-p_0-1} ||f_0||_{p_0}^{p_0} \mathrm{d}\lambda$$

$$= (2A_0)^{p_0} p \int_0^\infty \lambda^{p-p_0-1} \int_X |f(x)|^{p_0} |\chi_{A_\lambda}(x)|^{p_0} \mathrm{d}\mu(x)\mathrm{d}\lambda$$

$$
\begin{aligned}
&= (2A_0)^{p_0} p \int_0^\infty \int_X \lambda^{p-p_0-1} |f(x)|^{p_0} \chi_{\left(0, \frac{|f(x)|}{c}\right)}(\lambda) \mathrm{d}\mu(x) \mathrm{d}\lambda \\
&= (2A_0)^{p_0} p \int_X |f(x)|^{p_0} \left(\int_0^\infty \lambda^{p-p_0-1} \chi_{\left(0, \frac{|f(x)|}{c}\right)}(\lambda) \mathrm{d}\lambda \right) \mathrm{d}\mu(x) \\
&= (2A_0)^{p_0} p \int_X |f(x)|^{p_0} \left[\frac{\lambda^{p-p_0}}{p-p_0} \right]_0^{\frac{|f(x)|}{c}} \mathrm{d}\mu(x) \\
&= (2A_0)^{p_0} p \int_X |f(x)|^{p_0} \frac{|f(x)|^{(p-p_0)}}{(p-p_0) c^{(p-p_0)}} \mathrm{d}\mu(x) \\
&= \frac{(2A_0)^{p_0} p}{(p-p_0) c^{(p-p_0)}} \int_X |f(x)|^p \mathrm{d}\mu(x).
\end{aligned}
$$

That is, we have,

$$
p \int_0^\infty \lambda^{p-1} d_{Tf_0} \left(\frac{\lambda}{2} \right) \mathrm{d}\lambda \leq \frac{(2A_0)^{p_0} p}{(p-p_0) c^{(p-p_0)}} ||f||_p^p. \tag{2.5}
$$

For the second integral,

$$
\begin{aligned}
p \int_0^\infty \lambda^{p-1} d_{Tf_1} \left(\frac{\lambda}{2} \right) \mathrm{d}\lambda &\leq p \int_0^\infty \lambda^{p-1} \left(\frac{2A_0}{\lambda} ||f_1||_{p_1} \right)^{p_1} \mathrm{d}\lambda \\
&= (2A_1)^{p_1} p \int_0^\infty \lambda^{p-p_1-1} ||f_1||_{p_1}^{p_1} \mathrm{d}\lambda \\
&= (2A_1)^{p_1} p \int_0^\infty \lambda^{p-p_1-1} \int_X |f(x)|^{p_1} |\chi_{B_\lambda}(x)|^{p_1} \mathrm{d}\mu(x) \mathrm{d}\lambda \\
&= (2A_1)^{p_1} p \int_0^\infty \int_X \lambda^{p-p_1-1} |f(x)|^{p_1} \chi_{\left(\frac{|f(x)|}{c}, \infty\right)}(\lambda) \mathrm{d}\mu(x) \mathrm{d}\lambda \\
&= (2A_1)^{p_1} p \int_X |f(x)|^{p_1} \left[\frac{\lambda^{p-p_1}}{p-p_1} \right]_{\frac{|f(x)|}{c}}^{\infty} \mathrm{d}\mu(x) \\
&= (2A_1)^{p_1} p \int_X |f(x)|^{p_0} \frac{|f(x)|^{(p-p_1)}}{(p_1-p) c^{(p-p_1)}} \mathrm{d}\mu(x) \\
&= \frac{(2A_1)^{p_1} p}{(p_1-p) c^{(p_1-p)}} \int_X |f(x)|^p \mathrm{d}\mu(x).
\end{aligned}
$$

So we have,

$$p\int_0^\infty \lambda^{p-1} a_{Tf_1}(\frac{\lambda}{2})\mathrm{d}\lambda \leq \frac{(2A_1)^{p_1}p}{(p_1-p)c^{(p_1-p)}}||f||_p^p. \tag{2.6}$$

Now, from Inequalities (2.4), (2.5) and (2.6) we get,

$$\begin{aligned}||Tf||_p^p &\leq \frac{(2A_0)^{p_0}p}{(p-p_0)c^{(p-p_0)}}||f||_p^p + \frac{(2A_1)^{p_1}p}{(p_1-p)c^{(p_1-p)}}||f||_p^p \\ &= \left(\frac{(2A_0)^{p_0}p}{(p-p_0)c^{(p-p_0)}} + \frac{(2A_1)^{p_1}p}{(p_1-p)c^{(p_1-p)}}\right)||f||_p^p.\end{aligned}$$

We want to choose a c such that

$$\frac{(2A_0)^{p_0}p}{(p-p_0)c^{(p-p_0)}} = \frac{(2A_1)^{p_1}p}{(p_1-p)c^{(p_1-p)}}$$

From this we have,

$$c = \left(2^{p_0-p_1}\frac{A_0^{p_0}}{A_1^{p_1}}\frac{p_1-p}{p-p_0}\right)^{\frac{1}{p_1-p_0}}.$$

Now,

$$\begin{aligned}\frac{(2A_0)^{p_0}p}{(p-p_0)c^{(p-p_0)}} &= \frac{2^{p_0}A_0^{p_0}p}{(p-p_0)\left(2^{p_0-p_1}\frac{A_0^{p_0}}{A_1^{p_1}}\frac{p_1-p}{p-p_0}\right)^{\frac{p-p_0}{p_1-p_0}}} \\ &= 2^p A_0^{p_0\left(\frac{p_1-p}{p_1-p_0}\right)} A_1^{p_1\left(\frac{p-p_0}{p_1-p_0}\right)}\left(\frac{p-p_0}{p_1-p}\right)^{\frac{p-p_0}{p_1-p_0}}\frac{p}{(p-p_0)}.\end{aligned}$$

Let $t \in (0,1)$ such that $\frac{1}{p} = \frac{(1-t)}{p_0} + \frac{t}{p_1}$. Then using $\frac{p_1}{p} = \frac{p_1(1-t)+tp_0}{p_0}$ and $\frac{p_0}{p} = \frac{p_1(1-t)+tp_0}{p_1}$

$$p_0\left(\frac{p_1-p}{p_1-p_0}\right) = p(1-t)$$

and

$$p_1\left(\frac{p-p_0}{p_1-p_0}\right) = tp$$

and

$$\left(\frac{p-p_0}{p_1-p}\right)^{\frac{p-p_0}{p_1-p_0}}\frac{p}{(p-p_0)} = (p_1-p_0)^{-1}(t/p)^{\frac{(t-1)p_1}{p_1(1-t)+tp_0}}((1-t)/p_0)^{\frac{-tp_0}{p_1(1-t)+tp_0}}$$

using the above expressions we have

$$\frac{(2A_0)^{p_0}p}{(p-p_0)c^{(p-p_0)}} = 2^pA_0^{p(1-t)}A_1^{tp}(p_1-p_0)^{-1}(t/p)^{\frac{(t-1)p_1}{p_1(1-t)+tp_0}}((1-t)/p_0)^{\frac{-tp_0}{p_1(1-t)+tp_0}}$$

Therefore we have

$$||Tf||_p^p \leq \frac{(2A_0)^{p_0}p}{(p-p_0)c^{(p-p_0)}}||f||_p^p + \frac{(2A_1)^{p_1}p}{(p_1-p)c^{(p_1-p)}}||f||_p^p$$

$$= 2^{p+1} A_0^{p(1-t)} A_1^{tp} (p_1 - p_0)^{-1} (t/p)^{\frac{(t-1)p_1}{p_1(1-t)+tp_0}} ((1-t)/p_0)^{\frac{-tp_0}{p_1(1-t)+tp_0}} ||f|_p^p.$$

□

Next we prove another important interpolation theorem. We start with the following definition.

Definition 2.6 (Truncation of a function)**.** *If f is a measurable function, then truncation of f is any one of the functions g defined by letting $g(x) = f(x)$ if $r_1 < f(x) \le r_2$ and $g(x) = 0$ otherwise, where r_1 and r_2 are non-negative.*

Lemma 2.10 (Three Lines Lemma)**.** *Let F be a bounded continuous complex valued function on the closed strip*

$$S = \{x + iy \in \mathbb{C} : 0 \le x \le 1\},$$

that is analytic in the interior of S. If $|f(iy)| \le m_0$ and $|F(1+iy)| \le m_1$, for all $y \in \mathbb{R}$ then $|F(x+iy)| \le m_0^{1-x} m_1^x$ for all $z = x + iy \in S$.

Proof. The problem is reduced to the case $m_0 = 1 = m_1$ by considering the function $\frac{F(z)}{m_0^{1-z} m_1^z}$. Thus, suppose that $|F(iy)| \le 1$ and $|F(1+iy)| \le 1$ for all $y \in \mathbb{R}$. We want to show that $|F(z)| \le 1$, for all $z \in S$.
We observe that if we are able to prove that $\lim_{|y| \longrightarrow \infty} F(x+iy) = 0$ uniformly for $0 \le x \le 1$, then there is $y_0 > 0$ such that $|F(x+iy)| \le 1$ for $|y| > y_0$, while $|F(z)| \le 1$ on the boundary of the rectangle with vertices $iy_0, 1+iy_0, 1-iy_0, -iy_0$. Therefore, by using maximal principle, we conclude that $|F(z)| \le 1$ on the strip S.
Note that we can apply the above argument for each of the functions $F_n(z) = F(z)e^{(z^2-1)/n}$, for $n \in \mathbb{N}$, because

$$|F_n(z)| = |F(x+iy)|e^{-y^2/n+(x^2-1)/n} \le |F(z)|e^{-y^2/n} \longrightarrow 0$$

uniformly as $|y| \longrightarrow \infty$. Therefore, $|F_n(z)| \le 1$, for each $n \in \mathbb{N}$. Now the desired result follows from the fact that $F_n(z) \longrightarrow F(z)$ when $n \longrightarrow \infty$, for each $z \in S$.

□

Now we are ready to prove the following interpolation theorem.

Theorem 2.11 (Riesz-Thorin interpolation). *Suppose a linear operator is of strong type (p_i, q_i) with the operator norm M_i, for $i = 0, 1$. Then, T is of strong type (p, q) with its operator norm $M \leq M_0^{1-t} M_1^t$, where*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \text{ and } \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Proof. First we estimate $||Tf||_q$ for simple function f belongs to the domain of T. Since

$$||Tf||_q = \sup_{||g||_{q'}=1} \left| \int_Y (Tf(x))g(x) \, \mathrm{d}\nu \right|,$$

where g is simple function and q' is conjugate exponent of q it is enough to show that absolute value of each such integral is less than or equal to $M_0^{1-t} M_1^t ||f||_p$. Now, dividing by $||f||_p$, we can reduce the problem further to the case $||f||_p = 1$. Suppose $f = \sum_{j=1}^{m} a_j \chi_{E_j}$ and $g = \sum_{k=1}^{n} b_k \chi_{F_k}$ are such simple functions that satisfy all of the above conditions. Suppose $\alpha_j = 1/p_j, \beta_j = 1/q_j$, for $j = 0, 1$, and $\alpha = 1/p$ and $\beta = 1/q$. Also let $\alpha(z) = (1-z)\alpha_0 + z\alpha_1$ and $\beta(z) = (1-z)\beta_0 + z\beta_1$, for $z \in \mathcal{C}$. Now in the above expression of f and g if $a_j = |a_j|e^{i\theta_j}$ and $b_k = |b_k|e^{i\phi_k}$, we define

$$f_z = \sum_{j=1}^{m} |a_j|^{\alpha(z)/\alpha} e^{i\theta_j} \chi_{E_j}$$

and

$$g_z = \sum_{k=1}^{n} |b_k|^{(1-\beta(z))/(1-\beta)} e^{i\phi_k} \chi_{F_k}.$$

Let us define $F(z) := \int_N (Tf_z(x))g_z(x) \, \mathrm{d}\nu$. By using the linearity of T we also have

$$F(z) = \sum_{j,k=1}^{mn} |a_j|^{\alpha(z)/\alpha} |b_k|^{(1-\beta(z))/(1-\beta)} \gamma_{jk},$$

where $\gamma_{jk} = e^{i(\theta_j+\phi_k)} \int_N (T\chi_{E_j}(x))\chi_{F_k(x)} \, \mathrm{d}\nu$. Note that each term of this sum is bounded in the strip S, given in Lemma 2.10. Therefore the function F is bounded when restricted to this strip. Now, we prove that $|F(iy)| \leq M_0$ and

$|F(1+iy)| \leq M_1$ for all $y \in \mathbb{R}$. Then, the desired inequality

$$\left| \int_Y (Tf(x))g(x) \, \mathrm{d}x \right| \leq M_0^{1-t} M_1^t$$

follows from Lemma 2.10. First observe that $\alpha(iy) = \alpha_0 + iy(\alpha_1 - \alpha_0)$ and $1 - \beta(iy) = (1-\beta_0) - iy(\beta_1 - \beta_0)$. Hence, we have

$$|f_{iy}|^{p_0} = |e^{i\ arg\ f} |f|^{iy(\alpha_1 - \alpha_0)/\alpha} |f|^{(p/p_0)}|^{p_0} = |f|^p.$$

and

$$|g_{iy}|^{q_0'} = |e^{i\ arg\ g} |g|^{-iy(\beta_1 - \beta_0)/(1-\beta)} |g|^{(q'/q_0')}|^{q_0'} = |g|^{q'}.$$

Therefore, by using Hölder's inequality and the fact that T is of strong type (p_0, q_0) with operator norm M_0, we get

$$\begin{aligned} |F(iy)| \leq ||Tf_{iy}||_{q_0} ||g_{iy}||_{q_0'} &\leq M_0 ||f_{iy}||_{p_0} ||g_{iy}||_{q_0'} \\ &= M_0 ||f||_p^{(p/p_0)} ||g||_q^{(q'/q_0')} = M_0. \end{aligned}$$

With a similar argument it follows that $|F(1+iy)| \leq M_1$. Therefore $||Tf||_q \leq M_0^{1-t} M_1^t ||f||_p$ for all simple functions $f \in L^p(X)$. To prove the result for general function $f \in L^p(X)$, we shall show that we can find a sequence of simple function $(f_n)_{n \in \mathbb{N}}$ such that $||f_n - f||_p \longrightarrow 0$ and $Tf_n(x) \longrightarrow Tf(x)$ a.e. as $n \longrightarrow \infty$. We can assume that f is non-negative. We can also assume that $p_0 \leq p_1$. Let

$$f_0(x) = \begin{cases} f(x) & \text{when } f(x) > 1. \\ 0 & \text{when } f(x) \leq 1. \end{cases}$$

and $f_1 = f - f_0$. We have, $(f_0)^{p_0} \leq f^p$, and $(f_1)^{p_1} \leq f^p$. Let $(g_m)_{m \in \mathbb{N}}$ be a sequence of non-negative simple functions increases pointwise to f then by monotone convergence theorem, $||g_m - f||_p \longrightarrow 0$ as $m \longrightarrow \infty$. For the same reason, we also have $||\left(g_m\right)_0 - f_0||_p$ and $||\left(g_m\right)_1 - f_1||_p \longrightarrow 0$, as $m \longrightarrow \infty$. Since T is of strong type (p_0, q_0) and (p_1, q_1), we have $||T\left(g_m\right)_0 - Tf_0||_p \longrightarrow 0$ and $||T\left(g_m\right)_1 - Tf_1||_p \longrightarrow 0$, as $m \longrightarrow \infty$. So, there is are subsequences $\left(\left(g_{m_k}\right)_0\right)_{k \in \mathbb{N}}$ and $\left(\left(g_{m_k}\right)_1\right)_{k \in \mathbb{N}}$ such that $T\left(g_{m_k}\right)_0$ and $T\left(g_{m_n}\right)_1$ converge to Tf_0 and Tf_1 almost everywhere respectively. Now, let $f_k = \left(g_{m_n}\right)_0 + \left(g_{m_n}\right)_1$. Then, we have a sequence $(f_k)_{k \in \mathbb{N}}$ satisfying the desired properties: $\lim\limits_{k \longrightarrow \infty} ||f_k - f||_p = 0$ and $\lim\limits_{k \longrightarrow \infty} Tf_k(x) =$

$Tf_0(x) + Tf_1(x) = Tf(x)$ almost everywhere. This completes the proof. □

2.4 Schwartz Functions and tempered distribution

This section is dedicated to the study of rapidly decreasing functions. Most material presented here is available in [23], and we do not give detailed proofs of any results.

2.4.1 The space of Schwartz functions

Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we denote $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$. The first order partial derivative of a function f on $\mathbb{R}^n$ with respect to jth variable x_j is denoted by $\partial_j f$ while the mth order partial derivative with respect to jth variable is denoted by $\partial_j^m f$. A multi-index α is an ordered n-tuple of non-negative integers. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, $D^\alpha f$ denotes the derivative,

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 \cdots + \alpha_n$ denotes its size. For $x \in \mathbb{R}^n$ and a multi-index α we set $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Definition 2.7 (Schwartz function)**.** *A complex valued smooth function f is a Schwartz function or a rapidly decreasing function if for every pair of multi-indices α and β,*

$$\rho_{\alpha,\beta}(f) := \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty.$$

The set of all Schwartz class functions are denoted by $\mathcal{S}(\mathbb{R}^n)$. The quantities $\rho_{\alpha,\beta}$ are called Schwartz seminorms of f. The collection of seminorms $\{\rho_{\alpha,\beta}\}$ as given in Definition 2.7 give a topology on $\mathcal{S}(\mathbb{R}^n)$. The details of the construction of open sets using seminorms can be found in [23]. The concept of interest to us is the convergence of sequences in $\mathcal{S}(\mathbb{R}^n)$. We state it as a definition.

Definition 2.8. *Let f_k, f be in $\mathcal{S}(\mathbb{R}^n)$ for $k \in \mathbb{N}$. The sequence $(f_k)_{k\in\mathbb{N}}$ converges to f in $\mathcal{S}(\mathbb{R}^n)$ if for all multi-indices α and β we have*

$$\rho_{\alpha,\beta}(f_k - f) \longrightarrow 0$$

as $k \longrightarrow \infty$.

Remark 2.4. An equivalent definition of the Schwartz seminorms is given by

$$\rho_{\alpha,N} := \sup_{x\in\mathbb{R}^n} \left\{ \left(1+|x|^2\right)^N |D^\alpha f(x)| \right\},$$

for multiindex α and $N \in \mathbb{N} \cup \{0\}$. These seminorms are equivalent to that given in Definition 2.7 in the sense that they give the same topology on the space $\mathcal{S}(\mathbb{R}^n)$, and capture all essential properties of Schwartz functions.

The space $\mathcal{S}(\mathbb{R}^n)$ is a topological vector space, i.e., the operations $(f, g) \longmapsto f + g$, $(a, f) \longmapsto af$ and $f \longmapsto \partial^\alpha f$ are continuous for all $f, g \in \mathcal{S}(\mathbb{R}^n)$, $a \in \mathbb{C}$, and multi-indices α.

2.4.2 Tempered distributions

Since we have a topology (and a notion of convergence) on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, we can consider studying continuous linear functionals.

Definition 2.9 (Tempered distribution)**.** *The space of all bounded linear functionals is known as the space of tempered distribution, and is denoted by $\mathcal{S}'(\mathbb{R}^n)$.*

The action of a tempered distribution u on a function $f \in \mathcal{S}(\mathbb{R}^n)$ is represented by $\langle u, f\rangle = u(f)$. We give have a characterization for tempered distribution.

Proposition 2.12 ([23])**.** *A linear functional u on $\mathcal{S}(\mathbb{R}^n)$ is a tempered distribution if and only if there exists $C > 0$ and N, $m \in \mathbb{N}$ such that*

$$|\langle u, f\rangle| \le C \sum_{\substack{|\alpha|\le m \\ |\beta|\le N}} \rho_{\alpha,\beta}(f),$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$.

We know convolutions of two functions. Now, we define convolution of a Schwartz function with a tempered distribution. This is of interest to us in Chapter 5. First, we define, for a function $f : \mathbb{R}^n \to \mathbb{C}$, the function $\tilde{f} : \mathbb{R}^n \to \mathbb{C}$ as $\tilde{f}(x) = f(-x)$. Also, for a fixed $a \in \mathbb{R}^n$, we define $\tau_a f(x) = f(x - a)$. With this, we now make the following definition.

Definition 2.10. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $h \in \mathcal{S}(\mathbb{R}^n)$. Then, the convolution of u with h is a function $u * h : \mathbb{R}^n \to \mathbb{C}$, given by*

$$(u * h)(x) = \langle u, \tau_x \tilde{h} \rangle.$$

We say that a tempered distribution u coincides with a function h if we have

$$\langle u, f \rangle = \int_{\mathbb{R}^n} h(x) f(x) \, \mathrm{d}x$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$.

2.5 Fourier transform

In this section, we recall a few facts about the Fourier transform on $\mathbb{R}^n$.

Definition 2.11 (Fourier transform)**.** *For a given $f \in L^1(\mathbb{R}^n)$ function, Fourier transform of f is defined by*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \, \mathrm{d}x,$$

where $x \cdot \xi = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$. Here $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n$.

The following facts about Fourier transform can be found in [12]. If $f, g \in L^1(\mathbb{R}^n)$, then following are true.

1. Fourier transform is a linear operator, that is, for $\alpha, \beta \in \mathbb{C}$, we have the following

$$(\alpha f + \beta g)\widehat{\ } = \alpha \widehat{f} + \beta \widehat{g}.$$

2. Convolution of f, g is defined as
$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\ \mathrm{d}y.$$
Then we have $(f * g)\widehat{} = \hat{f}\hat{g}$.

3. Let for $h \in \mathbb{R}^n$, $\tau_h f(x) = f(x+h)$. Then
$$(\tau_h f)\widehat{}(\xi) = \hat{f}(\xi)e^{ih\cdot\xi}.$$

4. Let $\rho \in O(n)$, where $O(n)$ is the set of all orthogonal transformations on $\mathbb{R}^n$. Then,
$$(f(\rho\cdot))\widehat{}(\xi) = \hat{f}(\rho\xi).$$

5. If $g(x) = \lambda^{-n} f(\lambda^{-1}x)$, then $\hat{g}(\xi) = \hat{f}(\lambda\xi)$.

For Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ we have the following

1. The Fourier transform of a Schwartz function is a Schwartz function. Moreover, the Fourier transform as a map from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ is a continuous bijection of period 4.

2. If $f \in \mathcal{S}(\mathbb{R}^n)$ and P is a polynomial, then
$$(P(D)f)\widehat{} = P\hat{f},$$
$$(Pf)\widehat{} = P(-D)\hat{f},$$
where, D is the differential operator.

3. Fourier transform is a continuous linear one to one mapping from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$ whose inverse is also continuous and
$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)e^{ix\xi}\ \mathrm{d}\xi.$$

4. For $f \in \mathcal{S}(\mathbb{R}^n)$, $(\hat{f})\widehat{} = \tilde{f}$, where $\tilde{f}(x) = f(-x)$.

5. For $f, g \in \mathcal{S}(\mathbb{R}^n)$, we have $(f * g)^{\wedge} = \hat{f}\hat{g}$.

6. The following duality relation holds for $f, g \in \mathcal{S}(\mathbb{R}^n)$.
$$\int\limits_{\mathbb{R}^n} \hat{f}(x)\, g(x)\, \mathrm{d}x = \int\limits_{\mathbb{R}^n} f(x)\, \hat{g}(x)\, \mathrm{d}x.$$

We can also define the Fourier transform of a tempered distribution. The details and properties can be found in [23]. In our survey, we only require the definition, which we state here.

Definition 2.12 (Fourier transform of tempered distributions)**.** *Let $u \in \mathcal{S}'(\mathbb{R}^n)$. The Fourier transform $\hat{u}$ is defined by the tempered distribution.*
$$\langle \hat{u}, f\rangle = \langle u, \hat{f}\rangle,$$
for all $f \in \mathcal{S}(\mathbb{R}^n)$.

2.6 Integration on the sphere and Bessel function of the first kind

In this section we see some important results related to Bessel function of the first kind. The study of Bessel functions is often a part of a semester course on Special Functions. The natural way of development is through the study of solutions of an ordinary differential equation with non-constant coefficients. In this survey, we take a different approach with this study. Instead of studying differential equations, we consider the integration of functions defined on the unit sphere in $\mathbb{R}^n$, and develop some basic results concerning the Bessel function of the first kind.

2.6.1 Integration on the sphere

We are well versed with the integration on the Euclidean space $\mathbb{R}^n$. The theory of integration is developed through the Lebesgue measure, a natural way to capture "volume" of sets. In Harmonic Analysis, we often deal with universes other than the Euclidean ones, and therefore, we require the theory of measures and integration on non-Euclidean spaces. This theory is vast, and different structures have (somewhat) different theories. To exposit the complete idea is beyond the scope of this survey. Nonetheless, we exposit a very small part on the integration of functions that are defined on the unit sphere. We provide a rather heursitic argument, instead of a rigorous one.

The idea is that if we slice $\mathbb{S}^{n-1}$ by a hyperplane, we end up with a sphere inside the hyperplane (see Figure 2.1). Consequently, to integrate a function

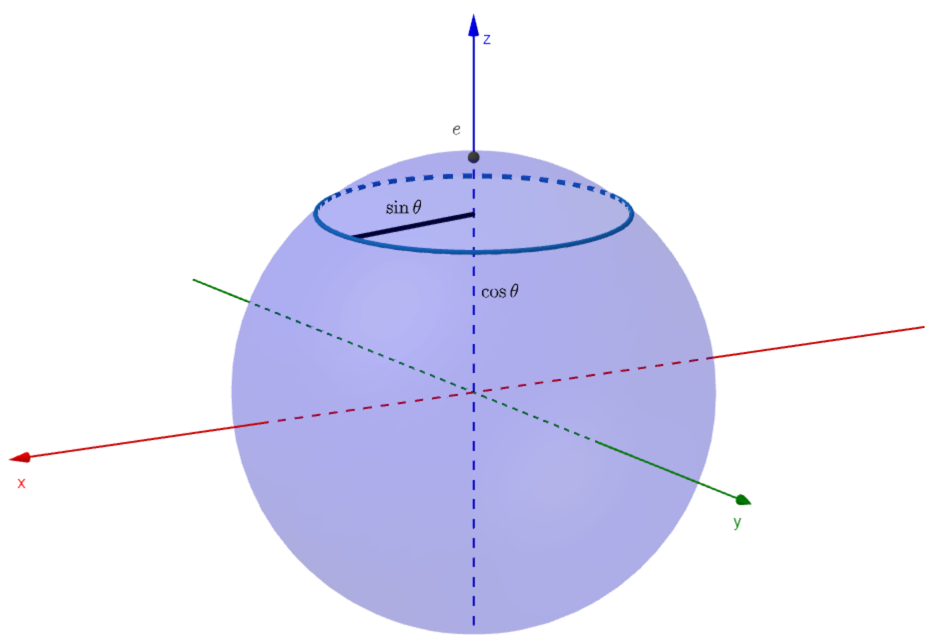

Figure 2.1: The slice of a sphere at a distance $\cos\theta$.

$f : \mathbb{S}^{n-1} \to \mathbb{C}$, it remains to "add up" all the integrals obtained on such "slices" made along a particular direction. Effectively, fixing a unit vector $e \in \mathbb{S}^{n-1}$, let us consider the sets $S_\theta = \{x' \in \mathbb{S}^{n-1} : e \cdot x' = \cos\theta\}$, formed by intersecting hyperplanes orthogonal to the vector e at a distance $\cos\theta$ from the origin. For a fixed θ, the integral of f over S_θ is again an integration over a sphere of a smaller dimension, and of a different radius. It is to be noticed that two spheres of different radii can be deformed into one another in a "smooth" fashion. In fact, one can do so by a simple rescaling! Hence, the measure on an n-dimensional sphere of radius, say R, must be R^n times the measure on the unit n-dimensional sphere. In the hyperplane $\langle e, x \rangle = \cos\theta$, we notice that the sphere S_θ has radius $\sin\theta$ (see Figure 2.1). On the other hand, for every $\theta \in [0,\pi]$, we get a slice (see Figure 2.2). Therefore, we have,

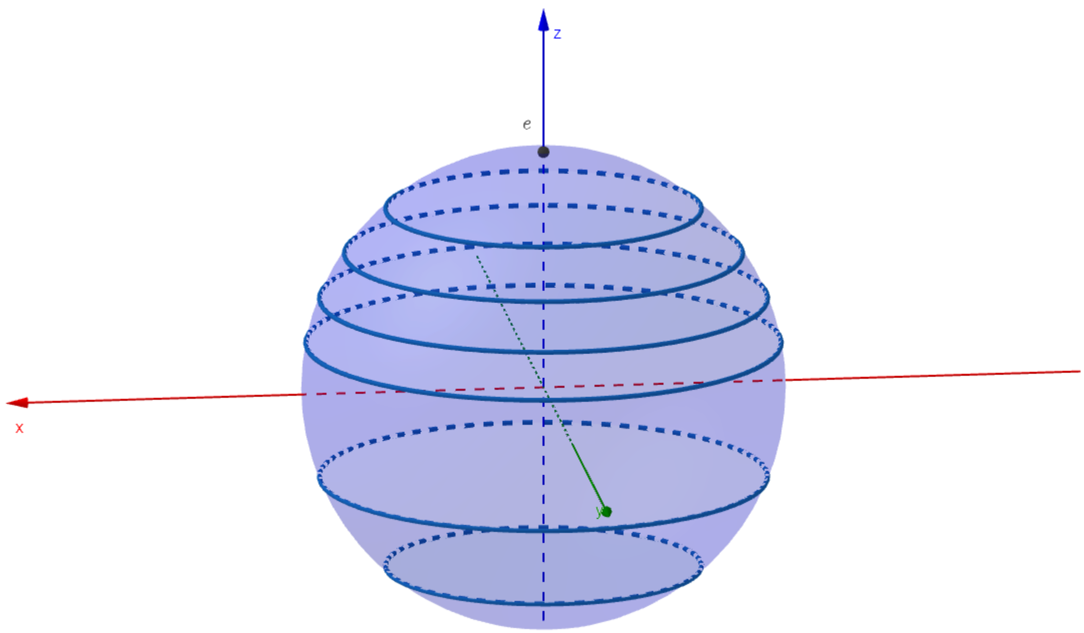

Figure 2.2: A sphere can be covered by slices through hyperplanes.

$$\begin{aligned}\int\limits_{\mathbb{S}^{n-1}} f(\sigma)\,\mathrm{d}\sigma &= \int\limits_0^\pi \int\limits_{S_\theta} f(\sigma(\theta,\omega))\ \mathrm{d}\omega\ \mathrm{d}\theta \\ &= \int\limits_0^\pi \int\limits_{\mathbb{S}^{n-2}} f(\sigma(\theta,\omega)) \sin^{n-2}\theta\ \mathrm{d}\omega\ \mathrm{d}\theta.\end{aligned}$$

Here, the point $\sigma \in \mathbb{S}^{n-1}$ depends on the distance θ of the hyperplane, and on a direction ω in the hyperplane. Of interest to us are the functions for which $f(\sigma(\theta,\omega))$ is independent of ω. From the argument we provided above, it is easy to see that in this case, we must have $f(\sigma) = \tilde{f}(\cos\theta)$, for an appropriate function $\tilde{f} : [0,\pi] \to \mathbb{C}$. Such functions are called "**radial**" functions on the sphere, and for them, we have

$$\int_{\mathcal{S}^{n-1}} f(\sigma)\ \mathrm{d}\sigma = \int_0^\pi \int_{\mathcal{S}^{n-2}} \tilde{f}(\cos\theta)\ \sin^{n-2}\theta \mathrm{d}\theta \mathrm{d}\omega$$

$$= |\mathbb{S}^{n-2}| \int_0^\pi \tilde{f}(\cos\theta) \cdot \sin^{n-2}\theta\ \mathrm{d}\theta.$$

With this understanding, we now proceed to look at the Bessel function of first kind.

2.6.2 Bessel Function of first kind

In this section, we follow the notation of [24]. The results given here can be found in the reference we mentioned. But we provide a few details that might of help to a new reader. For $k > -\frac{1}{2}$, Bessel function J_k is defined by the following expression

$$J_k(t) = \frac{(t/2)^k}{\Gamma[(2k+1)/2]\Gamma(1/2)} \int_{-1}^{1} e^{its}(1-s^2)^{(2k-1)/2}\ \mathrm{d}s$$

for $t > 0$.

If we develop the power series $\sum_{j=0}^{\infty}(its)^j/j!$ of e^{its}, for real $k > \frac{1}{2}$, we get

$$J_k(t) = \sum_{j=0}^{\infty}(-1)^j \frac{(t/2)^{k+2j}}{j!\Gamma(j+k+1)}. \tag{2.7}$$

Fourier transform of a radial function on $\mathbb{R}^n$, can be expressed in terms of Bessel function.

Theorem 2.13. *Let for $n \geq 2$, $f \in L^1(\mathbb{R}^n)$, be a radial function, i.e. $f(x) = f_0(|x|)$ for a.e. $x \in \mathbb{R}^n$. Then the Fourier transform of f, has the following form*

$$\widehat{f}(x) = F_0(r) = 2\pi r^{-[(n-2)/2]} \int_0^\infty f_0(s) J_{(n-2)/2}(2\pi r s) s^{n/2}\ \mathrm{d}s,$$

where, $r = |x|$.

Proof. By the definition of Fourier transform

$$\widehat{f}(x) = \int\limits_{\mathbb{R}^n} f(u)e^{-2\pi i\langle x,u\rangle}\ \mathrm{d}u = \int\limits_{0}^{\infty} f_0(s)\left(\int\limits_{\mathbb{S}^{n-1}} e^{-2\pi irs\langle x',u'\rangle}\ \mathrm{d}u'\right)s^{n-1}\ \mathrm{d}s. \quad (2.8)$$

The inner integral can be evaluate in the following way

$$\begin{aligned}\int\limits_{\mathbb{S}^{n-1}} e^{-2\pi irs\langle x',u'\rangle}\ \mathrm{d}u' &= \int\limits_{0}^{\pi}\int\limits_{\mathbb{S}^{n-1}} e^{-2\pi rscos\theta}\ \mathrm{d}v(sin\theta)^{n-2}\ \mathrm{d}\theta \\ &= \omega_{n-2}\int\limits_{-1}^{1} e^{-2\pi irs\xi}(1-\xi^2)^{(n-3)/2}\ \mathrm{d}\xi \\ &= \frac{2\pi^{(n-1)/2}\Gamma[(n-1)/2]\Gamma(1/2)}{\Gamma[(n-1)/2](\pi rs)^{-(n-2)/2}}J_{(n-2)/2}(2\pi rs) \\ &= 2\pi(rs)^{-(n-2)/2}J_{(n-2)/2}(2\pi rs).\end{aligned}$$

Substituting this in Equation (2.8) we get the desired result. □

Next, we see another useful result.

Theorem 2.14. *If $\nu > -\frac{1}{2}$ then*

$$J_{\nu+\gamma+1}(t) = \frac{t^{\gamma+1}}{2^{\gamma}\Gamma(\gamma+1)}\int\limits_{0}^{1} J_{\nu}(ts)s^{\nu+1}(1-s^2)^{\nu}\ \mathrm{d}s$$

whenever $\gamma > -1$ and $t > 0$.

Proof. Using Equation (2.7) we have

$$\int\limits_{0}^{1} J_{\nu}(ts)s^{\nu+1}(1-s^2)^{\nu}\ \mathrm{d}s = \int\limits_{0}^{1}\left(\sum_{j=0}^{\infty}(-1)^j\frac{(ts/2)^{\nu+2j}}{j!\Gamma(j+\nu+1)}\right)s^{\nu+1}(1-s^2)^{\gamma}\ \mathrm{d}s$$

Now substituting $s^2 = r$, we have

$$\int\limits_{0}^{1} J_{\nu}(ts)s^{\nu+1}(1-s^2)^{\nu}\ \mathrm{d}s = \sum_{j=0}^{\infty}(-1)^j\frac{(t/2)^{\nu+2j}}{j!\Gamma(j+\nu+1)}\frac{1}{2}\int\limits_{0}^{1} r^{\nu+j}(1-r)^{\gamma}\ \mathrm{d}r.$$

Using the well known relation $\Gamma(x)\Gamma(y) = \int\limits_{0}^{1} u^{x-1}(1-u)^{y-1}\ \mathrm{d}u$, we have

$$\begin{aligned}\int\limits_{0}^{1} J_{\nu}(ts)s^{\nu+1}(1-s^2)^{\nu}\ \mathrm{d}s &= \frac{2^{\gamma}\Gamma(\gamma+1)}{t^{\gamma+1}}\sum_{j=0}^{\infty}(-1)^j\frac{(t/2)^{\nu+\gamma+1+2j}}{j!\Gamma(\nu+\gamma+j+2)} \\ &= \frac{2^{\gamma}\Gamma(\gamma+1)}{t^{\gamma+1}}J_{\nu+\gamma+1}(t).\end{aligned}$$

This completes the proof. □

2.7 Approximation to Identity

In this section, we consider a special family of (compactly supported smooth) functions $\{\varphi_\epsilon\}_{\epsilon>0}$, called an approximation to identity. To motivate such a family, let us consider the space $L^1(\mathbb{R}^n)$. Then, Young's convolution inequality quickly tells us that $L^1(\mathbb{R}^n)$ is closed under taking convolutions with $\|f * g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}$, making it a Banach Algebra. However, this algebra lacks unity, i.e., there is no $f \in L^1(\mathbb{R}^n)$ such that for all $g \in L^1(\mathbb{R}^n)$, we have $f * g = g = g * f$. To see this, suppose on the contrary, $f \in L^1(\mathbb{R}^n)$ is the unity for convolution. Then, taking Fourier transforms, we obtain for all $g \in L^1(\mathbb{R}^n)$ that $\hat{g} = \hat{f}\hat{g}$. Consider the Gaussian $g(x) = e^{-x^2}$. It is known that $\hat{g}(\xi) = \sqrt{\pi}e^{-\frac{\xi^2}{4}}$. Therefore, we must have $\hat{f} \equiv 1$, which contradicts the Riemann-Lebesgue Lemma.

To understand the identity for convolution, let us go back to Schwartz functions. Let us consider the Dirac delta distribution $\delta \in \mathcal{S}'(\mathbb{R}^n)$, given by

$$\delta(f) = f(0).$$

The fact that δ is a tempered distribution easily follows from the convergence in $\mathcal{S}(\mathbb{R}^n)$. Now, we see that for any $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\delta * f(x) = \delta\left(\tau_x\tilde{f}\right) = \left(\tau_x\tilde{f}\right)(0) = \tilde{f}(-x) = f(x).$$

That is, the Dirac delta distribution is an identity for convolution. However, $\delta \notin L^1(\mathbb{R}^n)$, since it is not a function to begin with! The question, now, is that whether we can "approximate" the distribution δ with nice (preferably L^1) functions? To understand whether this is possible, we first observe that δ is a "point-mass distribution". That is, δ ignores all the points in $\mathbb{R}^n$ and focuses on 0 alone. In fact, this is the reason δ cannot be viewed as a true function (for otherwise, it would be zero almost everywhere). So, essentially, we require a collection of functions supported around 0 in a manner that their supports shrink but the size does not change. The following example deals with this construction.

Example 2.1. Consider the function $\varphi : \mathbb{R}^n \to \mathbb{C}$, defined as

$$\varphi(x) = \begin{cases} Ce^{-\frac{1}{1-|x|^2}}, & |x| < 1. \\ 0, & |x| \geq 1. \end{cases}$$

It is clear that $\varphi \in C_c^\infty(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$. Here, $C > 0$ is chosen such that $\|\varphi\|_{L^1(\mathbb{R}^n)} = 1$. We consider the dilations $\varphi_\epsilon = \epsilon^{-n}\varphi\left(\frac{x}{\epsilon}\right)$. The graphs in Figure 2.3 show that as $\epsilon \to 0$, the functions φ_ϵ concentrates more on the point 0, a desirable property for the Dirac distribution. Let us now quickly check if φ_ϵ can approximate δ (for Schwartz functions).

For $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\left| \int\limits_{\mathbb{R}^n} \varphi_\epsilon(x) f(x) \,\mathrm{d}x - f(0) \right| \leq \int\limits_{B(0,\epsilon)} \varphi_\epsilon(x) |f(x) - f(0)| \,\mathrm{d}x.$$

Since f is continuous at 0, for every $\eta > 0$, there is some $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ and $x \in B(0,\epsilon)$, we have $|f(x) - f(0)| < \eta$. Consequently, for $\epsilon < \epsilon_0$, we have,

$$\left| \int\limits_{\mathbb{R}^n} \varphi_\epsilon(x) f(x) \,\mathrm{d}x - f(0) \right| < \eta \int\limits_{\mathbb{R}^n} \varphi_\epsilon(x) \,\mathrm{d}x = \eta.$$

By a simple change of variables, we easily see that for any $x \in \mathbb{R}^n$,

$$\lim_{\epsilon \to 0} (\varphi_\epsilon * f)(x) = f(x) = (\delta * f)(x).$$

For this reason, we call the collection $\{\varphi_\epsilon\}_{\epsilon > 0}$ an approximation to identity.

Remark 2.5. It is to be noticed that the calculations in Example 2.1 follow through for L^1-functions easily by density of $\mathcal{S}(\mathbb{R}^n)$. Particularly, we have $\lim\limits_{\epsilon \to 0} \varphi_\epsilon * f = f$, for all $f \in L^1(\mathbb{R}^n)$.

Example 2.1 is not a unique way to construct an approximate identity. In fact, a more general definition holds. For details on the (uniform and pointwise) convergence, we refer the reader to [12].

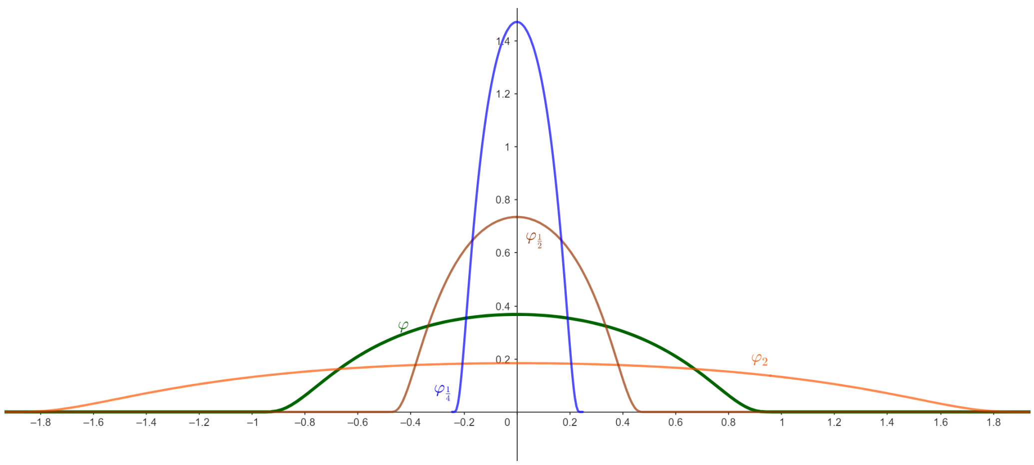

Figure 2.3: The functions φ_ϵ concentrate towards 0 as ϵ becomes smaller.

Definition 2.13 (Approximation to Identity)**.** *An approximate identity on* $\mathbb{R}^n$ *is a family* $\{\varphi_\epsilon\}_{\epsilon>0}$ *of* L^1*-functions such that*

1. *There is some* $C > 0$ *such that for all* $\epsilon > 0$*, we have* $\|\varphi_\epsilon\|_{L^1(\mathbb{R}^n)} \leq C$.

2. *For all* $\epsilon > 0$*, we have* $\int\limits_{\mathbb{R}^n} \varphi_\epsilon(x)\,\mathrm{d}x = 0$.

3. *For every* $\delta > 0$*, we have* $\lim\limits_{\epsilon\to 0} \int\limits_{|x|>\delta} \varphi_\epsilon(x)\,\mathrm{d}x = 0$.

Remark 2.6. It is clear that the family $\{\varphi_\epsilon\}_{\epsilon>0}$ in Example 2.1 is an approximation to identity. In the sequel, we use the construction of Example 2.1 without mentioning it explicitly.

2.8 Bochner Integral and related results

In this section our main aim to discuss about integration of functions that take values in a separable Banach space over $\mathbb{C}$. Throughout this section, B denotes a separable Banach space. The theory we are about to describe is analogous to Lebesgue's theory of integration, except for a few modifications. To begin integrating functions, we first require the notion of measurability. The results presented here are a part of the exposition found in [16].

Definition 2.14. *A function $F : \mathbb{R}^n \longrightarrow B$ is strongly measurable if for each $b' \in B'$ (the dual of B) the complex valued map $x \longmapsto \langle F(x), b' \rangle$ is measurable.*

Remark 2.7. Since we do not deal with any other notion of measurability of vector valued functions, we use "*measurable*" to mean strongly measurable.

Our goal is to define integration of functions $F : \mathbb{R}^n \to B$ in a manner analogous to the Lebesgue theory. One of the important compoenents in the Lebesgue theory is that whenever a function $f : \mathbb{R}^n \to \mathbb{C}$ is measurable, so is $|f| : \mathbb{R}^n \to [0, \infty)$. First, we get an analogue for this result. That is, we wish to prove that given a measurable function $F : \mathbb{R}^n \to B$, the real valued map $x \mapsto \|F(x)\|_B$ is also measurable. For the same, we begin with the following lemma.

Lemma 2.15. *Let B be a separable Banach space. Then, there exists a countable collection $\{b'_n\}_{n\in\mathbb{N}} \subseteq B'$ with $||b'_n||_{B'} = 1$, such that $\forall x \in B$, $||x||_B = \sup_{n\in\mathbb{N}}\{|b'_n(x)|\}$.*

Proof. Let $\mathcal{D} \subseteq B$ be countable dense subset. For a fixed $x_n \in \mathcal{D}$, we have $||x_n||_B = \sup_{||b'||_{B'}=1} \{|b'(x_n)|\}$. Therefore, there exists a sequence $(b'_{k,n})_{k\in\mathbb{N}}$ in B such that for all $k \in \mathbb{N}$, $||b'_{k,n}||_{B'} = 1$, and $|b'_{k,n}(x_n)| > ||x_n||_B - \frac{1}{k}$. Therefore, we have

$$||x_n||_B = \sup_{k\in\mathbb{N}}\{|b'_{k,n}(x_n)|\}.$$

For $x \in B \setminus \mathcal{D}$, we consider the following.
First, we note that $\forall n \in \mathbb{N}$, $\left(1 - \frac{1}{n}\right) ||x_n||_B < ||x_n||_B = \sup_{||b'||_{B'}=1} \{|b'(x_n)|\}$. Then, $\forall n \in \mathbb{N}, \exists b'_n \in B'$ with $||b'_n||_{B'} = 1$ such that $\left(1 - \frac{1}{n}\right) ||x_n||_B < |b'_n(x_n)|$. Let us fix a $0 < \delta < 1$, and $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \delta$ and $x_{n_0} \in B(x, \delta)$. Then, using the triangle inequality and the observation about $\{b'_n\}_{n\in\mathbb{N}}$, we get

$$\begin{aligned}
(1-\delta)||x||_B < \left(1 - \frac{1}{n_0}\right) ||x||_B &\leq \left(1 - \frac{1}{n_0}\right) ||x_{n_0}||_B + \left(1 - \frac{1}{n_0}\right) \delta \\
&< |b'_{n_0}(x_{n_0})| + \left(1 - \frac{1}{n_0}\right) \delta \\
&\leq |b'_{n_0}(x_{n_0})| + |b'_{n_0}(x_{n_0} - x)| + \left(1 - \frac{1}{n_0}\right) \delta
\end{aligned}$$

$$\leq |b'_{n_0}(x)| + \left(2 - \frac{1}{n_0}\right)\delta$$
$$< |b'_{n_0}(x)| + 2\delta.$$

That is, $\forall \delta \in (0,1)$, we have,

$$(1-\delta)||x||_B < |b'_{n_0}(x)| + 2\delta \leq \sup_{n\in\mathbb{N}} |b'_n(x)|.$$

Upon simplifying, we get for all $\delta \in (0,1)$,

$$\delta \geq \frac{||x||_B - \sup\limits_{n\in\mathbb{N}} |b'_n(x)|}{2 + ||x||_B}$$

Hence,

$$||x||_B \leq \sup_{n\in\mathbb{N}} |b'_n(x)|.$$

Also, $||x||_B \geq \sup\limits_{n\in\mathbb{N}}\{|b'_n(x)|\}$, so that $||x||_B = \sup\limits_{n\in\mathbb{N}}\{|b'_n(x)|\}$. So, the required countable collection is $\{b'_{k,n} : k \in \mathbb{N}, n \in \mathbb{N}\} \cup \{b'_n | n \in \mathbb{N}\}$, as constructed above. □

Remark 2.8. We call the countable collection $\{b_n\}_{n\in\mathbb{N}}$, constructed in Lemma 2.15, a "*norming sequence*".

Next, we see that measurable Banach valued functions can be approximated by simple functions. Indeed, we first look at the definition of a Banach valued simple function.

Definition 2.15 (Simple Function)**.** *A function $F : \mathbb{R}^n \to B$ is simple if there are distinct elements $b_1, \cdots, b_k \in B$ and disjoint measurable sets $A_1, \cdots, A_k \subseteq \mathbb{R}^n$ such that $F(x) = \sum\limits_{i=1}^{k} \chi_{A_i}(x)\, b_i$.*

Remark 2.9. We notice that a function $F : \mathbb{R}^n \to B$ is simple if and only if its range is a finite set.

Theorem 2.16. *Let B be a separable Banach space. $F : \mathbb{R}^n \longrightarrow B$ be strongly measurable function. Then there exists a sequence $(F_j)_{j\in\mathbb{N}}$ of simple functions such that $F_n \longrightarrow F$ pointwise.*

Proof. Let $\left\{b'_j\right\}_{j\in\mathbb{N}} \subseteq B'$ be a norming collection (see Lemma 2.15). We know that the real valued function $x \longmapsto ||f(x) - b||_B = \sup\limits_{n\in\mathbb{N}}\{|b'_n(f(x) - b)|\}$ is measurable, $\forall b \in B$. Let $\{b_n\}_{n\in\mathbb{N}} \subseteq B$ be a countable dense set with $b_1 = 0$. For $b \in B$, let $k(j,b) \in \mathbb{N}$ be the smallest number such that $1 \leq k(j,b) \leq j$, $||b_{k(j,b)}||_B \leq ||b||_B$, and $||b - b_{k(j,b)}||_B = \min\limits_{1\leq l\leq j} ||b - b_l||_B$. For $j \in \mathbb{N}$, define $\varphi_j : B \longrightarrow B$ as $\phi_j(b) = b_{k(j,b)}$. Then clearly, $\lim\limits_{j\longrightarrow\infty} ||\varphi_j(b) - b||_B = 0$, and for each $j \in \mathbb{N}$, we have $||\varphi_j(b)||_B \leq ||b||_B$. Define $F_j : \mathbb{R}^n \longrightarrow B$ as $F_j(x) = (\varphi_j \circ f)(x)$. Clearly, $F_j(\mathbb{R}^n) \subseteq \{b_1, b_2, \cdots b_j\}$, and hence is simple. Now, let us define

$$A_{k,j} := F_j^{-1}(b_k) = \left\{ x \in \mathbb{R}^n | \|F(x) - b_k\|_B < \min_{1\leq l\leq k} \|F(x) - b_l\| \right\}.$$

Therefore, $A_{k,j}$ is measurable and we have $F_j = \sum\limits_{k=1}^{j} \chi_{A_{k,j}} b_k$. Moreover, for $x \in \mathbb{R}^n$, we have

$$\lim_{j\longrightarrow\infty} ||F_j(x) - F(x)||_B = \lim_{j\longrightarrow\infty} ||\varphi_j(F(x)) - F(x)||_B = 0.$$

That is, $F_n \longrightarrow F$ pointwise. □

We now state the result we were aiming for.

Corollary 2.17. *The real valued function $x \longmapsto ||F(x)||_B$ is measurable whenever $F : \mathbb{R}^n \longrightarrow B$ is strongly measurable.*

Proof. We have from Theorem 2.16, $||F(x)||_B = \lim\limits_{j\longrightarrow\infty} ||F_j(x)||_B$. We notice that (in the notation of Theorem 2.16),

$$||F_j(x)||_B = \begin{cases} ||b_k||_B, & x \in A_{k,j}. \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $x \longmapsto \|F_j(x)\|_B$ is measurable for every $j \in \mathbb{N}$. So $x \longmapsto ||F(x)||_B$ is measurable. □

With the basic construction about Banach values measurable functions at hand, we now proceed to define L^p-spaces. This definition is a direct analogue of the usual L^p-spaces defined in Section 2.1. Here, we replace the absolute value $(|\cdot|)$ of $\mathbb{C}$ by the norm $(\|\cdot\|_B)$ of B.

Definition 2.16 (Bochner space)**.** *For $1 \leq p < \infty$, the Bochner spaces are defined as*

$$L^p(\mathbb{R}^n, B) := \left\{ F : \mathbb{R}^n \longrightarrow B \;\middle|\; F \text{ is measurable and } \int_{\mathbb{R}^n} ||F(x)||_B^p \,\mathrm{d}x < \infty \right\}.$$

Norm of a function F in $L^p(\mathbb{R}^n, B)$, $1 \leq p < \infty$, is defined as

$$||F||_{L^p(\mathbb{R}^n,B)} := \left(\int_{\mathbb{R}^n} ||F(x)||_B^p \,\mathrm{d}x \right)^{1/p}.$$

For $p = \infty$, we define

$$||F||_{L^\infty(\mathbb{R}^n,B)} := \sup_{x\in\mathbb{R}^n} ||F(x)||_B,$$

and the space $L^\infty(\mathbb{R}^n, B)$ as

$$L^\infty(\mathbb{R}^n, B) := \{F : \mathbb{R}^n \longrightarrow B \mid F \text{ is strongly measurable and } ||F||_\infty < \infty\}.$$

Remark 2.10. We notice that for a function $F : \mathbb{R}^n \to B$, we have

$$\|F\|_{L^p(\mathbb{R}^n,B)} = \|n_F\|_{L^p(\mathbb{R}^n)},$$

where, $n_F : \mathbb{R}^n \to \mathbb{R}$ is defined by $n_F(x) = \|F(x)\|_B$. From this observation, it is clear that $\|\cdot\|_{L^p(\mathbb{R}^n,B)}$ is a norm on $L^p(\mathbb{R}^n, B)$.

Remark 2.11. When $B = \mathbb{C}$, the Bochner spaces $L^p(\mathbb{R}^n, \mathbb{C})$ coincide with the usual Lebesgue spaces $L^p(\mathbb{R}^n)$, for any $1 \leq p \leq +\infty$.

Remark 2.12. The Bochner spaces $L^p(\mathbb{R}^n, B)$ enjoy all the "nice" properties similar to the Lebesgue spaces. Particularly, $L^p(\mathbb{R}^n, B)$ is a banach space. We refer the reader to [16] for a complete exposition on the subject.

We now wish to give meaning to the symbol $\int_{\mathbb{R}^n} F(x)\,\mathrm{d}x$, for a Banach valued function F. As with the case of complex valued functions, simple functions play an important role in this. However, we first give a more general definition. Let $f \in L^p(\mathbb{R}^n)$ and $b \in B$. Then, we can define a function $f \cdot b : \mathbb{R}^n \longmapsto B$ as $(f \cdot b)(x) = f(x)b$. Now, we notice that

$$||(f \cdot b)||_{L^p(\mathbb{R}^n,B)} = \left(\int_{\mathbb{R}^n} ||f(x)b||_B^p \,\mathrm{d}x \right)^{1/p} = ||f||_p ||b||_B.$$

That is, $f \cdot b \in L^p(\mathbb{R}^n, B)$. Let us define

$$L^p \otimes B := \operatorname{span}\left\{ f \cdot b \middle| f \in L^p\left(\mathbb{R}^n\right), b \in B \right\}.$$

Clearly, the collection of simple functions taking non-zero values on a finite measure set is included in $L^p \otimes B$. We now show that the space $L^p \otimes B$ is dense in $L^p\left(\mathbb{R}^n, B\right)$.

Proposition 2.18. *Let B be a separable Banach space. The collection of functions of the form $\sum\limits_{j=1}^{m} \chi_{E_j} b_j$, where $b_j \in B$ and $\{E_j\}_{j=1}^m$ are disjoint, measurable subsets of $\mathbb{R}^n$ with finite measure, is a dense subset of $L^p(\mathbb{R}^n, B)$ for any $1 \leq p < \infty$.*

Proof. Let $F \in L^p(\mathbb{R}^n, B)$. Then, we have,

$$\int\limits_{\mathbb{R}^n} ||F(x)||_B^p \, \mathrm{d}x < \infty.$$

Therefore, for any $\epsilon > 0$, there is a bounded subset $K_1 \subseteq \mathbb{R}^n$ such that,

$$\int\limits_{\mathbb{R}^n \setminus K_1} ||F(x)||_B^p \, \mathrm{d}x < \frac{\epsilon^p}{3}.$$

Let $\{b_j\}_{j\in\mathbb{N}} = B_0 \subseteq B$ be a countable dense subset, and for $b_j \in B_0$, let us define

$$\widetilde{B}(b_j, \epsilon) = \{b \in B : ||b - b_j||_B \leq \epsilon(3|K_1|)^{-1/p}\}.$$

By the density of B_0, we have

$$B = \bigcup_{j=1}^{\infty} \widetilde{B}(b_j, \epsilon).$$

Now, let $A_1 = \widetilde{B}(b_1, \epsilon)$, and $A_j = \widetilde{B}(b_j, \epsilon) \setminus \left(\bigcup\limits_{k=1}^{j-1} \widetilde{B}(b_k, \epsilon) \right)$, for $j \geq 2$. We notice that the collection $\{A_j\}_{j\in\mathbb{N}}$ is pairwise disjoint and

$$\bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} \widetilde{B}(b_j, \epsilon) = B$$

Now consider the collection $\{\widetilde{A_j}\}_{j\in\mathbb{N}}$, where

$$\widetilde{A}_j := A_j \cap F(K_1).$$

Then clearly,

$$\bigcup_{j=1}^{\infty} \widetilde{A}_j = F(K_1).$$

Now, let

$$K_1 \cap F^{-1}(\widetilde{A}_j) = E_j.$$

Since F is strongly measurable, using Corollary 2.17, we see that E_j is measurable. Also, $E_j \subseteq K_1$ and hence, is of finite measure. Note that

$$\bigcup_{j=1}^{\infty} E_j = K_1 \cap \left(\bigcup_{j=1}^{\infty} F^{-1}(\widetilde{A}_j) \right) = K_1 \cap \left(F^{-1}\left(F(K_1)\right)\right) = K_1,$$

since $K_1 \subseteq F^{-1}(F(K_1))$.

Since $\{A_j\}_{j\in\mathbb{N}}$ is a pairwise disjoint collection, so is the collection $\{E_j\}_{j\in\mathbb{N}}$. Now, we observe that

$$\sum_{j=1}^{\infty} \int_{E_j} \|F\left(x\right)\|_B^p \mathrm{d}x = \int_{K_1} ||F(x)||_B^p \, \mathrm{d}x \leq \int_{\mathbb{R}^n} \|F\left(x\right)\|_B^p \mathrm{d}x < \infty,$$

Therefore, there exists $m \in \mathbb{N}$ such that

$$\int_{\bigcup\limits_{j=m+1}^{\infty} E_j} ||F(x)||_B^p \, \mathrm{d}x < \frac{\epsilon^p}{3}.$$

Let us now define the simple function

$$F_m(x) = \sum_{j=1}^{m} \chi_{E_j}(x) b_j.$$

For any $b' \in B'$, and a fixed $j \in \mathbb{N}$ consider the function

$$\varphi_j(x) = \langle b', \chi_{E_j}(x) b_j \rangle = \begin{cases} 0, & \text{if } x \notin E_j. \\ b'(b_j). & \text{if } x \in E_j. \end{cases}$$

It is easy to verify that φ_j is meaurable. Hence, the function F_j is strongly measurable. Now let $x \in E_j$ for some $j = 1, 2, \cdots, m$. Then, $F(x) \in \tilde{A}_j \subseteq \widetilde{B}(b_j, \epsilon)$. Therefore, if $x \in E_j$, we have

$$||F(x) - b_j||_B^p < \frac{\epsilon}{(3|K_1|)^{-1/p}}. \tag{2.9}$$

Now, we have

$$\int_{\bigcup\limits_{j=1}^{m} E_j} \left|\left| F(x) - \sum_{j=1}^{m} \chi_{E_j}(x) b_j \right|\right|_B^p \mathrm{d}x = \sum_{j=1}^{m} \int_{E_j} ||F(x) - b_j||_B^p \, \mathrm{d}x$$

$$
\begin{aligned}
&< \sum_{j=1}^{m} \frac{\epsilon^p}{3|K_1|}|E_j| \\
&\leq \frac{\epsilon^p}{3|K_1|}|K_1| = \frac{\epsilon^p}{3}.
\end{aligned}
$$

Finally we see that

$$
\begin{aligned}
\int\limits_{\mathbb{R}^n} \left|\left|F(x) - \sum_{j=1}^{m} \chi_{E_j}(x) b_j\right|\right|_B^p \mathrm{d}x &= \int\limits_{\mathbb{R}^n \setminus K_1} ||F(x)||_B^p \mathrm{d}x + \int\limits_{\bigcup\limits_{j=m+1}^{\infty} E_j} ||F(x)||_B^p \\
&+ \int_{\bigcup\limits_{j=1}^{m} E_j} ||F(x) - \sum_{j=1}^{m} \chi_{E_j}(x) u_j||_B^p \mathrm{d}x \\
&< \frac{\epsilon^p}{3} + \frac{\epsilon^p}{3} + \frac{\epsilon^p}{3} = \epsilon^p.
\end{aligned}
$$

This completes the proof! □

The following corollary is immediate.

