Threshold of a stochastic two-species predator-prey model

M.Sc. Thesis

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"Threshold of a stochastic two-species predator-prey model"** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF MATHEMATICS**, **Indian Institute of Technology Indore**, is an authentic record of my work carried out during the period from July 2022 to June 2023 under the supervision of **Dr. Debopriya Mukhergee** Assistant Professor, Department of Mathematics, IIT Indore and **Dr. Bapan Ghosh**, Assistant Professor, Department of Mathematics, IIT Indore. The matter presented in this thesis has not been submitted for the award of any other degree of this or any other institute.

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Abstract

We consider a predator-prey model with the Beddington-DeAngelis functional response in a deterministic framework. Further, we include harvesting, and finally, we introduce stochasticity. First, we analyze key findings for the deterministic framework, including the system's equilibria and their stability. For the stochastic version, we use appropriate Lyapunov functions and integrate the knowledge of stochastic differential equations to establish the conditions under which both the prey and predator will become extinct. Subsequently, we determine the condition for the predator extinction while the prey continues to exist. Moreover, we explore the existence of unique positive global solutions for this model. Finally, we present some numerical simulations for time responses for the population.

Contents

CHAPTER 1

Intoduction

1.1 Single species models

The study of population dynamics often begins with basic models that describe the growth or decline of a single species over time. One of the foundational models in this area is the exponential growth model. The exponential growth model [8] describes the population growth of a single species in an environment with unlimited resources. The model is expressed with the following differential equation

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = ru(t),$$

where $u(t)$ represents the population size at time t, r signifies the intrinsic growth rate. Since in the case of a real ecosystem resources are limited, Pierre François Verhulst [12] proposed a modified model- called the logistic model, which is represented by the differential equation

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = ru(t)\left(1 - \frac{u(t)}{K}\right).$$

Here, K denotes the environment's carrying capacity. This model has the following properties:

(i) When the population is small, the growth rate is directly proportional to the population size.

(ii) However, as the population approaches a certain limit, the growth rate becomes smaller.

1.2 Basic predator-prey model

At the heart of ecological research lies the intricate relationship between predators and their prey, a dynamic interplay that fundamentally shapes ecosystems. Understanding this interaction is crucial for deciphering the complexities of natural systems and predicting their responses to environmental pressures. Predator-prey models provide a robust framework for simulating and analyzing the population dynamics of these key species. Through mathematical formulations and computational simulations, researchers can explore the nuances of predator-prey interactions, investigating how variables such as population densities, predation rates, and resource availability impact ecosystem stability and resilience. By delving into this captivating domain, we uncover insights into the mechanisms driving ecological dynamics and develop strategies for conservation and management vital to preserving biodiversity and ecosystem health.

We now introduce a predator-prey model to include interactions between two different species. This model considers the dynamics of two interacting populations: predators and prey. A classic example of a predator-prey model is the Lotka-Volterra model, proposed independently by Alfred J. Lotka [11] in 1925 and Vito Volterra [13] in 1926.

$$
\begin{cases}
\dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = ru(t) - au(t)v(t), & (1.1)\\
\dfrac{\mathrm{d}v(t)}{\mathrm{d}t} = sau(t)v(t) - mv(t), & (1.2)
\end{cases}
$$

Here, $u(t)$ and $v(t)$ are the density at time t of prey and predator, respectively. a is the predation rate (the rate at which predators consume prey), s signifies the conversion coefficient and m denotes the specific mortality rate of the predator. A predator-prey model provides insights into the complex interactions between species in an ecosystem, illustrating how changes in one population affect the dynamics of the other. This model has been widely studied and has important implications for ecological theory and conservation biology.

1.3 Generalisation of predator-prey model

The most general differential equation of predator-prey models is,

$$
\begin{cases}
\dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = ru(t)\left(1 - \dfrac{u(t)}{K}\right) - f(u(t), v(t))v(t), & (1.3)\\
\dfrac{\mathrm{d}v(t)}{\mathrm{d}t} = sf(u(t), v(t))v(t) - mv(t). & (1.4)
\end{cases}
$$

Here, $f(u(t), v(t))$ is some function of $u(t)$ and $v(t)$ known as functional response and based on that we can formulate several models. There are several types of functional response,

(i) In the Rosenzweig-MacArthur model

$$f(u(t), v(t)) = \frac{au(t)}{1 + ahu(t)}.$$

where h is handling time per prey item.

(ii) In the Beddington-DeAngelis functional response [3, 7]

$$f(u(t), v(t)) = \frac{au(t)}{1 + ahu(t) + b\tau v(t)}.$$

Here, b is a parameter representing the level of interference among predators and τ represents the time scale associated with predator interference.

(iii) In the Cosner functional response [6]

$$f(u(t), v(t)) = \frac{ae_0 u(t)}{1 + ae_0 u(t)v(t)}.$$

Here e_0 is the total encounter coefficient between the predator and the prey.

(iv) In ratio-dependent functional response model [2]

$$f(u, v) = \frac{au(t)}{qu(t) + v(t)},$$

where q is a parameter that modifies the effect of prey density on the functional response.

(v) In the hunting cooperation model [1, 4]

$$f(u(t), v(t)) = (a + \lambda v(t)).$$

As the predator density $v(t)$ increases, the predation rate increases linearly due to the term $\lambda v(t)$.

This thesis focuses on and derives results from the Beddington-DeAngelis functional response for the predator-prey model.

1.4 Preliminaries

We now present some preliminary information from references [5, 10].

Definitions

1. **Stopping Time:** A random variable τ with values in $\mathbb{N} \cup \{\infty\}$ is termed a stopping time (with respect to a filtration $\mathbb{F}$) if for each $n = 1, 2, \ldots$,

$$\{\tau = n\} \in \mathbb{F}_n.$$

2. M^2: Denoted by M^2, this class encompasses stochastic processes $f(t)$, $t \geq 0$, such that
$$E\left(\int_0^\infty |f(t)|^2 dt\right) < \infty.$$

3. M_T^2: For any $T > 0$, M_T^2 represents the space of stochastic processes $f(t)$, $t \geq 0$, such that
$$\mathbf{1}_{[0,T)} f \in M^2.$$

4. **Probability Space**: The triplet $(\Omega, \mathscr{F}, \mathbf{P})$ is referred to as a probability space, where
   (i) Ω denotes the sample space,
   (ii) $\mathscr{F}$ is the σ-field on Ω,
   (iii) $\mathbf{P}$ is the probability measure.

