Backreacted Black Holes

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by
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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled *Backreacted Black Holes* in the partial fulfilment of the requirements for the award of the degree of Master of Science and submitted in the discipline of Physics, Indian Institute of Technology Indore, is an authentic record of my own work carried out during the time period from August 2023 to May 2024 under the supervision of Dr. Debajyoti Sarkar, Assistant professor, Indian Institute of Technology Indore.

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Abstract

This report delves into the Arnowitt-Deser-Misner (ADM) formalism, focusing on its application in deriving the time evolution equation. By exploring the dynamics of gravitational fields within the framework of general relativity, particular attention is given to calculating the expectation value for the stress-energy tensor. Notably, this tensor encapsulates the Riemannian curvature tensor, necessitating the determination of Christoffel symbols for the Schwarzschild metric. Additionally, the report delves into the computation of the scalar curvature and extrinsic curvature tensor, energy density, and momentum density for the Schwarzschild metric. Our primary objective lies in determining the change in the metric tensor using the time evolution equation, shedding light on the dynamical evaluation of spacetime curvature. Through these analyses, a comprehensive understanding of the gravitational field dynamics is synthesized, offering valuable insights into Hawking radiation.

Acknowledgements

I want to extend a sincere obligation towards all the personages without whom the continuation of the project was not possible. I express my profound gratitude and deep regard to Dr. Debajyoti Sarkar, IIT Indore, for his guidance and enlightening discussions during the project. His valuable suggestions and feedback were of immense help. I sincerely acknowledge my senior Bhim Sen for his constant support and guidance during this semester.

Dedication

This dedication is extended to my parents in heartfelt appreciation for their unwavering and divine affection.

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Chapter 1

Introduction

General Relativity has proved to be one of the most elegant and successful physical theories since its first appearance in the paper The Field Equations of Gravitation on November 25, 1907 [1]. Albert Einstein's theory is based on the Equivalence Principle, which holds that all bodies are affected by gravity similarly and that physical laws are independent of reference frames, making it impossible to distinguish between the effects of a gravitational field and those of a uniform accelerating frame. These presumptions led to the distribution of matter being attributed to the geometry of spacetime itself, along with the premise that spacetime is a curved manifold structure defined by a metric tensor $g_{\mu\nu}$. The Einstein Field Equations in particular, specify this relation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.1}$$

where G is the gravitational constant, c is the speed of light, Λ is the cosmological constant, R is the scalar curvature, and $T_{\mu\nu}$ is the stress-energy tensor. We now use the geometrized unit system, with $G = c = 1$, and we set $\Lambda = 0$ to ignore the influence of the cosmological constant.

1.1 Einstein-Hilbert action

Given a region $\mathcal{V}$ of the spacetime manifold and a scalar function $\mathcal{L}\left(\psi, \partial_\alpha \psi\right)$, known as the Lagrangian density, which depends on the field variables ψ and their first derivatives $\partial_\alpha \psi$, the field equations of a field theory can be inferred using the Lagrangian formulation. We will only consider generic tensors of type $(\ r, s\ )$ (omitting the indices for brevity), even if the fields ψ could be of any type. Similar to how Newtonian mechanics is expressed in Lagrangian form, the action functional $\mathcal{S}[\psi]$ is defined as the integral [2]

$$\mathcal{S}[\psi] = \int_{\mathcal{V}} \mathcal{L}\left(\psi, \partial_\alpha \psi\right) \sqrt{-g}\, \mathrm{d}^4 x \tag{1.2}$$

where $\sqrt{-g}\, \mathrm{d}^4 x$ is the appropriate volume element and g is the (negative) determinant of the metric $g_{\mu\nu}$. Then, by ensuring that $\mathcal{S}[\psi]$ is stationary under an arbitrary variation $\delta\psi$ about the actual fields ψ_0, the field equations are recovered. A natural definition of variation for a smooth one-parameter family of field configurations ψ_λ arises from the derivative

$$\delta\psi = \left.\frac{\mathrm{d}\psi_\lambda}{\mathrm{d}\lambda}\right|_{\lambda=0} \tag{1.3}$$

This is what we require to vanish on the $\partial\mathcal{V}$ boundary of our spacetime region

$$\left.\delta\psi\right|_{\partial\mathcal{V}} = 0 \tag{1.4}$$

We now assume that the action functional is associated with a smooth tensor field χ of type (s, r) (thus dual to ψ).

$$\mathcal{S} = \int_{\mathcal{V}} \mathrm{d}^4 x \chi \psi \tag{1.5}$$

When it is implied that the indices of χ and ψ are contracted. The relation can be obtained by taking the derivative of $\mathcal{S}$ with regard to the parameter λ.

$$\delta\mathcal{S} \doteq \left.\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}\lambda}\right|_{\lambda=0} = \int_{\mathcal{V}} \mathrm{d}^4 x \chi \delta\psi \tag{1.6}$$

Consequently, the functional derivative is defined as the variation of $\mathcal{S}$ with regard to ψ about ψ_0.

$$\chi = \left.\frac{\delta\mathcal{S}}{\delta\psi}\right|_{\psi_0} \tag{1.7}$$

Which, given the action's stationarity, must disappear in the same way:

$$\chi = 0 \tag{1.8}$$

These relationships guarantee that the field equations included in identity 1.8 have a solution in ψ_0. Hilbert and Einstein originally examined the variational approach to general relativity in 1915, when they postulated the basic gravitational action:

$$\mathcal{S}_H = \frac{1}{16\pi} \int_{\mathcal{V}} R\sqrt{-g}\ \mathrm{d}^4 x \tag{1.9}$$

The Hilbert term is denoted by $\mathcal{S}_H$. Since the scalar curvature R is the only nontrivial scalar function that can be formed from the metric and its derivatives up to the second order, this is, in fact, the simplest gravitational action that can be imagined. The decision

$$\mathcal{L}_H \doteq \frac{1}{16\pi} R\sqrt{-g} \tag{1.10}$$

given the complexity of the other options, this proves to be quite persuasive. However, it also establishes a simple relationship between the Newtonian theory of gravitation and weak field limits. We will also include the contributions from the matter fields, indicated by ϕ, in the term in addition to $\mathcal{S}_H$.

$$\mathcal{S}_M = \int_{\mathcal{V}} \mathcal{L}_M \left(\phi, \partial_\alpha \phi; g_{\mu\nu}\right) \sqrt{-g}\ \mathrm{d}^4 x \tag{1.11}$$

To keep things simple, we'll suppose that $\mathcal{L}_M$ only depends on the field ϕ and its first derivatives, as well as the metric coefficients $g_{\mu\nu}$. The whole matter and Hilbert terms add up to the total action functional.

$$\mathcal{S} = \mathcal{S}_H + \mathcal{S}_M \tag{1.12}$$

Thus, we can show that the Einstein field equation 1.1 stems from the stationarity of $\mathcal{S}$ under arbitrary variations of $g_{\mu\nu}$.

1.2 Variation of the action

Let's start by focusing just on the Hilbert term. Using the variant of the inverse metric $\delta g^{\mu\nu}$ rather than $\delta g_{\mu\nu}$ will show to be more handy. This has no bearing whatsoever on the outcomes because the two variables are connected by

$$g^{\alpha\lambda} g_{\lambda\beta} = \delta^\alpha{}_\beta \quad \Longrightarrow \quad \delta g_{\mu\nu} = -g_{\mu\alpha} g_{\nu\beta} \delta g^{\alpha\beta} \tag{1.13}$$

As the variation may be brought under the integral sign, we can execute the variation of $\mathcal{S}_H$ (according to definition 1.6) by concentrating on the integrand, that is, the Hilbert Lagrangian density $\mathcal{L}_H$.

$$\begin{aligned}(16\pi)\delta\mathcal{L}_H &= \delta\left(g^{\mu\nu}R_{\mu\nu}\sqrt{-g}\right)\\ &= -\frac{\delta g}{2\sqrt{-g}}g^{\mu\nu}R_{\mu\nu} + \left(\delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right)\sqrt{-g}\end{aligned} \tag{1.14}$$

The variation of the metric determinant δg is provided by Jacobi's formula:

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu} = -g g_{\mu\nu}\delta g^{\mu\nu} \tag{1.15}$$

We can substitute δg in 1.14 by using the second form of this identity and remembering that $g < 0$.

$$(16\pi)\delta\mathcal{L}_H = \left[\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right]\sqrt{-g} \tag{1.16}$$

It is now evident that if $\delta R_{\mu\nu}$ vanishes, the field equations' solely gravitational component is regained. In the broader situation, however, this assumption need not hold since additional boundary terms result from the first derivatives of $\delta g^{\mu\nu}$ entering the variation $\delta R_{\mu\nu}$. In fact, we find that if we turn to the Palatini identity (as demonstrated in section A.2.3 of the Appendix).

$$\delta R_{\mu\nu} = \nabla_\rho\left(\delta\Gamma^\rho{}_{\mu\nu}\right) - \nabla_\mu\left(\delta\Gamma^\rho{}_{\rho\nu}\right) \tag{1.17}$$

The contravariant vector $V^\rho \doteq g^{\mu\nu}\delta\Gamma$ is introduced.The last term of equation 1.16 can be recast as a divergence utilizing $\nabla_\rho g_{\mu\nu} = 0$ of Levi-Civita connections and ${}^\rho{}_{\mu\nu} - g^{\rho\nu}\delta\Gamma^\mu{}_{\mu\nu}$ (whose explicit expressions are covered in Appendix, section A.2.2).

$$\begin{aligned}\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} &= \sqrt{-g}g^{\mu\nu}\left[\nabla_\rho\delta\Gamma^\rho{}_{\mu\nu} - \nabla_\mu\Gamma^\rho{}_{\rho\nu}\right]\\ &= \sqrt{-g}\nabla_\rho\left[g^{\mu\nu}\delta\Gamma^\rho{}_{\mu\nu} - g^{\rho\nu}\delta\Gamma^\mu{}_{\mu\nu}\right] \doteq \partial_\rho\left(\sqrt{-g}V^\rho\right)\end{aligned} \tag{1.18}$$

With the replacement of these results in action integral 1.9 and the reintroduction of the multiplicative constant (16π), the variation $\delta\mathcal{S}_H$ splits into the volume and boundary components using Stokes' theorem.

$$\delta\mathcal{S}_H = \frac{1}{16\pi}\int_{\mathcal{V}}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\sqrt{-g}\delta g^{\mu\nu}\mathrm{d}^4x + \frac{1}{16\pi}\oint_{\partial\mathcal{V}} V^{\mu}\mathrm{d}\sigma_{\mu}\mathrm{d}^3x \tag{1.19}$$

where the oriented volume element of the hypersurface $\partial\mathcal{V}$ is $\mathrm{d}\sigma_{\mu}$. We proceed as though the surface terms can be safely ignored, ignoring the second integral. The matter action 1.11 variation is now under consideration. Its dependence on $g_{\mu\nu}$ and the matter fields ϕ are as follows:

$$\begin{aligned}\delta\mathcal{S}_M &= \int_{\mathcal{V}}\left[\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}}\delta g^{\mu\nu}\sqrt{-g} + \mathcal{L}_M\delta\sqrt{-g}\right]\mathrm{d}^4x \\ &= \int_{\mathcal{V}}\left[\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} - \frac{1}{2}\mathcal{L}_M g_{\mu\nu}\right]\sqrt{-g}\delta g^{\mu\nu}\mathrm{d}^4x\end{aligned} \tag{1.20}$$

The stress-energy tensor $T_{\mu\nu}$ can be defined as follows:

$$T_{\mu\nu} \doteq -2\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} + \mathcal{L}_M g_{\mu\nu} \tag{1.21}$$

We observe that the variation of $\mathcal{S}$ as a whole is as follows:

$$\begin{aligned}\delta\mathcal{S} &= \int_{\mathcal{V}}\left[\frac{1}{16\pi}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} - \frac{1}{2}\mathcal{L}_M g_{\mu\nu}\right]\sqrt{-g}\delta g^{\mu\nu}\mathrm{d}^4x \\ &= \frac{1}{16\pi}\int_{\mathcal{V}}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - 8\pi T_{\mu\nu}\right]\sqrt{-g}\delta g^{\mu\nu}\mathrm{d}^4x\end{aligned} \tag{1.22}$$

The stationarity of $\mathcal{S}$ necessitates the integrand to be identically zero due to the arbitrariness of $\delta g^{\mu\nu}$, ultimately leading to the Einstein field equations.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \tag{1.23}$$

This, in accordance with the left side of 1.23, maybe rewritten in an equivalent form using the Einstein tensor $G_{\mu\nu}$:

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.24}$$

The fourdivergence represents the intended conservation of the stress-energy tensor $T_{\mu\nu}$.

$$\nabla_\mu T^{\mu\nu} = 0 \tag{1.25}$$

The Riemann curvature tensor $R^\rho{}_{\sigma\mu\nu}$ symmetries guarantee this, as do the Bianchi identities $\nabla_\mu G^{\mu\nu} = 0$. The invariance of the action under an infinitesimal translation of coordinates (see section A.1.8) can also be used to demonstrate this result.

Chapter 2

ADM Hamiltonian formulation of General Relativity

This section covers the Arnowitt, Deser, and Misner Hamiltonian formulation of general relativity [3], which is based on the gravitational action functional 2.59. Two main characteristics of the canonical formulation are significant:

- Time holds a privileged position among the coordinates (x^μ). In particular, the original four-dimensional description is replaced by the evolution of tensor fields on a spacelike three-dimensional hypersurface Σ.
- The system's time evolution is defined by Hamilton's equations, which are first-order differential equations in the time derivatives.

The problems arising from the redundancy of variables $g_{\mu\nu}$ are also clarified by the canonical form of general relativity. In fact, while this guarantees the theory's general covariance, it complicates the process of determining the bare minimum of information required to produce a consistent beginning value formulation. Since it is a requirement of the general relativity quantization program, this reduction to the independent dynamical modes of the gravitational field is extremely desirable. Actually, it is only possible to examine the relationship between the commutators in quantum mechanics and the Poisson brackets of the Hamiltonian theory when the unconstrained canonical variables are extracted from the corresponding total set.
We give a thorough derivation of Hamilton's equations and find the basic Poisson brackets between the limited variables in the sections that follow. The reader may consult ADM's 1962 article[5] for a formal discussion of the system's independent variable isolation.

2.1 Einstein-Hilbert action in $3+1$ formalism

Let us go back to the Hilbert Lagrangian density $\mathcal{L}_H = {}^4R\sqrt{-g}$, leaving out the unnecessary multiplication factor $(16\pi)^{-1}$ for the time being. The four-dimensional quantities R^4 and $\sqrt{-g}$ will be swapped out for their three-dimensional equivalents [3],

$$\mathcal{L}_H = \left[R^3 + K^2 + K^{ij}K_{ij} - 2\nabla_{\boldsymbol{n}}K - \frac{2}{N}D^iD_iN\right]N\sqrt{\gamma} \tag{2.1}$$

The action functional $\mathcal{S}_H$ was presented in Chapter 1 and is defined as the integral of $\mathcal{L}_H$ over a region $\mathcal{V}$ of the spacetime manifold. The $3+1$ dimensional decomposition divides this region into a family of hypersurfaces Σ_t, designated by the time t, allowing us continue the integration on $\mathcal{V}$:

$$\mathcal{S}_H = \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \left[R^3 + K^2 + K^{ij}K_{ij} - 2\nabla_{\boldsymbol{n}}K - \frac{2}{N}D^iD_iN\right]N\sqrt{\gamma}\mathrm{d}^3x \tag{2.2}$$

where t_1 and t_2 are generic lower and upper time limits. Before proceeding with our analysis, we shall unveil the divergences hidden in the last two terms of $\mathcal{L}_H$ by rewriting them in the following form:

$$\sqrt{\gamma}D^iD_iN = \sqrt{\gamma}D_i\left(\partial^iN\right) = \partial_i\left(\sqrt{\gamma}\partial^iN\right) \tag{2.3}$$

$$N\sqrt{\gamma}\nabla_{\boldsymbol{n}}K = N\sqrt{\gamma}n^\alpha\nabla_\alpha K = \partial_\alpha\left(\sqrt{\gamma}NKn^\alpha\right) + \sqrt{\gamma}NK^2 \tag{2.4}$$

When we substitute in the Lagrangian density $\mathcal{L}_H$, the sign of the term K^2 changes, leading us to

$$\begin{aligned}\mathcal{L}_H = {} & \left(R^3 - K^2 + K^{ij}K_{ij}\right)N\sqrt{\gamma} \\ & - 2\left[\partial_i\left(\sqrt{\gamma}\partial^iN\right) + \partial_\alpha\left(\sqrt{\gamma}NKn^\alpha\right)\right]\sqrt{\gamma}\end{aligned} \tag{2.5}$$

Temporarily recasting the two divergences in four-dimensional notation will prove to be more enlightening. To do this, we take a look at the generalized equation

$$R^4 = R^3 + K^2 - K_{\mu\nu}K^{\mu\nu} - 2^4R_{\mu\nu}n^\mu n^\nu$$

Then, use the contracted Ricci identity to substitute a commutator of spacetime connections for the final term:

$$R^4 = R^3 + K^2 - K_{\mu\nu}K^{\mu\nu} - 2n^\mu \left[\nabla_\alpha, \nabla_\mu\right] n^\alpha$$

Now, we use these equations and the orthogonality relation to rewrite the commutator in the form,

$$\begin{aligned} n^\mu \left[\nabla_\alpha, \nabla_\mu\right] n^\alpha &= n^\mu \left(\nabla_\alpha \nabla_\mu - \nabla_\mu \nabla_\alpha\right) n^\alpha \\ &= \nabla_\alpha \left(n^\mu \nabla_\mu n^\alpha - n^\alpha \nabla_\mu n^\mu\right) - \left(\nabla_\alpha n^\mu\right) \nabla_\mu n^\alpha + \left(\nabla_\alpha n^\alpha\right)^2 \\ &= \nabla_\alpha \left(n^\mu \nabla_\mu n^\alpha - n^\alpha \nabla_\mu n^\mu\right) - K^{\alpha\mu} K_{\alpha\mu} + K^2 \end{aligned}$$

When this result is substituted in the Lagrangian density $\mathcal{L}_H$, the contraction $K_{\mu\nu}K^{\mu\nu} = K_{ij}K^{ij}$ is obtained by adopting the Latin indices.

$$\mathcal{L}_H = \left(R^3 - K^2 + K_{ij}K^{ij}\right) N\sqrt{\gamma} - 2\sqrt{-g}\nabla_\alpha \left(n^\mu \nabla_\mu n^\alpha - n^\alpha \nabla_\mu n^\mu\right) \tag{2.6}$$

The "divergence-free" components of the two expressions 2.5 and 2.6 are the same, which suggests that the later words must be comparable. The next section will focus on examining the four-dimensional divergence portion in particular.