Corollary 2.19. *The space $L^p \otimes B$ is dense in the Bochner space $L^p(\mathbb{R}^n, B)$, for any $1 \leq p < \infty$.*

Now let us define an operator $I : L^1 \otimes B \longrightarrow B$ as follows. For $F = \sum\limits_{j=1}^{m} f_j \cdot b_j \in L^1 \otimes B$,

$$
I(F) := \sum_{j=1}^{m} \left(\int\limits_{\mathbb{R}^n} f_j(x) \mathrm{d}x \right) b_j. \tag{2.10}
$$

It is clear that I is linear. The following lemma says that this operator is continuous.

Lemma 2.20. *The operator I defined in Equation* (2.10) *is a bounded operator from $L^1 \otimes B$ to B.*

Proof. We have for any $F = \sum\limits_{j=1}^{m} f_j \cdot b_j \in L^1 \otimes B$,

$$
||I(F)||_B = \sup_{||b'||_{B'} \leq 1} \left| \left\langle b', \sum_{j=1}^{m} \left(\int\limits_{\mathbb{R}^n} f_j(x) \mathrm{d}x \right) b_j \right\rangle \right|
$$

$$
\begin{aligned}
&= \sup_{||b'||_{B'}\leq 1} \left| \int\limits_{\mathbb{R}^n} \left\langle b', \sum_{j=1}^{m} f_j(x) b_j \right\rangle \mathrm{d}x \right| \\
&\leq \sup_{||b'||_{B'}\leq 1} \left\{ \int\limits_{\mathbb{R}^n} \left| \left\langle b', \sum_{j=1}^{m} f_j(x) b_j \right\rangle \right| \mathrm{d}x \right\} \\
&\leq \sup_{||b'||_{B'}\leq 1} \left\{ \int\limits_{\mathbb{R}^n} ||b'||_{B'} \left|\left| \sum_{j=1}^{m} f_j(x) b_j \right|\right|_B \mathrm{d}x \right\} \\
&\leq ||F||_{L^1(\mathbb{R}^n,B)}.
\end{aligned}
$$

□

We have already observed that the space $L^1 \otimes B$ is a dense subset of $L^1(\mathbb{R}^n, B)$. Therefore the operator I can be extended uniquely to $L^1(\mathbb{R}^n, B)$. The unique extension of the operator I is known as **Bochner integral**, and for a function $F \in L^1(\mathbb{R}^n, B)$, we denote it by $\int\limits_{\mathbb{R}^n} F(x)\,\mathrm{d}x$. We now move on to prove a duality result for Bochner spaces. We require the following lemma.

Lemma 2.21. *Let $F \in L^p(\mathbb{R}^n, B)$. Then, for each $\epsilon > 0$, there exists a non-negative function $h \in L^{p'}(\mathbb{R}^n)$ with $||h||_{L^{p'}(\mathbb{R}^n)} \leq 1$ such that*

$$
||F||_{L^p(\mathbb{R}^n,B)} < \int\limits_{\mathbb{R}^n} h(x)||F(x)||_B \,\mathrm{d}x + \epsilon.
$$

Proof. Consider the real valued function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$, given by $\varphi(x) = ||F(x)||_B$. Note that

$$
||\varphi||_{L^p(\mathbb{R}^n)} = \left(\int\limits_{\mathbb{R}^n} |\varphi(x)|^p \,\mathrm{d}x \right)^{1/p} = \left(\int\limits_{\mathbb{R}^n} ||F(x)||_B^p \mathrm{d}x \right)^{1/p} = ||F||_{L^p(\mathbb{R}^n,B)}.
$$

That is, $\varphi \in L^p(\mathbb{R}^n)$. Also,

$$
||F||_{L^p(\mathbb{R}^n,B)} = ||\varphi||_{L^p(\mathbb{R}^n)} = \sup \left\{ \left| \int\limits_{\mathbb{R}^n} h(x)\varphi(x) \,\mathrm{d}x \right| : ||h||_{L^p(\mathbb{R}^n)} \leq 1 \right\}.
$$

Therefore, for $\epsilon > 0$, there exists a function $h' \in L^{p'}(\mathbb{R}^n)$ such that

$$
||F||_{L^p(B)} \leq \left| \int\limits_{\mathbb{R}^n} h'(x)||F(x)||_B \,\mathrm{d}x \right| + \frac{\epsilon}{2} \leq \int\limits_{\mathbb{R}^n} |h'(x)|||F(x)||_B \,\mathrm{d}x + \frac{\epsilon}{2} \tag{2.11}
$$

Note that as the function $h'(\cdot)||F(\cdot)||_B \in L^1(\mathbb{R}^n)$, there exists $R > 0$ such that

$$\int\limits_{\mathbb{R}^n\setminus B(0,R)} |h'(x)|||F(x)||_B \, \mathrm{d}x < \frac{\epsilon}{2}.$$

Therefore from Inequality (2.11), we have

$$\begin{aligned} ||F||_{L^p(B)} &\leq \int\limits_{B(0,R)} |h'(x)|||F(x)||_B \, \mathrm{d}x + \int\limits_{\mathbb{R}^n\setminus B(0,R)} |h'(x)|||F(x)||_B \, \mathrm{d}x + \frac{\epsilon}{2} \\ &\leq \int\limits_{B(0,R)} |h'(x)|||F(x)||_B \, \mathrm{d}x + \epsilon \\ &= \int\limits_{\mathbb{R}^n} |h'(x)|\chi_{B(0,R)}||F(x)||_B \, \mathrm{d}x + \epsilon. \end{aligned}$$

Hence, the desired function is $h(x) = |h'(x)|\chi_{B(0,R)}(x)$. □

We now present the duality result. This result is often used in this thesis, especially in the Littlewood-Paley theory (Chapter 6).

Theorem 2.22.

1. *Let B be a separable Banach space. Then for any $F \in L^p(\mathbb{R}^n, B)$, $1 \leq p < \infty$, we have*

$$||F||_{L^p(\mathbb{R}^n,B)} = \sup_{||G||_{L^{p'}(\mathbb{R}^n,B')}\leq 1} \left| \int\limits_{\mathbb{R}^n} \langle G(x), F(x)\rangle \, \mathrm{d}x \right|. \tag{2.12}$$

2. *The space $L^p(\mathbb{R}^n, B)$ is isometrically embed in $\left(L^{p'}(\mathbb{R}^n, B')\right)'$, where $1 \leq p \leq \infty$.*

Proof.

1. Let $F \in L^p(\mathbb{R}^n, B)$. Then for an $\epsilon > 0$ there exists a function $F_\epsilon(x) = \sum_{j=1}^{m_\epsilon} \chi_{E_j}(x) b_j \in L^p \otimes B$ (for some $m \in \mathbb{N}$), such that

$$||F_\epsilon - F||_{L^p(B)} < \frac{\epsilon}{2}.$$

Using Lemma 2.21, we have a non-negative function $h \in L^{p'}(\mathbb{R}^n)$ with $||h||_{L^{p'}(\mathbb{R}^n)} \leq 1$ such that

$$||F_\epsilon||_{L^p(B)} = \left(\int\limits_{\mathbb{R}^n} ||F_\epsilon(x)||_B^p \, \mathrm{d}x\right)^{1/p} < \int\limits_{\mathbb{R}^n} h(x)||F_\epsilon(x)||_B \, \mathrm{d}x + \frac{\epsilon}{4}. \quad (2.13)$$

Also, for $b_j \in B$, there exists $b'_j \in B'$ with $||b'_j||_{B'} = 1$, such that

$$||b_j||_B < |\langle b'_j, b_j\rangle| + \frac{\epsilon}{4(||h||_{L^{p'}(\mathbb{R}^n)} + 1)}. \quad (2.14)$$

We notice that given $b'_j \in B'$ with the property shown in Inequality (2.14), we define $\widetilde{b'_j} := e^{-i \arg\langle b_j, b'_j\rangle} b'_j$. Then, clearly, $\langle \widetilde{b'_j}, b_j\rangle = \left|\langle b'_j, b_j\rangle\right|$ and $\|\widetilde{b'_j}\|_{B'} = 1$. In the sequel, we abuse notation and write b'_j to denote $\tilde{b'_j}$. Hence, we have,

$$\|b_j\|_B < \langle b'_j, b_j\rangle + \frac{\epsilon}{4\left(1 + \|h\|_{L^{p'}(\mathbb{R}^n)}\right)}. \quad (2.15)$$

Let us now define the function

$$G(x) = \sum_{j=1}^m h(x)\chi_{E_j}(x)b'_j.$$

Now, we have,

$$\begin{aligned}
||G||_{L^{p'}(\mathbb{R}^n, B')}^{p'} &\leq \sum_{j=1}^m ||h\chi_{E_j} b'_j||_{L^{p'}(\mathbb{R}^n, B')}^{p'} \\
&= \sum_{j=1}^m \int\limits_{\mathbb{R}^n} ||h(x)\chi_{E_j}(x)b'_j||_{B'}^{p'} \, \mathrm{d}x \\
&= \sum_{j=1}^m \int\limits_{\mathbb{R}^n} |h(x)\chi_{E_j}(x)|^{p'} ||b'_j||_{B'}^{p'} \, \mathrm{d}x \\
&= \sum_{j=1}^m \int\limits_{\mathbb{R}^n} |h(x)\chi_{E_j}(x)|^{p'} \, \mathrm{d}x \\
&= \int\limits_{\bigcup_{j=1}^n E_j} |h(x)|^{p'} \, \mathrm{d}x \\
&\leq ||h||_{L^{p'}(\mathbb{R}^n)}^{p'} \leq 1.
\end{aligned}$$

We also notice that for any $x \in \mathbb{R}^n$,

$$\langle G(x), F_\epsilon(x)\rangle = \sum_{j,k=1}^{m_\epsilon} h(x)\,\chi_{E_j \cap E_k}(x)\,\langle b_j', b_k\rangle \in \mathbb{R},$$

and we have,

$$\begin{aligned}\int_{\mathbb{R}^n} \langle G(x), F_\epsilon(x)\rangle \,\mathrm{d}x &= \int_{\mathbb{R}^n} \left\langle \sum_{j=1}^{m} h(x)\chi_{E_j}(x)u_j^*, \sum_{j=1}^{m} \chi_{E_j}(x)u_j \right\rangle \mathrm{d}x \\ &= \int_{\mathbb{R}^n} h(x) \sum_{j=1}^{m} \chi_{E_j}(x)\langle u_j^*, u_j\rangle \,\mathrm{d}x.\end{aligned}$$

From Inequality (2.15), we get,

$$\int_{\mathbb{R}^n} \langle G(x), F_\epsilon(x)\rangle \,\mathrm{d}x > \int_{\mathbb{R}^n} h(x) \sum_{j=1}^{m_\epsilon} \chi_{E_j}(x) \left(||b_j||_B - \frac{\epsilon}{4(||h||_{L^1(\mathbb{R}^n)} + 1)} \right) \mathrm{d}x.$$

As $||F_\epsilon(x)||_B = ||\sum_{j=1}^{m_\epsilon} \chi_{E_j}(x) b_j||_B \leq \sum_{j=1}^{m_\epsilon} \chi_{E_j}(x)||b_j||_B$, we get

$$\begin{aligned}\int_{\mathbb{R}^n} \langle G(x), F_\epsilon(x)\rangle \,\mathrm{d}x > &\int_{\mathbb{R}^n} h(x)||F_\epsilon(x)||_B \,\mathrm{d}x \\ &- \frac{\epsilon}{4(||h||_{L^1(\mathbb{R}^n)} + 1)} \int_{\mathbb{R}^n} \sum_{j=1}^{m_\epsilon} h(x)\chi_{E_j}(x) \,\mathrm{d}x.\end{aligned}$$

From Inequality (2.13), we obtain,

$$\begin{aligned}\int_{\mathbb{R}^n} \langle G(x), F_\epsilon(x)\rangle \,\mathrm{d}x &\geq ||F_\epsilon||_{L^p(\mathbb{R}^n,B)} - \frac{\epsilon}{4} - \frac{||h||_{L^1(\mathbb{R}^n)}\epsilon}{4(||h||_{L^1(\mathbb{R}^n)} + 1)} \\ &\geq ||F_\epsilon||_{L^p(\mathbb{R}^n,B)} - \frac{\epsilon}{4} - \frac{\epsilon}{4} = ||F_\epsilon||_{L^p(\mathbb{R}^n,B)} - \frac{\epsilon}{2}.\end{aligned}$$

Hence,

$$||F_\epsilon||_{L^p(\mathbb{R}^n,B)} \leq \sup_{||G||_{L^{p'}} \leq 1} \left| \int_{\mathbb{R}^n} \langle G(x), F_\epsilon(x)\rangle \,\mathrm{d}x \right| + \frac{\epsilon}{2}. \tag{2.16}$$

Now, for $G \in L^{p'}(\mathbb{R}^n, B')$ with $\|G\|_{L^{p'}(\mathbb{R}^n,B')} \leq 1$, we have,

$$\begin{aligned}\left| \int_{\mathbb{R}^n} \langle G(x), F_\epsilon\rangle \,\mathrm{d}x \right| &\leq \int_{\mathbb{R}^n} |\langle G(x), F_\epsilon(x) - F(x)\rangle| \,\mathrm{d}x + \left| \int_{\mathbb{R}^n} \langle G(x), F(x)\rangle \,\mathrm{d}x \right| \\ &\leq \int_{\mathbb{R}^n} ||G(x)||_{B'}||F_\epsilon(x) - F(x)||_B \,\mathrm{d}x \\ &+ \sup_{||G||_{L^{p'}(\mathbb{R}^n,B')} \leq 1} \left| \int_{\mathbb{R}^n} \langle G(x), F(x)\rangle \,\mathrm{d}x \right|\end{aligned}$$

$$\leq ||G||_{L^{p'}(\mathbb{R}^n,B')}||F_\epsilon - F||_{L^p(\mathbb{R}^n,B)}$$
$$+ \sup_{||G||_{L^{p'}(\mathbb{R}^n,B')}\leq 1} \left| \int_{\mathbb{R}^n} \langle G(x), F(x)\rangle \, \mathrm{d}x \right|$$
$$\leq \frac{\epsilon}{2} + \sup_{||G||_{L^{p'}}\leq 1} \left| \int_{\mathbb{R}^n} \langle G(x), F(x)\rangle \, \mathrm{d}x \right|. \tag{2.17}$$

Therefore, from Inequalities (2.16) and (2.17), we get

$$||F_\epsilon||_{L^p(\mathbb{R}^n,B)} \leq \sup_{||G||_{L^{p'}}\leq 1} \left| \int_{\mathbb{R}^n} \langle G(x), F(x)\rangle \, \mathrm{d}x \right| + \epsilon.$$

Now, we observe that

$$\begin{aligned} ||F||_{L^p(\mathbb{R}^n,B)} &\leq ||F_\epsilon - F||_{L^p(\mathbb{R}^n,B)} + ||F_\epsilon||_{L^p(\mathbb{R}^n,B)} \\ &\leq \frac{\epsilon}{2} + \sup_{||G||_{L^{p'}(\mathbb{R}^n,B')}\leq 1} \left| \int_{\mathbb{R}^n} \langle G(x), F(x)\rangle \, \mathrm{d}x \right| + \epsilon \\ &= \sup_{||G||_{L^{p'}(\mathbb{R}^n,B')}\leq 1} \left| \int_{\mathbb{R}^n} \langle G(x), F(x)\rangle \, \mathrm{d}x \right| + \frac{3\epsilon}{2}. \end{aligned}$$

As $\epsilon > 0$ is arbitrary, we have

$$||F||_{L^p(\mathbb{R}^n,B)} \leq \sup_{||G||_{L^{p'}(\mathbb{R}^n,B')}\leq 1} \left| \int_{\mathbb{R}^n} \langle G(x), F(x)\rangle \, \mathrm{d}x \right|.$$

Now for $||G||_{L^{p'}(\mathbb{R}^n,B')} \leq 1$, by using Hölder's inequality, we get,

$$\left| \int_{\mathbb{R}^n} \langle G(x), F(x)\rangle \, \mathrm{d}x \right| \leq ||G||_{L^{p'}(\mathbb{R}^n,B')}||F||_{L^p(\mathbb{R}^n,B)} \leq ||F||_{L^p(\mathbb{R}^n,B)}.$$

This gives us

$$||F||_{L^p(\mathbb{R}^n,B)} = \sup_{||G||_{L^{p'}(\mathbb{R}^n,B')}\leq 1} \left| \int_{\mathbb{R}^n} \langle G(x), F(x)\rangle \, \mathrm{d}x \right|.$$

2. For $F \in L^p(\mathbb{R}^n, B)$, we define a linear functional $H_F : L^{p'}(\mathbb{R}^n, B') \longrightarrow \mathbb{C}$ by

$$H_F(G) = \int_{\mathbb{R}^n} \langle G(x), F(x)\rangle \mathrm{d}x.$$

We have

$$|H_F(G)| = \left| \int_{\mathbb{R}^n} \langle G(x), F(x)\rangle \mathrm{d}x \right| \leq \int_{\mathbb{R}^n} |\langle G(x), F(x)\rangle| \, \mathrm{d}x$$

$$\leq \int_{\mathbb{R}^n} ||G(x)||_{B'}||F(x)||_B \ \mathrm{d}x \leq ||G||_{L^{p'}(\mathbb{R}^n,B')}||f||_{L^p(\mathbb{R}^n,B)}.$$

Here, the last inequality is consequence of Hölder's inequality. Using the first part of this theorem we have

$$||H_F|| = \sup_{||G||_{L^{p'}(\mathbb{R}^n,B')}\leq 1} \left| \int_{\mathbb{R}^n} \langle G(x), F(x)\rangle \ \mathrm{d}x \right| = ||F||_{L^p(\mathbb{R}^n,B)}.$$

Thus the space $L^p(\mathbb{R}^n, B)$ is isometrically embed in $\left(L^{p'}(\mathbb{R}^n, B')\right)'$, for any $1 \leq p \leq \infty$.

□

Before we end this chapter, we present a remarkable fact about taking continuous operators inside an integral.

Proposition 2.23. *Let B be a reflexive separable Banach space and B^* be the dual space of B. Let $\Lambda \in B^*$. Then, for any $F \in L^1(B)$, we have,*

$$\Lambda\left(\int_{\mathbb{R}^n} F(x) \ \mathrm{d}x\right) = \int_{\mathbb{R}^n} \Lambda(F(x)) \ \mathrm{d}x. \tag{2.18}$$

Proof. Recall from Proposition 2.18 that $L^1 \otimes B$ is a dense subset of $L^1(B)$. First we prove the result in this dense set. Let $F \in L^1 \otimes B$. That is, $F(x) = \sum_{i=1}^{m} f_i(x)b_i$, where $f_i \in L^1(\mathbb{R}^n)$ and $b_i \in B$ for all $i = 1, 2, \cdots, m$. So,

$$\Lambda\left(\int_{\mathbb{R}^n} F(x) \ \mathrm{d}x\right) = \Lambda\left(\int_{\mathbb{R}^n} \sum_{i=1}^{m} f_i(x)b_i \ \mathrm{d}x\right)$$

Now by definition for Bochner integral, we get

$$\begin{aligned}
\Lambda\left(\int_{\mathbb{R}^n} F(x) \ \mathrm{d}x\right) &= \Lambda\left(\sum_{i=1}^{m}\left(\int_{\mathbb{R}^n} f_i(x) \ \mathrm{d}x\right) b_i\right) \\
&= \sum_{i=1}^{m}\left(\int_{\mathbb{R}^n} f_i(x) \ \mathrm{d}x\right) \Lambda(b_i) \\
&= \sum_{i=1}^{m}\int_{\mathbb{R}^n} f_i(x)\Lambda(b_i) \ \mathrm{d}x
\end{aligned}$$

$$= \int\limits_{\mathbb{R}^n} \Lambda(F(x)) \, \mathrm{d}x.$$

So the result is true when $F \in L^1 \otimes B$. The L.H.S of Equation (2.18) is the operator $\Lambda \circ I : L^1(\mathbb{R}^n, B) \longrightarrow \mathbb{C}$. Clearly, it is continuous. Consider the linear map $T : L^1(\mathbb{R}^n, B) \longrightarrow \mathbb{C}$, defined as $T(F) = \int\limits_{\mathbb{R}^n} (\Lambda \circ F)(y) \, \mathrm{d}y$. Then, $|T(F)| \leq \int\limits_{\mathbb{R}^n} |\Lambda(F(x))| \, \mathrm{d}x \leq \int\limits_{\mathbb{R}^n} ||\Lambda||_{B'} \cdot ||F(y)|| \, \mathrm{d}y \leq ||\Lambda||_{B'} \cdot ||F||_{L^1(\mathbb{R}^n, B)}$. Hence, T is continuous. Since $T = \Lambda \circ I$ on the dense set $L^1 \otimes B$, the equality follows on entire $L^1(\mathbb{R}^n, B)$.

□

We now see a special case of Proposition 2.23.

Corollary 2.24. *Let $B = \ell^r(\mathbb{C})$, for some $r > 1$. Let $F \in L^1(\ell^r)$. We can write $F(y) = (f_i(y))_{i \in \mathbb{N}}$, where $f_i \in L^1(\mathbb{R}^n)$. Then*

$$\int\limits_{\mathbb{R}^n} (f_i(y))_i \, \mathrm{d}y = \left(\int\limits_{\mathbb{R}^n} f_i(y) \, \mathrm{d}y \right)_i .$$

Proof. Suppose for $j \in \mathbb{N}$, $\Lambda_j \in \ell^{r'}$ such that for any $(x_i)_{i \in \mathbb{N}} \in l^r$, $\Lambda_j((x_i)_{i \in \mathbb{N}}) = x_j$. Now suppose $(x_i)_i \in l^r$ be such that

$$\int\limits_{\mathbb{R}^n} F(y) \, \mathrm{d}y = (x_i)_i.$$

Now by using above lemma

$$x_j = \Lambda_j((x_i)_i) = \Lambda_j \left(\int\limits_{\mathbb{R}^n} F(y) \, \mathrm{d}y \right) = \int\limits_{\mathbb{R}^n} \Lambda_j(F(y)) \, \mathrm{d}y = \int\limits_{\mathbb{R}^n} f_i(y) \, \mathrm{d}y.$$

Therefore we can say that

$$\int\limits_{\mathbb{R}^n} (f_i(y))_i \, \mathrm{d}y = \left(\int\limits_{\mathbb{R}^n} f_i(y) \, \mathrm{d}y \right)_i .$$

□

CHAPTER 3

Maximal Operators

In Chapter 1, we have seen that the study of averages (over intervals) is natural in differentiation theory. A natural generalization of these averages to higher dimensions are averages over balls centered at a given point. However, in higher dimensions, one can also look at averages over cubes centered at a point. This chapter is dedicated to the study of a few averaging operators and their corresponding maximal function, and forms a base for upcoming chapters. A maximal function corresponding to a collection of averages is understood as the "largest" average (around a given point). We begin with the Hardy-Littlewood maximal operator.

3.1 Hardy-Littlewood Maximal operator

Definition 3.1 (Hardy-Littlewood maximal function)**.** *Let $B_r = B(0, r)$ be the Euclidean ball of radius r centered at origin. Given $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy-*

Littlewood maximal function of f is defined as

$$Mf\left(x\right) := \sup_{r>0} \frac{1}{|B_r|} \int\limits_{B_r} |f\left(x-y\right)| dy.$$

The function Mf is defined in the extended real sense, i.e., $Mf(x) = +\infty$ is allowed.

Before we developing theory on Hardy-Littlewood maximal function let us see an easy example.

Example 3.1. Let $f = \chi_{[0,1]}$. For a given $\epsilon > 0$, we denote $M_\epsilon f(x)$ to be average of f over $(x-\epsilon, x+\epsilon)$. That is, we wish to find

$$Mf(x) = \sup_{\epsilon>0} M_\epsilon f(x),$$

for a given $x \in \mathbb{R}$.

$$M_\epsilon f(x) = \frac{1}{|(x-\epsilon, x+\epsilon)|} \int_{(x-\epsilon,x+\epsilon)} \chi_{[0,1]}(y) \text{ d}y = \frac{1}{2\epsilon} \int_{(x-\epsilon,x+\epsilon)\cap[0,1]} 1 \text{ d}y.$$

We consider the following cases.

1. First, let us take $x \in [0,1]$. Then, either $(x-\epsilon, x+\epsilon) \subseteq [0,1]$, or $x-\epsilon \in [0,1]$ but $x+\epsilon \notin [0,1]$, or $x-\epsilon \notin [0,1]$ but $x+\epsilon \in [0,1]$, or $[0,1] \subset (x-\epsilon, x+\epsilon)$. We compute the value of the above integral for each of these intervals. If $(x-\epsilon, x+\epsilon) \subseteq [0,1]$ then $(x-\epsilon, x+\epsilon) \cap [0,1] = (x-\epsilon, x+\epsilon)$. This case is illustrated in Figure 3.1.

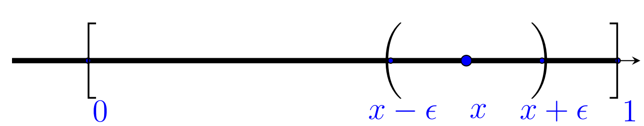

Figure 3.1: $(x-\epsilon, x+\epsilon) \subseteq [0,1]$

Therefore,

$$M_\epsilon f(x) = \frac{1}{|2\epsilon|} \int_{(x-\epsilon,x+\epsilon)} 1 \text{ d}y = \frac{2\epsilon}{2\epsilon} = 1.$$

If $x-\epsilon \in [0,1]$, and $x+\epsilon \notin [0,1]$ then $(x-\epsilon, x+\epsilon) \cap [0,1] = (x-\epsilon, 1]$. For an illustration, see Figure 3.2.

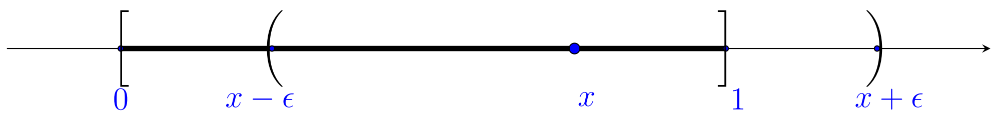

Figure 3.2: $(x-\epsilon, x+\epsilon) \cap [0,1] = (x-\epsilon, 1]$.

So,

$$M_\epsilon f(x) = \frac{1}{|2\epsilon|} \int_{(x-\epsilon,1]} 1 \text{ d}y = \frac{1-x+\epsilon}{2\epsilon}.$$

Note that $x+\epsilon > 1$ therefore $\epsilon > 1-x$. So, $1-x+\epsilon < 2\epsilon$. Therefore $M_\epsilon f(x) < 1$.

If $x-\epsilon \notin [0,1]$ and $x+\epsilon \in [0,1]$, then $(x-\epsilon, x+\epsilon) \cap [0,1] = [0, x+\epsilon)$. Therefore,

$$M_\epsilon f(x) = \frac{1}{|2\epsilon|} \int_{[0,x+\epsilon)} 1 \text{ d}y = \frac{x+\epsilon}{2\epsilon}.$$

Note that in this case $x-\epsilon < 0$ so $x < \epsilon$. Therefore $M_\epsilon f(x) < 1$ as well.

Lastly, if $[0,1] \subseteq (x-\epsilon, x+\epsilon)$, then $(x-\epsilon, x+\epsilon) \cap [0,1] = [0,1]$, as seen in Figure 3.3.

Hence, we must have $\epsilon > \frac{1}{2}$, and hence

$$M_\epsilon f(x) = \frac{1}{|2\epsilon|} \int_{[0,1]} 1 \text{ d}y = \frac{1}{2\epsilon} < 1.$$

Thus, $Mf(x) = \sup_{\epsilon>0} M_\epsilon f(x) = 1$, when $x \in [0,1]$.

2. Next, let us assume $x > 1$. Then, either $(x-\epsilon, x+\epsilon) \cap [0,1] = \emptyset$ or $x-\epsilon \in [0,1]$ or $[0,1] \subseteq (x-\epsilon, x+\epsilon)$. In the first case clearly $M_\epsilon f(x) = 0$. In the second case, $(x-\epsilon, x+\epsilon) \cap [0,1] = (x-\epsilon, 1]$. For an illustration, see Figure 3.4 We notice that, we must have $x-1 < \epsilon < x$.

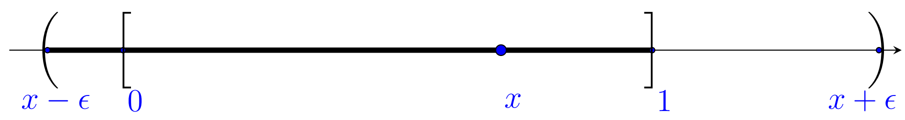

Figure 3.3: $[0,1] \subseteq (x-\epsilon, x+\epsilon)$

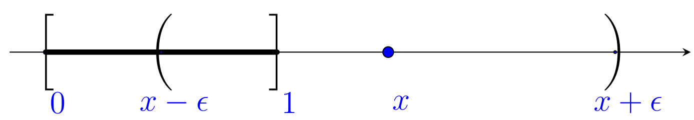

Figure 3.4: $(x-\epsilon, x+\epsilon) \cap [0,1] = (x-\epsilon, 1]$

Therefore, we have

$$M_\epsilon f(x) = \frac{1}{2\epsilon} \int_{(x-\epsilon,1]} 1 \text{ d}y = \frac{(1-x+\epsilon)}{2\epsilon}.$$

Note that $\frac{d}{d\epsilon}(M_\epsilon f(x)) = \frac{x-1}{2\epsilon^2} > 0$. Therefore, $M_\epsilon f(x)$ is an increasing function of ϵ. Hence, $M_\epsilon f(x) \leq M_x f(x) = \frac{1}{2x}$. That is, when $x - \epsilon \in [0,1]$ we have $M_\epsilon f(x) \leq \frac{1}{2x}$, with equality for $\epsilon = x$.

Lastly, if $[0,1] \subseteq (x-\epsilon, x+\epsilon)$, we have $\epsilon > x$, and $(x-\epsilon, x+\epsilon) \cap [0,1] = [0,1]$, as shown in Figure 3.5.

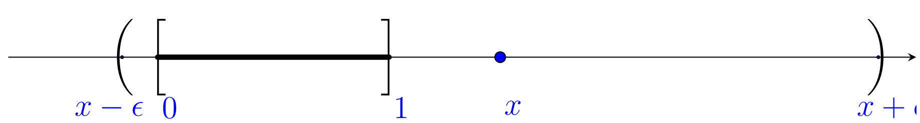

Figure 3.5: $[0,1] \subseteq (x-\epsilon, x+\epsilon)$

Then

$$M_\epsilon f(x) = \frac{1}{|2\epsilon|} \int_{[0,1]} 1 \text{ d}y = \frac{1}{2\epsilon} < \frac{1}{2x}.$$

Thus, we get $Mf(x) = \sup\limits_{\epsilon>0} M_\epsilon f(x) = \frac{1}{2x}$.

3. Finally we consider the case $x < 0$. Then either one of the following holds: $(x-\epsilon, x+\epsilon) \cap [0,1] = \emptyset$ or $x+\epsilon \in [0,1]$ or $[0,1] \subseteq (x-\epsilon, x+\epsilon)$. In the first case, it is clear that $M_\epsilon f(x) = 0$. In the second case, we have $(x-\epsilon, x+\epsilon) \cap [0,1] = [0, x+\epsilon)$ as seen in Figure 3.6. We notice that this gives $-x < \epsilon < 1-x$.

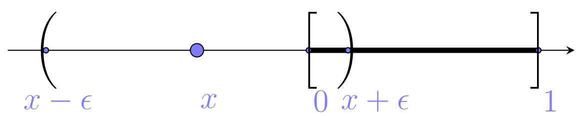

Figure 3.6: $(x-\epsilon, x+\epsilon) \cap [0,1] = [0, x+\epsilon)$

Therefore, we have
$$M_\epsilon f(x) = \frac{1}{2\epsilon} \int_{[0,x+\epsilon)} 1 \text{ d}y = \frac{x+\epsilon}{2\epsilon}.$$
We see that $\frac{d}{d\epsilon}(M_\epsilon f(x)) = \frac{-x}{2\epsilon^2} > 0$, since $x < 0$. Therefore, $M_\epsilon f(x)$ is an increasing function of ϵ. Hence, $M_\epsilon f(x) \leq M_{1-x} f(x) = \frac{1}{2(1-x)}$. That is, when $x+\epsilon \in [0,1]$, we have $M_\epsilon f(x) \leq \frac{1}{2(1-x)}$, with equality for $\epsilon = 1-x$. Lastly, when $[0,1] \subset (x-\epsilon, x+\epsilon)$, we have $\epsilon > 1-x$, and $(x-\epsilon, x+\epsilon) \cap [0,1] = [0,1]$. For an illustration, see Figure 3.7.

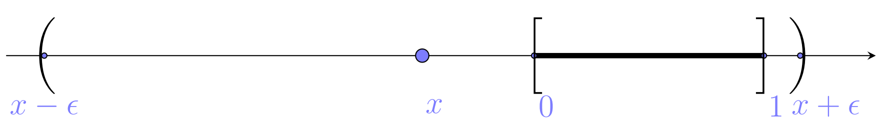

Figure 3.7: $[0,1] \subset (x-\epsilon, x+\epsilon)$

Hence, we get
$$M_\epsilon f(x) = \frac{1}{|2\epsilon|} \int_{[0,1]} 1 \text{ d}y = \frac{1}{2\epsilon} < \frac{1}{2(1-x)}.$$
Now, we have $Mf(x) = \sup_{\epsilon>0} M_\epsilon f(x) = \frac{1}{2(1-x)}$, for $x < 1$.

Combining the above observations we have

$$Mf(x) = \begin{cases} \frac{1}{2(1-x)}, & \text{if } x < 1. \\ 1, & \text{if } x \in [0,1]. \\ \frac{1}{2x}, & \text{if } x > 1. \end{cases}$$

Figure 3.8 shows the graph of the maximal function for the function $\chi_{[0,1]}$.

Figure 3.8: Maximal Function of $\chi_{[0,1]}$

Now, we see a first estimate related to Hardy-Littlewood maximal function. For a fixed $f \in L^1(\mathbb{R}^n)$, maximal function corresponding to the convolution of f with dilations of a radial, decreasing function is bounded by the Hardy-Littlewood maximal function of f.

Proposition 3.1. *Let φ be a function that is positive, radial, and decreasing (as a function on $(0,\infty)$). Then, for $f \in L^1(\mathbb{R}^n)$,*

$$\sup_{t>0} |\varphi_t * f(x)| \leq ||\varphi||_1 Mf(x), \tag{3.1}$$

where $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$.

Proof. Let us first assume that φ is a simple function , i.e., it can be written as

$$\varphi(x) = \sum_{j=1}^{\infty} a_j \chi_{B_{r_j}}(x),$$

with $a_j > 0$ and B_{r_j} is a ball centered at 0 of radius $r_j > 0$. Then,

$$\begin{aligned}
|\varphi * f(x)| &= \left| \sum_{j=1}^{k} a_j |B_{r_j}| \frac{1}{|B_{r_j}|} \chi_{B_{r_j}} * f(x) \right| \\
&= \left| \sum_{j=1}^{k} a_j |B_{r_j}| \frac{1}{|B_{r_j}|} \int_{B_{r_j}} f(x-y)\mathrm{d}y \right| \\
&\leq \sum_{j=1}^{k} a_j |B_{r_j}| Mf(x) \\
&= ||\varphi||_1 Mf(x).
\end{aligned}$$

Now for any $t > 0$, we have $\varphi_t(x) = t^{-n} \sum\limits_{j=1}^{k} a_j \chi_{B_{r_j}}(\frac{x}{t}) = t^{-n} \sum\limits_{j=1}^{k} a_j \chi_{B_{tr_j}}(x)$. Thus,

$$\begin{aligned}
|\varphi_t * f(x)| &= \left| \frac{1}{t^n} \sum_{j=1}^{k} a_j \chi_{B_{tr_j}} * f(x) \right| \\
&= \left| \frac{1}{t^n} \sum_{j=1}^{k} a_j |B_{tr_j}| \frac{1}{|B_{tr_j}|} \int_{B_{r_j}} f(x-y)\mathrm{d}y \right| \\
&\leq \frac{1}{t^n} \sum_{j=1}^{k} t^n |B_{tr_j}| Mf(x) \\
&= ||\varphi||_1 Mf(x).
\end{aligned}$$

Therefore, the result is true for simple functions. Now, let φ be any arbitrary function satisfying the hypothesis of the theorem. Then, φ can be approximated by an increasing sequence of simple functions, say $(\varphi_n)_{n\in\mathbb{N}}$, each satisfying the hypothesis of the theorem. Then for each $n \in \mathbb{N}$ we have

$$|\varphi_n * f(x)| \leq ||\varphi_n||_1 Mf(x) \leq ||\varphi||_1 Mf(x).$$

That is, $\forall n \in \mathbb{N}$,

$$\left| \int\limits_{\mathbb{R}^n} \varphi_n(y) f(x-y)\mathrm{d}y \right| \leq ||\varphi||_1 Mf(x).$$

By dominated convergence theorem, we have

$$\left| \int\limits_{\mathbb{R}^n} \varphi(y) f(x-y)\mathrm{d}y \right| \leq ||\varphi||_1 Mf(x).$$

That is, $|\varphi * f(x)| \leq ||\varphi||_1 Mf(x)$. Now any dilation φ_t is also positive, radial, decreasing function with the same L^1-norm as that of φ. Therefore, it also satisfies the same inequality. That is, for any $t > 0$, we have

$$|\varphi_t * f(x)| \leq ||\varphi||_1 Mf(x).$$

Hence, the result is proved. $\square$

Let us see a few more maximal functions that are equivalent to the Hardy-Littlewood maximal function.

Definition 3.2 (Cubic maximal function)**.** *Let Q_r be the cube $[-r,r]^n$. Then for $f \in L^1_{loc}(\mathbb{R}^n)$, the cubic maximal function is defined by*

$$M' f(x) := \sup_{r>0} \frac{1}{(2r)^n} \int_{Q_r} |f(x-y)|dy.$$

We observe that when $n = 1$, M and M' coincide. For $n > 1$, let us consider the cube $Q_r = [-r,r]^n$ and let D_1 and D_2 be the balls centred at 0 with radius r and $\sqrt{n}r$, respectively. Then, $D_1 \subseteq Q_r \subseteq D_2$ as seen in Figure 3.9. We have

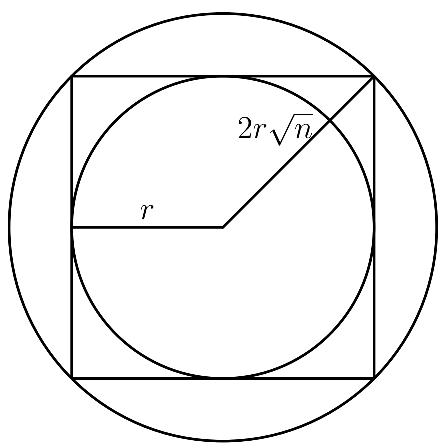

Figure 3.9: $D_1 \subseteq Q_r \subseteq D_2$

$|D_1| = a_n r^n$, $|Q_r| = (2r)^n$ and $|D_2| = a_n(\sqrt{n}r)^n$, where a_n is the volume of the unit ball and it only depends on n, the dimension of the space. Thus, we have the following

$$\frac{1}{(2r)^n}\int\limits_{D_1}|f(x-y)|\mathrm{d}x \leq \frac{1}{(2r)^n}\int\limits_{Q_r}|f(x-y)|\mathrm{d}x \leq \frac{1}{(2r)^n}\int\limits_{D_2}|f(x-y)|\mathrm{d}x.$$

That is,

$$\frac{a_n}{2^n|D_1|}\int\limits_{D_1}|f(x-y)|\mathrm{d}x \leq \frac{1}{(2r)^n}\int\limits_{Q_r}|f(x-y)|\mathrm{d}x \leq \frac{a_n(\sqrt{n})^n}{2^n|D_2|}\int\limits_{D_2}|f(x-y)|\mathrm{d}x.$$

So, we have,

$$\frac{a_n}{2^n}Mf(x) \leq M^{'}f(x) \leq \frac{a_n(\sqrt{n})^n}{2^n}Mf(x).$$

That is, there are positive constants c_n and C_n depending only on n, such that

$$c_nM^{'}f\left(x\right) \leq Mf\left(x\right) \leq C_nM^{'}f\left(x\right), \tag{3.2}$$

where, $c_n = \dfrac{2^n}{a_n(\sqrt{n})^n}$ and $C_n = \dfrac{2^n}{a_n}$. Due to Inequality (3.2) the operators M and $M^{'}$ are essentially interchangeable and we use whichever is more appropriate, depending on the situation.

There is another kind of maximal function where we are interested in the averages over the cubes containing the given point that might not necessarily be the center of the cube.

Definition 3.3 (Non-centered cubic maximal function)**.** *Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then non-centred cubic maximal function of f is defined by*

$$M^{''}f(x) := \sup_{x\in Q}\frac{1}{|Q|}\int_Q|f(y)|dy,$$

where Q is a cube in $\mathbb{R}^n$ containing x.

We now show that M and $M^{''}$ are equivalent. For the same, we first see that the set of all cubes containing a point x is larger than the set of cubes whose center is x. Therefore, directly by the definitions of $M^{'}$ and $M^{''}$ we have

$$M^{'}f(x) \leq M^{''}f(x). \tag{3.3}$$

Now let Q be any cube containing x with side length is a.

It is easy to see that the ball centered at x and with radius $\sqrt{n}a$ contains the cube Q (see Figure 3.10). Also, $|Q| = a^n$ and $|B(x, a\sqrt{n})| = C(a\sqrt{n})^n$. That

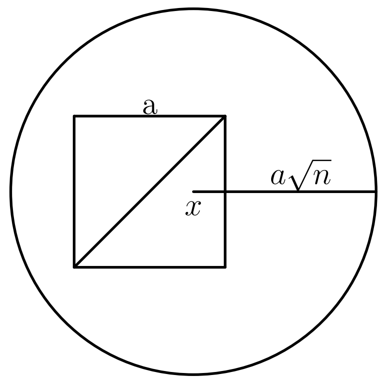

Figure 3.10: $Q \subseteq B(x, a\sqrt{n})$

is, $|Q| = \frac{|B(x,a\sqrt{n})|}{C(\sqrt{n})^n}$. Therefore, we have

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(y)|\ \mathrm{d}y &\leq \frac{1}{|Q|}\int_{B(x,a\sqrt{n})} |f(y)|\ \mathrm{d}y \\ &= \frac{C(\sqrt{n})^n}{|B(x,a\sqrt{n})|}\int_{B(x,a\sqrt{n})} |f(y)|\ \mathrm{d}y \\ &\leq C(\sqrt{n})^n Mf(x).\end{aligned}$$

Since the choice of Q is arbitrary we get $M^{''}f(x) \leq C(\sqrt{n})^n Mf(x)$. Using this observation along with Inequalities (3.2) and (3.3), we get constants C_1 and C_2 such that

$$C_1 Mf(x) \leq M^{''}f(x) \leq C_2 Mf(x). \tag{3.4}$$

In the sequel, the equivalence of these maximal operators becomes useful. Particularly, one can interchange them as per convenience, without disturbing the results. In the next chapter, the non-centered cubic maximal function helps us understand weighted boundedness of Hardy-Littlewood maximal function. Before that, however, let us see another kind of averaging operator.

3.2 Dyadic Maximal operator and Calderón-Zygmund decomposition

We begin with the dyadic decomposition of $\mathbb{R}^n$. We define the unit cube, open on right, to be the set $[0,1)^n$ and define $\mathcal{Q}_0$ be the collection of those cubes congruent to $[0,1)^n$ with vertices on $\mathbb{Z}^n$. By dylating this family by a factor of 2^{-k}, we end up with a collection $\mathcal{Q}_k$, $(k \in \mathbb{Z})$. That is, $\mathcal{Q}_k$ is the family of cubes, open on the right, whose vertices are adjacent points of the lattice $(2^{-k}\mathbb{Z})^n$. The cubes in $\bigcup\limits_{k\in\mathbb{Z}} \mathcal{Q}_k$ are called ***dyadic cubes***. We observe that each family $\mathcal{Q}_k$ is countable, since a cube in $\mathcal{Q}_k$ is uniquely determined by its vertices (that comes from a countable collection $(2^{-k}\mathbb{Z})^n$). Therefore, the total collection $\bigcup\limits_{k\in\mathbb{Z}} \mathcal{Q}_k$ of all dyadic cubes is also countable (being a countable union of countable sets). Figure 3.11 shows a part of the dyadic decomposition of $\mathbb{R}^n$.

From this construction we immediately get the following properties:

1. Given $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, there is a unique cube $Q \in \mathcal{Q}_k$ such that $x \in Q$.

2. Any two dyadic cubes are either disjoint or contained in one-another.

3. A dyadic cube in $\mathcal{Q}_k$ is contained in a unique cube of each family $\mathcal{Q}_j$, for $j < k$, and contains 2^n many dyadic cubes from the family $\mathcal{Q}_{k+1}$.

We now give dyadic averages of functions defined in $\mathbb{R}^n$ and the corresponding maximal operator. Later, this construction gives us an important technique called Calderón-Zygmund decomposition, that is used throughout our exposition.

Definition 3.4 (Dyadic average). *Given a function $f \in L^1_{loc}(\mathbb{R}^n)$, we define the dyadic average at level $k \in \mathbb{Z}$ as*

$$E_k f(x) := \sum_{Q\in\mathcal{Q}_k} \left(\frac{1}{|Q|} \int\limits_Q f(y)\mathrm{d}y \right) \chi_Q(x). \tag{3.5}$$

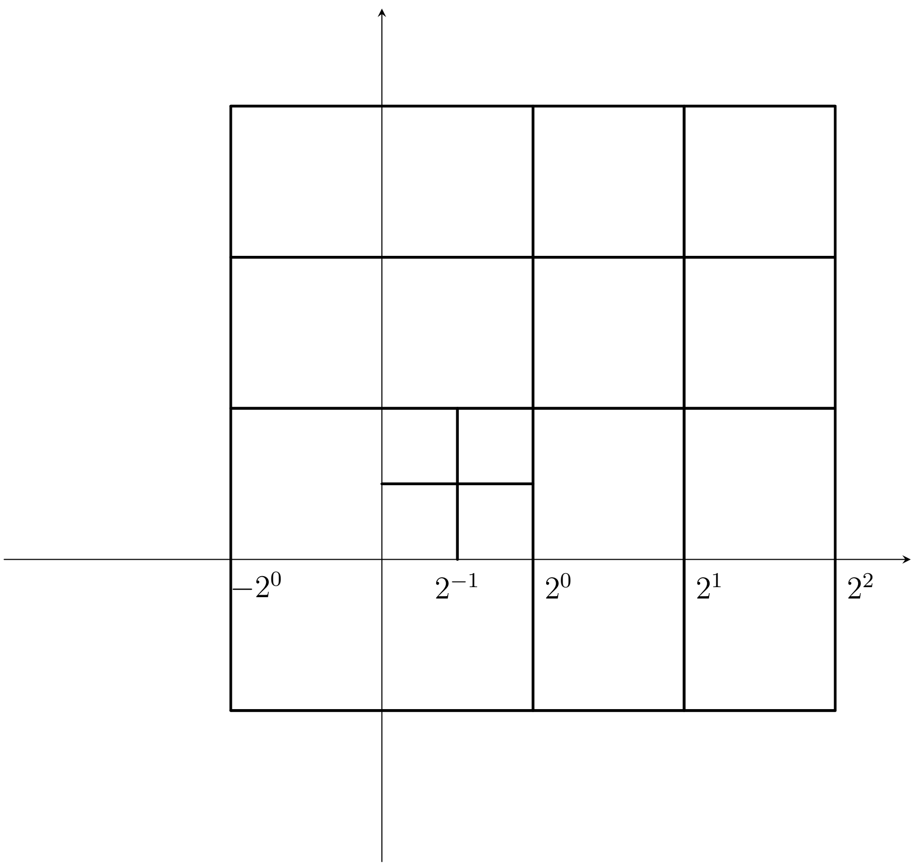

Figure 3.11: Dyadic decomposition of $\mathbb{R}^n$

It is easy to see that for a fixed $Q \in \mathcal{Q}_k$, $E_k f$ is constant, and equals the average of f over Q. The dyadic average satisfies the following fundamental lemma.

Lemma 3.2. *If Ω is a union of the cubes in $\mathcal{Q}_k$, then*

$$\int\limits_{\Omega} E_k f(x)\mathrm{d}x = \int\limits_{\Omega} f(x)\mathrm{d}x. \tag{3.6}$$

Proof. Let $\Omega = \bigcup\limits_{j\in I} Q_j$ for some index set $I \subseteq \mathbb{N}$. Here, $\forall j \in I$, $Q_j \in \mathcal{Q}_k$. Since Ω is a disjoint union, we have,

$$\int\limits_{\Omega} E_k f(x)\mathrm{d}x = \sum_{j\in I} \int\limits_{Q_j} E_k f(x)\mathrm{d}x$$

$$
\begin{aligned}
&= \sum_{j\in I}^{\infty} \int\limits_{Q_j} \sum_{Q\in\mathcal{Q}_k} \left(\int_Q f(y)\mathrm{d}y \right) \chi_Q(x)\mathrm{d}x \\
&= \sum_{j\in I} \int\limits_{Q_j} \left(\frac{1}{|Q_j|} \int\limits_{Q_j} f(y)\mathrm{d}y \right) \chi_{Q_j}(x)\mathrm{d}x \\
&= \sum_{j\in I} \left(\frac{1}{|Q_j|} \int\limits_{Q_j} f(y)\mathrm{d}y \right) \int\limits_{Q_j} \chi_{Q_j}(x)\mathrm{d}x \\
&= \sum_{j\in I}^{\infty} \int\limits_{Q_j} f(y)\mathrm{d}y \\
&= \int_{\bigcup\limits_{j\in I}^{\infty} Q_j} f(y)\mathrm{d}y \\
&= \int\limits_{\Omega} f(y)\mathrm{d}y.
\end{aligned}
$$

□

Now, we define the dyadic maximal function.

Definition 3.5 (Dyadic maximal function)**.** *Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then the dyadic maximal function is defined as*

$$M_d f(x) := \sup_{k\in\mathbb{Z}} |E_k f(x)|. \tag{3.7}$$

Before developing any further theory, let us see a simple example.

Example 3.2. Let $f = \chi_{[0,1]}$. In one dimensional situation, dyadic cubes are dyadic intervals. We keep the notations same. That is, Q denotes a dyadic interval. Let $Q \in \mathcal{Q}_k$ for some $k \in \mathbb{Z}$. We wish to evaluate $E_k f(x)$, for a given $x \in \mathbb{R}$. We have the following three cases

1. First suppose $x \in [0,1)$. Then one of the following may happen. Either $Q \cap [0,1] = \emptyset$ or $[0,1] \subseteq Q$ or $Q \subseteq [0,1]$. In the first case we readily see that $E_k f(x) = 0$. In the second case $Q \cap [0,1] = [0,1]$ therefore
$$E_k f(x) = \frac{1}{|Q|} \int\limits_{Q} \chi_{[0,1]}(y)\ \mathrm{d}y = \frac{1}{|Q|} \int\limits_{[0,1]} \chi_{[0,1]}(y)\ \mathrm{d}y = \frac{1}{|Q|} < 1.$$

Lastly, if $Q \subset [0,1]$, then $[0,1] \cap Q = Q$. Hence,

$$E_k f(x) = \frac{1}{|Q|} \int\limits_Q \chi_{[0,1]}(y)\ \mathrm{d}y = \frac{1}{|Q|} \int\limits_Q 1\ \mathrm{d}y = \frac{|Q|}{|Q|} = 1.$$

Consequently, for $|E_k f(x)| \leq 1$. Therefore if $x \in [0,1)$, we have

$$M_d f(x) = \sup_{k \in \mathbb{Z}} |E_k f(x)| = 1.$$

2. Next let us consider $x \geq 1$. Let $Q \in \mathcal{Q}_k$ such that $x \in Q$. Then either $Q \cap [0,1] = \emptyset$, or $\{1\}$, or $[0,1] \subseteq Q$. In the first case, when $Q \cap [0,1] = \emptyset$ or $\{1\}$, clearly $E_k f(x) = 0$. In the second case we have $Q \cap [0,1] = [0,1]$. Therefore,

$$E_k f(x) = \frac{1}{|Q|} \int\limits_Q \chi_{[0,1]}(y)\ \mathrm{d}y = \frac{1}{|Q|} \int\limits_{[0,1]} 1\ \mathrm{d}y = \frac{1}{|Q|} < 1.$$

That is, $E_k f(x) < 1$ when $x \geq 1$.

We notice that as the size of the dyadic cube Q increases, the value of $\frac{1}{|Q|} \int\limits_Q \chi_{[0,1]}(y)\ \mathrm{d}y = \frac{1}{|Q|} \int\limits_{[0,1]} 1\ \mathrm{d}y$ decreases. Therefore, we have to find smallest dyadic cube containing the point x and $[0,1] \subseteq Q$. Let $k \in \mathbb{Z}$ be such that $2^{-k} \leq x < 2^{-(k+1)}$. Taking logarithm, we get $k \leq \log_2 x < -(k+1)$. Therefore, $-k = [\log_2 x]$ where, $[\cdot]$ is the greatest integer function. So the desired cube is $[0, 2^{[\log_2 x]})$. For this choice of k, $E_k f(x) = \frac{1}{2^{[\log_2 x]}}$ and $M_d f(x) = \frac{1}{2^{[\log_2 x]}}$, for $x \geq 1$.

3. Finally let us consider $x < 0$. Let Q be a dyadic cube containing the point x. By construction of dyadic cubes given at the start of this section, it is clear that $Q \cap [0,1] = \emptyset$ or $\{0\}$. In either case $E_k f(x) = 0$, and hence $M_d f(x) = 0$, for $x < 0$. Therefore in this case $Q \cap [0,1] = \emptyset$ and hence, $M_d f(x) = 0$.

Combining the above analysis we have the following

$$M_d f(x) = \begin{cases} \frac{1}{2^{[\log_2 x]}}, & \text{if } x \geq 1. \\ 1, & \text{if } 0 \leq x < 1. \\ 0, & \text{if } x < 0. \end{cases}$$

Figure 3.12 gives the graph of $M_d\chi_{[0,1]}$.

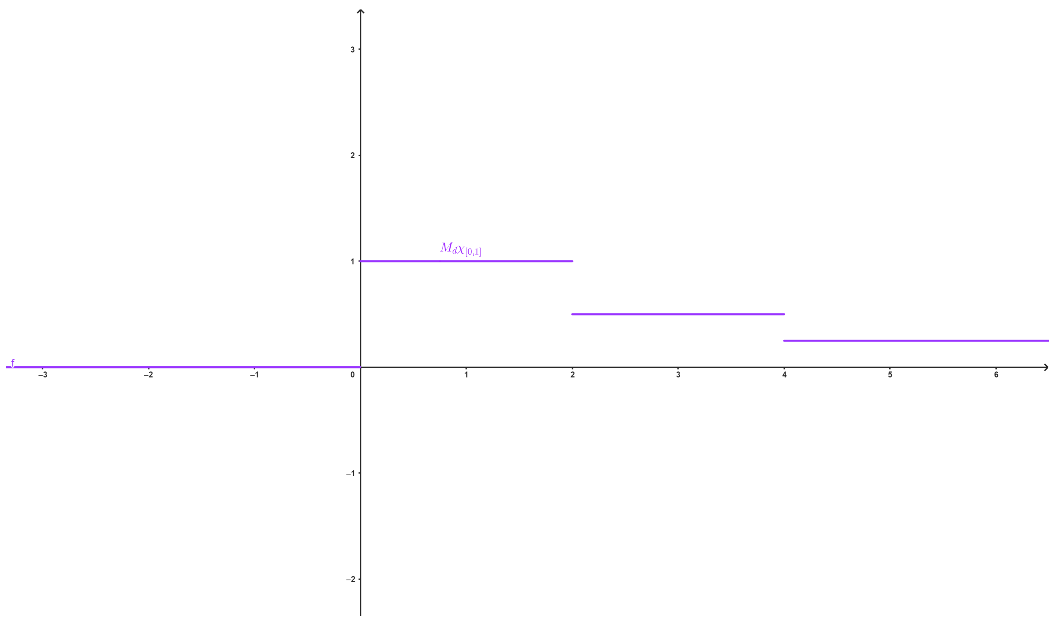

Figure 3.12: Dyadic Maximal Function of $\chi_{[0,1]}$

It is clear that M_d is not equivalent of M.

Next we see a weak type inequality for the dyadic maximal function. This plays a crucial role in the Calderón-Zygmund decomposition on $\mathbb{R}^n$, and also in the Lebesgue differentiation theorem.

Theorem 3.3. *The following are true.*

1. *The dyadic maximal function is weak* $(1,1)$.

2. *For any* $f \in L^1_{loc}(\mathbb{R}^n)$, $\lim_{k\longrightarrow\infty} E_k f(x) = f(x)$ *for a.e.* $x \in \mathbb{R}^n$.

Proof. Let us fix an $f \in L^1(\mathbb{R}^n)$. Due to the sublinearity of M_d, it suffices to show the result for positive functions, since general (complex real valued) function can be decomposed into a sum of positive functions.

We consider the following set, for a given $\lambda > 0$,

$$\Omega_k := \{x \in \mathbb{R}^n : E_k f(x) > \lambda \text{ and } E_j f(x) \leq \lambda \text{ for any } j < k\}.$$

Equivalently, $x \in \Omega_k$ if $E_k f(x)$ is the first conditional expectation of f which is greater than λ. Note that since $f \in L^1(\mathbb{R}^n)$, $\int_Q f(x)\ \mathrm{d}x \leq \int_{\mathbb{R}^n} f(x)\ \mathrm{d}x < \infty$ for every $Q \in \mathcal{Q}_k$. So $\frac{1}{|Q_k|}\int_{Q_k} f(x)\ \mathrm{d}x \leq \frac{||f||_1}{|Q_k|}$. Also $|Q_k| = (2^{-k})^n \longrightarrow \infty$ as $k \longrightarrow -\infty$. Therefore as $\lim\limits_{k \longrightarrow -\infty} \frac{1}{|Q_k|}\int_{Q_k} f(y)\mathrm{d}y = 0$ Hence, by well ordering principle, for a given $\lambda > 0$ there exists s smallest $k \in \mathbb{Z}$ such that $E_k f(x) > \lambda$ For any $x \in \Omega_k$, there exists exactly one cube $Q \in \mathcal{Q}_k$ such that $x \in Q$. $E_k f(x) = \frac{1}{|Q|}\int_Q f(y)\mathrm{d}y > \lambda$. That is, the entire cube Q is inside Ω_k. Therefore, Ω_k can be written as disjoint union of cubes in $\mathcal{Q}_k$.

We also notice that if $k_1 \neq k_2$, then $\Omega_{k_1} \cap \Omega_{k_2} = \emptyset$. To see this, without loss of generality let us assume that $k_1 < k_2$, and $\exists\ x \in \Omega_{k_1} \cap \Omega_{k_2}$. Then, $E_{k_2} f(x) > \lambda$ and $E_j f(x) \leq \lambda$ when $j < k_2$. Particularly $E_{k_1} \leq \lambda$, which contradicts the definition of Ω_{k_1}. Therefore $\bigcup\limits_{k \in \mathbb{Z}} \Omega_k$ is a disjoint union dyadic cubes.

To prove the weak $(1, 1)$ boundedness of M_d, we need to estimate the size of the set $H := \{x \in \mathbb{R}^n : M_d f(x) > \lambda\}$ for a given $\lambda > 0$. Now, we claim the following

$$H = \bigcup_{k \in \mathbb{Z}} \Omega_k.$$

If $x \in H$ then by the definition of $M_d f(x)$, for any $x \in H$, $\exists\ k' \in \mathbb{Z}$ such that $E_{k'} f(x) > \lambda$. Now if $k \in \mathbb{Z}$ is the minimum of all such k' then $E_k f(x) > \lambda$ and $E_j f(x) \leq \lambda\ \forall j < k$. That is $x \in \Omega_k$, and hence $x \in \bigcup\limits_{k \in \mathbb{Z}} \Omega_k$. So, $H \subseteq \bigcup\limits_{k \in \mathbb{Z}} \Omega_k$. Conversely, suppose $x \in \bigcup\limits_{k \in \mathbb{Z}} \Omega_k$. Then, $\exists\ k' \in \mathbb{Z}$ such that $x \in \Omega_{k'}$. By the definition of $\Omega_{k'}$, $E_{k'} f(x) > \lambda$. Hence, $M_d f(x) = \sup\limits_{k \in \mathbb{Z}} E_k f(x) > \lambda$. That is, $H \supseteq \bigcup\limits_{k \in \mathbb{Z}} \Omega_k$, and our claim is proved. We have seen that $\bigcup\limits_{k \in \mathbb{Z}} \Omega_k$ is a disjoint union of dyadic cubes. Thus, from Lemma 3.2, we have

$$\begin{aligned}
|\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}| &= \left| \bigcup_{k\in\mathbb{Z}} \Omega_k \right|. \\
&= \sum_k |\Omega_k| \\
&\leq \sum_k \frac{1}{\lambda} \int_{\Omega_k} E_k f(x)\mathrm{d}x \\
&= \frac{1}{\lambda} \sum_k \int_{\Omega_k} f(x)\mathrm{d}x \\
&\leq \frac{1}{\lambda} ||f||_1.
\end{aligned}$$

This proves (1).