5. **Complete Filtered Probability Space**: Represented by $\mathfrak{A} = (\Omega, \mathscr{F}, \mathbb{F}, \mathbf{P})$, it consists of a complete probability space with filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ satisfying:
   (i) $\mathbf{P}$ is complete on $(\Omega, \mathscr{F})$,
   (ii) For each $t \geq 0$, $\mathscr{F}_t$ contains all $(\mathscr{F}, \mathbf{P})$-null sets,
   (iii) The filtration $\mathbb{F}$ is right-continuous.

Theorem 1.1. *Let $\xi(t)$ be an Itô process. Suppose $F(t,x)$ is a real-valued function with continuous partial derivatives $F_t(t,x)$, $F_x(t,x)$, $F_{xx}(t,x)$ for all $t \geq 0$ and $x \in \mathbb{R}$. Additionally, assume $b(t)F_x(t,\xi(t)) \in M_T^2$ for all $T \geq 0$. Then $F(t,\xi(t))$ is an Itô process such that:*

$$dF(t,\xi(t)) = \left(F_t(t,\xi(t)) + F_x(t,\xi(t))a(t) + \frac{1}{2}F_{xx}(t,\xi(t))b(t)^2\right)dt + F_x(t,\xi(t))b(t)d\mathcal{B}(t).$$

Here, $\xi(t)$ is any Itô process satisfying:

$$d\xi(t) = a(t)dt + b(t)d\mathcal{B}(t).$$

Theorem 1.2. *Suppose f and g are Lipschitz continuous functions from $\mathbb{R}$ to $\mathbb{R}$, i.e., there exists a constant $C > 0$ such that for any $x, y \in \mathbb{R}$:*

$$|f(x) - f(y)| \leq C|x - y|$$

$$|g(x) - g(y)| \leq C|x - y|$$

Furthermore, let ξ_0 be $\mathscr{F}_0$-measurable and square-integrable random variable. Then the initial value problem (I.V.P):

$$\begin{cases} d(\xi(t)) = f(\xi(t))dt + g(\xi(t))d\mathcal{B}(t) \\ \xi(0) = \xi_0 \end{cases}$$

has a solution $\xi(t)$, $t \geq 0$, within the class of Itô processes.
The solution is unique in the sense that if $\eta(t)$, $t \geq 0$, is another Itô process, then $\xi(t)$ and $\eta(t)$ are identical almost surely.

CHAPTER 2

Predator prey model with Beddington-DeAngelis functional response

2.1 The model

The Beddington-DeAngelis model is a fundamental system in the realm of predator-prey dynamics, offering a sophisticated framework to understand the intricacies of these ecological relationships. Proposed by Peter J. Beddington and Donald L. DeAngelis, this model introduces a refined approach to capturing the dynamics of predator-prey interactions by integrating factors such as predator functional response and prey density-dependent mortality. Through its elegant mathematical formulations, the Beddington-DeAngelis model provides a nuanced depiction of how predator and prey populations coexist and fluctuate in response to changing environmental conditions. Studying the mechanisms revealed by this model provides researchers with valuable insights into ecosystem stability and resilience. This knowledge informs conservation and management strategies aimed at preserving biodiversity and ecosystem integrity. The Beddington-DeAngelis model is

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = ru(t)\left(1 - \dfrac{u(t)}{K}\right) - \dfrac{au(t)v(t)}{1 + ahu(t) + b\tau v(t)}. & (2.1) \\ \dfrac{\mathrm{d}v(t)}{\mathrm{d}t} = \dfrac{sau(t)v(t)}{1 + ahu(t) + b\tau v(t)} - mv(t) - \gamma v^2(t). & (2.2) \end{cases}$$

Here, γ represents the inter-specific competition rate of predators.

Including the concept of harvesting in ecological models is crucial for gaining a more complete understanding of natural ecosystems. When extending the Beddington-DeAngelis model to account for harvesting, we enrich its applicability to a broader range of ecological scenarios. By introducing harvesting, we acknowledge the human impact on natural systems, as many ecosystems are subject to the exploitation of resources. This extension allows us to explore the dynamic interplay between predator-prey interactions and human activities such as fishing or hunting, shedding light on the complex dynamics that emerge when natural and Anthropocene forces intersect. Ultimately, this refined model not only enhances our theoretical understanding of ecological systems but also provides valuable insights for sustainable management and conservation practices in the face of increasing human pressures on the environment. The model under harvesting reads as,

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = ru(t)\left(1 - \dfrac{u(t)}{K}\right) - \dfrac{au(t)v(t)}{1 + ahu(t) + b\tau v(t)} - q_1e_1u(t), & (2.3) \\ \dfrac{\mathrm{d}v(t)}{\mathrm{d}t} = \dfrac{sau(t)v(t)}{1 + ahu(t) + b\tau v(t)} - mv(t) - \gamma v^2(t) - q_2e_2v(t). & (2.4) \end{cases}$$

Here, q_1 and q_2 represent the catchability coefficients of prey and predator species, respectively, while e_1 and e_2 denote the harvesting efforts applied to prey and predator species, respectively.

2.2 Equilibrium

If f(t) is a solution of a differential equation and is constant, it is called an equilibrium.

We can say a system of prey-predator model is in equilibrium if there is no change in their density(number). We can obtain it by taking $\frac{du(t)}{dt} = \frac{du(t)}{dt} = 0$ i.e.

$$\begin{cases} u(t)\left(r\left(1-\dfrac{u(t)}{K}\right)-\dfrac{av(t)}{1+ahu(t)+b\tau v(t)}-q_1e_1\right)=0, & (2.5)\\ v(t)\left(\dfrac{sau(t)}{1+ahu(t)+b\tau v(t)}-m-\gamma v(t)-q_2e_2\right)=0. & (2.6)\end{cases}$$

The model has the following equilibrium

(i) $(0,0)$ is trivial equilibrium.

(ii) There is only one boundary equilibrium

$$\left(\frac{K}{r}\left(r-q_1e_1\right),0\right)$$

only if $r > q_1e_1$

(iii) Due to the highly non-linear nature of the system (2.5)-(2.6), determining the equilibrium in terms of parameters is challenging. We'll compute the co-existing equilibrium numerically in the subsequent paragraphs.