2.1.1 Boundary terms in the $3+1$ Lagrangian density

The contribution of $\mathcal{S}_H$, which needs to be added to the gravitational action's boundary term $\mathcal{S}_B$, will now be discussed. Reintroducing the multiplicative constant $(16\pi)^{-1}$ and applying Stokes' theorem, we obtain

$$\begin{aligned} &-\frac{1}{8\pi}\int_{\mathcal{V}} \sqrt{-g}\nabla_\alpha \left(n^\mu \nabla_\mu n^\alpha - n^\alpha \nabla_\mu n^\mu\right) \mathrm{d}^4x = \\ &= -\frac{1}{8\pi}\oint_{\partial\mathcal{V}} \varepsilon \left(n^\mu \nabla_\mu n^\alpha - n^\alpha \nabla_\mu n^\mu\right) \sqrt{|h|} r_\alpha \mathrm{d}^3x \end{aligned}$$

where $\mathrm{d}\sigma_\alpha = \varepsilon r_\alpha \sqrt{|h|}\mathrm{d}^3x$ is the oriented volume element on $\partial\mathcal{V}$, and r_α indicates the unit normal to $\partial\mathcal{V}$. Assuming $\partial\mathcal{V}$ to represent the union of two spacelike hypersurfaces Σ_{t_1} and Σ_{t_2} (where $t_2 > t_1$) connected by a timelike hypersurface $\mathcal{T}$ will allow us to proceed further. Given that n_α and $\varepsilon = n_\alpha n^\alpha = -1$ are the unit normals on Σ_{t_2}, the surface integral's contribution from Σ_{t_2} is

$$-\frac{1}{8\pi}\int_{\Sigma_{t_2}} \varepsilon \left(n^\mu \nabla_\mu n^\alpha - n^\alpha \nabla_\mu n^\mu\right) \sqrt{|h|} r_\alpha \mathrm{d}^3x = \frac{1}{8\pi}\int_{\Sigma_{t_2}} K\sqrt{|h|}\mathrm{d}^3x \tag{2.7}$$

where $K = -\nabla_\alpha n^\alpha$ and $h > 0$ denote the determinant of the induced metric on $\partial\mathcal{V}$. In a similar vein, Σ_{t_1} contributes

$$-\frac{1}{8\pi}\int_{\Sigma_{t_1}} \varepsilon\left(n^\mu \nabla_\mu n^\alpha - n^\alpha \nabla_\mu n^\mu\right)\sqrt{|h|}r_\alpha \mathrm{d}^3x = -\frac{1}{8\pi}\int_{\Sigma_{t_1}} K\sqrt{|h|}\mathrm{d}^3x \tag{2.8}$$

where the negative orientation of Σ_{t_1} with regard to the future-directed normal, $r_\alpha = -n_\alpha$, is represented by the minus sign. The related integrals over Σ_{t_1} and Σ_{t_2} contained in $\mathcal{S}_B$ are seen to be cancelled out by 2.7 and 2.8. However, the remainder of $\mathcal{S}_B$ is not neutralized by the contribution from $\mathcal{T}$. Actually, it provides

$$\begin{aligned}
&-\frac{1}{8\pi}\int_{\mathcal{T}} \varepsilon\left(n^\mu \nabla_\mu n^\alpha - n^\alpha \nabla_\mu n^\mu\right)\sqrt{|h|}r_\alpha \mathrm{d}^3x \\
&\quad = -\frac{1}{8\pi}\int_{\mathcal{T}} \left(n^\mu \nabla_\mu n^\alpha\right) r_\alpha \sqrt{|h|}\mathrm{d}^3x = \frac{1}{8\pi}\int_{\mathcal{T}} n^\mu n^\alpha \left(\nabla_\mu r_\alpha\right)\sqrt{|h|}\mathrm{d}^3x
\end{aligned} \tag{2.9}$$

Because r_α has a spacelike nature, we have employed the orthogonality relation $n^\alpha r_\alpha = 0$ in the second line. Given that $\varepsilon = 1$ on $\mathcal{T}$, we obtain $\mathcal{S}_B$ by combining the integral 2.9 with the last term.

$$\begin{aligned}
&\tfrac{1}{8\pi}\textstyle\int_{\mathcal{T}} n^\mu n^\alpha \left(\nabla_\mu r_\alpha\right)\sqrt{|h|}\mathrm{d}^3x - \tfrac{1}{8\pi}\int_{\mathcal{T}} K\sqrt{|h|}\mathrm{d}^3x \\
&\quad = \tfrac{1}{8\pi}\textstyle\int_{\mathcal{T}} \left(n^\mu n^\nu + g^{\mu\nu}\right)\nabla_\mu r_\nu \sqrt{|h|}\mathrm{d}^3x
\end{aligned} \tag{2.10}$$

The last integral of 2.9 can be made simpler by introducing a foliation of $\mathcal{T}$ by the two surfaces S_t, each of which corresponds to the Σ_t boundary:

$$S_t = \partial\Sigma_t$$

The extrinsic curvature tensor of S_t can be defined as follows by viewing S_t as a two-hypersurface contained in the three-dimensional space Σ_t.

$$\kappa_{ij} \doteq -D_i r_j + r_i D_{\boldsymbol{r}} r_j \tag{2.11}$$

where r^i is the normal to S_t and i, j denote the coordinates of Σ_t. The only difference between this and the defining relation of $K_{\mu\nu}$ is a sign caused by $\varepsilon = 1$. The scalar curvature is obtained by contracting κ_{ij} with the induced metric h^{ij}:

$$\kappa \doteq \kappa_{ij}h^{ij} = -h^{ij}D_i r_j = -D_i r^i \tag{2.12}$$

where r^i is the normal to S_t and i, j denote the coordinates of Σ_t. The only difference between this and the defining relation of $K_{\mu\nu}$ is a sign caused by $\varepsilon = 1$. The scalar curvature is obtained by contracting κ_{ij} with the induced metric h^{ij}:

$$\begin{aligned}\kappa &= -D_\mu r^\mu = -g^{\mu\nu} \left(\delta^\alpha{}_\mu + n^\alpha n_\mu\right) \left(\delta^\beta{}_\nu + n^\beta n_\nu\right) \nabla_\alpha r_\beta \\ &= -\nabla_\alpha r^\alpha - n^\alpha n^\beta \nabla_\alpha r_\beta = -\left(g^{\alpha\beta} + n^\alpha n^\beta\right) \nabla_\alpha r_\beta\end{aligned} \tag{2.13}$$

We observe that exactly κ with the opposite sign is contained in the integral 2.10. Similarly, the relationship between the determinants γ and g can also be used to rewrite h as the product of the determinant σ of the induced metric on S_t and the lapse function N:

$$\sqrt{|h|} = N\sqrt{\sigma} \tag{2.14}$$

These findings put us in a position to recast $\mathcal{S}$'s total boundary term as a surface integral on S_t:

$$\mathcal{S}_B = -\frac{1}{8\pi} \int_{t_1}^{t_2} \mathrm{d}t \oint_{S_t} \kappa N \sqrt{\sigma} \mathrm{d}^2 x \tag{2.15}$$

It is possible to rewrite the nondynamical term $\mathcal{S}_0$ in terms of κ_0, which is the extrinsic curvature of S_t contained in flat space. Consequently, in $3+1$ formalism, the gravitational action becomes

$$\begin{aligned}\mathcal{S}_G = \frac{1}{16\pi} \int_{t_1}^{t_2} \mathrm{d}t \Bigg[&\int_{\Sigma_t} \left(R - K^2 + K^{ij} K_{ij}\right) N \sqrt{\gamma} \mathrm{d}^3 x \\ &-2 \oint_{S_t} \left(\kappa - \kappa_0\right) N \sqrt{\sigma} \mathrm{d}^2 x \Bigg]\end{aligned} \tag{2.16}$$

2.2 The Hamiltonian formalism

Now that most of the theory's mathematical foundation has been established, the opportunity to work on the Hamiltonian formalism dissertation has presented itself. In particular, we will investigate the action $\mathcal{S} = \mathcal{S}_G$ and ignore the matter contribution $\mathcal{S}_M$, concentrating on the vacuum situation. We will now conceal the immaterial multiplicative constant $(16\pi)^{-1}$ found in action 2.16 in accordance with the ADM notation. The first basic finding is that $\mathcal{S}$ is dependent on the shift and lapse functions N, N^i, $\gamma_{ij}, \dot{\gamma}_{ij}$, and their spatial derivatives. Given that the action integral does not contain the time derivatives of N, N^i, the lapse and shift functions,

They are not included in the collection of dynamic variables, even though they are four configuration variables. We will, in fact, demonstrate that N and N^i function as four Lagrange multipliers, each of which generates a constraint equation. Let's indicate the gravitational Lagrangian with L:

$$L = \int_{\Sigma_t} \left(R^3 - K^2 + K^{ij}K_{ij}\right) N\sqrt{\gamma}\mathrm{d}^3x - 2\oint_{S_t} \left(\kappa - \kappa_0\right) N\sqrt{\sigma}\mathrm{d}^2x \tag{2.17}$$

The first integral is the volume part of L, which we label as L_0. The corresponding Lagrangian density is:

$$\mathcal{L}_0 = \left(R^3 - K^2 + K^{ij}K_{ij}\right) N\sqrt{\gamma} \tag{2.18}$$

The partial derivative of the Lagrangian with regard to $\dot{q}$ represents the canonically conjugate momentum p for each configuration variable q in Hamiltonian mechanics. In the same way, π, the canonical momentum density, is defined as

$$\pi \doteq \frac{\partial \mathcal{L}}{\partial \dot{q}} \tag{2.19}$$

The Hamiltonian density $\mathcal{H}$ is then recovered by performing the Legendre transformation of $\mathcal{L}$, with π as the dual variables:

$$\mathcal{H} \doteq \sum_q \pi\dot{q} - \mathcal{L}$$

Due to the aforementioned absence of $\dot{N}$ and $\dot{N}^i$ in 2.17, the corresponding momenta π_N and π_{N^i} vanish:

$$\pi_N \doteq \frac{\partial \mathcal{L}}{\partial \dot{N}} = 0 \quad \pi_{N^i} \doteq \frac{\partial \mathcal{L}}{\partial \dot{N}^i} = 0 \tag{2.20}$$

Therefore, we are left with the six independent momenta π^{ij} conjugate to the components of γ_{ij} :

$$\pi^{ij} \doteq \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ij}} \tag{2.21}$$

We first note that the boundary term of the Lagrangian (2.17) is independent of the time derivative $\dot{\gamma}_{ij}$ before attempting to determine the explicit expression of π^{ij}. As a result, we simply need to assess the subsequent partial derivatives.

$$\frac{\partial R}{\partial \dot{\gamma}_{ij}} = 0 \quad \frac{\partial K_{rs}}{\partial \dot{\gamma}_{ij}} = -\frac{1}{2N}\delta^i{}_r \delta^j{}_s \tag{2.22}$$

which follow from the absence of $\dot{\gamma}_{ij}$ in the three-dimensional scalar curvature R and from the explicit form of K_{ij}, given by equation 3.57 . Combining these results, we obtain:

$$\begin{aligned}\pi^{ij} &= -\frac{\sqrt{\gamma}}{2}\left(\gamma^{rk}\gamma^{sl} - \gamma^{rs}\gamma^{kl}\right)\left(\delta^i{}_r \delta^j{}_s K_{kl} + \delta^i{}_k \delta^j{}_l K_{rs}\right) \\ &= -\frac{\sqrt{\gamma}}{2}\left(2K^{ij} - 2\gamma^{ij}K\right) = \sqrt{\gamma}\left(K\gamma^{ij} - K^{ij}\right)\end{aligned} \tag{2.23}$$

Observe that when $\sqrt{\gamma}^W$ enters the expression with $W = 1$, π^{ij} is a contravariant tensor density of weight 1. By using the metric γ_{ij} to reduce the indices of π^{ij}, the covariant version π_{ij} can be obtained. The momenta π^{ij} are given in another equivalent form in the 1962 work by Arnowitt, Deser, and Misner [5]. In our case, this form comes from equation 2.17 and the connection $\sqrt{-g} = N\sqrt{\gamma}$:

$$\pi^{ij} = \sqrt{-g}\left({}^4\Gamma^0{}_{pq} - {}^4\Gamma^0{}_{kl}\gamma^{kl}\gamma_{pq}\right)\gamma^{ip}\gamma^{jq} \tag{2.24}$$

We may also dispense with K^{ij} and K and raising the indices:

$$\pi^{ij} = \frac{\sqrt{\gamma}}{2N}\left[2\gamma^{ij}D_k N^k - D^i N^j - D^j N^i + \left(\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl}\right)\dot{\gamma}_{kl}\right] \tag{2.25}$$

On the other hand, we will rewrite the extrinsic curvature tensor and $\dot{\gamma}_{ij}$ as functions of γ_{ij} and π_{ij} since the Hamiltonian is a functional of the configuration variables and their conjugate momenta. Let's calculate the trace of π^{ij} to do this:

$$\pi \doteq \gamma_{ij}\pi^{ij} = 2\sqrt{\gamma}K \tag{2.26}$$

We then combine 2.23 and 2.26 to obtain the desired inversion:

$$K^{ij} = \frac{1}{2\sqrt{\gamma}} \left(\pi\gamma^{ij} - 2\pi^{ij}\right) \tag{2.27}$$

$$K = \frac{\pi}{2\sqrt{\gamma}} \tag{2.28}$$

$$\dot{\gamma}_{ij} = D_i N_j + D_j N_i - \frac{N}{\sqrt{\gamma}} \left(\pi\gamma_{ij} - 2\pi_{ij}\right) \tag{2.29}$$

This allows us to rewrite the volume part $\mathcal{L}_0$ of the Lagrangian density as a function of the canonical variables:

$$\mathcal{L}_0 = N\sqrt{\gamma}R + \frac{N}{\sqrt{\gamma}} \left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) \tag{2.30}$$

We denote by $\mathcal{H}_0$ the Hamiltonian density corresponding to $\mathcal{L}_0$, namely $\mathcal{H}_0 \doteq \pi^{ij}\dot{\gamma}_{ij} - \mathcal{L}_0$. By means of equations 2.29 and 2.30 we can replace $\dot{\gamma}_{ij}$ and $\mathcal{L}_0$, thus arriving at

$$\begin{aligned} \mathcal{H}_0 &= 2\pi^{ij} D_i N_j - N\sqrt{\gamma}R + \frac{N}{\sqrt{\gamma}} \left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2}\right) \\ &= 2D_i \left(\pi^{ij} N_j\right) - 2N_j D_i \pi^{ij} - N\sqrt{\gamma}R + \frac{N}{\sqrt{\gamma}} \left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2}\right) \end{aligned} \tag{2.31}$$

where the covariant derivative of a tensor density, defined in Appendix (section A.1.2), is denoted by $D_i\pi^{ij}$. The integral of $\mathcal{H}_0$ over Σ_t and the contribution of $\kappa - \kappa_0$, which was computed in the previous section, are combined to derive the entire Hamiltonian H.

$$H = \int_{\Sigma_t} \mathcal{H}_0 \, \mathrm{d}^3x + 2\oint_{S_t} \left(\kappa - \kappa_0\right) N\sqrt{\sigma}\mathrm{d}^2x \tag{2.32}$$

Let H_Σ and H_S denote respectively the volume and boundary parts of H, such that $H \doteq H_\Sigma + H_S$. Since the divergence $2D_i\left(\pi^{ij}N_j\right)$ contained in $\mathcal{H}_0$ gives rise to a surface integral, it must be added to H_S, leaving only the true volume
terms in H_Σ :

$$H_\Sigma = \int_{\Sigma_t} \left[-2N_j D_i \pi^{ij} - N\sqrt{\gamma}R + \frac{N}{\sqrt{\gamma}} \left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2}\right)\right] \mathrm{d}^3x \tag{2.33}$$

$$H_S = 2\oint_{S_t} \left[N\left(\kappa - \kappa_0\right) + N_i \frac{\pi^{ij}}{\sqrt{\gamma}} r_j\right] \sqrt{\sigma}\mathrm{d}^2x \tag{2.34}$$

Following the ADM notation, we shall rewrite H_Σ to emphasize the role of N and N^i. If we define the quantities

$$\begin{aligned} R^0 &= -\sqrt{\gamma}R - \frac{1}{\sqrt{\gamma}}\left(\frac{\pi^2}{2} - \pi^{ij}\pi_{ij}\right) \\ R^i &= -2D_j\pi^{ij} \end{aligned} \tag{2.36}$$

we immediately see that the volume term takes on the simple form:

$$H_\Sigma = \int_{\Sigma_t} \left[NR^0 + N_i R^i\right] \mathrm{d}^3x \tag{2.37}$$

or equivalently

$$H_\Sigma = \int_{\Sigma_t} N_\mu R^\mu \mathrm{d}^3x \tag{2.38}$$

when the notation $N = N_0$ was chosen. It appears from the strange equation of H_Σ that the shift and lapse functions act as Lagrangian multipliers. We will demonstrate this and the identical vanishing of H_Σ in the ensuing sections. In fact, four constraint equations that require $\mathcal{S}_G$ to be stationary lead to the requirement that R^0 and R^i be zero.