To prove (2), let us first consider $f \in C(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ be fixed. So, for a given $\epsilon > 0, \exists\, \delta > 0$ such that $\forall y \in \mathbb{R}^n$ with $|y - x| < \delta$, we have $|f(x) - f(y)| < \epsilon$. For cubes $Q_k \in \mathcal{Q}_k$ $(k \in \mathbb{Z})$, we have the following relation between their diameters

$$\mathrm{diam}(Q_k) = 2^{-k}\mathrm{diam}(Q_0) = 2^{-k}\sqrt{n},$$

where $Q_0 = [0, 1)^n \in \mathcal{Q}_0$ is the unit cube in $\mathbb{R}^n$. By the Archimedean property, $\exists\; k_0 \in \mathbb{N}$ such that $2^{k_0} \geq k_0 > \dfrac{\sqrt{n}}{\delta}$. So $\delta > 2^{-k_0}\sqrt{n}$. Therefore $\forall k \geq k_0$, $\mathrm{diam}(Q_k) = 2^{-k}\sqrt{n} \leq 2^{-k_0}\sqrt{n} < \delta$. That is for any $Q_k \in \mathcal{Q}_k$(with $k > k_0$) with $x \in Q_k$, $\forall y \in Q_k$ we have $|y - x| \leq 2^{-k}\sqrt{n} < \delta$, and hence $|f(x) - f(y)| < \epsilon$. Now $\forall k \geq k_0$,

$$\begin{aligned}
|E_k f(x) - f(x)| &= \left| \sum_{Q\in\mathcal{Q}_k} \left(\frac{1}{|Q|} \int_Q f(y)\mathrm{d}y \right) \chi_Q(x) - f(x) \right| \\
&= \left| \frac{1}{|Q_k|} \int_{Q_k} f(y)\mathrm{d}y - f(x) \right| \\
&= \left| \frac{1}{|Q_k|} \int_{Q_k} (f(y) - f(x))\mathrm{d}y \right| \\
&\leq \frac{1}{|Q_k|} \int_{Q_k} |f(y) - f(x)|\mathrm{d}y
\end{aligned}$$

$$< \frac{1}{|Q_k|} \int\limits_{Q_k} \epsilon \mathrm{d}y = \epsilon.$$

Hence, $\lim\limits_{k \longrightarrow \infty} E_k f(x) \longrightarrow f(x)$ for any $x \in \mathbb{R}^n$ whenever $f \in C(\mathbb{R}^n)$. Now, consider the set $\mathcal{S} := \{f \in L^1(\mathrm{R}^n) : E_k f(x) \longrightarrow f(x) \text{ a.e.}\}$. It is clear from above that $C(\mathbb{R}^n) \subseteq \mathcal{S}$. We also know from part (1) that $M_d f(x)$ is weak $(1, 1)$. Hence by Theorem 2.7 the set $\mathcal{S}$ is closed and dense in $L^1(\mathbb{R}^n)$. As a result, $\mathcal{S} = L^1(\mathbb{R}^n)$. So the result is true if for L^1 functions. To complete the proof, note that if $f \in L^1_{loc}(\mathbb{R}^n)$ then $f\chi_Q \in L^1(\mathbb{R}^n)$ for any cube $Q \in \mathcal{Q}_0$. Hence, (2) holds for almost every $x \in Q$, and so for almost every $x \in \mathbb{R}^n$. □

We now give an important result related to the dyadic decomposition of $\mathbb{R}^n$, called the Calderón-Zygmund decomposition. This theorem allows us to decompose a given function in $L^1(\mathbb{R}^n)$ into two parts, called "good" and "bad" parts. This technique forms the crux for many proofs involving weak type inequalities.

Theorem 3.4. *Given a non-negative function $f \in L^1(\mathbb{R}^n)$, and given a $\lambda > 0$, there exists a sequence $\{Q_j\}_{j \in \mathbb{N}}$ of disjoint dyadic cubes such that*

1. *$f(x) \leq \lambda$ for almost every $x \notin \bigcup\limits_{j \in \mathbb{N}} Q_j$,*
2. *$\left| \bigcup\limits_{j \in \mathbb{N}} Q_j \right| \leq \frac{1}{\lambda} ||f||_1$, and*
3. *For every $j \in \mathbb{N}$, $\lambda < \frac{1}{|Q_j|} \int\limits_{Q_j} f(y) \mathrm{d}y \leq 2^n \lambda$.*

Proof. For a fixed $\lambda > 0$, let us consider the set

$$\Omega_k := \{x \in \mathbb{R}^n : E_k f(x) > \lambda \text{ and } E_j f(x) \leq \lambda \text{ if } j < k\}.$$

As in Theorem 3.3, $\bigcup\limits_{k \in \mathbb{Z}} \Omega_k$ is a disjoint union of dyadic cubes. Now consider the family $\{Q_j\}_{j \in \mathbb{N}}$ of dyadic cubes in $\bigcup\limits_{k \in \mathbb{Z}} \Omega_k$. That is, $\bigcup\limits_{k \in \mathbb{Z}} \Omega_k = \bigcup\limits_{j \in \mathbb{N}} Q_j$. So,

$$\begin{aligned} \left| \bigcup_{j \in \mathbb{N}} Q_j \right| &\leq \sum_{k \in \mathbb{Z}} |\Omega_k| \\ &\leq \sum_{k \in \mathbb{Z}} \frac{1}{\lambda} \int\limits_{\Omega_k} E_k f(y) \mathrm{d}y \end{aligned}$$

$$= \frac{1}{\lambda} \sum_{k\in\mathbb{Z}} \int\limits_{\Omega_k} f(y)\mathrm{d}y$$
$$\leq \frac{1}{\lambda} ||f||_1.$$

This proves the second part of the theorem.

We notice that if $x \notin \bigcup\limits_{j\in\mathbb{N}} Q_j$, then for every $k \in \mathbb{Z}$, $E_k f(x) \leq \lambda$. Since $f \in L^1(\mathbb{R}^n)$, by the second part of Theorem 3.3, $E_k f(x) \longrightarrow f(x)$, as $k \longrightarrow \infty$, for almost every $x \in \mathbb{R}^n$. We therefore have $f(x) \leq \lambda$ almost every $x \notin \bigcup\limits_{j\in\mathbb{N}} Q_j$.

Lastly, by definition of the sets Ω_k, the average of f over Q_j is greater than λ. This is the first inequality in (3). Now suppose $x \in \Omega_k$. Then by the definition of the sets Ω_k, $E_{k-1} f(x) \leq \lambda$. If $Q^{'}$ is the unique cube in $\mathcal{Q}_{k-1}$ containing x then $\frac{1}{|Q'|}\int_{Q'} f(x)\mathrm{d}x \leq \lambda$. Now,

$$\frac{1}{|Q_j|}\int\limits_{Q_j} f(y)\mathrm{d}y \leq \frac{|Q^{'}|}{|Q_j||Q'|}\int\limits_{Q^{'}} f(y)\mathrm{d}y \leq 2^n\lambda.$$

This proves (3). □

The decomposition of $\mathbb{R}^n$ given by the previous theorem allows us to decompose the function f as the sum of two functions, g and b, defined by

$$g(x) = \begin{cases} f(x), & \text{if } x \notin \bigcup\limits_j Q_j. \\ \frac{1}{|Q_j|}\int\limits_{Q_j} f(y)\mathrm{d}y, & \text{if } x \in Q_j. \end{cases} \tag{3.8}$$

And

$$b(x) = \sum_j b_j(x),$$

where,

$$b_j(x) = \left(f(x) - \frac{1}{|Q_j|}\int_{Q_j} f(y)\mathrm{d}y \right) \chi_{Q_j}(x) = f(x) - g(x).$$

Then $g(x) \leq 2^n\lambda$ almost everywhere and b_j is supported on Q_j with $\int\limits_{\mathbb{R}^n} b_j(x)\mathrm{d}x = 0$. The functions g and b defined above are called "good" and "bad" parts respectively of the given function f. It is easy to see that the function g has the same integral as that of f.

Lemma 3.5. *If f is a non negative locally integrable function, then*

$$|\{x \in \mathbb{R}^n : M'f(x) > 4^n\lambda\}| \leq 2^n |\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}|. \tag{3.9}$$

Proof. As in Theorem 3.4, we form the decomposition

$$\{x \in \mathbb{R}^n : M_d f(x) > \lambda\} = \bigcup_{j \in \mathbb{N}} Q_j,$$

where each Q_j is a dyadic cube. Let $2Q_j$ be the cube with the same center as Q_j and whose sides are twice as long. We claim that

$$\{x \in \mathbb{R}^n : M'f(x) > 4^n\lambda\} \subseteq \bigcup_{j \in \mathbb{N}} 2Q_j.$$

Let us fix $x \notin \bigcup_j 2Q_j$ and let Q be any cube centred at x. Let $l(Q)$ denote the length of the cube Q and $k \in \mathbb{Z}$ such that $2^{k-1} \leq l(Q) < 2^k$. Then Q intersects m many dyadic cubes in $\mathcal{Q}_k$; say them $R_1, R_2, \cdots, R_m$. Observe that $m \leq 2^n$. We have explained this fact taking $k = 0$ and $n = 2$ in Figure 3.13.

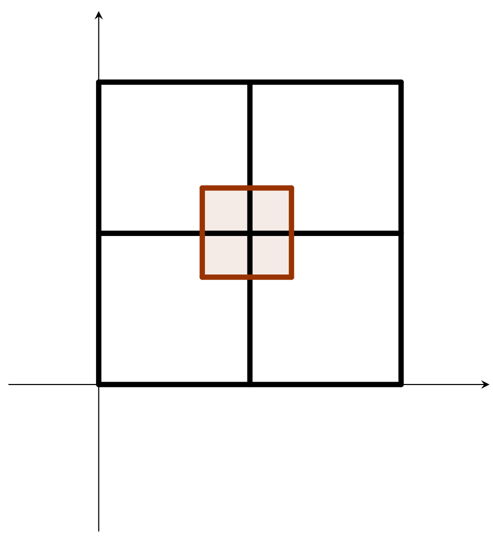

Figure 3.13: Q intersects 4 dyadic cubes

None of these cubes is contained in any of the $Q_j's$, for otherwise we would have $x \in \bigcup_{j \in \mathbb{N}} 2Q_j$. Hence, the average of f on each R_i is at most λ, and so

$$\frac{1}{|Q|}\int_Q f(y)\mathrm{d}y = \frac{1}{|Q|}\sum_{i=1}^{m}\int_{Q \bigcap R_i} f(y)\mathrm{d}y$$

$$\leq \sum_{i=1}^{m} \frac{2^{kn}}{|Q||R_i|} \int_{R_i} f(y)\mathrm{d}y$$
$$\leq 2^n m\lambda$$
$$\leq 4^n\lambda.$$

As the above inequality is true for any cube Q centered at x, we have $M'f(x) \leq 4^n\lambda$. Therefore, $x \notin \{y \in \mathbb{R}^n : M'f(y) > 4^n\lambda\}$. That is, we have

$$\{x \in \mathbb{R}^n : M'f(x) > 4^n\lambda\} \subseteq \bigcup_j 2Q_j.$$

Therefore,

$$\begin{aligned}
|\{x \in \mathbb{R}^n : M'f(x) > 4^n\lambda\}| &\leq |\bigcup_j 2Q_j| \\
&\leq 2^n|\bigcup_j Q_j| \\
&= 2^n|\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}|.
\end{aligned}$$

□

3.3 $L^p - L^p$ boundedness of Hardy-Littlewood Maximal operator

In this section we wish to study the L^p boundedness of the Hardy-Littlewood maximal operator. The idea we use is to get weak type boundedness for $p = 1$, and strong boundedness for $p = \infty$. For $1 < p < \infty$, the result then follows from the Marcinkiewicz interpolation theorem. As mentioned earlier we can replace Hardy-Littlewood operator by the cubic maximal operator, and the boundedness result won't change. In what follows, we use the centered cubic maximal function M'. With the help of the Lemma 3.5, we now prove weak $(1, 1)$ boundedness of cubic maximal function.

Theorem 3.6. *The maximal operator M' is weak $(1, 1)$.*

Proof. From Lemma 3.5 we have,

$$|\{x \in \mathbb{R}^n : M'f(x) > \lambda\}| \leq 2^n |\{x \in \mathbb{R}^n : M_d f(x) > 4^{-n}\lambda\}|.$$

Using the weak $(1,1)$ inequality of the dyadic maximal operator (Theorem 3.3), we get

$$\begin{aligned} |\{x \in \mathbb{R}^n : M'f(x) > \lambda\}| &\leq 2^n |\{x \in \mathbb{R}^n : M_d f(x) > 4^{-n}\lambda\}| \\ &\leq 2^n \frac{1}{4^{-n}\lambda} ||f||_1 \\ &= \frac{8^n}{\lambda} ||f||_1. \end{aligned}$$

Hence M' is weak $(1,1)$. □

Remark 3.1. We have shown that the maximal operators M and $M^{'}$ are essentially interchangeable due to Inequality (3.2). Therefore we can say that

$$\{x \in \mathbb{R}^n : Mf(x) > \lambda\} \subseteq \left\{x \in \mathbb{R}^n : M'f(x) > \frac{\lambda}{C_n}\right\}.$$

Now, using Theorem 3.6,

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \leq \left|\left\{x \in \mathbb{R}^n : M'f(x) > \frac{\lambda}{C_n}\right\}\right| \leq \frac{C_n 8^n}{\lambda} ||f||_1.$$

That is, the Hardy-Littlewood maximal operator M is weak $(1,1)$.

An important consequence of the weak $(1,1)$ inequality is the Lebesgue differentiation Theorem. It is a generalization, to higher dimensions, of the first fundamental theorem of calculus.

Corollary 3.7 (Lebesgue's differentiation)**.** *For any $f \in L^1_{loc}(\mathbb{R}^n)$ we have*

$$\lim_{r \longrightarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\ \mathrm{d}y = f(x),$$

for almost every x in $\mathbb{R}^n$.

Proof. Let $f \in C(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ be fixed. Then for given $\epsilon > 0$, $\exists\, r_0 > 0$ such that for any $r < r_0$, and $y \in B(x,r)$ we have $|f(x) - f(y)| < \epsilon$. Now,

$$\begin{aligned} \left|\frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\ \mathrm{d}y - f(x)\right| &= \left|\frac{1}{|B(x,r)|} \int_{B(x,r)} (f(y) - f(x))\ \mathrm{d}y\right| \\ &\leq \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)|\ \mathrm{d}y < \epsilon. \end{aligned}$$

Therefore for $f \in C(\mathbb{R}^n)$, we have

$$\lim_{r \longrightarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \mathrm{d}y = f(x),$$

for every $x \in \mathbb{R}^n$. We know from Theorem 3.6 that the Hardy-Littlewood maximal operator is weak $(1,1)$. Hence, by using a similar argument as given in the second part of the Theorem 3.3 we have the desired result. □

With the weak $(1,1)$ inequality of the Hardy-Littlewood maximal operator we now see its strong L^p-boundedness. As discussed in the beginning of this section, the result is an easy consequence of the Marcinkiewicz interpolation theorem.

Theorem 3.8. *The operator M is strong (p,p), for $1 < p \leq \infty$.*

Proof. First, notice that for any $f \in L^\infty(\mathbb{R}^n)$, we have for any $r > 0$, $\frac{1}{|B_r|} \int\limits_{B_r} |f(x-y)|\, dy \leq \sup\limits_{r>0} \frac{1}{|B_r|} \int\limits_{B_r} ||f||_\infty \mathrm{d}y = ||f||_\infty$. Hence $Mf(x) \leq ||f||_\infty$ for almost every $x \in \mathbb{R}^n$. That is, $||Mf||_\infty \leq ||f||_\infty$. We have already shown that the operator M is weak $(1,1)$. Hence by using Marcinkiewicz Interpolation Theorem, we have that M is strong (p,p), for any $1 < p \leq \infty$. □

It is natural to ask whether Lemma 3.5 can be used to get strong L^1-boundedness of M. The next result shows that it is not possible.

Lemma 3.9. *For $f(\neq 0) \in L^1(\mathbb{R}^n)$, $Mf \notin L^1(\mathbb{R}^n)$.*

Proof. Let $f(\neq 0) \in L^1(\mathbb{R}^n)$. Define $f_j = f\chi_{B(0,j)}$. Then, $f_j \longrightarrow f$ in $L^1(\mathbb{R}^n)$ and therefore we can find j_0 such that, $f_{j_0} \not\equiv 0$. Now note that $B(0,j_0) \subset B(x,|x|+j_0)$. This is shown in Figure 3.14.

Therefore,

$$\begin{aligned} Mf(x) \geq Mf_{j_0}(x) &= \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f_{j_0}(y)|\, \mathrm{d}y \\ &\geq \frac{1}{|B(x,|x|+j_0)|} \int_{B(x,|x|+j_0)} |f_{j_0}(y)|\, \mathrm{d}y \end{aligned}$$

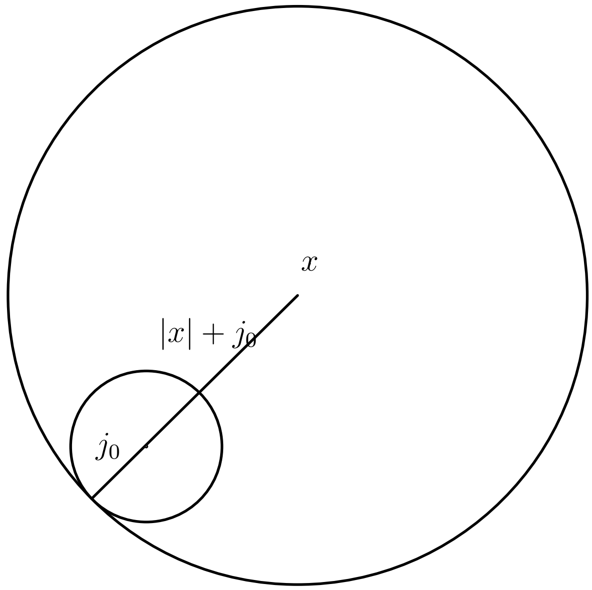

Figure 3.14: $B(0, j_0) \subset B(x, |x| + j_0)$

$$= \frac{1}{B(x, |x| + j_0)} \int_{B(x,|x|+j_0)\cap B(0,j_0)} |f(y)| \, \mathrm{d}y$$
$$= c_n(|x| + j_0)^{-n} \int_{B(0,j_0)} |f(y)|\mathrm{d}y$$
$$= c_n(|x| + j_0)^{-n} ||f_{j_0}||_{L^1(\mathbb{R}^n)}.$$

This implies that

$$\int_{\mathbb{R}^n} Mf(x)\mathrm{d}x \geq c_n||f_{j_0}||_1 \int_{\mathbb{R}^n} (|x| + j_0)^{-n}\mathrm{d}x = \infty.$$

Hence $Mf \notin L^1(\mathbb{R}^n)$. □

It is also natural to ask about the admissible range of $q \geq 1$ such that $M : L^p(\mathbb{R}^n) \longrightarrow L^q(\mathbb{R}^n)$ is bounded.

We discuss it in the next result.

Proposition 3.10. *The operator M is not bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ unless $p = q$.*

Proof. Let $f \in L^p(\mathbb{R}^n)$ and let $f_\lambda(x) := f(\lambda x)$, for a fixed $\lambda > 0$. Then,

$$||f_\lambda||_p = \left(\int\limits_{\mathbb{R}^n} |f(\lambda x)|^p \mathrm{d}x \right)^{1/p}.$$

Taking $\lambda x = y$ we get

$$||f_\lambda||_p = \lambda^{-n/p} \left(\int\limits_{\mathbb{R}^n} |f(y)|^p \mathrm{d}x \right)^{1/p} = \lambda^{-n/p} ||f||_p.$$

Note that

$$\begin{aligned} Mf_\lambda(x) &= \sup_{R>0} \frac{1}{|B(0,R)|} \int_{B(0,R)} |f_\lambda(x-y)| \mathrm{d}y \\ &= \sup_{R>0} \frac{1}{|B(0,R)|} \int_{B(0,R)} |f(\lambda x - \lambda y)| \mathrm{d}y \\ &= Mf(\lambda x) \\ &= (Mf)_\lambda(x). \end{aligned}$$

Therefore $||Mf_\lambda(x)||_q = \lambda^{-n/q} ||Mf||_q$. If M is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, we must have for all $\lambda > 0$,

$$||Mf_\lambda||_q \leq C ||f_\lambda||_p.$$

That is,

$$\lambda^{-n/q} ||Mf||_q \leq C \lambda^{-n/p} ||f||_p.$$

Equivalently, for all $\lambda > 0$ we require

$$\lambda^{n(1/p-1/q)} \leq C \frac{||f||_p}{||Mf||_q} < \infty.$$

However, this is only possible when $p = q$. □

3.4 Rectangular Maximal Operator

In this section we discuss about a special type of maximal operator called Rectangular maximal operator. In the next chapter we see some special weights related to the rectangular maximal operator that are ultimately useful in our study of Littlewood-Paley theory. In this section we only give its definition.

First by a rectangle in $\mathbb{R}^n$ we mean the set $R(h_1, \cdots, h_2) = [-h_1, h_1] \times \cdots \times$

$[-h_n, h_n]$, where $h_1, \cdots, h_n$ are non-negative real numbers.

Definition 3.6 (Strong maximal function)**.** *Let $f \in L^1_{loc}(\mathbb{R}^n)$, strong maximal function of f is defined by*

$$M_s f(x) := \sup_{h_1,\cdots,h_n>0} \frac{1}{|R(h_1,\cdots,h_n)|} \int\limits_{R(h_1,\cdots,h_n)} |f(x-y)| \, \mathrm{d}y.$$

We notice that any cube Q, is a rectangle. Hence, the strong maximal function M_s is pointwise larger than M'. That is, $\forall x \in \mathbb{R}^n$, we have $M'f(x) \leq M_s f(x)$. It can be shown that the strong maximal operator M_s is bounded on $L^p(\mathbb{R}^n)$ for $p > 1$ but M_s is not weak $(1,1)$. We refer the reader to [7] for further details on this topic.

3.5 Sharp maximal operator and BMO space

This section is dedicated to the study of another useful maximal operator. The idea is to capture the mean deviation of a given function from its average behaviour. For a given $f \in L^1_{loc}(\mathbb{R}^n)$, we denote the average of f on a cube Q by f_Q, that is,

$$f_Q = \frac{1}{|Q|} \int\limits_Q f(y) \, \mathrm{d}y.$$

This lead us to the definition of the sharp maximal function.

Definition 3.7 (Sharp Maximal function)**.** *Let $f \in L^1_{loc}(\mathbb{R}^n)$. We define the sharp maximal function of f by*

$$M^{\#} f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int\limits_Q |f(y) - f_Q| \, \mathrm{d}y,$$

where the supremum is taken over all cubes Q containing x.

We collect all the functions which do not deviate far away from their average. We call this collection the BMO space.

Definition 3.8 (BMO space)**.** *Let $f \in L^1_{loc}(\mathbb{R}^n)$. We say f has bounded mean oscillation if the function $M^{\#}f$ is bounded. The space of such functions is denoted by BMO$(\mathbb{R}^n)$. That is,*

$$BMO(\mathbb{R}^n) = \{f \in L^1_{loc}(\mathbb{R}^n) : M^{\#}f \in L^\infty(\mathbb{R}^n)\}.$$

First we notice that BMO$(\mathbb{R}^n) \neq \emptyset$, since for any constant function "C", we have $M^{\#}C = 0$. Also we see that BMO$(\mathbb{R}^n)$ is a vector space. Indeed is $f_1, f_2 \in$ BMO$(\mathbb{R}^n)$ and $\alpha, \beta \in \mathbb{C}$, then,

$$\begin{aligned}
&M^{\#}(\alpha f_1 + \beta f_2)\\
&= \sup_{x \ni Q} \frac{1}{|Q|} \int\limits_Q \left| \alpha f_1(y) + \beta f_2(y) - \left(\frac{1}{|Q|} \int\limits_Q (\alpha f_1(z) + \beta f_2(z)) \, \mathrm{d}z \right) \right| \mathrm{d}y\\
&\leq \alpha M^{\#} f_1(x) + \beta M^{\#} f_2(x).
\end{aligned}$$

We can define a norm on BMO$(\mathbb{R}^n)$ by

$$||f||_* = ||M^{\#}f||_\infty.$$

The function $||\cdot||_*$ is not a true norm since it can not separate constant function from one another. However, by taking equivalence classes of functions defined upto addition of constant, $||\cdot||_*$ become a norm. We begin with the following easy property of $M^{\#}$ and $||\cdot||_*$.

Proposition 3.11. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then,*

$$\frac{1}{2}||f||_* \leq \sup_Q \inf_{a \in \mathbb{C}} \frac{1}{|Q|} \int\limits_Q |f(y) - a| \, \mathrm{d}y \leq ||f||_*, \tag{3.10}$$

$$M^{\#}(|f|)(x) \leq 2M^{\#}f(x). \tag{3.11}$$

Proof. We have for any $a \in \mathbb{C}$ we can write

$$\int\limits_Q |f(x) - f_Q| \, \mathrm{d}x \leq \int\limits_Q |f(x) - a| \, \mathrm{d}x + \int\limits_Q |a - f_Q| \, \mathrm{d}x. \tag{3.12}$$

We notice that,

$$|a - f_Q| = \left| a - \frac{1}{|Q|} \int\limits_Q f(x) \, \mathrm{d}x \right|$$

$$= \left| \frac{1}{|Q|} \int_Q (a - f(x)) \, \mathrm{d}x \right|$$

$$\leq \frac{1}{|Q|} \int_Q |a - f(x)| \, \mathrm{d}x.$$

Therefore,

$$\int_Q |a - f_Q| \, \mathrm{d}x \leq \int_Q \left(\frac{1}{|Q|} \int_Q |a - f(x)| \, \mathrm{d}x \right) \, \mathrm{d}y = \int_Q |a - f(x)| \, \mathrm{d}x.$$

Using the above inequality, Inequality (3.12) becomes

$$\int_Q |f(x) - f_Q| \, \mathrm{d}x \leq 2 \int_Q |f(x) - a| \, \mathrm{d}x.$$

Now, dividing both side by $|Q|$ we get

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| \, \mathrm{d}x \leq \frac{2}{|Q|} \int_Q |f(x) - a| \, \mathrm{d}x. \tag{3.13}$$

Note that $\forall a \in \mathbb{C}$, Inequality (3.13) is true and left hand side of this inequality is free from a. Therefore, we have

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| \, \mathrm{d}x \leq 2 \inf_{a \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(x) - a| \, \mathrm{d}x.$$

Taking supremum over all cubes containing a point y we get

$$\sup_{Q \ni y} \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, \mathrm{d}x \leq 2 \sup_{Q \ni y} \inf_{a \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(x) - a| \, \mathrm{d}x.$$

Therefore, by the definition of $M^{\#}$, we have,

$$M^{\#} f(y) \leq 2 \sup_{Q \ni y} \inf_{a \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(x) - a| \, \mathrm{d}x.$$

Hence,

$$||f||_* = ||M^{\#} f(y)||_\infty \leq 2 \sup_{Q \ni y} \inf_{a \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(x) - a| \, \mathrm{d}x. \tag{3.14}$$

Now in Inequality (3.14), taking $a = f_Q$, we get

$$\frac{1}{2} ||f||_* \leq \sup_{Q \ni y} \inf_{a \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(x) - a| \, \mathrm{d}x \leq \sup_{Q \ni y} \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, \mathrm{d}x$$

$$= M^{\#} f(y)$$

$$\leq ||f||_*.$$

The fact that $M^{\#}|f|(x) \leq 2M^*f(x)$, follows from above. $\square$

We now get a relation between L^p-norm of $M_d f$ and $M^{\#}f$, for $f \in L^p(\mathbb{R}^n)$. For the same we required the following "good-λ inequality".

Lemma 3.12. *If $f \in L^{p_0}(\mathbb{R}^n)$ for some $1 \leq p_0 < \infty$, then for any $\gamma > 0$ and $\lambda > 0$,*

$$|\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda, M^{\#}f(x) \leq \gamma\lambda\}| \leq 2^n\gamma|\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}|$$

Proof. With out loss of generality we may assume that f is non-negative. Let λ, $\gamma > 0$ be fixed. Let us form the Calderón-Zygmund decomposition of the function f at the height λ. Then by Theorem 3.3 we have

$$\{x \in \mathbb{R}^n : M_d f(x) > \lambda\} = \bigcup_{k\in\mathbb{Z}} \Omega_k = \bigcup_{j\in\mathbb{N}} Q_j,$$

where $\{Q_j\}_{j\in\mathbb{N}}$ is a family of disjoint dyadic cubes. Suppose $Q_{j_0} \in \mathcal{Q}_{k_0}$ be one of such cube for some $k_0 \in \mathbb{Z}$. Clearly, $Q_{j_0} \subseteq \Omega_{k_0}$. We first prove that

$$|\{x \in Q_{j_0} : M_d f(x) > 2\lambda, M^{\#}f(x) \leq \gamma\lambda\}| \leq 2^n\gamma|Q_{j_0}|.$$

Suppose Q' be the dyadic cube whose sides are twice as long and $Q_{j_0} \subset Q'$. Then $Q' \in \mathcal{Q}_{k_0-1}$. Hence by the definition of the set Ω_{k_0}, $E_{k_0-1}f(x) \leq \lambda$, for all $x \in Q_{j_0}$. Therefore

$$f_{Q'} = \frac{1}{|Q'|}\int\limits_{Q'} f(y)\ \mathrm{d}y \leq \lambda.$$

Further, if $x \in Q_{j_0}$ and $M_d f(x) > 2\lambda$ then $M_d(f\chi_{Q_{j_0}})(x) > 2\lambda$. This can be proved as follows. If $M_d f(x) > 2\lambda$ there exists a dyadic cube $Q_{j_1} \in \mathcal{Q}_{k_1}$ with $x \in Q_{j_1}$ such that

$$\frac{1}{|Q_{j_1}|}\int\limits_{Q_{j_1}} f(y)\ \mathrm{d}y > 2\lambda > \lambda. \tag{3.15}$$

Therefore, by the definition of Ω_{k_0}, we have $k_1 \geq k_0$. So, $Q_{j_1} \subseteq Q_{j_0}$. We have the following

$$\frac{1}{|Q_{j_1}|}\int\limits_{Q_{j_1}} f(y)\ \mathrm{d}y = \frac{1}{|Q_{j_1}|}\int\limits_{Q_{j_1}} f\chi_{Q_{j_0}}(y)\ \mathrm{d}y > 2\lambda.$$

Hence, if $x \in Q_{j_0}$ and $M_d f(x) > 2\lambda$ we have $M_d(f\chi_{Q_{j_0}})(x) > 2\lambda$. Now for $x \in \{y \in Q_{j_0} : M_d f(y) > 2\lambda\}$, we have

$$M_d(f\chi_{Q_{j_0}})(x) \leq M_d(f - f_{Q'})\chi_{Q_{j_0}}(x) + f_{Q'} M_d(\chi_{Q_{j_0}})(x).$$

Since $\forall x \in Q_{j_0}$, $M_d(\chi_{Q_{j_0}})(x) = 1$, from the above inequality, we get

$$M_d(f - f_{Q'})\chi_{Q_{j_0}}(x) \geq M_d(f\chi_{Q_{j_0}})(x) - f_{Q'} > 2\lambda - \lambda = \lambda.$$

Therefore,

$$\{x \in Q_{j_0} : M_d f(x) > 2\lambda\} \subseteq \{x \in Q_{j_0} : M_d((f - f_{Q'})\chi_{Q'})(x) > \lambda\}.$$

From the weak $(1,1)$ inequality of dyadic maximal function we have for any $x \in Q_{j_0}$,

$$\begin{aligned}
|\{x \in Q_{j_0} : M_d((f - f_{Q'})\chi_{Q_{j_0}})(x) > \lambda\}| &\leq \frac{1}{\lambda} \int\limits_{Q_{j_0}} |f(y) - f_{Q'}| \,\mathrm{d}y \\
&\leq \frac{2^n |Q_{j_0}|}{\lambda} \frac{1}{|Q'|} \int\limits_{Q'} |f(y) - f_{Q'}| \,\mathrm{d}y \\
&= \frac{2^n |Q_{j_0}|}{\lambda} M^{\#} f(x).
\end{aligned}$$

Therefore,

$$|\{x \in Q_{j_0} : M_d((f - f_{Q'})\chi_{Q_{j_0}})(x) > \lambda\}| \leq \frac{2^n |Q_{j_0}|}{\lambda} \inf_{x \in Q_{j_0}} M^{\#} f(x).$$

Hence,

$$|\{x \in Q_{j_0} : M_d f(x) > 2\lambda\}| \leq \frac{2^n |Q_{j_0}|}{\lambda} \inf_{x \in Q_{j_0}} M^{\#} f(x).$$

As $\{x \in Q_{j_0} : M_d f(x) > 2\lambda, M^{\#} f(x) \leq \gamma\lambda\} \subseteq \{x \in Q_{j_0} : M_d f(x) > 2\lambda\}$ we have

$$\begin{aligned}
|\{x \in Q_{j_0} : M_d f(x) > 2\lambda, M^{\#} f(x) \leq \gamma\lambda\}| &\leq |\{x \in Q_{j_0} : M_d f(x) > 2\lambda\}| \\
&\leq \frac{2^n |Q_{j_0}|}{\lambda} \inf_{x \in Q_{j_0}} M^{\#} f(x) \\
&\leq \frac{2^n |Q_{j_0}|}{\lambda} \gamma\lambda \\
&\leq 2^n \gamma |Q_{j_0}|.
\end{aligned}$$

Note that

$$\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda, M^{\#} f(x) \leq \gamma\lambda\}$$

$$= \left\{ x \in \bigcup_{j\in\mathbb{N}} Q_j : M_d f(x) > 2\lambda, M^{\#} f(x) \leq \gamma\lambda \right\}$$
$$\cup \left\{ x \notin \bigcup_{j\in\mathbb{N}} Q_j : M_d f(x) > 2\lambda, M^{\#} f(x) \leq \gamma\lambda \right\}.$$

If $x \notin \bigcup_{j\in\mathbb{N}} Q_j$ then $M_d f(x) \leq \lambda$. So, the set $\left\{ x \notin \bigcup_{j\in\mathbb{N}} Q_j : M_d f(x) > 2\lambda, M^{\#} f(x) \leq \gamma\lambda \right\} = \emptyset$. Therefore,

$$\begin{aligned} &|\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda, M^{\#} f(x) \leq \gamma\lambda\}| \\ &\qquad = \left| \left\{ x \in \bigcup_{j\in\mathbb{N}} Q_j : M_d f(x) > 2\lambda, M^{\#} f(x) \leq \gamma\lambda \right\} \right|. \end{aligned}$$

Because $\bigcup_{j\in\mathbb{N}} Q_j$ is a disjoint union, we get

$$\begin{aligned} &|\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda, M^{\#} f(x) \leq \gamma\lambda\}| \\ &\qquad = \sum_{j\in\mathbb{N}} \left| \left\{ x \in Q_j : M_d f(x) > 2\lambda, M^{\#} f(x) \leq \gamma\lambda \right\} \right| \\ &\qquad \leq \sum_{j\in\mathbb{N}} 2^n \gamma |Q_j| \\ &\qquad = 2^n \gamma \sum_{j\in\mathbb{N}} |Q_j| \\ &\qquad = 2^n \gamma |\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}|. \end{aligned}$$

□

With the help of above lemma we can show that L^p norm of M_d bounded above by L^p norm of $M^{\#}$.

Lemma 3.13. *If $1 \leq p_0 \leq p < \infty$ and $f \in L^{p_0}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |M_d f(x)|^p \, \mathrm{d}x \leq C \int_{\mathbb{R}^n} |M^{\#} f(x)|^p \, \mathrm{d}x.$$

Proof. For any $N > 0$, let us first define the following

$$I_N = \int_0^N p\lambda^{p-1} |\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}| \, \mathrm{d}\lambda. \tag{3.16}$$

Note that I_N is finite, because

$$\begin{aligned}I_N &= \int_0^N p\lambda^{p_0-1-p_0+p}|\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}| \, \mathrm{d}\lambda \\ &\leq \frac{N^{p-p_0}p}{p_0} \int_0^N p_0 \lambda^{p_0-1}|\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}| \, \mathrm{d}\lambda.\end{aligned}$$

Since $f \in L^{p_0}(\mathbb{R}^n)$ and M_d is strong (p_0, p_0),

$$||M_d f||_{p_0}^{p_0} = p_0 \int_0^\infty \lambda^{p_0-1}|\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}| \, \mathrm{d}\lambda < \infty.$$

Hence, $I_N < \infty$. Now doing a substitution $\lambda = 2s$, in Equation (3.16) we get

$$I_N = 2^p \int_0^{N/2} ps^{p-1}|\{x \in \mathbb{R}^n : M_d f(x) > 2s\}| \, \mathrm{d}s.$$

Equivalently,

$$I_N = 2^p \int_0^{N/2} p\lambda^{p-1}|\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda\}| \, \mathrm{d}\lambda. \tag{3.17}$$

Note that

$$\begin{aligned}\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda\} \subset \{x &\in \mathbb{R}^n : M_d f(x) > 2\lambda, M^{\#} f(x) \leq \gamma\lambda\} \\ &\cup \{x \in \mathbb{R}^n : M^{\#} f(x) > \gamma\lambda\}.\end{aligned}$$

Therefore,

$$\begin{aligned}|\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda\}| < |\{x &\in \mathbb{R}^n : M_d f(x) > 2\lambda, M^{\#} f(x) \leq \gamma\lambda\}| \\ &+ |\{x \in \mathbb{R}^n : M^{\#} f(x) > \gamma\lambda\}|.\end{aligned}$$

Using the above inequality in Equation (3.17), we get,

$$\begin{aligned}I_N &= 2^p \int_0^{N/2} p\lambda^{p-1}|\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda\}| \, \mathrm{d}\lambda \\ &\leq 2^p \int_0^{N/2} p\lambda^{p-1}|\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda, M^{\#} f(x) \leq \gamma\lambda\}| \, \mathrm{d}\lambda \\ &+ \int_0^{N/2} p\lambda^{p-1}|\{x \in \mathbb{R}^n : M^{\#} f(x) > \gamma\lambda\}| \, \mathrm{d}\lambda.\end{aligned}$$

By using Lemma 3.12, for the first part of the right hand side and doing a change of variable in the second part of right hand side we arrive at

$$\begin{aligned} I_N &= 2^{n+p}\gamma \int_0^{N/2} p\lambda^{p-1}|\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}| \, \mathrm{d}\lambda \\ &+ \frac{2^p}{\gamma^p} \int_0^{\gamma N/2} p\lambda^{p-1}|\{x \in \mathbb{R}^n : M^{\#} f(x) > \lambda| \, \mathrm{d}\lambda \\ &= 2^{n+p}\gamma I_{N/2} + \frac{2^p}{\gamma^p} \int_0^{\gamma N/2} p\lambda^{p-1}|\{x \in \mathbb{R}^n : M^{\#} f(x) > \lambda| \, \mathrm{d}\lambda \\ &\le 2^{n+p}\gamma I_N + \frac{2^p}{\gamma^p} \int_0^{\gamma N/2} p\lambda^{p-1}|\{x \in \mathbb{R}^n : M^{\#} f(x) > \lambda| \, \mathrm{d}\lambda. \end{aligned}$$

Now choosing $\gamma = \frac{1}{2^{1+p+n}}$, we get

$$\frac{1}{2} I_N \le \frac{2^p}{\gamma^p} \int_0^{\gamma N/2} p\lambda^{p-1}|\{x \in \mathbb{R}^n : M^{\#} f(x) > \lambda| \, \mathrm{d}\lambda.$$

We have $\gamma N/2 = N/2^{2+p+n} < N$. So, we have

$$I_N \le \frac{2^{p+1}}{\gamma^p} \int_0^{N} p\lambda^{p-1}|\{x \in \mathbb{R}^n : M^{\#} f(x) > \lambda| \, \mathrm{d}\lambda.$$

Note that

$$\int_0^{N} p\lambda^{p-1}|\{x \in \mathbb{R}^n : M^{\#} f(x) > \lambda| \, \mathrm{d}\lambda \le ||M^{\#} f||_p^p.$$

So for all $N > 0$, we have

$$I_N = \int_0^{N} p\lambda^{p-1}|\{x \in \mathbb{R}^n : M^{\#} f(x) > \lambda| \, \mathrm{d}\lambda \le \frac{2^{p+1}}{\gamma^p} ||M^{\#} f||_p^p.$$

By taking $N \longrightarrow \infty$, we get,

$$||M_d f||_p^p = \int_0^{\infty} p\lambda^{p-1}|\{x \in \mathbb{R}^n : M^{\#} f(x) > \lambda| \, \mathrm{d}\lambda \le \frac{2^{p+1}}{\gamma^p} ||M^{\#} f||_p^p.$$

$\square$

CHAPTER 4

Muckenhoupt weights

In the previous chapter, we have seen a variety of maximal functions. We have also seen that the Hardy-Littlewood maximal function is L^p-bounded. This chapter deals with boundedness of these operators on weighted Lebesgue spaces. Our goal here is to characterize all positive (measurable) functions w on $\mathbb{R}^n$ such that the Hardy-Littlewood maximal operator M is bounded on $L^p(w)$, for any $1 \leq p < \infty$. We won't worry about the case L^∞, since $L^\infty(w) = L^\infty(\mathbb{R}^n)$, for any positive measurable function w.

The study of such weights started more than half a century ago, when Rosenblum in [20] first gave a condition on such weight functions. However, the author studied the condition in a very specific context of Fourier series. Muckenhoupt, in [19], characterized the condition in the one-dimensional case. Muckenhoupt's work was generalized to higher dimensions by Coifman and Fefferman (see [5]). A lot of related work and surveys can be found in [25]. In this chapter, we exposit some of these works on the general Euclidean space $\mathbb{R}^n$.

4.1 A Weighted Norm Inequality

In this section we see that for a given positive function w maximal operator is bounded from $L^p(Mw)$ space into $L^p(w)$ space. We start with a few simple lemmata.

Lemma 4.1. *If $f \in L^1(Mw)$ then $f_j = f\chi_{B(0,j)}$ is a sequence of integrable function which increases pintwise to f.*

Proof. Since $f \in L^1(Mw)$, we have

$$\int\limits_{\mathbb{R}^n} |f(x)|Mw(x)\ \mathrm{d}x < \infty.$$

First, we prove that there exists a constant $C > 0$ such that $Mw(x) > C$, $\forall x \in B(0,j)$. If this is not the case, then $\forall\ C > 0$, $\exists\ x \in B(0,j)$ (depending on C) such that $Mw(x) < C$. Therefore there is a sequence $(x_m)_{m\in\mathbb{N}}$ such that $Mw(x_m) \longrightarrow 0$ as $m \longrightarrow \infty$. As $B(0,j)$ is a pre-compact set, $(x_m)_{m\in\mathbb{N}}$ has a convergent subsequence. By passing on to the subsequence, we may assume that $x_m \longrightarrow x_0$ as $m \longrightarrow \infty$. Therefore, for all $\epsilon > 0$, $\exists\ n_1 \in \mathbb{N}$ such that, $\forall m > n_1$, $Mw(x_m) < \epsilon$. Notice that for a fixed $r > 0$, $\exists\ n_0 \in \mathbb{N}$ such that $B(x_0,r) \subseteq B(x_m,2r)$. Now suppose $n_2 = \max\{n_0,n_1\}$. Then $\forall m > n_2$,

$$\frac{1}{|B(x_m,2r)|}\int\limits_{B(x_m,2r)} w(x)\ \mathrm{d}x < \epsilon.$$

Now we notice that

$$\begin{aligned}\frac{1}{|B(x_m,2r)|}\int\limits_{B(x_m,2r)} w(x)\ \mathrm{d}x &\geq \frac{1}{C(2r)^n}\int\limits_{B(x_0,r)} w(x)\ \mathrm{d}x\\ &= \frac{1}{2^n|B(x_0,r)|}\int\limits_{B(x_0,r)} w(x)\ \mathrm{d}x.\end{aligned}$$

Therefore,

$$\frac{1}{2^n|B(x_0,r)|}\int\limits_{B(x_0,r)} w(x)\ \mathrm{d}x < \epsilon.$$

As this is true for any $\epsilon > 0$,

$$\int_{B(x_0,r)} w(x)\ \mathrm{d}x = 0.$$

Since $|B(x,r)| \neq 0$, we have that $w(x) = 0$ for a.e. $x \in B(x_0,r)$. Since r was arbitrary we conclude that $w(x) = 0$ for a.e. $x \in \mathbb{R}^n$. However, this is a contradiction. Therefore $\exists\ C > 0$ such that $Mw(x) > C$, $\forall x \in B(0,j)$. Now we have

$$\int_{\mathbb{R}^n} |f(x)|\chi_{B(0,j)}(x)\ \mathrm{d}x = \int_{B(0,j)} |f(x)|\frac{Mw(x)}{Mw(x)}\ \mathrm{d}x \leq \frac{1}{C}\int_{B(0,j)} |f(x)|Mw(x)\ \mathrm{d}x < \infty.$$

□

Lemma 4.2. *Let w be a non-negative function in $L^1(\mathbb{R}^n)$. If for some $x_0 \in \mathbb{R}^n$, we have $Mw(x_0) > 0$, then $Mw(x) > 0, \forall x \in \mathbb{R}^n$.*

Proof. As $Mw(x_0) > 0$ there is $r_0 > 0$ such that $\frac{1}{|B(x_0,r_0)|}\int_{B(x_0,r_0)} w(y)\ \mathrm{d}y > 0$. Let $x \in \mathbb{R}^n$ be arbitrary. Then, there is $r_1 > 0$ such that $B(x_0,r_0) \subset B(x,r_1)$. Therefore, $\int_{B(x,r_1)} w(y)\ \mathrm{d}y > \int_{B(x_0,r_0)} w(y)\ \mathrm{d}y > 0$. Hence, $\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)} w(y)\ \mathrm{d}y > 0$

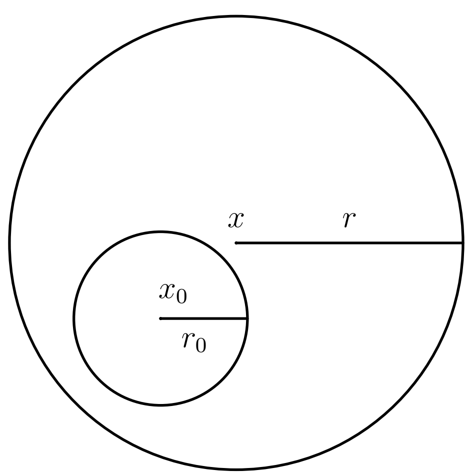

Figure 4.1: $B(x_0,r_0) \subset B(x,r_1)$

and

$$Mw(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} w(y)\ \mathrm{d}y > \frac{1}{|B(x,r_1)|} \int\limits_{B(x,r_1)} w(y)\ \mathrm{d}y > 0,$$

$\square$

We are now in a position to give a weighted norm inequality for the Hardy-Littlewood Maximal function.

Theorem 4.3. *If w is a non-negative, measurable function and $1 < p < \infty$, then there exists a constant $C_p > 0$ such that*

$$\int\limits_{\mathbb{R}^n} (Mf(x))^p\, w(x)\mathrm{d}x \leq C_p \int\limits_{\mathbb{R}^n} |f(x)|^p Mw(x)\mathrm{d}x.$$

Furthermore,

$$\int_{\{x:Mf(x)>\lambda\}} w(x)\mathrm{d}x \leq \frac{C_1}{\lambda} \int_{\mathbb{R}^n} |f(x)|Mw(x)\mathrm{d}x.$$

Proof. We show that $||Mf||_{L^\infty(w)} \leq ||f||_{L^\infty(Mw)}$ and that the weak $(1,1)$ inequality holds; the strong (p,p) inequality then follows from the Marcinkiewicz interpolation theorem. We have the following cases:

Case 1. $Mw \equiv 0$. Then, for any $r > 0$, we have

$$\frac{1}{|B_r|} \int\limits_{B_r} w\,(x-y)\,dy \leq Mw(x).$$

By Lebesgue differentiation theorem, for almost every $x \in \mathbb{R}^n$, we have

$$0 \leq w(x) = \lim_{r\longrightarrow 0^+} \frac{1}{|B_r|} \int\limits_{B_r} w\,(x-y)\,\mathrm{d}y \leq Mw(x) = 0.$$

Thus, we have $w(x) = 0$ for a.e. $x \in \mathbb{R}^n$. So, in this case the theorem holds trivially.

Case 2 : $Mw(x_0) > 0$ for some $x \in \mathbb{R}^n$. Due to Lemma 4.2, $Mw(x) > 0$. Here, we first show that $||Mf||_{L^\infty(w)} \leq ||f||_{L^\infty(Mw)}$. If $a > ||f||_{L^\infty(Mw)}$, then $Mw(\{x \in \mathbb{R}^n : |f(x)| > a\}) = 0$. This means $\int_{\{x\in\mathbb{R}^n:|f(x)|>a\}} Mw(x)\mathrm{d}x = 0$. As $Mw(x) > 0, \forall x \in \mathbb{R}^n$, we have $|\{x \in \mathbb{R}^n : |f(x)| > a\}| = 0$. Therefore, $|f(x)| \leq a$, almost everywhere. So, $Mf(x) \leq a$, almost everywhere. Hence, $|\{x \in \mathbb{R}^n : Mf(x) > a\}| = 0$. This gives us $\int_{\{x\in\mathbb{R}^n:Mf(x)>a\}} w(x)\mathrm{d}x = 0$. Equivalently,

$w(\{x \in \mathbb{R}^n : Mf(x) > a\}) = 0$, and hence, $||Mf||_{L^\infty(w)} \leq a$.

Now we can conclude that $||Mf||_{L^\infty(w)} \leq ||f||_{L^\infty(Mw)}$.

Next, we show the weak $(1,1)$ inequality. To prove this we may assume, due to Lemma 4.1, that f is non-negative and $f \in L^1(\mathbb{R}^n)$. If $(Q_j)_{j\in\mathbb{N}}$ is the Calderon-Zygmund decomposition of f at a height $\lambda > 0$, then by Theorem 3.4 we have the following

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j} f(x)\mathrm{d}x \leq 2^n\lambda. \tag{4.1}$$

Now as we proved in Lemma 3.5,

$$\{x \in \mathbb{R}^n : M'f(x) > 4^n\lambda\} \subset \bigcup_{j\in\mathbb{N}} 2Q_j.$$

Therefore,

$$w(\{x \in \mathbb{R}^n : M'f(x) > 4^n\lambda\}) \leq \sum_{j=1}^{\infty} w(2Q_j).$$

Then, we have,

$$\int_{\{x\in\mathbb{R}^n : M'f(x)>4^n\lambda\}} w(x)\mathrm{d}x \leq \sum_{j=1}^{\infty} w(2Q_j) = \sum_{j=1}^{\infty} 2^n|Q_j|\frac{1}{|2Q_j|}\int_{2Q_j} w(x)\mathrm{d}x.$$

From Inequality (4.1), we have $|Q_j| < \frac{1}{\lambda}\int\limits_{Q_j} f(y)\mathrm{d}y$. Thus,

$$\begin{aligned}
\int_{\{x\in\mathbb{R}^n : M'f(x)>4^n\lambda\}} w(x)\mathrm{d}x &\leq \sum_{j=1}^{\infty}\frac{2^n}{\lambda}\int_{Q_j} f(y)\left(\frac{1}{|2Q_j|}\int_{2Q_j} w(x)\mathrm{d}x\right)\mathrm{d}y \\
&\leq \frac{2^n}{\lambda}\sum_{j=1}^{\infty}\int_{Q_j} f(y)M^{''}w(y)\mathrm{d}y \\
&\leq \frac{2^nC}{\lambda}\int_{\bigcup\limits_{j\in\mathbb{N}} Q_j} f(y)Mw(y)\mathrm{d}y \\
&\leq \frac{2^nC}{\lambda}\int_{\mathbb{R}^n} f(y)Mw(y)\mathrm{d}y.
\end{aligned}$$

So we have $w(\{x \in \mathbb{R}^n : M'f(x) > 4^n\lambda\}) \leq \frac{2^nC}{\lambda}\int_{\mathbb{R}^n} f(y)Mw(y)\mathrm{d}y$. We know that $Mf(x) \leq C_nM'f(x)$. Hence, $w(\{x \in \mathbb{R}^n : M'f(x) > 4^n\lambda\}) \geq w(\{x \in \mathbb{R}^n : Mf(x) > C_n4^n\lambda\})$. Therefore,

$$w(\{x \in \mathbb{R}^n : Mf(x) > C_n4^n\lambda\}) \leq \frac{2^nC}{\lambda}\int_{\mathbb{R}^n} f(y)Mw(y)\mathrm{d}y.$$

Now if we replace $C_n 4^n \lambda$ by λ, we have

$$w(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \leq \frac{8^n C C_n}{\lambda} \int_{\mathbb{R}^n} f(y) Mw(y) \mathrm{d}y.$$

This shows that M is weak $(1,1)$ with weights Mw on domain and w on co-domain. The result now follows from Marcinkiewicz interpolation theorem. $\square$

Note that if for a positive function w, $Mw(x) \leq Cw(x)$ for some $C > 0$, almost everywhere then the operator M is bounded on $L^p(w)$ space. This is a sufficient condition on w. The natural question now is "what are the necessary conditions for the same?"

4.2 Definition and Properties of A_p weights

In this section we characterize the non-negative, locally integrable functions w such that the Hardy-Littlewood maximal operator is bounded on the space $L^p(w)$. To simplify our notation, throughout this section we replace our earlier definition of *Hardy-Littlewood maximal function* with non-centered cubic maximal function (Definition 3.3). Abusing notation, we use M to denote the non-centered cubic maximal function. Due to its equivalence with Hardy-Littlewood maximal function, all results discussed here follow for Hardy-Littlewood maximal function as well. We want to find a necessary condition on w for which M is bounded on $L^p(w)$ space. We know that strong boundedness implies weak boundedness. Therefore let us first assume that Mf satisfies a weighted, weak-type inequality,

$$w(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p w(x) \mathrm{d}x, \tag{4.2}$$

Let f be a non-negative function and Q be a cube such that $f(Q) := \int_Q f(x)\mathrm{d}x > 0$. Fix $0 < \lambda < \dfrac{f(Q)}{|Q|}$. If $x \in Q$ then $\lambda < \frac{1}{|Q|} \int\limits_Q f(y)\mathrm{d}y \leq \sup\limits_{Q' \ni x} \dfrac{1}{|Q'|} \int\limits_{Q'} (f\chi_Q)(y)\mathrm{d}y = M(f\chi_Q)(x)$. Therefore $Q \subset \{x \in \mathbb{R}^n : M(f\chi_Q)(x) > \lambda\}$. In Inequality (4.2), we replace f by $f\chi_Q$ to get

$$w(Q) \leq \frac{C}{\lambda^p} \int\limits_Q |f(x)|^p w(x) \mathrm{d}x. \tag{4.3}$$

As this holds for all such λ satisfying $0 < \lambda < \dfrac{f(Q)}{|Q|}$, it follows that

$$w(Q)\left(\frac{f(Q)}{|Q|}\right)^p \leq C\int\limits_Q |f(x)|^p w(x)\mathrm{d}x. \tag{4.4}$$

Now for a given measurable set $S \subset Q$, let $f = \chi_S$. Then, Inequality (4.4) becomes

$$w(Q)\left(\frac{|S|}{|Q|}\right)^p \leq Cw(S). \tag{4.5}$$

Remark 4.1. The same condition can be obtained for balls replacing M with the Hardy-Littlewood maximal function.

From Inequality (4.5) we immediately deduce the following:

1. The weight w is either identically 0 or $w > 0$ a.e. To see this we consider the following:
   If w is not positive almost everywhere, then $w = 0$ on a set of positive measure S. Then by Inequality (4.5), for every cube Q containing S, $w(Q) = 0$. So $w \equiv 0$ a.e.

2. The weight w is either locally integrable or $w = \infty$ a.e.

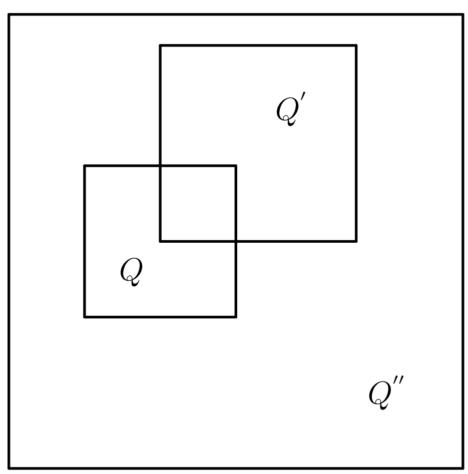

Figure 4.2: $Q, Q' \subseteq Q$”

If w is not locally integrable, assume $w(Q) = \infty$ for some cube Q, then the same is true for any larger cube containing the cube Q due to the monotonicity of measure. Let Q' be any cube and Q'' be a cube containing the both Q and Q'. Then $w(Q'') = \infty$. Now from Inequality (4.5), we have the following

$$w(Q'')\left(\frac{|Q'|}{|Q''|}\right)^p \leq Cw(Q').$$

Therefore $w(Q') = \infty$. Now let S be any set of positive measure and Q_0 be a cube such that $S \subseteq Q_0$. Since $w(Q_0) = \infty$, using Ineqaulity (4.5), we have $w(S) = \infty$. That is, $w = \infty$ a.e.

To deduce the necessary conditions for weak (p, p) boundedness of M, we consider two cases.

Case 1: $p = 1$. In this case Inequality (4.5) becomes

$$\frac{w(Q)}{|Q|} \leq C\frac{w(S)}{|S|}.$$

Let $a = \inf\{w(x) : x \in Q\}$, where "inf" is the essential infimum, that is, excluding a set of measure zero. We claim that for each $\epsilon > 0$ there exists $S_\epsilon \subseteq Q$ with $|S_\epsilon| > 0$ such that $w(x) \leq a + \epsilon$ for any $x \in S_\epsilon$. Suppose this is not true. Then $\exists\ \epsilon > 0$ such that for all $S \subset Q$ with $|S| > 0$ we have $w(x) > a + \epsilon$ for any $x \in S$. But this is same as saying $a + \epsilon$ is an essential lower bound of w on Q. Since this cannot happen, our claim is true. Hence for all $\epsilon > 0$ we have,

$$\begin{aligned}\frac{w(Q)}{|Q|} &\leq C\frac{w(S_\epsilon)}{|S_\epsilon|}\\ &= \frac{C}{|S_\epsilon|}\int\limits_{S_\epsilon} w(x)\ \mathrm{d}x.\\ &\leq \frac{C}{|S_\epsilon|}\int\limits_{S_\epsilon} (a+\epsilon)\ \mathrm{d}x.\\ &= C(a+\epsilon).\end{aligned}$$

Since $\epsilon > 0$, is arbitrary, for any cube Q,

$$\frac{w(Q)}{|Q|} \leq C\inf_{x\in Q} w(x) \leq Cw(x),\ \text{for a.e } x \in Q \tag{4.6}$$

Inequality (4.6) called the A_1 **condition**, and we refer to the weights which satisfy it as A_1 **weights**. Condition (4.6) is equivalent to

$$Mw(x) \leq Cw(x), \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{4.7}$$

Clearly, Inequality (4.7) implies Inequality (4.6). Conversely suppose that Inequality (4.6) holds and let x be such that $Mw(x) > Cw(x)$. Then there exists a cube Q with rational vertices such that $w(Q)/|Q| > Cw(x)$. Therefore, x lies in a subset of Q of measure zero. Taking the union over all such cubes (with rational vertices), we have that $Mw(x) > Cw(x)$ holds only on a set of measure 0 in $\mathbb{R}^n$.
Case 2: $1 < p < \infty$. In Inequality (4.4), let $f = w^{1-p'}\chi_Q$. Then,

$$w(Q)\left(\frac{1}{|Q|}\int\limits_Q w^{1-p'}(x)\mathrm{d}x\right)^p \leq C\int\limits_Q w^{1-p'}(x)\mathrm{d}x.$$

Equivalently,

$$\left(\frac{1}{|Q|}\int\limits_Q w(x)\mathrm{d}x\right)\left(\frac{1}{|Q|}\int\limits_Q w^{1-p'}(x)\mathrm{d}x\right)^{p-1} \leq C, \tag{4.8}$$

where C is independent of Q.

Condition (4.8) is called A_p **condition** and the weights that satisfy it are called A_p **weights**.

The following properties of A_p weights are consequences of the definition.

Proposition 4.4.

1. *$A_p \subseteq A_q$, for any $1 \leq p < q$.*
2. *For $p > 1$, $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$.*
3. *If $w_0,\ w_1 \in A_1$ then $w_0 w_1^{1-p} \in A_p$.*

Proof.

1. Let us first assume that $1 = p < q$. As $w \in A_1$ we have

$$\frac{w(Q)}{|Q|} \leq Cw(x), \text{ for a.e } x \in Q.$$

Since $1-q' \leq 0$ we get
$$\left(\frac{w(Q)}{|Q|}\right)^{1-q'} \geq Cw(x)^{1-q'}.$$
Therefore,
$$\int\limits_Q \left(\frac{w(Q)}{|Q|}\right)^{1-q'} \mathrm{d}x \geq C \int\limits_Q w(x)^{1-q'} \mathrm{d}x.$$
This implies,
$$\left(\frac{w(Q)}{|Q|}\right)^{1-q'} \geq \frac{C}{|Q|} \int\limits_Q w(x)^{1-q'} \mathrm{d}x,$$
and we have
$$\left(\frac{w(Q)}{|Q|}\right)^{(1-q')(q-1)} \geq C \left(\frac{1}{|Q|} \int\limits_Q w(x)^{1-q'} \mathrm{d}x\right)^{(q-1)}.$$
Because $(q-1)(q'-1)=1$, we arrive at
$$\left(\frac{w(Q)}{|Q|}\right)^{-1} \geq C \left(\frac{1}{|Q|} \int\limits_Q w(x)^{1-q'} \mathrm{d}x\right)^{(q-1)}.$$
Therefore,
$$\left(\frac{1}{|Q|} \int\limits_Q w(x)^{1-q'} \mathrm{d}x\right)^{(q-1)} \left(\frac{w(Q)}{|Q|}\right) \leq C.$$
Hence, $w \in A_q$ and $A_1 \subseteq A_q$. Now suppose $p > 1$. As $p < q$ we have, $q' < p'$. So, $\frac{q'-1}{p'-1} < 1$. Let $s > 1$ be such that $\dfrac{q'-1}{p'-1} + \dfrac{1}{s} = 1$. Then, by using Hölder's inequality,
$$\begin{aligned}
\left(\frac{1}{|Q|} \int\limits_Q w(x)^{1-q'} \mathrm{d}x\right)^{(q-1)} &\leq \frac{1}{|Q|^{q-1}} \left(\int\limits_Q w(x)^{1-p'}\ \mathrm{d}x\right)^{\frac{1}{p'-1}} |Q|^{\frac{q-1}{s}} \\
&= \left(|Q|^{\frac{1}{s}-1}\right)^{(q-1)} \left(\int\limits_Q w(x)^{1-p'}\ \mathrm{d}x\right)^{\frac{1}{p'-1}} \\
&= \left(\frac{1}{|Q|} \int\limits_Q w(x)^{1-p'}\ \mathrm{d}x\right)^{p-1}.
\end{aligned}$$

Therefore,

$$\frac{w(Q)}{|Q|}\left(\frac{1}{|Q|}\int\limits_Q w(x)^{1-q'}\mathrm{d}x\right)^{(q-1)} \leq \frac{w(Q)}{|Q|}\left(\frac{1}{|Q|}\int\limits_Q w(x)^{1-p'}\mathrm{d}x\right)^{p-1} \leq C.$$

Hence, $w \in A_q$.