2.3 Stability analysis

Assume,

$$f(u,v) = ru(t)\left(1-\frac{u(t)}{K}\right)-\frac{au(t)v(t)}{1+ahu(t)+b\tau v(t)}-q_1e_1u(t), \tag{2.7}$$

$$g(u,v) = \frac{sau(t)v(t)}{1+ahu(t)+b\tau v(t)}-mv(t)-\gamma v^2(t)-q_2e_2v(t). \tag{2.8}$$

Then Jacobian matrix is defined as

$$J(u,v) = \begin{bmatrix} \frac{\partial f(u,v)}{\partial u} & \frac{\partial f(u,v)}{\partial v} \\ \frac{\partial g(u,v)}{\partial u} & \frac{\partial g(u,v)}{\partial v} \end{bmatrix},$$

where

$$\frac{\partial f(u,v)}{\partial u} = r\left(1 - \frac{2u(t)}{K}\right) - \frac{av(t)(1 + ahu(t) + b\tau v(t) - ahu^2(t))}{(1 + ahu(t) + b\tau v(t))^2} - q_1 e_1,$$

$$\frac{\partial f(u,v)}{\partial v} = -\frac{au(t)(1 + ahu(t) + b\tau v(t) - b\tau v^2(t))}{(1 + ahu(t) + b\tau v(t))^2},$$

$$\frac{\partial g(u,v)}{\partial u} = \frac{asv(t)(1 + ahu(t) + b\tau v(t) - ahu^2(t))}{(1 + ahu(t) + b\tau v(t))^2},$$

$$\frac{\partial g(u,v)}{\partial v} = \frac{asu(t)(1 + ahu(t) + b\tau v(t) - b\tau v^2(t))}{(1 + ahu(t) + b\tau v(t))^2} - m - 2\gamma v(t) - q_2 e_2.$$

Local stability of equilibrium points

If we have an equilibrium point (u^*, v^*).The point (u^*, v^*) is stable if both the eigenvalue of $J(u^*, v^*)$ is less than 0.

If both the eigenvalue of $J(u^*, v^*)$ is greater than 0 then the equilibrium is locally unstable.

If one of the eigenvalues of $J(u^*, v^*)$ is greater than 0 and the other is less than 0, it is a saddle point.

2.4 Simulation results for stability analysis

This section will graphically analyze the stability model under different conditions. By varying key parameters and initial conditions, we will explore the dynamic behaviour of the system through numerical simulations and phase plane diagrams.

Example 2.1: Suppose the parameter in (2.3)-(2.4) as $r = 5, K = 43, b =$

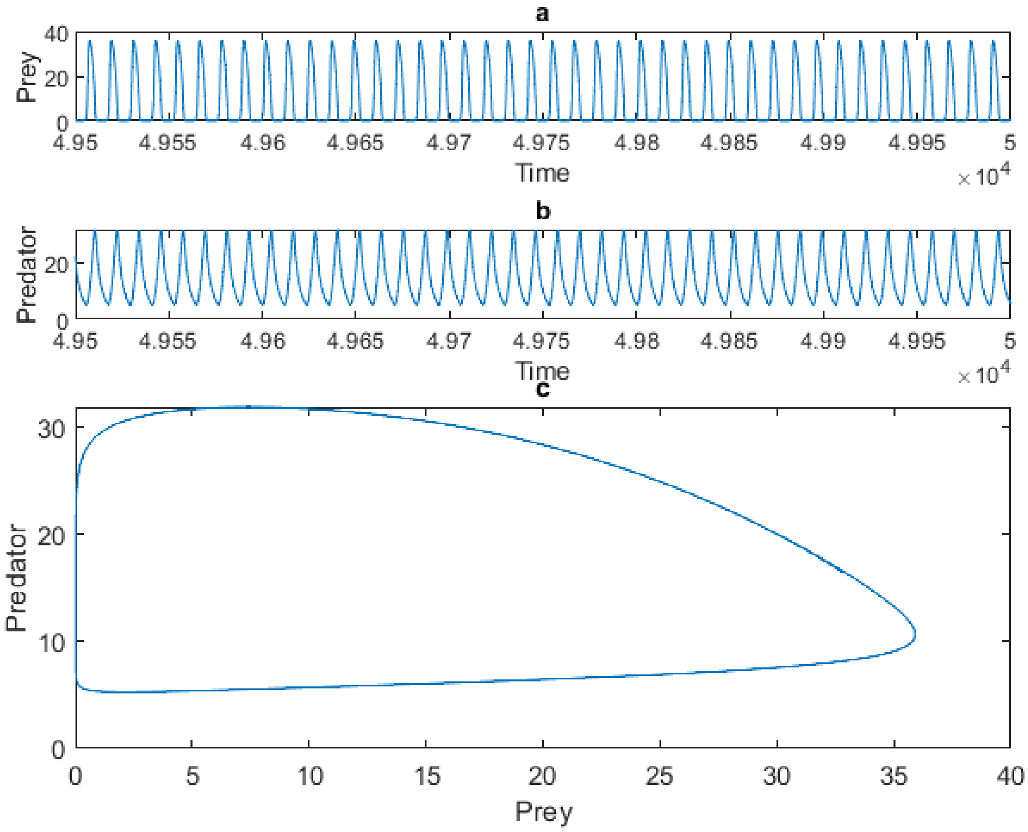


Figure 2.1: Subplot (a) show that changes in prey concerning time. Subplot (b) shows the change in predators concerning time. Subplot (c) is a phase portrait of prey and predator.

$0, a = 0.4, h = 0.3, \tau = 0.5, q_1 = 0.1, e_1 = 0.2, s = 0.3, m = 0.15, \gamma = 0.01, q_2 = 0.05, e_2 = 0.04$, with initial conditions $u(0) = 32.77$ and $v(0) = 16.51$. We will now determine the co-existing equilibrium and evaluate its stability using the eigenvalue approach. The co-existing equilibrium is $(0.0013, 13.4005)$, with the eigenvalues of its Jacobian matrix being $0.4111 \pm 0.8056\text{i}$. Hence the equilibrium is unstable. In Figure 2.1, we notice oscillations in the prey and predator populations.

Example 2.2: Consider the parameter as $r = 5, K = 43, b = 0.3, a = 0.4, h =$

$0.3, \tau = 0.5, q_1 = 0.1, e_1 = 0.2, s = 0.3, m = 0.15, \gamma = 0.01, q_2 = 0.05, e_2 = 0.04$, with initial conditions $u(0) = 23.8$ and $v(0) = 2.6$. Then for this example equilibrium point is $(32.2732, 27.6991)$ and the eigenvalues of its Jacobian matrix, which are -0.0051 ± 0.7502i. By varying the value of b from 0 to 0.3, a different solution is obtained also the nature of equilibrium is changed. Figure 2.2 illustrates this observation, where the real parts of the eigenvalues are negative and the eigenvalues are complex, indicating a stable equilibrium.