2.3 Parametric form of the canonical equations

As we continue our study, we will first introduce the idea of the canonical equations' parametric form [4]. To keep things simple, let's look at how a system with a finite number M of degrees of freedom acts:

$$\mathcal{S} = \int_{t_1}^{t_2} \mathrm{d}t L(q,\dot{q},t) = \int_{t_1}^{t_2} \mathrm{d}t \left(\sum_{k=1}^{M} p_k\dot{q}_k - H(p,q,t)\right) \tag{2.39}$$

Since the time t is the only coordinate for which there is no definition of a conjugate momentum, it is distinguished from the other configuration variables of the system. To overcome this asymmetry, a new arbitrary parameter τ can be introduced, allowing t and its conjugate momentum p_t to be promoted to the set of dynamical variables. The so-called action in the parameterized form can be obtained by directly substituting 2.39 with the notational change $t = q_{M+1}$ and letting the configuration variables $\{q_k\}_{k=1}^{M+1}$ become functions of τ.

$$\tilde{\mathcal{S}} = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \tilde{L}\left(q_1, \ldots, q_{M+1}; q_1', \ldots, q_{M+1}'\right) \tag{2.40}$$

where the derivative with respect to τ is denoted by a prime. The modified Lagrangian $\tilde{L}$ is related to L through the equation

$$\tilde{L}\left(q_1, \ldots, q_{M+1}; q_1', \ldots, q_{M+1}'\right) = L\left(q_1, \ldots, q_{M+1}; \frac{q_1'}{q_{M+1}'}, \ldots, \frac{q_M'}{q_{M+1}'}\right) q_{M+1}'$$

Thus, using the conventional technique, the momentum $p_t = p_{M+1}$ associated with the time $t = q_{M+1}$ can be defined. It turns out to be just minus the Hamiltonian H:

$$\begin{aligned} p_{M+1} &\doteq \frac{\partial \tilde{L}}{\partial q_{M+1}'} = L - \left(\sum_{k=1}^{M} \frac{\partial L}{\partial \dot{q}_k} \frac{q_k'}{\left(q_{M+1}'\right)^2}\right) q_{M+1}' \\ &= L - \sum_{k=1}^{M} p_k \dot{q}_k = -H \end{aligned} \tag{2.41}$$

As a result, the new $2M + 2$-dimensional phase space contains q_{M+1} and p_{M+1}. The Lagrangian $\tilde{L}$ has an interesting quality that we will now concentrate on it is a homogeneous function of the first order in the variables $q_1', \ldots, q_{M+1}'$. If we calculate $\tilde{L}$'s partial derivatives in relation to q_k', that is

$$\frac{\partial \tilde{L}}{\partial q_k'} = \frac{\partial L}{\partial \dot{q}_k} \quad \frac{\partial \tilde{L}}{\partial q_{M+1}'} = \tilde{L} - \dot{q}_k \frac{\partial L}{\partial \dot{q}_k}$$

then we see that the following relation holds:

$$\sum_{k=1}^{M+1} \frac{\partial \tilde{L}}{\partial q_k'} q_k' = \tilde{L} \tag{2.42}$$

Euler's theorem on homogeneous functions can then be used to demonstrate our point. We can demonstrate that the action in parameterized form becomes after substituting the $M + 1$ momenta p_k for the partial derivatives in equation 2.42.

$$\tilde{\mathcal{S}} = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \left(\sum_{k=1}^{M+1} p_k q_k' \right) \tag{2.43}$$

while the Hamiltonian $\tilde{H}$ of the extended system vanishes identically:

$$\tilde{H} \doteq \sum_{k=1}^{M+1} p_k q_k' - \tilde{L} = 0 \tag{2.44}$$

This remarkable characteristic, which the volume term H_Σ (equation 2.38) falls into, inspires our digression on the parameterized form. Specifically, we may use the Lagrangian multiplier method to recast the parameterized action 2.43 in an insightful manner. Given that p_{M+1} is constrained by the equation $p_{M+1} = -H$ (2.41), a relationship between the $M+1$ conjugate momenta must exist that compromises the independence of the canonical variables. This constraint can be expressed clearly by employing an auxiliary function in the action integral.

$$C\left(q_1, \ldots, q_{M+1}; p_1, \ldots, p_{M+1}\right) = p_{M+1} + H \tag{2.45}$$

and a Lagrangian multiplier $\lambda = \lambda(\tau)$, which remains unspecified due to the arbitrariness of τ. Hence, the parameterized action becomes

$$\tilde{\mathcal{S}} = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \left(\sum_{k=1}^{M+1} p_k q_k' - \lambda C \right) \tag{2.46}$$

The constraint equation $C = 0$ (corresponding to the identity 2.41) and the $M+1$ canonical equations of motion are obtained, respectively, from independent variations of λ and q_k. This indicates that action 2.46 preserves all of the original system's informational content. Furthermore, regardless of the nature of L, the extended system is conservative since $\tilde{L}$ and C do not directly depend on τ. By introducing four new external parameters τ^μ and the same number of configuration variables $q^{M+1+\mu} = x^\mu\left(\tau^\alpha\right)$, together with their corresponding momenta $p_{M+1+\mu}$, the above procedure may be extended to the case of a field theory with M degrees of freedom. To connect $p_{M+1}, \ldots, p_{M+4}$ with the Hamiltonian and momentum densities of the field, the extra four constraint equations $C^\mu = 0$ and Lagrangian multipliers $\lambda^\mu\left(\tau^\alpha\right)$ are needed. The parameter formalism is relevant because it may be used to "reduce" the parameterized action $\tilde{S}$ to its canonical form, thus reversing this process. This involves inserting the constraint equations into $\tilde{S}$ after specifying coordinate conditions, which fix the arbitrary parameters τ^μ. The system's intrinsic degrees of freedom will then become visible due to the lessened action. In fact, the Lagrangian's volume term L_Σ (derived from the Hamiltonian H_Σ, equation 2.38) manifests as a parameterized form Lagrangian, whereby N_μ and R^μ serve as Lagrange multipliers and restrictions, respectively.

$$L_\Sigma \doteq \pi^{ij}\dot{\gamma}_{ij} - H_\Sigma = \pi^{ij}\dot{\gamma}_{ij} - N_\mu R^\mu \tag{2.47}$$

By implication, we aim to prove that N_μ truly behaves as Lagrange multipliers when the action variation is considered.

2.3.1 Variation of the lapse function

Let us return to the gravitational action 2.16 (ignoring the constant factor $(16\pi)^{-1}$), here reproduced for convenience:

$$\mathcal{S}_G = \int_{t_1}^{t_2} \mathrm{d}t \left[\int_{\Sigma_t} \left(R - K^2 + K^{ij}K_{ij}\right) N\sqrt{\gamma}\mathrm{d}^3x - 2\oint_{S_t} (\kappa - \kappa_0)\, N\sqrt{\sigma}\mathrm{d}^2x \right]$$

We require the variation δN to vanish on the boundary, keeping in mind the definitions from section 1.1. This suggests that since there are no derivatives of N, we can safely ignore the surface integral over S_t. The volume term variation is simple to understand:

$$\begin{aligned} \frac{\delta \mathcal{S}}{\delta N} &= \sqrt{\gamma}\left(R - K^2 + K^{ij}K_{ij}\right) + N\sqrt{\gamma}\left(-\frac{2}{N}\right)\left(-K^2 + K^{ij}K_{ij}\right) \\ &= \sqrt{\gamma}\left(R + K^2 - K^{ij}K_{ij}\right) \end{aligned} \tag{2.48}$$

Next, we extreme the activity by setting the value of equation 2.48 to zero. This provides the vacuum case Hamiltonian constraint, which is when $E = 0$. The relation is obtained by substituting the conjugate momenta (using 2.26 and 2.27) for K_{ij}.

$$R^0 = 0 \tag{2.49}$$

which shows that N is truly a Lagrangian multiplier, as R^0 is not affected by the variation of N.

2.3.2 Variation of the shift functions

Because of the covariant derivatives $D_i N_j$ in the extrinsic curvature tensor, there are some nuances in the subsequent variation with regard to the shift functions. Thus, we explicitly examine a smooth one-parameter family $(N_i)_\lambda$ of shift functions to start this proof. Upon computing the derivative of the volume term of $\mathcal{S}_G$ with respect to λ, we derive the following:

$$\begin{aligned}
\left.\frac{\mathrm{d}S}{\mathrm{d}\lambda}\right|_{\lambda=0} &= \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \left[R - K^2 + K^{ij}K_{ij}\right] N\sqrt{\gamma}\mathrm{d}^3x \\
&= 2\int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} N\sqrt{\gamma}\left(-K\gamma^{ij} + K^{ij}\right) \left.\frac{\mathrm{d}K_{ij}}{\mathrm{d}\lambda}\right|_{\lambda=0} \mathrm{d}^3x \\
&= -2\int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \pi^{ij} D_i \left(\left.\frac{\mathrm{d}N_j}{\mathrm{d}\lambda}\right|_{\lambda=0}\right) \mathrm{d}^3x
\end{aligned}$$

Using the symmetry of K_{ij}, we identified the equation 2.23 of the conjugate momenta π^{ij} in the passage from the second to the third line. We rewrite the integrand after recalling the notion of the covariant derivative of a tensor density.

$$\pi^{ij} D_i \left(\left.\frac{\mathrm{d}N_j}{\mathrm{d}\lambda}\right|_{\lambda=0}\right) = D_i \left(\pi^{ij}\left.\frac{\mathrm{d}N_j}{\mathrm{d}\lambda}\right|_{\lambda=0}\right) - \left(D_i\pi^{ij}\right)\left.\frac{\mathrm{d}N_j}{\mathrm{d}\lambda}\right|_{\lambda=0}$$

Upon substitution in the integral (resorting to the definition of δN_i), the divergence can be reduced to a boundary term:

$$\begin{aligned}
\left.\frac{\mathrm{d}S}{\mathrm{d}\lambda}\right|_{\lambda=0} =& 2\int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \left(D_i\pi^{ij}\right)\delta N_j \ \mathrm{d}^3x \\
&- 2\int_{t_1}^{t_2} \mathrm{d}t \oint_{S_t} \frac{\pi^{ij}}{\sqrt{\gamma}} r_i \delta N_j \sqrt{\sigma}\mathrm{d}^2x
\end{aligned}$$

The surface integral vanishes as $\delta N_i = 0$ on the boundary. Hence, we can discard this term and demand $\mathcal{S}_G$ to be stationary, using the notion of functional derivative 1.7

$$\frac{\delta\mathcal{S}}{\delta N_j} = 2D_i\pi^{ij} = 0$$

thereby leading to the three constraint equations:

$$R^i = 0 \tag{2.50}$$

These match the $p_i = 0$ momentum restrictions 3.53 in the vacuum. They make up the four constraint equations of the system stated before, together with 2.49. This outcome completes the evidence and suggests that when the restrictions are applied, the volume term H_Σ(2.38) vanishes in the same way:

$$H_\Sigma = 0 \tag{2.51}$$

It should be noted that these requirements do not entail the disappearance of H_S. The relationship between the value of H_S in asymptotically flat spacetime and the idea of the system's energy—which is, generally speaking, distinct from zero—will actually be covered in the section that follows.

2.4 Hamilton's equations

The twelve Hamilton equations that represent the time development of the canonical variables γ_{ij} and π^{ij} may now be found:

$$\dot{\gamma}_{ij} = \frac{\delta H}{\delta \pi^{ij}} \tag{2.52}$$

$$\dot{\pi}^{ij} = -\frac{\delta H}{\delta \gamma_{ij}} \tag{2.53}$$

To this end, we rewrite the total gravitational action $\mathcal{S}_G$ (equation 2.16) in terms of the canonical variables, preserving only the term $\pi^{ij}\dot{\gamma}_{ij}$:

$$\begin{aligned}
\mathcal{S}_G &= \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \left(\pi^{ij}\dot{\gamma}_{ij} - \mathcal{H}\right) \mathrm{d}^3x \\
&= \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \left[\pi^{ij}\dot{\gamma}_{ij} + 2N_j D_i \pi^{ij} + N\sqrt{\gamma}R - \frac{N}{\sqrt{\gamma}}\left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2}\right)\right] \mathrm{d}^3x \\
&\quad - 2\int_{t_1}^{t_2} \mathrm{d}t \oint_{S_t} \left[N\left(\kappa - \kappa_0\right) + N_i \frac{\pi^{ij}}{\sqrt{\gamma}} r_j\right] \sqrt{\sigma}\mathrm{d}^2x
\end{aligned} \tag{2.54}$$

We require that the variation of the configuration variables vanishes on the boundary $S_t = \partial\Sigma_t$, namely:

$$\delta N|_{S_t} = \delta N_i|_{S_t} = \delta\gamma_{ij}|_{S_t} = 0 \tag{2.55}$$

The conjugate momenta, which are handled as independent variables, won't be constrained, though. Accordingly, the variation of H with respect to N and N_i is identical to the variation of the Lagrangian carried out in sections 2.3.1 and 2.3.2, resulting in the four constraint equations 2.49 and 2.50, up to an inconsequential overall sign. Rather, the variants of γ_{ij} and π^{ij} necessitate a more involved analysis, which we continue in turn in the ensuing paragraphs.

2.4.1 Variation of the conjugate momenta

Starting with the second set of equations 2.53, we recover them by setting the variation of $\mathcal{S}_G$ with respect to π^{ij} to zero. Specifically, we start by looking at the second term from the 2.54 volume integral:

$$\begin{aligned}\mathcal{P} &\doteq \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \left(2N_j D_i \pi^{ij}\right) \mathrm{d}^3x \\ &= 2\int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \left[D_i\left(N_j\pi^{ij}\right) - \pi^{ij} D_i N_j\right] \mathrm{d}^3x\end{aligned} \tag{2.56}$$

We now transform the total covariant derivative in a divergence and then apply Stokes' theorem:

$$\mathcal{P} = 2\int_{t_1}^{t_2} \mathrm{d}t \oint_{S_t} N_i \frac{\pi^{ij}}{\sqrt{\gamma}} r_j \sqrt{\sigma} \mathrm{d}^2x - 2\int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \pi^{ij} D_i N_j \ \mathrm{d}^3x$$

The final portion of the boundary term of $\mathcal{S}_G$ is eliminated by the first integral of $\mathcal{P}$, leaving only a surface integral that is independent of π^{ij}. As a result, $\mathcal{S}_G$ variation decreases to:

$$\delta_\pi \mathcal{S}_G = \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \delta\pi^{ij} \left[\dot{\gamma}_{ij} - 2D_i N_j - \frac{N}{\sqrt{\gamma}}\left(2\pi_{ij} - \pi\gamma_{ij}\right)\right] \mathrm{d}^3x$$

Because of the stationarity of $\mathcal{S}_G$ and the arbitrariness of $\delta\pi^{ij}$, the argument of the first integral vanishes. In order to do the functional derivative and suppress $\delta\pi^{ij}$, we will replace $2D_iN_j$ with its symmetrization, $D_iN_j + D_jN_i$. This leads us to the relationship:

$$\frac{\delta \mathcal{S}_G}{\delta \pi^{ij}} = \dot{\gamma}_{ij} - D_i N_j - D_j N_i - \frac{N}{\sqrt{\gamma}} \left(2\pi_{ij} - \pi \gamma_{ij}\right) = 0 \tag{2.57}$$

or equivalently

$$\dot{\gamma}_{ij} = \frac{\delta H}{\delta \pi^{ij}} = D_i N_j + D_j N_i - \frac{N}{\sqrt{\gamma}} \left(2\pi_{ij} - \pi \gamma_{ij}\right) \tag{2.58}$$

Equation 2.58 is obtained by substituting the extrinsic curvature K_{ij} for π^{ij} and its trace.

$$\dot{\gamma}_{ij} = D_i N_j + D_j N_i - 2N K_{ij}$$

As we can see, the change of π^{ij} yields the relation 2.57, which uses the shift and lapse functions to fix the temporal evolution of the three-dimensional metric.