2. To prove the second part we have the following. As $p > 1$,

$$\begin{aligned} w^{1-p'} \in A_{p'} &\Leftrightarrow \frac{w^{1-p'}(Q)}{|Q|}\left(\frac{1}{|Q|}\int\limits_Q w(x)^{(1-p')(1-p)}\ \mathrm{d}x\right)^{p'-1} \leq C. \\ &\Leftrightarrow \left(\frac{w^{1-p'}(Q)}{|Q|}\right)^{p-1}\left(\frac{1}{|Q|}\int\limits_Q w(x)\ \mathrm{d}x\right)^{(p'-1)(p-1)} \leq C. \\ &\Leftrightarrow \left(\frac{w^{1-p'}(Q)}{|Q|}\right)^{p-1}\left(\frac{1}{|Q|}\int\limits_Q w(x)\ \mathrm{d}x\right) \leq C. \\ &\Leftrightarrow w \in A_p. \end{aligned}$$

3. Since $w_0 \in A_1$, we have the following

$$\frac{w_0(Q)}{|Q|} \leq Cw_0(x),\ \text{for a.e } x \in Q.$$

So we have for a.e. $x \in Q$,

$$w_0(x)^{-1} \leq C\left(\frac{w_0(Q)}{|Q|}\right)^{-1}. \tag{4.9}$$

Similarly, as $w_1 \in A_1$ we have for a.e. $x \in Q$,

$$w_1(x)^{-1} \leq C\left(\frac{w_1(Q)}{|Q|}\right)^{-1}. \tag{4.10}$$

Now, we have,

$$\begin{aligned} &\left(\frac{1}{|Q|}\int\limits_Q w_0(x)w_1(x)^{1-p}\mathrm{d}x\right)\left(\frac{1}{|Q|}\int\limits_Q w_0(x)^{1-p'}w_1(x)^{(1-p)(1-p')}\mathrm{d}x\right)^{p-1} \\ &= \left(\frac{1}{|Q|}\int\limits_Q w_0(x)w_1(x)^{1-p}\mathrm{d}x\right)\left(\frac{1}{|Q|}\int\limits_Q w_0(x)^{1-p'}w_1(x)\mathrm{d}x\right)^{p-1} \\ &\leq \left(\frac{1}{|Q|}\int\limits_Q w_0(x)\left(C\frac{w_1(Q)}{|Q|}\right)^{1-p}\mathrm{d}x\right) \times \end{aligned}$$

$$\left(\frac{1}{|Q|}\int\limits_{Q}\left(C\frac{w_0(Q)}{|Q|}\right)^{1-p'}w_1(x)\mathrm{d}x\right)^{p-1}$$
$$\leq\left(C\frac{w_1(Q)}{|Q|}\right)^{1-p}\frac{w_0(Q)}{|Q|}\left(\frac{w_1(Q)}{|Q|}\right)^{p-1}\left(C\frac{w_0(Q)}{|Q|}\right)^{-1}$$
$$\leq C.$$

That is, $w_0w_1^{1-p}\in A_p$.

□

4.3 Characterization of A_p weights

We have seen that the A_p condition is necessary for M to be weak type (p,p), $1\leq p<\infty$. We now see its sufficiency.

Theorem 4.5. *For $1\leq p<\infty$, the weak (p,p) inequality*

$$w(\{x\in\mathbb{R}^n : Mf(x)>\lambda\})\leq\frac{C}{\lambda^p}\int\limits_{\mathbb{R}^n}|f(x)|^p w(x)\mathrm{d}x$$

holds if and only if $w\in A_p$.

Proof. We proved the necessity of the A_p condition in Section 1.2 .

Now suppose that $p>1$ and $w\in A_p$. Given a function $f\in L^p(w)$, we first show that Inequality (4.4) holds and so Inequality (4.5) also holds. By Hölder's inequality, we have

$$\left(\frac{1}{|Q|}\int\limits_{Q}|f(x)|\ \mathrm{d}x\right)^{p}\leq\frac{1}{|Q|^p}\left(\int\limits_{Q}|f(x)|^p w(x)\ \mathrm{d}x\right)^{\frac{p}{p}}\left(\int\limits_{Q}w(x)^{-\frac{p'}{p}}\ \mathrm{d}x\right)^{\frac{p}{p'}}$$
$$=\frac{1}{|Q|^p}\left(\int\limits_{Q}|f(x)|^p w(x)\ \mathrm{d}x\right)\left(\int\limits_{Q}w(x)^{1-p'}\ \mathrm{d}x\right)^{p-1}$$
$$\leq C\left(\frac{1}{|Q|}\int\limits_{Q}|f(x)|^p w(x)\ \mathrm{d}x\right)\frac{|Q|}{w(Q)},$$

Where the last inequality is a consequence of the A_p-condition. This shows that

$$w(Q)\left(\frac{f(Q)}{|Q|}\right)^p \leq C \int\limits_Q |f(x)|^p w(x)\ \mathrm{d}x.$$

So, Inequality (4.5) holds. We may assume without loss of generality that f is non-negative. We form the Calderón-Zygmund decomposition of f at height $4^{-n}\lambda$ to get a collection of disjoint cubes $(Q_j)_{j\in\mathbb{N}}$ such that $f(Q_j) > 4^{-n}\lambda|Q_j|$. Then by a similar argument as in the proof of Lemma 3.5, we can show that

$$\{x \in \mathbb{R}^n : Mf(x) > \lambda\} \subseteq \bigcup_{j\in\mathbb{N}} 3Q_j.$$

Here we dilate the cubes by a factor of 3 instead of 2 because M is a non-centered maximal operator.

Therefore,

$$w(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \leq \sum_{j\in\mathbb{N}} w(3Q_j).$$

As $Q_j \subset 3Q_j$, using Inequality (4.4), we get

$$w(3Q_j)\left(\frac{|Q_j|}{|3Q_j|}\right)^p \leq Cw(Q_j).$$

$$\Rightarrow w(3Q_j) \leq C3^{np}w(Q_j).$$

Therefore, we have

$$w(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \leq C3^{np}\sum_{j\in\mathbb{N}} w(Q_j).$$

Now from Inequality (4.4),

$$\begin{aligned} w(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) &\leq C3^{np}\sum_j \left(\frac{|Q_j|}{f(Q_j)}\right)^p \int\limits_{Q_j} |f(x)|^p w(x)\ \mathrm{d}x \\ &\leq C3^{np}\left(\frac{4^n}{\lambda}\right)^p \int\limits_{\mathbb{R}^n} |f(x)|^p w(x)\ \mathrm{d}x. \end{aligned}$$

This shows that M is weak (p,p) with respect to w, when $p > 1$.

Now, suppose $w \in A_1$. We have shown that $Mw(x) \leq Cw(x)$ almost every $x \in \mathbb{R}^n$. So, by using Theorem 4.3 we get

$$\int_{\{x:Mf(x)>\lambda\}} w(x)\mathrm{d}x \leq \frac{C_1}{\lambda}\int_{\mathbb{R}^n} |f(x)|Mw(x)\mathrm{d}x \leq \frac{C_1}{\lambda}\int_{\mathbb{R}^n} |f(x)|w(x)\mathrm{d}x.$$

Therefore, M is weak $(1,1)$ whenever $w \in A_1$. $\square$

4.3.1 Strong-type inequalities with weights

We have seen that the operator M is weak (p,p) on $L^p(w)$ if and only if $w \in A_p$ for $1 \le p < \infty$. If $w \in A_p$, is it true that M is strong (p,p) on $L^p(w)$? Here we are going to find an affirmative answer to this question. Before going to derive the weighted strong type inequalities, let us first prove the following lemma. It gives the relation between $L^\infty(\mathbb{R}^n)$ and $L^\infty(w)$, for $w \in A_p$.

Proposition 4.6. *If $w \in A_p$, for $1 < p < \infty$, then $L^\infty(w) = L^\infty(\mathbb{R}^n)$ with equality of the norms.*

Proof. We show that $w(E) = 0$ if and only if $|E| = 0$. First, suppose $w(E) = 0$. Let Q be a cube such that $E \subseteq Q$ and $w(Q) > 0$. In Theorem 4.5, we have shown that Inequality (4.5) is true when $w \in A_p$. So, we have

$$w(Q)\left(\frac{|E|}{|Q|}\right)^p \le Cw(E) = 0.$$

As $w(Q) > 0$, we must have $|E| = 0$. If E is unbounded, we can write $E = \bigcup_{j\in\mathbb{N}} E \cap Q_j$, for disjoint cubes Q_j. By the above observation, whenever $w(E) = 0$, we also have $w(E \cap Q_j) = 0$ and hence $|E \cap Q_j| = 0$. $\therefore |E| = \sum_{j\in\mathbb{N}} |E \cap Q_j| = 0$.

Conversely, If $|E| = 0$ then by definition $w(E) = 0$.

Now suppose $f \in L^\infty(\mathbb{R}^n)$ and let $a = ||f||_\infty$. So $|\{x \in \mathbb{R}^n : |f(x)| > a\}| = 0$. Therefore $w(\{x \in \mathbb{R}^n : |f(x)| > a\}) = 0$. This implies $f \in L^\infty(w)$ and $||f||_{L^\infty(w)} \le a$. Similarly we can show that $||f||_{L^\infty(w)} \ge ||f||_\infty$. This completes the proof! □

Corollary 4.7. *The maximal operator M is bounded on $L^\infty(w)$, for any $w \in A_p$ for any $1 \le p < \infty$.*

Proof. The proof follows from the fact that M is bounded on $L^\infty(\mathbb{R}^n)$ together with Proposition 4.6. □

The following theorem gives a partial answer to our question of characterizing weights that make M bounded on $L^p(w)$.

Theorem 4.8. *If $1 \leq q < p < \infty$ and $w \in A_q$ then M is strong (p, p).*

Proof. As $w \in A_q$, from Theorem 4.5 we know that M is weak (q, q). We have already seen that $||Mf||_{L^\infty(w)} \leq ||f||_{L^\infty(w)}$. By Marcinkiewicz interpolation theorem we have M is strong (p, p) on $L^p(w)$. $\square$

To complete our desired characterization, it is now enough to show that given a $w \in A_p$, there exists $1 < q < p$, such that $w \in A_q$. Essentially we ask whether $A_p = \bigcup_{q<p} A_q$? Notice that from Proposition 4.4 we already have $\bigcup_{q<p} A_q \subseteq A_p$, we now prove the other inclusion.

For this we require the reverse Hölder inequality. We begin with the following lemma.

Lemma 4.9. *Let $w \in A_p$, $1 \leq p < \infty$. Then for every $0 < \alpha < 1$, there exists $0 < \beta < 1$ such that given a cube Q and $S \subset Q$ with $|S| \leq \alpha |Q|$, $w(S) \leq \beta w(Q)$.*

Proof. We know that if $w \in A_p$, then for any cube Q and any measurable subset S of Q we have

$$w(Q) \left(\frac{|S|}{|Q|} \right)^p \leq C w(S).$$

As $Q \setminus S \subseteq Q$, and all of these sets have finite measure, we also have,

$$w(Q) \left(1 - \frac{|S|}{|Q|} \right)^p \leq C \left(w(Q) - w(S) \right).$$

Since $|S| \leq \alpha |Q|$, we have $(1 - \alpha)^p \leq \left(1 - \frac{|S|}{|Q|} \right)^p$. Therefore, we get

$$w(Q) \left(1 - \alpha \right)^p \leq C \left(w(Q) - w(S) \right).$$

Simplifying,

$$w(S) \leq \frac{C - (1 - \alpha)^p}{C} w(Q).$$

Choosing $C > 1$, we get $0 < \beta = 1 - \frac{(1 - \alpha)^p}{C} < 1$. The desired result now follows. $\square$

Theorem 4.10 (Reverse Hölder Inequality). *Let $w \in A_p$, $1 \leq p < \infty$. Then there exists constants C and $\epsilon > 0$, depending only on p and the A_p constant of w, such that for any cube Q,*

$$\left(\frac{1}{|Q|}\int_Q w(x)^{1+\epsilon}\ \mathrm{d}x\right)^{\frac{1}{(1+\epsilon)}} \leq \frac{C}{|Q|}\int_Q w(x)\ \mathrm{d}x. \tag{4.11}$$

Proof. Fix a cube Q and form the Calderón-Zygmund decompositions of w with respect to Q at heights given by the following increasing sequence $\frac{w(Q)}{|Q|} = \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots$; where $\lambda_k' s$ are chosen later. For each λ_k we get a family of disjoint cubes $\{Q_{k,j}\}_{k,j\in\mathbb{N}}$ such that

$$w(x) \leq \lambda_k \text{ if } x \notin \Omega_k = \bigcup_j Q_{k,j},$$

and

$$\lambda_k < \frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} w(x)\mathrm{d}x \leq 2^n\lambda_k. \tag{4.12}$$

From Theorem 3.3, we have $\Omega_k = \bigcup_{j\in\mathbb{N}}\Omega'_{k,j}$, where $\Omega'_{k,j} = \{x \in \mathbb{R}^n : E_{k,j}w(x) > \lambda_k \text{ and } E_{k,i}w(x) \leq \lambda_k, \forall i < j\}$. For each j, $\Omega'_{k,j} = \bigcup_{m\in\mathbb{N}} Q^{(m)}_{k,j}$ where $Q^{(m)}_{k,j}$ are disjoint dyadic cubes.

Suppose $x \in \Omega_{k+1}$. Then $\exists\ l \in \mathbb{Z}$ such that $x \in \Omega'_{k+1,l} = \bigcup_{m\in\mathbb{N}} Q^{(m)}_{k+1,l}$. This implies $E_{k+1,l}w(x) > \lambda_{k+1}$. That is, $\frac{1}{|Q^{(m)}_{k+1,l}|}\int_{Q^{(m)}_{k+1,l}} w(x)\mathrm{d}x > \lambda_{k+1} > \lambda_k$ for some dyadic cube $Q^m_{k+1,l}$. So $\exists\ j \leq l$ such that $E_{k,j}w(x) > \lambda_k$ and for any $i < j$, $E_{k,i}w(x) \leq \lambda_k$. Hence, $x \in \Omega_k$. Therefore $\Omega_{k+1} \subseteq \Omega_k$. If we fix Q_{k,j_0} from the Calderón-Zygmund decomposition at the height λ_k, then $Q_{k,j_0} \cap \Omega_{k+1}$ is union of cubes $\{Q_{k+1,i}\}_{i\in I}$ from the decomposition at height λ_{k+1}. Therefore,

$$|Q_{k,j_0} \cap \Omega_{k+1}| = \sum_{i\in I}|Q_{k+1,i}|.$$

Using the first inequality of (4.12), we get

$$|Q_{k,j_0} \cap \Omega_{k+1}| \leq \frac{1}{\lambda_{k+1}}\sum_{i\in\mathbb{N}}\int_{Q_{k+1,i}} w(x)\mathrm{d}x \leq \frac{1}{\lambda_{k+1}}\int_{Q_{k,j_0}} w(x)\mathrm{d}x.$$

Now using second inequality of (4.12), we get

$$|Q_{k,j_0} \cap \Omega_{k+1}| \leq \frac{2^n \lambda_k}{\lambda_{k+1}} |Q_{k,j_0}|.$$

Now, let us fix $\alpha < 1$ and choose the λ_k's so that $\dfrac{2^n \lambda_k}{\lambda_{k+1}} = \alpha$; that is $\lambda_k = (2^n \alpha^{-1})^k w(Q)/|Q|$. Then $|Q_{k,j_0} \cap \Omega_{k+1}| \leq \alpha |Q_{k,j_0}|$. By Lemma 4.9, there exists $\beta < 1$ such that $w(Q_{k,j_0} \cap \Omega_{k+1}) \leq \beta w(Q_{k,j_0})$. Now,

$$\begin{aligned} w\left(\left(\bigcup_{j \in \mathbb{N}} Q_{k,j}\right) \cap \Omega_{k+1}\right) &= w\left(\bigcup_{j \in \mathbb{N}} (Q_{k,j} \cap \Omega_{k+1})\right) \\ &= \sum_{j \in \mathbb{N}} w(Q_{k,j} \cap \Omega_{k+1}) \leq \beta w(\cup Q_{k,j}). \end{aligned}$$

Therefore we have $w(\Omega_{k+1}) \leq \beta w(\Omega_k)$. Iterating this inequality we get $w(\Omega_k) \leq \beta^k w(\Omega_0)$. Similarly, $|\Omega_k| \leq \alpha^k |\Omega_0|$. Hence, by downward monotone convergence theorem we have,

$$\left| \bigcap_{k \in \mathbb{N}} \Omega_k \right| = \lim_{k \longrightarrow \infty} |\Omega_k| = 0.$$

Therefore,

$$\begin{aligned} \frac{1}{|Q|} \int\limits_Q w(x)^{1+\epsilon} \, \mathrm{d}x &= \frac{1}{|Q|} \int\limits_{Q \setminus \Omega_0} w(x)^{1+\epsilon} \, \mathrm{d}x + \frac{1}{|Q|} \int\limits_{\Omega_0} w(x)^{1+\epsilon} \, \mathrm{d}x \\ &= \frac{1}{|Q|} \int\limits_{Q \setminus \Omega_0} w(x)^{1+\epsilon} \, \mathrm{d}x + \frac{1}{|Q|} \sum_{k=0}^{\infty} \int_{\Omega_k \setminus \Omega_{k+1}} w(x)^{1+\epsilon} \, \mathrm{d}x \\ &\leq \frac{1}{|Q|} \int\limits_{Q \setminus \Omega_0} w(x) \lambda_0^{\epsilon} \, \mathrm{d}x + \frac{1}{|Q|} \sum_{k=0}^{\infty} \int_{\Omega_k \setminus \Omega_{k+1}} w(x) \lambda_{k+1}^{\epsilon} \, \mathrm{d}x \\ &= \lambda_0^{\epsilon} \frac{w(Q)}{|Q|} + \frac{1}{|Q|} \sum_{k=0}^{\infty} \lambda_{k+1}^{\epsilon} w(\Omega_k) \\ &\leq \lambda_0^{\epsilon} \frac{w(Q)}{|Q|} + \frac{1}{|Q|} \sum_{k=0}^{\infty} (2^n \alpha^{-1})^{(k+1)\epsilon} \lambda_0^{\epsilon} \beta^k w(\Omega_0). \end{aligned}$$

Now we fix $\epsilon > 0$ such that $(2^n \alpha^{-1})^{\epsilon} \beta < 1$, then the series $\sum\limits_{k=0}^{\infty} (2^n \alpha^{-1})^{(k+1)\epsilon} \beta^k$ converges to $\frac{(2^n \alpha^{-1})^{\epsilon}}{1 - (2^n \alpha^{-1})^{\epsilon} \beta}$. As $w(\Omega_0) \leq w(Q)$, we have that

$$\frac{1}{|Q|} \int\limits_Q w(x)^{1+\epsilon} \, \mathrm{d}x = \lambda_0^{\epsilon} \frac{w(Q)}{|Q|} + C \lambda_0^{\epsilon} \frac{w(Q)}{|Q|} = C \left(\frac{w(Q)}{|Q|} \right)^{1+\epsilon}. \tag{4.13}$$

This completes the proof. $\square$

As a corollary to the reverse Hölder inequality, we get the following properties of A_p weights.

Corollary 4.11.

1. *$A_p = \bigcup_{q<p} A_q$, $1 < p < \infty$.*

2. *If $w \in A_p$, $1 \le p < \infty$, then $\exists\ \epsilon > 0$ such that $w^{1+\epsilon} \in A_p$.*

3. *If $w \in A_p$, $1 < p < \infty$, then there exists $\delta > 0$ such that given a cube Q and $S \subseteq Q$,*
$$\frac{w(S)}{w(Q)} \le C\left(\frac{|S|}{|Q|}\right)^{\delta}.$$

Proof.

1. It is clear from Proposition 4.4, that $\bigcup\limits_{q<p} A_q \subseteq A_p$. Now suppose $w \in A_p$. We need to find some $q < p$ such that $w \in A_q$. Again by Proposition 4.4, if $w \in A_p$ we have $w^{1-p'} \in A_{p'}$. Therefore from Theorem 4.10 $\exists\ \epsilon > 0$ such that
$$\left(\frac{1}{|Q|}\int\limits_Q w^{(1-p')(1+\epsilon)}(x)\ \mathrm{d}x\right)^{\frac{1}{(1+\epsilon)}} \le \frac{C}{|Q|}\int\limits_Q w(x)^{1-p'}\ \mathrm{d}x.$$
Choosing a $q > 1$ such that $(p'-1)(1+\epsilon) = (q'-1)$. This is possible since $1 + (p'-1)(1+\epsilon) > 1$. Then we observe that $q < p$, and $\dfrac{q'-1}{p'-1} = 1+\epsilon$. Therefore from Equation (4.13,) we have
$$\left(\frac{1}{|Q|}\int\limits_Q w^{(1-q')}(x)\ \mathrm{d}x\right)^{\frac{p'-1}{q'-1}} \le \frac{C}{|Q|}\int\limits_Q w(x)^{1-p'}\mathrm{d}x.$$
As $p-1 > 0$, we have the following
$$\left(\frac{1}{|Q|}\int\limits_Q w^{(1-q')}(x)\mathrm{d}x\right)^{\frac{(p'-1)(p-1)}{q'-1}} \le C^{p-1}\left(\frac{1}{|Q|}\int\limits_Q w^{1-p'}(x)\mathrm{d}x\right)^{p-1}.$$

We know that $w \in A_p$ and $(p'-1)(p-1) = 1$, the above inequality becomes

$$\left(\frac{1}{|Q|}\int\limits_Q w^{(1-q')}(x)\mathrm{d}x\right)^{\frac{1}{q'-1}} \leq C\frac{|Q|}{w(Q)}.$$

Since $(q-1)(q'-1) = 1$,

$$\left(\frac{1}{|Q|}\int\limits_Q w^{(1-q')}(x)\mathrm{d}x\right)^{q-1} \leq C\frac{|Q|}{w(Q)}.$$

This is the same as,

$$\frac{w(Q)}{|Q|}\left(\frac{1}{|Q|}\int\limits_Q w^{(1-q')}(x)\mathrm{d}x\right)^{q-1} \leq C,$$

which further implies $w \in A_q$.

2. First consider the case $p > 1$. Let us choose $\epsilon > 0$ so small, such that w and $w^{1-p'}$ both satisfy the reverse Hölder Inequality (4.11) Since $p-1>0$, we have,

$$\left(\frac{1}{|Q|}\int\limits_Q w^{(1-p')(1+\epsilon)}(x)\mathrm{d}x\right)^{\frac{p-1}{(1+\epsilon)}} \leq C^{p-1}\left(\frac{1}{|Q|}\int\limits_Q w^{1-p'}(x)\mathrm{d}x.\right)^{p-1}.$$

Also, $w \in A_p$. Therefore, we get

$$\begin{aligned}\left(\frac{1}{|Q|}\int\limits_Q w^{(1-p')(1+\epsilon)}(x)\mathrm{d}x\right)^{\frac{p-1}{(1+\epsilon)}} &\leq C^{p-1}\left(\frac{1}{|Q|}\int\limits_Q w^{1-p'}(x)\mathrm{d}x.\right)^{p-1}\\ &\leq C\frac{|Q|}{w(Q)}.\end{aligned}$$

From the reverse Hölder inequality for w, we get

$$\left(\frac{1}{|Q|}\int\limits_Q w^{(1-p')(1+\epsilon)}(x)\mathrm{d}x\right)^{p-1} \leq C\left(\frac{|Q|}{w(Q)}\right)^{1+\epsilon} \leq \frac{C}{\frac{1}{|Q|}\int\limits_Q w^{1+\epsilon}(x)\mathrm{d}x}.$$

Therefore, we have

$$\left(\frac{1}{|Q|}\int\limits_Q w^{1+\epsilon}(x)\mathrm{d}x\right)\left(\frac{1}{|Q|}\int\limits_Q w^{(1-p')(1+\epsilon)}(x)\mathrm{d}x\right)^{p-1} \leq C.$$

This implies $w^{1+\epsilon} \in A_p$.

If $p = 1$, then for any cube Q and a.e $x \in Q$, $\frac{1}{|Q|}\int\limits_Q w(y)\mathrm{d}y \leq Cw(x)$. Now by reverse Hölder inequality, we get some $\epsilon > 0$ for which,

$$\frac{1}{|Q|}\int\limits_Q w^{1+\epsilon}(x)\mathrm{d}x \le C\left(\frac{1}{|Q|}\int\limits_Q w(x)\mathrm{d}x\right)^{1+\epsilon} \le Cw^{1+\epsilon}(x).$$

Therefore, for any cube Q and for a.e $x \in Q$

$$\frac{1}{|Q|}\int\limits_Q w^{1+\epsilon}(x)\ \mathrm{d}x \le Cw^{1+\epsilon}(x),$$

which implies $w^{1+\epsilon} \in A_1$.

3. Let $S \subseteq Q$ and let w satisfy reverse Hölder inequality with $\epsilon > 0$. Then, by using Hölder inequality and reverse Hölder inequality we have

$$\begin{aligned} wS = \int\limits_Q \chi_S w(x)\ \mathrm{d}x &\le \left(\int\limits_Q w^{1+\epsilon}(x)\ \mathrm{d}x\right)^{1/(1+\epsilon)} |S|^{\epsilon/(1+\epsilon)} \\ &\le Cw(Q)\left(\frac{|S|}{|Q|}\right)^{\epsilon/(1+\epsilon)}. \end{aligned}$$

This gives desired inequality by choosing $\delta = \epsilon/(1+\epsilon)$.

□

We are now ready to characterize all weights that make M bounded on $L^p(w)$.

Theorem 4.12. *The maximal operator M is bounded on $L^p(w)$ if and only if $w \in A_p$ for $1 < p < \infty$.*

Proof. Using the first part of the Corollary 4.11, we can say that if $w \in A_p$, $p > 1$, then there exists $1 \le q < p$ such that $w \in A_q$. Now we can use Theorem 4.8 to conclude the result. □

We have seen that M is bounded on $L^p(w)$ spaces if and only if $w \in A_p$. It is indeed true that constant functions are A_p weights. In next section, we see a way to construct A_p weights with the help of Hardy-Littlewood maximal operator M. This can give a variety of non-trivial examples.

4.4 Construction of A_1 weights

In this section, we construct A_1 weight with the help of Hardy-Littlewood maximal function. The result presented here is due to Coifman and Rochberg ([6]). This, combined with Proposition 4.4, lets us construct a few A_p weights for all $1 \leq p < \infty$. We start with the following lemma.

Lemma 4.13. *Given an operator S which is weak $(1,1)$ and $0 < \gamma < 1$, and a set E of finite measure, there exists a constant C depending only on γ such that*

$$\int_E |Sf(x)|^\gamma \, \mathrm{d}x \leq C|E|^{1-\gamma}||f||_1^\gamma.$$

Proof. For $f \in L^p(X, \nu)$ and $0 < p < \infty$, we have

$$||f||_p^p = p \int_0^\infty \lambda^{p-1} d_f(\lambda) \mathrm{d}\lambda,$$

where $d_f(\lambda) = \{x \in \mathbb{R}^n : |f(x)| > \lambda\}$. Therefore,

$$\int_E |Sf(x)|^\gamma \, \mathrm{d}x = \gamma \int_0^\infty \lambda^{\gamma-1} |\{x \in E : |Sf(x)| > \lambda\}| \, \mathrm{d}\lambda.$$

Since S is weak $(1,1)$, we have $|\{x \in E : |Sf(x)| > \lambda\}| \leq \dfrac{C}{\lambda}||f||_1$. Also, $\{x \in E : |Sf(x)| > \lambda\} \subset E$, so that $|\{x \in E : |Sf(x)| > \lambda\}| \leq |E|$. Therefore, $|\{x \in E : |Sf(x)| > \lambda\}| \leq \min\left(|E|, \dfrac{C}{\lambda}||f||_1\right)$. Hence,

$$\begin{aligned}
\int_E |Sf(x)|^\gamma \, \mathrm{d}x &\leq \gamma \int_0^\infty \lambda^{\gamma-1} \min\left(|E|, \frac{C}{\lambda}||f||_1\right) \, \mathrm{d}\lambda \\
&= \gamma \int_0^{C\frac{||f||_1}{|E|}} \lambda^{\gamma-1}|E| \, \mathrm{d}\lambda + \gamma \int_{C\frac{||f||_1}{|E|}}^\infty C\lambda^{\gamma-2}||f||_1 \, \mathrm{d}\lambda \\
&= \lambda|E| \left[\frac{\lambda^\gamma}{\gamma}\right]_0^{C\frac{||f||_1}{|E|}} + \gamma C||f||_1 \left[\frac{\lambda^{\gamma-1}}{\gamma-1}\right]_{C\frac{||f||_1}{|E|}}^\infty \\
&= \frac{(C||f||_1)^\gamma}{|E|^{1-\gamma}} \left(1 + \frac{\gamma}{1-\gamma}\right) \\
&= C||f||_1^\gamma |E|^{1-\gamma}.
\end{aligned}$$

□

We also require an easy inequality concerning power of sums of positive numbers.

Lemma 4.14. *Let a, b, $c \geq 0$ be such that $c \leq a + b$. Then for any $0 \leq \alpha < 1$ we have $c^\alpha \leq a^\alpha + b^\alpha$.*

Proof. Let $\beta = \frac{1}{\alpha}$. Then clearly $\beta > 1$. We first prove that $(a+b)^\beta \geq a^\beta + b^\beta$. Let us consider the following function $G : [1, \infty) \longrightarrow \mathbb{R}$, defined as $G(r) = (a+b)^r - a^r - b^r$. Note that $G(r)$ can be written as, $G(r) = a^r\{(1 + b/a)^r - (1 + (b/a)^r)\}$. Now we show that $G(r) \geq 0$. To this end, consider another function H of $1 \leq r < \infty$, defined as $H(r) = (1 + x)^r - (1 + x^r)$, for a fixed $x > 0$. Note that $H'(r) = (1 + x)^r log(1 + x) - x^r logx \geq 0$. Therefore H is an increasing function of r. So, $H(r) \geq H(1)$for any $r \geq 1$. This clearly implies $H(r) \geq 0$ for any $r \geq 1$. Hence by substituting $x = b/a$ we get $\{(1 + b/a)^r - (1 + (b/a)^r)\} \geq 0$. That is $G(r) \geq 0$. So, $(a + b)^r \geq a^r + b^r$ for any $r \geq 1$. By taking $a_1 = a^\alpha$ and $b_1 = b^\alpha$ in place of a and b respectively, we get $a^\alpha + b^\alpha \geq (a + b)^\alpha$. As a, b and c are non-negative with $a + b \geq c$ we have $(a + b)^\alpha \geq c^\alpha$. Therefore we have $a^\alpha + b^\alpha \geq (a + b)^\alpha \geq c^\alpha$. Hence the result. □

We now discuss the construction of A_1 weights. In fact, we see that up to a multiplication by bounded function, this is only way to produce A_1 weights.

Theorem 4.15.

1. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ be such that $Mf(x) < \infty$ a.e. If $0 \leq \delta < 1$, then $w(x) = Mf(x)^\delta$ is an A_1 weight whose A_1 constant depends only on δ.*

2. *Conversely, if $w \in A_1$, then there exists $f \in L^1_{loc}(\mathbb{R}^n)$, $0 \leq \delta < 1$, and K a function on $\mathbb{R}^n$, with $K, \frac{1}{K} \in L^\infty(\mathbb{R}^n)$, such that $w(x) = K(x)Mf(x)^\delta$.*

Proof.

1. It suffices to show that there exists a constant C such that for every cube Q and almost every $x \in Q$,
$$\frac{1}{|Q|}\int\limits_Q (Mf(x))^\delta \, \mathrm{d}x \le CMf(x)^\delta.$$
Now fix cube Q and decompose f as $f = f_1 + f_2$, where $f_1 = f\chi_{2Q}$. Then $Mf(x) \le Mf_1(x) + Mf_2(x)$ and by using Lemma 4.14, we get,
$$Mf(x)^\delta \le Mf_1(x)^\delta + Mf_2(x)^\delta.$$
As M is weak $(1,1)$, from Lemma 4.13 we have,
$$\begin{aligned}\frac{1}{|Q|}\int\limits_Q (Mf_1(x))^\delta \, \mathrm{d}x &\le \frac{C_\delta}{|Q|}|Q|^{1-\delta}||f_1||_1^\delta \\ &= C_\delta\left(\frac{1}{|Q|}\int\limits_{\mathbb{R}^n} f_1(x)\, \mathrm{d}x\right)^\delta \\ &= C_\delta\left(\frac{1}{|Q|}\int\limits_{2Q} f(x)\, \mathrm{d}x\right)^\delta \\ &= 2^{n\delta}C_\delta\left(\frac{1}{|2Q|}\int\limits_{2Q} f(x)\, \mathrm{d}x\right)^\delta \\ &\le 2^{n\delta}C_\delta Mf(x)^\delta.\end{aligned}$$
To estimate Mf_2, we see that if $y \in Q$ and R is a cube with $y \in R$ and $\int\limits_R |f_2(x)|\, \mathrm{d}x > 0$, then we must have $l(R) > \frac{1}{2}l(Q)$, where $l(\cdot)$ denotes the side length of a cube. To see this, assume if possible $l(R) \le \frac{1}{2}l(Q)$.
$$\begin{aligned}f_2 &= f - f_1 \\ &= f - f\chi_{2Q}.\end{aligned}$$
As $x \in R \Rightarrow x \in 2Q$. $f_2(x) = 0$, $\forall x \in R$. This implies $\int\limits_R |f(x)|\, \mathrm{d}x = 0$, which is a contradiction! Hence, there exists a constant c_n depending only on n, such that if $x \in Q$ then $x \in c_nR$. Therefore,
$$\frac{1}{|R|}\int\limits_R |f_2(x)|\, \mathrm{d}x \le \frac{c_n^n}{|c_nR|}\int\limits_{c_nR} |f_2(x)|\, \mathrm{d}x \le c_n^n Mf(x),$$

and so, $Mf_2(y) \le c_n^n Mf(x)$ for any $y \in Q$. Thus,

$$\frac{1}{|Q|}\int_Q Mf_2(y)^\delta \, \mathrm{d}y \le c_n^{n\delta} Mf(x)^\delta.$$

Now, for any cube Q,

$$\begin{aligned}\frac{1}{|Q|}\int_Q Mf(x)^\delta \, \mathrm{d}x &\le \frac{1}{|Q|}\int_Q (Mf_1(x))^\delta \, \mathrm{d}x + \frac{1}{|Q|}\int_Q (Mf_2(x))^\delta \, \mathrm{d}x \\ &\le 2^{n\delta} C_\delta Mf(x)^\delta + c_n^{n\delta} Mf(x)^\delta \\ &= CMf(x)^\delta.\end{aligned}$$

This proves (1).

2. For $w \in A_1$, by reverse Hölder inequality $\exists\ \epsilon > 0$ such that

$$\left(\frac{1}{|Q|}\int_Q w^{1+\epsilon}(x) \, \mathrm{d}x\right)^{\frac{1}{1+\epsilon}} \le \frac{C}{|Q|}\int_Q w(x) \, \mathrm{d}x.$$

And for any cube Q, for a.e $x \in Q$, we have

$$\frac{1}{|Q|}\int_Q w(x) \, \mathrm{d}x \le Cw(x).$$

Therefore for a.e $x \in Q$, we get

$$\left(\frac{1}{|Q|}\int_Q w^{1+\epsilon}(x) \, \mathrm{d}x\right)^{\frac{1}{1+\epsilon}} \le Cw(x),$$

which implies for a.e $x \in \mathbb{R}^n$,

$$\left(Mw^{1+\epsilon}(x)\right)^{\frac{1}{1+\epsilon}} \le Cw(x).$$

Let $f = w^{1+\epsilon}$ and $\delta = \frac{1}{1+\epsilon}$, so we have $Mf(x)^\delta \le cw(x)$ for a.e $x \in \mathbb{R}^n$. By Lebesgue differentiation theorem $w^{1+\epsilon}(x) \le Mw^{1+\epsilon}(x) = Mf(x)$, so $w(x) \le (Mf(x))^\delta$. Therefore, for a.e $x \in \mathbb{R}^n$, $w(x) \le (Mf(x))^\delta \le Cw(x)$. Now let $K(x) = \frac{w(x)}{Mf(x)^\delta}$. Note that $K(x) \le 1$ and $K^{-1}(x) \le C$ for a.e $x \in \mathbb{R}^n$. So K, $K^{-1} \in L^\infty(\mathbb{R}^n)$. Finally, we have $w(x) = K(x)Mf(x)^\delta$.

$\square$

4.5 An Extrapolation Theorem

In this section we discuss a remarkable result due to Rubio de Francia ([21]) that deals with boundedness of operators on Muckenhoupt weighted L^p spaces. We recall that we have seen "interpolation" results, wherein we can discuss boundedness of linear operators between two "end points". We see that "extrapolation" is also possible! Particularly, if it is known that an operator T is bounded on $L^p(w)$ for a fixed $p > 1$ and each $w \in A_p$, then it becomes bounded on all L^r-spaces with all Muckenhoupt weights.

Theorem 4.16 (Rubio de Francia [21])**.** *Fix, $1 < r < \infty$. If T is a bounded operator on $L^r(w)$ for any $w \in A_r$, with operator norm depending only on the A_r constant of w, then T is bounded on $L^p(w)$, for any $1 < p < \infty$, and $w \in A_p$.*

Proof. We first show that if $1 < q < r$ and $w \in A_1$ then T is bounded on $L^q(w)$. By Theorem 4.15 we know that the function $(Mf)^{\frac{r-q}{r-1}} \in A_1$ since $r - q < r - 1$, and by Proposition 4.4, $w(Mf)^{q-r} \in A_r$. Therefore,

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n} |Tf(x)|^q w(x)\ \mathrm{d}x \\
&= \int\limits_{\mathbb{R}^n} |Tf(x)|^q (Mf(x))^{-(r-q)q/r} (Mf(x))^{(r-q)q/r} w(x)\ \mathrm{d}x \\
&= \int\limits_{\mathbb{R}^n} |Tf(x)|^q (Mf(x))^{-(r-q)q/r} (Mf(x))^{(r-q)q/r} w(x)^{q/r} w(x)^{r-q/r}\ \mathrm{d}x \\
&\le \left(\int\limits_{\mathbb{R}^n} |Tf(x)|^r w(x) (Mf(x))^{q-r}\ \mathrm{d}x \right)^{q/r} \left(\int\limits_{\mathbb{R}^n} (Mf(x))^q w(x)\ \mathrm{d}x \right)^{(r-q)/r} \\
&\le C \left(\int\limits_{\mathbb{R}^n} |f(x)|^r w(x) (Mf(x))^{q-r}\ \mathrm{d}x \right)^{q/r} \left(\int\limits_{\mathbb{R}^n} |f(x)|^q w(x)\ \mathrm{d}x \right)^{(r-q)/r}.
\end{aligned}$$

The last inequality follows because T is bounded operator on $L^r\left((w)(Mf)^{q-r}\right)$ and M is strong (q,q) on $L^q(w)$, for $w \in A_q$. We also have that $|f(x)| \le |Mf(x)|$ a.e. and $q - r < 0$. So, $Mf(x)^{q-r} \le |f(x)|^{q-r}$ a.e. The above inequality now become

$$\begin{aligned}
&\int_{\mathbb{R}^n} |Tf(x)|^q w(x)\, \mathrm{d}x \\
&\leq C \left(\int_{\mathbb{R}^n} |f(x)|^r w(x) |f(x)|^{q-r}\, \mathrm{d}x \right)^{q/r} \left(\int_{\mathbb{R}^n} |f(x)|^q w(x)\, \mathrm{d}x \right)^{(r-q)/r} \\
&= C \int_{\mathbb{R}^n} |f(x)|^q w(x)\, \mathrm{d}x.
\end{aligned}$$

We now show that, given any $1 < p < \infty$ and $1 < q < \min(p, r)$, T is bounded on $L^p(w)$ for $w \in A_{\frac{p}{q}}$. The desired result follows at once from this: given $w \in A_p$, by Corollary 4.13 there exists $q > 1$ such that $w \in A_{\frac{p}{q}}$ and so T is bounded on $L^p(w)$.

Let us fix a $w \in A_{\frac{p}{q}}$. Then, by duality there exists a non-negative $u \in L^{(p/q)'}(w)$ with $||u||_{(p/q)'} = 1$ such that

$$\left(\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, \mathrm{d}x \right)^{q/p} = \int_{\mathbb{R}^n} |Tf(x)|^q w(x) u(x)\, \mathrm{d}x.$$

For any $s > 1$, $wu \leq (M(wu)^s)^{1/s}$, and $(M(wu)^s)^{1/s} \in A_1$. This can be shown in the following way: let $K \subset \mathbb{R}^n$ be any compact set. Then,

$$\begin{aligned}
\int_K w(x)^s u(x)^s \mathrm{d}x &\leq \int_K w(x)^{s-1} u(x)^s w(x)\, \mathrm{d}x \\
&\leq \left(\int_K w(x)^{(s-1)p'} w(x)\, \mathrm{d}x \right)^{1/p'} \left(\int_K u(x)^{(p/q)'} w(x) \mathrm{d}x \right)^{\frac{s}{(p/q)'}}
\end{aligned}$$

As $u \in L^{(p/q)'}(w)$, $\left(\int_K u(x)^{(p/q)'} w(x) \mathrm{d}x \right)^{\frac{s}{(p/q)'}} < \infty$. Since $w \in A_{\frac{p}{q}}$, choosing a proper s we have $w^{1+(s-1)p'} \in A_{\frac{p}{q}}$. Therefore $w^{1+(s-1)p'} \in L^1_{loc}(\mathbb{R}^n)$. So $(wu)^s \in L^1_{loc}(\mathbb{R}^n)$. Now by using Theorem 4.15, we have $(M(wu)^s)^{1/s} \in A_1$. Therefore, by the first part of the proof we have

$$\int_{\mathbb{R}^n} |Tf(x)|^q w(x) u(x)\, \mathrm{d}x$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}^n} |Tf(x)|^q (M(wu)^s(x))^{1/s}\, \mathrm{d}x \\
&\le C \int_{\mathbb{R}^n} |f(x)|^q (M(wu)^s)^{1/s} \\
&= C \int_{\mathbb{R}^n} |f(x)|^q w(x)^{p/q} (M(wu)^s(x))^{1/s} w(x)^{-p/q}\, \mathrm{d}x \\
&\le C \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, \mathrm{d}x \right)^{q/p} \left(\int_{\mathbb{R}^n} (M(wu)^s)^{(p/q)'/s} w(x)^{(1-(p/q)')}\, \mathrm{d}x \right)^{1/(p/q)'} .
\end{aligned}
$$

Since $w \in A_{p/q}$, by Proposition 4.4, $w^{1-(p/q)' \in A_{(p/q)'}}$. Therefore, If we take s sufficiently close to 1, $w^{1-(p/q)'} \in A_{(p/q)'/s}$. Hence by Theorem 4.12 the second integral is bounded by

$$C \int_{\mathbb{R}^n} (wu)^{(p/q)'} w^{1-(p/q)'}\, \mathrm{d}x < \infty.$$

So, we have

$$
\begin{aligned}
\left(\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, \mathrm{d}x \right)^{q/p} &= \int_{\mathbb{R}^n} |Tf(x)|^q w(x) u(x)\, \mathrm{d}x \\
&\le C \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, \mathrm{d}x \right)^{q/p} .
\end{aligned}
$$

□

4.6 Strong A_p weights

We have introduced the strong maximal function in Chapter 3. This operator satisfies weighted norm inequalities with weights analogous to A_p weights. These weights are known as "strong" A_p weights. The collection of all such weights is denoted as A_p^*. The strong A_p condition is the following: $w \in A_p^*$, for $1 < p < \infty$, if for any rectangle R with sides parallel to the co-ordinate axes,

$$\left(\frac{1}{|R|} \int_R w(x)\, \mathrm{d}x \right) \left(\frac{1}{|R|} \int_R w(x)^{1-p'}\, \mathrm{d}x \right)^{p-1} \le C,$$

where C is independent of R. A non-negative locally integrable function $w \in A_1^*$ if for almost every $x \in \mathbb{R}^n$,

$$M_s w(x) \leq C w(x).$$

We have seen that Hardy-Littlewood maximal operator is bounded on $L^p(w)$ space, for $1 < p < \infty$. Similar result is true for the strong maximal function.

Theorem 4.17 ([7]). *For $1 < p < \infty$, M_s is bounded on $L^P(w)$ if and only if $w \in A_p^*$.*

Another interesting fact about "strong" A_p weights is given by the following result. It gives a connection of the n-dimensional strong A_p weights with the one-dimensional A_p weights. The result is of importance in proving the higher dimensional analogue of the Marcinkiewicz multiplier theorem (see Chapter 6).

Theorem 4.18 ([7]). *If $w \in A_p^*$, $1 < p < \infty$, then for each $i \in \{1, 2, \cdots, n\}$, $w(x_1, \cdots, x_{i-1}, \cdot, x_{i+1}, \cdots, x_n)$ satisfies one-dimensional A_p condition with a uniform constant.*

CHAPTER 5

Calderón-Zygmund Theory

In this chapter, we begin our study of translation invariant operators. As mentioned earlier, we are interested in operators of convolution type. That is, operators T that act as $Tf = K * f$ for a fixed kernel K. While in general, the kernel K can be a distribution (generalized function), we deal with only those kernels that come from locally integrable functions.

Our aim is to get sufficient conditions on K that make the operator $T : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$ bounded. To this end, we first look at a prototypical operator, namely the Hilbert transform.

5.1 Hilbert transform

In this section we deal with a basic convolution type operator. As mentioned in the introduction of this chapter, our goal is to find sufficient conditions on the kernel K so that T becomes L^p bounded. Let us, for some time, become more restrictive and demand that T also commutes with dilations. That is , if for $\lambda > 0$,

$\delta_\lambda f(x) = \lambda^{-n} f(x/\lambda)$ is the dilation of f by λ, we require $\delta_\lambda Tf = T\delta_\lambda f$. Let us try and use the definitions of T and δ_λ to get some preliminary observations on K. We write that

$$\begin{aligned}(T\delta_\lambda f)(x) &= (K * \delta_\lambda f)(x)\\ &= \int\limits_{\mathbb{R}^n} K(x-y)(\delta_\lambda f)(y)\ \mathrm{d}y\\ &= \int\limits_{\mathbb{R}^n} K(x,y) f(y/\lambda)\lambda^{-n}\ \mathrm{d}y\\ &= \int\limits_{\mathbb{R}^n} K(x-\lambda) f(y)\ \mathrm{d}y.\end{aligned}$$

On the other hand $(\delta_\lambda Tf)(x) = \lambda^{-n} \int\limits_{\mathbb{R}^n} K(\frac{x}{\lambda} - y) f(y)\ \mathrm{d}y$. It is clear that if $K(\lambda z) = \lambda^{-n} K(z)$, then T commutes with dilation. Functions satisfying such a relation are called "homogeneous of degree $-n$". Particularly , if T commutes with dilations have kernel of the form $K(x) = K\left(||x|| \frac{x}{||x||}\right) = \frac{K(x/||x||)}{||x||^n}$. Therefore the typical convolution operators that commutes with dilations have kernel of the form $K(x) = \frac{\Omega(x/||x||)}{||x||^n}$, where Ω is a function defined on $\mathbb{S}^{n-1}$. Such operators are called *singular integrals*, and their boundedness was first studied by Calderón and Zygmund in [4].

If we restrict our attention to $\mathbb{R}$, then $\mathbb{S}^0 = \{\pm 1\}$. One example of this desired kernel in this case is $K(x) = \frac{1}{x} = \frac{sgn(x)}{|x|}$. We now begin studying this kernel.

5.1.1 The principal value of $1/x$

We wish to integrate functions against $\frac{1}{x}$. However in general, this is not possible since $\frac{1}{x}$ has a non-integrable singularity at 0. Therefore, any integral involving $\frac{1}{x}$ is an improper integral. We are interested in its Cauchy principle value. We denote by p.v.$\frac{1}{x}$ the principle value distribution of $\frac{1}{x}$, defined as

$$\text{p.v.}\frac{1}{x}(\varphi) = \lim_{\epsilon \longrightarrow 0} \int_{|x|>\epsilon} \frac{\varphi(x)}{x}\ \mathrm{d}x, \tag{5.1}$$

for $\varphi \in \mathcal{S}(\mathbb{R})$. We first show that this expression is well defined. To see this, we first rewrite Equation (5.1) in the following way.

$$\text{p.v.}\frac{1}{x}(\varphi) = \lim_{\epsilon \longrightarrow 0} \int_{\epsilon<|x|<1} \frac{\varphi(x)}{x}\ \mathrm{d}x + \int_{|x|>1} \frac{\varphi(x)}{x}\ \mathrm{d}x.$$

Now we see that since the function $\frac{\varphi(0)}{x}$ is odd, we have $\int_{\epsilon<|x|<1} \frac{\varphi(0)}{x}\ \mathrm{d}x = 0$. Therefore,

$$\text{p.v.}\frac{1}{x}(\varphi) = \lim_{\epsilon \longrightarrow 0} \int_{\epsilon<|x|<1} \frac{\varphi(x)-\varphi(0)}{x}\ \mathrm{d}x + \int_{|x|>1} \frac{\varphi(x)}{x}\ \mathrm{d}x.$$

We know that φ is smooth. Hence by the mean value theorem, we get,

$$\text{p.v.}\frac{1}{x}(\varphi) = \int_{|x|<1} \varphi^{'}(\xi(x))\ \mathrm{d}x + \int_{|x|>1} \frac{\varphi(x)}{x}\ \mathrm{d}x,$$

where, $\xi(x) = t_x \cdot x$, for some $t_x \in (0,1)$. Therefore

$$\begin{aligned}
\left|\text{p.v.}\frac{1}{x}(\varphi)\right| &\leq \int_{|x|<1} |\varphi^{'}(\xi(x))|\ \mathrm{d}x + \left|\int_{|x|>1} \frac{\varphi(x)}{x}\ \mathrm{d}x\right| \\
&\leq \int_{|x|<1} |\varphi^{'}(\xi(x))|\ \mathrm{d}x + \left|\int_{|x|>1} \frac{x\varphi(x)}{x^2}\ \mathrm{d}x\right| \\
&\leq \int_{|x|<1} |\varphi^{'}(\xi(x))|\ \mathrm{d}x + ||x\varphi||_\infty \int_{|x|>1} \frac{1}{x^2}\ \mathrm{d}x \\
&\leq C\left(||\varphi^{'}||_\infty + ||x\varphi||_\infty\right) < \infty.
\end{aligned}$$

So we get that p.v.$\frac{1}{x}$ is a tempered distribution.

Next proposition we see that p.v.$\frac{1}{x}$ can be seen as a (distributional) limit of certain "nice" function.

Proposition 5.1. *Consider $Q_t(x) = \frac{1}{\pi}\frac{x}{x^2+\pi^2}$. Then, $\forall \varphi \in \mathcal{S}(\mathbb{R})$, we have*

$$\lim_{t \longrightarrow 0} \int_{\mathbb{R}} Q_t(x)\varphi(x)\ \mathrm{d}x = \frac{1}{\pi} p.v.\frac{1}{x}(\varphi)$$

.

Proof. For each $\epsilon > 0$, the functions $\psi_\epsilon(x) = x^{-1}\chi_{\{|x|>\epsilon\}}$ is bounded and define a tempered distribution in the following way

$$\psi_\epsilon(\varphi) = \int_{\mathbb{R}} \psi_\epsilon(x)\phi(x)\mathrm{d}x = \int_{\{|x|>\epsilon\}} \frac{\varphi(x)}{x}\ \mathrm{d}x$$

for $\varphi \in \mathcal{S}(\mathbb{R})$. By the definition of p.v.$\frac{1}{x}$, we have $\forall \varphi \in \mathcal{S}(\mathbb{R})$,

$$\lim_{\epsilon \longrightarrow 0} \psi_\epsilon(\varphi) = \text{p.v.}\frac{1}{x}(\varphi).$$

Therefore it suffices to prove that $\forall \varphi \in \mathcal{S}'(\mathbb{R})$,

$$\lim_{t \longrightarrow 0} \left(Q_t - \frac{1}{\pi}\psi_t \right)(\varphi) = 0.$$

Let us fix $\phi \in \mathcal{S}(\mathbb{R})$. Then,

$$\begin{aligned}(\pi Q_t - \psi_t)(\varphi) &= \int_{\mathbb{R}} \frac{x\varphi(x)}{t^2+x^2}\mathrm{d}x - \int_{|x|>t} \frac{\varphi(x)}{x}\mathrm{d}x \\ &= \int_{|x|<t} \frac{x\varphi(x)}{t^2+x^2}\mathrm{d}x + \int_{|x|>t} \left(\frac{x}{t^2+x^2} - \frac{1}{x} \right) \varphi(x)\mathrm{d}x.\end{aligned}$$

By substituting $x = ty$, we obtain,

$$(\pi Q_t - \psi_t)(\varphi) = \int_{|y|<1} \frac{y\varphi(ty)}{1+y^2}\mathrm{d}y - \int_{|y|>1} \frac{\varphi(ty)}{y(1+y^2)}\mathrm{d}y.$$

Now we see that $\lim\limits_{t\longrightarrow 0} \frac{x\varphi(tx)}{1+x^2} = \frac{x\varphi(0)}{1+x^2}$ and $\lim\limits_{t\longrightarrow 0} \frac{\varphi(tx)}{x(1+x^2)} = \frac{\varphi(0)}{x(1+x^2)}$. Also, we notice that, $\forall x \in (-1,1)$ we have $\frac{x\varphi(tx)}{1+x^2} \leq \frac{C}{1+x^2}$ and for $|x| > 1$ we have $\left|\frac{\varphi(tx)}{x(1+x^2)}\right| \leq \frac{C}{(1+x^2)}$. We know that $\frac{1}{1+x^2}$ is integrable on $\mathbb{R}$. Hence by using the dominated convergence theorem, we get

$$\begin{aligned}\lim_{t\longrightarrow 0}(\pi Q_t - \psi_t)(\varphi) &= \lim_{t\longrightarrow 0} \int_{|y|<1} \frac{y\varphi(ty)}{1+y^2}\mathrm{d}y - \lim_{t\longrightarrow 0} \int_{|y|>1} \frac{\varphi(ty)}{y(1+y^2)}\mathrm{d}y \\ &= \int_{|y|<1} \frac{y\varphi(0)}{1+y^2}\mathrm{d}y - \int_{|y|>1} \frac{\varphi(0)}{y(1+y^2)}\mathrm{d}y.\end{aligned}$$

Since the integrand of both the integrals above are odd functions on a symmetric domain, we have,

$$\lim_{t\longrightarrow 0}(\pi Q_t - \psi_t)(\varphi) = 0.$$

□

5.1.2 Definition and properties of Hilbert transform

As a consequence Proposition 5.1, we define Hilbert transform.

Definition 5.1. *Let $f \in \mathcal{S}(\mathbb{R})$, then we define its Hilbert transform by one of the following expressions:*

$$Hf(x) = \frac{1}{\pi} \lim_{\epsilon \longrightarrow 0} \int_{|y|>\epsilon} \frac{f(x-y)}{y}\, \mathrm{d}y, \tag{5.2}$$

Equivalently, we can define $Hf = p.v.\frac{1}{x} * f = \lim\limits_{t\longrightarrow 0} Q_t * f$.

Next, we see that Hilbert transform can be defined in terms of Fourier transform.

Proposition 5.2. *For $f \in \mathcal{S}(\mathbb{R})$ we have*

$$\widehat{Hf}(\xi) = -isgn(\xi)\widehat{f}(\xi). \tag{5.3}$$

Remark 5.1. Proposition 5.2 is the starting point of our study of multipliers. In chapter 6 we deal with operator T that satisfy $\widehat{Tf} = m\hat{f}$, for a "nice" function m. Indeed Hilbert transform is a prototypical example.

Next, we see a few basic properties of the Hilbert transform. All of them are direct consequence of definition and Proposition 5.2.

Lemma 5.3. *Let $f, g \in L^2(\mathbb{R})$. Then, we have the following*

1. *$H(Hf) = -f$,*
2. *$\widetilde{Hf} = -H\widetilde{f}$, where $\widetilde{f}(x) = f(-x)$,*
3. *$\int\limits_{\mathbb{R}} Hf(x) \cdot g(x)\mathrm{d}x = \int\limits_{\mathbb{R}} f(x) \cdot Hg(x)\mathrm{d}x$.*

Proof.

1. Taking the Fourier transform

$$\begin{aligned}\widehat{H(Hf)} &= -i\mathrm{sgn}(\xi)\widehat{Hf}(\xi) \\ &= (-i\mathrm{sgn}(\xi))^2\widehat{f}(\xi) \\ &= -\widehat{f}(\xi)\end{aligned}$$

Since the Fourier transform on $L^2(\mathbb{R})$ is an isometry, we have, $H(Hf) = -f$.

2. By definition,

$$\begin{aligned}\widetilde{Hf}(x) &= Hf(-x) \\ &= \frac{1}{\pi}\lim_{\epsilon \longrightarrow 0}\int_{|y|>\epsilon} \frac{f(-x-y)}{y}\,\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi} \lim_{\epsilon \longrightarrow 0} \int_{|y|>\epsilon} \frac{\widetilde{f}(x+y)}{y} \, \mathrm{d}y \\
&= \frac{1}{\pi} \lim_{\epsilon \longrightarrow 0} \int_{|z|>\epsilon} \frac{\widetilde{f}(x-z)}{-z} \, \mathrm{d}z \\
&= -H\widetilde{f}(x).
\end{aligned}
$$

3. Using the duality of Fourier transform, and the second part of this Lemma we have the following.

$$
\begin{aligned}
\int_{\mathbb{R}} Hf(x)g(x) \, \mathrm{d}x &= \int_{\mathbb{R}} \widehat{(Hf)}(x)\widehat{\widehat{g}}(x) \, \mathrm{d}x \\
&= \int_{\mathbb{R}} -i\mathrm{sgn}(\xi)\widehat{f}(\xi)\widehat{\widehat{g}}(\xi) \, \mathrm{d}\xi \\
&= \int_{\mathbb{R}} \widehat{f}(\xi)\widehat{(H\widetilde{g})}(\xi) \, \mathrm{d}\xi \\
&= \int_{\mathbb{R}} \hat{\hat{f}}(\xi)(H\widetilde{g})(\xi) \, \mathrm{d}\xi \\
&= -\int_{\mathbb{R}} f(-\xi)\widetilde{Hg})(\xi) \, \mathrm{d}\xi \\
&= -\int_{\mathbb{R}} f(\xi)(Hg)(\xi) \, \mathrm{d}\xi.
\end{aligned}
$$

□

5.1.3 $L^p - L^p$ boundedness of Hilbert transform

We have already seen that Hilbert transform is an isometry on $L^2(\mathbb{R})$. We now ask whether it is bounded on $L^p(\mathbb{R})$ for $p \geq 1$.

Theorem 5.4. *For $f \in \mathcal{S}(\mathbb{R})$, the following assertions are true:*

1. *H is weak $(1,1)$. That is, $\exists C > 0$such that $\forall \lambda > 0$.*

$$|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}| \leq \frac{C}{\lambda}||f||_1.$$

2. *H is strong (p,p), for any $1 < p < \infty$. That is $\exists C_p > 0$ such that*

$$||Hf||_p \leq C_p||f||_p.$$

Proof.