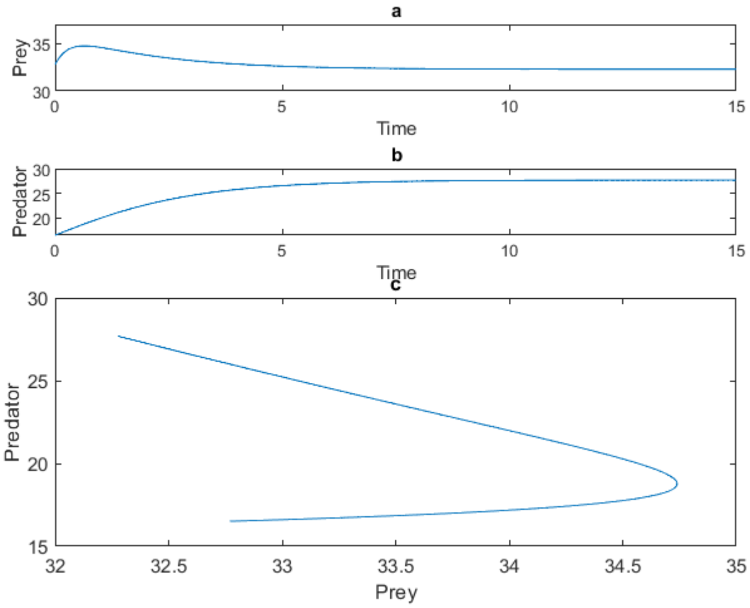


Figure 2.2: Subplot (a) show that changes in prey concerning time. Subplot (b) shows the change in predators concerning time. Subplot (c) is a phase portrait of prey and predator.

CHAPTER 3

Stochastic version of the Beddington-DeAngelis model

Various random disturbances are ubiquitous in real ecosystems, causing birth rates, carrying capacities, competition coefficients, death rates, and other parameters within the system to exhibit random fluctuations to some extent. Consequently, researchers have investigated the effects of environmental stochasticity on population dynamics. Understanding how ecosystems evolve in response to stochastic influences is particularly significant when compared to deterministic models. The impact of white noise on population dynamics underscores the need for comprehensive studies that account for stochasticity in ecological modelling.

$$\begin{cases} \mathrm{d}u(t) = \left(ru(t)\left(1-\dfrac{u(t)}{K}\right) - \dfrac{au(t)v(t)}{1+ahu(t)+b\tau v(t)} - q_1e_1u(t)\right)\mathrm{d}t + \sigma_1 u(t)\mathrm{d}\mathcal{B}_1(t), & (3.1)\\ \mathrm{d}v(t) = \left(\dfrac{sau(t)v(t)}{1+ahu(t)+b\tau v(t)} - mv(t) - \gamma v^2(t) - q_2e_2v(t)\right)\mathrm{d}t + \sigma_2 v(t)\mathrm{d}\mathcal{B}_2(t). & (3.2) \end{cases}$$

Here, $\mathcal{B}_1(t)$ and $\mathcal{B}_2(t)$ are Brownian motions defined on complete probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbf{P})$ and σ_1 and σ_2 are intensity of white noise.

This chapter is structured as follows: Firstly, we shall find conditions so that the prey and predator are extinct and persistent for system (3.1)-(3.2) for any given initial conditions. Subsequently, we demonstrate the existence of a unique global positive solution and finally, we will see some examples based on these results.

3.1 Extinction and persistence

In this section, we will present sufficient conditions for the extinction and persistence of prey and predator as described by equations (3.1)-(3.2).

Definition 1[14]

(1) The species $u(t)$ and $v(t)$ are considered extinct if $\lim_{t\to\infty} u(t) = 0$ and $\lim_{t\to\infty} v(t) = 0$ almost surely (a.s.).

(2) The species $u(t)$ and $v(t)$ are considered persistent in the mean if $\liminf_{t\to\infty} u(t) > 0$ and $\liminf_{t\to\infty} v(t) > 0$ almost surely (a.s.).

3.1.1 Extinction

Theorem 3.1. *Suppose* $(u(t), v(t))$ *be the solution of* $(3.1) - (3.2)$ *with given initial condition* $(u(0), v(0)) \in \mathbb{R}_+^2$ *then the following are true.*

(i) If $r < q_1e_1 + \frac{1}{2}\sigma_1^2$ *then, prey will extinct.*

(ii) If $s < h(m + q_2e_2 + \frac{1}{2}\sigma_2^2)$ *then, the predator will extinct.*

Proof (i) By letting the Lyapunov function as ln$u$ applying Itô formula we get,

$$\mathrm{d}\ \ln u = \left(r\left(1 - \frac{u}{K}\right) - \frac{av}{1 + ahu + b\tau v} - q_1e_1 - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma_1\mathrm{d}\mathcal{B}_1(t).$$

By performing the integration from 0 to t on both sides, we obtain,

$$\ln u(t) = \ln u(0) + rt - \int_0^t \frac{ru(p)}{K}\mathrm{d}p - \int_0^t \frac{av(p)}{1 + ahu(p) + b\tau v(p)}dp - q_1e_1t - \frac{1}{2}\sigma_1^2 t + \sigma_1\mathcal{B}_1(t)$$

$$\leq \ln u(0) + (r - q_1e_1 - \frac{1}{2}\sigma^2)t + \sigma_1\mathcal{B}_1(t).$$

Note that,

$$\lim_{t\to\infty}\frac{\mathcal{B}_1(t)}{t} = 0.$$

So we get,

$$\text{lim sup}_{t\to\infty}\frac{\ln(u(t)}{t} \leq r - q_1e_1 - \frac{1}{2}\sigma_1^2.$$

Therefore by the given condition

$$\text{lim sup}_{t\to\infty}\frac{\ln(u(t))}{t} \leq r - q_1e_1 - \frac{1}{2}\sigma_1^2 < 0.$$

So,

$$\text{lim}_{t\to\infty}u(t) = 0 \text{ a.s.}$$

(ii) Let us consider the Lyapunov function as ln$v$ and apply It$\hat{o}$ formula,

$$\text{d } \ln v = \left(\frac{sau}{1+ahu+b\tau v} - m - q_2e_2 - \gamma v - \frac{1}{2}\sigma_2^2\right)\text{d}t + \sigma_2\text{d}\mathcal{B}_2(t).$$

By performing the integration from 0 to t on both sides, we obtain,

$$\ln v(t) = \ln v(0) + \int_0^t \frac{sau(p)}{1+ahu(p)+b\tau v(p)}\text{d}p - mt - q_2e_2t - \int_0^t \gamma v(p)\text{d}p - \frac{1}{2}\sigma_2^2 t + \sigma_2\mathcal{B}_2(t)$$

$$\ln v(t) \leq \ln v(0) + \left(\frac{s}{h} - m - q_2e_2 - \frac{1}{2}\sigma^2\right)t + \sigma_2\mathcal{B}_2(t)$$

which implies

$$\text{lim sup}_{t\to\infty}\frac{\ln(v(t))}{t} \leq \frac{s}{h} - m - q_2e_2 - \frac{1}{2}\sigma_2^2 < 0 \text{ a.s.}$$

From here we get,

$$\text{lim}_{t\to\infty}v(t) = 0 \text{ a.s.}$$

This proves part (ii) of Theorem 3.1.