2.4.2 Variation of the metric

Since this variation entails more than the previous one, we will move cautiously. First, we look at how the Hamiltonian density $\mathcal{H}_\Sigma$ varies in terms of the volume term:

$$\delta_\gamma \mathcal{H}_\Sigma = \delta_\gamma \left[-2N_j D_i \pi^{ij} - N\sqrt{\gamma} R + \frac{N}{\sqrt{\gamma}} \left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2}\right)\right] \tag{2.59}$$

Note that Jacobi's formula yields the variation of γ.

$$\delta \gamma = \gamma \gamma^{ab} \delta \gamma_{ab} \tag{2.60}$$

whereas it is simple to calculate the last term included in parenthesis:

$$\delta_\gamma \left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2}\right) = \left(2\pi^a{}_i \pi^{ib} - \pi \pi^{ab}\right) \delta \gamma_{ab} \tag{2.61}$$

The relations given in sections A.2.2 and A.2.3 of the Appendix, which we previously met in Chapter 1, are required for the variation of $N\sqrt{\gamma}R$. By using G^{ab} to represent the three-dimensional contravariant Einstein tensor, we may get

$$\delta[-N\sqrt{\gamma}R] = N\sqrt{\gamma}\left(R^{ab} - \frac{1}{2}\gamma^{ab} R\right)\delta\gamma_{ab} - N\sqrt{\gamma} D_a \delta V^a$$

$$= N\sqrt{\gamma}G^{ab}\delta\gamma_{ab} + \sqrt{\gamma}\delta V^a D_a N - \sqrt{\gamma}D_a\left(N\delta V^a\right) \tag{2.62}$$

In order to simplify the calculations, we present the two variables.

$$\delta\mathcal{B}_1 \doteq -2\delta_\gamma D_i\left(\pi^{ij}N_j\right)$$
$$\delta\mathcal{B}_2 \doteq -D_a\left(2\pi^{ab}N^c\delta\gamma_{bc} - \pi^{bc}N^a\delta\gamma_{bc}\right)$$

As a result, we can now express the variation of the 2.59 first term as

$$\delta_\gamma\left(-2N_jD_i\pi^{ij}\right) = 2\delta_\gamma\left(\pi^{ij}D_iN_j\right) + \delta\mathcal{B}_1 \tag{2.63}$$

We can express $2\delta_\gamma\left(\pi^{ij}D_iN_j\right)$ in terms of $\delta\gamma_{ij}$ and $\delta\mathcal{B}_2$ by using the relation A.21 from the Appendix:

$$\begin{aligned}\delta_\gamma\left(2\pi^{ij}D_iN_j\right) &= -2\pi^{ij}N_a\delta\Gamma^a{}_{ij} = -2\pi^{ij}N_a\left(\gamma^{ab}D_i\delta\gamma_{jb} - \frac{1}{2}\gamma^{ab}D_b\delta\gamma_{ij}\right)\\ &= D_a\left(2\pi^{ab}N^c - \pi^{bc}N^a\right)\delta\gamma_{bc} + \delta\mathcal{B}_2\end{aligned} \tag{2.64}$$

The preceding equation becomes $D_i\pi^{ij} = 0$ because to the three restrictions 2.50.

$$\delta_\gamma\left(2\pi^{ij}D_iN_j\right) = \left(2\pi^{ab}D_aN^c - \pi^{bc}D_aN^a\right)\delta\gamma_{bc} + \delta\mathcal{B}_2 \tag{2.65}$$

Equations 2.60 to 2.65, when combined, yield the variation $\delta_\gamma\mathcal{H}_\Sigma$.

$$\begin{aligned}\delta_\gamma\mathcal{H}_\Sigma = &\left(2\pi^{ab}D_aN^c - \pi^{bc}D_aN^a\right)\delta\gamma_{bc} + N\sqrt{\gamma}G^{ab}\delta\gamma_{ab}\\ &+ \frac{N}{\sqrt{\gamma}}\left[-\frac{1}{2}\left(\pi_{cd}\pi^{cd} - \frac{\pi^2}{2}\right)\gamma^{ab} + 2\pi^a{}_c\pi^{bc} - \pi\pi^{ab}\right]\delta\gamma_{ab}\\ &+ \delta\mathcal{B}_1 + \delta\mathcal{B}_2 + \sqrt{\gamma}\delta V^aD_aN\end{aligned} \tag{2.66}$$

By integrating $\delta_\gamma\mathcal{H}_\Sigma$ over the hypersurface Σ_t, the variation of H_Σ is obtained. Specifically, the final three terms in equation 2.66 result in a surface integral, which we represent by δB. But since there is no derivative of $\delta\gamma_{ab}$ and $\delta\gamma_{ab}|_{S_t} = 0$ in the integral of $\delta\mathcal{B}_1 + \delta\mathcal{B}_2$, the sole non-vanishing boundary contribution is provided by

$$\delta B \doteq \int_{\Sigma_t}\left(\delta\mathcal{B}_1 + \delta\mathcal{B}_2 + \sqrt{\gamma}\delta V^aD_aN\right)\mathrm{d}^3x = -\oint_{S_t}N\delta V^ar_a\sqrt{\sigma}\mathrm{d}^2x \tag{2.67}$$

Using the logic from section 2.4.1 and the relation 6.25 applied to the three-dimensional case simplifies the contraction $\delta V^a r_a$ in 2.67.

$$\delta V^a r_a = -\sigma^{bc}\delta\gamma_{bc,a}r^a = -2\sigma^{bc}D_b r_c = 2\kappa \tag{2.68}$$

where the induced metric on S_t extended to Σ_t is $\sigma^{ab} = r^a r^b + \gamma^{ab}$. Equation 2.34's variation of H_S shows that $\delta B + \delta_\gamma H_S$ vanishes when compared to 2.67.

$$\delta_\gamma H_S = 2\oint_{S_t} N\delta\kappa\sqrt{\sigma}\mathrm{d}^2x = -\delta B \tag{2.69}$$

This suggests that we may safely ignore H_S and that the variation of H is limited to the remaining terms of H_Σ. Now, using the relation A.26 from the Appendix, we can easily recast the product $\delta V^a D_a N$ found in the first line of equation 2.66:

$$\begin{aligned}\delta V^a D_a N &= \gamma^{ab}\gamma^{cd}\left(D_a\delta\gamma_{bc} - D_c\delta\gamma_{ab}\right)D_d N\\ &= D_a\left[\left(\gamma^{ab}D^c N - \gamma^{bc}D^a N\right)\delta\gamma_{bc}\right] - \left(D^a D^b N - \gamma^{ab}D_c D^c N\right)\delta\gamma_{ab}\end{aligned}$$

A further division by components enables the displacement of the divergence resulting from the constraint $\delta\gamma_{ab}|_{S_t} = 0$. Consequently, we obtain

$$\delta V^a D_a N = -\left(D^a D^b N - \gamma^{ab}D_c D^c N\right)\delta\gamma_{ab} \tag{2.70}$$

In the end, the combination of equations 2.66 to 2.70 results in $2\pi^{bc}N^a$ following the symmetrization of the indices a and b.

$$\begin{aligned}\frac{\delta\mathcal{H}}{\delta\gamma_{ab}} =& D_c\left(\pi^{ac}N^b + \pi^{bc}N^a - \pi^{ab}N^c\right) + N\sqrt{\gamma}\left(R^{ab} - \frac{1}{2}\gamma^{ab}R\right)\\ & - \sqrt{\gamma}\left(D^a D^b N - \gamma^{ab}D_c D^c N\right) - \frac{N}{2\sqrt{\gamma}}\left(\pi_{cd}\pi^{cd} - \frac{1}{2}\pi^2\right)\gamma^{ab}\\ & + \frac{2N}{\sqrt{\gamma}}\left(\pi^a{}_c\pi^{bc} - \frac{1}{2}\pi\pi^{ab}\right)\end{aligned} \tag{2.71}$$

By definition, requiring the action to remain stationary recovers the equations of motion. When the product $\pi^{ij}\dot{\gamma}_{ij}$ is integrated by parts with regard to the time coordinate, the outcome is

$$\begin{aligned}\delta\mathcal{S}_G &= \delta_\gamma \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \left(\pi^{ij}\dot{\gamma}_{ij} - \mathcal{H}\right) \mathrm{d}^3x \\ &= -\int_{t_1}^{t_2} \mathrm{d}t \left[\int_{\Sigma_t} \delta\gamma_{ij} \left(\dot{\pi}^{ij} + \frac{\delta\mathcal{H}}{\delta\gamma_{ij}}\right) \mathrm{d}^3x\right] = 0 \end{aligned} \tag{2.72}$$

The second set of Hamilton equations is provided by this variant because of the arbitrary nature of $\delta\gamma_{ij}$:

$$\dot{\pi}^{ij} = -\frac{\delta\mathcal{H}}{\delta\gamma_{ij}} \tag{2.73}$$

The important findings from the previous chapters, specifically the explicit form of 2.73 and the four constraints 2.49, 2.50, and Hamilton's equations 2.58 for $\dot{\gamma}_{ij}$, should be summarized as follows:

$$R^0 = -\sqrt{\gamma}R - \frac{1}{\sqrt{\gamma}}\left(\frac{\pi^2}{2} - \pi^{ij}\pi_{ij}\right) = 0 \tag{2.74}$$

$$R^i = -2D_j\pi^{ij} = 0 \tag{2.75}$$

$$\dot{\gamma}_{ij} = D_iN_j + D_jN_i - \frac{N}{\sqrt{\gamma}}\left(2\pi_{ij} - \pi\gamma_{ij}\right) \tag{2.76}$$

$$\begin{aligned}\dot{\pi}^{ij} = &-N\sqrt{\gamma}\left(R^{ij} - \frac{1}{2}\gamma^{ij}R\right) + \frac{N}{2\sqrt{\gamma}}\left(\pi_{cd}\pi^{cd} - \frac{\pi^2}{2}\right)\gamma^{ij} \\ &-\frac{2N}{\sqrt{\gamma}}\left(\pi^{ic}\pi_c{}^j - \frac{1}{2}\pi\pi^{ij}\right) + \sqrt{\gamma}\left(D^iD^jN - \gamma^{ij}D_cD^cN\right) \\ &+D_c\left(\pi^{ij}N^c\right) - \pi^{ic}D_cN^j - \pi^{jc}D_cN^i \end{aligned} \tag{2.77}$$

2.5 Poisson brackets

Given two differentiable functions $f\left(q_k, p_k, t\right)$ and $g\left(q_k, p_k, t\right)$ in classical Hamiltonian mechanicsThe Poisson bracket of f and g, of the canonical variables q_k, p_k, with $k \in \{1, \ldots, M\}$, is defined as the function

$$\{f, g\} = \sum_{k=1}^{M}\left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}\right) \tag{2.78}$$

Consequently, the Poisson bracket can be viewed as an anticommutative, bilinear binary operation operating on the space of functions that are dependent on time and phase space. Additionally, it fulfils the equations for any three functions of this kind, $f, g, andh$.

$$
\begin{aligned}
\{fg, h\} &= f\{g, h\} + \{f, h\}g && (2.79)\\
\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} &= 0 && (2.80)
\end{aligned}
$$

known as the Jacobi identity and Leibniz's rule, respectively. Hamilton's equations of motion can be rewritten as follows using this definition:

$$
\begin{aligned}
\dot{q}_k &= \frac{\partial H}{\partial p_k} = \{q_k, H\} && (2.81)\\
\dot{p}_k &= -\frac{\partial H}{\partial q_k} = \{p_k, H\} && (2.82)
\end{aligned}
$$

Therefore, generally speaking, any function $f(q_k, p_k, t)$ has its time evolution determined by

$$
\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\} + \frac{\partial f}{\partial t} \tag{2.83}
$$

By analogy with equation 2.83, the Poisson bracket for a field theory can be defined. Specifically, we obtain the following by taking into account the whole time derivative of a differentiable function $f = f(\gamma_{ij}, \pi^{ij}, t)$.

$$
\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\delta f}{\delta \gamma_{ij}}\dot{\gamma}_{ij} + \frac{\delta f}{\delta \pi^{ij}}\dot{\pi}^{ij} + \frac{\partial f}{\partial t}\\
&= \frac{\delta f}{\delta \gamma_{ij}}\frac{\delta \mathcal{H}}{\delta \pi^{ij}} - \frac{\delta f}{\delta \pi^{ij}}\frac{\delta \mathcal{H}}{\delta \gamma_{ij}} + \frac{\partial f}{\partial t} && (2.84)
\end{aligned}
$$

Thus, if the Poisson bracket is introduced, it can be recast in the well-known form 2.83.

$$
\{f, g\} \doteq \frac{\delta f}{\delta \gamma_{ij}}\frac{\delta g}{\delta \pi^{ij}} - \frac{\delta f}{\delta \pi^{ij}}\frac{\delta g}{\delta \gamma_{ij}} \tag{2.85}
$$

By using this concept, we can determine the system's fundamental Poisson brackets among its canonical variables:

$$
\begin{aligned}
&\{\gamma_{ij}, \gamma_{kl}\} = 0 \\
&\left\{\pi^{ij}, \pi^{kl}\right\} = 0 \\
&\left\{\gamma_{ij}, \pi^{kl}\right\} = \delta_i{}^k \delta_j{}^l
\end{aligned}
\tag{2.86}
$$

They bear a strong resemblance to those that surface in classical mechanics. But since the limitations apply to the canonical coordinates γ_{ij} and π^{ij}.

2.6 General relativity as initial-value problem

The formulation of the constrained initial-value problem, which is commonly employed in numerical general relativity calculations, is reviewed briefly in this section. From this, a set of evolution equations for the gravitational variables is derived, where the source is the classical matter stress tensor. With a shift vector β^i and a general metric expressed in terms of the lapse N [4],

$$
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + \gamma_{ij} \left(dx^i + \beta^i dt\right) \left(dx^j + \beta^j dt\right), \tag{2.87}
$$

where the spatial indices are $i, j = 1, 2, 3$, and the spacetime indices are $\mu, \nu = 0, \cdots, 3$. Σ is a spacelike hypersurface with a fixed t slice. The intrinsic Riemann tensor on Σ and terms involving the extrinsic curvature K_{ij} of Σ contained in spacetime can be separated out from the Riemann tensor. The timelike unit normal to Σ, which we define as n^μ, is

$$
n_\mu = -N \frac{\partial t}{\partial x^\mu} \tag{2.88}
$$

It is possible to define an orthogonal projector that projects into Σ's tangent space.

$$
\gamma^\alpha_\beta = \delta^\alpha_\beta + n^\alpha n_\beta. \tag{2.89}
$$

The custom of using a minus sign to define the extrinsic curvature

$$
K_{\alpha\beta} = -\gamma^\mu_\alpha \gamma^\nu_\beta \nabla_\mu n_\nu \tag{2.90}
$$

There exists a special Levi-Civita connection D_i linked with the induced metric. Consequently, the hypersurface's intrinsic curvature tensor is defined and denoted by R^k_{lij}. Another name for the comparable Ricci scalar is the surface's Gaussian curvature.

Similarly, using n^μ and the orthogonal projector, the stress-energy tensor can be broken down into components tangent to Σ and components normal to Σ.

$$E = T_{\mu\nu} n^\mu n^\nu, \quad p_\alpha = T_{\mu\nu} n^\mu \gamma^\nu_\alpha, \quad S_{\alpha\beta} = T_{\mu\nu} \gamma^\mu_\alpha \gamma^\nu_\beta \tag{2.91}$$

It is customary to create a rescaled timelike normal before expressing the Einstein equations as an initial-value problem.

$$m_\mu = N n_\mu \tag{2.92}$$

which is dt's dual. The new time coordinate has the value $t + \delta t$ if we translate each point on Σ by $m^\mu \delta t$. Then, employing Lie derivatives with respect to time, the Einstein equations (in trace-reversed form) can be expressed as [3].

$$\partial_t h_{ij} = -2NK_{ij} + \nabla_i \beta^j + \nabla_j \beta^i \tag{2.93}$$

$$\partial_t K_{ij} = -D_i D_j N + N \left(R_{ij} + KK_{ij} - 2K_{ik}K^k_j + 4\pi G_N \left((S - E)\gamma_{ij} - 2S_{ij} \right) \right) \tag{2.94}$$

$$\begin{aligned}
R &= \text{Ricci Scalar for 3D spatial dimention} \\
K &= \text{Trace of the extrinsic curvature} \\
K^{ij} &= \text{extrinsic curvature tensor} \\
N &= \text{ lapse function} \\
h &= \text{ determinant of the spatial metric} \\
S &= \text{Trace of momentum density tensor} \\
E &= \text{Energy density}
\end{aligned}$$

together with the constraints

$$R + K^2 - K_{ij}K^{ij} = 16\pi G_N E$$
$$D_j K_i^j - D_i K = 8\pi G_N p_i$$

The Bianchi identities ensure that the restrictions are met on succeeding timeslices if the stress-energy tensor satisfies $\nabla_\mu T^{\mu\nu} = 0$.

In order to find a semiclassical stress tensor $\langle T_{\mu\nu} \rangle$ that we can subsequently put into these equations, we will attempt to define an initial-value problem in the following sections.

