Application of Petz map to the bulk reconstruction problem in AdS/CFT

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by
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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled *Application of Petz map to the bulk reconstruction problem in AdS/CFT* in the partial fulfilment of the requirements for the award of the degree of Master of Science and submitted in the discipline of Physics, Indian Institute of Technology Indore, is an authentic record of my own work carried out during the time period from August 2023 to May 2024 under the supervision of Dr. Debajyoti Sarkar, Assistant professor, Indian Institute of Technology Indore.

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Abstract

The global bulk reconstruction procedure in the context of the AdS/CFT correspondence was accomplished during the first decade of the 21st century. This global reconstruction technique can be utilized to perform entanglement wedge reconstruction in Rindler coordinates, where access is limited to the entanglement wedge within the Cauchy slice and the corresponding boundary sub-region. Recently, entanglement wedge reconstruction has been achieved using a universal quantum recovery channel, but only for true free fields. We have observed that the recovery channel for a maximally mixed reference state corresponds to a first-order change in the excited state modular Hamiltonian, accompanied by a scaling factor dimension of code space.

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Dedication

This dedication is extended to my parents in heartfelt appreciation for their unwavering and divine affection.

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Chapter 1

Introduction

The AdS/CFT correspondence, a pioneering concept in theoretical physics, provides a fundamental synthesis of ideas from two dissimilar domains: gravity and quantum field theory. At its root, it is a remarkable idea that unites our understanding of the cosmos on both the smallest and largest sizes. Conceived by Juan Maldacena in 1997, this correspondence states that a theory of gravity in Anti-de Sitter space (AdS), a negatively curved spacetime, is mathematically equivalent to a quantum field theory residing on the frontier of that space. This extreme duality has attracted the attention of mathematicians and physicists alike, providing insights into the holographic principle, the quantum nature of gravity, and even the inner workings of fundamental particles.

In this paper, Chapter 1 and Chapter 2 provided a brief overview of Anti-de Sitter space, conformal field theory, and the AdS/CFT correspondence, followed by a discussion on bulk reconstruction. The primary objective was to understand how bulk reconstruction has been achieved using quantum channels, as proposed by *Cotlar et al* [1]. To grasp this concept, an understanding of the functioning of quantum channels was necessary, which was covered in Chapter 3. Chapters 4 and Chapter 5 delved into the entanglement wedge reconstruction, achieved through the Petz map and the twirled Petz map. Subsequently, the research question addressed the connection between the recovery channel and the *Sarosi-Ugajin* [2] formula for the first-order change in the excited state modular Hamiltonian. Ultimately, the conclusion drawn was that a new definition of the ground state modular Hamiltonian could be formulated based on this connection.

1.1 Anti-de-Sitter Space

The maximally symmetric spacetime with negative cosmological constant is Anti-de Sitter spacetime or AdS spacetime for short. $(d+1)$ dimensional Anti-de Sitter space, AdS_{d+1} for short, may be embedded into $(d+2)$ dimensional Minkowski spacetime $(X^0, X^1, \ldots, X^d, X^{d+1}) \in \mathbb{R}^{d,2}$, with metric $\bar{\eta} = diag(-, +, +, \ldots, +, -)$, i.e.

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + \cdots + (dX^d)^2 - (dX^{d+1})^2 = \bar{\eta}_{MN} dX^M dX^N$$

where $M, N \in \{0, \ldots, d+1\}$. In particular, AdS_{d+1} is given by the hypersurface

$$\bar{\eta}_{MN} dX^M dX^N = -(dX^0)^2 + \sum_{i=1}^{d} (dX^i)^2 - (dX^{d+1})^2 = -L^2, \tag{1.1}$$

inside $\mathbb{R}^{d,2}$. The Anti-de Sitter space's radius of curvature is represented by L in 1.1. Note that the hypersurface represented by 1.1 is invariant under $O(d, 2)$ transformations performed on $\mathbb{R}^{d,2}$ in the normal way. Stated otherwise, AdS_{d+1} has an isometry group of $O(d, 2)$. Like $(d + 1)$-dimensional Minkowski spacetime, $O(d, 2)$ contain $(d + 1)(d + 2)/2$ Killing generators. As a result, Anti-de Sitter space has maximal symmetry as well.