3.1.2 Persistent in mean

Lemma 3.2. *[9] Suppose a S.D.E*

$$\begin{cases} dY(t) = Y(t)[\alpha - \beta Y(t)]\,dt + \sigma Y(t)\,d\mathcal{B}(t) & (3.3) \\ Y(0) = Y_0 > 0 & (3.4) \end{cases}$$

with α, β *and* σ *are positive and* $\alpha - \frac{1}{2}\sigma^2 \geq 0$. *Assume* $Y(t)$ *be solution of* $(3.3) - (3.4)$ *then,*

$(i) limsup_{t\to\infty} \dfrac{lnY(t)}{t} = 0,$ *a.s.*

$(ii) lim_{t\to\infty} \dfrac{1}{t}\displaystyle\int_0^t Y(s)ds = \dfrac{\alpha - \frac{1}{2}\sigma^2}{\beta}$ *a.s.*

Lemma 3.3. *The species* $u(t)$ *is said to be persistent in the mean if*

$$lim_{t\to\infty} \frac{1}{t}\int_0^t u(s)ds = c,$$

where c is some positive constant.

Proof On contrary suppose $u(t)$ is not persistent then $\lim_{t\to\infty} u(t) = 0$. By definition of limit, there exists a time T such that

$$u(t) < \epsilon \text{ for all } t \geq T$$

Then,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t u(s)\mathrm{d}s \leq \lim_{t\to\infty} \frac{1}{t}\int_0^t \epsilon \ \mathrm{d}s = \lim_{t\to\infty} \frac{1}{t}\ (\epsilon\ t) = \epsilon.$$

This is a contradiction to the given condition because ϵ is arbitrarily small. Which proves our required result.

Theorem 3.4. *Let* $(u(t), v(t))$ *be the solution of system* $(3.1) - (3.2)$ *with initial condition* $(u(0), v(0))$ *with* $\frac{s}{h} < m + \frac{1}{2}\sigma_2^2 + q_2 e_2$ *and* $r > \frac{1}{2}\sigma_1^2$ *then the predator will extinct but prey will be persistent.*

Proof If $\frac{s}{h} < m + \frac{1}{2}\sigma_2^2 + q_2 e_2$ holds then, by Theorem 3.1 we can conclude predator will extinct. So for every $\epsilon > 0$ there exists a time T such that $v(t) < \epsilon$, for all $t > T$. For $t > T$ equation 3.1 become,

$$\begin{aligned}\mathrm{d}u(t) &= u(t)\left(r\left(1-\frac{u(t)}{K}\right)-\frac{av(t)}{1+ahu(t)+b\tau v(t)}\right)+\sigma_1 u(t)\mathrm{d}\mathcal{B}_1(t)\\ &\geq u(t)\left(r\left(1-\frac{u(t)}{K}\right)-\frac{a\epsilon}{1+ahu(t)+b\tau v(t)}\right)+\sigma_1 u(t)\mathrm{d}\mathcal{B}_1(t)\\ &\geq u(t)\left(r-\frac{ru(t)}{K}-\frac{a\epsilon}{1}\right)+\sigma_1 u(t)\mathrm{d}\mathcal{B}_1(t).\end{aligned}$$

Letting, $\alpha = r - a\epsilon$ and $\beta = \dfrac{r}{K}$ in Lemma 3.2 we get,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t u(s)ds \geq \frac{K\left(r-a\epsilon-\frac{1}{2}\sigma_1^2\right)}{r}.$$

As ϵ is arbitrarily small, letting $\epsilon \to 0$ we get,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t u(s)ds \geq \frac{K\left(r-\frac{1}{2}\sigma_1^2\right)}{r}. \tag{3.5}$$

Again, from equation (3.1)

$$\begin{aligned}\mathrm{d}u(t) &= u(t)\left(r\left(1-\frac{u(t)}{K}\right)-\frac{av(t)}{1+ahu(t)+b\tau v(t)}\right)+\sigma_1 u(t)\mathrm{d}\mathcal{B}_1(t)\\ &\leq u(t)\left(r\left(1-\frac{u(t)}{K}\right)\right)+\sigma_1 u(t)\mathrm{d}\mathcal{B}_1(t)\\ &= u(t)\left(r-\frac{ru(t)}{K}\right)+\sigma_1 u(t)\mathrm{d}\mathcal{B}_1(t).\end{aligned}$$

Letting, $\alpha = r$ and $\beta = \frac{r}{K}$ in Lemma 3.2 we get,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t u(s)ds \leq \frac{K\left(r-\frac{1}{2}\sigma_1^2\right)}{r}. \tag{3.6}$$

By equation (3.7) – (3.8)

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t u(s)ds = \frac{K\left(r-\frac{1}{2}\sigma_1^2\right)}{r}.$$

By Lemma 3.3 we can say $u(t)$ is persistent in the mean.

3.2 Global positive solution

In the population growth model existence of a positive solution is widely acknowledged as a foundational element for exploring various dynamic behaviours within any model. This section aims to prove that the system described in equations (3.1)-(3.2) possesses a unique global solution.

Lemma 3.5. *Let a, s, b, τ, u, v and be any non-negative number and h is positive then,*

(i) $\dfrac{asuv}{1 + ahu + b\tau v} \leq u + \dfrac{sv}{h},$

(ii) $\dfrac{av}{1 + ahu + b\tau v} \leq \ av.$

Proof (i) If a, s, u or v are zero then it is trivial. Suppose these are non-zero then, we have

$$\frac{asuv}{1 + ahu + b\tau v} \leq \frac{asuv}{ahu + b\tau v}.$$

If we can show

$$\frac{asuv}{ahu + b\tau v} \leq u + \frac{sv}{h}$$

then we are done. Let us proceed with the proof. Note that, as a, s, b, τ are non negative then

$$as \leq as + b\tau$$

Now, multiply by uv adding the positive quantity $ahu^2 + b\tau \frac{s}{h} v^2$ in the right hand side of the above inequality,

$$asuv \leq (as\frac{h}{h} + b\tau)uv + ahu^2 + b\tau\frac{s}{h}v^2.$$

By re-arranging the above terms,

$$asuv \leq ahu(u + s\frac{v}{h}) + b\tau v(u + s\frac{v}{h}).$$

Now, dividing both sides by $ahu + b\tau v$, We get,

$$\frac{asuv}{ahu + b\tau v} \leq u + \frac{sv}{h}.$$

This is our required result.