2.7 Summary of ADM formalism

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In the following, our goal will be to formulate an initial-value problem to determine a semiclassical stress tensor $\langle T_{\mu\nu} \rangle$ which we can then insert into these equations.M formalism The ADM method decomposes spacetime into space and time, splitting the metric into spatial and temporal components, facilitating a Hamiltonian description of gravity.

$$\mathcal{L}_G = \left[R^3 + K^2 + K^{ij}K_{ij} - 2\nabla_i K - \frac{2}{N}\nabla^i \nabla_i N \right] N\sqrt{h}$$

R^3 = Ricci scalar for 3D spatial dimension

K^2 = Trace of the scalar extrinsic curvature squared

$K^{ij}K_{ij}$ = Trace of the extrinsic curvature tensor squared

N = Lapse function

h = Determinant of the spatial metric

S = Trace of momentum density tensor

E = Energy density

As Ricci scalar in 3+1 dimension has the form of

$$\mathrm{R}^4 = R^3 + K^2 - K_{\mu\nu}K^{\mu\nu} - 2n^\mu \left[\nabla_\alpha, \nabla_\mu\right] n^\alpha$$

Now let us construct the Hamiltonian of 3+1 D spacetime.

$$\mathcal{H} \doteq \sum_q \pi \dot{q} - \mathcal{L}_G$$

Here the conjugate variables are spatial 3D metric h_{ij} and π_{ij} In Hamiltonian mechanics, each configuration variable q is associated with a canonically conjugate momentum p, given by the partial derivative of the Lagrangian with respect to $\dot{q}$. Similarly, the canonical momentum density π^{ij} is defined as

$$\pi^{ij} \doteq \frac{\partial \mathcal{L}_G}{\partial \dot{h}_{ij}}$$

We are now in the position to determine the twelve Hamilton equations that describe the time evolution of the canonical variables

$$\dot{h}_{ij} = \frac{\delta H}{\delta \pi^{ij}}$$
$$\dot{\pi}^{ij} = -\frac{\delta H}{\delta \gamma_{ij}}$$
$$\pi^{ij} = \sqrt{h}\left(K h^{ij} - K^{ij}\right)$$

After long calculations, we end up with these time evaluation equations

$$\partial_t h_{ij} = -2N K_{ij} + \nabla_i \beta^j + \nabla_j \beta^i$$
$$\partial_t K_{ij} = -\nabla_i \nabla_j N + N\left(R_{ij} + K K_{ij} - 2K_{ik} K_j^k + 4\pi G\left((S-E)h_{ij} - 2S_{ij}\right)\right)$$

Chapter 3

Semiclassical expansion

We will find that the Einstein equations are modified by terms up to fourth order in time derivatives due to quantum fluctuations of scalar fields. Treating Einstein's gravity as effective field theory also leads to higher-derivative corrections, where we anticipate terms like R^2 and $R_{\mu\nu}R^{\mu\nu}$ to occur in the effective action, with coefficients to be matched to experiment. Unphysical solutions to field equations are generally introduced by terms of high derivative order. Concurrently, there exist limitations on the physical solutions arising from expanding the equations of motion through a derivative expansion for a more comprehensive theory. The stress-energy tensor's expectation value is expressed as follows: [4, 5, 6, 7].

$$\begin{aligned}
\langle T_{\mu\nu}\rangle \to \langle T_{\mu\nu}\rangle& \\
+\frac{\log M^2}{2(2\pi)^2}\Bigg(g_{\mu\nu}&\left(\frac{1}{8}m^4+\frac{1}{4}\left(\xi-\frac{1}{6}\right)m^2R-\frac{1}{2}\left(\xi^2-\frac{1}{3}\xi+\frac{1}{40}\right)\Box R+\frac{1}{8}\left(\xi-\frac{1}{6}\right)^2R^2\right. \\
&\left.-\frac{1}{720}R_{\lambda\rho}R^{\lambda\rho}+\frac{1}{720}R_{\sigma\tau\lambda\rho}R^{\sigma\tau\lambda\rho}\right)-\frac{1}{2}\left(\xi-\frac{1}{6}\right)m^2R_{\mu\nu}+\frac{1}{2}\left(\xi^2-\frac{1}{3}\xi+\frac{1}{30}\right)R_{;\mu\nu} \\
&-\frac{1}{120}\Box R_{\mu\nu}-\frac{1}{2}\left(\xi-\frac{1}{6}\right)^2RR_{\mu\nu}+\frac{1}{90}R^{\lambda}{}_{\mu}R_{\lambda\nu}-\frac{1}{180}R^{\lambda\rho}R_{\lambda\mu\rho\nu}-\frac{1}{180}R^{\lambda\sigma\tau}{}_{\mu}R_{\lambda\sigma\tau\nu}\Bigg).
\end{aligned} \tag{3.1}$$

We note this shift ambiguity has a vanishing trace in the conformally coupled case, $m^2 = 0$ and $\xi = 1/6$, so it does not change the expression for the trace anomaly.

3.1 Schwarzschild spacetime with Standard coordinates

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1}dr^2 + r^2 d\Omega \tag{3.2}$$

where $d\Omega = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on the two-sphere. Using (r, θ, ϕ) as the coordinates of each hypersurface Σ_t, the three-dimensional metric γ_{ij} becomes

$$\gamma_{ij} = \mathrm{diag}\left[\left(1-\frac{2m}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right] \tag{3.3}$$

We've seen that the Christoffel symbols in terms of the metric are given by

$$\Gamma^m_{ij} = \frac{1}{2}g^{ml}\left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}\right) \tag{3.4}$$

This expression involves calculating the inverse metric tensor g^{ml} and doing a lot of sums to find each Christoffel symbol. Often, an easier way is to exploit the relation between the Christoffel symbols and the geodesic equation. The geodesic equation is (where a dot above a symbol means the derivative with respect to τ):

$$g_{aj}\ddot{x}^j + \left(\partial_i g_{aj} - \frac{1}{2}\partial_a g_{ij}\right)\dot{x}^j\dot{x}^i = 0 \tag{3.5}$$

The following equation is formally equivalent to this:

$$\ddot{x}^m + \Gamma^m{}_{ij}\dot{x}^j\dot{x}^i = 0 \tag{3.6}$$

The method for calculating the Christoffel symbols is to work out the terms above, divide them by g_{aj}, and then compare the result term by term with the following terms. By doing this we are able to read off the $\Gamma^m{}_{ij}$ as the coefficients of $\dot{x}^j\dot{x}^i$. We can use this technique to work out the $\Gamma^m{}_{ij}$ for the Schwarzschild metric, which is

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{3.7}$$

First, take $a = \phi$ in geodesic equation. Since the Schwarzschild metric doesn't depend on $\phi, \partial_\phi g_{ij} = 0$ for all elements. Further, since the Schwarzschild metric is diagonal, $g_{\phi j}$ is restricted to $g_{\phi\phi}$, so the equation becomes

$$g_{\phi\phi}\ddot{\phi} + \partial_i g_{\phi\phi}\dot{\phi}\dot{x}^i = 0 \tag{3.8}$$

Since $g_{\phi\phi} = r^2 \sin^2\theta$ there are 2 non-zero derivatives, so this equation expands to

$$\begin{aligned} r^2\sin^2\theta\ddot{\phi} + 2r\sin^2\theta\dot{\phi}\dot{r} + 2r^2\sin\theta\cos\theta\dot{\phi}\dot{\theta} = 0 \\ \ddot{\phi} + \frac{2}{r}\dot{\phi}\dot{r} + 2\cot\theta\dot{\phi}\dot{\theta} = 0 \end{aligned} \tag{3.9}$$

By comparing this with the previous equation, we can read off the Christoffel symbols to have the following relations:

$$\begin{aligned} \Gamma^\phi_{r\phi} + \Gamma^\phi_{\phi r} &= \frac{2}{r} \\ \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} &= \frac{1}{r} \\ \Gamma^\phi_{\theta\phi} + \Gamma^\phi_{\phi\theta} &= 2\cot\theta \\ \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} &= \cot\theta \end{aligned} \tag{3.10}$$

Here, we've used the symmetry of the Christoffel symbols. Because no other terms appear in the equation, all the other ${\Gamma^\phi}_{ij}$ are zero, so the complete set is where the rows are labelled t, r, θ and ϕ from top to bottom, and the columns the same order from left to right:

$$\Gamma^\phi_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{bmatrix} \tag{3.11}$$

Now consider $a = \theta$ in the geodesic equation. This time, one of the g_{ij} does depend on θ, so we will get a contribution from the $\partial_\theta g_{\phi\phi}$ term. We get

$$\begin{aligned} r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - \frac{1}{2}r^2(2\sin\theta\cos\theta)\dot{\phi}^2 = 0 \\ \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0 \end{aligned} \tag{3.12}$$

Again, we can read off the symbols to get

$$\Gamma^{\theta}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \tag{3.13}$$

For $a = r$ in the geodesic equation. We get

$$\begin{aligned} 0 =& \left(1 - \frac{2GM}{r}\right)^{-1} \ddot{r} - \frac{2GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-2} \dot{r}^2 - \\ & \frac{1}{2}\left[-\frac{2GM}{r^2}\dot{t}^2 - \frac{2GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-2}\dot{r}^2 + 2r\dot{\theta}^2 + 2r\sin^2\theta\dot{\phi}^2\right] \\ 0 =& \ddot{r} + \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)\dot{t}^2 - \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1}\dot{r}^2 - \\ & r\left(1 - \frac{2GM}{r}\right)\dot{\theta}^2 - r\sin^2\theta\left(1 - \frac{2GM}{r}\right)\dot{\phi}^2 \end{aligned} \tag{3.14}$$

Comparing terms, we get

$$\Gamma^{r}{}_{ij} = \begin{bmatrix} \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & -\frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r\left(1 - \frac{2GM}{r}\right) & 0 \\ 0 & 0 & 0 & -r\sin^2\theta\left(1 - \frac{2GM}{r}\right) \end{bmatrix} \tag{3.15}$$

Finally, for $a = t$ the metric is again independent of t so the situation is a lot simpler:

$$\begin{aligned} -\left(1 - \tfrac{2GM}{r}\right)\ddot{t} - \tfrac{2GM}{r^2}\dot{r}\dot{t} = 0 \\ \ddot{t} + \tfrac{2GM}{r^2}\left(1 - \tfrac{2GM}{r}\right)^{-1}\dot{r}\dot{t} = 0 \end{aligned} \tag{3.16}$$

Let's employ Natural units where G equals 1.

Consequently, the six non-vanishing Christoffel symbols $\Gamma^{i}{}_{jk}$ related to γ_{ij} are

$$\begin{array}{ll} \Gamma^{r}_{rr} = -\frac{M}{r(r-2M)} & \Gamma^{r}_{\theta\theta} = -(r-2M) \\ \Gamma^{r}_{\phi\phi} = -(r-2M)\sin^2\theta & \Gamma^{\theta}_{r\theta} = \frac{1}{r} \\ \Gamma^{\phi}_{r\phi} = \frac{1}{r} & \Gamma^{\phi}_{\theta\phi} = \cot\theta \end{array}$$

The Riemann curvature tensor is given by the following expression

$$R^{\lambda}_{\mu\nu\sigma} = \partial_\mu \Gamma^{\lambda}_{\nu\sigma} - \partial_\nu \Gamma^{\lambda}_{\mu\sigma} + \Gamma^{\lambda}_{\mu\alpha}\Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\lambda}_{\nu\alpha}\Gamma^{\alpha}_{\mu\sigma}$$

This formula relates the components of the Riemann curvature tensor to the first and second derivatives of the Christoffel symbols.

The non-zero terms of the Stress Energy Tensor for Schwarschild metric is

$$\langle T_{\mu\gamma}\rangle = \tfrac{\log M_n^2}{2(2\pi)^2}\left(g_{\mu\nu}\tfrac{1}{8}m_0^4 + g_{\mu\gamma}\tfrac{1}{720}R_{\sigma\tau\lambda\rho}R^{\sigma\tau\lambda\rho} + \tfrac{1}{180}R^{\sigma\tau\lambda}_{\mu}R_{\sigma\tau\lambda\gamma}\right)$$

Let's find the expression of each term for Schwarzschild metric,

$$\begin{aligned}
&R_{\sigma\tau\lambda\rho}R^{\sigma\tau\lambda\rho} \\
&= 8\left(\frac{4M^2}{r^6}\right) + 16\left(\frac{M^2}{r^6}\right) \\
&= \frac{48M^2}{r^6} \\
&R^{\lambda\sigma\tau}_{\mu}R_{\lambda\sigma\tau\gamma} \\
&= \frac{-4M^2(M-r)}{r^7} - \frac{M^2(2M-r)}{r^7} + \frac{4M^2(2M-r)}{r^7} \\
&+ \frac{M^2(2M-r)}{r^7} + \frac{M^2(2M-r)}{r^7} - \frac{M^2(2M-r)}{r^7} \\
&= 0
\end{aligned}$$

3.2 Symmetries of Riemannian curvature tensor

The Riemannian curvature tensor is a mathematical object that describes the curvature of a Riemannian manifold. It has the following symmetries:

- Skew-symmetry in the first two and the last two indices:

$$R_{ijkl} = -R_{jikl} = -R_{ijlk}$$

- Symmetry in the pair of pairs of indices:

$$R_{ijkl} = R_{klij}$$

- Cyclic relation or first Bianchi identity:

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0$$

These symmetries reduce the number of independent components of the curvature tensor from n^4 to $\frac{1}{12}n^2(n^2-1)$, where n is the dimension of the manifold.

The geometric meaning of these symmetries can be summarized as follows:

The skew-symmetry reflects the failure of parallel transport to be commutative.
The symmetry in the pair of pairs of indices reflects the curvature tensor's invariance under the parallelogram's change of orientation.
The cyclic relation reflects the Jacobi identity for the Lie bracket of vector fields.

3.3 Dynamical equations for Schwarzschild metric

1. **Lapse Function (N):** - The lapse function represents the local rate of proper time experienced by observers following the coordinate lines of the chosen time coordinate. It measures the "stretching" or "compression" of proper time between nearby spatial hypersurfaces. In the Schwarzschild metric, the lapse function is given by:

$$N = \sqrt{1 - \frac{2GM}{c^2 r}}$$

This function ensures that time coordinates are chosen such that they are orthogonal to the spatial hypersurfaces, and it determines the proper time experienced by stationary observers.

2. **Shift Vector (β^i):** - The shift vector describes how the spatial coordinates evolve from one hypersurface to the next in a foliation of spacetime. It represents the spatial part of the change in coordinates between consecutive spatial slices. In the Schwarzschild metric, the shift vector components are zero in all spatial directions:

$$\beta^r = 0, \quad \beta^\theta = 0, \quad \beta^\phi = 0$$

This indicates that there is no "drift" or displacement of spatial coordinates as we move from one spatial slice to another in Schwarzschild spacetime. These functions are essential in defining the slicing of spacetime into spatial hypersurfaces and formulating the Hamiltonian formalism of General Relativity in the ADM framework. They play a crucial role in understanding the dynamics of the gravitational field and matter distributions. The dynamical equations for Schwarchild metric is given by

$$\begin{aligned}
\partial_t h_{ij} &= -2NK_{ij} + \nabla_i \beta^j + \nabla_j \beta^i \\
\partial_t K_{ij} &= -D_i D_j N + N\left(R_{ij} + KK_{ij} - 2K_{ik}K_j^k + 4\pi G_N\left((S-E)\gamma_{ij} - 2S_{ij}\right)\right)
\end{aligned} \tag{3.17}$$

We have found all the terms required for evaluating the coupled ADM equations. The expression of the expectation value of Stress Energy Tensor of Schwarzschild metric is given by,

$$\begin{aligned}
\langle T_{11}\rangle &= \frac{\log M_n^2}{8\pi^{2_n}}\left(\left(1-\frac{2GM}{r}\right)\left(\frac{1}{8}m_0^4 + \frac{46M^2}{720\left(r^6\right)}\right)\right) \\
\langle T_{22}\rangle &= \frac{\log M_n^2}{8\pi^2}\left(\frac{1}{\left(1-\frac{2GM}{r}\right)}\left(\frac{1}{8}m_0^4 + \frac{46M^2}{720r^6}\right)\right) \\
\langle T_{33}\rangle &= \frac{\log M_n^2}{8\pi^2}\left(r^2\left(\frac{1}{8}m_0^4 + \frac{46M^2}{720r^6}\right)\right) \\
\langle T_{44}\rangle &= \frac{\log M_n^2}{8\pi^2}\left(r^2\sin^2\theta\left(\frac{1}{8}m_0^4 + \frac{46M^2}{720r^6}\right)\right)
\end{aligned}$$

Where M_n^2 is the normalization factor. As we have got the expression of the Stress Energy Tensor, we can derive the Energy and Momentum flux density Tensor using projection operators,

$$S = S_{11} + S_{22} + S_{33} + S_{44}$$
$$S_{11} = T_{\mu r}\gamma_1^{\mu}\gamma_1^{\nu}$$
$$S_{11} = 4T_{11}$$
$$S_{11} = 4\frac{\log M_n^2}{8\pi^2}\left(\left(1-\frac{2M}{r}\right)\left(\frac{1}{8}m^4+\frac{46M^2}{720\left(r^6\right)}\right)\right)$$
$$S_{22} = T_{11}\gamma_2^2\gamma_2^2$$
$$S_{22} = T_{22}$$
$$S_{22} = \frac{\log M_n^2}{8\pi^2}\left(\frac{1}{(1-\frac{2M}{r})}\left(\frac{1}{8}m^4+\frac{46M^2}{720r^6}\right)\right)$$
$$S_{33} = T_{33}\gamma_3^3\gamma_3^3$$
$$S_{33} = T_{33}$$
$$S_{33} = \frac{\log M_n^2}{8\pi^2}\left(r^2\left(\frac{1}{8}m^4+\frac{46M^2}{720r^6}\right)\right)$$
$$S_{44} = T_{44}\gamma_4^4\gamma_4^4$$
$$S_{44} = \frac{\log M_n^2}{8\pi^2}\left(r^2\sin^2\theta\left(\frac{1}{8}m^4+\frac{46M^2}{720r^6}\right)\right)$$
$$S = 4T_{11} + T_{22} + T_{33} + T_{44}$$
$$S = \frac{\log M_n^2}{8\pi^2}\left(\frac{1}{8}m_0^4+\frac{46M}{720r^6}\right)\left(4\left(1-\frac{M}{r}\right)+\frac{1}{(1-\frac{2M}{r})}+r^2+r^2\sin\theta\right)$$