1. Let $\lambda > 0$ be fixed and $f \in \mathcal{S}(\mathbb{R})$ be non-negative. We form the Calderón-Zygmund decomposition of f at the height λ. This gives a sequence of disjoint intervals $\{I_j\}_{j\in\mathbb{N}}$ such that

$$f(x) \leq \lambda \text{ for a.e. } x \notin \Omega = \bigcup_{j=1}^{\infty} I_j, \tag{5.4}$$

$$|\Omega| \leq \frac{1}{\lambda}||f||_1, \tag{5.5}$$

$$\lambda < \frac{1}{|I_j|}\int_{I_j} f(x)\mathrm{d}x \leq 2\lambda. \tag{5.6}$$

Given this decomposition of $\mathbb{R}$, we now decompose $f = g + b$, where g and b are defined by

$$g(x) = \begin{cases} f(x), & \text{if } x \notin \bigcup_j Q_j. \\ \frac{1}{|Q_j|}\int\limits_{Q_j} f(y)\mathrm{d}y, & \text{if } x \in Q_j. \end{cases}$$

And,

$$b(x) = \sum_{j=1}^{\infty} b_j(x), \tag{5.7}$$

where,

$$b_j(x) = \left(f(x) - \frac{1}{|I_j|}\int_{I_j} f(y)\mathrm{d}y\right)\chi_{I_j}(x). \tag{5.8}$$

Then $g(x) \leq 2\lambda$ almost everywhere, and b_j is supported on I_j and has zero integral : Since H is linear, we have $\{x \in \mathbb{R} : |Hf(x)| > \lambda\} \subseteq \{x \in \mathbb{R} : |Hg(x)| > \lambda/2\} \cup \{x \in \mathbb{R} : |Hb(x)| > \lambda/2\}$. So, we have,

$$\begin{aligned} |\{x \in \mathbb{R} : |Hf(x)| > \lambda\}| &\leq |\{x \in \mathbb{R} : |Hg(x)| > \lambda/2\}| \\ &\quad + |\{x \in \mathbb{R} : |Hb(x)| > \lambda/2\}|. \end{aligned} \tag{5.9}$$

We estimate the first term of the above inequality in the following way:

$$|\{x \in \mathbb{R} : |Hg(x)| > \lambda/2\}| \leq (2/\lambda)^2 \int_{\mathbb{R}} |Hg(x)|^2\mathrm{d}x$$

Using the fact that $||Hg||_2 = ||g||_2$ we can write

$$|\{x \in \mathbb{R} : |Hg(x)| > \lambda/2\}| \leq (2/\lambda)^2 \int\limits_{\mathbb{R}} |g(x)|^2 \mathrm{d}x.$$

Since $|g(x)| \leq 2\lambda$, almost everywhere, the above inequality becomes

$$|\{x \in \mathbb{R} : |Hg(x)| > \lambda/2\}| \leq \frac{8}{\lambda} \int\limits_{\mathbb{R}} |g(x)| \, \mathrm{d}x. \tag{5.10}$$

As $\int\limits_{\mathbb{R}} g(x)\mathrm{d}x = \int\limits_{\mathbb{R}} f(x)\mathrm{d}x$ and f is non-negative, we get

$$|\{x \in \mathbb{R} : |Hg(x)| > \lambda/2\}| \leq \frac{8}{\lambda} \int\limits_{\mathbb{R}} f(x) \, \mathrm{d}x = \frac{8}{\lambda}||f||_1. \tag{5.11}$$

To estimate the "bad part" in Inequality (5.9), we consider the following. Let $2I_j$ be the interval with the same center as I_j and twice the length, and let $\Omega^* = \bigcup\limits_{j=1}^{\infty} 2I_j$. Then, from Inequality (5.5), we have

$$|\Omega^*| \leq 2\left|\bigcup_{j=1}^{\infty} I_j\right| = 2|\Omega| \leq \frac{2}{\lambda}||f||_1 \tag{5.12}$$

Now, we see that,

$$\begin{aligned} |\{x \in \mathbb{R} : |Hb(x)| > \lambda/2\}| &\leq |\{x \in \Omega^* : |Hb(x)| > \lambda/2\}| \\ &\quad + |\{x \notin \Omega^* : |Hb(x)| > \lambda/2\}| \\ &\leq |\Omega^*| + |\{x \notin \Omega^* : |Hb(x)| > \lambda/2\}| \end{aligned}$$

From Inequality (5.12), we get

$$\begin{aligned} |\{x \in \mathbb{R} : |Hb(x)| > \lambda/2\}| &\leq \frac{2}{\lambda}||f||_1 + |\{x \notin \Omega^* : |Hb(x)| > \lambda/2\}| \\ &\leq \frac{2}{\lambda}||f||_1 + \frac{2}{\lambda}\int_{\mathbb{R}\setminus\Omega^*} |Hb(x)|\mathrm{d}x. \end{aligned}$$

We know that $\sum\limits_{j=1}^{\infty} b_j$ and $\sum\limits_{j=1}^{\infty} Hb_j$ converge to b and Hb in L^2.Thus, there is a subsequence $T_{n_k} = \sum\limits_{j=1}^{n_k} Hb_j$ such that $T_{n_k} \longrightarrow Hb$ pointwise almost everywhere. Therefore, for almost every $x \in \mathbb{R}$, we have

$$\begin{aligned} |Hb(x)| &= \left|\lim_{k\longrightarrow\infty} \sum_{j=1}^{n_k} Hb_j(x)\right| \\ &= \lim_{k\longrightarrow\infty} \left|\sum_{j=1}^{n_k} Hb_j(x)\right| \end{aligned}$$

$$\leq \lim_{k\longrightarrow\infty} \sum_{j=1}^{n_k} |Hb_j(x)|$$
$$\leq \sum_{j=1}^{\infty} |Hb_j(x)|.$$

Therefore,

$$\int_{\mathbb{R}\setminus\Omega^*} |Hb(x)| \leq \int_{\mathbb{R}\setminus\Omega^*} \sum_{j=1}^{\infty} |Hb_j(x)| \, \mathrm{d}x.$$

Now we have

$$|\{x \in \mathbb{R} : |Hb(x)| > \lambda/2\}| \leq \frac{2}{\lambda}||f||_1 + \frac{2}{\lambda}\int_{\mathbb{R}\setminus\Omega^*} \sum_{j=1}^{\infty} |Hb_j(x)| \, \mathrm{d}x. \quad (5.13)$$

Since $2I_j \subseteq \Omega^*$. We have

$$\int\limits_{\mathbb{R}\setminus\Omega^*} |Hb_j(x)| \, \mathrm{d}x \leq \int\limits_{\mathbb{R}\setminus 2I_j} |Hb_j(x)| \, \mathrm{d}x,$$

for every $j \in \mathbb{N}$. Therefore to complete the proof of the weak $(1,1)$ inequality it suffices to show that

$$\sum_{j=1}^{\infty} \int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)|\mathrm{d}x \leq C||f||_1.$$

Note that, Since $x \notin 2I_j$ and $\mathrm{supp}(b_j) \subseteq I_j$, we have

$$Hb_j(x) = \int_{I_j} \frac{b_j(y)}{x-y}\mathrm{d}y.$$

Suppose the center of I_j is c_j. Then,

$$\int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)|\mathrm{d}x = \int_{\mathbb{R}\setminus 2I_j} \left| \int_{I_j} \frac{b_j(y)}{x-y} \, \mathrm{d}y \right| \mathrm{d}x.$$

Since b_j has zero integral, we have $\int_{I_j} \frac{b_j(y)}{x-c_j}\mathrm{d}y = 0$, and we can write

$$\int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)|\mathrm{d}x = \int_{\mathbb{R}\setminus 2I_j} \left| \int_{I_j} b_j(y) \left(\frac{1}{x-y} - \frac{1}{x-c_j} \right) \, \mathrm{d}y \right| \mathrm{d}x$$
$$\leq \int_{\mathbb{R}\setminus 2I_j} \int_{I_j} |b_j(y)| \frac{|y-c_j|}{|x-y||x-c_j|} \, \mathrm{d}y \, \mathrm{d}x.$$

Now, by applying Fubini's theorem, we arrive at

$$\int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)|\mathrm{d}x \leq \int_{I_j} |b_j(y)| \left(\int_{\mathbb{R}\setminus 2I_j} \frac{|y-c_j|}{|x-y||x-c_j|}\mathrm{d}x \right) \mathrm{d}y.$$

As $y \in I_j$ and $x \in 2I_j$, we have $|y-c_j| < \frac{|I_j|}{2}$ and $|x-y| > \frac{|x-c_j|}{2}$. Therefore

$\frac{|y-c_j|}{|x-y||x-c_j|} < \frac{|I_j|}{|x-c_j|^2}$. we have

$$\int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)|\mathrm{d}x \leq \int_{I_j} |b_j(y)| \left(\int_{\mathbb{R}\setminus 2I_j} \frac{|I_j|}{|x-c_j|^2}\mathrm{d}x \right) \mathrm{d}y. \tag{5.14}$$

Let $2I_j = (a, b)$ where $a, b \in \mathbb{R}$ with $a < b$. Then

$$\begin{aligned}
\int_{\mathbb{R}\setminus 2I_j} \frac{|I_j|}{|x-c_j|^2}\mathrm{d}x &= \int_{-\infty}^{a} \frac{|I_j|}{|c_j-x|^2}\mathrm{d}x + \int_{b}^{\infty} \frac{|I_j|}{|x-c_j|^2} \\
&= \frac{|I_j|}{c_j-a} + \frac{|I_j|}{b-c_j} \\
&= \frac{|I_j|(b-a)}{(c_j-a)(b-c_j)} \\
&= 2,
\end{aligned}$$

$$\int_{\mathbb{R}\setminus 2I_j} \frac{|I_j|}{|x-c_j|^2}\ \mathrm{d}x = 2. \tag{5.15}$$

where the last inequality follows from the fact that $(b-a) = 2|I_j|$ and $(c_j - a) = (b - c_j) = |I_j|$. Therefore from Inequality (5.14), we get

$$\int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)|\mathrm{d}x \leq 2\int_{I_j} |b_j(y)|\mathrm{d}y. \tag{5.16}$$

Now note that

$$\begin{aligned}
\int_{I_j} |b_j(y)|\mathrm{d}y &= \int_{I_j} \left| \left(f(y) - \frac{1}{|I_j|}\int_{I_j} f(x)\mathrm{d}x \right) \chi_{I_j}(y) \right| \mathrm{d}y \\
&\leq \int_{I_j} |f(y)|\mathrm{d}y + \int_{I_j} |f(y)|\mathrm{d}y \\
&= 2\int_{I_j} |f(y)|\mathrm{d}y.
\end{aligned}$$

That is, using the above observation in Inequality (5.16) we get

$$\int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)|\mathrm{d}x \leq 4\int_{I_j} |f(y)|\mathrm{d}y.$$

Therefore

$$\begin{aligned}
\sum_{j}^{\infty} \int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)|\mathrm{d}x &\leq 4\sum_{j=1}^{\infty} \int_{I_j} |f(y)|\mathrm{d}y \\
&= 4\int_{\bigcup_{j=1}^{\infty} I_j} |f(y)|\mathrm{d}y
\end{aligned}$$

$$\leq 4 \int_{\mathbb{R}} |f(y)| \mathrm{d}y$$

$$= 4||f||_1.$$

So from (5.13)

$$|\{x \in \mathbb{R} : |Hb(x)| > \lambda/2\}| \leq \frac{2}{\lambda}||f||_1 + \frac{8}{\lambda}||f||_1 = \frac{10}{\lambda}||f||_1. \tag{5.17}$$

Now by using Inequalities (5.9), (5.10) and (5.17) we have,

$$|\{x \in \mathbb{R} : |Hf(x)| > \lambda| \leq \left(\frac{8}{\lambda} + \frac{10}{\lambda}\right) ||f||_1 = \frac{18}{\lambda}||f||_1.$$

This completes the proof of weak $(1,1)$ boundedness of the Hilbert Transform.

2. We have shown that H is weak $(1,1)$ and strong $(2,2)$. Therefore by Marcinkiewicz interpolation theorem we have strong (p,p) inequality for $1 < p < 2$. Immediately we see that H can be extended to $L^p(\mathbb{R})$ for $1 < p \leq 2$, and is in fact bounded. If $p > 2$, then $p' < 2$ and we have

$$||Hf||_p = \sup\left\{\left|\int_{\mathbb{R}} Hf(x) \cdot g(x) \mathrm{d}x\right| : ||g||_{p'} \leq 1\right\}.$$

Now by using part (3) of Lemma (5.3) we have

$$\begin{aligned} ||Hf||_p &= \sup\left\{\left|\int_{\mathbb{R}} f(x) \cdot Hg(x) \mathrm{d}x\right| : ||g||_{p'} \leq 1\right\} \\ &\leq ||f||_p \sup\{||Hg||_{p'} : ||g||_{p'} \leq 1\} \\ &\leq C_{p'}||f||_p. \end{aligned}$$

in the last inequality we have used the fact that $H : L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})$ is bounded for $1 < p \leq 2$.

□

Remark 5.2. Due to the strong (p,p) inequalities of the Hilbert transform $\forall 1 < p < \infty$ we can continuously extend H to $L^p(\mathbb{R})$. We now see that it is also possible when $p = 1$.

Let $f \in L^1(\mathbb{R})$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{S}(\mathbb{R})$ that converges to f in $L^1(\mathbb{R})$. By using weak $(1,1)$ inequality of H, we have for any $\epsilon > 0$,

$$|\{x \in \mathbb{R} : |(Hf_n - Hf_m)(x)| > \epsilon\}| \leq \frac{1}{\epsilon}||f_n - f_m||_1.$$

Since $(f_n)_{n\in\mathbb{N}}$ is Cauchy in $L^1(\mathbb{R})$, $|\{x \in \mathbb{R} : |(Hf_n - Hf_m)(x)| > \epsilon\}| \longrightarrow 0$ as $m, n \longrightarrow \infty$. So, $(Hf_n)_{n\in\mathbb{N}}$ is Cauchy sequence in measure. Therefore $(Hf_n)_{n\in\mathbb{N}}$ converges to a measurable function in measure which we define to be the Hilbert transform of f. However, we do not claim that $Hf \in L^1(\mathbb{R})$ for $f \in L^1(\mathbb{R})$.

5.1.4 Pointwise convergence of truncated integrals

We have already seen that the definition of Hilbert transform can be extended to the functions in $L^p(\mathbb{R})$. However, this extension gives us no idea of point evaluation of Hf as a function. Here we see that Equation (5.2) is true for almost every $x \in \mathbb{R}$ for any $f \in L^p(\mathbb{R})$. To this end, we notice that the function $\frac{1}{y}\chi_{\{|y|>\epsilon\}} \in L^q(\mathbb{R})$, for any $1 < q \leq \infty$. Hence the function

$$H_\epsilon f(x) = \frac{1}{\pi}\int_{|y|>\epsilon} \frac{f(x-y)}{y}\mathrm{d}y, \tag{5.18}$$

is well defined for any $f \in L^p(\mathbb{R})$, when $1 \leq p < \infty$. Now we see that

$$\begin{aligned}
\widehat{\left(\frac{1}{y}\chi_{\{|y|>\epsilon\}}\right)}(\xi) &= \lim_{N\longrightarrow\infty}\int_{\epsilon<|y|<N} \frac{e^{-2\pi i y\xi}}{y}\ \mathrm{d}y \\
&= \lim_{N\longrightarrow\infty}\int_{\epsilon<|y|<N} \frac{\sin(2\pi y\xi)}{y}\mathrm{d}y \\
&= -2i\mathrm{sgn}(\xi)\lim_{N\longrightarrow\infty}\int_{2\pi\epsilon|\xi|}^{2\pi N|\xi|} \frac{\sin(t)}{t}\mathrm{d}t.
\end{aligned}$$

The last integral is uniformly bounded. We know that $H_\epsilon f = \left(\frac{1}{y}\chi_{\{|y|>\epsilon\}}\right) * f$. Hence using Plancheral theorem, strong $(2,2)$ inequality for H_ϵ holds with the constant independent of ϵ. We can also prove the weak $(1,1)$ inequality essentially to the proof of Theorem 5.4.

Now we see that when $f \in L^p(\mathbb{R})$, is fixed then the family $(H_\epsilon f)$ converges to Hf in $L^p(\mathbb{R})$ if $p > 1$ and in measure if $p = 1$. Indeed, if we fix a sequence

$(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}(\mathbb{R})$ converging to f in L^p, then we have

$$\begin{aligned} ||H_\epsilon f - Hf||_p &\leq ||H_\epsilon f - H_\epsilon f_n||_p + ||H_\epsilon f_n - Hf_n||_p + ||Hf_n - Hf||_p \\ &\leq C||f_n - f||_p + ||H_\epsilon f_n - Hf_n||_p + C||f_n - f||_p. \end{aligned}$$

Now, since $f_n \in \mathcal{S}(\mathbb{R})$, we have,

$$\begin{aligned} Hf_n - H_\epsilon f_n &= \lim_{\eta \longrightarrow 0} \int\limits_{|y|>\eta} \frac{f(x-y)}{y}\,\mathrm{d}y - \int\limits_{|y|>\epsilon} \frac{f(x-y)}{y}\,\mathrm{d}y \\ &= \lim_{\eta \longrightarrow 0} \int\limits_{\eta<|y|<\epsilon} \frac{f(x-y)}{y}\,\mathrm{d}y \\ &= \lim_{\eta \longrightarrow 0} \int\limits_{\eta<|y|<\epsilon} \frac{f(x-y)-f(x)}{|y|}\,\mathrm{d}y, \end{aligned}$$

where the last equality follows from the fact that

$$\int\limits_{\eta<|y|<\epsilon} \frac{1}{y}\,\mathrm{d}y = 0, \forall \eta > 0.$$

$$\begin{aligned} \therefore |Hf_n - H_\epsilon f_n| &\leq \lim_{\eta \longrightarrow 0} \int\limits_{\eta<|y|<\epsilon} \frac{|f(x-y)-f(x)|}{|y|}\,\mathrm{d}y \\ &\lim_{\eta \longrightarrow 0} \int\limits_{\eta<|y|<\epsilon} \frac{|y| \sup\limits_{z\in(x-\epsilon,x+\epsilon)} |f'(x)|}{|y|}\,\mathrm{d}y \\ &= 2\epsilon \sup_{z\in(x-\epsilon,x+\epsilon)} |f'(z)|. \end{aligned}$$

Particularly, for $\epsilon < 1$, we have

$$\sup_{z\in(x-\epsilon,x+\epsilon)} |f'(z)| \leq \sup_{z\in(x-1,x+1)} |f'(z)| \in \mathcal{S}(\mathbb{R}).$$

$$\begin{aligned} \therefore ||Hf_n - H_\epsilon f_n||_p^p &= \int\limits_{\mathbb{R}^n} (2\epsilon)^p \left(\sup_{z\in(x-\epsilon,x+\epsilon)} |f'(z)| \right)^p \mathrm{d}x \\ &= (\alpha\epsilon)^p \left|\left| \sup_{z\in(x-\epsilon,x+\epsilon)} |f'(z)| \right|\right|_p^p. \end{aligned}$$

That is

$$||H_\epsilon f - Hf||_p \leq C\left(||f_n - f||_p + \epsilon\right).$$

Now, given $\eta > 0$, choose $n_0 \in \mathbb{N}$ such that $||f_{n_0} - f||_p < \frac{\eta}{2(1+C)}$ and $\epsilon_0 =$

$\frac{1}{1+C}\min\{1,\frac{\eta}{2}\}$. Then, $\forall 0<\epsilon<\epsilon_0$,

$$||H_\epsilon f - Hf||_p < C\left(\frac{\eta}{2(1+C)}+\frac{\eta}{2(1+C)}\right) < \eta.$$

$\therefore \lim\limits_{\epsilon\longrightarrow 0} H_\epsilon f = Hf$ in the L^p-norm.

Definition 5.2. *Let $f\in L^p(\mathbb{R})$. We define a maximal operator H^* as*

$$H^*f(x) = \sup_{\epsilon>0}|H_\epsilon f(x)|.$$

We want to show that H^* is strong (p,p) for any ,$1<p<\infty$. To that end, we need the following results.

Lemma 5.5. *If $f\in\mathcal{S}(\mathbb{R})$ then $H^*f(x)\le M(Hf)(x)+CMf(x)$ for some constant $C>0$, independent of f.*

Proof. We show that for each $\epsilon>0$, H_ϵ satisfies the inequality with a constant independent of ϵ. Let $\varphi\in\mathcal{S}(\mathbb{R})$ be non-negative, even, decreasing on $(0,\infty)$, supported on $\{x\in\mathbb{R}:|x|\le\frac{1}{2}\}$ with $||\varphi||_1=1$. Let $\varphi_\epsilon(x)=\epsilon^{-n}\varphi(x/\epsilon)$. Then,

$$\frac{1}{y}\chi_{\{|y|>\epsilon\}}(y) = \left(\text{p.v.}\frac{1}{x}*\varphi_\epsilon\right)(y) + \left[\frac{1}{y}\chi_{\{|y|>\epsilon\}}(y) - \left(\text{p.v.}\frac{1}{x}*\varphi_\epsilon\right)(y)\right]. \tag{5.19}$$

Let us find a pointwise estimate for the term $\left[\frac{1}{y}\chi_{\{|y|>\epsilon\}} - \left(\text{p.v.}\frac{1}{x}*\varphi_\epsilon\right)(y)\right]$. It is sufficient to find the estimate when $\epsilon=1$ since it is follows for any other ϵ by dilation.

If $|y|>1$, then,

$$\left|\frac{1}{y}-\left(\text{p.v.}\frac{1}{x}*\varphi\right)(y)\right| = \left|\frac{1}{y}-\lim_{\delta\longrightarrow 0}\int_{|x|>\delta}\frac{\varphi(x)}{y-x}\mathrm{d}x\right|.$$

Since φ is supported on $\{x\in\mathbb{R}:|x|\le\frac{1}{2}\}$, we have

$$\begin{aligned}\left|\frac{1}{y}-\left(\text{p.v.}\frac{1}{x}*\varphi\right)(y)\right| &= \left|\frac{1}{y}-\int_{|x|<\frac{1}{2}}\frac{\varphi(x)}{y-x}\mathrm{d}x\right|\\ &= \left|\int_{|x|<\frac{1}{2}}\left(\frac{1}{y}-\frac{\varphi(x)}{y-x}\right)\mathrm{d}x\right|\\ &\le \int_{|x|<\frac{1}{2}}\frac{\varphi(x)|x|}{|y||y-x|}\mathrm{d}x.\end{aligned}$$

As $|x|<\frac{1}{2}$, we have

$$\left|\frac{1}{y}-\left(\text{p.v.}\frac{1}{x}*\varphi\right)(y)\right| \le \frac{1}{2y^2}\int_{|x|<\frac{1}{2}}\frac{\varphi(x)}{|1-x/y|}\mathrm{d}x.$$

Also since $|x| < \frac{1}{2}$, and $|y| > 1$, we have $|\frac{x}{y}| < \frac{1}{2}$ so $|1 - x/y| > \frac{1}{2}$. Therefore,

$$\left|\frac{1}{y} - \left(\text{p.v.}\frac{1}{x} * \varphi\right)(y)\right| \leq \frac{2}{2y^2}\int_{|x|<\frac{1}{2}} \varphi(x)\mathrm{d}x = \frac{1}{y^2},$$

since $||\varphi||_1 = 1$. That is, in this case,

$$\left|\frac{1}{y}\chi_{\{|y|>\epsilon\}} - \left(\text{p.v.}\frac{1}{x} * \varphi_\epsilon\right)(y)\right| \leq \frac{C}{1+y^2}, \tag{5.20}$$

for some different constant C. If $|y| < 1$, then,

$$\left|\frac{1}{y}\chi_{\{|y|>1\}} - (\text{p.v.}\frac{1}{x} * \varphi)(y)\right| = \left| - \lim_{\delta\longrightarrow 0}\int_{\delta<|x|<2} \frac{\varphi(y-x)}{x}\mathrm{d}x + \int_{|x|>2}\frac{\varphi(y-x)}{x}\right|.$$

For $|y| < 1$ and $|x| > 2$ we have $|y - x| > 1$. Therefore $\varphi(y - x) = 0$. So, $\int_{|x|>2}\frac{\varphi(y-x)}{x} = 0$. Also $\int_{|x|<2}\frac{1}{x}\,\mathrm{d}x = 0$. Hence, using the mean value theorem, we get

$$\begin{aligned}
\left|\frac{1}{y}\chi_{\{|y|>1\}} - (\text{p.v.}\frac{1}{x} * \varphi)(y)\right| &= \left|\lim_{\delta\longrightarrow 0}\int_{\delta<|x|<2}\frac{\varphi(y-x)-\varphi(y)}{x}\mathrm{d}x\right| \\
&= \left|\lim_{\delta\longrightarrow 0}\int_{\delta<|x|<2}\varphi^{'}(\xi(x))\mathrm{d}x\right| \\
&= \left|\int_{|x|<2}\varphi^{'}(\xi(x))\mathrm{d}x\right| \\
&\leq \int_{|x|<2}|\varphi^{'}(\xi(x))|\mathrm{d}x \\
&\leq C||\varphi^{'}||_\infty.
\end{aligned}$$

Hence, in this case, we have

$$\left|\frac{1}{y}\chi_{\{|y|>\epsilon\}} - \left(\text{p.v.}\frac{1}{x} * \phi_\epsilon\right)(y)\right| \leq C. \tag{5.21}$$

Note that if $|y| > 1$ then $\frac{1}{2} < \frac{y^2}{1+y^2} < 1$. Therefore we have $\frac{C}{y^2} < \frac{2C}{1+y^2}$. For $|y| < 1$ we have $1 < 1 + y^2 < 2$ therefore $\frac{1}{2} < \frac{1}{1+y^2}$. So, $C < \frac{2C}{1+y^2}$. Therefore from Inequality (5.21), we have,

$$\left|\frac{1}{y}\chi_{\{|y|>\epsilon\}} - \left(\text{p.v.}\frac{1}{x} * \varphi_\epsilon\right)(y)\right| \leq \frac{C}{1+y^2}.$$

Now, from Equation (5.19), we arrive at

$$\frac{1}{y}\chi_{\{|y|>\epsilon\}} \leq \left(\text{p.v.}\frac{1}{x} * \varphi_\epsilon\right)(y) + \frac{C}{1+y^2}.$$

Now taking convolution with f, we get

$$\left(\frac{1}{y}\chi_{\{|y|>\epsilon\}} * f\right)(x) \leq \left(\left(\text{p.v.}\frac{1}{x} * \varphi_\epsilon\right) * f\right)(x) + \left(\frac{C}{1+y^2} * f\right)(x). \tag{5.22}$$

Now,

$$\left(\left(\text{p.v.}\frac{1}{x} * \varphi_\epsilon\right) * f\right)(x) = \left(\text{p.v.}\frac{1}{x} * f * \varphi_\epsilon\right)(x) = (Hf * \varphi_\epsilon)(x).$$

By using Proposition 3.1, we have conclude

$$\left|\left(\left(\phi_\epsilon * \text{p.v.}\frac{1}{x}\right) * f\right)(x)\right| \leq M(Hf(x)).$$

Also, we have,

$$\left|\left(\frac{C}{1+y^2} * f\right)(x)\right| \leq CMf(x).$$

Therefore from Inequality (5.22), we get

$$|H_\epsilon f(x)| = \left|\left(\frac{1}{y}\chi_{\{|y|>\epsilon\}} * f\right)(x)\right| \leq M(Hf(x)) + CMf(x).$$

This completes the proof. □

We now ready to show that H^* is bounded on $L^p(\mathbb{R})$, for $1 < p < \infty$.

Theorem 5.6. *The operator H^* is strong (p, p) and weak $(1, 1)$.*

Proof. From Lemma 5.5, we get for $1 < p < \infty$,

$$||H^* f||_p \leq ||M(Hf)||_p + C||Mf||_p.$$

By using strong (p, p) boundedness of M and H, we get,

$$||H^* f||_p \leq C||f||_p.$$

That is, H^* is strong (p, p) where $1 < p < \infty$.

Now we show that H^* is weak $(1, 1)$. It is enough to consider $f \geq 0$. For a fix $\lambda > 0$, form the Calderón-Zygmund decomposition of f at height λ. Then we can write f as

$$f = g + b = g + \sum_{j=1}^{\infty} b_j.$$

Where g and b_j are as mentioned in Theorem 5.4. Now $H^* f \leq H^* g + H^* b$. Therefore we have

$$|\{x \in \mathbb{R} : H^* f(x) > \lambda\}| \leq \left|\left\{x \in \mathbb{R} : H^* g(x) > \frac{\lambda}{2}\right\}\right| + \left|\left\{x \in \mathbb{R} : H^* b(x) > \frac{\lambda}{2}\right\}\right|. \tag{5.23}$$

As H^* is strong $(2, 2)$, using an argument similar to that in the Theorem 5.4, we get

$$\left|\left\{x \in \mathbb{R} : H^*g(x) > \frac{\lambda}{2}\right\}\right| \leq C\frac{8}{\lambda}||f||_1. \tag{5.24}$$

Let I_j, $2I_j$, Ω and Ω^* are as in the Theorem 5.4. We get a similar inequality as before from the Theorem 5.4. That is,

$$|\{x \in \mathbb{R} : |H^*b(x)| > \lambda/2\}| \leq \frac{2}{\lambda}||f||_1 + |\{x \notin \Omega^* : |H^*b(x)| > \lambda/2\}| \tag{5.25}$$

Now we estimate the second term of Inequality (5.25). To make the notation less cumbersome, we replace $\frac{\lambda}{2}$ by λ and show that

$$|\{x \notin \Omega^* : |H^*b(x)| > \lambda\}| \leq \frac{C}{\lambda}||f||_1. \tag{5.26}$$

Let $x \notin \Omega^*$ be fixed and $\epsilon > 0$. Then one of the following holds:

1. $(x - \epsilon, x + \epsilon) \cap I_j = I_j$;

2. $(x - \epsilon, x + \epsilon) \cap I_j = \emptyset$;

3. $x - \epsilon \in I_j$ or $x + \epsilon \in I_j$.

For case (1), if $|y| > \epsilon$, then $x - y > x + \epsilon$ or $x - y < x - \epsilon$. Therefore, $x - y \notin (x - \epsilon, x + \epsilon)$. So $x - y \notin I_j$. Since b_j is supported on I_j, we must have $b_j(x - y) = 0$ on $|y| > \epsilon$. Therefore

$$H_\epsilon b_j(x) = \int_{|y|>\epsilon} \frac{b_j(x-y)}{y} \mathrm{d}y = 0.$$

Now for the second case we show that $Hb_j = H_\epsilon b_j$. If this is not true, then $\exists \eta > 0$ such that for any $\tau > 0$, $\exists\ 0 < \delta < \tau$,

$$\left|H_\epsilon b_j - \int_{|y|>\delta} \frac{b_j(x-y)}{y} \mathrm{d}y\right| > \eta.$$

Choosing $\tau = \epsilon$, we have

$$\left|H_\epsilon b_j - \int_{|y|>\delta} \frac{b_j(x-y)}{y} \mathrm{d}y\right| = 0,$$

which is a contradiction! Therefore $Hb_j = H_\epsilon b_j$, when $(x - \epsilon, x + \epsilon) \cap I_j = \emptyset$. Now we have

$$H_\epsilon b_j(x) = \int_{|y|>\epsilon} \frac{b_j(x-y)}{y} \mathrm{d}y.$$

As $|y| > \epsilon$ and is supported on I_j, we have

$$H_\epsilon b_j(x) = \int_{I_j} \frac{b_j(y)}{x-y} \mathrm{d}y.$$

Also, $\int_{I_j} b_j(y)\mathrm{d}y = 0$. So

$$H_\epsilon b_j(x) = \int_{I_j} \left(\frac{b_j(y)}{x-y} - \frac{b_j(y)}{x-c_j} \right) \mathrm{d}y.$$

Therefore,

$$|H_\epsilon b_j(x)| \le \int_{I_j} \left| \frac{1}{x-y} - \frac{1}{x-c_j} \right| |b_j(y)| \mathrm{d}y. \tag{5.27}$$

As $|y-c_j| < \frac{|I_j|}{2}$ and $|x-y| > \frac{|x-c_j|}{2}$,

$$|H_\epsilon b_j(x)| \le \int_{I_j} \frac{|I_j|}{|x-c_j|^2} |b_j(y)| \mathrm{d}y = \frac{|I_j|}{|x-c_j|^2} ||b_j||_1. \tag{5.28}$$

For case (3), as $x \notin \Omega^*$, $I_j \subset (x-3\epsilon, x+3\epsilon)$ can be shown with a figure, (see above) and for all $y \in I_j$, $|x-y| > \frac{\epsilon}{3}$. Therefore,

$$|H_\epsilon b_j(x)| \le \int_{I_j} \frac{|b_j(y)|}{|x-y|} \mathrm{d}y \le \frac{3}{\epsilon} \int_{x-3\epsilon}^{x+3\epsilon} |b_j(y)| \mathrm{d}y. \tag{5.29}$$

If we sum over all j's , using Inequalities (5.28) and (5.29) we have,

$$\begin{aligned} |H_\epsilon b(x)| &\le \sum_{j=1}^{\infty} |H_\epsilon b_j(x)| \\ &\le \sum_{j=1}^{\infty} \frac{|I_j|}{|x-c_j|^2} ||b_j||_1 + \frac{3}{\epsilon} \sum_{j=1}^{\infty} \int_{x-3\epsilon}^{x+3\epsilon} |b_j(y)| \ \mathrm{d}y \end{aligned}$$

For the first term of the above inequality, the sum runs over all j's for which case (2) holds. For the second term, sum is running over j's for which case (3) holds. Now by Equation (5.7) and the fact that $(I_j)_{j \in \mathbb{N}}$ is a pairwise disjoint collection, we have

$$|H_\epsilon b(x)| \le \sum_{j=1}^{\infty} \frac{|I_j|}{|x-c_j|^2} ||b_j||_1 + \frac{3}{\epsilon} \int_{x-3\epsilon}^{x+3\epsilon} |b(y)| \ \mathrm{d}y$$

Note that $\frac{1}{6\epsilon} \int_{x-3\epsilon}^{x+3\epsilon} |b(y)| \mathrm{d}y \le CMb(x)$ we have

$$|H_\epsilon b(x)| \le \sum_{j=1}^{\infty} \frac{|I_j|}{|x-c_j|^2} ||b_j||_1 + CMb(x).$$

It follows from this that,

$$|\{x \notin \Omega^* : H^* b(x) > \lambda\}|$$

$$\leq \left|\left\{x \notin \Omega^* : \sum_{j=1}^{\infty} \frac{|I_j|}{|x-c_j|^2}||b_j||_1 > \frac{\lambda}{2}\right\}\right| + \left|\left\{x \in \mathbb{R} : Mb(x) > \frac{\lambda}{2C}\right\}\right|$$
$$\leq \frac{2}{\lambda}\int_{\mathbb{R}\backslash\Omega^*} \sum_{j=1}^{\infty} \frac{|I_j|}{|x-c_j|^2}||b_j||_1 \, \mathrm{d}x + \frac{C'}{\lambda}||b||_1.$$

The last inequality follows from the fact that M is weak $(1,1)$. As $\mathbb{R}\backslash\Omega^* \subset \mathbb{R}\backslash 2I_j$ we have,

$$|\{x \notin \Omega^* : H^*b(x) > \lambda\}| \leq \sum_{j=1}^{\infty} \int_{\mathrm{R}\backslash 2I_j} \frac{|I_j|}{|x-c_j|^2}||b_j||_1 + \frac{C'}{\lambda}||b||_1.$$

From Equation (5.15) we get

$$|\{x \notin \Omega^* : H^*b(x) > \lambda\}| \leq \frac{4}{\lambda}\sum_{j=1}^{\infty}||b_j||_1 + \frac{C'}{\lambda}\sum_{j=1}^{\infty}||b_j||_1$$
$$\leq \frac{C"}{\lambda}||b||_1 \leq \frac{C}{\lambda}||f||_1.$$

Therefore,

$$\left|\left\{x \notin \Omega^* : H^*b(x) > \frac{\lambda}{2}\right\}\right| \leq \frac{C_1}{\lambda}||f||_1.$$

This completes the proof. □

We have seen that if $f \in L^p(\mathbb{R})$ then $H_\epsilon f$ converges to Hf in L^p norm. The following theorem shows that, we also have pointwise convergence.

Theorem 5.7. *Given $f \in L^p(\mathrm{R})$, for $1 \leq p < \infty$, we have*

$$Hf(x) = \lim_{\epsilon \longrightarrow 0} H_\epsilon f(x) \ \ a.e\ x \in \mathbb{R}. \tag{5.30}$$

Proof. Since $H_\epsilon f$ converges to Hf in $L^p(\mathbb{R})$, there exists a subsequence $\{H_{\epsilon_k} f\}$ such that Equation(5.30) holds. We only need to show that $\lim_{\epsilon \longrightarrow 0} H_\epsilon f(x)$ exists for almost every $x \in \mathbb{R}$. From Theorem 5.6, H^* is weak (p,p) for $1 \leq p < \infty$. Therefore by using the Theorem 2.7 we have the set $\{f \in L^p(\mathbb{R}) : \lim_{\epsilon \longrightarrow 0} H_\epsilon f(x)$ exists a.e$\}$ is closed. For the functions in $\mathcal{S}(\mathbb{R})$ this limit exists almost everywhere. Therefore $\mathcal{S}(\mathbb{R}) \subseteq \{f \in L^p(\mathbb{R}) : \lim_{\epsilon \longrightarrow 0} H_\epsilon f(x)$ exists a.e$\}$. Since $\mathcal{S}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, we must have $\{f \in L^p(\mathbb{R}) : \lim_{\epsilon \longrightarrow 0} H_\epsilon f(x)$ exists a.e $\} = L^p(\mathbb{R})$. □

5.2 Calderón-Zygmund operator

Let us now observe a few things from our study of Hilbert transform. To show L^p boundedness, it was enough to show L^2 boundedness and weak $(1,1)$ inequality. The desired result then followed by interpolation argument. For the L^2 boundedness of Hilbert transform, all that was needed was an observation about its Fourier transform. On the other hand, for weak $(1,1)$ bound, Calderón-Zygmuund decomposition together with certain properties of the function $\frac{1}{x}$, played an important role.

In this section we pick up the study of general convolution type operator, $Tf = K * f$. Again our goal is to see if T is bounded on $L^p(\mathbb{R}^n)$. To this end, we try to mimic the proof of boundedness of Hilbert transform. The conditions required to do so are given in the next theorem.

Theorem 5.8 (Calderön-Zygmund)**.** *Let K be locally integrable function on $\mathbb{R}^n \setminus \{0\}$ such that its Fourier transform is a function on $\mathbb{R}^n$, and*

$$|\widehat{K}(\xi)| \leq A.$$

Moreover, assume that there is some $B > 0$ such that for all $y \in \mathbb{R}^n$, we have,

$$\int_{|x|>2|y|} |K(x-y) - K(x)| \, \mathrm{d}x \leq B. \tag{5.31}$$

Then for any $1 < p < \infty$, we have a constant $C_p > 0$ such that $\forall f \in L^p(\mathbb{R}^n)$,

$$||K * f||_p \leq C_p ||f||_p.$$

Further, we also have a constant $C > 0$ such that $\forall f \in L^1(\mathbb{R}^n)$,

$$|\{x \in \mathbb{R}^n : |K * f(x)| > \lambda\}| \leq \frac{C}{\lambda} ||f||_1.$$

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$, and let $Tf = K * f$. First we show that T is bounded on $L^2(\mathbb{R}^n)$. Note that, since $\widehat{K}(\xi)$ is bounded, using the Plancheral theorem, we get,

$$||Tf||_2 = ||K * f||_2 = ||\widehat{K * f}||_2 = ||\widehat{K}(\xi)\widehat{f}(\xi)||_2 \leq A||\widehat{f}||_2 = A||f||_2.$$

Next, we prove weak $(1,1)$ boundedness of T. As before we employ the Calderón-Zygmund decomposition. Let $\lambda > 0$ be fixed and $f \in L^1(\mathbb{R}^n)$ be positive. This gives a sequence of disjoint dyadic cubes $\{Q_j\}_{j\in\mathbb{N}}$ such that

$$f(x) \leq \lambda \text{ for a.e. } x \notin \Omega = \bigcup_{j=1}^{\infty} Q_j,$$

$$|\Omega| \leq \frac{1}{\lambda}||f||_1, \tag{5.32}$$

and

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j} f(x)\ \mathrm{d}x \leq 2^n\lambda. \tag{5.33}$$

Given this decomposition of $\mathbb{R}^n$, we write $f = g + b$, where,

$$g(x) = \begin{cases} f(x), & \text{if } x \notin \bigcup\limits_j Q_j. \\ \frac{1}{|Q_j|}\int\limits_{Q_j} f(y)\mathrm{d}y, & \text{if } x \in Q_j. \end{cases}$$

And,

$$b(x) = \sum_{j=1}^{\infty} b_j(x), \tag{5.34}$$

where,

$$b_j(x) = \left(f(x) - \frac{1}{|Q_j|}\int_{Q_j} f(y)\mathrm{d}y\right)\chi_{Q_j}(x).$$

We know that for almost every $x \in \mathbb{R}^n$, we have

$$g(x) \leq 2^n\lambda. \tag{5.35}$$

Also, we have that b_j is supported on Q_j and has zero integral. Using the linearity of T we get,

$$\begin{aligned} \left|\left\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\right\}\right| &\leq \left|\left\{x \in \mathbb{R}^n : |Tg(x)| > \frac{\lambda}{2}\right\}\right| \\ &\quad + \left|\left\{x \in \mathbb{R}^n : |Tb(x)| > \frac{\lambda}{2}\right\}\right|. \end{aligned} \tag{5.36}$$

From the L^2-boundedness of T, we have

$$\left|\left\{x \in \mathbb{R}^n : |Tg(x)| > \frac{\lambda}{2}\right\}\right| \leq \left(\frac{2}{\lambda}\right)^2 \int\limits_{\mathbb{R}^n} |Tg(x)|^2\ \mathrm{d}x \leq \frac{4A}{\lambda^2}\int\limits_{\mathbb{R}^n} |g(x)|^2\ \mathrm{d}x$$

Using Inequality (5.35) we have

$$\begin{aligned} \left|\left\{x \in \mathbb{R}^n : |Tg(x)| > \frac{\lambda}{2}\right\}\right| &\leq \frac{2^{n+2}A\lambda}{\lambda^2}\int\limits_{\mathbb{R}^n} |g(x)|\ \mathrm{d}x \\ &\leq \frac{2^{n+2}A}{\lambda}\int\limits_{\mathbb{R}^n} |f(x)|\ \mathrm{d}x. \end{aligned}$$

Last inequality follows from the fact that $||g||_1 = ||f||_1$. So we have

$$|\{x \in \mathbb{R}^n : |Tg(x)| > \frac{\lambda}{2}\}| \leq \frac{2^{n+2}A}{\lambda} \int\limits_{\mathbb{R}^n} |f(x)| \, \mathrm{d}x. \tag{5.37}$$

Now let Q_j^* be the same center as Q_j where sides are $2\sqrt{n}$ times larger and let

$$\Omega^* = \bigcup_{j \in \mathbb{N}} Q_j^*.$$

Then,

$$|\Omega^*| \leq 2^n n^{n/2} |\Omega| \leq \frac{2^n n^{n/2}}{\lambda} ||f||_1.$$

Now,

$$\begin{aligned}
&\left|\left\{x \in \mathbb{R}^n : |Tb(x)| > \frac{\lambda}{2}\right\}\right| \\
&\leq \left|\left\{x \in \Omega^* : |Tb(x)| > \frac{\lambda}{2}\}| + |\{x \notin \Omega^* : |Tb(x)| > \frac{\lambda}{2}\right\}\right| \\
&\leq |\Omega^*| + \left|\left\{x \notin \Omega^* : |Tb(x)| > \frac{\lambda}{2}\right\}\right| \\
&\leq \frac{2^n n^{n/2}}{\lambda} ||f||_1 + \frac{2}{\lambda} \int_{\mathbb{R}^n \setminus \Omega^*} |Tb(x)| \, \mathrm{d}x \\
&\leq \frac{2^n n^{n/2}}{\lambda} ||f||_1 + \frac{2}{\lambda} \int_{\mathbb{R}^n \setminus \Omega^*} \sum_{j=1}^{\infty} |Tb_j(x)| \, \mathrm{d}x \\
&\leq \frac{2^n n^{n/2}}{\lambda} ||f||_1 + \frac{2}{\lambda} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n \setminus \Omega^*} |Tb_j(x)| \, \mathrm{d}x \\
&\leq \frac{2^n n^{n/2}}{\lambda} ||f||_1 + \frac{2}{\lambda} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n \setminus Q_j^*} |Tb_j(x)| \, \mathrm{d}x
\end{aligned}$$

That is we get

$$\left|\left\{x \in \mathbb{R}^n : |Tb(x)| > \frac{\lambda}{2}\right\}\right| \leq \frac{2^n n^{n/2}}{\lambda} ||f||_1 + \frac{2}{\lambda} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n \setminus Q_j^*} |Tb_j(x)| \, \mathrm{d}x \tag{5.38}$$

Now, we observe that

$$Tb_j(x) = K * b_j(x) = \int\limits_{\mathbb{R}^n} K(x-y) b_j(y) \, \mathrm{d}y = \int\limits_{Q_j} K(x-y) b_j(y) \, \mathrm{d}y.$$

Here the last inequality follows from the fact that b_j is supported on Q_j. Now, because b_j has zero integral, we have

$$\int\limits_{Q_j} K(x-c_j) b_j(y) \, \mathrm{d}y = 0,$$

where c_j is the center of the cube Q_j. Therefore,

$$Tb_j(x) = \int_{Q_j} [K(x-y) - K(x-c_j)]b_j(y)\ \mathrm{d}y.$$

This implies

$$|Tb_j(x)| \leq \int_{Q_j} |K(x-y) - K(x-c_j)||b_j(y)|\ \mathrm{d}y.$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus Q_j^*} |Tb_j(x)|\ \mathrm{d}x &\leq \int_{\mathbb{R}^n \setminus Q_j^*} \left(\int_{Q_j} |K(x-y) - K(x-c_j)||b_j(y)|\ \mathrm{d}y \right)\ \mathrm{d}x \\
&= \int_{Q_j} \left(\int_{\mathbb{R}^n \setminus Q_j} |K(x-y) - K(x-c_j)|\ \mathrm{d}x \right) |b_j(y)|\ \mathrm{d}y. \qquad (5.39)
\end{aligned}$$

It is easily seen that

$$\mathbb{R}^n \setminus Q_j^* \subseteq \{x \in \mathbb{R}^n : |x - c_j| > 2|y - c_j|\}.$$

Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus Q_j} |K(x-y) - K(x-c_j)|\ \mathrm{d}x \\
&\leq \int_{|x-c_j|>2|y-c_j|} |K(x-y) - K(x-c_j)|\ \mathrm{d}x \\
&\leq \int_{|x-c_j|>2|y-c_j|} |K((x-c_j) - (y-c_j)) - K(x-c_j)|\ \mathrm{d}x \\
&\leq B
\end{aligned}$$

The last inequality follows from the Condition (5.31) in the hypothesis of the theorem and a simple change of variable. Now from Inequality (5.39) we have,

$$\int_{\mathbb{R}^n \setminus Q_j^*} |Tb_j(x)|\ \mathrm{d}x \leq B \int_{Q_j} |b_j(y)|\ \mathrm{d}y.$$

Therefore from Inequality (5.38) we get

$$\left| \left\{ x \in \mathbb{R}^n : |Tb(x)| > \frac{\lambda}{2} \right\} \right| \leq \frac{2^n n^{n/2}}{\lambda} ||f||_1 + \frac{2B}{\lambda} \sum_{j=1}^{\infty} \int_{Q_j} |b_j(y)|\ \mathrm{d}y. \qquad (5.40)$$

Now, we see that

$$\begin{aligned}\int_{Q_j} |b_j(x)| \,\mathrm{d}x &\leq \int_{Q_j} \left| \left(f(x) - \frac{1}{|Q_j|} \int_{Q_j} f(y) \,\mathrm{d}y \right) \chi_{Q_j}(x) \right| \mathrm{d}x \\ &\leq \int_{Q_j} |f(x)| \,\mathrm{d}x + \left(\frac{1}{|Q_j|} \int_{Q_j} |f(y)| \,\mathrm{d}y \right) |Q_j| \\ &\leq 2 \int_{Q_j} |f(y)| \,\mathrm{d}y.\end{aligned}$$

Using above observations in Inequality (5.40) we get

$$\begin{aligned}\left| \left\{ x \in \mathbb{R}^n : |Tb(x)| > \frac{\lambda}{2} \right\} \right| &\leq \frac{2^n n^{n/2}}{\lambda} ||f||_1 + \frac{4B}{\lambda} \sum_{j=1}^{\infty} \int_{Q_j} |f(y)| \,\mathrm{d}y \\ &\leq \left(\frac{2^n n^{n/2}}{\lambda} + \frac{4B}{\lambda} \right) ||f||_1.\end{aligned}$$

Using the above inequality and Inequality (5.37) in Inequality (5.36) we conclude that T is weak $(1,1)$. Note that we already proved that T is strong $(2,2)$. Therefore by using Marcinkiewicz interpolation theorem, T is bounded on $L^p(\mathbb{R}^n)$, for any $1 < p \leq 2$. To prove the result for $p \geq 2$ we use a duality arguments. For the same, let us first study the transpose of T. For $f, g \in \mathcal{S}(\mathbb{R})$, whence we can use Fubini's theorem, we have,

$$\begin{aligned}\int_{\mathbb{R}^n} Tf(x)g(x) \,\mathrm{d}x &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x-y)f(y) \,\mathrm{d}y \right) g(x) \,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x-y)g(x) \,\mathrm{d}x \right) f(y) \,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \widetilde{K}(y-x)g(x) \,\mathrm{d}x \right) f(y) \,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \left(\widetilde{K} * g \right)(y) f(y) \,\mathrm{d}y,\end{aligned}$$

where $\widetilde{K}(x) = K(-x)$. Therefore transpose of the operator T is $T^t g = \widetilde{K} * g$. We show that $\widetilde{K}$ satisfies the hypothesis given in the theorem. Since $\widehat{\widetilde{K}} = \widetilde{\widehat{K}}$, we have for all $\xi \in \mathbb{R}^n$,

$$|\widehat{\widetilde{K}}(\xi)| = |\widehat{K}(-\xi)| \leq A.$$

Also,

$$\begin{aligned}\int_{|x|>2|y|} |\widetilde{K}(x-y) - \widetilde{K}(x)| \, \mathrm{d}x &= \int_{|x|>2|y|} |K(y-x) - K(-x)| \, \mathrm{d}x \\ &= \int_{|x|>2|y|} |K(x+y) - K(x)| \, \mathrm{d}x \leq B.\end{aligned}$$

Therefore from the observation above, T^t is also bounded on $L^p(\mathbb{R}^n)$, for any $1 < p \leq 2$. Thus for $p > 2$, we have

$$\begin{aligned}||Tf||_p &= \sup\left\{\left|\int\limits_{\mathbb{R}^n} Tf(x)g(x) \, \mathrm{d}x\right| : ||g||_{p'} \leq 1\right\} \\ &= \sup\left\{\left|\int\limits_{\mathbb{R}^n} f(x)T^t g(x) \, \mathrm{d}x\right| : ||g||_{p'} \leq 1\right\}.\end{aligned}$$

Using Hölder's inequality, we get,

$$\begin{aligned}||Tf||_p &\leq \sup\left\{||f||_p ||T^t g||_{p'} : ||g||_{p'} \leq 1\right\} \\ &\leq ||f||_p \sup\left\{||T|| ||g||_{p'} : ||g||_{p'} \leq 1\right\} \\ &\leq C||f||_p.\end{aligned}$$

This completes the proof. □

The Condition (5.31) known as **Hörmander condition** . It is to be noticed that Hörmander condition is crucial to the proof of Theorem 5.8, while the condition on Fourier transform of K for our boundedness result. It is the Hörmander condition that gives way to the proof. This generalises the idea! If a convolution type operator is known to be bounded, then the only condition we require is the Hörmander condition. However, we observe that it has nothing to do with convolution kernel! In fact, it can be written for function $K(x, y)$ of two variables. This observation motivates us to generalize convolution-type operator satisfying the hypothesis of Theorem 5.8. Let us begin by formally defining a kernel. We denote

$$\Delta := \{(x, x) \in (\mathbb{R}^n \times \mathbb{R}^n) | x \in \mathbb{R}^n\}$$

.

Definition 5.3 (Standard Kernel)**.** *A function $K : (\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta \longrightarrow \mathbb{C}$ is a standard kernel if there exists $\delta > 0$ and $C > 0$ such that $\forall (x, y) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta$, we have*

$$|K(x,y)| \leq \frac{C}{|x-y|^n}, \tag{5.41}$$

$$|K(x,y) - K(x,z)| \leq C\frac{|y-z|^\delta}{|x-y|^{n+\delta}} \quad \textit{if} \quad |x-y| > 2|y-z|, \tag{5.42}$$

$$|K(x,y) - K(w,y)| \leq C\frac{|x-w|^\delta}{|x-y|^{n+\delta}} \quad \textit{if} \quad |x-y| > 2|x-w|. \tag{5.43}$$

Now we can define generalized Calderón-Zygmund operator in the following way.

Definition 5.4 (Generalized Calderón-Zygmund Operator)**.** *An operator T is generalized Calderón-Zygmund operator if*

1. *T is bounded on $L^2(\mathbb{R}^n)$, and*

2. *there exists a standard kernel K such that for $f \in L^2(\mathbb{R}^n)$ with compact support, whenever $x \notin$ supp (f),*

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y) \; \mathrm{d}y.$$

Before we start studying the L^p-boundedness of Generalized Caldeón-Zygmund operators, we consider the following easy result about their transpose. It is helpful for the duality argument we wish to employ.

Lemma 5.9. *Let T be a generalized Calderón-Zygmund operator with kernel K. Then, its transpose T^t is also a generalized Calderón-Zygmund operator with kernel $\tilde{K}$. Here, we define $\tilde{K}(x,y) = K(y,x)$.*

Proof. Since T is a Calderón-Zygmund operator, T is bounded on $L^2(\mathbb{R}^n)$. It is, therefore, clear that T^t is bounded on $L^2(\mathbb{R}^n)$. Now, suppose $f, g \in L^2(\mathbb{R}^n)$ are of compact supports such that $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset$. Then, we have,

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)\, T^t g(x)\, \mathrm{d}x &= \int_{\mathbb{R}^n} Tf(x)\, g(x)\, \mathrm{d}x \\
&= \int_{\operatorname{supp}(g)} Tf(x)\, g(x)\, \mathrm{d}x \\
&= \int_{\operatorname{supp}(g)} \left(\int_{\mathbb{R}^n} K(x,y)\, f(y)\, \mathrm{d}y \right) g(x)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} \tilde{K}(y,x)\, g(x)\, \mathrm{d}x \right) \mathrm{d}y.
\end{aligned}$$

Now, fix $g \in L^2(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp}(g)$. We wish to show that

$$T^t g(x) = \int_{\mathbb{R}^n} \tilde{K}(x,y)\, g(y)\, \mathrm{d}y.$$

For this, consider an approximation to identity $\{\varphi_\epsilon\}_{\epsilon>0}$, where $\operatorname{supp}(\varphi_\epsilon) \subseteq B(0,\epsilon)$. For instance, one may consider the family defined in Example 2.1. It is then clear that $\operatorname{supp}(\tau_x \tilde{\varphi}_\epsilon) \subseteq B(x,\epsilon)$, where τ_x is the translation by x and $\tilde{\varphi}(z) = \varphi(-z)$. Since $\operatorname{supp}(g)$ is compact, there is some $\epsilon > 0$ such that $\operatorname{supp}(\varphi_\epsilon) \cap \operatorname{supp}(g) = \emptyset$. Consequently, we have

$$\begin{aligned}
T^t g(x) &= \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \varphi_\epsilon(x-y)\, T^t g(z)\, \mathrm{d}z \\
&= \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \varphi_\epsilon(x-z) \left(\int_{\mathbb{R}^n} \tilde{K}(z,y)\, g(y)\, \mathrm{d}y \right) \mathrm{d}z \\
&= \int_{\mathbb{R}^n} \tilde{K}(x,y)\, g(y)\, \mathrm{d}y.
\end{aligned}$$

□

We now generalize Theorem 5.8 to certain operators.

Theorem 5.10. *Let T be bounded operator on $L^2(\mathbb{R}^n)$, and let K be a function on*

$(\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta$ such that if $f \in L^2(\mathbb{R}^n)$ has compact support, then for $x \notin supp(f)$,

$$Tf(x) = \int\limits_{\mathbb{R}^n} K(x,y) f(y)\ \mathrm{d}y.$$

Further, suppose K also satisfies

$$\int_{|x-y|>2|y-z|} |K(x,y) - K(x,z)| \leq C \tag{5.44}$$

and,

$$\int_{|x-y|>2|x-w|} |K(x,y) - K(w,y)| \leq C. \tag{5.45}$$

Then T is weak $(1,1)$ and strong (p,p), for any $1 < p < \infty$.

Proof. With the help of similar arguments, as those used in the proof of Theorem 5.8, forming Calderön-Zygmund decomposition for function f at a height λ and using the fact that T is bounded on $L^2(\mathbb{R}^n)$, it can be proved that for the good part we have

$$\left| \left\{ x \in \mathbb{R}^n : |Tg(x)| > \frac{\lambda}{2} \right\} \right| \leq \frac{C}{\lambda} \int\limits_{\mathbb{R}^n} |f(x)|\ \mathrm{d}x.$$

Similarly for the bad part, we have,

$$\left| \left\{ x \in \mathbb{R}^n : |Tb(x)| > \frac{\lambda}{2} \right\} \right| \leq \frac{2^n n^{n/2}}{\lambda} ||f||_1 + \frac{2}{\lambda} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n \setminus Q_j^*} |Tb_j(x)|\ \mathrm{d}x.$$

Now we note that if $x \notin \operatorname{supp}(b_j) = Q_j$, we have

$$Tb_j(x) = \int\limits_{\mathbb{R}^n} K(x,y) b_j(y)\ \mathrm{d}y = \int\limits_{Q_j} K(x,y) b_j(y)\ \mathrm{d}y.$$

Since b_j has zero integral, we can write

$$Tb_j(x) = \int\limits_{Q_j} (K(x,y) - K(x,c_j)) b_j(y)\ \mathrm{d}y.$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus Q_j^*} |Tb_j(x)|\ \mathrm{d}x &= \int_{\mathbb{R}^n \setminus Q_j^*} \left| \int_{Q_j} (K(x,y) - K(x,c_j)) b_j(y)\ \mathrm{d}y \right| \mathrm{d}x \\
&\leq \int_{\mathbb{R}^n \setminus Q_j^*} \int_{Q_j} |K(x,y) - K(x,c_j)| |b_j(y)|\ \mathrm{d}y\ \mathrm{d}x \\
&= \int\limits_{Q_j} |b_j(y)| \left(\int_{\mathbb{R}^n \setminus Q_j^*} |K(x,y) - K(x,c_j)| \mathrm{d}x \right) \mathrm{d}y.
\end{aligned}$$

Now, we know from the proof of the Theorem 5.8 that $\mathbb{R}^n \setminus Q_j^* \subseteq \{x \in \mathbb{R}^n :$

$|x-c_j| > 2|y-c_j|\}$. Therefore, using condition (5.44), we have

$$\int_{\mathbb{R}^n\setminus Q_j^*} |K(x,y)-K(x,c_j)|\mathrm{d}x \le C.$$

Hence,

$$\int_{\mathbb{R}^n\setminus Q_j^*} |Tb_j(x)|\ \mathrm{d}x \le C\int_{Q_j} |b_j(y)|\ \mathrm{d}y.$$

The rest of the arguments to prove T is weak $(1,1)$ are verbatim to the proof of Theorem 5.8.

It is given that T is strong $(2,2)$. Therefore by Marcinkeicz Interpolation theorem T is strong (p,p) for $1<p\le 2$. To prove T is strong (p,p) for $p>2$, we use the duality argument. From Lemma 5.9, we know that T^t is a generalized Calderón-Zygmund operator with kernel $\tilde{K}$. From Condition (5.45), we have,

$$\int_{|x-y|>2|x-w|} |\widetilde{K}(y,x)-\widetilde{K}(y,w)|\ \mathrm{d}y = \int_{|x-y|>2|x-w|} |K(x,y)-K(w,y)|\ \mathrm{d}y \le C.$$

Therefore $\widetilde{K}$ satisfies the condition (5.44) and hence T^t is weak $(1,1)$. As T is bounded on $L^2(\mathbb{R}^n)$, T^t is also bounded $L^2(\mathbb{R}^n)$. Therefore by Marcinkeicz Interpolation theorem T^t is bounded on $L^p(\mathbb{R}^n)$, for $1<p\le 2$. Now the fact that T is bounded on $L^p(\mathbb{R}^n)$ for $p>2$ follows from duality arguments, used in Theorem 5.8. □

We now see that a standard kernel satisfies the hypotheses of Theorem 5.10. Consequently generalized Calderoï-Zygmund operator are weak $(1,1)$ and strong (p,p) for any $1<p<\infty$.

Lemma 5.11. *A standard kernel K satisfies conditions (5.44) and (5.45).*

Proof. We note that

$$\begin{aligned}\int_{|x-y|>2|y-z|} |K(x,y)-K(x,z)|\ \mathrm{d}x &\le C\int_{|x-y|>2|y-z|} \frac{|y-z|^\delta}{|x-y|^{n+\delta}}\ \mathrm{d}x\\ &\le C|y-z|^\delta \int_{\mathbb{S}^{n-1}}\int_{r>2|y-z|} \frac{r^{n-1}\mathrm{d}r}{r^{n+\delta}}\mathrm{d}\sigma\\ &\le C|y-z|^\delta|\mathcal{S}^{n-1}|\int_{r>2|y-z|}\frac{\mathrm{d}r}{r^{1+\delta}}\end{aligned}$$

$$= C.$$

Condition (5.45) is proved similarly. □

Now we can prove the following important result.

Theorem 5.12. *A generalized Calderón-Zygmund operator T is bounded on $L^p(\mathbb{R}^n)$, for $1 < p < \infty$ and is weak $(1,1)$.*

Proof. Note that, with the help of Lemma 5.11, it is clear that T satisfies all the hypothesis of Theorem 5.10. That completes the proof. □

5.3 Weighted inequalities for Calderón-Zygmund operators

In this section we study the boundedness of (generalized) Calderön-Zygmund operators on weighted L^p spaces. We keep our focus on Muckenhoupt weights, which we have studied in Chapter 4. The following Lemma is crucial to us.