(ii) As $(1 + ahu + b\tau v) \geq 1$ we have,

$$\frac{av}{1 + ahu + b\tau v} \leq \frac{av}{1 + ahu + b\tau v}(1 + ahu + b\tau v) = av.$$

This proves the second part of the Lemma.

Now, we will prove the main theorem of this section, with the help of Lemma 3.5.

Theorem 3.6. *System (3.1)-(3.2) possesses a unique solution* $(u(t), v(t)) \in \mathbb{R}_+^2$ *for* $t \geq 0$ *for any initial value* $(u(0), v(0)) \in \mathbb{R}_+^2$*. Furthermore, it is guaranteed that* $(u(t), v(t))$ *will always remain within* $\mathbb{R}_+^2$ *with a probability of one.*

Proof The system satisfies all the prerequisites of the existence and uniqueness theorem. Hence, for any given initial condition $(u(0), v(0)) \in \mathbb{R}_+^2$, our system possesses a unique local solution $(u(t), v(t)) \in \mathbb{R}_+^2$ on the interval $[0, \tau_e)$, where τ_e denotes the explosion time. To prove this theorem, it is essential to demonstrate $\tau_e = \infty$ almost surely. Let n_0 be a sufficiently large non-negative number such that $(u(0), v(0)) \in [\frac{1}{n_0}, n_0]^2$. For $n \in \mathbb{N}$ with $n > n_0$, we define the stopping time τ_n as:

$$\tau_n = \inf\left\{t \in [0, \tau_e) : u(t) \notin \left(\frac{1}{n}, n\right) \text{ or } v(t) \notin \left(\frac{1}{n}, n\right)\right\}.$$

Clearly, $\{\tau_n\}$ is an increasing sequence. Denote $\lim_{n\to\infty} \tau_n = \tau_\infty$, and we know that $\tau_\infty \leq \tau_e$. To achieve our aim, we must show $\tau_\infty = \infty$ almost surely.

We can prove this by the contradiction method. If $\tau_\infty = \infty$ a.s. does not hold. Then there must exists a positive number T and a very small positive constant ϵ such that $\mathbb{P}\{\tau_\infty \leq T\} \geq \epsilon$. Therefore, there exists an integer $n_1 \geq n_0$ such that $\mathbb{P}\{\tau_n \leq T\} \geq \epsilon$. Next, let us consider the Lyapunov function $F : \mathbb{R}_+^2 \to \overline{\mathbb{R}}_+$ as follows:

$$F(u, v) = (u - 1 - \ln u) + (v - 1 - \ln v).$$

Applying It$\hat{o}$ formula we get the non-Brownian motion part as,

$$
\begin{aligned}
LF &= u\left(1-\frac{1}{u}\right)\left(r\left(1-\frac{u}{K}\right)-\frac{av}{1+ahu+b\tau v}-q_1e_1\right)+\frac{\sigma_1^2}{2}\\
&\quad +v\left(1-\frac{1}{v}\right)\left(\frac{sau}{1+ahu+b\tau v}-m-\gamma v-q_2e_2\right)+\frac{\sigma_2^2}{2}\\
&= (u-1)\left(\frac{r}{K}(K-u)-\frac{av}{1+ahu+b\tau v}-q_1e_1+\frac{\sigma_1^2}{2}\right.\\
&\quad +(v-1)\left(\frac{sau}{1+ahu+b\tau v}-\gamma v-m-q_2e_2\right)+\frac{\sigma_2^2}{2}\\
&= \frac{-r}{K}u^2+\frac{a}{1+ahu+b\tau v}\Big(su(v-1)-a(u-1)\Big)-q_1e_1(u-1)-(m+q_2e_2)(v-1)-\gamma v^2\\
&\quad +(r+\frac{r}{K}-q_1e_1)u+(\gamma-m-q_2e_2)v-r+q_1e_1+m+q_2e_2+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}\\
&\le \frac{asuv}{1+ahu+b\tau v}+\frac{av}{1+ahu+b\tau v}-\frac{r}{K}u^2+\frac{r}{K}u-\gamma v^2+\gamma v+q_1e_1+m+q_2e_2+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}.
\end{aligned}
$$

By Lemma 3.5, we conclude that

$$
\begin{aligned}
LF &\le u+\frac{sv}{h}+av+-\frac{r}{K}u^2+\frac{r}{K}u-\gamma v^2+\gamma v+q_1e_1+m+q_2e_2+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}\\
&= -\frac{r}{K}u^2+\left(\frac{r}{K}+1\right)u-\gamma v^2+\left(\gamma+\frac{s}{h}+a\right)v+c_0,
\end{aligned}
$$

where $c_0 = q_1e_1+m+q_2e_2+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}$.
Let $c_1=\sup\Big(-\frac{r}{K}u^2+\Big(\frac{r}{K}+1\Big)u-\gamma v^2+\Big(\gamma+\frac{s}{h}+a\Big)v+c_0\Big)$. It follows that $LF\le c_1$. Therefore by It$\hat{o}$ formula,

$$\mathrm{d}F \le c_1\mathrm{d}t+\sigma_1(u-1)\mathrm{d}\mathcal{B}_1(t)+\sigma_1(v-1)\mathrm{d}\mathcal{B}_2(t).$$

Now integrating from 0 to $(T\wedge\tau_e)$ and taking expectation we get,

$$\mathbb{E}\Big(F(u(T\wedge\tau_e),v(T\wedge\tau_e))\Big)\le F\Big(u(0),v(0)\Big)+c_1(T\wedge\tau_e). \tag{3.7}$$

Define $\Omega_n=\{\tau_n\le T\}$ for $n\ge n_1$ then for every $\omega\in\Omega_n$, there is an observation that either $u(\tau_n,\omega)$ or $v(\tau_n,\omega)$ must be equal to $\frac{1}{n}$ or n. Therefore, we get:

$$F(u(\tau_n,\omega),v(\tau_n,\omega))\ge(n-1-\ln n)\wedge(\frac{1}{n}-1-\ln\frac{1}{n})$$