The value of Energy density tensor is calculated as follows,

$$E = T_{\mu\nu}n^{\mu}n^{\nu}$$
$$E = T_{11}n^{11} + T_{12}n^1n^2 + T_{21}n^2n^1 + T_nn^2n^2$$
$$E = T_{11}n^1n^1$$
$$E = \left(n^1\right)^2\frac{\log M_n^2}{8\pi^2}\left(\frac{1-2M}{r}\left(\frac{1}{8}m_0^4+\frac{46M^2}{(720)\left(r^6\right).}\right)\right.$$
$$E = \frac{\log M_n^2}{8\pi^2}\left(\frac{1}{8}m_0^4+\frac{46M^2}{720\left(r^6\right)}\right)$$

3.4 Evaluation of extrinsic curvature tensor and scalar curvature tensor

Let us evaluate the scalar extrinsic curvature K, embedded in the three-dimensional hypersurface Σ_t, using $K = -D_i n^i$ (we write n^i instead of r^i to avoid confusion) since the components of the normal unit vector n^i pointing outside 3D spacetime.

$$n_i = \left(\sqrt{\frac{r}{r-2M}}, 0, 0\right)_i \Longrightarrow n^i = \left(\sqrt{\frac{r-2M}{r}}, 0, 0\right)^i \tag{3.18}$$

we obtain

$$\begin{aligned}
K &= -\gamma^{ij}\left(\partial_i n_j - \Gamma^a{}_{ij} n_a\right) \\
&= -\frac{r-2M}{r}\partial_r\left(\sqrt{\frac{r}{r-2M}}\right) + \sqrt{\frac{r}{r-2M}}\left[-\frac{M}{r^2} - 2\frac{r-2M}{r^2}\right] \\
&= -\frac{2}{r}\sqrt{\frac{r-2M}{r}}
\end{aligned}$$

The scalar curvature K_0 referred to the embedding in a flat spacetime can be effortlessly recovered by the relation $K_0 = K|_{M=0}$ (or equivalently by using the Christoffel symbols associated to the background metric). Hence we have

$$K_0 = -\frac{2}{r} \tag{3.19}$$

Let us use a two-sphere of radius r to identify the boundary S_t. This suggests that $\sigma = r^2 \sin\theta$ is the only possible value for the determinant σ of the induced metric on S_t. Now that we have substituted these quantities in the integral and calculated the limit, we may evaluate M_{ADM}:

$$\begin{aligned}
M_{ADM} &= \frac{1}{8\pi}\lim_{S_t\to\infty}\oint_{S_t}\left(K - K_0\right)\sqrt{\sigma}\mathrm{d}^2x \\
&= -\frac{1}{4\pi}\lim_{r\to\infty}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \frac{1}{r}\left(\sqrt{\frac{r-2M}{r}} - 1\right) r^2 \sin\theta \\
&= -\lim_{r\to\infty} r\left(\sqrt{\frac{r-2M}{r}} - 1\right) = M
\end{aligned}$$

For a body of mass m that is spherically symmetric, this is the anticipated outcome. The standard coordinates of the Schwarzschild spacetime result in a negligible result concerning the ADM momentum.

$$\mathcal{P}_i^{ADM} = 0 \tag{3.20}$$

We take into consideration the extrinsic curvature tensor to demonstrate this:

$$K_{ij} = -\nabla_j n_i = {}^4\Gamma^\mu{}_{ij} n_\mu = -N^4\Gamma^0{}_{ij} \tag{3.21}$$

The extrinsic curvature tensor values that are non zero are

$$\begin{aligned}
K_{11} &= -\nabla_1 n_1 \\
&= \partial_t \sqrt{\frac{r - 2M(t)}{r}} \\
&= \frac{1}{2}\left(\frac{r-2M}{r}\right)^{-1/2}\left(-\frac{2dM}{dt}\right)\frac{1}{r} \\
K_{11} &= -\left(\frac{r-2M}{r}\right)^{-1/2}\left(\frac{\hbar c^4}{M^2 15360\pi G^2}\right)\frac{1}{r}
\end{aligned}$$

The expression of first coupled ADM equation,

$$\begin{aligned}
\partial_t h_{11} &= -\frac{2}{r}\left(\frac{\hbar c^4}{15360\pi G^2}\right)\left(\frac{1}{M_i^2}\right) - 2N\left(\partial_t K_{11}\right)t \\
\partial_t K_{11} &= -\nabla_1\nabla_1\sqrt{1-\frac{2M}{r}} - \sqrt{1-\frac{2M}{r}}\left(4\pi G(S-E)\left(-\left(1-\frac{2M}{r}\right)+S_{11}\right)\right)
\end{aligned}$$

The second set of coupled ADM equations are,

$$\begin{aligned}
\partial_t h_{22} &= -2\sqrt{1-\frac{2M}{r}}\nabla_2(0) = 0 \\
\partial_t K_{22} &= \nabla_2\nabla_2\sqrt{1-\frac{2M}{r}} - \sqrt{1-\frac{2M}{r}}\left(-4\pi G(S-E)\left(\frac{1}{1-\frac{2M}{r}}\right) - 8\pi G S_{22}\right)
\end{aligned}$$

The last coupled ADM equations are,

$$\begin{aligned}
\partial_t h_{21} &= \frac{2M}{r^2} - 2\sqrt{1-\frac{2M}{r}}\left(\partial_t K_{21}\right)t \\
\partial_t K_{21} &= -\nabla_2\nabla_1\sqrt{1-\frac{2M}{r}}
\end{aligned}$$

Finally, we ended up with the dynamical equation of the Schwarzschild metric, which, when solved, can give the time evolution of the metric. The three non-zero components in the time evolution expression show only the radial and time components of the Schwarzschild metric changes, which is well known to us by Hawking Radiation.

3.5 Discussion

After calculating how black holes work, We figured out a special equation that tells us how the space around a black hole changes over time. We used something called ADM formalism to find this equation. It's like a key that helps us understand how gravity and matter interact in the space around black holes. The expectation value of stress-energy tensor is found in the paper Semiclassical Dynamics of Hawking Radiation by Lowe, David A. and Thorlacius [7].

This equation is super important because it helps us see how the shape of space near a black hole evolves as time goes on. By solving this equation, we better understand how the black hole's surroundings change and what that means for how black holes behave. We have three parts that don't stay the same as time evolves. These parts tell us how the distance from the centre of the black hole and time itself changes over time. This point is important, especially when we think about something called Hawking radiation.

Hawking radiation is a big idea in physics that says black holes can give off energy and get smaller over time. Our equation helps us understand this by showing how space and time near the black hole change over time.

So, by figuring out this equation, we've learned how black holes evolve through ADM equations. It's like solving a puzzle piece by piece to see the big picture of the universe.

Chapter 4

Mathematica File

All the terms that are found for the Schwarschild metric are verified using Mathematica software named RTGC. The following pages contain the result of it.

In[ ]:= `Quit`

In[ ]:= `<< EDCRGTCcode.m`

SetDelayed: Tag Laplacian in $\nabla^2_{\text{UpList}\_:\{\}}$ x_ is Protected.

SetDelayed: Tag Classify in Classify[x_] is Protected.

In[ ]:= `Coordinates = {t, r, θ, ϕ}`

Out[ ]= {t, r, θ, ϕ}

In[ ]:= Metric = DiagonalMatrix$\left[\left\{-\left(1-\frac{2GM}{r}\right), \left(1-\frac{2GM}{r}\right)^{-1}, r^2, r^2 \text{Sin}[\theta]^2\right\}\right]$

Out[ ]= $\left\{\left\{-1+\frac{2GM}{r}, 0, 0, 0\right\}, \left\{0, \frac{1}{1-\frac{2GM}{r}}, 0, 0\right\}, \left\{0, 0, r^2, 0\right\}, \left\{0, 0, 0, r^2 \text{Sin}[\theta]^2\right\}\right\}$

In[ ]:= `RGtensors[Metric, Coordinates]`

$$\text{gdd} = \begin{pmatrix} -1+\frac{2GM}{r} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2GM}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \text{Sin}[\theta]^2 \end{pmatrix}$$

$$\text{LineElement} = -\frac{r\, d[r]^2}{2GM-r} + \frac{(2GM-r)\, d[t]^2}{r} + r^2 d[\theta]^2 + r^2 d[\phi]^2 \text{Sin}[\theta]^2$$

$$\text{gUU} = \begin{pmatrix} \frac{r}{2GM-r} & 0 & 0 & 0 \\ 0 & -\frac{2GM-r}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{\text{Csc}[\theta]^2}{r^2} \end{pmatrix}$$

gUU computed in 0.004432 sec

Gamma computed in 0.001311 sec

Riemann(dddd) computed in 0.001763 sec

Riemann(Uddd) computed in 0.001445 sec

Ricci computed in 0.000088 sec

Weyl computed in $7.\times10^{-6}$ sec

Ricci Flat

Out[ ]= All tasks completed in 0.011598 seconds

In[ ]:= `R`

Out[ ]= 0

In[ ]:= **Rdddd**

Out[ ]=

$$\left\{\left\{\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\left\{\left\{0,-\frac{2GM}{r^3},0,0\right\},\left\{\frac{2GM}{r^3},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\}\right\},\left\{\left\{0,0,-\frac{GM(2GM-r)}{r^2},0\right\},\{0,0,0,0\},\left\{\frac{GM(2GM-r)}{r^2},0,0,0\right\},\{0,0,0,0\}\right\},\left\{\left\{0,0,0,-\frac{GM(2GM-r)\,\mathrm{Sin}[\theta]^2}{r^2}\right\},\{0,0,0,0\},\{0,0,0,0\},\left\{\frac{GM(2GM-r)\,\mathrm{Sin}[\theta]^2}{r^2},0,0,0\right\}\right\}\right\},\right.$$

$$\left\{\left\{\left\{0,\frac{2GM}{r^3},0,0\right\},\left\{-\frac{2GM}{r^3},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\}\right\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\left\{\{0,0,0,0\},\left\{0,0,\frac{GM}{2GM-r},0\right\},\left\{0,-\frac{GM}{2GM-r},0,0\right\},\{0,0,0,0\}\right\},\left\{\{0,0,0,0\},\left\{0,0,0,\frac{GM\,\mathrm{Sin}[\theta]^2}{2GM-r}\right\},\{0,0,0,0\},\left\{0,-\frac{GM\,\mathrm{Sin}[\theta]^2}{2GM-r},0,0\right\}\right\}\right\},$$

$$\left\{\left\{\left\{0,0,\frac{GM(2GM-r)}{r^2},0\right\},\{0,0,0,0\},\left\{-\frac{GM(2GM-r)}{r^2},0,0,0\right\},\{0,0,0,0\}\right\},\left\{\{0,0,0,0\},\left\{0,0,-\frac{GM}{2GM-r},0\right\},\left\{0,\frac{GM}{2GM-r},0,0\right\},\{0,0,0,0\}\right\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\left\{\{0,0,0,0\},\{0,0,0,0\},\left\{0,0,0,2GMr\,\mathrm{Sin}[\theta]^2\right\},\left\{0,0,-2GMr\,\mathrm{Sin}[\theta]^2,0\right\}\right\}\right\},$$

$$\left.\left\{\left\{\left\{0,0,0,\frac{GM(2GM-r)\,\mathrm{Sin}[\theta]^2}{r^2}\right\},\{0,0,0,0\},\{0,0,0,0\},\left\{-\frac{GM(2GM-r)\,\mathrm{Sin}[\theta]^2}{r^2},0,0,0\right\}\right\},\left\{\{0,0,0,0\},\left\{0,0,0,-\frac{GM\,\mathrm{Sin}[\theta]^2}{2GM-r}\right\},\{0,0,0,0\},\left\{0,\frac{GM\,\mathrm{Sin}[\theta]^2}{2GM-r},0,0\right\}\right\},\left\{\{0,0,0,0\},\{0,0,0,0\},\left\{0,0,0,-2GMr\,\mathrm{Sin}[\theta]^2\right\},\left\{0,0,2GMr\,\mathrm{Sin}[\theta]^2,0\right\}\right\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}\right\}\right\}$$

In[ ]:=
```
RUUUU = Raise[RUddd, 2, 3, 4]
```

Out[ ]=

$$\begin{aligned}
&\Big\{\Big\{\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\},\\
&\Big\{\Big\{0, -\frac{2\,G\,M}{r^3}, 0, 0\Big\}, \Big\{\frac{2\,G\,M}{r^3}, 0, 0, 0\Big\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\Big\},\\
&\Big\{\Big\{0, 0, -\frac{G\,M}{(2\,G\,M-r)\,r^4}, 0\Big\}, \{0, 0, 0, 0\}, \Big\{\frac{G\,M}{(2\,G\,M-r)\,r^4}, 0, 0, 0\Big\}, \{0, 0, 0, 0\}\Big\},\\
&\Big\{\Big\{0, 0, 0, -\frac{G\,M\,\mathrm{Csc}[\theta]^2}{(2\,G\,M-r)\,r^4}\Big\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \Big\{\frac{G\,M\,\mathrm{Csc}[\theta]^2}{(2\,G\,M-r)\,r^4}, 0, 0, 0\Big\}\Big\}\Big\},\\
&\Big\{\Big\{\Big\{0, \frac{2\,G\,M}{r^3}, 0, 0\Big\}, \Big\{-\frac{2\,G\,M}{r^3}, 0, 0, 0\Big\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\Big\},\\
&\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\},\\
&\Big\{\{0, 0, 0, 0\}, \Big\{0, 0, \frac{G\,M\,(2\,G\,M-r)}{r^6}, 0\Big\}, \Big\{0, -\frac{G\,M\,(2\,G\,M-r)}{r^6}, 0, 0\Big\}, \{0, 0, 0, 0\}\Big\},\\
&\Big\{\{0, 0, 0, 0\}, \Big\{0, 0, 0, \frac{G\,M\,(2\,G\,M-r)\,\mathrm{Csc}[\theta]^2}{r^6}\Big\},\\
&\quad \{0, 0, 0, 0\}, \Big\{0, -\frac{G\,M\,(2\,G\,M-r)\,\mathrm{Csc}[\theta]^2}{r^6}, 0, 0\Big\}\Big\}\Big\},\\
&\Big\{\Big\{\Big\{0, 0, \frac{G\,M}{(2\,G\,M-r)\,r^4}, 0\Big\}, \{0, 0, 0, 0\}, \Big\{-\frac{G\,M}{(2\,G\,M-r)\,r^4}, 0, 0, 0\Big\}, \{0, 0, 0, 0\}\Big\},\\
&\Big\{\{0, 0, 0, 0\}, \Big\{0, 0, -\frac{G\,M\,(2\,G\,M-r)}{r^6}, 0\Big\}, \Big\{0, \frac{G\,M\,(2\,G\,M-r)}{r^6}, 0, 0\Big\}, \{0, 0, 0, 0\}\Big\},\\
&\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\},\\
&\Big\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \Big\{0, 0, 0, \frac{2\,G\,M\,\mathrm{Csc}[\theta]^2}{r^7}\Big\}, \Big\{0, 0, -\frac{2\,G\,M\,\mathrm{Csc}[\theta]^2}{r^7}, 0\Big\}\Big\}\Big\},\\
&\Big\{\Big\{\Big\{0, 0, 0, \frac{G\,M\,\mathrm{Csc}[\theta]^2}{(2\,G\,M-r)\,r^4}\Big\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \Big\{-\frac{G\,M\,\mathrm{Csc}[\theta]^2}{(2\,G\,M-r)\,r^4}, 0, 0, 0\Big\}\Big\},\\
&\Big\{\{0, 0, 0, 0\}, \Big\{0, 0, 0, -\frac{G\,M\,(2\,G\,M-r)\,\mathrm{Csc}[\theta]^2}{r^6}\Big\},\\
&\quad \{0, 0, 0, 0\}, \Big\{0, \frac{G\,M\,(2\,G\,M-r)\,\mathrm{Csc}[\theta]^2}{r^6}, 0, 0\Big\}\Big\},\\
&\Big\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \Big\{0, 0, 0, -\frac{2\,G\,M\,\mathrm{Csc}[\theta]^2}{r^7}\Big\}, \Big\{0, 0, \frac{2\,G\,M\,\mathrm{Csc}[\theta]^2}{r^7}, 0\Big\}\Big\},\\
&\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}\Big\}\Big\}
\end{aligned}$$

In[ ]:=
```
Contract[Outer[Times, Rdddd, RUUUU], {1, 5}, {2, 6}, {3, 7}, {4, 8}]
```