Lemma 5.13. *If T is a Calderón-Zygmund operator, then for each $s > 1$, we have,*

$$M^{\#}(Tf)(x) \leq C_s M(|f|^s)(x)^{1/s}.$$

Proof. Fix $s > 1$ and $x \in \mathbb{R}^n$. Let g be an arbitrary cube Q containing x. If we can find $a_0 \in \mathbb{C}$, such that

$$\frac{1}{|Q|} \int_Q |Tf(y) - a_0| \text{ d}y \leq CM(|f|^s)(x)^{1/s}, \tag{5.46}$$

then by Proposition 3.11, we would have

$$\frac{1}{2}||Tf||_* \leq \sup_Q \inf_{a \in \mathbb{C}} \frac{1}{|Q|} \int_Q |Tf(y) - a| \text{ d}y \leq CM(|f|^s)(x)^{1/s}.$$

Now for such a cube let $f_1 = f\chi_{Q^*}$. Where $Q^* = (4\sqrt{n}+1)Q$. We write f as $f = f_1 + f_2$, and let $a_0 = Tf_2(x)$. Then

$$\begin{aligned}\frac{1}{|Q|}\int\limits_Q |Tf(y) - a_0|\ \mathrm{d}y &= \frac{1}{|Q|}\int\limits_Q |Tf(y) - Tf_2(x)|\ \mathrm{d}y\\ &= \frac{1}{|Q|}\int\limits_Q |Tf_1(y) + Tf_2(y) - Tf_2(x)|\ \mathrm{d}y\\ &\le \frac{1}{|Q|}\int\limits_Q |Tf_1(y)|\ \mathrm{d}y + \frac{1}{|Q|}\int\limits_Q |Tf_2(y) - Tf_2(x)|\ \mathrm{d}y.\end{aligned}$$

Using Hölder inequality for exponent s and s', we have

$$\begin{aligned}\frac{1}{|Q|}\int\limits_Q |Tf_1(y)|\ \mathrm{d}y &\le \left(\frac{1}{|Q|}\int\limits_Q |Tf_1(y)|^s\ \mathrm{d}y\right)^{1/s}\left(\frac{1}{|Q|}\int\limits_Q 1^{s'}\mathrm{d}y\right)^{1/s'}\\ &= \left(\frac{1}{|Q|}\int\limits_Q |Tf_1(y)|^s\ \mathrm{d}y\right)^{1/s}.\end{aligned}$$

Since T is a Calderön-Zygmund operator, it is bounded on $L^s(\mathbb{R}^n)$. Therefore we get

$$\begin{aligned}\frac{1}{|Q|}\int\limits_Q |Tf_1(y)|\ \mathrm{d}y &\le C\left(\frac{1}{|Q|}\int\limits_Q |f_1(y)|^s\right)^{1/s}\\ &\le C\left(\frac{(4\sqrt{n}+1)^n}{|Q^*|}\int\limits_{Q^*} |f(y)|^s\ \mathrm{d}y\right)^{1/s}.\end{aligned}$$

The last inequality follows from the fact that $f_1 = f\chi_{Q^*}$. We also have

$$\frac{1}{|Q^*|}\int\limits_{Q^*} |f(y)|^s\ \mathrm{d}y \le M(|f|^s)(x).$$

That is,

$$\frac{1}{|Q|}\int\limits_Q |Tf_1(y)|\ \mathrm{d}y \le C(4\sqrt{n}+1)^{n/s} M(|f|^s)(x)^{1/s}. \tag{5.47}$$

Also because T is a Calderón-Zygmund operator there is some standard kernel K such that

$$\frac{1}{|Q|}\int\limits_Q |Tf_2(y) - Tf_2(x)|\ \mathrm{d}y$$

$$= \frac{1}{|Q|} \int\limits_Q \left| \int\limits_{\mathbb{R}^n} K(y,z) f_2(z) \; \mathrm{d}z - \int\limits_{\mathbb{R}^n} K(x,z) f_2(z) \; \mathrm{d}z \right| \; \mathrm{d}y.$$

Since f_2 is supported on $\mathbb{R}^n \setminus Q^*$, we have,

$$\frac{1}{|Q|} \int\limits_Q |Tf_2(y) - Tf_2(x)| \; \mathrm{d}y \leq \frac{1}{|Q|} \int\limits_Q \left(\int_{\mathbb{R}^n \setminus Q^*} |K(y,z) - K(x,z)| f_2(z) \; \mathrm{d}z \right) \; \mathrm{d}y.$$

Using the fact that K is a standard kernel, we get some $\delta > 0$ such that, for $|x - z| > 2|x - y|$,

$$|K(y,z) - K(x,z)| \leq C \frac{|y - x|^\delta}{|x - z|^{n+\delta}}. \tag{5.48}$$

We claim that $\mathbb{R}^n \setminus Q^* \subseteq \{z \in \mathbb{R}^n : |x - z| > 2|x - y|, \text{ for } x, y \in Q\}$.

Suppose l be the length of each side of the cube Q. As $x, y \in Q$, $2|x - y| \leq 2l\sqrt{n}$. Now if $z \in \mathbb{R}^n \setminus Q^*$ then $2l\sqrt{n} \leq |x - z|$. Hence for all $x, y \in Q$ and $z \in \mathbb{R}^n \setminus Q^*$, $2|x - y| \leq |x - z|$ (See Figure 5.1). This proves our claim.

Now, we get

$$\frac{1}{|Q|} \int\limits_Q |Tf_2(y) - Tf_2(x)| \; \mathrm{d}y \leq C \frac{1}{|Q|} \int\limits_Q \left(\int_{\mathbb{R}^n \setminus Q^*} \frac{|y - x|^\delta}{|x - z|^{n+\delta}} |f(z)| \; \mathrm{d}z \right) \; \mathrm{d}y. \tag{5.49}$$

Note that $\mathbb{R}^n \setminus Q^* \subseteq \bigcup\limits_{k=0}^{\infty} A_k$ where $A_k := \{z \in \mathbb{R}^n : 2^k 2\sqrt{n}l < |x - z| < 2^{k+1} 2\sqrt{n}l \}$. Note that the sets A_k's are disjoint. (See Figure 5.2 for the case when x is at one of the corner point of Q.)

Therefore from Inequality(5.49), we arrive at

$$\frac{1}{|Q|} \int\limits_Q |Tf_2(y) - Tf_2(x)| \; \mathrm{d}y \leq C \frac{1}{|Q|} \int\limits_Q \left(\sum_{k=0}^{\infty} \int\limits_{A_k} \frac{|y - x|^\delta}{|x - z|^{n+\delta}} |f(z)| \; \mathrm{d}z \right) \; \mathrm{d}y.$$

For each $k \in \mathbb{N} \cup \{0\}$, we have $A_k \subseteq B_k$, where $B_k = B(x, 2^{k+1} 2l\sqrt{n})$. As for $z \in A_k$, $|x - z| > 2^k 2l\sqrt{n}$ we have

$$\frac{1}{|Q|} \int\limits_Q |Tf_2(y) - Tf_2(x)| \; \mathrm{d}y \leq C \frac{1}{|Q|} \int\limits_Q \left(\sum_{k=0}^{\infty} \int\limits_{B_k} \frac{|y - x|^\delta}{(2^k 2l\sqrt{n})^{n+\delta}} |f(z)| \; \mathrm{d}z \right) \; \mathrm{d}y.$$

Further, since $x, y \in Q$, $|x - y| \leq l\sqrt{n} < 2l\sqrt{n}$. We have

$$\frac{1}{|Q|} \int\limits_Q |Tf_2(y) - Tf_2(x)| \; \mathrm{d}y$$

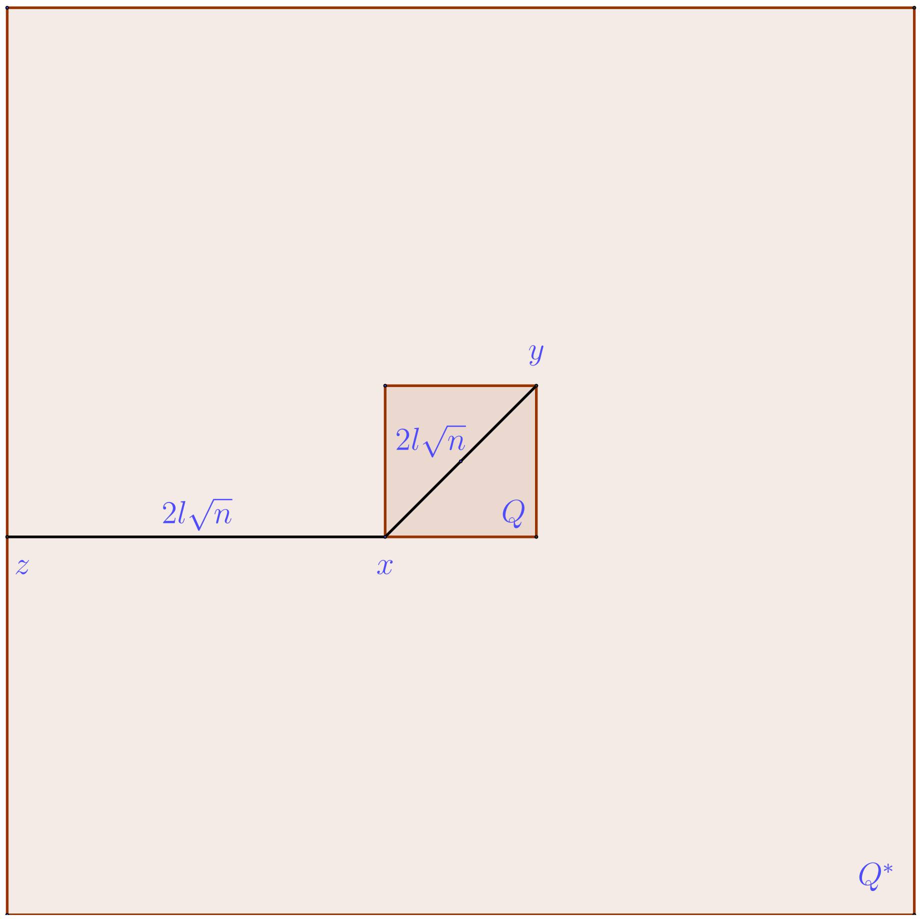

Figure 5.1: $\mathbb{R}^n \setminus Q^* \subseteq \{z \in \mathbb{R}^n : |x-z| > 2|x-y|,\ \text{for } x, y \in Q\}$.

$$
\begin{aligned}
&\leq C\frac{1}{|Q|}\int_Q \left(\sum_{k=0}^{\infty}\int_{B_k} \frac{(2l\sqrt{n})^{\delta}}{(2^k 2l\sqrt{n})^{n+\delta}}|f(z)|\ \mathrm{d}z\right)\ \mathrm{d}y \\
&= C\frac{1}{|Q|}\int_Q \left(\sum_{k=0}^{\infty}\frac{2^n}{(2^{k+1}2l\sqrt{n})^n 2^{k\delta}}\int_{B_k}|f(z)|\ \mathrm{d}z\right)\ \mathrm{d}y \\
&\leq C\frac{2^n}{|Q|}\int_Q \left(\sum_{k=0}^{\infty}\frac{1}{2^{k\delta}|B_k|}\int_{B_k}|f(z)|\mathrm{d}z\right)\mathrm{d}y
\end{aligned}
$$

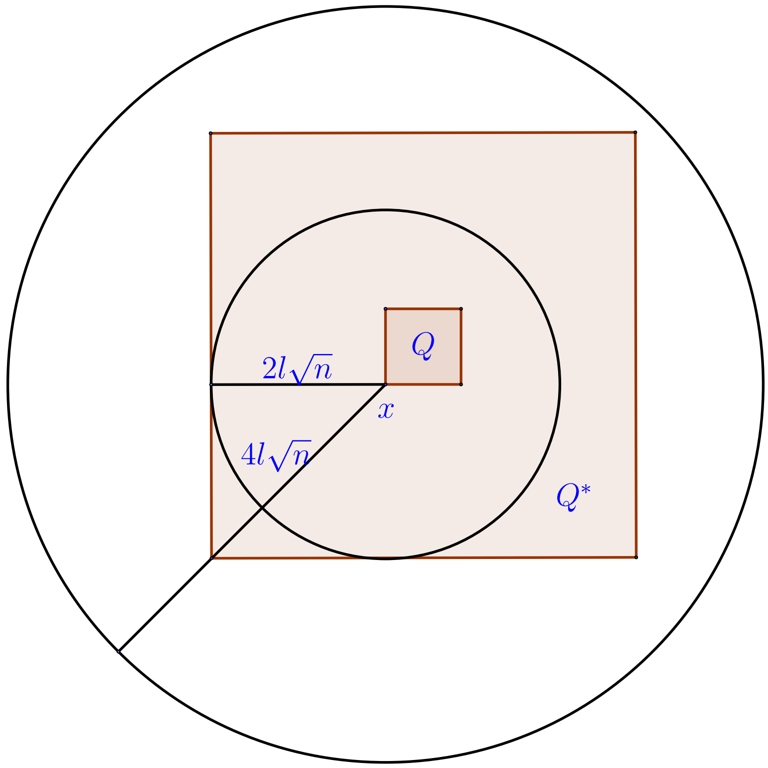

Figure 5.2: $A_0 = \{z \in \mathbb{R}^n : 2l\sqrt{n} < |x - z| < 4l\sqrt{n}\}$

$$\leq C\frac{2^n}{|Q|}\int_Q Mf(x)\left(\sum_{k=0}^{\infty}\frac{1}{2^{k\delta}}\right)\,\mathrm{d}y.$$

Here we have used that Hardy-Littlewood maximal function and non-centered cubic maximal function are equivalent. We know the series $\sum_{k=0}^{\infty}\frac{1}{2^{k\delta}}$ converges. So we have

$$\frac{1}{|Q|}\int_Q |Tf_2(y) - Tf_2(x)|\,\mathrm{d}y \leq C\frac{2^n}{|Q|}\int_Q Mf(x)\,\mathrm{d}y \leq C2^n Mf(x). \tag{5.50}$$

Using Hölder's inequality, we get

$$Mf(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)|\,\mathrm{d}y \leq M(|f|^s)(x)^{1/s}.$$

Therefore from (5.50) we have

$$\frac{1}{|Q|}\int\limits_Q |Tf_2(y) - Tf_2(x)|\ \mathrm{d}y \le 2^n C M(|f|^s)(x)^{1/s}. \tag{5.51}$$

This completes the proof. □

Lemma 5.14. *Let $w \in A_p$, and $1 \le p_0 \le p < \infty$, and $f \in L^{p_0}(w)$ then for all $\gamma > 0$ and $\lambda > 0$, there exists $\delta > 0$, such that*

$$w(\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda, M^{\#} f(x) \le \gamma\lambda\}) \le C\gamma^\delta w(\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}).$$

Proof. We recall from Theorem 3.3 that $\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}$ can be written as disjoint union of dyadic cubes Q. Because of a similar argument, used in the proof of Lemma 3.12, it is enough to show

$$w(\{x \in Q : M_d f(x) > 2\lambda, M^{\#} f(x) \le \gamma\lambda\}) \le C\gamma^\delta w(\{x \in Q : M_d f(x) > \lambda\}).$$

As $w \in A_p$, then by using A_∞ condition, there exists $\delta > 0$ such that, for any measurable subset S of the cube Q, we have

$$\frac{w(S)}{w(Q)} \le C\left(\frac{|S|}{|Q|}\right)^\delta. \tag{5.52}$$

Let $S := \{x \in Q : M_d f(x) > 2\lambda, M^{\#} f(x) \le \gamma\lambda\}$. From Lemma 3.12 $|S| \le 2^n\gamma|Q|$. That is $\frac{|S|}{|Q|} \le 2^n\gamma$. Therefore, from Inequality (5.52) we have

$$\frac{w(S)}{w(Q)} \le C(2^n\gamma)^\delta.$$

□

Lemma 5.15. *Let $w \in A_p$, and $1 \le p_0 \le p < \infty$. If f is such that $M_d f \in L^{p_0}(w)$, then*

$$\int\limits_{\mathbb{R}^n} |M_d f(x)|^p w(x)\ \mathrm{d}x \le C \int\limits_{\mathbb{R}^n} |M^{\#} f(x)|^p w(x)\ \mathrm{d}x.$$

Proof. For $N \in \mathbb{N}$, consider

$$I_N := \int\limits_0^N p\lambda^{p-1} w(\{x \in \mathbb{R}^n : M_d f(x) > \lambda\})\ \mathrm{d}\lambda.$$

Note that for each $N \in \mathbb{N}$,

$$I_N = \frac{p}{p_0}\int\limits_0^N p_0\lambda^{p_0-p_0+p-1} w(\{x \in \mathbb{R}^n : M_d f(x) > \lambda\})\ \mathrm{d}\lambda$$

$$\leq \frac{pN^{p-p_0}}{p_0} \int\limits_0^N p_0 \lambda^{p_0-1} w(\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}) \, \mathrm{d}\lambda$$

$$\leq \frac{pN^{p-p_0}}{p_0} ||M_d f||^{p_0}_{L^{p_0}(w)} < \infty.$$

By doing a change of variable ($\lambda = 2\lambda'$ and writing the expression in terms of λ), we get

$$I_N = 2^p \int\limits_0^{N/2} p\lambda^{p-1} w(\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda\}) \, \mathrm{d}\lambda. \tag{5.53}$$

Since

$$\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda\} \subseteq \{x \in \mathbb{R}^n : M_d f(x) > 2\lambda, M^{\#} f(x) \leq \gamma\lambda\} \\ \cup \{x \in \mathbb{R}^n : M^{\#} f(x) > \gamma\lambda\},$$

Equation (5.53) becomes

$$I_N \leq 2^p \int\limits_0^{N/2} p\lambda^{p-1} w(\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda, M^{\#} f(x) \leq \gamma\lambda\}) \, \mathrm{d}\lambda \\ + 2^p \int\limits_0^{N/2} p\lambda^{p-1} w(\{x \in \mathbb{R}^n : M^{\#} f(x) > \gamma\lambda\}) \, \mathrm{d}\lambda.$$

Now using Lemma 5.14 and similar arguments as in Lemma 3.12, desired result is proved. □

We now see a few preliminary properties of Calderön-Zygmund operator on weighted L^p space. The results that follow lead us to weighted boundedness of Calderön-Zygmund operator.

Lemma 5.16. *If T is a Calderón-Zygmund operator, then for any $w \in A_p$, with $1 < p < \infty$, $Tf \in L^p(w)$ for any compactly supported bounded function f.*

Proof. Let $w \in A_p$ be fixed and $\mathrm{supp}(f) \subseteq B(0, R)$. We observe that,

$$\int\limits_{\mathbb{R}^n} |Tf(x)|^p w(x) \, \mathrm{d}x = \int_{|x|<2R} |Tf(x)|^p w(x) \, \mathrm{d}x + \int_{|x|\geq 2R} |Tf(x)|^p w(x) \, \mathrm{d}x. \tag{5.54}$$

Now for any $\epsilon > 0$, as $\frac{1}{1+\epsilon} + \frac{\epsilon}{1+\epsilon} = 1$, using Hölder's inequality

$$\int_{|x|<2R} |Tf(x)|^p w(x) \, \mathrm{d}x$$

$$\leq \left(\int_{|x|<2R} w(x)^{1+\epsilon}\ \mathrm{d}x\right)^{1/(1+\epsilon)} \left(\int_{|x|<2R} |Tf(x)|^{p(1+\epsilon)/\epsilon}\ \mathrm{d}x\right)^{\epsilon/(1+\epsilon)}. \qquad (5.55)$$

By using reverse Hölder inequality we can choose $\epsilon_0 > 0$, such that the first integral in the right hand side of the above inequality is finite. Note that $q = \frac{p(1+\epsilon_0)}{\epsilon_0} > 1$. Hence $Tf \in L^q(\mathbb{R}^n)$. Therefore the last integral in the right hand side of Inequality (5.55), is also finite. That is,

$$\int_{|x|<2R} |Tf(x)|^p w(x)\ \mathrm{d}x < \infty. \qquad (5.56)$$

Now to complete the proof, we show that other integral in Equation (5.54) is also finite. Since T is a Calderön-Zygmund operator, there exists a standard kernel K such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\ \mathrm{d}y,$$

whenever $x \notin \mathrm{supp}(f)$. As $\mathrm{supp}(f) \subseteq B(0,R)$, for any $x \in \mathbb{R}^n$ with $|x| > 2R$, we have

$$|Tf(x)| = \left|\int_{\mathbb{R}^n} K(x,y)f(y)\ \mathrm{d}y\right| \leq \int_{\mathbb{R}^n} |K(x,y)f(y)|\ \mathrm{d}y.$$

Since K is a standard kernel, $|K(x,y)| \leq \frac{C}{|x-y|^n}$ for some C independent of $x, y \in \mathbb{R}^n$. Moreover, since f is supported inside $B(0,R)$, we have

$$|Tf(x)| \leq C \int_{|y|<R} \frac{|f(y)|}{|x-y|^n}\ \mathrm{d}y \leq C||f||_\infty \int_{|y|<R} \frac{\mathrm{d}y}{|x-y|^n}. \qquad (5.57)$$

Now note that if $|x| > 2R$ and $|y| < R$ then $\frac{|y|}{|x|} < \frac{1}{2}$. This implies $1 - \frac{|y|}{|x|} > \frac{1}{2}$. Therefore,

$$|x-y| > |\ |x| - |y|\ | = |x|\left|\ 1 - \frac{|y|}{|x|}\ \right| > \frac{|x|}{2}.$$

From Inequality (5.57), we get

$$|Tf(x)| \leq 2^n C||f||_\infty \int_{|y|<R} \frac{dy}{|x|^n} = \frac{2^n C||f||_\infty}{|x|^n}|B(0,R)| = \frac{C}{|x|^n}.$$

Now,

$$\begin{aligned}\int_{|x|>2R} |Tf(x)|^p w(x)\ \mathrm{d}x &\leq C \int_{|x|>2R} \frac{w(x)}{|x|^{np}}\ \mathrm{d}x \\ &= \sum_{k=1}^{\infty} \int_{2^k R<|x|<2^{k+1}R} \frac{w(x)}{|x|^{np}}\ \mathrm{d}x\end{aligned}$$

$$\leq C \sum_{k=1}^{\infty} (2^k R)^{-np} \int_{|x|<2^{k+1}R} w(x)\, \mathrm{d}x$$
$$= C \sum_{k=1}^{\infty} (2^k R)^{-np} w\left(B(0, 2^{k+1}R)\right).$$

As $w \in A_p$, there exists $q < p$ such that $w \in A_q$. By the A_q condition for any measurable subset S of a cube Q, we have

$$w(Q)\left(\frac{|S|}{|Q|}\right)^q \leq Cw(S).$$

This is true for balls also. Hence,

$$w(B(0, 2^{k+1}R))\left(\frac{|B(0, 2^k R)|}{|B(0, 2^{k+1}R)|}\right)^q \leq Cw\left(B(0, 2^k R)\right),$$

which implies

$$w(B(0, 2^{k+1}R)) \leq 2^{nq} w\left(B(0, 2^k R)\right).$$

Applying the same inequality for the ball $B(0, 2^k R)$ and continuing this we get

$$w(B(0, 2^{k+1}R)) \leq 2^{(k+1)nq} w\left(B(0, R)\right).$$

That is, with a new constant C depending on n, R, w we have

$$w\left(B(0, 2^{k+1}R)\right) \leq C 2^{knq}.$$

From the above observation, we have,

$$\int_{|x|>2R} |Tf(x)|^p w(x)\, \mathrm{d}x \leq C \sum_{k=1}^{\infty} (2^k R)^{-np} 2^{knq} = CR^{-np} \sum_{k=1}^{\infty} \frac{1}{2^{kn(p-q)}}.$$

As $p > q$ the series $\sum\limits_{k=1}^{\infty} \frac{1}{2^{kn(p-q)}} < \infty$. Therefore,

$$\int_{|x|>2R} |Tf(x)|^p w(x)\, \mathrm{d}x < \infty.$$

$\square$

We are now ready to prove the main result of this section.

Theorem 5.17. *If T is a Calderón-Zygmund operator, then for any $w \in A_p$, $1 < p < \infty$, T is bounded on $L^p(w)$.*

Proof. Let $w \in A_p$ be fixed. It is enough to prove this result for compactly supported bounded function because these functions are dence in $L^p(w)$. By

Corollary 2.7 we can find an $s > 1$ such that $w \in A_{p/s}$. Now by the Lebesgue differentiation theorem, $Tf(x) \leq M_d(Tf)(x)$ for a.e. $x \in \mathbb{R}^n$. Therefore,

$$\int\limits_{\mathbb{R}^n} |Tf(x)|^p w(x)\ \mathrm{d}x \leq \int\limits_{\mathbb{R}^n} |M_d(Tf(x))|^p w(x)\ \mathrm{d}x. \tag{5.58}$$

Using Lemma 5.16, $Tf \in L^p(w)$ and hence $M_d(Tf) \in L^p(w)$. Thus from Lemma 5.15, we get

$$\int\limits_{\mathbb{R}^n} |Tf(x)|^p w(x)\ \mathrm{d}x \leq C \int\limits_{\mathbb{R}^n} |M^{\#}(Tf(x))|^p w(x)\ \mathrm{d}x.$$

Now, by using Lemma 5.13, we obtain

$$\int\limits_{\mathbb{R}^n} |Tf(x)|^p w(x)\ \mathrm{d}x \leq C \int\limits_{\mathbb{R}^n} M(|f|^s)(x)^{p/s} w(x)\ \mathrm{d}x.$$

As $w \in A_{p/s}$ and M is bounded on $L^{p/s}(w)$ we have

$$\int\limits_{\mathbb{R}^n} |Tf(x)|^p w(x)\ \mathrm{d}x \leq C \int\limits_{\mathbb{R}^n} |f(x)|^p w(x)\ \mathrm{d}x.$$

□

Theorem 5.17 gives strong weighted boundedness of Calderón-Zygmund operators for $1 < p < \infty$. We now see the weak $(1,1)$ boundedness of Calderón-Zygmund operator with respect to A_1 weights.

Theorem 5.18. *Let T be a Calderón-Zygmund operator and let $w \in A_1$. Then, for every $f \in L^1(\mathbb{R}^n)$,*

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda} \int\limits_{\mathbb{R}^n} |f(x)| w(x)\ \mathrm{d}x.$$

Proof. We form the Calderón -Zygmund decomposition of f at the height $\lambda > 0$. This gives a sequence of disjoint dyadic cubes $\{Q_j\}_{j \in \mathbb{N}}$ such that

$$f(x) \leq \lambda \text{ for a.e. } x \notin \Omega = \bigcup_{j=1}^{\infty} Q_j, \tag{5.59}$$

$$|\Omega| \leq \frac{1}{\lambda} ||f||_1, \tag{5.60}$$

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} f(x)\ \mathrm{d}x \leq 2^n \lambda. \tag{5.61}$$

Given this decomposition of $\mathbb{R}$, We write $f = g + b$ where

$$g(x) = \begin{cases} f(x), & \text{if } x \notin \bigcup_j Q_j. \\ \frac{1}{|Q_j|} \int_{Q_j} f(y) \mathrm{d}y, & \text{if } x \in Q_j. \end{cases}$$

And

$$b(x) = \sum_{j=1}^{\infty} b_j(x), \tag{5.62}$$

where,

$$b_j(x) = \left(f(x) - \frac{1}{|Q_j|} \int_{Q_j} f(y) \mathrm{d}y \right) \chi_{Q_j}(x).$$

We recall for almost every $x \in \mathbb{R}^n$, we have

$$g(x) \leq 2^n \lambda. \tag{5.63}$$

Further, we also have,

$$\begin{aligned} w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq w & \left(\left\{ x \in \mathbb{R}^n : |Tg(x)| > \frac{\lambda}{2} \right\} \right) \\ & + w \left(\left\{ x \in \mathbb{R}^n : Tb(x) > \frac{\lambda}{2} \right\} \right). \end{aligned} \tag{5.64}$$

First, we estimate the "good part". We notice that

$$w(\{x \in \mathbb{R}^n : |Tg(x)| > \lambda\}) \leq \int_{\mathbb{R}^n} \frac{|Tg(x)|^2}{\lambda^2} w(x) \, \mathrm{d}x.$$

As $A_1 \subseteq A_2$, and T is strong $(2,2)$ with w as a weight, we have,

$$w(\{x \in \mathbb{R}^n : |Tg(x)| > \lambda\}) \leq \frac{C}{\lambda^2} \int_{\mathbb{R}^n} |g(x)|^2 w(x) \, \mathrm{d}x. \tag{5.65}$$

Note that for a particular $j \in \mathbb{N}$,

$$\begin{aligned} \int_{Q_j} |g(x)| w(x) \, \mathrm{d}x &\leq \int_{Q_j} \left(\frac{1}{|Q_j|} \int_{Q_j} |f(y)| \, \mathrm{d}y \right) w(x) \, \mathrm{d}x \\ &= \frac{1}{|Q_j|} \int_{Q_j} |f(y)| \, \mathrm{d}y \int_{Q_j} w(x) \, \mathrm{d}x \\ &= \int_{Q_j} |f(y)| \frac{w(Q_j)}{|Q_j|} \, \mathrm{d}y \\ &= \int_{Q_j} |f(y)| \frac{w(Q_j)}{|Q_j|} \, \mathrm{d}y. \end{aligned}$$

As $w \in A_1$, for almost every $x \in Q_j$, we have

$$\frac{w(Q_j)}{|Q_j|} \leq Cw(x).$$

That is,

$$\int\limits_{Q_j} |g(x)|w(x)\ \mathrm{d}x \leq \int\limits_{Q_j} |f(y)|w(y)\ \mathrm{d}y.$$

Now,

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} |g(x)|w(x)\ \mathrm{d}x &= \int_{\mathbb{R}^n \setminus \bigcup\limits_{j\in\mathbb{N}} Q_j} |g(x)|w(x)\ \mathrm{d}x + \int_{\bigcup\limits_{j\in\mathbb{N}} Q_j} |g(x)|w(x)\ \mathrm{d}x \\
&= \int_{\mathbb{R}^n \setminus \bigcup\limits_{j\in\mathbb{N}} Q_j} |f(x)|w(x)\ \mathrm{d}x + \sum_{j\in\mathbb{N}} \int\limits_{Q_j} |g(x)|w(x)\ \mathrm{d}x \\
&\leq \int_{\mathbb{R}^n \setminus \bigcup\limits_{j\in\mathbb{N}} Q_j} |f(x)|w(x)\ \mathrm{d}x + \sum_{j\in\mathbb{N}} \int\limits_{Q_j} |f(x)|w(x)\ \mathrm{d}x \\
&= \int\limits_{\mathbb{R}^n} |f(x)|w(x)\ \mathrm{d}x
\end{aligned}$$

Using the above observation in Inequality (5.65) we have

$$w(\{x \in \mathbb{R}^n : |Tg(x)| > \lambda\}) \leq \frac{2^n C}{\lambda} \int\limits_{\mathbb{R}^n} |f(x)|w(x)\ \mathrm{d}x. \tag{5.66}$$

Now we estimate the "bad part". We denote by Q_j^* the cube with the same centre c_j (as that of Q_j) whose sides are $2\sqrt{n}$ times longer. Then,

$$\begin{aligned}
w(\{x \in \mathbb{R}^n : Tb(x) > \lambda\}) &\leq w\left(\left\{x \notin \bigcup_{j\in\mathbb{N}} Q_j^* : |Tb(x)| > \lambda\right\}\right) \\
&\quad + w\left(\left\{x \in \bigcup_{j\in\mathbb{N}} Q_j^* : |Tb(x)| > \lambda\right\}\right) \\
&\leq w(\{x \notin \bigcup_{j\in\mathbb{N}} Q_j^* : |Tb(x)| > \lambda\}) + w(\bigcup_{j\in\mathbb{N}} Q_j^*) \qquad (5.67)
\end{aligned}$$

We can find a constant depending on n such that

$$w\left(\bigcup_{j\in\mathbb{N}} Q_j^*\right) \leq C \sum_{j\in\mathbb{N}} w(Q_j) = C \sum_{j\in\mathbb{N}} \frac{w(Q_j)}{|Q_j|} |Q_j|$$

Using Inequality (5.61), we get,

$$|Q_j| \leq \frac{1}{\lambda} \int\limits_{Q_j} |f(y)|\ \mathrm{d}y.$$

Therefore, we have the following

$$w\left(\bigcup_{j\in\mathbb{N}} Q_j^*\right) \le \frac{C}{\lambda}\sum_{j\in\mathbb{N}} \frac{w(Q_j)}{|Q_j|}\int\limits_{Q_j} |f(y)|\ \mathrm{d}y = \frac{C}{\lambda}\sum_{j\in\mathbb{N}}\int\limits_{Q_j} |f(y)|\frac{w(Q_j)}{|Q_j|}\ \mathrm{d}y.$$

As $w \in A_1$ we have

$$w\left(\bigcup_{j\in\mathbb{N}} Q_j^*\right) \le \frac{C}{\lambda}\sum_{j\in\mathbb{N}}\int\limits_{Q_j} |f(y)|w(y)\ \mathrm{d}y \le \frac{C}{\lambda}\int\limits_{\mathbb{R}^n} |f(y)|w(y)\ \mathrm{d}y. \tag{5.68}$$

Now using Inequality(5.68) in Inequality (5.67), we get

$$w(\{x\in\mathbb{R}^n : Tb(x) > \lambda\}) \le w(\{x \notin \bigcup_{j\in\mathbb{N}} Q_j^* : |Tb(x)| > \lambda\}) + \frac{C}{\lambda}\int\limits_{\mathbb{R}^n} |f(y)|w(y)\ \mathrm{d}y. \tag{5.69}$$

To estimate $w(\{x \notin \bigcup_{j\in\mathbb{N}} Q_j^* : |Tb(x)| > \lambda\})$, we notice that

$$\begin{aligned} w(\{x \notin \bigcup_{j\in\mathbb{N}} Q_j^* : |Tb(x)| > \lambda\}) &\le \frac{C}{\lambda}\int_{\mathbb{R}^n\setminus\bigcup_{j\in\mathbb{N}} Q_j^*} |Tb(x)|w(x)\ \mathrm{d}x \\ &\le \int_{\mathbb{R}^n\setminus\bigcup_{j\in\mathbb{N}} Q_j^*}\sum_{j=1}^{\infty} |Tb_j(x)|w(x)\ \mathrm{d}x \\ &= \sum_{j=1}^{\infty}\int_{\mathbb{R}^n\setminus\bigcup_{j\in\mathbb{N}} Q_j^*} |Tb_j(x)|\ \mathrm{d}x. \end{aligned}$$

As $\mathbb{R}^n \setminus \bigcup_{j\in\mathbb{N}} Q_j^* \subset \mathbb{R}^n \setminus Q_j^*$ we have the following

$$w(\{x \notin \bigcup_{j\in\mathbb{N}} Q_j^* : |Tb(x)| > \lambda\}) \le \sum_{j=1}^{\infty}\int_{\mathbb{R}^n\setminus Q_j^*} |Tb_j(x)|w(x)\ \mathrm{d}x.$$

Note that $b_j \in L^2(\mathbb{R}^n)$ and it is supported on Q_j. Therefore, there exists a standard kernel K such that

$$w\left(\left\{x \notin \bigcup_{j\in\mathbb{N}} Q_j^* : |Tb(x)| > \lambda\right\}\right) \le \frac{C}{\lambda}\sum_{j=1}^{\infty}\int_{\mathbb{R}^n\setminus Q_j^*} w(x)\left|\int\limits_{Q_j} K(x,y)b_j(y)\ \mathrm{d}y\right|\ \mathrm{d}x. \tag{5.70}$$

As b_j has zero integral on Q_j, from Inequality (5.70) we arrive at,

$$\begin{aligned} &w(\{x \notin \bigcup_{j\in\mathbb{N}} Q_j^* : |Tb(x)| > \lambda\}) \\ &\le \frac{C}{\lambda}\sum_{j=1}^{\infty}\int_{\mathbb{R}^n\setminus Q_j^*} w(x)\left|\int\limits_{Q_j} [K(x,y) - K(x,c_j)]b_j(y)\ \mathrm{d}y\right|\ \mathrm{d}x \end{aligned}$$

$$\leq \frac{C}{\lambda}\sum_{j=1}^{\infty}\int_{\mathbb{R}^n\setminus Q_j^*} w(x)\int\limits_{Q_j} |K(x,y)-K(x,c_j)||b_j(y)|\ \mathrm{d}y\ \mathrm{d}x.$$

By the use of Fubini's theorem, we get

$$w(\{x\notin\bigcup_{j\in\mathbb{N}} Q_j^* :|Tb(x)|>\lambda\})$$
$$\leq \frac{C}{\lambda}\sum_{j=1}^{\infty}\int\limits_{Q_j}\left(\int_{\mathbb{R}^n\setminus Q_j^*} w(x)|K(x,y)-K(x,c_j)|\ \mathrm{d}x\right)|b_j(y)|\ \mathrm{d}y. \tag{5.71}$$

Since K is a standard kernel, there is some $C>0$ and $\delta>0$ such that for all $x\in\mathbb{R}^n$ satisfying $|x-y|>2|y-c_j|$, we have

$$|K(x,y)-K(x,c_j)|\leq\frac{|y-c_j|^{\delta}}{|x-c_j|^{n+\delta}}.$$

Therefore from Inequality (5.71), we have

$$w\left(\left\{x\notin\bigcup_{j\in\mathbb{N}}Q_j^* : |Tb(x)|>\lambda\right\}\right)$$
$$\leq \frac{C}{\lambda}\sum_{j=1}^{\infty}\int\limits_{Q_j}\left(\int_{\mathbb{R}^n\setminus Q_j^*} w(x)\frac{|y-c_j|^{\delta}}{|x-c_j|^{n+\delta}}\mathrm{d}x\right)|b_j(y)|\ \mathrm{d}y. \tag{5.72}$$

If a be the length of the cube Q_j and $y\in Q_j$ then $|y-c_j|\leq a\sqrt{n}$.

$$\int_{\mathbb{R}^n\setminus Q_j^*} w(x)\frac{|y-c_j|^{\delta}}{|x-c_j|^{n+\delta}}\mathrm{d}x\leq\int_{\mathbb{R}^n\setminus Q_j^*}\frac{(a\sqrt{n})^{\delta}}{|x-c_j|^{n+\delta}}w(x)\ \mathrm{d}x.$$

Now let $B_k:=B(x,2^k a\sqrt{n})$ and $A_k:=\{x\in\mathbb{R}^n : 2^k a\sqrt{n}<|x-c_j|<2^{k+1}a\sqrt{n}\}$. Then,

$$\begin{aligned}\int_{\mathbb{R}^n\setminus Q_j^*} w(x)\frac{|y-c_j|^{\delta}}{|x-c_j|^{n+\delta}}\mathrm{d}x &\leq \sum_{k=0}^{\infty}(a\sqrt{n})^{\delta}\int_{A_k}\frac{w(x)}{|x-c_j|^{n+\delta}}\ \mathrm{d}x\\ &\leq (a\sqrt{n})^{\delta}\sum_{k=0}^{\infty}\int_{B_k}\frac{w(x)}{(2^k a\sqrt{n})^{n+\delta}}\ \mathrm{d}x\\ &\leq \sum_{k=0}^{\infty}\frac{2^n a\sqrt{n}}{(2^{k+1}a\sqrt{n})^{n+\delta}}\int_{B_k} w(x)\ \mathrm{d}x\\ &\leq C2^n\sum_{k=0}^{\infty}\frac{1}{2^{k\delta}|B_k|}\int_{B_k} w(x)\ \mathrm{d}x\\ &\leq C2^n Mw(y)\sum_{k=0}^{\infty}\frac{1}{2^{k\delta}}.\end{aligned}$$

Note that the series appearing in the last inequality is convergent. Therefore there is a constant C depending on n such that

$$\int_{\mathbb{R}^n\setminus Q_j^*} w(x)\frac{|y-c_j|^\delta}{|x-c_j|^{n+\delta}}\mathrm{d}x \leq CMw(x). \tag{5.73}$$

Using Inequality (5.73) in Inequality (5.71) we get

$$w\left(\left\{x \notin \bigcup_{j\in\mathbb{N}} Q_j^* : |Tb(x)| > \lambda\right\}\right) \leq \frac{C}{\lambda}\sum_{j=1}^{\infty}\int_{Q_j} Mw(y)|b_j(y)|\ \mathrm{d}y.$$

As $w \in A_1$, $Mw(y) \leq Cw(y)$ for a.e. $y \in \mathbb{R}^n$. So the above inequality becomes

$$\begin{aligned} w(\{x \notin \bigcup_{j\in\mathbb{N}} Q_j^* : |Tb(x)| > \lambda\}) &\leq \frac{C}{\lambda}\sum_{j=1}^{\infty}\int_{Q_j} w(y)|b_j(y)|\ \mathrm{d}y \\ &\leq \frac{C}{\lambda}\sum_{j=1}^{\infty}\int_{Q_j} w(y)|b(y)|\ \mathrm{d}y \\ &\leq \frac{C}{\lambda}\int_{\mathbb{R}^n} w(y)|b(y)|\ \mathrm{d}y. \end{aligned}$$

Therefore we get

$$w(\{x \notin \bigcup_{j\in\mathbb{N}} Q_j^* : |Tb(x)| > \lambda\}) \leq \frac{C}{\lambda}\int_{\mathbb{R}^n} |b(y)|w(y)\ \mathrm{d}y \tag{5.74}$$

Now note that $f = g+b$. Therefore $|b| \leq |f|+|g|$, which further implies $|b| \leq 2|f|$, because $|g| \leq |f|$. Therefore from Inequality (5.74), we have

$$w(\{x \notin \bigcup_{j\in\mathbb{N}} Q_j^* : |Tb(x)| > \lambda\}) \leq \frac{2C}{\lambda}\int_{\mathbb{R}^n} |f(y)|w(y)\ \mathrm{d}y. \tag{5.75}$$

Now combining Inequalities (5.66) and (5.75) and using them in Inequality (5.64), we complete the proof! $\square$

CHAPTER 6

Littlewood-Paley Theory and Multipliers

The main aim of this chapter is to study multipliers on L^p-spaces. As discussed earlier, multipliers are translation invariant operators that are well-behaved with Fourier transform. We see that the study becomes a lot easier when we have tools due to Littlewood and Paley. The two authors wished to derive certain boundedness results for not just one functions, but rather a sequence of functions. In their paper (see [18]), they get the results for functions defined on $\mathbb{R}$ using complex analysis techniques.

However, the techniques of complex analysis are of no help in higher dimensions! Keeping this in mind, Calderón with Benedek and Panzone ([1]), derived the results for functions that take values in a Banach space. This approach led the authors to easily generalize Littlewood and Paley's work to higher dimensions. Apart from the study of boundedness of certain linear operators, Littlewood-Paley theory is also useful in the study of some exotic function spaces. We refer the reader to [10] for further details on this topic.

We are interested only from the viewpoint of boundedness of certain operators. We describe Calderón's approach in the chapter. We start by generalizing the Calderón-Zygmund theorem to Banach-valued functions.

6.1 Calderón-Zygmund Theorem for Banach-Valued Functions

We begin by giving a vector valued analogue of the Calderón-Zygmund Theorem (Theorem 5.10 in chapter 5). Let A and B be two Banach spaces and $\mathcal{L}(A, B)$ be the space of all bounded linear operators from A to B. Let K be a function defined from the set $(\mathbb{R}^n \times \mathbb{R}^n) \setminus \triangle$ to the space $\mathcal{L}(A, B)$, where $\Delta := \{(x, x) : x \in \mathbb{R}^n\}$ is the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$. Let T be an operator such that for a compactly supported function $f \in L^\infty(\mathbb{R}^n, A)$, we have

$$Tf(x) = \int\limits_{\mathbb{R}^n} K(x, y)f(y)\, \mathrm{d}y,$$

whenever $x \notin \operatorname{supp}(f)$.

Theorem 6.1. *Let A and B be reflexive Banach spaces and T be an operator as defined above, Let $T : L^r(\mathbb{R}^n, A) \longrightarrow L^r(\mathbb{R}^n, B)$ be bounded for some $1 < r < \infty$, and the above mentioned function K satisfies the following two conditions:*

$$\int_{|x-y|>2|y-z|} ||K(x, y) - K(x, z)||_{\mathcal{L}(A,B)}\, \mathrm{d}x \le C. \tag{6.1}$$

$$\int_{|x-y|>2|x-w|} ||K(x, y) - K(w, y)||_{\mathcal{L}(A,B)}\, \mathrm{d}y \le C. \tag{6.2}$$

Then the operator T is bounded from $L^p(\mathbb{R}^n, A)$ to $L^p(\mathbb{R}^n, B)$ for all $1 < p < \infty$ and for $p = 1$ it is weak $(1, 1)$.

Proof. Given $f : \mathbb{R}^n \longrightarrow A$, a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined as $\varphi(x) = ||f(x)||_A$. Now, we form the Calderoń-Zygmund decomposition at the height $\lambda > 0$ for φ. Then we get a collection of dyadic cubes $\{Q_j\}_{j\in\mathbb{N}}$. Then we have

$$\varphi(x) = \varphi_g(x) + \varphi_b(x),$$

where

$$\varphi_g(x) = \begin{cases} \varphi(x), & \text{if } x \notin \bigcup\limits_j Q_j. \\ \frac{1}{|Q_j|} \int\limits_{Q_j} \varphi(y)\mathrm{d}y, & \text{if } x \in Q_j. \end{cases}$$

And

$$\varphi_b(x) = \sum_{j=1}^{\infty} \varphi_{b_j}(x),$$

where

$$\varphi_{b_j}(x) = \left(\varphi(x) - \frac{1}{|Q_j|} \int\limits_{Q_j} \varphi(y)\ \mathrm{d}y \right) \chi_{Q_j}.$$

Let $\Omega = \bigcup\limits_{j \in \mathbb{N}} Q_j$. We recall that the following properties hold

$$|\Omega| \leq \frac{1}{\lambda} ||\varphi||_1, \tag{6.3}$$

$$|\varphi_g(x)| \leq 2^n \lambda \text{ a.e x} \in \mathbb{R}^n, \tag{6.4}$$

$$\lambda < \frac{1}{|Q_j|} \int\limits_{Q_j} \varphi(x)\ \mathrm{d}x \leq 2^n \lambda. \tag{6.5}$$

Now we write the function f as $f = g + b$, where

$$g(x) = \begin{cases} f(x), & \text{if } x \notin \bigcup\limits_j Q_j. \\ \frac{1}{|Q_j|} \int\limits_{Q_j} f(y)\mathrm{d}y, & \text{if } x \in Q_j. \end{cases}$$

$$b = \sum_{j=1}^{\infty} b_j,$$

where,

$$b_j(x) = \left(f(x) - \frac{1}{|Q_j|} \int\limits_{Q_j} f(y)\ \mathrm{d}y \right) \chi_{Q_j}(x).$$

Here the integrals are understood in the Bochner sense. Note that if $x \notin \Omega$, then

$$||g(x)||_A = ||f(x)||_A = \varphi_g(x) \leq 2^n \lambda.$$

If $x \in Q_j$ then

$$||g(x)||_A \leq \frac{1}{|Q_j|} \int\limits_{Q_j} ||f(x)||_A\ \mathrm{d}x = \frac{1}{|Q_j|} \int\limits_{Q_j} \varphi(x)\ \mathrm{d}x \leq 2^n \lambda.$$

Therefore for any $x \in \mathbb{R}^n$,

$$||g(x)||_A \leq 2^n \lambda. \tag{6.6}$$

Also, we have

$$\int_{\mathbb{R}^n} b_j(x)\ \mathrm{d}x = 0.$$

First, we prove that T is weak $(1,1)$. Note that $Tf(x) = Tg(x) + Tb(x)$. Then, we have,

$$\begin{aligned} |\{x \in \mathbb{R}^n : ||Tg(x)||_B > \lambda\}| &\leq |\{x \in \mathbb{R}^n : ||Tg(x)||_B > \lambda/2\}| \\ &\quad + |\{x \in \mathbb{R}^n : ||Tb(x)||_B > \lambda/2\}|. \end{aligned} \tag{6.7}$$

Since the operator T is bounded from $L^r(\mathbb{R}^n, A)$ to $L^r(\mathbb{R}^n, B)$,

$$|\{x \in \mathbb{R}^n : ||Tg(x)||_B > \lambda/2\}| \leq C\frac{2^r}{\lambda^r}\int_{\mathbb{R}^n} ||g(x)||_A^r\ \mathrm{d}x.$$

Using the Inequality (6.4) we obtain

$$|\{x \in \mathbb{R}^n : ||Tg(x)||_B > \lambda/2\}| \leq C\frac{2(2^n\lambda)^{r-1}}{\lambda^r}\int_{\mathbb{R}^n} ||g(x)||_A\ \mathrm{d}x \leq \frac{C}{\lambda}\int_{\mathbb{R}^n} ||f(x)||_A\ \mathrm{d}x. \tag{6.8}$$

Now let Q_j^* be a cube with same center as Q_j and whose sides are $2\sqrt{n}$ times larger, and let

$$\Omega^* := \bigcup_{j\in\mathbb{N}} Q_j^*.$$

Then, it is easy to see that $|\Omega^*| \leq C|\Omega|$. Also, we have $||\varphi||_1 = ||f||_{L^1(\mathbb{R}^n,A)}$. So that by using Inequality (6.3), we get

$$|\Omega^*| \leq \frac{C}{\lambda}||f||_{L^1(\mathbb{R}^n,A)}$$

where C depends only on 'n'. Now,

$$\begin{aligned} |\{x \in \mathbb{R}^n : ||Tb(x)||_B > \lambda/2\}| &\leq |\{x \in \Omega^* : ||Tb(x)||_B > \lambda/2\}| \\ &\quad + |\{x \notin \Omega^* : ||Tb(x)||_B > \lambda/2\}| \\ &\leq |\Omega^*| + |\{x \notin \Omega^* : ||Tb(x)||_B > \lambda/2\}| \\ &\leq \frac{C}{\lambda}||f||_{L^1(A)} + \frac{2}{\lambda}\int_{\mathbb{R}^n\setminus\Omega^*} ||Tb(x)||_B\ \mathrm{d}x. \end{aligned}$$

So we have

$$|\{x \in \mathbb{R}^n : ||Tb(x)||_B > \lambda/2\}| \leq \frac{C}{\lambda}||f||_{L^1(A)} + \frac{2}{\lambda}\int_{\mathbb{R}^n\setminus\Omega^*} ||Tb(x)||_B \, \mathrm{d}x. \tag{6.9}$$

Now, To estimate the second term in the R.H.S of Inequality (6.9), we consider the following

$$\int\limits_{\mathbb{R}^n\setminus\Omega^*} ||Tb(x)||_B \, \mathrm{d}x \leq \int\limits_{\mathbb{R}^n\setminus\Omega^*} \sum_{j=1}^{\infty} ||Tb_j(x)||_B \, \mathrm{d}x = \sum_{j=1}^{\infty} \int\limits_{\mathbb{R}^n\setminus\Omega^*} ||Tb_j(x)||_B \, \mathrm{d}x.$$

Last equality is an easy consequence of Fubini's theorem for a non-negative integrand. We know that $\mathrm{supp}(b_j) \subseteq Q_j \subseteq \Omega^*$. Hence, from the definition of T, we recall that

$$\begin{aligned}\int\limits_{\mathbb{R}^n\setminus\Omega^*} ||Tb(x)||_B \, \mathrm{d}x &\leq \sum_{j=1}^{\infty} \int\limits_{\mathbb{R}^n\setminus\Omega^*} \left|\left| \int\limits_{\mathbb{R}^n} K(x,y)b_j(y) \, \mathrm{d}y \right|\right|_B \mathrm{d}x \\ &= \sum_{j=1}^{\infty} \int\limits_{\mathbb{R}^n\setminus\Omega^*} \left|\left| \int\limits_{Q_j} K(x,y)b_j(y) \, \mathrm{d}y \right|\right|_B \mathrm{d}x.\end{aligned}$$

Now, since $\mathbb{R}^n \setminus \Omega^* \subseteq \mathbb{R}^n \setminus Q_j$, we get

$$\int\limits_{\mathbb{R}^n\setminus\Omega^*} ||Tb(x)||_B \, \mathrm{d}x \leq \sum_{j=1}^{\infty} \int\limits_{\mathbb{R}^n\setminus Q_j^*} \left|\left| \int\limits_{Q_j} K(x,y)b_j(y) \, \mathrm{d}y \right|\right|_B \mathrm{d}x.$$

For $z \in \mathbb{R}^n$ with $z \neq x$, we have $\int\limits_{Q_j} K(x,z)b_j(y) \, \mathrm{d}y = 0$. So, we can write

$$\begin{aligned}\int\limits_{\mathbb{R}^n\setminus\Omega^*} ||Tb(x)||_B \, \mathrm{d}x &\leq \sum_{j=1}^{\infty} \int\limits_{\mathbb{R}^n\setminus Q_j^*} \left|\left| \int\limits_{Q_j} [K(x,y) - K(x,z)]b_j(y) \, \mathrm{d}y \right|\right|_B \mathrm{d}x \\ &\leq \sum_{j=1}^{\infty} \int\limits_{\mathbb{R}^n\setminus Q_j^*} \int\limits_{Q_j} ||[K(x,y) - K(x,z)]b_j(y)||_B \, \mathrm{d}y \, \mathrm{d}x \\ &\leq \sum_{j=1}^{\infty} \int\limits_{\mathbb{R}^n\setminus Q_j^*} \int\limits_{Q_j} ||K(x,y) - K(x,z)||_{\mathcal{L}(A,B)} ||b_j(y)||_A \, \mathrm{d}y \, \mathrm{d}x \\ &\leq \sum_{j=1}^{\infty} \int\limits_{Q_j} \int\limits_{\mathbb{R}^n\setminus Q_j^*} ||K(x,y) - K(x,z)||_{\mathcal{L}(A,B)} ||b_j(y)||_A \, \mathrm{d}y \, \mathrm{d}x.\end{aligned}$$

Now recall (Theorem 5.8, chapter 5) that $\mathbb{R}^n \setminus Q_j^* \subseteq \{x \in \mathbb{R}^n : |x-y| > 2|y-z|\}$.

Therefore by using Condition (6.1), we get

$$\int_{\mathbb{R}^n\setminus\Omega^*} ||Tb(x)||_B \, \mathrm{d}x \leq C\sum_{j=1}^{\infty}\int_{Q_j} ||b_j(y)||_A \, \mathrm{d}y. \tag{6.10}$$

Now, we observe that

$$\begin{aligned}\int_{Q_j} ||b_j(y)||_A \, \mathrm{d}y &= \int_{Q_j} \left|\left| \left(f(x) - \frac{1}{|Q_j|}\int_{Q_j} f(y) \, \mathrm{d}y \right) \chi_{Q_j}(x) \right|\right|_A \mathrm{d}x \\ &\leq 2\int_{Q_j} ||f(x)||_A \, \mathrm{d}x.\end{aligned}$$

Using the above in Inequality (6.10), we get

$$\int_{\mathbb{R}^n\setminus\Omega^*} ||Tb(x)||_B \, \mathrm{d}x \leq C\sum_{j=1}^{\infty}\int_{Q_j} ||f(x)||_A \, \mathrm{d}x \leq C\int_{\mathbb{R}^n} ||f(x)||_A \, \mathrm{d}x = C||f||_{L^1(\mathbb{R}^n,A)}.$$

Therefore from Inequality (6.9), we have

$$|\{x\in\mathbb{R}^n : ||Tb(x)||_B > \lambda/2\}| \leq \frac{C}{\lambda}||f||_{L^1(\mathbb{R}^n,A)} + \frac{2C}{\lambda}||f||_{L^1(\mathbb{R}^n,A)} = \frac{C}{\lambda}||f||_{L^1(\mathbb{R}^n,A)}. \tag{6.11}$$

Therefore using Inequalities (6.11) and (6.8) in Inequality (6.7), we obtain,

$$|\{x\in\mathbb{R}^n : ||Tg(x)||_B > \lambda\}| \leq \frac{C}{\lambda}||f||_{L^1(\mathbb{R}^n,A)}.$$

This proves that the operator T is weak $(1,1)$. As T is bounded from $L^r(\mathbb{R}^n, A)$ to $L^r(B)$ for some $r > 1$ by using Marcinkiewicz interpolation T is bounded from $L^p(\mathbb{R}^n, A)$ to $L^p(\mathbb{R}^n, B)$ for all $1 < p \leq r$. For all $p > r$ we use a duality argument. As A is reflexive $L^{r'}(\mathbb{R}^n, A')$ and $(L^r(\mathbb{R}^n, A))'$ are isomorphic to each other. Now let T^t be the transpose of the operator T, so for any $F \in L^{r'}(B')$ and $g \in L^r(A)$,

$$(T^tF)(g) = F(Tg) = \int_{\mathbb{R}^n} \langle F(x), Tg(x)\rangle \, \mathrm{d}x.$$

Now suppose $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset$, then we have

$$\begin{aligned}\int_{\mathbb{R}^n} \langle F(x), Tg(x)\rangle \, \mathrm{d}x &= \int_{\mathbb{R}^n} \left\langle F(x), \int_{\mathbb{R}^n} K(x,y)g(y) \, \mathrm{d}y \right\rangle \mathrm{d}x \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \langle F(x), K(x,y)g(y)\rangle \, \mathrm{d}y \, \mathrm{d}x.\end{aligned}$$

Let $K^t(x,y)$ be the transpose of the operator $K(x,y)$. Then,

$$\int_{\mathbb{R}^n} \langle F(x), Tg(x)\rangle \, \mathrm{d}x = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \langle K^t(x,y)F(x), g(y)\rangle \, \mathrm{d}y \, \mathrm{d}x.$$

Let $\widetilde{K}(x,y) = K^t(x,y)$. Therefore,

$$\int_{\mathbb{R}^n} \langle F(x), Tg(x)\rangle \, \mathrm{d}x = \int_{\mathbb{R}^n} \left\langle \int_{\mathbb{R}^n} \widetilde{K}(y,x)F(x) \, \mathrm{d}x, g(y) \right\rangle \mathrm{d}y.$$

Now using similar arguments as that in Lemma 5.9, we have

$$T^tF(x) = \int_{\mathbb{R}^n} \widetilde{K}(x,y)F(y) \, \mathrm{d}y.$$

At once we get,

$$\begin{aligned} &\int_{|x-y|>2|y-z|} ||\widetilde{K}(x,y) - \widetilde{K}(x,z)||_{\mathcal{L}(B',A')} \, \mathrm{d}x \\ &= \int_{|x-y|>2|y-z|} ||K^t(y,x) - K^t(z,x)||_{\mathcal{L}(B',A')} \mathrm{d}x \\ &= \int_{|x-y|>2|y-z|} ||K(y,x) - K(z,x)||_{\mathcal{L}(A,B)} \mathrm{d}x. \end{aligned}$$

By using Condition (6.2), we have

$$\int_{|x-y|>2|y-z|} ||\widetilde{K}(x,y) - \widetilde{K}(x,z)||_{\mathcal{L}(B',A')} \, \mathrm{d}x \le C.$$

So the operator T^t is also weak $(1,1)$. As the operator T is bounded from $L^r(\mathbb{R}^n, A)$ to $L^r(\mathbb{R}^n, B)$, T^t is bounded from $L^{r'}(B')$ to $L^{r'}(A')$. Therefore by Marcinkiewicz interpolation theorem T^t is bounded from $L^p(B')$ to $L^p(A')$ for $1 < p < r'$. Now suppose $p > r$. Then $p' < r'$, and

$$\begin{aligned} ||Tf||_{L^p(B)} &= \sup\left\{ \left| \int_{\mathbb{R}^n} \langle G(x), Tf(x)\rangle \, \mathrm{d}x \right| : ||G||_{L^{p'}(\mathbb{R}^n.B)} \le 1 \right\} \\ &= \sup\left\{ \left| \int_{\mathbb{R}^n} \langle T^tG(x), f(x)\rangle \, \mathrm{d}x \right| : ||G||_{L^{p'}(\mathbb{R}^n,B)} \le 1 \right\} \\ &= \sup\left\{ ||T^tG||_{L^{p'}(\mathbb{R}^n,A')} ||f||_{L^p(\mathbb{R}^n,A)} : ||G||_{L^{p'}(\mathbb{R}^n,B')} \le 1 \right\} \\ &\le ||f||_{L^p(\mathbb{R}^n,A)} \sup\left\{ ||T^t|| ||G||_{L^{p'}(\mathbb{R}^n,A')} : ||G||_{L^{p'}(\mathbb{R}^n,B')} \le 1 \right\} \\ &\le C||f||_{L^p(A)}. \end{aligned}$$

Hence the result. □

Now we see a few applications of the vector valued analogue of the Calderón-Zygmund theorem. In this section we mainly focus on some important inequalities of some vector valued operators. These inequalities are often used in upcoming sections.