Anti-de Sitter space has a conformal boundary. For large X^M, the hyperboloid given by 1.1 approaches the light-cone in $\mathbb{R}^{d,2}$ given by

$$\bar{\eta}_{MN} dX^M dX^N = -(dX^0)^2 + \sum_{i=1}^{d} (dX^i)^2 - (dX^{d+1})^2 = 0 \tag{1.2}$$

Therefore, we may define a 'boundary' of Anti-de Sitter space by the set of all lines on the light-cone 1.2 originating from $0 \in \mathbb{R}^{d,2}$. In a more fancy notation, the conformal boundary of AdS_{d+1}, denoted by ∂AdS_{d+1}, is given by the set of points

$$\partial AdS_{d+1} = \{[X] | X \in \mathbb{R}^{d,2}, X \neq 0, \bar{\eta}_{MN} dX^M dX^N = 0\}, \tag{1.3}$$

where we identify $[X]$ with $[\tilde{X}]$ if $(X^0, X^1, \ldots, X^{d+1}) = \lambda(\tilde{X}^0, \tilde{X}^1, \ldots, \tilde{X}^{d+1})$ for a real number λ. To see the topology of conformal boundary ∂AdS_{d+1} , we can represent any element $[X]$ of ∂AdS_{d+1} by the points X satisfying

$$\sum_{i=1}^{d} (X^i)^2 = 1 \tag{1.4}$$

Since X also has to satisfy 1.2, we further obtain

$$(X^0)^2 + (X^{d+1})^2 = 1 \tag{1.5}$$

How should we consider the space ∂AdS_{d+1}? It turns out that ∂AdS_{d+1} is a compactification of d-dimensional Minkowski space-time. To verify this, consider a point $X \neq 0$ satisfying 1.2. Introducing coordinates (u, v) by

$$u = X^{d+1} + X^d$$
$$v = X^{d+1} - X^d$$

we may rewrite 1.2 as

$$uv = \eta_{\mu\nu} X^\mu X^\nu \tag{1.6}$$

where the values of μ and ν are taken from $0, \ldots, d-1$, and the diagonal matrix $\eta_{\mu\nu}$ has entries $diag(-1, 1, \ldots, 1)$. We can re-scale the X whenever $\nu \neq 0$, ensuring that $\nu \neq 0$. Solving 1.6 for u requires knowing X^μ and $\mu \in 0, \ldots, d-1$. Minkowski spacetime with dimension d is thus obtained for $\nu \neq 0$. In d-dimensional Minkowski spacetime, we introduced infinities to the points with $\nu = 0$. Upon examining 1.6, we can observe that our Minkowski spacetime has a light cone added to it. To define conformal transformations, this is required. This also clarifies why d-dimensional Minkowski spacetime is a conformal compactification of ∂AdS_{d+1}. In the next section, we will study Conformal Field Theory.

1.2 Conformal Field Theory

Conformal field theory (CFTs) are actually relativistic quantum field theory with *Poincaré* symmetry and some scaling symmetry[3]

$$x'^\mu = \lambda x^\mu$$

and *special conformal transformations*

$$x^{\mu'} = \frac{x^\mu + a^\mu x^2}{1 + 2a_\nu x^\nu + a^2 x^2}$$

In conformal field theory, primary operators, which are local operators transforming as

$$e^{iD\alpha} \mathcal{O}(x) e^{-iD\alpha} = e^{\alpha\Delta}(e^\alpha x)$$

$$e^{iK_\mu a^\mu}\mathcal{O}(0)e^{-iK_\mu a^\mu} = \mathcal{O}(0)$$

The quantity Δ is known as the *scaling dimension* of $\mathcal{O}$. By iteratively applying derivatives to a primary operator $\mathcal{O}$, we can obtain its *descendant operators*. The scaling dimensions of these operators are determined by adding Δ to the number of derivatives applied. Descendants are never themselves primary unless they vanish.