Out[ ]=

$$\frac{48\,G^2\,M^2}{r^6}$$

In[ ]:= **RUUUd = Lower[RUUUU, 4]**

Out[ ]= $\Big\{\Big\{\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},$
$\Big\{\Big\{0,\frac{2\,G\,M}{(2\,G\,M-r)\,r^2},0,0\Big\},\Big\{\frac{2\,G\,M\,(2\,G\,M-r)}{r^4},0,0,0\Big\},\{0,0,0,0\},\{0,0,0,0\}\Big\},$
$\Big\{\Big\{0,0,-\frac{G\,M}{(2\,G\,M-r)\,r^2},0\Big\},\{0,0,0,0\},\Big\{\frac{G\,M}{r^5},0,0,0\Big\},\{0,0,0,0\}\Big\},$
$\Big\{\Big\{0,0,0,-\frac{G\,M}{(2\,G\,M-r)\,r^2}\Big\},\{0,0,0,0\},\{0,0,0,0\},\Big\{\frac{G\,M\,\mathrm{Csc}[\theta]^2}{r^5},0,0,0\Big\}\Big\}\Big\},$
$\Big\{\Big\{\Big\{0,-\frac{2\,G\,M}{(2\,G\,M-r)\,r^2},0,0\Big\},\Big\{-\frac{2\,G\,M\,(2\,G\,M-r)}{r^4},0,0,0\Big\},\{0,0,0,0\},\{0,0,0,0\}\Big\},$
$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},$
$\Big\{\{0,0,0,0\},\Big\{0,0,\frac{G\,M\,(2\,G\,M-r)}{r^4},0\Big\},\Big\{0,\frac{G\,M}{r^5},0,0\Big\},\{0,0,0,0\}\Big\},$
$\Big\{\{0,0,0,0\},\Big\{0,0,0,\frac{G\,M\,(2\,G\,M-r)}{r^4}\Big\},\{0,0,0,0\},\Big\{0,\frac{G\,M\,\mathrm{Csc}[\theta]^2}{r^5},0,0\Big\}\Big\}\Big\},$
$\Big\{\Big\{\Big\{0,0,\frac{G\,M}{(2\,G\,M-r)\,r^2},0\Big\},\{0,0,0,0\},\Big\{-\frac{G\,M}{r^5},0,0,0\Big\},\{0,0,0,0\}\Big\},$
$\Big\{\{0,0,0,0\},\Big\{0,0,-\frac{G\,M\,(2\,G\,M-r)}{r^4},0\Big\},\Big\{0,-\frac{G\,M}{r^5},0,0\Big\},\{0,0,0,0\}\Big\},$
$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},$
$\Big\{\{0,0,0,0\},\{0,0,0,0\},\Big\{0,0,0,\frac{2\,G\,M}{r^5}\Big\},\Big\{0,0,-\frac{2\,G\,M\,\mathrm{Csc}[\theta]^2}{r^5},0\Big\}\Big\}\Big\},$
$\Big\{\Big\{\Big\{0,0,0,\frac{G\,M}{(2\,G\,M-r)\,r^2}\Big\},\{0,0,0,0\},\{0,0,0,0\},\Big\{-\frac{G\,M\,\mathrm{Csc}[\theta]^2}{r^5},0,0,0\Big\}\Big\},$
$\Big\{\{0,0,0,0\},\Big\{0,0,0,-\frac{G\,M\,(2\,G\,M-r)}{r^4}\Big\},\{0,0,0,0\},\Big\{0,-\frac{G\,M\,\mathrm{Csc}[\theta]^2}{r^5},0,0\Big\}\Big\},$
$\Big\{\{0,0,0,0\},\{0,0,0,0\},\Big\{0,0,0,-\frac{2\,G\,M}{r^5}\Big\},\Big\{0,0,\frac{2\,G\,M\,\mathrm{Csc}[\theta]^2}{r^5},0\Big\}\Big\},$
$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}\Big\}\Big\}$

In[ ]:= **Contract[Outer[Times, RUUUd, Rdddd], {1, 5}, {2, 6}, {3, 7}]**

Out[ ]= $\Big\{\Big\{\frac{12\,G^2\,M^2\,(2\,G\,M-r)}{r^7},0,0,0\Big\},\Big\{0,-\frac{12\,G^2\,M^2}{(2\,G\,M-r)\,r^5},0,0\Big\},$
$\Big\{0,0,\frac{12\,G^2\,M^2}{r^4},0\Big\},\Big\{0,0,0,\frac{12\,G^2\,M^2\,\mathrm{Sin}[\theta]^2}{r^4}\Big\}\Big\}$

In[ ]:= **hdd = {{0, 0, 0, 0}, {0, $\frac{1}{1-\frac{2GM}{r}}$, 0, 0}, {0, 0, r^2, 0}, {0, 0, 0, r^2 Sin[θ]^2}}**

Out[ ]= $\left\{\{0,0,0,0\},\left\{0,\frac{1}{1-\frac{2GM}{r}},0,0\right\},\{0,0,r^2,0\},\{0,0,0,r^2\,\mathrm{Sin}[\theta]^2\}\right\}$

In[ ]:= **nU = {$\frac{-1}{\mathrm{Sqrt}\left[1-\frac{2GM}{r}\right]}$, 0, 0, 0}**

Out[ ]= $\left\{-\frac{1}{\sqrt{1-\frac{2GM}{r}}},0,0,0\right\}$

In[ ]:= **nd = Contract[Outer[Times, nU, gdd], {1, 2}]**

Out[ ]= $\left\{\frac{-2GM+r}{r\sqrt{\frac{-2GM+r}{r}}},0,0,0\right\}$

In[ ]:= **γdU = {{2, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}**

Out[ ]= {{2, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

In[ ]:= **covD[nd]**

Out[ ]= $\left\{\{0,0,0,0\},\left\{-\frac{GM}{r^2\sqrt{\frac{-2GM+r}{r}}},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\}\right\}$

In[ ]:= **KKdd = -Contract[Outer[Times, γdU, γdU, covD[nd]], {2, 5}, {4, 6}]**

Out[ ]= $\left\{\{0,0,0,0\},\left\{\frac{2GM}{r^2\sqrt{\frac{-2GM+r}{r}}},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\}\right\}$

Appendix

A.1 Definitions

A.1.1 Covariant derivative or connection

Assume that the manifold $\mathcal{M}$ is differentiable. A map from the tensor fields of rank(r, s) to the tensor fields $(r, s+1)$ such that: is known as a covariant derivative (or connection) ∇.

1. ∇ is linear, that is, $\nabla(T+S) = \nabla T + \nabla S$, where T, S are the same-rank tensor fields.
2. $\nabla(fT) = \mathrm{d}f \otimes T + f\nabla T$, where f is scalar field and $\mathrm{d}f$ is the $(0,1)$ tensor with components $\partial_\mu f$.
3. given the bases $\{\boldsymbol{e}_\mu\}$ and $\{\boldsymbol{\theta}^\mu\}$ of the tangent and cotangent spaces $T_p(\mathcal{M})$, $T_p^*(\mathcal{M})$, it satisfies

$$\nabla \boldsymbol{e}_\mu = \Gamma^\alpha{}_{\beta\mu} \boldsymbol{\theta}^\beta \otimes \boldsymbol{e}_\alpha \tag{A.1}$$

Here $\Gamma^\lambda{}_{\mu\nu}$ are representing connection coefficients. Specifically, ∇ is considered a metric connection if the following relation holds for a given metric $g_{\mu\nu}$ on $\mathcal{M}$:

$$\nabla g_{\mu\nu} = 0 \tag{A.2}$$

The Christoffel symbols in this instance are the connection coefficients $\Gamma^\lambda{}_{\mu\nu}$, which are ascertained by the equation

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} \left(g_{\alpha\nu,\mu} + g_{\mu\alpha,\nu} - g_{\mu\nu,\alpha} \right) \tag{A.3}$$

A.1.2 Tensor density

In the last section, we presented the covariant derivative ∇, which is a map from tensors of rank (r, s) to $(r, s+1)$ tensors. Nonetheless, we can broaden its applicability to the class of tensor densities, which are defined to streamline the computations.

$$\mathcal{T}^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s} = \sqrt{|g|}^W T^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s} \tag{A.4}$$

Where W is a real number, which is known as the weight of the tensor density, and $T^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s}$ is a tensor of type $(r, s), g$ is the determinant of the metric $g_{\mu\nu}$. The conventional derivation is thus simply generalized to the covariant derivative of $\mathcal{T}^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s}$.

A.1.3 Curvature tensors

The Riemann curvature tensor $R^\rho{}_{\sigma\mu\nu}$ determines the curvature of a manifold in its entirety. As per the sign convention, we define it as follows:

$$R^\rho{}_{\sigma\mu\nu} = \Gamma^\rho{}_{\sigma\nu,\mu} - \Gamma^\rho_{\sigma,\nu} + \Gamma^\rho_{\sigma\lambda\mu\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\sigma\mu} \tag{A.5}$$

The contraction of the first and third indices in A.4 yields the Ricci tensor:

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} = \frac{1}{\sqrt{|g|}}\partial_\lambda \left[\sqrt{|g|}\Gamma^\lambda{}_{\mu\nu}\right] - \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda_{\rho\nu} - \partial_\mu\partial_\nu \ln \sqrt{|g|} \tag{A.7}$$

Ultimately, the scalar curvature can be obtained by contracting $R_{\mu\nu}$ using the inverse metric $g^{\mu\nu}$.

$$R = g^{\mu\nu} R_{\mu\nu} \tag{A.8}$$

A.1.4 Lie derivative

Assume that the manifold $\mathcal{M}$ is differentiable. Given an open subset $I \subset \mathbb{R}$ and a regular vector field $X = X^\mu \partial_\mu$ on $\mathcal{M}$, we define the integral curve of X as

$$\begin{aligned} \alpha_p : I &\longrightarrow \mathcal{M} \\ s &\longmapsto \alpha_p(s) \end{aligned} \tag{A.10}$$

such that

$$\alpha_p(0) = p \tag{A.11}$$

$$\forall s_0 \in I \quad \left.\frac{\mathrm{d}\alpha_p}{\mathrm{d}s}\right|_{s_0} = \dot{\alpha}_p\left(s_0\right) = X_{s_0}\left(\alpha_p\right) \tag{A.12}$$

An open subset is $U \subset \mathcal{M}$. Every integral curve has a natural relationship with the map.

$$\phi_s^X : U \longrightarrow \mathcal{M} \tag{A.13}$$

$$p \longmapsto \alpha_p(s) \tag{A.14}$$

called the flow along X, such that

$$\forall s_0 \in I \quad \left.\frac{\mathrm{d}\alpha_p}{\mathrm{d}s}\right|_{s_0} = \dot{\alpha}_p\left(s_0\right) = X_{s_0}\left(\alpha_p\right) \tag{A.15}$$

The flow ϕ_s^X has the following properties:

1. $\phi_0^X(p) = \alpha_p(0) = p \quad \Longrightarrow \quad \phi_0^X = \mathbb{I}$
2. $\phi_s^X \circ \phi_t^X = \phi_{s+t}^X \quad \forall s, t \in \mathbb{R}$
3. ϕ_s^X is a diffeomorphism and $\left[\phi_s^X\right]^{-1} = \phi_{-s}^X$

This map allows us to define, at a point p, the Lie derivative of a differentiable tensor field T of rank (m, n) along X.

$$\left[\mathcal{L}_X(T)\right]_p = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left[\left(\phi_{-s}^X\right)_* T_{\phi_s^X(p)}\right] \tag{A.14}$$

Which in components reads

$$\begin{aligned}
&\left[\mathcal{L}_X(T)\right]^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} = X^\lambda \partial_\lambda T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} \\
&- T^{\lambda \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} \partial_\lambda X^{\mu_1} - \ldots - T^{\mu_1 \ldots \lambda}{}_{\nu_1 \ldots \nu_n} \partial_\lambda X^{\mu_n} \\
&+ T^{\mu_1 \ldots \mu_m}{}_{\lambda \ldots \nu_n} \partial_{\nu_1} X^\lambda + \ldots + T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \lambda} \partial_{\nu_n} X^\lambda
\end{aligned} \tag{A.17}$$

In the event that the connection ∇ is torsion-free, that is, if the Christoffel symbols in the final two indices exhibit symmetry

$$\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu} \tag{A.18}$$

It is possible to rewrite equation A.17 by substituting the covariant counterparts ∇_μ for the partial derivatives ∂_μ

$$\begin{aligned}
[\mathcal{L}_X(T)]^{\mu_1\ldots\mu_m}_{\nu_1\ldots\nu_n} =& X^{\lambda}\nabla_{\lambda}T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} \\
&- T^{\lambda\ldots\mu_m}{}_{\nu_1\ldots\nu_n}\nabla_{\lambda}X^{\mu_1} - \ldots - T^{\mu_1\ldots\lambda}{}_{\nu_1\ldots\nu_n}\nabla_{\lambda}X^{\mu_m} \\
&+ T^{\mu_1\ldots\mu_m}{}_{\lambda\ldots\nu_n}\nabla_{\nu_1}X^{\lambda} + \ldots + T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\lambda}\nabla_{\nu_n}X^{\lambda}
\end{aligned} \tag{A.19}$$

From the definition and the component relation A.17, the primary characteristics of the Lie derivative are readily ascertained:

1. the Lie derivative of a tensor field T of rank(m, n) is a tensor field of rank (m, n).
2. $\mathcal{L}_X(T)$ is linear both in X and in T.
3. the Lie derivative satisfies the Leibniz rule

$$\mathcal{L}_X(T \otimes S) = \mathcal{L}_X(T) \otimes S + T \otimes \mathcal{L}_X(S)$$

4. If f is a scalar field, $\mathcal{L}_X(f) = X(f)$.

A.2 Variation with respect to the metric

A.2.1 Christoffel symbols

To keep things brief, we will refer to the fluctuation of the metric $g_{\mu\nu}$ in some of the intermediary sections as $\theta_{\mu\nu} = \delta g_{\mu\nu}$.

$$
\begin{aligned}
\delta\Gamma_{\lambda\mu\nu} =& \frac{1}{2}\left(\delta g_{\lambda\nu,\mu} + \delta g_{\mu\lambda,\nu} - \delta g_{\mu\nu,\lambda}\right) \\
=& \frac{1}{2}\left(\nabla_\mu\theta_{\lambda\nu} + \nabla_\nu\theta_{\mu\lambda} - \nabla_\lambda\theta_{\mu\nu}\right) \\
&+ \frac{1}{2}\left[\Gamma^\sigma{}_{\mu\lambda}\theta_{\sigma\nu} + \Gamma^\sigma{}_{\mu\nu}\theta_{\lambda\sigma} + \Gamma^\sigma{}_{\nu\mu}\theta_{\sigma\lambda} + \Gamma^\sigma{}_{\nu\lambda}\theta_{\mu\sigma} - \Gamma^\sigma{}_{\lambda\mu}\theta_{\sigma\nu} - \Gamma^\sigma{}_{\lambda\nu}\theta_{\mu\sigma}\right] \\
&= \frac{1}{2}\left(\nabla_\mu\delta g_{\lambda\nu} + \nabla_\nu\delta g_{\mu\lambda} - \nabla_\lambda\delta g_{\mu\nu}\right) + \Gamma^\sigma{}_{\mu\nu}\delta g_{\sigma\lambda}
\end{aligned}
\tag{A.20}
$$

It follows that

$$
\begin{aligned}
\delta\Gamma^\rho{}_{\mu\nu} &= \delta g^{\rho\lambda}\Gamma_{\lambda\mu\nu} + g^{\rho\lambda}\delta\Gamma_{\lambda\mu\nu} \\
&= -g^{\rho\lambda}\Gamma^\sigma{}_{\mu\nu}\theta_{\sigma\lambda} + \frac{1}{2}g^{\rho\lambda}\left(\nabla_\mu\theta_{\lambda\nu} + \nabla_\nu\theta_{\mu\lambda} - \nabla_\lambda\theta_{\mu\nu}\right) + g^{\rho\lambda}\Gamma^\sigma{}_{\mu\nu}\theta_{\sigma\lambda} \\
&= \frac{1}{2}g^{\rho\lambda}\left(\nabla_\mu\delta g_{\lambda\nu} + \nabla_\nu\delta g_{\mu\lambda} - \nabla_\lambda\delta g_{\mu\nu}\right)
\end{aligned}
\tag{A.21}
$$

We shall also consider the contracted version of A.21 :

$$
\begin{aligned}
\delta\Gamma^\mu{}_{\mu\nu} &= \delta g^{\mu\lambda}\Gamma_{\lambda\mu\nu} + g^{\mu\lambda}\delta\Gamma_{\lambda\mu\nu} \\
&= -g^{\alpha\mu}g^{\beta\lambda}\theta_{\alpha\beta}\Gamma_{\lambda\mu\nu} + g^{\mu\lambda}\left[\frac{1}{2}\left(\nabla_\mu\theta_{\lambda\nu} + \nabla_\nu\theta_{\mu\lambda} - \nabla_\lambda\theta_{\mu\nu}\right) + \Gamma^\sigma{}_{\mu\nu}\theta_{\sigma\lambda}\right] \\
&= -\Gamma^\beta{}_{\mu\nu}g^{\alpha\mu}\theta_{\alpha\beta} + \frac{1}{2}g^{\lambda\mu}\nabla_\nu\theta_{\lambda\mu} + \Gamma^\beta{}_{\mu\nu}g^{\alpha\mu}\theta_{\alpha\beta} \\
&= \frac{1}{2}g^{\lambda\mu}\nabla_\nu\delta g_{\lambda\mu}
\end{aligned}
\tag{A.22}
$$

A.2.2 The vector δV^ρ

Let us consider the vector

$$
\delta V^\rho \doteq g^{\mu\nu}\delta\Gamma^\rho{}_{\mu\nu} - g^{\rho\nu}\delta\Gamma^\mu{}_{\mu\nu} \tag{A.23}
$$