Theorem 6.2. *Let T be a convolution operator with kernel K, which is bounded on $L^2(\mathbb{R}^n)$. Assume that the kernel K satisfies the Hörmander condition,*

$$\int\limits_{|x|>2|y|} |K(x-y) - K(x)| \ \mathrm{d}x \leq C.$$

Then for any $1 < r, p < \infty$ we have,

$$\left|\left| \left(\sum_{j=1}^{\infty} |Tf_j|^r \right)^{1/r} \right|\right|_p \leq C_{p,r} \left|\left| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{1/r} \right|\right|_p .$$

Moreover, for $p = 1$,

$$\left| \left\{ x \in \mathbb{R}^n : \left(\sum_{j=1}^{\infty} |Tf_j|^r \right)^{1/r} > \lambda \right\} \right| \leq \frac{C_r}{\lambda} \left|\left| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{1/r} \right|\right|_1 .$$

Proof. To prove the result we show that the vector-valued operator which associates to each sequence $(f_j)_{j\in\mathbb{N}}$ the sequence $(Tf_j)_{j\in\mathbb{N}}$ satisfies the hypothesis of Theorem 6.1 with $A = B = \ell^r$. If we denote the operator by τ then, using Corollary 2.24, we have,

$$\begin{aligned}
\tau\left((f_i)_{i\in\mathbb{N}}\right) = (K * f_i)_{i\in\mathbb{N}} &= \left(\int\limits_{\mathbb{R}^n} K(x-y) f_i(y) \ \mathrm{d}y \right)_{i\in\mathbb{N}} \\
&= \int\limits_{\mathbb{R}^n} \left(K(x-y) f_i(y)\right)_{i\in\mathbb{N}} \ \mathrm{d}y \\
&= \int\limits_{\mathbb{R}^n} \overline{K}(x,y) \left(f_i(y)\right)_{i\in\mathbb{N}} \ \mathrm{d}y,
\end{aligned}$$

where, $\overline{K}(x,y) = K(x-y)I$ and I is the identity operator on ℓ^r. First we prove that τ is bounded on $L^p(\mathbb{R}^n, \ell^r)$ when $p = r$.

$$||\tau((f_j)_{j\in\mathbb{N}})||_r^r = \int\limits_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} |Tf_j(x)|^r \right)^{r/r} \ \mathrm{d}x$$

$$= \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} |Tf_j(x)|^r \, \mathrm{d}x$$
$$= \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |Tf_j(x)|^r \, \mathrm{d}x.$$

In the last equality, we have employed Fubini's theorem. From the hypothesis on T, it is evident from the Calderón-Zygmund Theorem (Theorem 5.10, in chapter 5) that T is bounded on $L^p(\mathbb{R}^n)$, $\forall p$ satisfying $1 < p < \infty$. Hence, we have,

$$||\tau((f_j)_{j\in\mathbb{N}})||_r^r \leq C \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |f_j(x)|^r \, \mathrm{d}x = C||(f_j)_{j\in\mathbb{N}}||_r^r.$$

Now we notice

$$\int_{|x-y|>2|y-z|} ||\overline{K}(x,y) - \overline{K}(x,z)||_{\mathcal{L}(\ell^r,\ell^r)} \, \mathrm{d}x$$
$$= \int_{|x-y|>2|y-z|} |K(x-y) - K(x-z)|||I||_{\mathcal{L}(\ell^r,\ell^r)} \, \mathrm{d}x.$$

Taking $x = x' + y$ and $z - y = y'$, we get

$$\int_{|x-y|>2|y-z|} ||\overline{K}(x,y) - \overline{K}(x,z)||_{\mathcal{L}(\ell^r,\ell^r)} \, \mathrm{d}x = \int_{|x'|>2|y'|} |K(x') - K(x'-y')| \, \mathrm{d}x' \leq C.$$

Similarly, it can be shown that

$$\int_{|x-y|>2|x-w|} ||\overline{K}(x,y) - \overline{K}(w,y)||_{\mathcal{L}(\ell^r,\ell^r)} \, \mathrm{d}y \leq C.$$

Hence the desired result follows directly by using Theorem 6.1. □

Now we prove a matrix analogue of Theorem 6.2.

Theorem 6.3. *Let T be as defined in Theorem 6.2. Then for any $1 < r < \infty$ and p, $1 < p < \infty$.*

$$\left|\left| \left(\sum_{j,k\in\mathbb{Z}}^{\infty} |Tf_{j,k}|^r \right)^{1/r} \right|\right|_p \leq C_{p,r} \left|\left| \left(\sum_{j,k\in\mathbb{Z}}^{\infty} |f_{j,k}|^r \right)^{1/r} \right|\right|_p.$$

Moreover, for $p = 1$,

$$\left| \left\{ x \in \mathbb{R}^n : \left(\sum_{j,k\in\mathbb{Z}}^{\infty} |Tf_{j,k}|^r \right)^{1/r} > \lambda \right\} \right| \leq \frac{C_r}{\lambda} \left|\left| \left(\sum_{j,k\in\mathbb{Z}}^{\infty} |f_{j,k}|^r \right)^{1/r} \right|\right|_1.$$

Proof. As in the previous theorem, the operator τ is defined in the following way

$$\tau\left((f_{j,k})_{j,k\in\mathbb{Z}}\right)=\int_{\mathbb{R}^n}\overline{K}(x,y)\left((f_{j,k}(y))_{j,k\in\mathbb{Z}}\right)\,\mathrm{d}y.$$

Here $\overline{K}(x,y)=K(x-y)I$, where I is the identity operator on $\ell^r(\mathbb{Z}\times\mathbb{Z})$. With exactly the same reasoning as in Theorem 6.2, it can be shown that τ is bounded on $L^r\left(\mathbb{R}^n,\ell^r(\mathbb{Z}\times\mathbb{Z})\right)$. Again, using a similar argument as in Theorem 6.2, we see

$$\begin{aligned}&\int_{|x-y|>2|y-z|}||\overline{K}(x,y)-\overline{K}(x,z)||_{\mathcal{L}(\ell^r(\mathbb{Z}\times\mathbb{Z}),\ell^r(\mathbb{Z}\times\mathbb{Z}))}\,\mathrm{d}x\\ &=\int_{|x'|>2|y'|}|K(x')-K(x'-y')|\,\mathrm{d}x'<C.\end{aligned}$$

Finally, from Theorem 6.1, we conclude the result. $\square$

We now give a few applications of Theorem 6.2 and Theorem 6.3. The results discussed below are useful in our study of Littlewood-Paley Theory.

Corollary 6.4. *Let $\{I_j\}_{j\in\mathbb{N}}$ be a sequence of intervals on the real line, finite or infinite, and let $\{S_j\}_{j\in\mathbb{N}}$ be the sequence of operators defined by $(S_jf)\widehat{\ }(\xi)=\chi_{I_j}(\xi)\widehat{f}(\xi)$. Then for any $1<r,p<\infty$,*

$$\left|\left|\left(\sum_{j=1}^{\infty}|S_jf_j|^r\right)^{1/r}\right|\right|_p\le C_{p,r}\left|\left|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{1/r}\right|\right|_p.$$

Proof. We recall from Chapter 5 that if I_j is the interval (a_j,b_j) for some $a_j<b_j$, Then the operator S_j can be written as

$$S_j=\frac{i}{2}\left(M_{a_j}HM_{-a_j}-M_{b_j}HM_{-b_j}\right),$$

where for any $a\in\mathbb{R}$, $M_af(x)=e^{2\pi iax}f(x)$ and H is the Hilbert transform. We Notice that $|M_af(x)|=|f(x)|,\forall x\in\mathbb{R}$. Therefore,

$$\begin{aligned}|S_jf_j(x)|&\le\frac{1}{2}\left(|M_{a_j}HM_{-a_j}f_j(x)|+|M_{b_j}HM_{-b_j}f_j(x)|\right)\\ &=\frac{1}{2}\left(|HM_{-a_j}f_j(x)|+|HM_{-b_j}f_j(x)|\right).\end{aligned}$$

We know that the Hilbert transform is a Calderón-Zygmund operator of convolution type. Hence, by Theorem (6.2) we have $\forall 1 < p, r < \infty$,

$$\left\| \left(\sum_{j=1}^{\infty} |HM_{-a_j} f_j|^r \right)^{1/r} \right\|_p \leq C_{p,r} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{1/r} \right\|_p . \tag{6.12}$$

Similarly,

$$\left\| \left(\sum_{j=1}^{\infty} |HM_{-b_j} f_j|^r \right)^{1/r} \right\|_p \leq C_{p,r} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{1/r} \right\|_p . \tag{6.13}$$

As we have already seen that

$$|S_j f_j(x)| \leq \frac{1}{2} \left(|HM_{-a_j} f_j(x)| + |HM_{-b_j} f_j(x)| \right).$$

Therefore, we get,

$$\left\| \left(\sum_{j=1}^{\infty} |S_j f_j|^r \right)^{1/r} \right\|_p \leq \left\| \left(\sum_{j=1}^{\infty} \left(|HM_{-a_j} f_j| + |HM_{-b_j} f_j| \right)^r \right)^{1/r} \right\|_p .$$

Using the triangle inequality of ℓ^r and $L^p(\mathbb{R}^n)$, we arrive at

$$\left\| \left(\sum_{j=1}^{\infty} |S_j f_j|^r \right)^{1/r} \right\|_p \leq \left\| \left(\sum_{j=1}^{\infty} |HM_{-a_j} f_j|^r \right)^{1/r} \right\|_p + \left\| \left(\sum_{j=1}^{\infty} |HM_{-b_j} f_j|^r \right)^{1/r} \right\|_p .$$

By using Inequalities (6.12) and (6.13) we conclude

$$\left\| \left(\sum_{j=1}^{\infty} |S_j f_j|^r \right)^{1/r} \right\|_p \leq C_{p,r} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{1/r} \right\|_p .$$

□

The matrix valued analogue of Corollary 6.4 follows easily from the techniques discussed above and Theorem 6.3. We do not include the proof here since it is essentially verbatim to that of Corollary 6.4.

Corollary 6.5. *Let $\{I_j\}$ be a sequence of intervals on the real line, finite or infinite, and let $\{S_j\}$ be the sequence of operators defined by $(S_j f)\widehat{\ }(\xi) = \chi_{I_j}(\xi)\widehat{f}(\xi)$. Then for $1 < r, p < \infty$,*

$$\left\| \left(\sum_{j,k \in \mathbb{Z}} |S_j f_{j,k}|^r \right)^{1/r} \right\|_p \leq C_{p,r} \left\| \left(\sum_{j,k \in \mathbb{Z}} |f_{j,k}|^r \right)^{1/r} \right\|_p .$$

Until now we have seen a sequence of operators or a matrix -valued operator (τ), associated to a nice bounded linear operator $T : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$. Next, we consider the case when we have a sequence $\{T_j\}_{j \in \mathbb{N}}$ of "nice" operator. In the result that follows, the Muckenhoupt class A_2 plays an important role.

Theorem 6.6. *Let $\{T_j\}_{j\in\mathbb{N}}$ be a sequence of linear operators that are bounded on $L^2(w)$ for any $w \in A_2$ with constants that are uniform in j and which depend only on the A_2 constant of w. Then for all $1 < p < \infty$, we have,*

$$\Bigg|\Bigg|\left(\sum_{j=1}^{\infty}|T_jf_j|^2\right)^{1/2}\Bigg|\Bigg|_p \le C\Bigg|\Bigg|\left(\sum_{j=1}^{\infty}|f_j|^2\right)^{1/2}\Bigg|\Bigg|_p.$$

Proof. When $p = 2$, we have

$$\Bigg|\Bigg|\left(\sum_{j=1}^{\infty}|T_jf_j|^2\right)^{1/2}\Bigg|\Bigg|_2^2 = \int_{\mathbb{R}^n}\sum_{j=1}^{\infty}|T_jf_j(x)|^2\mathrm{d}x.$$

Since the integrand is non-negative, we can use Fubini's theorem and obtain

$$\Bigg|\Bigg|\left(\sum_{j=1}^{\infty}|T_jf_j|^2\right)^{1/2}\Bigg|\Bigg|_2^2 = \sum_{j=1}^{\infty}\int_{\mathbb{R}^n}|T_jf_j(x)|^2\mathrm{d}x.$$

From the hypothesis of the theorem, the operators T_j are bounded on $L^2(w)$ space for any $w \in A_2$ with the constant uniform in j and since constant function 1 is in A_2, we get by an application of Fubini's theorem,

$$\begin{aligned}\Bigg|\Bigg|\left(\sum_{j=1}^{\infty}|T_jf_j|^2\right)^{1/2}\Bigg|\Bigg|_2^2 \le C\sum_{j=1}^{\infty}\int_{\mathbb{R}^n}|f_j(x)|^2\mathrm{d}x &= \int_{\mathbb{R}^n}\sum_{j=1}^{\infty}|f_j(x)|^2\mathrm{d}x\\ &= C\Bigg|\Bigg|\left(\sum_{j=1}^{\infty}|f_j|^2\right)^{1/2}\Bigg|\Bigg|_2^2.\end{aligned}$$

This prove the result when $p = 2$. For $p > 2$, we notice that

$$\Bigg|\Bigg|\left(\sum_{j=1}^{\infty}|T_jf_j|^2\right)^{1/2}\Bigg|\Bigg|_p^2 = \left(\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|T_jf_j(x)|^2\right)^{p/2}\mathrm{d}x\right)^{2/p} = \Bigg|\Bigg|\sum_{j=1}^{\infty}|T_jf_j|^2\Bigg|\Bigg|_{p/2}.$$

By duality of L^p spaces, there exists a non-negative function $u \in L^{(p/2)'}(\mathbb{R}^n)$ with $||u||_{L^{(p/2)'}(\mathbb{R}^n)} = 1$ such that

$$\Bigg|\Bigg|\left(\sum_{j=1}^{\infty}|T_jf_j|^2\right)^{1/2}\Bigg|\Bigg|_p^2 = \int_{\mathbb{R}^n}\sum_{j=1}^{\infty}|T_jf_j(x)|^2u(x)\ \mathrm{d}x. \tag{6.14}$$

Now if $0 < \delta < 1$ then $M(u^{1/\delta})^\delta \in A_1$, with the A_1 constant depending only on δ. As $A_1 \subseteq A_2$, $M(u^{1/\delta})^\delta \in A_2$. As $u(x) \le (Mu^{1/\delta}(x))^\delta$ a.e $x \in \mathbb{R}^n$, from Equation

(6.14), we have

$$\left\|\left(\sum_{j=1}^{\infty}|T_jf_j|^2\right)^{1/2}\right\|_p^2 \le \int\limits_{\mathbb{R}^n}\sum_{j=1}^{\infty}|T_jf_j(x)|^2\left(Mu^{1/\delta}(x)\right)^{\delta}\ \mathrm{d}x.$$

Now, using Fubini's theorem, and the fact that all T_j's are bounded on $L^2(w)$ with constants uniform in j, we have

$$\begin{aligned}\left\|\left(\sum_{j=1}^{\infty}|T_jf_j|^2\right)^{1/2}\right\|_p^2 &\le C\sum_{j=1}^{\infty}\int\limits_{\mathbb{R}^n}|f_j(x)|^2\left(Mu^{1/\delta}(x)\right)^{\delta}\mathrm{d}x\\ &\le \int\limits_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_j(x)|^2\right)\left(Mu^{1/\delta}(x)\right)^{\delta}\ \mathrm{d}x.\end{aligned}$$

Now let us choose $0<\delta<1$ such that $\delta(p/2)'>1$. Applying Hölder's inequality with exponents $p/2$ and $(p/2)'$ we have

$$\begin{aligned}&\int\limits_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_j(x)|^2\right)\left(Mu^{1/\delta}(x)\right)^{\delta}\ \mathrm{d}x\\ &\le C\left(\int\limits_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_j(x)|^2\right)^{p/2}\mathrm{d}x\right)^{2/p}\left(\int\limits_{\mathbb{R}^n}\left(M(u^{1/\delta})(x)\right)^{\delta(p/2)'}\mathrm{d}x\right)^{1/(p/2)'}.\end{aligned}$$

As M is bounded on $L^{\delta(p/2)'}(\mathbb{R}^n)$,

$$\begin{aligned}\int\limits_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_j|^2\right)\left(Mu^{1/\delta}(x)\right)^{\delta}\ \mathrm{d}x &\le C\left\|\left(\sum_{j=1}^{\infty}|f_j|^2\right)^{1/2}\right\|_p^2||u^{1/\delta}||_{\delta(p/2)'}^{\delta}\\ &= \left\|\left(\sum_{j=1}^{\infty}|f_j|^2\right)^{1/2}\right\|_p^2\cdot||u||_{(p/2)'}.\end{aligned}$$

Since $||u||_{\delta(p/2)'}=1$, we have

$$\left\|\left(\sum_{j=1}^{\infty}|T_jf_j|^2\right)^{1/2}\right\|_p^2 \le C\left\|\left(\sum_{j=1}^{\infty}|f_j|^2\right)^{1/2}\right\|_p^2. \tag{6.15}$$

This prove the theorem for $p>2$. As T_j is bounded on $L^2(w)$, T_j^t is also bounded on $L^2(w)$. Therefore for $p>2$, T_j^t also satisfies Inequality 6.15. Using this fact and Hölder's inequality, we have for $p<2$,

$$\left\|\left(\sum_{j=1}^{\infty}|T_jf_j|^2\right)^{1/2}\right\|_p$$

$$= \sup\left\{\Big|\sum_{j=1}^{\infty}\int_{\mathbb{R}^n} T_j f_j(x)\cdot g_j(x)\ \mathrm{d}x\Big| : \Big\|\Big(\sum_{j=1}^{\infty}|g_j|^2\Big)^{1/2}\Big\|_{p'} \le 1\right\}$$
$$= \sup\left\{\Big|\sum_{j=1}^{\infty}\int_{\mathbb{R}^n} f_j(x)\cdot T_j^t g_j(x)\ \mathrm{d}x\Big| : \Big\|\Big(\sum_{j=1}^{\infty}|g_j|^2\Big)^{1/2}\Big\|_{p'} \le 1\right\}$$
$$\le \sup\left\{\sum_{j=1}^{\infty}\int_{\mathbb{R}^n} |f_j(x)|\cdot |T_j^t g_j(x)|\ \mathrm{d}x : \Big\|\Big(\sum_{j=1}^{\infty}|g_j|^2\Big)^{1/2}\Big\|_{p'} \le 1\right\}$$
$$\le \sup\left\{\int_{\mathbb{R}^n}\sum_{j=1}^{\infty} |f_j(x)|\cdot |T_j^t g_j(x)|\ \mathrm{d}x : \Big\|\Big(\sum_{j=1}^{\infty}|g_j|^2\Big)^{1/2}\Big\|_{p'} \le 1\right\}$$
$$\le \sup\left\{\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^2\Big)^{1/2}\Big\|_{p}\cdot\Big\|\Big(\sum_{j=1}^{\infty}|T_j^t g_j|^2\Big)^{1/2}\Big\|_{p'} : \Big\|\Big(\sum_{j=1}^{\infty}|g_j|^2\Big)^{1/2}\Big\|_{p'} \le 1\right\}$$
$$\le C_p\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^2\Big)^{1/2}\Big\|_{p}\cdot \sup\left\{\Big\|\Big(\sum_{j=1}^{\infty}|g_j|^2\Big)^{1/2}\Big\|_{p'} : \Big\|\Big(\sum_{j=1}^{\infty}|g_j|^2\Big)^{1/2}\Big\|_{p'} \le 1\right\}$$
$$\le C_p\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^2\Big)^{1/2}\Big\|_{p}.$$

This completes the proof! □

6.2 Littlewood-Paley theory

Now we are ready to study boundedness results for certain vector-valued operators. The results so developed are of direct use in the study of multipliers. We begin with the following construction.

For $j \in \mathbb{Z}$, let $\Delta_j = (-2^{j+1}, -2^j] \cup [2^j, 2^{j+1})$ and define an operator S_j as

$$(S_j f)\widehat{\ }(\xi) = \chi_{\Delta_j}(\xi)\widehat{f}(\xi),$$

where $j \in \mathbb{Z}$. If $f \in L^2(\mathbb{R})$, then by using Plancharal theorem we have,

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|S_j f|^2\Big)^{1/2}\Big\|_2^2 = \int_{\mathbb{R}}\sum_{j\in\mathbb{Z}}|S_j f(x)|^2\ \mathrm{d}x$$
$$= \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}|S_j f(x)|^2\ \mathrm{d}x$$

$$= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} |(S_j f)\widehat{\ }(\xi)|^2 \,\mathrm{d}\xi$$

$$= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} |\chi_{\Delta_j}(\xi)\widehat{f}(\xi)|^2 \,\mathrm{d}\xi$$

$$= \sum_{j\in\mathbb{Z}} \int_{\Delta_j} |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi$$

$$= \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi$$

$$= ||f||_2^2.$$

So we have

$$\left|\left| \left(\sum_{j\in\mathbb{Z}} |S_j f|^2 \right)^{1/2} \right|\right|_2 = ||f||_2. \tag{6.16}$$

In this section our main aim is to prove Littlewood-Paley theorem which states that $\left|\left| \left(\sum_{j\in\mathbb{Z}} |S_j f|^2 \right)^{1/2} \right|\right|_2$ and $||f||_2$ are comparable. We first consider the smooth analogue of S_j. We start with $\psi \in \mathcal{S}(\mathbb{R})$, with $0 \le \psi \le 1$, having support in $1/2 \le |\xi| \le 4$, which is equal to 1 on $1 \le |\xi| \le 2$. We define

$$\psi_j(\xi) = \psi(2^{-j}\xi)$$

and

$$(\tilde{S}_j f)\widehat{\ }(\xi) = \psi_j(\xi)\widehat{f}(\xi). \tag{6.17}$$

Note that by using the definitions of S_j and $\tilde{S}_j$, we have

$$(S_j\tilde{S}_j f)\widehat{\ }(\xi) = \chi_{\Delta_j}(\xi)(\tilde{S}_j f)\widehat{\ }(\xi) = \chi_{\Delta_j}(\xi)\psi_j(\xi)\widehat{f}(\xi)$$

Now, for $\xi \in \Delta_j$, $\psi_j(\xi) = 1$. Therefore

$$(S_j\tilde{S}_j f)\widehat{\ }(\xi) = \chi_{\Delta_j}(\xi)\widehat{f}(\xi) = (S_j f)\widehat{\ }(\xi).$$

That is, $S_j\tilde{S}_j = S_j$. Keeping the above construction in mind, we prove the following result.

Theorem 6.7. *For any $1 < p < \infty$, there exists a constant $C_p > 0$ such that for* *a $f \in L^p(\mathbb{R})$, we have,*

$$\left|\left| \left(\sum_{j=1}^{\infty} |\tilde{S}_j f|^2 \right)^{1/2} \right|\right|_p \leq C_p ||f||_p. \tag{6.18}$$

Proof. Let $\Phi \in \mathcal{S}(\mathbb{R})$ be such that $\widehat{\Phi} = \psi$ and let $\Phi_j(x) = 2^j \Phi(2^j x)$. Then note that

$$\widehat{\Phi_j}(\xi) = 2^j \widehat{\Phi(2^j \cdot)}(\xi) = \widehat{\Phi}(\xi/2^j) = \psi(\xi/2^j) = \psi_j(\xi).$$

Therefore for $f \in \mathcal{S}(\mathbb{R})$

$$(\tilde{S}_j f)\widehat{\ }(\xi) = \psi_j(\xi)\widehat{f}(\xi) = \widehat{\Phi}(\xi)\widehat{f}(\xi) = (\Phi_j * f)\widehat{\ }(\xi).$$

So, $\tilde{S}_j f = \Phi_j * f$. To prove the theorem we have to show that the operator T which maps a function f to the sequence $\left(\tilde{S}_j f\right)_{j\in\mathbb{Z}}$ is bounded from $L^p(\mathbb{R})$ to the space $L^p(\mathbb{R}, \ell^2)$, for $1 < p < \infty$. Now

$$\begin{aligned}(Tf)(y) = \left(\tilde{S}_j f\right)_{j\in\mathbb{Z}} = (\Phi_j * f(y))_{j\in\mathbb{Z}} &= \left(\int_{\mathbb{R}} \Phi_j(x-y) f_j(y)\ \mathrm{d}y \right)_{j\in\mathbb{Z}} \\ &= \int_{\mathbb{R}} \left(\Phi_j(x-y) f(y)\right)_{j\in\mathbb{Z}}\ \mathrm{d}y.\end{aligned}$$

In the last equation, we have used Corollary 2.24. Let $K : \mathbb{R} \times \mathbb{R} \longrightarrow \mathcal{L}(\mathbb{C}, l^2)$ be the kernel of the operator T. Then K is defined in the following way

$$K(x,y)(z) = (\Phi_j(x-y)z)_{j\in\mathbb{Z}},$$

where $(x, y) \in \mathbb{R} \times \mathbb{R}$ and $z \in \mathbb{C}$. We show that the operator T satisfies all the conditions of Theorem 6.2. First we note that T is bounded from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}, \ell^2)$. Indeed,

$$\begin{aligned}\left|\left| \left(\sum_{j\in\mathbb{Z}} |\tilde{S}_j f|^2 \right)^{1/2} \right|\right|_2^2 &= \int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} |\tilde{S}_j f(x)|^2\ \mathrm{d}x \\ &= \int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} |(\tilde{S}_j f)\widehat{\ }(\xi)|^2\ \mathrm{d}\xi \\ &= \int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} |\psi_j(\xi)|^2 |\widehat{f}(\xi)|^2\ \mathrm{d}\xi.\end{aligned}$$

We claim that for any $\xi \in \mathbb{R}$, at most three of ψ_j's are non zero. To see this, first we observe that for any $\xi \in \mathbb{R}$ there is a unique $j(\xi) \in \mathbb{Z}$, such that $|\xi| \in$

$[2^{j(\xi)}, 2^{j(\xi)+1})$. As for any $j \in \mathbb{Z}$, support of $\psi_j(\xi)$ is $[2^{j-1}, 2^{j+2})$, $\psi_j(\xi) = 0$ for $j < j(\xi) - 1$ and $j > j(\xi) + 1$. Therefore,

$$\left|\left| \left(\sum_{j\in\mathbb{Z}} |\tilde{S}_j f|^2 \right)^{1/2} \right|\right|_2^2 = \int_{\mathbb{R}} \left(|\psi_{j(\xi)-1}(\xi)|^2 + |\psi_{j(\xi)}(\xi)|^2 + |\psi_{j(\xi)+1}(\xi)|^2 \right) |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi$$

$$\leq 3 \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi = 3||f||_2^2.$$

Now we show that the kernel K satisfies Hörmander condition. That is,

$$\int_{|x-y|>2|y-z|} ||K(x,y) - K(x,z)||_{\mathcal{L}(\mathbb{R},\ell^2)} \, \mathrm{d}x \leq C.$$

As $||K(x,y)||_{\mathcal{L}(\mathbb{R},\ell^2)} = ||\left(\Phi_j(x-y)\right)_{j\in\mathbb{Z}}||_{\ell^2}$, we have the following.

$$\int_{|x-y|>2|y-z|} ||K(x,y) - K(x,z)||_{\mathcal{L}(\mathbb{R},\ell^2)} \, \mathrm{d}x$$

$$= \int_{|x-y|>2|y-z|} ||\left(\Phi_j(x-y) - \Phi_j(x-z)\right)_{j\in\mathbb{Z}}||_{\ell^2} \, \mathrm{d}x.$$

Now doing a change of variable as we did in Theorem 6.2, we have

$$\int_{|x-y|>2|y-z|} ||K(x,y) - K(x,z)||_{\mathcal{L}(\mathbb{R},\ell^2)} \, \mathrm{d}x$$

$$\leq \int_{|x|>2|y|} ||\left(\Phi_j(x+y) - \Phi_j(x)\right)_{j\in\mathbb{Z}}||_{\ell^2} \, \mathrm{d}x.$$

Hence, we observe that

$$\begin{aligned} ||\{\Phi_j(x+y) - \Phi_j(x)\}||_{l^2} &= \left(\sum_{j\in\mathbb{Z}} |\Phi_j(x+y) - \Phi_j(x)|^2 \right)^{1/2} \\ &\leq \left(\sum_{j\in\mathbb{Z}} |y|^2 \sup_{0\leq t\leq 1} \Phi_j'(x+ty)|^2 \right)^{1/2} \\ &\leq |y| \sup_{0\leq t\leq 1} \left(\sum_{j\in\mathbb{Z}} |\Phi_j(x+ty)|^2 \right)^{1/2}. \end{aligned}$$

So, we have

$$\int_{|x-y|>2|y-z|} ||K(x,y) - K(x,z)||_{\mathcal{L}(\mathbb{R},\ell^2)} \, \mathrm{d}x$$

$$\leq |y| \int\limits_{|x|>2|y|} \sup_{0\leq t\leq 1} \left(\sum_{j\in\mathbb{Z}} |\Phi_j(x+ty)|^2 \right)^{1/2} \mathrm{d}x. \tag{6.19}$$

Now, since $\ell^1 \subseteq \ell^2$,

$$\left(\sum_{j\in\mathbb{Z}} |\Phi_j'(x)|^2 \right)^{1/2} \leq \sum_{j\in\mathbb{Z}} |\Phi_j'(x)| \leq \sum_{j\in\mathbb{Z}} 2^{2j} |\Phi'(2^j x)|.$$

As $\Phi \in \mathcal{S}(\mathbb{R})$, we have some $C > 0$ such that $|\Phi'(x)| \leq C$ and $|x|^3 |\Phi'(x)| \leq C$, $\forall x \in \mathbb{R}$. Therefore, we have,

$$|\Phi'(x)| \leq C \min(1, |x|^{-3}).$$

Now, for $x \in \mathbb{R}$ there is some $i \in \mathbb{Z}$ such that $2^{-i} \leq |x| < 2^{-i+1}$. Then,

$$\left(\sum_{j\in\mathbb{Z}} |\Phi_j'(x)|^2 \right)^{1/2} \leq \sum_{j<i} 2^{2j} |\Phi'(2^j x)| + \sum_{j\geq i} 2^{2j} |\Phi'(2^j x)|.$$

Note that when $j < i$ we have $|x| \leq 2^{-i+1} \leq 2^{-j}$. Therefore $(2^i|x|)^{-1} \geq 1$ and we have

$$\sum_{j<i} 2^{2j} |\Phi'(2^j x)| \leq C \sum_{j<i} 2^{2j} = C 2^{2i} \leq C|x|^{-2}.$$

Further when $j \geq i$, then we have $(2^j|x|)^{-3} \leq 1$.

$$\sum_{j\geq i} 2^{2j} |\Phi'(2^j x)| \leq C \sum_{j\geq i} 2^{2j} (2^j|x|)^{-3} = C|x|^{-3} \sum_{j\geq i} 2^{-j} = C 2^{-i} |x|^{-3} \leq C|x|^{-2}.$$

Therefore we can say that

$$\left(\sum_{j\in\mathbb{Z}} |\Phi_j'(x)|^2 \right)^{1/2} \leq C|x|^{-2}.$$

From Inequality (6.19) we get

$$\int\limits_{|x-y|>2|y-z|} ||K(x,y) - K(x,z)||_{\mathcal{L}(\mathbb{R},l^2)} \, \mathrm{d}x \leq C|y| \int\limits_{|x|>2|y|} \sup_{0\leq t\leq 1} \frac{1}{|x+ty|^2} \, \mathrm{d}x. \tag{6.20}$$

For $|x| > 2|y|$, we have

$$|x+ty| \geq |x| - t|y| \geq |x| - t\frac{|x|}{2} \geq |x| - \frac{|x|}{2} = \frac{|x|}{2}.$$

Therefore,

$$\sup_{0\leq t\leq 1} \frac{1}{|x+ty|^2} \leq \frac{4}{|x|^2}.$$

So, from Inequality (6.20), we have

$$\int\limits_{|x-y|>2|y-z|} ||K(x,y)-K(x,z)||_{\mathcal{L}(\mathbb{R},l^2)}\ \mathrm{d}x \leq 4C|y| \int\limits_{|x|>2|y|} \frac{1}{|x|^2}\ \mathrm{d}x \leq C.$$

The result now readily follows from Theorem 2.23. □

Using the fact that $S_j\tilde{S}_j = S_j$, and using above theorem we can prove Inequality (6.18), replacing $\tilde{S}_j$ by S_j. In fact, a stronger claim can be proved.

Theorem 6.8 (Littlewood-Paley). *For any $1 < p < \infty$, there exists positive constants c_p and C_p such that for all $f \in L^p(\mathbb{R})$,*

$$c_p||f||_p \leq \left|\left| \left(\sum_{j\in\mathbb{Z}} |S_jf|^2 \right)^{1/2} \right|\right|_p \leq C_p||f||_p.$$

Proof. From Equation (6.16) it is clear that the result holds for $p = 2$. Using the identity $S_j\tilde{S}_j = S_j$ and from Theorems 6.4 and 6.7 we get for $1 < p < \infty$,

$$\begin{aligned}
\left|\left| \left(\sum_{j\in\mathbb{Z}} |S_jf|^2 \right)^{1/2} \right|\right|_p &= \left|\left| \left(\sum_{j\in\mathbb{Z}} |S_j\tilde{S}_jf|^2 \right)^{1/2} \right|\right|_p \\
&\leq C\left|\left| \left(\sum_{j\in\mathbb{Z}} |\tilde{S}_jf|^2 \right)^{1/2} \right|\right|_p \\
&\leq C_p||f||_p.
\end{aligned}$$

Therefore, we have

$$\left|\left| \left(\sum_{j\in\mathbb{Z}} |S_jf|^2 \right)^{1/2} \right|\right|_p \leq C_p||f||_p. \tag{6.21}$$

To prove the other inequality we first remark that, inner product on $L^2(\mathbb{R},\ell^2)$ is defined as

$$\langle F,G\rangle_{L^2(\mathbb{R},\ell^2)} = \int\limits_{\mathbb{R}} \langle F(x),G(x)\rangle_{\ell^2}\ \mathrm{d}x$$

Now, consider $F = (S_jf)_{j\in\mathbb{Z}}$ and $G = (S_jg)_{j\in\mathbb{Z}}$, for $f,g \in L^2(\mathbb{R})$. Then from Equation (6.16), and the Polarization identity, we have

$$\begin{aligned}
\langle F,G\rangle_{L^2(\mathbb{R},\ell^2)} = \frac{1}{4}\Big(&||F+G||^2_{L^2(\mathbb{R},\ell^2)} - ||F-G||^2_{L^2(\mathbb{R},\ell^2)} \\
&+i||F+iG||^2_{L^2(\mathbb{R},\ell^2)} - i||F-iG||^2_{L^2(\mathbb{R},\ell^2)}\Big)
\end{aligned}$$

$$= \frac{1}{4}\left(||f+g||_2^2 - ||f-g||_2^2 + i||f+ig||_2^2 - i||f-ig||_2^2\right)$$
$$= \langle f, g\rangle_{L^2(\mathbb{R})}.$$

That is, for any $f, g \in L^2(\mathbb{R})$, we have,

$$\int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} S_j f(x)\overline{S_j g(x)}\ \mathrm{d}x = \int_{\mathbb{R}} f(x)\overline{g(x)}\ \mathrm{d}x. \tag{6.22}$$

Particularly, Equation (6.22) holds for all $f, g \in \mathcal{S}(\mathbb{R})$.

Now for $p \neq 2$, we have the following for $f \in \mathcal{S}(\mathbb{R})$.

$$\begin{aligned}||f||_p &= \sup\left\{\left|\int_{\mathbb{R}} f(x)\overline{g(x)}\ \mathrm{d}x\right| : ||g||_{p'} \leq 1, g \in \mathcal{S}(\mathbb{R})\right\}\\ &= \sup\left\{\left|\int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} S_j f(x)\overline{S_j g(x)}\ \mathrm{d}x\right| : ||g||_{p'} \leq 1, g \in \mathcal{S}(\mathbb{R})\right\}\\ &\leq \sup\left\{\int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} |S_j f(x)||S_j g(x)|\ \mathrm{d}x : ||g||_{p'} \leq 1, g \in \mathcal{S}(\mathbb{R})\right\}.\end{aligned}$$

By using Hölder's inequality for the sum, we get

$$\begin{aligned}&||f||_p\\ &\leq \sup\left\{\int_{\mathbb{R}} \left(\sum_{j\in\mathbb{Z}} |S_j f(x)|^2\right)^{1/2} \left(\sum_{j\in\mathbb{Z}} |S_j g(x)|^2\right)^{1/2} \mathrm{d}x : ||g||_{p'} \leq 1, g \in \mathcal{S}(\mathbb{R})\right\}\end{aligned}$$

Further, from Hölder's inequality for exponents p and p' we have

$$\begin{aligned}&||f||_p\\ &\leq \sup\left\{\left|\left| \left(\sum_{j\in\mathbb{Z}} |S_j f|^2\right)^{1/2} \right|\right|_p \left|\left| \left(\sum_{j\in\mathbb{Z}} |S_j g(x)|^2\right)^{1/2} \right|\right|_{p'} : ||g||_{p'} \leq 1, g \in \mathcal{S}(\mathbb{R})\right\}.\end{aligned}$$

Now by using Inequality (6.21), we have

$$\begin{aligned}||f||_p &\leq \left|\left| \left(\sum_{j\in\mathbb{Z}} |S_j f(x)|^2\right)^{1/2} \right|\right|_p \sup\left\{C_p'||g||_{p'} : ||g||_{p'} \leq 1, g \in \mathbb{S}(\mathbb{R})\right\}\\ &\leq C_p\left|\left| \left(\sum_{j\in\mathbb{Z}} |S_j f(x)|^2\right)^{1/2} \right|\right|_p.\end{aligned}$$

So, there is a constant c_p such that

$$\left|\left| \left(\sum_{j\in\mathbb{Z}} |S_j f|^2\right)^{1/2} \right|\right|_p \geq c_p||f||_p. \tag{6.23}$$

□

We now wish to extend Theorems 6.7 and 6.8 to $\mathbb{R}^n$. First we prove an analogue of Theorem 6.7. Then we consider characteristic functions of products of dyadic intervals on the coordinate axes and define an operator similar to S_j, but taking functions defined on $\mathbb{R}^n$ as its input.

Theorem 6.9. *Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\psi(0) = 0$, let $\widehat{(S_j f)}(\xi) = \psi(2^{-j}\xi)\widehat{f}(\xi)$. Then for $1 < p < \infty$, there is a constant $C_p > 0$ such that $\forall f \in L^p(\mathbb{R}^n)$, we have*

$$\left|\left| \left(\sum_{j\in\mathbb{Z}} |S_j f|^2 \right)^{1/2} \right|\right|_p \leq C_p ||f||_p. \tag{6.24}$$

Furthermore if for all $\xi \neq 0$

$$\sum_{j\in\mathbb{Z}} |\psi(2^{-j}\xi)|^2 = C, \tag{6.25}$$

then, we also have a constant $C'_p > 0$ such that $\forall f \in L^p(\mathbb{R}^n)$.

$$||f||_p \leq C'_p \left|\left| \left(\sum_{j\in\mathbb{Z}} |S_j f|^2 \right)^{1/2} \right|\right|_p. \tag{6.26}$$

Proof. Note that as $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $\psi(0) = 0$, $\sum\limits_{j\in\mathbb{Z}} \psi(2^{-j}\xi) \leq C$. Therefore,

$$\begin{aligned} \left|\left| \left(\sum_{j\in\mathbb{Z}} |S_j f|^2 \right)^{1/2} \right|\right|_2^2 &= \int\limits_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} |\psi_j(\xi)|^2 |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \\ &\leq \int\limits_{\mathbb{R}^n} C|\widehat{f}(\xi)|^2 \, \mathrm{d}\xi = C||f||_2. \end{aligned}$$

Now following similar steps as in Theorem 6.7 we can prove Inequality (6.24). If for all $\xi \neq 0$

$$\sum_{j\in\mathbb{Z}} |\psi(2^{-j}\xi)|^2 = C,$$

we have

$$\left|\left| \left(\sum_{j\in\mathbb{Z}} |S_j f|^2 \right)^{1/2} \right|\right|_2^2 = C||f||_2.$$

Therefore with similar argument as in Theorem 6.8, we have for $f, g \in \mathcal{S}(\mathbb{R}^n)$.

$$\int\limits_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} S_j f(x)\overline{S_j g(x)} \, \mathrm{d}x = C \int\limits_{\mathbb{R}^n} f(x)\overline{g(x)} \, \mathrm{d}x.$$

Now Inequality (6.26) follows from the last part stated in the last part of Theorem 6.8. □

We now generalize Theorem 6.7 for functions in $L^p(\mathbb{R}, \ell^2)$. We keep the same notations as in Theorem 6.7.

Lemma 6.10. *For $1 < p < \infty$, there exists a constant $C_p > 0$ such that for any $(f_k)_{k\in\mathbb{Z}} \in L^p(\mathbb{R}, \ell^2)$, we have,*

$$\left|\left| \left(\sum_{j,k\in\mathbb{Z}} |\tilde{S}_j f_k|^2 \right)^{1/2} \right|\right|_p \leq C_p \left|\left| \left(\sum_{k\in\mathbb{Z}} |f_k|^2 \right)^{1/2} \right|\right|_p$$

Proof. First we show that for $p = 2$, the operator $(f_k)_{k\in\mathbb{Z}} \longmapsto \left(\tilde{S}_j f_k\right)_{j,k\in\mathbb{Z}}$ is bounded. Using the Plancheral theorem, we have

$$\begin{aligned} \left|\left| \left(\sum_{j,k\in\mathbb{Z}} |\tilde{S}_j f_k|^2 \right)^{1/2} \right|\right|_2^2 &= \sum_{j,k\in\mathbb{Z}} \int_{\mathbb{R}} |\tilde{S}_j f_k(x)|^2 \,\mathrm{d}x \\ &= \int_{\mathbb{R}} \sum_{j,k\in\mathbb{Z}} |(\tilde{S}_j f_k)\widehat{}(\xi)|^2 \,\mathrm{d}\xi \\ &= \int_{\mathbb{R}} \sum_{j,k\in\mathbb{Z}} |\psi_j(\xi)|^2 |\widehat{f_k}(\xi)|^2 \,\mathrm{d}\xi. \end{aligned}$$

Now arguing as in Theorem 6.7, we get

$$\left|\left| \left(\sum_{j,k\in\mathbb{Z}} |\tilde{S}_j f_k|^2 \right)^{1/2} \right|\right|_2 \leq 3 \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |f_k(\xi)|^2 \,\mathrm{d}\xi = 3 \left|\left| \left(\sum_{k\in\mathbb{Z}} |f_k|^2 \right)^{1/2} \right|\right|_2^2.$$

Let $K : \mathbb{R} \times \mathbb{R} \longrightarrow \mathcal{L}\left(\ell^2, \ell^2(\mathbb{Z} \times \mathbb{Z})\right)$ be kernel of the operator $(f_k)_{k\in\mathbb{Z}} \longmapsto \left(\tilde{S}_j f_k\right)_{j,k\in\mathbb{Z}}$. That is, for $(x, y) \in \mathbb{R} \times \mathbb{R}$, we have

$$K(x, y)\left((u_k)_{k\in\mathbb{Z}}\right) = \left(\Phi_j(x - y) u_k\right)_{j,k\in\mathbb{Z}}.$$

Note that

$$\begin{aligned} ||K(x, y)\left((u_k)_{k\in\mathbb{Z}}\right)||^2_{\ell^2(\mathbb{Z}\times\mathbb{Z})} &= \sum_{j,k\in\mathbb{Z}} |\Phi_j(x - y)|^2 |u_k|^2 \\ &= \sum_{j\in\mathbb{Z}} |\Phi_j(x - y)|^2 \sum_{k\in\mathbb{Z}} |u_k|^2 \\ &= ||\left(\Phi_j(x - y)\right)_{j\in\mathbb{Z}}||^2_{\ell^2} ||\left(u_k\right)_{k\in\mathbb{Z}}||^2_{\ell^2}. \end{aligned}$$

Therefore

$$||K(x,y)|| = ||\left(\Phi_j(x-y)\right)_{j\in\mathbb{Z}}||_{\ell^2}.$$

By the exact same arguments as in Theorem 6.7, we see that K satisfies the Hörmander condition

$$\int\limits_{|x-y|>2|y-z|} ||K(x,y)-K(x,z)||\,\mathrm{d}x \le C.$$

The lemma now follows from Theorem 6.2. $\square$

We are now ready to state the Littlewood-Paley theorem for functions defined on $\mathbb{R}^2$. To make the theorem less cumbersome, we make the following constructions and notations here. Let f be a complex valued function defined on $\mathbb{R}^2$. Let us define the operators

$$\left(S_j^1 f\right)\widehat{}\,(\xi_1,\xi_2) = \chi_{\Delta_j}(\xi_1)\widehat{f}(\xi_1,\xi_2)$$

and

$$\left(S_k^2 f\right)\widehat{}\,(\xi_1,\xi_2) = \chi_{\Delta_k}(\xi_2)\widehat{f}(\xi_1,\xi_2).$$

We use $(f)_{x_2}(x_1) = f(x_1,x_2)$ and $f_{x_1}(x_2) = f(x_1,x_2)$. If $\mathcal{F}_1$ is the one dimensional Fourier transform, then

$$\mathcal{F}_1\left(\tilde{S}_j(f)_{x_2}(\cdot)\right)(\xi_1) = \psi(2^{-j}\xi_1)\widehat{(f)_{x_2}}(\xi_1) = \mathcal{F}_1\left(\tilde{S}_j^1 f\right)(\xi_1,x_2).$$

Similar statement can be made about restriction of f to the second variable. We now see that Lemma 6.10 also holds for functions of several variables where $\tilde{S}_j$ acts only on one of the variables.

Lemma 6.11. *There is a constant $C_p > 0$ such that $\forall (f_k)_{k\in\mathbb{Z}} \in L^p(\mathbb{R},\ell^2)$, we have*

$$\Bigg|\Bigg|\left(\sum_{j,k\in\mathbb{Z}} |\tilde{S}_j f|^2\right)^{1/2}\Bigg|\Bigg|_p \le C_p \Bigg|\Bigg|\left(\sum_{k\in\mathbb{Z}} |f_k|^2\right)^{1/2}\Bigg|\Bigg|_p.$$

Proof. We notice that,

$$\Bigg|\Bigg|\left(\sum_{j,k\in\mathbb{Z}} |\tilde{S}_j f|^2\right)^{1/2}\Bigg|\Bigg|_p \le C_p \Bigg|\Bigg|\left(\int\limits_{\mathbb{R}^2}\left(\sum_{j,k\in\mathbb{Z}} |\tilde{S}_j f(x)|^2\right)^{p/2}\mathrm{d}x\right)^{1/p}$$

$$= \left(\int\limits_{\mathbb{R}} \left[\int\limits_{\mathbb{R}} \left(\sum_{j,k\in\mathbb{Z}} |\tilde{S}_j f(x_1,x_2)|^2 \right)^{p/2} \mathrm{d}x_1 \right]^{p/p} \mathrm{d}x_2 \right)^{1/p}$$

$$= \left(\int\limits_{\mathbb{R}} \left|\left| \left(\sum_{j,k\in\mathbb{Z}} |(\tilde{S}_j f_k)_{x_2}(\cdot)|^2 \right)^{1/2} \right|\right|_p^p \mathrm{d}x_2 \right)^{1/p}$$

$$= \left(\int\limits_{\mathbb{R}} \left|\left| \left(\sum_{j,k\in\mathbb{Z}} |\tilde{S}_j (f_k)_{x_2}(\cdot)|^2 \right)^{1/2} \right|\right|_p^p \mathrm{d}x_2 \right)^{1/p}.$$

As $(f_k)_{x_2}$ is a one variable function, using Lemma 6.10 we have

$$\left|\left| \left(\sum_{j,k\in\mathbb{Z}} |\tilde{S}_j f|^2 \right)^{1/2} \right|\right|_p \leq C_p \left(\int\limits_{\mathbb{R}} \left|\left| \left(\sum_{k\in\mathbb{Z}} |(f_k)_{x_2}(\cdot)|^2 \right)^{1/2} \right|\right|_p^p \mathrm{d}x_2 \right)^{1/p}$$

$$= C_p \left(\int\limits_{\mathbb{R}^2} \left(\sum_{k\in\mathbb{Z}} |f_k(x)|^2 \right)^{p/2} \right)^{1/p} \mathrm{d}x$$

$$= C_p \left|\left| \left(\sum_{k\in\mathbb{Z}} |f_k|^2 \right)^{1/2} \right|\right|_p.$$

□

Theorem 6.12 (Littlewood-Paley in $\mathbb{R}^2$). *Let $1 < p < \infty$. Then there exist positive constants c_p and C_p such that $\forall f \in L^p(\mathbb{R}^2)$, we have,*

$$c_p ||f||_p \leq \left|\left| \left(\sum_{j,k\in\mathbb{Z}} |S_j^1 S_k^2 f|^2 \right)^{1/2} \right|\right|_p \leq C_p ||f||_p.$$

Proof. Note that $S_j^1 = S_j^1 \tilde{S}_j^1$. Therefore

$$\left|\left| \left(\sum_{j,k\in\mathbb{Z}} |S_j^1 S_k^2 f|^2 \right)^{1/2} \right|\right|_p = \left|\left| \left(\sum_{j,k\in\mathbb{Z}} |S_j^1 \tilde{S}_j^1 S_k^2 f|^2 \right)^{1/2} \right|\right|_p.$$

By using Theorem 6.5, we have

$$\left|\left| \left(\sum_{j,k\in\mathbb{Z}} |S_j^1 \tilde{S}_j^1 S_k^2 f|^2 \right)^{1/2} \right|\right|_p \leq C \left|\left| \left(\sum_{j,k\in\mathbb{Z}} |\tilde{S}_j^1 S_k^2 f|^2 \right)^{1/2} \right|\right|_p.$$

From Lemma 6.11 we obtain

$$\left|\left| \left(\sum_{j,k\in\mathbb{Z}} |\tilde{S}_j^1 S_k^2 f|^2 \right)^{1/2} \right|\right|_p \leq C_p \left|\left| \left(\sum_{j,k\in\mathbb{Z}} |S_k^2 f|^2 \right)^{1/2} \right|\right|_p.$$

Finally, using Theorem 6.8 we arrive at

$$\left|\left|\left(\sum_{j,k\in\mathbb{Z}} |S_k^2 f|^2\right)^{1/2}\right|\right|_p \leq C_p||f||_p,$$

which proves

$$\left|\left|\left(\sum_{j,k\in\mathbb{Z}} |S_j^1 S_k^2 f|^2\right)^{1/2}\right|\right|_p \leq C_p||f||_p.$$

Now to prove other inequality we use the fact that

$$\int_{\mathbb{R}^2} f(x)\overline{g(x)}\ \mathrm{d}x = \int_{\mathbb{R}^2} \sum_{j,k\in\mathbb{Z}} S_j^1 S_k^2 f(x) \cdot \overline{S_j^1 S_k^2 g(x)}\ \mathrm{d}x,$$

for all $f, g \in \mathcal{S}(\mathbb{R}^2)$. Indeed this follows from the L^2-estimate and the Polarization identities of $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2, \ell^2(\mathbb{Z}\times\mathbb{Z}))$. The result now follows from the duality argument given in the proof of Theorem 6.8. □

6.3 Multipliers

Suppose $m \in L^\infty(\mathbb{R}^n)$. We define an operator T_m on $L^2(\mathbb{R}^n)$ by

$$\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi). \tag{6.27}$$

By Plancherel theorem, $T_m f$ is well defined and bounded on $L^2(\mathbb{R}^n)$. In fact we have the following result.

Theorem 6.13. *The operator $T_m : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is bounded with operator norm $||m||_\infty$.*

Proof. We notice that,

$$||T_m f||_2 = ||\widehat{T_m f}||_2 = ||m(\xi)\widehat{f}(\xi)||_2 = |m(\xi)|||\widehat{f}(\xi)||_2 \leq ||m||_\infty ||f||_2.$$

Therefore T_m is bounded on $L^2(\mathbb{R}^n)$ and $||T_m||_2 \leq ||m||_\infty$. Now let $\epsilon > 0$ and let A be a subset of $\{x \in \mathbb{R}^n : |m(x)| > ||m||_\infty - \epsilon\}$ whose measure is finite and positive. Let $f \in L^2(\mathbb{R}^n)$ such that $\widehat{f} = \chi_A$. Now,

$$||T_m f||_2^2 = ||m(\xi)\widehat{f}(\xi)||_2^2 = \int_{\mathbb{R}^n} |m(\xi)|^2 |\widehat{f}(\xi)|^2\ \mathrm{d}\xi$$

$$= \int_A |m(\xi)|^2 \, d\xi$$

$$\geq (||m||_\infty - \epsilon)^2 |A|$$

$$= (||m||_\infty - \epsilon)^2 ||f||_2^2.$$

Since $\epsilon > 0$ is arbitrary, we have $||T_m||_2 \geq ||m||_\infty$. Therefore, $||T_m||_2 = ||m||_\infty$. $\square$

Definition 6.1. *A function $m \in L^\infty(\mathbb{R}^n)$ is an L^p-multiplier if the operator T_m is bounded on $L^p(\mathbb{R}^n)$.*

Now let us see some examples of multipliers.

Example 6.1. We have seen that the Hilbert transform has the following expression

$$\widehat{Hf}(\xi) = -i \operatorname{sgn}(\xi) \widehat{f}(\xi),$$

So, for $f \in L^2(\mathbb{R})$, $m(\xi) = -i \operatorname{sgn}(\xi)$ is a multiplier for the operator H. As we have also seen that H is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$, m is a multiplier on $L^p(\mathbb{R})$ as well.

Definition 6.2. *Let $-\infty \leq a < b \leq \infty$, we define $m_{a,b}(\xi) = \chi_{(a,b)}(\xi)$. We define an operator $S_{a,b}$ associated with this multiplier as*

$$\widehat{S_{a,b}f}(\xi) = \chi_{(a,b)}(\xi) \widehat{f}(\xi).$$

We see that $\chi_{(a,b)}$ is a multiplier on $L^p(\mathbb{R})$ for any $1 \leq p \leq \infty$. For the same, let us first define modulations.

Definition 6.3 (Modulation)**.** *For $a \in \mathbb{R}$, modulation by 'a' is the operator M_a is defined as*

$$M_a f(x) = e^{2\pi i a x} f(x).$$

Lemma 6.14. *The multiplier of the operator iM_aHM_{-a} is sgn($\xi - a$).*

Proof. Using the definition of the operator M_a and the definition of Hilbert transform we have the following

$$(M_aHM_{-a}f)\widehat{\ }(\xi) = \left(e^{2\pi i a \cdot} HM_{-a}f\right)\widehat{\ }(\xi)$$

$$
\begin{aligned}
&= (HM_{-a}f(x))\widehat{\ }(\xi - a) \\
&= -i \ \mathrm{sgn}(\xi - a)\widehat{M_{-a}f}(\xi - a) \\
&= -i \ \mathrm{sgn}(\xi - a)\widehat{f}(\xi).
\end{aligned}
$$

Therefore the multiplier of the operator iM_aHM_{-a} is $\mathrm{sgn}(\xi - a)$. □

Using Lemma 6.14 we get an equivalent expression for the operator $S_{a,b}$ in terms of Hilbert transform.

Theorem 6.15. *On $L^2(\mathbb{R})$ the operator $S_{a,b}$ can be written as*

$$S_{a,b} = \frac{i}{2}\left(M_aHM_{-a} - M_bHM_{-b}\right),$$

provided $-\infty < a < b < \infty$. In the other cases, we have,

$$
\begin{aligned}
S_{-\infty,b} &= I + \frac{i}{2}M_bHM_{-b}, \ \textit{for } b < \infty. \\
S_{a,\infty} &= \frac{i}{2}M_aHM_{-a} + I, \ \textit{for } a > -\infty. \\
S_{-\infty,\infty} &= I.
\end{aligned}
$$

Here, I is the identity operator on $L^2(\mathbb{R}^n)$.

Proof. We notice from Lemma 6.14 that

$$
\begin{aligned}
\widehat{S_{a,b}f}(\xi) &= \chi_{(a,b)}(\xi)\widehat{f}(\xi) \\
&= \frac{1}{2}\left(\mathrm{sgn}(\xi - a) - \mathrm{sgn}(\xi - b)\right)\widehat{f}(\xi) \\
&= \frac{i}{2}\left(M_aHM_{-a}f - M_bHM_{-b}f\right)\widehat{\ }(\xi).
\end{aligned}
$$

Now, let us assume that $b = \infty$. We notice that

$$
\chi_{a,\infty}(\xi)\widehat{f}(\xi) = \begin{cases} 0 & \text{if } \xi \le a. \\ \widehat{f}(\xi) & \text{if } \xi > a. \end{cases}
$$

Therefore we can write the following

$$
\begin{aligned}
(\widehat{S_{a,\infty}f})(\xi) &= \chi_{a,\infty}(\xi)\widehat{f}(\xi) \\
&= \frac{1}{2}\Big(\mathrm{sgn}(\xi - a)\widehat{f}(\xi) + \widehat{f}(\xi)\Big) \\
&= \frac{i}{2}\left(M_aHM_{-a}f\right)^{\wedge} + \widehat{f}(\xi).
\end{aligned}
$$

Therefore we have

$$S_{a,\infty} = \frac{i}{2} M_a H M_{-a} + I.$$

The proof for the case $S_{-\infty,b}$ is similar. The last case when $a = -\infty$ and $b = -\infty$, since $\chi_{(-\infty,\infty)} \equiv 1$. $\square$

Theorem 6.16. *The operator $S_{a,b}$ is bounded on $L^p(\mathbb{R})$, where $1 < p < \infty$.*

Proof. First we see that M_a is an isometry on $L^p(\mathbb{R})$. Indeed, we have

$$||M_a f(x)||_p^p = \int_{\mathbb{R}} |e^{2\pi i a x} f(x)|^p \, \mathrm{d}x = \int_{\mathbb{R}} |f(x)|^p \, \mathrm{d}x = ||f||_p^p.$$

Therefore $||M_a|| = 1$. Now using Theorem 6.15 and the fact that H is strong (p,p) we have the following

$$\begin{aligned} ||S_{a,b}f||_p &\le \frac{1}{2}\left(||M_a H_{-a} f||_p + ||M_b H M_{-b}||_p\right) \\ &\le \frac{1}{2}\left(||H M_{-a} f||_p + ||H M_{-b}||_p\right) \\ &\le \frac{C_p}{2}\left(||M_{-a} f||_p + ||M_{-b}||_p\right) \\ &\le C_p ||f||_p. \end{aligned}$$

We have proved the case when $-\infty < a < b < \infty$. As the operators $M_a H M_{-a}$ and I are bounded, $S_{a,\infty}$ is also bounded. $\square$

We next look at some consequence of the definition of multipliers. The following result gives a way to generate "new" multipliers from a given multiplier.

Proposition 6.17. *If m is a multiplier on $L^p(\mathbb{R}^n)$, then the functions defined by*

1. *$m(\xi + a)$, for $a \in \mathbb{R}^n$;*

2. *$m(\lambda \xi)$, for $\lambda > 0$; and*

3. *$m(\rho \xi)$, for $\rho \in O(n)$ (orthogonal group),*

are multipliers on $L^p(\mathbb{R}^n)$ with the same norm as that of m.

Proof. We begin by studying translation of m.

Let $m^{'}(\xi) = m(\xi - a)$. It is clear that $m' \in L^{\infty}(\mathbb{R}^n)$, for a fixed $a \in \mathbb{R}^n$. Then, by definition of a multipliers we have

$$\widehat{(T_{m'}f)}(\xi) = m^{'}(\xi)\widehat{f}(\xi).$$

Therefore $\widehat{(T_{m'}f)}(\xi + a) = m^{'}(\xi + a)\widehat{f}(\xi + a) = m(\xi)\widehat{f}(\xi + a)$. As $\widehat{e^{2\pi i a \cdot}f}(\xi) = \widehat{f}(\xi + a)$, we have

$$\begin{aligned}\widehat{(T_{m'}f)}(\xi + a) &= m(\xi)(\widehat{e^{2\pi i a \cdot}f})(\xi) \\ &= m(\xi)(\widehat{M_a f(x)})(\xi) \\ &= (T_m(M_a f))\widehat{\ }(\xi).\end{aligned}$$

Therefore, $\widehat{(T_{m'}f)}(\xi) = (T_m(M_a f))\widehat{\ }(\xi - a)$. Now using the properties of Fourier transform we get

$$\widehat{(T_{m'}f)}(\xi) = ((M_{-a}T_m M_a)(f))\widehat{\ }(\xi).$$

Taking the inverse Fourier transform, we have

$$T_{m'} = M_{-a}T_m M_a. \tag{6.28}$$

Similarly,

$$T_m = M_a T_{m'} M_{-a}. \tag{6.29}$$

As $||M_a|| = ||M_{-a}|| = 1$, on any $L^p(\mathbb{R}^n)$, we have from Equitation (6.28) and (6.29) that $||T_{m'}|| \leq ||T_m||$ and $||T_m|| \leq ||T_{m'}||$. That is, m' is an L^p-multiplier with $||T_{m'}|| = ||T_m||$.

Next, we consider dilations, $m_1(\xi) = m(\lambda\xi)$ for $\lambda > 0$. Again it is easy to see that $m_1 \in L^{\infty}(\mathbb{R}^n)$. Then, by definition, we have,

$$\widehat{(T_{m_1}f)}(\xi) = m_1(\xi)\widehat{f}(\xi).$$

Therefore,

$$\widehat{(T_{m_1}f)}(\xi/\lambda) = m(\xi)\widehat{f}(\xi/\lambda)$$

We know that $(\lambda^n f(\lambda\cdot))\widehat{\ }(\xi) = \widehat{f}(\xi/\lambda)$. Hence, we must have

$$\widehat{(T_{m_1}f)}(\xi/\lambda) = m(\xi)\,(\lambda^n f(\lambda\cdot))\widehat{\ }(\xi).$$

Let $g(x) = \lambda^n f(\lambda x)$. Then,

$$\widehat{(T_{m_1} f)}(\xi/\lambda) = m(\xi)\widehat{(g(x))}(\xi) = \widehat{(T_m g)}(\xi).$$

Therefore,

$$\begin{aligned}\widehat{(T_{m_1} f)}(\xi) &= \widehat{(T_m g)}(\lambda\xi)\\ &= \left(\lambda^{-n} T_m g(\lambda^{-1}\cdot)\right)\widehat{}\\ &= \left(\lambda^{-n} T_m \left(\lambda^n f(\lambda^{-1}\lambda\cdot)\right)\right)\widehat{}(\xi)\\ &= \widehat{(T_m f)}(\xi).\end{aligned}$$

Therefore we have $T_{m_1} f = T_m f$. Clearly, m is an multiplier with $||T_{m_1}||_p = ||T_m||_p$.

Lastly we look at the action of the orthogonal group on multipliers. Let $\rho \in O(n)$, and $m_2(\xi) = m(\rho\xi)$. We have that $m_2 \in L^\infty(\mathbb{R}^n)$ and

$$\widehat{(T_{m_2} f)}(\xi) = m_2(\xi)\widehat{f}(\xi).$$

Therefore,

$$\begin{aligned}\widehat{(T_{m_2} f)}(\rho^{-1}\xi) &= m(\xi)\widehat{f(\rho^{-1}\cdot)}(\xi)\\ &= \widehat{(T_m g)}(\xi),\end{aligned}$$

where $g(x) = f(\rho^{-1}x)$. Consider the map $\varphi : O \longrightarrow GL(L^p(\mathbb{R}^n))$ defined by $\varphi(\rho)f(x) = f(\rho^{-1}x)$. We wish that $\forall \rho \in O(n)$, $\varphi(\rho) : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$ is an isometry. Indeed, we have

$$\begin{aligned}||\varphi(\rho)f||_p^p &= \int\limits_{\mathbb{R}^n} |f(\rho^{-1}x)|^p \, \mathrm{d}x\\ &= \int\limits_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x = ||f||_p^p.\end{aligned}$$

That is for all $\rho \in O(n)$, the operator norm $||\varphi(\rho)|| = 1$, on any $L^p(\mathbb{R}^n)$.