A fundamental characteristic of CFTs is the *state-operator correspondence*, which is the set of primary operators and their progeny at any given point x being in one-to-one correspondence with a full basis of the Hilbert space of the CFT quantized on $\mathbb{S}^{d-1}$. By calculating the route integral on a Euclidean solid ball centred on the operator, the map from operators to states is defined. It is invertible because, given a state on the ball's boundary, we can dilate the ball to a point by defining an operator that, when we scale the ball back up, would create that state. Moreover, if the operator we are interested in has dimension Δ, the state on $\mathbb{S}^{d-1}$ will have energy $\Delta + E_0$. The ground state energy is denoted by E_0. Depending on the dimension, we may or may not be able to use a local counter term to establish $E_0 = 0$. The derivation of this relation involves the observation that if polar coordinates in the Euclidean origin region are taken and then transformed into $\rho = e^\tau$, we have

$$ds^2 = d\rho^2 + \rho^2 d\Omega_{d-1}^2 = e^{2\tau}(d\tau^2 + d\Omega_{d-1}^2), \tag{1.7}$$

so dilations $\rho' = e^\alpha \rho$ are equivalent to Euclidean cylinder time translations $\tau' = \tau + \alpha$.

1.3 AdS/CFT correspondence

The AdS/CFT correspondence, proposed by Juan Maldacena [4] in 1997, is a remarkable duality that relates a quantum theory of gravity in Anti-de Sitter (AdS) space to a conformal field theory (CFT) living on the boundary of that space. This correspondence provides a remarkable connection between two seemingly disparate theories, allowing insights from one side to be translated to the other. It not only provides insights into the behavior of strongly coupled quantum systems via classical gravitational descriptions but also sheds light on fundamental aspects of quantum gravity itself. Moreover, the correspondence has found applications in various areas of physics, from condensed matter systems to black hole physics and quantum information theory, continuing to inspire new avenues of research and exploration. As previously observed, a conformal boundary is located at the boundary of AdS (or bulk). When considering a cylinder, the volume inside represents a three-dimensional space, while the boundary represents a two-dimensional space. If the

AdS is located within the volume or bulk of the cylinder, the CFT is situated on its boundary. It remains unchanged regardless of the resizing of grid dimensions. The introduction of duality causes the two-dimensional border space to transform into a three-dimensional space. The original space is flat, while the new space has negative curvature, namely a hyperbolic, anti-de Sitter, or AdS space. The conformal field theory in the initial space without gravity, but it transforms into a comprehensive quantum theory of gravity when extended to higher-dimensional space. This refers to the AdS/CFT duality.

The authors of the paper [5] assert that there exists a correspondence between any conformal field theory on $\mathbb{R} \otimes \mathbb{S}^{d-1}$ and a theory of quantum gravity in an asymptotically AdS_{d+1} spacetime. This takes us to how the observable on both sides (border and bulk) are mapped. The solution is in the AdS/CFT dictionary. It can be viewed as an isomorphism between the Hilbert spaces:

$$\phi : \mathcal{H}_{AdS} \longrightarrow \mathcal{H}_{CFT}$$

where ϕ is the map between the bulk and the boundary. We obtain the extrapolate dictionary by evaluating the limit as r approaches infinity of $r^{\Delta}\phi_i(r, t, \Omega)$, which equals $\mathcal{O}_i(t, \Omega)$. where r is the radial bulk direction, Δ is the scaling dimension of conformal primary $\mathcal{O}_i$. This might be understood as we are nearing the boundary from the bulk limit, which is the extrapolate dictionary formalism. One such extended dictionary formalism is the HKLL prescription in which a bulk field is equal to a boundary operator being smeared over a causal diamond using a kernel. We shall explore more about HKLL in the coming sections.

Chapter 2

Bulk reconstruction

Bulk reconstruction in the AdS/CFT correspondence is the process of extracting information about the dynamics of spacetime and gravity within the Anti-de Sitter (AdS) bulk from the behavior of a conformal field theory (CFT) residing on its boundary. In other words, it's the procedure of reconstructing what's happening in the "bulk" gravitational theory from observations made solely in the "boundary" CFT. Bulk reconstruction was developed by Alex Hamilton, Danial Kabat, Gilad Lifschytz and David A. Lowe in [6, 7, 8, 9], in their name Bulk reconstruction is often called as HKLL reconstruction or HKLL procedure. In this chapter, We will study HKLL reconstruction.

2.1 HKLL reconstruction

We will work in AdS_D in *Poincaré* coordinates, with metric

$$ds^2 = \frac{R^2}{Z^2}(-dT^2 + |dX|^2 + dZ^2), \tag{2.1}$$

here R is the AdS radius. The coordinate range over $0 < Z < \infty$, $-\infty < T < \infty$, and $X \in \mathbb{R}^{d-1}$ where $d = D - 1$.