Which appears in the variation of R. This is the same as

$$\delta V^{\rho} = \left(g^{\mu\nu}\delta g^{\rho\lambda} - g^{\rho\nu}\delta g^{\mu\lambda}\right)\Gamma_{\lambda\mu\nu} + \left(g^{\mu\nu}g^{\rho\lambda} - g^{\rho\nu}g^{\mu\lambda}\right)\delta\Gamma_{\lambda\mu\nu} \tag{A.24}$$

With the variation $\delta\Gamma_{\lambda\mu\nu}$, we concentrate on the second product and use the notation $\theta_{\mu\nu,\lambda} = \delta g_{\mu\nu,\lambda}$:

$$\begin{aligned}\left(g^{\mu\nu}g^{\rho\lambda} - g^{\rho\nu}g^{\mu\lambda}\right)\delta\Gamma_{\lambda\mu\nu} &= \frac{1}{2}\left[\left(g^{\mu\nu}g^{\rho\lambda} - g^{\rho\nu}g^{\mu\lambda}\right)\left(\delta g_{\lambda\nu,\mu} + \delta g_{\mu\lambda,\nu} - \delta g_{\mu\nu,\lambda}\right)\right] \\ &= g^{\mu\nu}g^{\rho\lambda}\delta g_{\lambda\mu,\nu} - g^{\mu\nu}g^{\rho\lambda}\delta g_{\mu\nu,\lambda}\end{aligned}$$

It follows that

$$\delta V^{\rho} = \left(g^{\mu\nu}\delta g^{\rho\lambda} - g^{\rho\nu}\delta g^{\mu\lambda}\right)\Gamma_{\lambda\mu\nu} + g^{\mu\nu}g^{\rho\lambda}\left(\delta g_{\lambda\mu,\nu} - \delta g_{\mu\nu,\lambda}\right) \tag{A.25}$$

We may rewrite δV^{ρ} by introducing the covariant derivatives of $\delta g_{\mu\nu}$ using equations A.21 and A.22:

$$\begin{aligned}\delta V^{\rho} &= \frac{1}{2}g^{\mu\nu}g^{\rho\lambda}\left(\nabla_{\mu}\delta g_{\lambda\nu} + \nabla_{\nu}\delta g_{\mu\lambda} - \nabla_{\lambda}\delta g_{\mu\nu}\right) - \frac{1}{2}g^{\rho\nu}g^{\lambda\mu}\nabla_{\nu}\delta g_{\lambda\mu} \\ &= g^{\mu\nu}g^{\rho\lambda}\left(\nabla_{\mu}\delta g_{\lambda\nu} - \nabla_{\lambda}\delta g_{\mu\nu}\right)\end{aligned} \tag{A.26}$$

A.2.3 Curvature

Let us examine the $\delta\Gamma^{\lambda}{}_{\mu\nu}$ fluctuation of the Christoffel symbols caused by a metric variation. $\delta\Gamma^{\lambda}{}_{\mu\nu}$ is a tensor of rank $(1, 2)$ since it represents the difference of two connections. Selecting the local inertial frame will cause the Christoffel symbols to disappear.

$$\Gamma^{\lambda}{}_{\mu\nu} \overset{*}{=} 0 \tag{A.27}$$

It is highlighted that the equality holds in a Lorentz frame by the sign $\overset{*}{=}$. We can now describe the variation of the Riemann curvature tensor $R^{\rho}{}_{\sigma\mu\nu}$ by using equation A.27 and substituting ∇_{μ} for the partial derivatives ∂_{μ}.

$$\delta R^{\rho}_{\sigma\mu\nu} \overset{*}{=} \delta\left[\Gamma^{\rho}{}_{\sigma\nu,\mu} - \Gamma^{\rho}{}_{\sigma\mu,\nu}\right] \overset{*}{=} \nabla_{\mu}\delta\Gamma^{\rho}_{\sigma\nu} - \nabla_{\nu}\delta\Gamma^{\rho}{}_{\sigma\mu} \tag{A.28}$$

Since the quantity on the left is tensorial, the equality must hold true regardless of the reference frame. Thus, $=$ can be substituted for $\overset{*}{=}$ to obtain the Palatini identity:

$$\delta R^{\rho}{}_{\sigma\mu\nu} = \nabla_{\mu}\delta\Gamma^{\rho}{}_{\sigma\nu} - \nabla_{\nu}\delta\Gamma^{\rho}{}_{\sigma\mu} \tag{A.29}$$

Moreover, the Ricci curvature tensor is used in its contracted version:

$$\delta R_{\mu\nu} = \nabla_{\lambda}\delta\Gamma^{\lambda}{}_{\mu\nu} - \nabla_{\mu}\delta\Gamma^{\lambda}_{\lambda\nu} \tag{A.30}$$

The computation of the variation δR can be easily accomplished thanks to the Palatini identity:

$$\begin{aligned}
\delta R &= -R^{\mu\nu}\delta g_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \\
&= -R^{\mu\nu}\delta g_{\mu\nu} + g^{\mu\nu}\left[\nabla_{\lambda}\delta\Gamma^{\lambda}{}_{\mu\nu} - \nabla_{\mu}\delta\Gamma^{\lambda}{}_{\lambda\nu}\right] \\
&= -R^{\mu\nu}\delta g_{\mu\nu} + \nabla_{\lambda}\delta V^{\lambda}
\end{aligned} \tag{A.31}$$

where δV^{λ} was defined in equation A.23. Substituting the relation A.26 we obtain:

$$\begin{aligned}
\nabla_{\lambda}\delta V^{\lambda} &= g^{\mu\nu}g^{\rho\lambda}\nabla_{\rho}\left(\nabla_{\mu}\delta g_{\lambda\nu} - \nabla_{\lambda}\delta g_{\mu\nu}\right) \\
&= \nabla^{\mu}\nabla^{\nu}\delta g_{\mu\nu} - \nabla^{\lambda}\nabla_{\lambda}\delta \ln|g|
\end{aligned} \tag{A.32}$$

Therefore we have

$$\delta R = -R^{\mu\nu}\delta g_{\mu\nu} + \nabla^{\mu}\nabla^{\nu}\delta g_{\mu\nu} - \nabla^{\lambda}\nabla_{\lambda}\delta \ln|g| \tag{A.33}$$

The Riemann curvature tensor is given by the following expression

$$R^{\lambda}_{\mu\nu\sigma} = \partial_{\mu}\Gamma^{\lambda}_{\nu\sigma} - \partial_{\nu}\Gamma^{\lambda}_{\mu\sigma} + \Gamma^{\lambda}_{\mu\alpha}\Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\lambda}_{\nu\alpha}\Gamma^{\alpha}_{\mu\sigma}$$

We already know that the formula relates the components of the Riemann curvature tensor to the first and second derivatives of the Christoffel symbols, so the values of Riemann curvature tensor are,

$$
\begin{array}{l|l}
R^1_{212} = \dfrac{2M}{r^2\,(r-2M)} & R^1_{221} = \dfrac{2M}{r^2\,(r-2M)} \\
R^1_{313} = -\dfrac{M}{r} & R^1_{331} = +\dfrac{M}{r} \\
R^1_{414} = -\dfrac{M(\sin\theta)^2}{r} & R^1_{441} = \dfrac{M(\sin\theta)^2}{r} \\
R^2_{112} = -\dfrac{2M\,(2M-r)}{r^4} & R^2_{121} = \dfrac{2M\,(2M-r)}{r^4} \\
R^2_{323} = -\dfrac{M}{r} & R^2_{332} = \dfrac{M}{r} \\
R^2_{424} = -\dfrac{M(\sin\theta)^2}{r} & R^2_{442} = \dfrac{M(\sin\theta)^2}{r} \\
R^3_{113} = \dfrac{M\,(2M-r)}{r^4} & R^3_{131} = \dfrac{M\,(r-2M)}{r^4} \\
R^3_{223} = \dfrac{M}{r^2\,(r-2GM)} & R^3_{232} = \dfrac{M}{r^2\,(2M-r)} \\
R^3_{434} = \dfrac{2M(\sin\theta)^2}{r} & R^3_{443} = \dfrac{2M(\sin\theta)^2}{r} \\
R^4_{114} = \dfrac{M\,(2M-r)}{r^4} & R^4_{141} = \dfrac{M\,(r-2M)}{r4} \\
R^4_{224} = \dfrac{M}{r^2\,(r-2M)} & R^4_{242} = \dfrac{M}{r^2\,(2M-r)} \\
R^4_{334} = -\dfrac{2GM}{r} & R^4_{343} = \dfrac{2M}{r}
\end{array}
$$

By lowering the first index of Riemann curvature tensor, we arrive at the following expressions,

$$R_{1212} = \frac{2M}{r^3}$$
$$R_{1313} = \frac{M(2M-r)}{r^2}$$
$$R_{1221} = -\frac{2M}{r^3}$$
$$R_{1414} = \frac{M(2M-r)(\sin\theta)^2}{r^2} \quad \Bigg| \quad R_{1331} = \frac{-M(2M-r)}{r^2}$$
$$R_{2112} = \frac{-2M}{r^3} \quad \Bigg| \quad R_{1441} = \frac{-M(2M-r)(\sin\theta)^2}{r^2}$$
$$R_{2121} = \frac{2M}{r^3}$$
$$R_{2323} = -\frac{M}{2M-r}$$
$$R_{2442} = \frac{-M(\sin\theta)^2}{2M-r} \quad \Bigg| \quad R_{2332} = \frac{M}{2M-r}$$
$$R_{3113} = \frac{-M(2M-r)}{r^2} \quad \Bigg| \quad R_{2442} = \frac{M(\sin\theta)^2}{2M-r}$$
$$R_{3131} = \frac{M(2M-r)}{r^2}.$$
$$R_{3223} = \frac{+M}{2M-r} \quad \Bigg| \quad R_{3232} = \frac{-M}{2M-r}$$
$$R_{3434} = -2Mr(\sin\theta)^2 \quad | \quad R_{3443} = 2Mr(\sin\theta)^2$$
$$R_{4114} = -\frac{M(2m-r)(\sin\theta)^2}{r^2} \Bigg| \quad R_{4141} = \frac{M(2M-r)(\sin\theta)^2}{r^2}$$
$$R_{4224} = \frac{M(\sin\theta)^2}{2M-r} \quad \Bigg| \quad R_{4242} = \frac{-M(\sin\theta)^2}{2M-r}$$
$$R_{4334} = 2Mr(\sin\theta)^2 \quad | \quad R_{4343} = -2Mr(\sin\theta)^2$$

By raising all the indices, we end up with the following expressions,

$$
\begin{array}{l|l}
R^{1212} = \dfrac{2M}{r^3} & R^{1221} = -\dfrac{2M}{r^3} \\
R^{1313} = \dfrac{M}{(2M-r)r^4} & R^{1331} = \dfrac{-M}{(2M-r)r^4} \\
R^{1414} = \dfrac{M(\operatorname{cosec}\theta)^2}{(2M-r)r^4} & R^{1441} = -\dfrac{M(\operatorname{cosec}\theta)^2}{(2M-r)r^4} \\
R^{2112} = -\dfrac{2M}{r^3} & R^{2121} = \dfrac{2M}{r^3} \\
R^{2323} = \dfrac{-M(2M-r)}{r^6} & \\
R^{2332} = \dfrac{M(2M-r)}{r^6} & \\
R^{2424} = \dfrac{-M(2M-r)(\operatorname{cosec}\theta)^2}{r^6} & R^{2442} = \dfrac{M(2M-r)(\operatorname{cosec}\theta)^2}{r^6} \\
R^{3113} = \dfrac{-M}{(2M-r)r^4} & R^{3131} = \dfrac{M}{(2M-r)r^4} \\
R^{3223} = \dfrac{M(2M-r)}{r^6} & R^{3232} = \dfrac{-M(2M-r)}{r^6} \\
R^{3434} = -\dfrac{2M(\operatorname{cosec}\theta)^2}{r^7} & R^{3443} = \dfrac{2M(\operatorname{cosec}\theta)^2}{r^7} \\
R^{4114} = \dfrac{-M(\operatorname{cosec}\theta)^2}{(2M-r)r^4} & R^{4141} = \dfrac{M(\operatorname{cosec}\theta)^2}{(2M-r)r^4} \\
R^{4224} = \dfrac{M(2M-r)(\operatorname{cosec}\theta)^2}{r^6} & R^{4242} = -\dfrac{M(2M-r)(\operatorname{cosec}\theta)^2}{r^6} \\
R^{4334} = \dfrac{2M(\operatorname{cosec}\theta)^2}{r^7} & R^{4343} = -\dfrac{2M(\operatorname{cosec}\theta)^2}{r^7}
\end{array}
$$

A.2.3 Calculation and Expression of ADM equations

The expression of the first set of ADM equations becomes,

$$
\begin{aligned}
\partial_t h_{11} &= -2N\left(K_{11}\right)\\
&= \frac{-2}{r}\left(\frac{r-2M}{r}\right)^{1/2}\left(\frac{r-2M}{r}\right)^{-1/2}\left(\frac{\hbar c^9}{15360\pi G^2}\right)\left(\frac{1}{M^2}\right)\\
\partial_t h_{11} &= -\frac{2}{r}\left(\frac{\hbar c^4}{15360\pi G^2}\right)\left(\frac{1}{M_i^2}\right) - 2N\left(\partial_t K_{11}\right)t
\end{aligned}
$$

The expression of second set ADM equations becomes

$$
\begin{aligned}
&\partial_t K_{11} = -\nabla_1\nabla_1\sqrt{1-\frac{2M}{r}} - \sqrt{1-\frac{2M}{r}}\left(4\pi G(S-E)\left(-\left(1-\frac{2M}{r}\right)+S_{11}\right)\right.\\
&\nabla_1\sqrt{1-\frac{2M}{r}} = +\frac{1}{2}\left(\frac{r-2M}{r}\right)^{-1/2}\left(2\cdot\frac{dM}{dt}\right)\frac{1}{r}\\
&\nabla_1\nabla_1\sqrt{1-\frac{2M}{r}} = \frac{1}{r}\partial_t\left(\frac{1}{\sqrt{1-\frac{2M}{r}}}\right)\frac{dM}{dt} + \frac{d^2M}{dt^2}\left(\frac{1}{\sqrt{1-\frac{2M}{r}}}\right)\\
&\nabla_1\nabla_1\sqrt{1-\frac{2M}{r}} = \frac{1}{r}\left(-\frac{1}{2}\frac{1}{\left(1-\frac{2M}{r}\right)^{3/2}}\left(\frac{dM}{dt}\right)^2\right) + \frac{d^2M}{dt^2}\left(\frac{1}{\sqrt{1-\frac{2m}{r}}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
S &= \frac{\log M^2}{8\pi^2}\left(\frac{1}{8}m_0^4 + \frac{46M}{720r^6}\right)\left(4\left(1-\frac{M}{r}\right) + \frac{1}{\left(1-\frac{2GM}{r}\right)} + r^2 + r^2\sin\theta\right)\\
E &= \frac{\log M^2}{8\pi^2}\left(\frac{1}{8}m_0^4 + \frac{46M^2}{720\left(4r^6\right)}\right)\\
S_{11} &= 4\frac{\log M^2}{8\pi^2}\left(\left(1-\frac{2M}{r}\right)\left(\frac{1}{8}m_0^4 + \frac{46M^2}{720\left(4r^6\right)}\right)\right)
\end{aligned}
$$

The second set of coupled ADM equations are

$$\partial_t h_{22} = -2\sqrt{1-\frac{2M}{r}}\nabla_2(0) = 0$$

$$\partial_t K_{22} = \nabla_2\nabla_2\sqrt{1-\frac{2M}{r}} - \sqrt{1-\frac{2M}{r}}\left(-4\pi G(S-E)\left(\frac{1}{1-\frac{2M}{r}}\right) - 8\pi G S_{22}\right)$$

where,

$$\nabla_2\nabla_2\sqrt{1-\frac{2M}{r}} = \frac{1}{2}\left(1-\frac{2M}{r}\right)^{-1/2} + \frac{4M}{r^3} - \frac{4M^2}{r^4} - \frac{1}{4}\left(1-\frac{2M}{r}\right)^{-3/2}$$

$$S_{22} = \frac{\log M^2}{8\pi^2}\left(\frac{1}{\left(1-\frac{2M}{r}\right)}\left(\frac{1}{8}m_0^4 + \frac{46M^2}{720r^6}\right)\right)$$

The last coupled ADM equations are

$$\partial_t h_{21} = -2\sqrt{1-\frac{2M}{r}}\nabla_2\sqrt{1-\frac{2M}{r}}$$

$$\partial_t h_{21} = \frac{2M}{r^2} - 2\sqrt{1-\frac{2M}{r}}\left(\partial_t K_{21}\right)t$$

$$\partial_t K_{21} = -\nabla_2\nabla_1\sqrt{1-\frac{2M}{r}} + \sqrt{1-\frac{2M}{r}}(4\pi)\left((S-E)(0) - 2S_{21}\right)$$

where,

$$\nabla_2\nabla_1\sqrt{1-\frac{2M}{r}} = \frac{dM}{dt}\left(\left(1-\frac{2M}{r}\right)^{-3/2}\frac{M}{r^2} - \left(1-\frac{2M}{r}\right)^{-1/2}\frac{1}{r^2}\right), S_{21} = 0$$

$$\partial_t K_{21} = \frac{dM}{dt}\left(\left(1-\frac{2M}{r}\right)^{-3/2}\frac{M}{r^2} - \left(1-\frac{2M}{r}\right)^{-1/2}\frac{1}{r^2}\right)$$

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