If $n \to \infty$, we get,

$$F(u(\tau_n, \omega), v(\tau_n, \omega)) \geq \infty \tag{3.8}$$

By equation (3.9) – (3.10) we get,

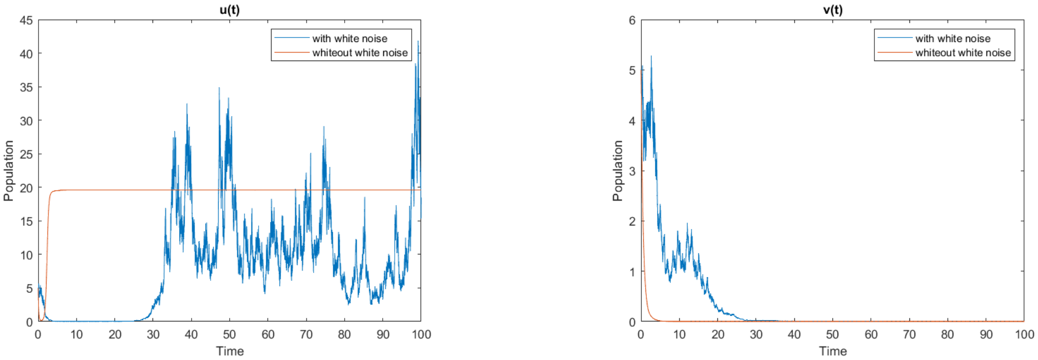
$$\infty > F\Big(u(0), v(0)\Big) + c_1(T \wedge \tau_e) \geq F(u(\tau_n, \omega), (v(\tau_n, \omega))) \geq \infty, \tag{3.9}$$

which is a contradiction. Thus $\tau_\infty = \infty$ must be true, this proved the theorem.

3.3 Numerical simulation

We have derived some theoretical results in the previous sections, such as sufficient conditions for the extinction of prey and predators. We will see some examples of conditions for the persistence of prey and extinction of predators to show how it works. By numerical simulation, we will see how it goes concerning time.

Example 3.1 Fix the parameters $r = 0.5, K = 20, b = 0.05, a = 0.4, h = 0.3, \tau = 0.5, q_1 = 0.1, e_1 = 0.1, s = 0.03, m = 0.15, \gamma = 0.01, q_2 = 0.05, e_2 = 0.1, \sigma_1 = 0.5, \sigma_2 = 0.3$ with initial conditions $u(0) = 5$ and $v(0) = 5$. Here $\frac{s}{h} = 0.1$ and $m + q_2e_2 + \frac{1}{2}\sigma_2^2 = 0.2$ which means $\frac{s}{h} < m + q_2e_2 + \frac{1}{2}\sigma_2^2$ and $\frac{1}{2}\sigma_1^2 = 0.125$ so, $r > \frac{1}{2}\sigma_1^2$. So, Theorem 3.4 implies predators will be extinct but prey will persist.

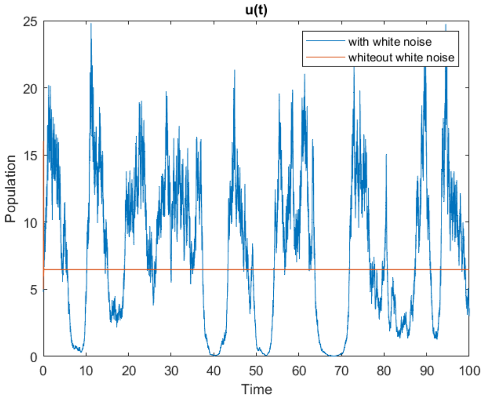


(a) Nature of prey with and without white noise in example 3.1

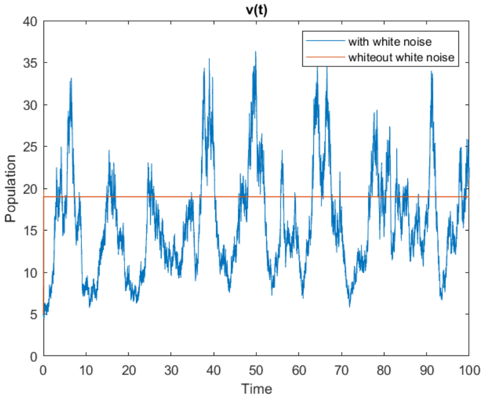
(b) Nature of predator with and without white noise in example 3.1

Figure 3.1: Time series comparison chart of Example 3.1

Example 3.2 Fix the parameters $r = 5, K = 20, b = 0.05, a = 0.4, h = 0.3, \tau = 0.5, q_1 = 0.1, e_1 = 0.1, s = 0.3, m = 0.15, \gamma = 0.01, q_2 = 0.05, e_2 = 0.1, \sigma_1 = 0.5, \sigma_2 = 0.3$ with initial conditions $u(0) = 5$ and $v(0) = 5$. Here $\frac{s}{h} = 1$ and $m + q_2e_2 + \frac{1}{2}\sigma_2^2 = 0.2$ which means $\frac{s}{h} \not< m + q_2e_2 + \frac{1}{2}\sigma_2^2$.Also $q_1e_1 + \frac{1}{2}\sigma_1^2 = 0.135$ that's why $r \not< q_1e_1 + \frac{1}{2}\sigma_1^2$. Which does not satisfy the condition for Theorem 3.1 and both prey and predator are in persistence.



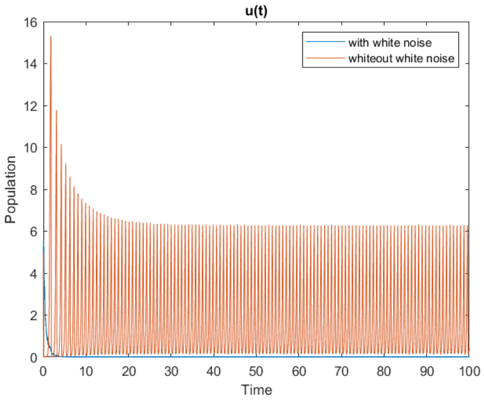
(a) Nature of prey with and without white noise in example 3.2



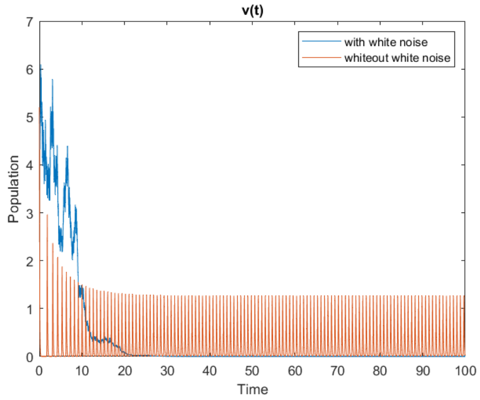
(b) Nature of predator with and without white noise in example 3.2

Figure 3.2: Time series comparison chart of Example 3.2

Example 3.3 Fix the parameters $r = 0.1, K = 20, b = 0.05, a = 0.4, h = 0.3, \tau = 0.5, q_1 = 0.1, e_1 = 0.1, s = 0.3, m = 0.15, \gamma = 0.01, q_2 = 0.05, e_2 = 0.1, \sigma_1 = 0.5, \sigma_2 = 0.3$ with initial conditions $u(0) = 5$ and $v(0) = 5$. Here $\frac{s}{h} = 1$ and $m + q_2 e_2 + \frac{1}{2}\sigma_2^2 = 0.2$ which means $\frac{s}{h} \not< m + q_2 e_2 + \frac{1}{2}\sigma_2^2$. Also $q_1 e_1 + \frac{1}{2}\sigma_1^2 = 0.135$ that's why $r < q_1 e_1 + \frac{1}{2}\sigma_1^2$. This satisfies the condition for Theorem 3.1 for the extinction of prey. But both prey and predators are both extinct.