We consider a free scalar field of mass m in this background. Normalizable solutions to the free wave equation $(-\Box + m^2)\phi = 0$ can be expanded in a complete set of modes

$$\phi(T, X, Z) = \int_{|\omega|>|k|} d\omega d^{d-1}k a_\omega e^{-i\omega T} e^{ik.T} Z^{d/2} J_\nu(\sqrt{\omega^2 - k^2}Z) \tag{2.2}$$

The Bessel function has order $\nu = \Delta - d/2$ where $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2R^2}$ is the conformal dimension of the corresponding operator. In *Poincaré* coordinates, we define the boundary field by

$$\phi_0^{Poincare}(T,X) = \lim_{Z\to 0}\frac{1}{Z^\Delta}\phi(T,X,Z)$$
$$= \frac{1}{2^\nu\Gamma(\nu+1)}\int_{|\omega|>|k|} d\omega d^{d-1}k a_\omega e^{-i\omega T}e^{ik.X}(\omega^2-k^2)^{\nu/2} \tag{2.3}$$

Note that

$$a_{\omega K} = \frac{2^\nu\Gamma(\nu+1)}{(2\pi)^d(\omega^2-k^2)^{\nu/2}}\int dT d^{d-1}X e^{i\omega T}e^{-ik.X}\phi_0^{Poincare}(T,X). \tag{2.4}$$

Substituting this back into the bulk mode expansion 2.2, we obtain an expression for the bulk field in terms of the boundary field, namely

$$\phi(T,X,Z) = \int dT' d^{d-1}X' k(T',X'|T,X,Z)\phi_0^{Poincare}(T',X') \tag{2.5}$$

where

$$k(T',X'|T,X,Z) = \frac{2^\nu\Gamma(\nu+1)}{(2\pi)^d(\omega^2-k^2)^{\nu/2}}\int_{|\omega|>|k|} d\omega d^{d-1}k e^{-i\omega(T-T')}e^{ik.(X-X')}$$
$$Z^{d/2}J_\nu(\sqrt{\omega^2-k^2}Z)/(\omega^2-k^2)^{\nu/2} \tag{2.6}$$

one generically obtains a smearing function with support on the entire boundary of the *Poincaré* patch. In the following, we will improve on this by constructing smearing functions that manifest the property that local bulk operators go over to local boundary operators as the bulk point approaches the boundary.

2.1.1 Poincare mode sum

Consider a field in AdS_3. The *Poincaré* mode sum 2.5 reads.

$$\phi(T,X,Z) = \frac{2^\nu\Gamma(\nu+1)}{4\pi^2}\int_{|\omega|>|k|} d\omega dk \frac{ZJ_\nu(\sqrt{\omega^2-k^2}Z)}{(\omega^2-k^2)^{\nu/2}}$$
$$\times\Big(\int dT'dX' e^{-i\omega(T-T')}e^{ik(X-X')}\phi_0^{Poincare}(T',X')\Big)$$

The *Poincaré* boundary field has no Fourier components with $|\omega| < |k|$, so provided we perform the T' and X' integrals first, we can subsequently integrate over ω and k without restriction. Thus

$$\phi(T, X, Z) = 2^{\nu}\Gamma(\nu+1)\int d\omega dk e^{-i\omega T} e^{ikX} \frac{Z J_{\nu}(\sqrt{\omega^2 - k^2}Z)}{(\omega^2 - k^2)^{\nu/2}} \phi_0^{Poincare}(\omega, k) \tag{2.7}$$

where $\phi_0^{Poincare}(\omega, k)$ is the Fourier transform of the boundary field. We now use the two integrals

$$\int_0^{2\pi} d\theta e^{-ir\omega \sin\theta - kr\cos\theta} = 2\pi J_0(r\sqrt{\omega^2 - k^2}) \tag{2.8}$$

$$\int_0^1 r dr(1 - r^2)^{\nu-1} J_0(br) = 2^{\nu-1}\Gamma(\nu) b^{-\nu} J_{\nu}(b) \tag{2.9}$$

to obtain

$$\frac{J_{\nu}(\sqrt{\omega^2 - k^2}Z)}{(\omega^2 - k^2)^{\nu/2}} = \frac{1}{\pi(2Z)^{\nu}\Gamma(\nu)} \int_{T'^2 + Y'^2 < Z^2} dT' dY' (Z^2 - T'^2 - Y'^2)^{\nu-1} e^{-i\omega T'} e^{-kY'} \tag{2.10}$$