Now, we have

$$\begin{aligned}\widehat{(T_{m_2} f)}(\xi) &= \widehat{(T_m g)}(\rho\xi)\\ &= (T_m g(\rho\cdot))\widehat{}(\xi)\\ &= \left(\phi(\rho^{-1})T_m g(\cdot)\right)\widehat{}(\xi)\end{aligned}$$

$$= \left(\left(\phi(\rho^{-1})T_m\phi(\rho)\right) f\right)\widehat{}\,(\xi).$$

Therefore $T_{m_2} = \phi(\rho^{-1})T_m\phi(\rho)$. This also gives $T_m = (\varphi(\rho^{-1}))^{-1}T_{m_2}(\varphi(\rho))^{-1}$. As $\varphi(\rho)$ is an isometry on $L^p(\mathbb{R}^n)$, we have $||T_m|| \leq ||T_{m_2}||$ and $||T_{m_2}|| \leq ||T_m||$. Hence m_2 is an L^p-multiplier with $||T_m||_p = ||T_{m_2}||_p$. □

6.3.1 The Hörmander multiplier theorem

In this section we deal with the space

$$L^2_a(\mathbb{R}^2) := \{g \in L^2(\mathbb{R}) : (1 + |\cdot|^2)^{a/2}\widehat{g} \in L^2(\mathbb{R}^n) \in L^2(\mathbb{R}^n)\}.$$

This space is known as the Sobolev space of regularity 'a'. The origin of Sobolev spaces lies with the distribution theory, where we generalize the notion of classical derivatives to functions that might not be continuous. For details on distribution we refer to [23]. We would like to remark here that if $g \in L^2(\mathbb{R}^n)$ is smooth and $a \in \mathbb{N}$ is an even number, then we notice that $(1 + ||\cdot||^2)^{a/2}\widehat{g} = ((I + \Delta)^{a/2}g)^a$, where $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian on $\mathbb{R}^n$. Essentially, the Sobolev space $L^2_a(\mathbb{R}^n)$ consists of functions in $L^2(\mathbb{R}^n)$ that are a times "differentiable" in some sense. Similarly, we can define Sobolev spaces of $L^p(\mathbb{R}^n)$-functions. However that is out of scope of this thesis. For more details we refer the readers to [11]. The Sobolev norm of the function g is defined by

$$||g||_{L^2_a} = \left(\int\limits_{\mathbb{R}^n} |(1 + |\xi|^2)^{a/2}\widehat{g}(\xi)|^2 \, \mathrm{d}\xi\right)^{1/2}.$$

We have the following easy result for Sobolev functions.

Proposition 6.18. *If $a > n/2$ and $g \in L^2_a(\mathbb{R}^n)$ then $\widehat{g} \in L^1(\mathbb{R}^n)$.*

Proof. We have,

$$\int\limits_{\mathbb{R}^n} |\widehat{g}(\xi)| \, \mathrm{d}\xi = \int\limits_{\mathbb{R}^n} (1 + |\xi|^2)^{a/2}\widehat{g}(\xi)(1 + |\xi|^2)^{-a/2} \, \mathrm{d}\xi$$

$$\leq \left(\int\limits_{\mathbb{R}^n} |(1 + |\xi|^2)^{a/2}\widehat{g}(\xi)|^2 \, \mathrm{d}(\xi)\right)^{1/2} \left(\int\limits_{\mathbb{R}^n} (1 + |\xi|^2)^{-a} \, \mathrm{d}\xi\right)^{1/2}$$

$$\leq C_a ||g||_{L^2_a}.$$

We see from the polar decomposition of $\mathbb{R}^n$ that

$$\int_{\mathbb{R}^n} (1 + ||\xi||^2)^{-a} \, \mathrm{d}\xi = C \int_0^\infty \frac{r^{n-1}}{(1+r^2)^a} \, \mathrm{d}r < \infty,$$

since the total power of r towards ∞ is $n - 1 - a < -1$. □

To prove the main result of this section we need to prove a weighted norm inequality.

Lemma 6.19. *Let $m \in L^2_a(\mathbb{R}^n)$, $a > n/2$, and let $\lambda > 0$. Define the operator $(T_\lambda f)\widehat{}(\xi) = m(\lambda\xi)\widehat{f}(\xi)$. Then for a positive function u defined in $\mathbb{R}^n$, we have*

$$\int_{\mathbb{R}^n} |T_\lambda f(x)|^2 u(x) \, \mathrm{d}x \leq C \int_{\mathbb{R}^n} |f(x)|^2 M u(x) \, \mathrm{d}x,$$

where the constant C is independent of u and λ and M is the Hardy-Littlewood maximal operator.

Proof. Let $K \in L^2(\mathbb{R}^n)$ be such that $\widehat{K} = m$ then note that $(1 + |x|^2)^{a/2} K(x) = R(x) \in L^2(\mathbb{R}^n)$. We have $(\lambda^{-n} K(\lambda^{-1} \cdot))\widehat{} = \widehat{K}(\lambda \cdot) = m(\lambda \cdot)$. Let K_1 be such that $K_1(x) = \lambda^{-n} K(\lambda^{-1} x)$. Therefore

$$(T_\lambda f)\widehat{}(\xi) = \widehat{K}(\lambda\xi)\widehat{f}(\xi) = (K_1 * f)\widehat{}(\xi).$$

Therefore, we have,

$$\begin{aligned}
&\int_{\mathbb{R}^n} |T_\lambda f(x)|^2 u(x) \, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} |K_1 * f(x)|^2 u(x) \, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \frac{\lambda^{-n} R(\lambda^{-1}(x-y))}{(1 + |\lambda^{-1}(x-y)|^2)^{a/2}} f(y) \, \mathrm{d}y \right|^2 u(x) \, \mathrm{d}x \\
&\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\lambda^{-n} R(\lambda^{-1}(x-y))|^2 \, \mathrm{d}y \right) \left(\int_{\mathbb{R}^n} \frac{|f(y)|^2}{(1 + |\lambda^{-1}(x-y)|^2)^a} \, \mathrm{d}y \right) u(x) \, \mathrm{d}x.
\end{aligned}$$

In the last Inequality, we have employed the Hölder inequality with $p = p' = 2$.

$$||m||^2_{L^2_a} = \lambda^n \int_{\mathbb{R}^n} |R(\lambda^{-1} x)|^2 \, \mathrm{d}x.$$

Therefore

$$\int_{\mathbb{R}^n} |T_\lambda f(x)|^2 u(x)\ \mathrm{d}x \le \lambda^{-n} ||m||^2_{L^2_a} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|f(y)|^2}{(1+|\lambda^{-1}(x-y)|^2)^a}\ \mathrm{d}y \right) u(x)\ \mathrm{d}x.$$

Since the integrand is non-negative by applying Fubini's theorem, we obtain

$$\int_{\mathbb{R}^n} |T_\lambda f(x)|^2 u(x)\ \mathrm{d}x \le \lambda^{-n} ||m||^2_{L^2_a} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{u(x)}{(1+|\lambda^{-1}(x-y)|^2)^a}\ \mathrm{d}x \right) |f(y)|^2\ \mathrm{d}y. \tag{6.30}$$

The inner integration can be written as

$$\int_{\mathbb{R}^n} \frac{u(x)}{(1+|\lambda^{-1}(x-y)|)^a}\ \mathrm{d}x = (\varphi_\lambda * u)(y).$$

where, $\varphi(x) = \frac{1}{1+|x|^2}$ and $\varphi_\lambda(x) = \lambda^{-n}\varphi(\lambda^{-1}x)$. Note the φ is positive, radial, and decreasing (as a function on $(0,\infty)$), and integrable. Therefore we have

$$|\varphi * u(x)| \le ||\varphi||_1 Mu(x)$$

for a.e $x \in \mathbb{R}^n$. Now, Inequality (6.30) gives

$$\begin{aligned}
\int_{\mathbb{R}^n} |T_\lambda f(x)|^2 u(x)\ \mathrm{d}x &\le ||m||^2_{L^2_a} \int_{\mathbb{R}^n} ||\varphi||_1 Mu(y)|f(y)|^2\ \mathrm{d}y \\
&\le C_a ||m||^2_{L^2_a} \int_{\mathbb{R}^n} |f(y)|^2 Mu(y)\ \mathrm{d}y \\
&= C \int_{\mathbb{R}^n} |f(y)|^2 Mu(y)\ \mathrm{d}y.
\end{aligned}$$

□

To prove Hörmander's multiplier theorem, we require the following construction. let $\psi \in C^\infty(\mathbb{R}^n)$ be a function which is radial and supported on the annulus $1/2 \le |\xi| \le 2$ such that

$$\sum_{j\in\mathbb{Z}} |\psi(2^j\xi)|^2 = 1,$$

when $\xi \neq 0$.

We now prove the main result of this section due to Hörmander (see [15]).

Theorem 6.20 (Hörmander)**.** *Let ψ be as defined above and let m be such that* *for some $a > n/2$,*

$$\sup_{j\in\mathbb{Z}} ||m(2^j\cdot)\psi||_{L^2_a} < \infty.$$

Then the operator T associated with the multiplier m is bounded on $L^p(\mathbb{R}^n)$, for any $1 < p < \infty$.

Proof. First define the family of operators $\{S_j\}_{j\in\mathbb{Z}}$ by $(S_jf)\widehat{\ }(\xi) = \psi(2^{-j}\xi)\widehat{f}(\xi)$. Then by our choice of ψ, Inequality (6.26) holds. Now let $\widetilde{\psi} \in C_c^\infty(\mathbb{R}^n)$ be supported on $1/4 \leq |\xi| \leq 4$ and equal to 1 on $1/2 \leq |\xi| \leq 2$. Now, let us define $\tilde{S}_j$ by $(\tilde{S}_jf)\widehat{\ }(\xi) = \tilde{\psi}(2^{-j}\xi)\widehat{f}(\xi)$. Then note that

$$(S_j\tilde{S}_jf)\widehat{\ }(\xi) = \psi(2^{-j}\xi)\tilde{\psi}(2^{-j}\xi)\widehat{f}(\xi) = \psi(2^{-j}\xi)\widehat{f}(\xi) = (S_jf)\widehat{\ }(\xi).$$

Therefore $S_j\tilde{S}_j = S_j$. Note that the family $\left(\tilde{S}_j\right)_{j\in\mathbb{Z}}$ satisfies Inequality (6.24). Since $(S_j)_{j\in\mathbb{Z}}$ satisfies Inequality (6.26), we have

$$||Tf||_p \leq C\left|\left|\left(\sum_{j\in\mathbb{Z}} |S_jTf|^2\right)^{1/2}\right|\right|_p = C\left|\left|\left(\sum_{j\in\mathbb{Z}} |S_j\tilde{S}_jTf|^2\right)^{1/2}\right|\right|_p.$$

Now we observe that

$$(\tilde{S}_jTf)\widehat{\ }(\xi) = \tilde{\psi}(2^{-j}\xi)(Tf)\widehat{\ }(\xi) = \tilde{\psi}(2^{-j}\xi)m(\xi)\widehat{f}(\xi) = m(\xi)(\tilde{S}_jf)\widehat{\ }(\xi) = (T\tilde{S}_jf)\widehat{\ }(\xi).$$

Therefore we have,

$$||Tf||_p \leq C\left|\left|\left(\sum_{j\in\mathbb{Z}} |S_jT\tilde{S}_jf|^2\right)^{1/2}\right|\right|_p. \tag{6.31}$$

Let $\tilde{S}_jf = g_j$. Since the multiplier of the operator S_jT is $\psi(2^{-j}\xi)m(\xi)$, by the hypothesis of this theorem and Lemma 6.19, there exists a constant C independent of j such that for any positive u defined on $\mathbb{R}^n$, we have,

$$\int_{\mathbb{R}^n} |S_jTg_j(x)|^2u(x)\ \mathrm{d}x \leq C\int_{\mathbb{R}^n} |g_j(x)|^2Mu(x)\ \mathrm{d}x. \tag{6.32}$$

Now, for $p > 2$, we see that

$$\left|\left|\left(\sum_{j\in\mathbb{Z}} |S_jTg_j|^2\right)^{1/2}\right|\right|_p^2 = \left|\left|\sum_{j\in\mathbb{Z}} |S_jTg_j|^2\right|\right|_{p/2}.$$

So there exists $u \in L^{(p/2)'}(\mathbb{R}^n)$ with $u \geq 0$ and $||u||_{(p/2)'} = 1$, such that

$$\left|\left|\left(\sum_{j\in\mathbb{Z}} |S_jTg_j|^2\right)^{1/2}\right|\right|_p^2 = \int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}} |S_jTg_j(x)|u(x)\ \mathrm{d}x.$$

Using Inequality (6.32), we obtain

$$\left|\left|\left(\sum_{j\in\mathbb{Z}}|S_jTg_j|^2\right)^{1/2}\right|\right|_p^2 \leq C\int\limits_{\mathbb{R}^n}|g_j(x)|^2Mu(x)\ \mathrm{d}x.$$

Now, applying Hölder's Inequality for exponents $p/2$ and $(p/2)'$ and using the fact that M is bounded on $L^{(p/2)'}(\mathbb{R}^n)$, we have

$$\left|\left|\left(\sum_{j\in\mathbb{Z}}|S_jTg_j|^2\right)^{1/2}\right|\right|_p^2 \leq C\left|\left|\left(\sum_{j\in\mathbb{Z}}|g_j|^2\right)^{1/2}\right|\right|_p^2||u||_{(p/2)'} \leq C\left|\left|\left(\sum_{j\in\mathbb{Z}}|g_j|^2\right)^{1/2}\right|\right|_p^2.$$

From Inequality(6.31), we get,

$$||Tf||_p \leq C\left|\left|\left(\sum_{j\in\mathbb{Z}}|g_j|^2\right)^{1/2}\right|\right|_p = C\left|\left|\left(\sum_{j\in\mathbb{Z}}|\tilde{S}_jf|^2\right)^{1/2}\right|\right|_p.$$

Since the family $\left(\tilde{S}_j\right)_{j\in\mathbb{N}}$, satisfies Inequality (6.24) we have $||Tf||_p \leq C||f||_p$. This completes the proof for $p > 2$. When $p < 2$, we use a duality argument. We want to see the transpose of operator T . Notice that, for $f, g \in \mathcal{S}(\mathbb{R}^n)$, using duality of Fourier transform, we get,

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} Tf(x)g(x)\ \mathrm{d} &= \int\limits_{\mathbb{R}^n} Tf(x)\hat{\hat{g}}(-x)\ \mathrm{d}x \\
&= \int\limits_{\mathbb{R}^n} \widehat{Tf}(x)\widehat{g}(-x)\ \mathrm{d}x \\
&= \int\limits_{\mathbb{R}^n} m(x)\hat{f}(x)\hat{g}(-x)\ \mathrm{d}x \\
&= \int\limits_{\mathbb{R}^n} \hat{f}(-x)m(-x)\hat{g}(x)\ \mathrm{d}x \\
&= \int\limits_{\mathbb{R}^n} \hat{f}(-x)\widehat{T^tg}(x)\ \mathrm{d}x \\
&= \int\limits_{\mathbb{R}^n} \hat{\hat{f}}(-x)T^tg(x)\ \mathrm{d}x \\
&= \int\limits_{\mathbb{R}^n} f(x)T^tg(x)\ \mathrm{d}x.
\end{aligned}$$

Here transpose of operator T, is defined by

$$\widehat{T^tg}(x) = m(-x)\widehat{g}(x).$$

Therefore T^t is also bounded for $p > 2$. Now for $p < 2$,

$$\begin{aligned}||Tf||_p &= \sup\left\{\left|\int_{\mathbb{R}^n} Tf(x)g(x)\ \mathrm{d}x\right| : ||g||_{p'} \leq 1\right\}\\ &\leq \sup\{||f||_p||T^t g||_{p'} : ||g||_{p'} \leq 1\}\\ &\leq C_p||f||_p\{||g||_{p'} : ||g||_{p'} \leq 1\}\\ &\leq C_p||f||_p.\end{aligned}$$

□

6.3.2 The Marcinkiewicz multiplier theorem

In this section, we study multipliers associated to dyadic intervals on $\mathbb{R}$. The theorem was originally due to Marcinkiewicz. Here, we follow the ideas of Duoandikoetxea ([7]).

Theorem 6.21. *Let m be a bounded function which has uniformly bounded variation on each dyadic interval in $\mathbb{R}$. Then m is a multiplier on $L^p(\mathbb{R})$, $1 < p < \infty$.*

Proof. Let T be the operator associated with the multiplier m, i.e, $(Tf)\widehat{}(\xi) = m(\xi)\widehat{f}(\xi)$. Let T_j be the operator associated with the multiplier $m\chi_{I_j}$. We consider the case $I_j = (2^j, 2^{j+1})$; The case when $I_j = (-2^{j+1}, 2^j)$ can be handled in exactly the same way. We do not provide the details of the latter case here. Note that for $\xi \in I_j$, we have,

$$m(2^j) + \int_{2^j}^{\xi} \mathrm{d}m(t) = m(2^j) + m(\xi) - m(2^j) = m(\xi).$$

Therefore,

$$(m\chi_{I_j})(\xi) = m(2^j) + \int_{2^j}^{\xi} \mathrm{d}m(t).$$

This gives

$$(m\chi_{I_j})(\xi)\widehat{f}(\xi) = m(2^j)\chi_{I_j}(\xi)\widehat{f}(\xi) + \int_{2^j}^{\xi} \widehat{f}(\xi)\ \mathrm{d}m(t). \tag{6.33}$$

We claim that

$$T_j f(x) = m(2^j) S_j f(x) + \int_{2^j}^{2^{j+1}} S_{t,2^{j+1}} f(x)\ \mathrm{d}m(t), \tag{6.34}$$

where $(S_{t,2^{j+1}} f)\widehat{}(\xi) = \chi_{(t,2^{j+1})}(\xi)\widehat{f}(\xi)$. To this end, we observe that

$$\begin{aligned}\int_{2^j}^{\xi} \widehat{f}(\xi)\ \mathrm{d}m(t) &= \int_{2^j}^{2^{j+1}} \chi_{(2^j,\xi)}(t)\widehat{f}(\xi)\ \mathrm{d}m(t)\\ &= \int_{2^j}^{2^{j+1}} \chi_{(t,2^{j+1})}(\xi)\widehat{f}(\xi)\ \mathrm{d}m(t)\\ &= \int_{2^j}^{2^{j+1}} (S_{t,2^{j+1}} f)\widehat{}(\xi)\ \mathrm{d}m(t).\end{aligned}$$

We denote

$$(\tau f)\widehat{}(\xi) = \int_{2^j}^{2^{j+1}} (S_{t,2^{j+1}} f)\widehat{}(\xi)\ \mathrm{d}m(t).$$

Now, considering $f \in \mathcal{S}(\mathbb{R})$, we can use Fubini's theorem to get,

$$\begin{aligned}\tau f(x) &= \int_{\mathbb{R}} \int_{2^j}^{2^{j+1}} (S_{t,2^{j+1}} f)\widehat{}(\xi)\ \mathrm{d}m(t) e^{ix\xi}\ \mathrm{d}\xi\\ &= \int_{2^j}^{2^{j+1}} \int_{\mathbb{R}} \chi_{(t,2^{j+1})}(\xi)\widehat{f}(\xi) e^{ix\xi}\ \mathrm{d}\xi \mathrm{d}m(t)\\ &= \int_{2^j}^{2^{j+1}} \int_{\mathbb{R}} (S_{t,2^{j+1}} f)\widehat{}(\xi) e^{ix\xi}\ \mathrm{d}\xi\ \mathrm{d}m(t)\\ &= \int_{2^j}^{2^{j+1}} S_{t,2^{j+1}} f(x)\ \mathrm{d}m(t).\end{aligned}$$

Therefore by taking inverse Fourier transform both side in Equation (6.33) and from the definition of S_j, Equation (6.34) is proved. Now we show that for any $w \in A_2$, S_j and $S_{t,2^{j+1}}$ are bounded on $L^2(w)$. The operator S_j can be written as

$$S_j = \frac{i}{2}\left(M_{2^j} H M_{-2^j} - M_{2^{j+1}} H M_{-2^{j+1}}\right)$$

where for any $a \in \mathbb{R}$, $M_a f(x) = e^{2\pi i a x} f(x)$. We recall that the Hilbert Transform

H, is a Calderón-Zygmund operator, and hence is bounded on $L^2(w)$, for any $w \in A_2$. Since $||M_a||_{L^2(w)} = 1$, for any $a \in \mathbb{R}$, the operator S_j is bounded on $L^2(w)$, for any $w \in A_2$. Similarly, $S_{t,2^{j+1}}$ can be written as

$$S_{t,2^{j+1}} = \frac{i}{2}\left(M_t H M_{-t} - M_{2^{j+1}} H M_{-2^{j+1}}\right).$$

Hence $S_{t,2^{j+1}}$ is also bounded on $L^2(w)$ $\forall w \in A_2$. Now, we have,

$$\left|\left|\int_{2^j}^{2^{j+1}} S_{t,2^{j+1}} f \ \mathrm{d}m(t)\right|\right|_{L^2(w)} = \left(\int_{\mathbb{R}} \left|\int_{2^j}^{2^{j+1}} S_{t,2^{j+1}} f(\xi) \mathrm{d}m(t)\right|^2 w(\xi)\ \mathrm{d}\xi\right)^{1/2}.$$

Using Minkowski's integral inequality, we have,

$$\begin{aligned}\left|\left|\int_{2^j}^{2^{j+1}} S_{t,2^{j+1}} f(\xi)\ \mathrm{d}m(t)\right|\right|_{L^2(w)} &\le \int_{2^j}^{2^{j+1}} \left(\int_{\mathbb{R}} |S_{t,2^{j+1}} f(\xi)|^2 w(\xi)\ \mathrm{d}\xi\right)^{1/2} \mathrm{d}m(t)\\ &= \int_{2^j}^{2^{j+1}} ||S_{t,2^{j+1}} f||_{L^2(w)}\ \mathrm{d}m(t).\end{aligned}$$

As the operator $S_{t,2^{j+1}}$ is bounded on $L^2(w)$ we obtain

$$\begin{aligned}\left|\left|\int_{2^j}^{2^{j+1}} S_{t,2^{j+1}} f(\xi)\ \mathrm{d}m(t)\right|\right|_{L^2(w)} &\le C||f||_{L^2(w)} \int_{2^j}^{2^{j+1}} \mathrm{d}m(t)\\ &\le CV(m)||f||_{L^2(w)},\end{aligned}$$

where $V(m)$ is the total variation of m. Combining the above observations, we get

$$\begin{aligned}||T_j f||_{L^2(w)} &\le |m(2^j)|||S_j f||_{L^2(w)} + \left|\left|\int_{2^j}^{2^{j+1}} S_{t,2^{j+1}} f(\xi)\ \mathrm{d}m(t)\right|\right|_{L^2(w)}\\ &\le C||m||_\infty ||f||_{L^2(w)} + CV(m)||f||_{L^2(w)}\\ &\le C||f||_{L^2(w)}.\end{aligned}$$

From Theorem 6.8, we have

$$||Tf||_p \le C\left|\left|\left(\sum_{j\in\mathbb{Z}} |S_j T f|^2\right)^{1/2}\right|\right|_p.$$

Hence, we observe that

$$(S_j T f)\widehat{\ }(\xi) = \chi_{I_j}(\xi) m(\xi) \widehat{f}(\xi) = \chi_{I_j}(\xi) m(\xi) \chi_{I_j}(\xi) \widehat{f}(\xi)$$

$$= \chi_{I_j}(\xi)m(\xi)(S_jf)\widehat{}(\xi) = (T_jS_jf)\widehat{}(\xi).$$

That is, $S_jT = T_jS_j$. Therefore

$$||Tf||_p \leq C\left|\left|\left(\sum_{j\in\mathbb{Z}} |T_jS_jf|^2\right)^{1/2}\right|\right|_p.$$

Since T_j is bounded on $L^2(w)$ for any $w \in A_2$, from Theorem 6.6, we get

$$\left|\left|\left(\sum_{j\in\mathbb{Z}} |T_jS_jf|^2\right)^{1/2}\right|\right|_p \leq C||f||_p.$$

This completes the proof. □

We now generalize Theorem 6.21 to higher dimensions. We state the result only for $\mathbb{R}^2$, where the notations are bit less cumbersome. The general case of $\mathbb{R}^n$ follows in a similar fashion. For details, we refer the reader to [12] and [7].

Theorem 6.22. *Suppose m is a bounded function on $\mathbb{R}^2$, twice differentiable in each quadrant, such that*

$$\sup_{j\in\mathbb{Z}} \int\limits_{I_j} \left|\frac{\partial m}{\partial t_1}(t_1,t_2)\right| \,\mathrm{d}t_1 < \infty,$$

$$\sup_{j\in\mathbb{Z}} \int\limits_{I_j} \left|\frac{\partial m}{\partial t_2}(t_1,t_2)\right| \,\mathrm{d}t_2 < \infty,$$

$$\sup_{j\in\mathbb{Z}} \int_{I_i\times I_j} \left|\frac{\partial^2 m}{\partial t_1 \partial t_2}(t_1,t_2)\right| \mathrm{d}t_1\mathrm{d}t_2 < \infty,$$

where I_j is dyadic interval in $\mathbb{R}$. Then m is a multiplier on $L^p(\mathbb{R}^2)$, for any $1 < p < \infty$.

Proof. We restrict our attention to dyadic intervals in $\mathbb{R}_+$. That is we only analyse the first quadrant of $\mathbb{R}^n$. Other cases can be handled exactly similar way. We take $I_i = (2^i, 2^{i+1})$ and $I_j = (2^j, 2^{j+1})$. Now, for a fixed $(\xi_1, \xi_2) \in I_i \times I_j$. We claim that

$$m(\xi_1,\xi_2) = \int\limits_{2^i}^{\xi_1}\int\limits_{2^j}^{\xi_2} \frac{\partial^2 m}{\partial t_1 \partial t_2}\mathrm{d}t_1 \,\mathrm{d}t_2 + \int\limits_{2^i}^{\xi_1} \frac{\partial m}{\partial t_1}(t_1, 2^j)\,\mathrm{d}t_1$$

$$+\int\limits_{2^j}^{\xi_2} \frac{\partial m}{\partial t_2}(2^i, t_2)\ \mathrm{d}t_2 + m(2^i, 2^j). \tag{6.35}$$

Indeed, if $(\xi_1, \xi_2) \in I_i \times I_j$, then

$$\int\limits_{2^i}^{\xi_1}\int\limits_{2^j}^{\xi_2} \frac{\partial^2 m}{\partial^2 t_1 \partial^2 t_2}\mathrm{d}t_1\ \mathrm{d}t_2 = m(\xi_1, \xi_2) - m(\xi_1, 2^j) - \int\limits_{2^j}^{\xi_2} \frac{\partial m}{\partial t_2}(2^i, t_2)\ \mathrm{d}t_2,$$

$$\text{and} \int\limits_{2^i}^{\xi_1} \frac{\partial m}{\partial t_1}(t_1, 2^j)\ \mathrm{d}t_1 = m(\xi, 2^j) - m(2^i, 2^j).$$

By substituting the above observations in the RHS of (6.35) our claim is proved.

Now we consider the operators

$$(S^1_{I_i} f)\widehat{\ }(\xi) = \chi_{I_j}(\xi_1)\widehat{f}(\xi),$$
$$(S^2_{I_j})\widehat{\ }(\xi) = \chi_{I_j}(\xi_2)\widehat{f}(\xi).$$

Then

$$(S^1_{I_i} S^2_{I_j} f)\widehat{\ }(\xi) = \chi_{I_i}\xi_1\chi_{I_j}(\xi_2)\widehat{f}(\xi) = \chi_{I_i}\chi_{I_j}(\xi_1, \xi_2)\widehat{f}(\xi).$$

Now multiplying both sides of Equation (6.35) by $\chi_{I_i \times I_j}(\xi_1, \xi_2)$, we have

$$\begin{aligned} m(\xi_1, \xi_2)\chi_{I_i \times I_j} &= \int\limits_{I_i}\int\limits_{I_j} \chi_{(2^i,\xi_1)}(t_1)\chi_{(2^j,\xi_2)}(t_2)\frac{\partial^2 m}{\partial t_1 \partial t_2}\ \mathrm{d}t_1\ \mathrm{d}t_2 \\ &+ \int\limits_{I_i} \chi_{(2^i,\xi_1)}(t_1)\frac{\partial m}{\partial t_1}(t_1, 2^j)\ \mathrm{d}t_1 + \int\limits_{I_j} \chi_{(2^j,\xi_2)}(t_2)\frac{\partial m}{\partial t_2}(2^i, t_2)\ \mathrm{d}t_2 \\ &+ m(2^i, 2^j)\chi_{I_i \times I_j}(\xi_1, \xi_2). \end{aligned}$$

Now multiplying both sides by $\widehat{f}(\xi)$, we have

$$\begin{aligned} (T_{i,j} f)\widehat{\ }(\xi) = \int\limits_{I_i \times I_j} &\left(S^1_{t_1, 2^{i+1}} S^2_{t_2, 2^{j+1}} f\right)\widehat{\ }(\xi)\frac{\partial^2 m}{\partial t_1 \partial t_2}(t_1, t_2)\ \mathrm{d}t_1\ \mathrm{d}t_2 \\ &+ \int\limits_{I_i} \left(S^1_{t_1, 2^{i+1}} f\right)\widehat{\ }(\xi_1)\frac{\partial m}{\partial t_1}(t_1, 2^{i+1})\ \mathrm{d}t_1 \\ &+ \int\limits_{I_j} \left(S^1_{t_2, 2^{i+1}} f\right)\widehat{\ }(\xi_2)\frac{\partial m}{\partial t_2}(t_{2^j}, 2^{j+1})\ \mathrm{d}t_{t_2} \\ &+ (S^1_i S^2_j f)\widehat{\ }(\xi) m(2^i, 2^j). \end{aligned}$$

Now taking inverse Fourier transform and arguing as in Theorem 6.21 we get,

$$\begin{aligned}(T_{i,j}f)(x) = \int\limits_{I_i\times I_j} \left(S^1_{t_1,2^{i+1}}S^2_{t_2,2^{j+1}}f\right)(x)\frac{\partial^2 m}{\partial t_1\partial t_2}(t_1,t_2)\ \mathrm{d}t_1\ \mathrm{d}t_2 \\ + \int\limits_{I_i}\left(S^1_{t_1,2^{i+1}}f\right)(x)\frac{\partial m}{\partial t_1}(t_1,2^{i+1})\ \mathrm{d}t_1 \\ + \int\limits_{I_j}\left(S^1_{t_2,2^{i+1}}f\right)(x)\frac{\partial m}{\partial t_2}(t_{2^j},2^{j+1})\ \mathrm{d}t_2 \\ + (S^1_iS^2_jf)(x)m(2^i,2^j).\end{aligned}$$

Now we want to show that the operator $T_{i,j}$ is bounded on $L^2(w)$ space for any $w\in A_2^*$. First we notice that

$$\left(S^1_{t,2^{i+1}}f\right)^{\widehat{}}(\xi) = \chi_{t,2^{i+1}}(\xi_1)\widehat{f}(\xi).$$

Therefore we can write

$$\int\limits_{\mathbb{R}^2} S^1_{t,2^{i+1}}f(x)e^{-ix\xi}\ \mathrm{d}x = \chi_{t,2^{i+1}}(\xi_1)\int\limits_{\mathbb{R}^2} f(x)e^{-ix\xi}\ \mathrm{d}x.$$

This can be written as

$$\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} S^1_{t,2^{i+1}}f(x)e^{-ix_1\xi_1}\mathrm{d}x_1e^{-ix_2\xi_2}\mathrm{d}x_2 = \chi_{t,2^{i+1}}(\xi_1)\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} f(x)e^{-ix_1\xi_1}\mathrm{d}x_1e^{-ix_2\xi_2}\ \mathrm{d}x_2.$$

Equivalently,

$$\mathcal{F}_1\left(\mathcal{F}_1\left(S^1_{t,2^{i+1}}f(\cdot,x_2)(\xi_1)\right)\right)(\xi_2) = \mathcal{F}_1\left(\mathcal{F}_1\left(\chi_{(t,2^{i+1})}(\xi_1)f(\cdot,x_2)\right)(\xi_1)\right)(\xi_2),$$

where $\mathcal{F}_1$ denote the one dimensional Fourier transform. Equivalently,

$$\mathcal{F}_1\left(S^1_{t,2^{i+1}}f(\cdot,x_2)(\xi_1)\right) = \mathcal{F}_1\left(\chi_{(t_1,2^{i+1})}(\xi_1)f(\cdot,x_2)\right)(\xi_1).$$

We know that the one dimensional operator $S^1_{t,2^{i+1}}$ operator is bounded on $L^2(w)$ space for any $w\in A_2(\mathbb{R})$. Now suppose $w\in A_2^*$. Then $w(\cdot,x_2)\in A_2(\mathbb{R})$ and $w(x_1,\cdot)\in A_2(\mathbb{R})$. Therefore,

$$\begin{aligned}||S^1_{t_1,2^{i+1}}f||_{L^2(w)} &= \left(\int\limits_{\mathbb{R}^2}|S^1_{t_1,2^{i+1}}f(x_1,x_2)|^2w(x_1,x_2)\ \mathrm{d}x\right)^{1/2} \\ &= \left(\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}|S^1_{t_1,2}f(x_1,x_2)|^2w(x_1,x_2)\ \mathrm{d}x_1\ \mathrm{d}x_2\right)^{1/2}\end{aligned}$$

$$\leq C\left(\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} |f(x_1,x_2)|^2 w(x_1,x_2)\ \mathrm{d}x_1\ \mathrm{d}x_2\right)^{1/2}$$

$$= C||f||_{L^2(w)}.$$

Similarly, $S^2_{t_2,2^{j+1}}$, S^1_j and S^2_k operators are also bounded on $L^2(w)$, for any $w \in A_2^*$. Using these facts we show that $T_{i,j}$ is bounded on $L^2(w)$, for any $w \in A_2^*$. First note that, using Minkowski's integral inequality, we obtain,

$$\left|\left|\int\limits_{I_i\times I_j} \left(S^1_{t_1,2^{i+1}}S^2_{t_2,2^{j+1}}f\right)(x)\frac{\partial^2 m}{\partial t_1\partial t_2}(t_1,t_2)\ \mathrm{d}t_1\ \mathrm{d}t_2\right|\right|_{L^2(w)}$$

$$= \left(\int\limits_{\mathbb{R}^2}\left|\int\limits_{I_i\times I_j} \left(S^1_{t_1,2^{i+1}}S^2_{t_2,2^{j+1}}f\right)(x)\frac{\partial^2 m}{\partial t_1\partial t_2}\ \mathrm{d}t_1\ \mathrm{d}t_2\right|^2 w(x)\ \mathrm{d}x\right)^{1/2}$$

$$\leq \int\limits_{I_i\times I_j}\left|\frac{\partial^2 m}{\partial t_1\partial t_2}(t_1,t_2)\right|\left(\int\limits_{\mathbb{R}^2}\left|S^1_{t_1,2^{i+1}}S^2_{t_2,2^{j+1}}f(x)\right|^2 w(x)\ \mathrm{d}x\right)^{1/2}\ \mathrm{d}t_1\ \mathrm{d}t_2$$

$$= \int\limits_{I_i\times I_j}\left|\frac{\partial^2 m}{\partial t_1\partial t_2}(t_1,t_2)\right|\left(\int\limits_{\mathbb{R}^2}\left|S^1_{t_1,2^{i+1}}S^2_{t_2,2^{j+1}}f(x)\right|^2 w(x_1,x_2)\ \mathrm{d}x_1\ \mathrm{d}x_2\right)^{1/2}\ \mathrm{d}t_1\ \mathrm{d}t_2$$

$$= \int\limits_{I_i\times I_j}\left|\frac{\partial^2 m}{\partial t_1\partial t_2}(t_1,t_2)\right|\left(\int\limits_{\mathbb{R}^2}\left|S^2_{t_2,2^{j+1}}f(x)\right|^2 w(x_1,x_2)\ \mathrm{d}x_1\ \mathrm{d}x_2\right)^{1/2}\ \mathrm{d}t_1\ \mathrm{d}t_2$$

$$= \int\limits_{I_i\times I_j}\left|\frac{\partial^2 m}{\partial t_1\partial t_2}(t_1,t_2)\right|\left(\int\limits_{\mathbb{R}^2}|f(x)|^2 w(x)\ \mathrm{d}x\right)^{1/2}\ \mathrm{d}t_1\ \mathrm{d}t_2.$$

From the above hypothesis of this theorem,

$$\sup_{j\in\mathbb{Z}}\int_{I_i\times I_j}\left|\frac{\partial^2 m}{\partial t_1\partial t_2}(t_1,t_2)\right|\mathrm{d}t_1\mathrm{d}t_2 < \infty.$$

We have

$$\left|\left|\int\limits_{I_i\times I_j} \left(S^1_{t_1,2^{i+1}}S^2_{t_2,2^{j+1}}f\right)(x)\frac{\partial^2 m}{\partial t_1\partial t_2}(t_1,t_2)\ \mathrm{d}t_1\ \mathrm{d}t_2\right|\right|_{L^2(w)} \leq C||f||_{L^2(w)}.$$

Similarly, we can show that,

$$\left|\left|\int\limits_{I_i} \left(S^1_{t_1,2^{i+1}}f\right)(x)\frac{\partial m}{\partial t_1}(t_1,2^{i+1})\ \mathrm{d}t_1\right|\right|_{L^2(w)} \leq C||f||_{L^2(w)},$$

$$\left|\left|\int\limits_{I_j} \left(S^1_{t_2,2^{i+1}}f\right)(x)\frac{\partial m}{\partial t_2}(t_{2^j},2^{j+1})\ \mathrm{d}t_2\right|\right|_{L^2(w)} \le C||f||_{L^2(w)}.$$

and

$$\left|\left|(S^1_iS^2_jf)(x)m(2^i,2^j)\right|\right| \le C||m||_\infty||f||_{L^2(w)}.$$

Using all these facts it is easy to see that $T_{i,j}$ is bounded on $L^2(w)$, for any $w\in A_2^*$. Now using the Littlewood Paley theory on $\mathbb{R}^2$ we have

$$||Tf||_p \le C\left|\left|\left(\sum_{i,j}|S^1_iS^2_jTf|^2\right)^{1/2}\right|\right|_p.$$

Also, from the definition of S^1_j, S^2_j, T and T_j, we can show that

$$S^1_iS^2_jTf = T_{i,j}S^1_iS^2_jf.$$

Therefore

$$||Tf||_p \le C\left|\left|\left(\sum_{i,j}|S^1_iS^2_jTf|^2\right)^{1/2}\right|\right|_p \le C\left|\left|\left(\sum_{i,j}|T_{i,j}S^1_iS^2_jf|^2\right)^{1/2}\right|\right|_p.$$

Now by using Theorem 6.6 with an obvious change for A_2^* weights we can say that

$$\left|\left|\left(\sum_{i,j}|T_{i,j}S^1_iS^2_jf|^2\right)^{1/2}\right|\right|_p \le \left|\left|\left(\sum_{i,j}|S^1_iS^2_jf|^2\right)^{1/2}\right|\right|_p.$$

Lastly, by using Theorem 6.4, we have

$$\left|\left|\left(\sum_{i,j}|S^1_iS^2_jf|^2\right)^{1/2}\right|\right|_p \le ||f||_p.$$

This completes the proof! □

6.3.3 Bochner-Riesz multipliers

In this section we discuss about the operator

$$(T^af)\widehat{\ }(\xi) = (1-|\xi|^2)^a_+\widehat{f}(\xi),$$

where $a>0$ and $A_+=\max(A,0)$. Such operators were first introduced by Bochner in [3] and arise naturally in the study of multipliers associated to balls in $\mathbb{R}^n$. For details we refer the reader to [12]. We start with the following construction. Let us choose functions $\varphi_k\in\mathbb{C}^\infty_c(\mathbb{R})$ which are supported on $[1-$

$2^{k+1}, 1-2^{-k-1})$ such that $0 \leq \varphi_k \leq 1$ and $|D^\beta \varphi_k| \leq C_\beta 2^{k\beta}$ for $\beta \in \mathbb{N} \cup \{0\}$ (where C_β is independent of k), and for $1/2 \leq t \leq 1$, we have

$$\sum_{k=1}^{\infty} \varphi_k(t) = 1.$$

We also define φ_0 by

$$\varphi_0(t) = 1 - \sum_{k=1}^{\infty} \varphi_k(t),$$

for $0 \leq t < \frac{1}{2}$ and $\phi_0(t) = 0$ for $t \geq 1/2$. For $0 \leq |\xi| \leq 1$, we have

$$\sum_{k=0}^{\infty} \varphi_k(|\xi|) = 1.$$

Since $(1-|\xi|^2)_+^a$ survives exactly when $0 \leq |\xi| \leq 1$, we have

$$(1-|\xi|^2)_+^a = \sum_{k=0}^{\infty} (1-|\xi|^2)^a \varphi_k(|\xi|).$$

Now we define another sequence of functions by

$$\tilde{\varphi}_k(|\xi|) = 2^{ka}(1-|\xi|^2)^a \varphi_k(|\xi|).$$

So, we can write

$$(1-|\xi|^2)_+^a = \sum_{k=0}^{\infty} 2^{-ka} \tilde{\varphi}_k(|\xi|).$$

Therefore we can decompose the operator T^a as

$$T^a f = \sum_{k=0}^{\infty} 2^{-ka} T_k f,$$

where,

$$(T_k f)\widehat{\ }(\xi) = \tilde{\phi}_k(|\xi|)\widehat{f}(\xi).$$

The behaviour of the operator T_k is discussed in the following lemma.

Lemma 6.23. *Given $0 < \delta < 1$, let φ be a function on $\mathbb{R}$ which is supported on $(1-4\delta, 1-\delta)$ such that $0 \leq \varphi \leq 1$ and $|D^\beta \phi| \leq C\delta^{-|\beta|}$ for any $\beta \in \mathbb{N}$. Then for any $\epsilon > 0$ the operator T_δ associated with the multiplier $\varphi(|\xi|)$, satisfies*

$$||T_\delta f||_p \leq C_\epsilon \delta^{-(\frac{n-1}{2}+\epsilon)\left|\frac{2}{p}-1\right|} ||f||_p.$$

Proof. Let K be such that $\widehat{K}(\xi) = \varphi(|\xi|)$. Let $a \in \mathbb{N}$ be even. We claim that

$$||(1+|\cdot|^a)K||_2 \leq C\delta^{\frac{1}{2}-a}. \tag{6.36}$$

Note that

$$||(1+|x|^a)K||_2 = ||((1+|\cdot|)K)\widehat{}||_2.$$

As for any polynomial P, $(Pf)\widehat{}(\xi) = P(-D)\widehat{f}(\xi)$,

$$||((1+|\cdot|^a)K)\widehat{}||_2 = C||(I+(-\Delta)^{a/2})(Q(||\cdot||))||_2.$$

It is easy to verify using binomial expansion that

$$1+|x|^a = C\left(1+\sum_{|\beta|=a} x^\beta\right).$$

We have,

$$||((1+|\cdot|^a)K)\widehat{}||_2 \le C\left(||\varphi(|\cdot|)||_2 + \sum_{|\beta|=a} ||D^\beta(\varphi(|\cdot|))||_2\right).$$

Now we notice that

$$||\varphi(|\cdot|)||_2^2 = \int_{1-4\delta<|\xi|<1-\delta} |\varphi(|\xi|)|^2 \, \mathrm{d}\xi \le \int_{1-4\delta<|\xi|<1-\delta} 1 \, \mathrm{d}\xi.$$

depending on the value of delta there are two cases. First let us consider $1-4\delta \ge 0$. That is $\delta \le \frac{1}{4}$. Then using polar decomposition of $\mathbb{R}^n$, we have

$$\begin{aligned}
&\int_{1-4\delta<|\xi|<1-\delta} 1 \, \mathrm{d}\xi \\
&= \int_{\mathbb{S}^{n-1}} \int_{1-4\delta<r<1-\delta} r^{n-1} \, \mathrm{d}r \, \mathrm{d}u \\
&= \frac{|\mathbb{S}^{n-1}|}{n}\{(1-\delta)^n - (1-4\delta)^n\} \\
&\le \frac{|\mathbb{S}^{n-1}|}{n}(1-\delta-1+4\delta)\sum_{i=1}^{n-1}(1-\delta)^{n-1-i}(1-4\delta)^i \\
&\le C(3\delta),
\end{aligned}$$

where the second last inequality follows from that fact that $(1-\delta) \le 1$ and $(1-4\delta) \le 1$. In the other case when $1-4\delta < 0$, we have

$$\int_{1-4\delta<|\xi|<1-\delta} 1 \, \mathrm{d}\xi = \int_{\mathbb{S}^{n-1}} \int_{0<r<1-\delta} r^{n-1} \, \mathrm{d}r \, \mathrm{d}u = |\mathbb{S}^{n-1}|\frac{(1-\delta)^n}{n}.$$

We observe that for $\frac{d}{d\delta}(\frac{(1-\delta)^n}{\delta}) < 0$,and hence $\frac{(1-\delta)^n}{\delta}$ is decreasing on $(\frac{1}{4}, 1)$. Therefore,

$$\frac{(1-\delta)^n}{\delta} \le \frac{(1-\frac{1}{4})^n}{\frac{1}{4}} = C.$$

So $(1-\delta)^n \le C\delta$. This gives,

$$\int_{1-4\delta<|\xi|<1-\delta} 1 \, \mathrm{d}\xi \le C\delta.$$

Hence we get $||\varphi||_2 \le C\delta^{1/2}$. Now,

$$\begin{aligned} ||D^\beta \phi(|\cdot|)||_2^2 &= \int_{1-4\delta<|\xi|<1-\delta} |D^\beta \varphi(|\xi|)|^2 \, \mathrm{d}\xi \\ &\le C \int_{1-4\delta<|\xi|<1-\delta} \delta^{-2|\beta|} \, \mathrm{d}\xi \\ &= C\delta^{-2a} \int_{1-4\delta<|\xi|<1-\delta} 1 \, \mathrm{d}\xi \\ &\le C\delta^{1-2a}. \end{aligned}$$

Therefore,

$$\sum_{|\beta|=a} ||D^{-\beta}\phi(|\cdot|)||_2 \le C\delta^{\frac{1}{2}-a}.$$

Therefore we have

$$||(1+|\cdot|^2)K||_2 \le C\left(\delta^{1/2} + \delta^{-a+\frac{1}{2}}\right) = C\delta^{1/2}\left(1+\delta^{-a}\right) \le C\delta^{1/2} 2\delta^{-a} = C\delta^{-a+\frac{1}{2}}.$$

Here we use the fact that $\delta^{-a} > 1$. This proves our claim. Now we prove that Inequality (6.36) is true for any $a > 0$. Let $s > 1$ be such that $as \in \mathbb{N}$ is even. Now using the concavity of the function $x \longmapsto x^{1/s}$, we have,

$$(1+|x|^a) \le C\left(1+|x|^{as}\right)^{1/s}.$$

Therefore,

$$\begin{aligned} ||(1+|x|^a)K||_2 &\le C||(1+|x|^{as})^{1/s}K||_2 \\ &\le C||(1+|\cdot|^{as})K||_2^{1/s}||K||_2^{1/s'}. \end{aligned}$$

Since as is an even positive integer, we know from Inequality (6.36) that

$$||(1+|\cdot|^{as})K||_2^{1/s} \le C\delta^{\frac{1}{2s}-a}.$$

As $||K||_2 = ||\phi(|\cdot|)||_2 \le C\delta^{1/2}$, we have,

$$||(1+|\cdot|^{as})K||_2^{1/s}||K||_2^{1/s'} \le C\delta^{-a+\frac{1}{2s}}\delta^{\frac{1}{2s'}} \le C\delta^{-a+\frac{1}{2}}.$$

Taking $a = \frac{n}{2} + \epsilon$, we obtain by using Hölder's inequality,

$$||K||_1 = ||K(1+|x|^a)(1+|x|^a)^{-1}||_1 \le ||K(1+|x|^a)||_2||(1+|x|^a)^{-1}||_2.$$

Since $||(I+|x|^a)^{-1}||_2 \leq C_\epsilon$, we have

$$||K||_1 \leq C_\epsilon \delta^{-a+\frac{1}{2}} \leq C_\epsilon \delta^{-(\frac{n-1}{2}+\epsilon)}.$$

We also have $(T_\delta f)\widehat{}(\xi) = \varphi(|\xi|)\widehat{f}(\xi) = \widehat{K}(\xi)\widehat{f}(\xi) = (K * f)\widehat{}(\xi)$. That is,

$$T_\delta f = K * f.$$

Therefore $||T_\delta f||_2 = ||(T_\delta f)\widehat{}||_2 = ||\phi(|\cdot|)\widehat{f}||_2 \leq C||f||_2$. Also, using Young's convolution inequality we get

$$||T_\delta f||_1 \leq C||K||_1||f||_1 \leq C_\epsilon \delta^{-(\frac{n-1}{2}+\epsilon)},$$

and

$$||T_\delta f||_\infty \leq C||K||_1||f||_\infty \leq C_\epsilon \delta^{-(\frac{n-1}{2}+\epsilon)}||f||_\infty.$$

The result now follows from Riesz-Thorin interpolation for $p=1$ and $p=2$ together with $p=2$ and $p=\infty$. □

Lemma 6.24. *If m is a function with compact support which is a multiplier on $L^p(\mathbb{R}^n)$ for some p then $\widehat{m} \in L^p(\mathbb{R}^n)$.*

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$ with $\widehat{f}(x) = 1$ for $x \in \operatorname{supp}(m)$. Now

$$(T_m f)\widehat{}(\xi) = m(\xi)\widehat{f}(\xi) = m(\xi).$$

Note that

$$\widehat{m}(\xi) = (\widetilde{T_m f})(\xi) = T_m f(-\xi).$$

As $T_m f \in L^p(\mathbb{R}^n)$, $\widehat{m} \in L^p(\mathbb{R}^n)$. □

Lemma 6.25. *The Fourier transform of $(1-|\xi|^2)_+^a$ is*

$$K^a(x) = \pi^{-a}\Gamma(a+1)|x|^{-\frac{n}{2}-a}J_{\frac{n}{2}+a}(2\pi|x|),$$

where

$$J_\nu(t) = \frac{(\frac{t}{2})^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1} e^{its}(1-s^2)^{\nu-\frac{1}{2}}\,\mathrm{d}s,$$

is the Bessel function of the first kind.

Proof. First note that

$$\int_{\mathbb{R}^n} (1-|\xi|^2)_+^a \,\mathrm{d}\xi = \int_{0\le|\xi|\le 1} (1-|\xi|^2)^a \,\mathrm{d}\xi = |\mathbb{S}^{n-1}| \int_{0\le s\le 1} (1-s^2)^a s^{n-1} \,\mathrm{d}s < \infty.$$

That is, $(1-|\xi|^2)_+^a \in L^1(\mathbb{R}^n)$.

Therefore,

$$\begin{aligned}
\big((1-|\cdot|^2)^a\big)\widehat{\ }(x) &= \int_{0\le|\xi|\le 1} (1-|\xi|^2)_+^a e^{-2\pi i\langle x,\xi\rangle} \,\mathrm{d}\xi \\
&= \int_{\mathbb{S}^{n-1}} \int_{0\le s\le 1} (1-s^2)^a e^{-2\pi i\langle x,su\rangle} s^{n-1} \,\mathrm{d}s \,\mathrm{d}u \\
&= \int_{0\le s\le 1} (1-s^2)^a s^{n-1} \int_{\mathbb{S}^{n-1}} e^{-2\pi i\langle x,su\rangle} \,\mathrm{d}u \,\mathrm{d}s \\
&= \int_{0\le s\le 1} s^{n-1}(1-s^2)^a 2\pi(|x|s)^{-(\frac{n-1}{2})} J_{\frac{n-2}{2}}(2\pi|x|s) \,\mathrm{d}s \\
&= 2\pi|x|^{(1-\frac{n}{2})} \int_{0\le s\le 1} (1-s^2)^a s^{n/2} J_{\frac{n-2}{2}}(2\pi|x|s) \,\mathrm{d}s \\
&= 2\pi|x|^{(1-\frac{n}{2})} \frac{2^a\Gamma(a+1)}{(2\pi|x|)^{a+1}} J_{\frac{n-2}{2}}(2\pi|x|).
\end{aligned}$$

Here we used the following. If $r=|x|$ and $x'=\frac{x}{|x|}$ and $u'=\frac{u}{|u|}$ we have

$$\int_{\mathbb{S}^{n-1}} e^{-2\pi i x u} \,\mathrm{d}u' = \int_{\mathbb{S}^{n-1}} e^{-2\pi i|x||u|\langle x',u'\rangle} \,\mathrm{d}u' = 2\pi(|x||u|)^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(2\pi|x||u|).$$

□

We are now in a position to give the main result of this section.

Theorem 6.26. *The Bochner-Riesz multipliers T^a satisfy the following:*

1. *If $a > \frac{n-1}{2}$ then T^a is bounded on $L^p(\mathbb{R}^n)$, for any $1 \le p \le \infty$.*

2. *If $0 < a \le \frac{n-1}{2}$ then T^a is bounded on $L^p(\mathbb{R}^n)$ if*
$$\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{a}{n-1},$$
and is not bounded if
$$\left|\frac{1}{p}-\frac{1}{2}\right| \ge \frac{2a+1}{2n}.$$

Proof. First note that using Lemma 6.23 for any $\epsilon > 0$, we have

$$||T_k f||_p \leq C_\epsilon 2^{-(k+1)\{-(\frac{n-1}{2}+\epsilon)\}|\frac{2}{p}-1|}||f||_p \leq C_\epsilon 2^{k(\frac{n-1}{2}+\epsilon)|\frac{2}{p}-1|}||f||_p.$$

$$||T^a f||_p \leq \sum_{k=0}^{\infty} ||T_k f||_p 2^{-ka} \leq C_\epsilon \sum_{k=0}^{\infty} 2^{k(\frac{n-1}{2}+\epsilon)|\frac{2}{p}-1|-ka}||f||_p.$$

T^a is bounded if $k(\frac{n-1}{2}+\epsilon)|\frac{2}{p}-1| - ka < 0$, which implies

$$\left(\frac{n-1}{2}+\epsilon\right)\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{a}{2}.$$

As $\epsilon > 0$ is arbitrary, we have

$$\left(\frac{n-1}{2}\right)\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{a}{2}.$$

That is T^a is bounded if $\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{a}{n-1}$. This proves the first part of (2). We observe that if $a > \frac{n-1}{2}$, then $\frac{a}{n-1} > \frac{1}{2} < \frac{a}{n-1}$. Hence (1) is proved. Now we recall a few asymptotic properties of Bessel function. As $t \longrightarrow 0$, we have $J_\nu(t) \leq Mt^\nu$, for some constant M, and as $t \longrightarrow \infty$, we have

$$M_1 t^{-\frac{1}{2}} \leq J_\nu(t) \leq M_2 t^{-\frac{1}{2}},$$

for constants $M_1,\, M_2 > 0$.

Using these facts, we have for $|x| \longrightarrow 0$,

$$|K^a(x)| \leq C.$$

And for $|x| \longrightarrow \infty$, we have,

$$C_1|x|^{-(\frac{n+1}{2}+a)} \leq |K^a(x)| \leq C_2|x|^{-(\frac{n-1}{2}+a)}.$$

Now,

$$\begin{aligned}
||K^a||_p^p &= \int_{\mathbb{R}^n} |K^a(x)|^p \,\mathrm{d}x \\
&= \int_{0\leq|x|\leq1} |K^a(x)|^p \,\mathrm{d}x + \int_{|x|>1} |K^a(x)|^p \,\mathrm{d}x \\
&\geq \int_{0\leq|x|\leq1} |K^a(x)|^p \,\mathrm{d}x + C\int_{|x|>1} |x|^{-(\frac{n-1}{2}+a)p} \,\mathrm{d}x \\
&\geq \int_{0\leq|x|\leq1} |K^a(x)|^p \,\mathrm{d}x + C|\mathbb{S}^{n-1}|\int_{r>1} r^{-(\frac{n+1}{2}+a)p} r^{n-1} \,\mathrm{d}r
\end{aligned}$$

$$= \int\limits_{0\leq|x|\leq 1} |K^a(x)|^p \; \mathrm{d}x + C \int\limits_{r>1} r^{-(\frac{n+1}{2}+a)p+n-1} \; \mathrm{d}r.$$

Therefore $||K^a||_p^p < \infty$ only if $(\frac{n+1}{2} + a)p - n + 1 > 1$ that is if $p > \frac{2n}{n+1+2a}$.

Now by using Lemma 6.24, T^a can be bounded on $L^p(\mathbb{R}^n)$, only if $p > \frac{2n}{n+1+2a}$. Suppose on the contrary that T^a is bounded on $L^p(\mathbb{R}^n)$, for

$$\left|\frac{1}{p} - \frac{1}{2}\right| \geq \frac{2a+1}{2n}. \tag{6.37}$$

There are two possibilities. When $\frac{1}{p} - \frac{1}{2} > 0$, Inequality (6.37) implies

$$\frac{1}{p} \leq \frac{2n}{n+1+2a}.$$

This is clearly a contradiction!

In the other case when $\frac{1}{p} - \frac{1}{2} < 0$, we have $p' \leq \frac{2n}{n+1+2a}$. We note that if $K(x)$ is the kernel of T^a, then transpose $(T^a)^*$ has kernel $\widetilde{K}$, where $\widetilde{K}(x) = K(-x)$. So the multiplier of $(T^a)^*$ is $\widehat{\widetilde{K}}(\xi)$. Fourier transform of the multiplier of $(T^a)^*$, is $\hat{\hat{\widetilde{K}}}(x) = K(x) = K^a(-x)$. Note that by our assumption $(T^a)^*$ is bounded for $p' \leq \frac{2n}{n+1+2a}$. But $K^a(\cdot) \in L^p(\mathbb{R}^n)$ only when $p > \frac{2n}{n+1+2a}$. This gives contradiction and completes the proof.

$\square$

CHAPTER 7

Conclusion

This thesis is a brief survey of a few techniques commonly used in Harmonic Analysis. We have seen the importance of averaging operators, and their techniques in understanding the L^p-boundedness of certain translation-invariant operators. The techniques presented open the doors for a graduate level study as well as research. Now, we mention few of those directions one can pursue.

We have seen in Chapter 6 three types of multiplier operators. In Section 6.3.3, we commented that the study of multipliers associated to the characteristic functions of balls is difficult. This is one of the directions of study one can take up. Indeed, the study of multipliers is a vast subject in itself.

In Chapter 5, we have seen singular integral operators. The Calderón-Zygmund theory answers most of the questions that may arise in the study of singular integrals of convolution type. Therefore, the next step would be to see singular integrals of non-convolution type. These are typically difficult to study, since we no longer have translation-invariance at hand. Nonetheless, much development is done in this direction. One often starts by defining a special tempered distribution, called the $T(1)$-distribution, and studies its boundedness. The theory so developed, what is called the "$T(1)$-theory", answers certain questions on the boundedness of singular integrals of non-convolution type. A more general theory, called the "$T(b)$-theory" is also developed in accordance to this. We refer the reader to [11] for preliminary details on the topic.

Speaking of the theory of boundedness of translation-invariant operators, we notice that the Muckenhoupt class (A_p) of weights have been more than helpful in deriving a variety of weighted and unweighted results. We believe that this class finds its use in the study of several other exotic operators. For instance Duoandikoetxea et al. in [8] have used A_p weights in the study of a maximal function associated to the k-plane transform. The k-plane transform is a natural integral transform that comes in the study of densities using only the average values along k-dimensional planes. It is interesting to see the use of A_p weights in the study of boundedness of similar kind of operators.

Another branch of study one may pursue is the multilinear Harmonic analysis, where, as the name suggests, we look at multilinear operators (taking more than one input) and ask similar questions about boundedness. Several theories about such operators has been developed till date. We refer the reader to [11] for a brief introduction to the topic. Some of the interpolation results are also known for multilinear operators (see for instance, [13] and the references therein). However, we believe much work can be done here.

Before closing the thesis, we would like to mention one last direction of work. Our study only deals with operators defined on the Euclidean space. It is natural to ask whether similar theory can be developed on non-Euclidean spaces. There are several challenges that one may face if one tries to develop this theory verbatim. First, to talk about (Hardy-Littlewood) averaging operators, one would require "balls". So, one must work with a metric structure that has a compatible measure. Riemannian manifolds are easy examples of such spaces. Apart from this, the major difficulty that one faces is that on general non-Euclidean spaces, one may not have the notion of "cubes". Consequently, the Calderón-Zygmund decomposition, that has been the main ingredient of many proofs, no longer holds. This forces one to work only with metric balls. However, the non-Euclidean metric balls might not be as well-behaved as the Euclidean ones. For instance, in the Euclidean space $\mathbb{R}^n$, we have, what we call the "measure-doubling" phenomenon. That is, the measure of the ball of radius $2r$ (centered at any point) in the Euclidean space is a constant (2^n) times that of the ball of radius r. Such a phenomenon is not expected in other non-Euclidean spaces. For instance, in the hyperbolic space, the volume of a ball increases exponentially, while in the sphere, it increases like sine. Due to the troublesome nature of these spaces, similar theory on these spaces has not been developed yet. One would like to see if new techniques can be developed for these spaces that give analogous results.

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