(a) Nature of prey with and without white noise in example 3.3

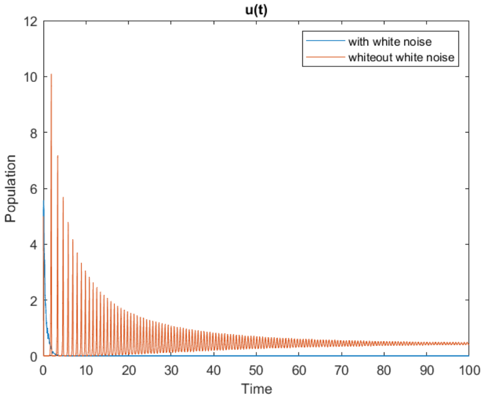


(b) Nature of predator with and without white noise in example 3.3

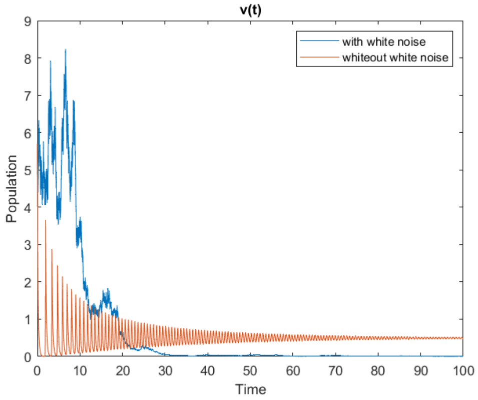
Figure 3.3: Time series comparison chart of Example 3.3

Example 3.4 Fix the parameters $r = 0.2, K = 20, b = 0.05, a = 0.4, h = 0.3, \tau =$

$0.5, q_1 = 0.1, e_1 = 0.1, s = 0.3, m = 0.04, \gamma = 0.01, q_2 = 0.05, e_2 = 0.1, \sigma_1 = 0.5, \sigma_2 = 0.3$ with initial conditions $u(0) = 5$ and $v(0) = 5$. Here $\frac{s}{h} = 1$ and $m + q_2e_2 + \frac{1}{2}\sigma_2^2 = 0.09$ which means $\frac{s}{h} \nless m + q_2e_2 + \frac{1}{2}\sigma_2^2$. Also $q_1e_1 + \frac{1}{2}\sigma_1^2 = 0.135$ that's why $\nless q_1e_1 + \frac{1}{2}\sigma_1^2$. This does not satisfy the condition for Theorem 3.1 regarding the extinction of prey and predators. Even after that, both prey and predators are still extinct.



(a) Nature of prey with and without white noise in example 3.4



(b) Nature of predator with and without white noise in example 3.4

Figure 3.4: Time series comparison chart of Example 3.4

CHAPTER 4

Future plans and conclusion

4.1 Conclusion

The thesis concludes with the following results:

1. For any given initial condition, the system (3.1)-(3.2) has a unique solution $(u(t), v(t)) \in \mathbb{R}^2_+$, and this solution belongs to $\mathbb{R}^2_+$ with probability one.
2. There exists a condition under which the prey will go extinct.
3. There exists a condition under which the predator will go extinct.
4. Additionally we get a condition where the predator will go extinct but the prey will persist.

4.2 Future plans

4.2.1 Bedington-DeAngelis model with white noise and Lévy jumps

In this thesis, we study we studied the model that includes only white noise but in future, we aim to find results on the model that also includes Lévy jumps the SDE for such model is

$$\begin{aligned}
\mathrm{d}u(t) =& \left(ru(t)\left(1 - \frac{u(t)}{K}\right) - \frac{au(t)v(t)}{1 + ahu(t) + b\tau v(t)} - q_1 e_1 u(t) \right)\mathrm{d}t + \sigma_1 u(t)\mathrm{d}\mathcal{B}_1(t) \\
&+ \int_Z F_1(u(t^-), z)\widetilde{\mathcal{N}_1}(dt, dz), \\
\mathrm{d}v(t) =& \left(\frac{sau(t)v(t)}{1 + ahu(t) + b\tau v(t)} - mv(t) - \gamma v^2(t) - q_2 e_2 v(t) \right)\mathrm{d}t + \sigma_2 v(t)\mathrm{d}\mathcal{B}_2(t) \\
&+ \int_Z F_2(v(t^-), z)\widetilde{\mathcal{N}_2}(dt, dz).
\end{aligned}$$

Here, $u(t^-)$ and $v(t^-)$ represents the left limit of $u(t)$ and $v(t)$, respectively. Here $\widetilde{\mathcal{N}_i};\ \ i = 1, 2$ represent time-homogeneous compensated Poisson random measure on Z, a measurable subspace of $\mathbb{R}^2$

4.2.2 To study large deviation principle

The large deviations principle delves into the probabilities of infrequent occurrences that are exceedingly unlikely, often characterized by exponential rarity, with respect to specific parameters. These parameters could encompass factors like the number of random elements within a system, the duration of observation for a stochastic system, the intensity of noise impacting a dynamic system, or the temperature within a chemical reaction. This theoretical framework finds applications across a spectrum of scientific disciplines, including queuing theory, statistics, finance, and engineering. Particularly within statistical physics, it is gaining prominence for analyzing systems in both equilibrium and nonequilibrium states. Within this domain, parallels can be drawn between well-established concepts like entropy and free energy in statistical physics and the more technically named concepts within large deviation theory, such as the rate function and the scaled cumulant generating function.

Our objective is to investigate large deviations in the context of the stochastic two-species predator-prey model. We will consider Gaussian randomness, encompassing both additive and multiplicative noise types. Our approach involves employing the contraction principle argument and leveraging weak convergence to establish a Freidlin-Wentzell type large deviation principle for both the additive and multiplicative noise scenarios.

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