Inserting this into 2.7, one gets

$$\begin{aligned} \phi(T, X, Z) = \frac{\nu}{\pi} \int_{T'^2 + Y'^2 < Z^2} \Big(\frac{Z^2 - T'^2 - Y'^2}{Z}\Big)^{\nu-1} \\ \times \int d\omega dk e^{-i\omega(T+T')} e^{iK(X+iY')} \phi_0^{Poincare}(\omega, k) \end{aligned} \tag{2.11}$$

We identify the second line of 2.11 as $\phi_0^{Poincare}(T + T', X + iY')$, so we can write (recall $\nu = \Delta - 1$)

$$\phi(T, X, Z) = \frac{\Delta - 1}{\pi} \int_{T'^2 + Y'^2 < Z^2} \Big(\frac{Z^2 - T'^2 - Y'^2}{Z}\Big)^{\nu-2} \phi_0^{Poincare}(T + T', X + iY') \tag{2.12}$$

We have managed to represent the bulk field in the (real T, imaginary X) plane as an integral over a disc with radius Z. This is a bulk reconstruction equation, where ϕ is a bulk field and $\phi_0^{Poincare}$ is the boundary field in *Poincaré* coordinates. Our primary objective is to comprehend the bulk reconstruction using recovery channel technique used by *Cotler et al* [1]. In the following chapter what is recovery channel and how it works.

Chapter 3

Quantum recovery channel

The Petz transpose map or Petz recovery channel has been a ubiquitous tool in quantum information theory and has been at the forefront of research within this field. Originally discovered by D. Petz in the 1980s [10, 11, 12], it was further rediscovered within a different context in quantum error correction [13] and within quantum statistical mechanics [14].

3.1 Petz recovery channel

A von Neumann algebra is a algebra of bounded operators on a Hilbert space. Let $\mathcal{N} : \mathcal{M} \to \mathcal{M}_0$ be a channel between the von Neumann algebras $\mathcal{M}$ and $\mathcal{M}_0$. $\mathcal{M}$ acts on Hilbert space $\mathcal{H}$ and $\mathcal{M}_0$ acts on Hilbert space $\mathcal{H}_0$. Assume that the input state σ is normal and corresponding output state $\mathcal{N}(\sigma)$ is faithful and normal. Then there exists a unique channel $\mathcal{P}_{\sigma,\mathcal{N}} : \mathcal{M}_0 \to \mathcal{M}$ characterized by the relation,

$$\langle\langle A, \mathcal{N}^\dagger(B_0)\rangle\rangle_\sigma = \langle\langle \mathcal{P}^\dagger_{\sigma,\mathcal{N}}(A), B_0\rangle\rangle_{\mathcal{N}(\sigma)} \tag{3.1}$$

In above relation in the left hand side all quantity belongs to $\mathcal{M}$ and in the right hand side all quantity belongs to $\mathcal{M}_0$. Dagger of a channel changes the direction of the channel.

$$A, \mathcal{N}^\dagger(B_0), \sigma \in \mathcal{M} \text{ and } \mathcal{P}^\dagger_{\sigma,\mathcal{N}}(A), B_0, \mathcal{N}(\sigma) \in \mathcal{M}_0$$

$$\mathcal{N} : \mathcal{M} \to \mathcal{M}_0$$

$$\mathcal{N}^\dagger : \mathcal{M}_0 \to \mathcal{M}$$

$$\mathcal{P}_{\sigma,\mathcal{N}} : \mathcal{M}_0 \to \mathcal{M}$$

$$\mathcal{P}^\dagger_{\sigma,\mathcal{N}} : \mathcal{M} \to \mathcal{M}_0$$

So the relation 3.1 is connecting to different Hilbert space $\mathcal{H}$ and $\mathcal{H}_0$. Weighted Hilbert-Schmidt inner product is defined for bounded operators a and b and state ζ as

$$\langle\langle a, b\rangle\rangle_\zeta \equiv \langle Tr[a^\dagger \zeta^{1/2} b \zeta^{1/2}]\rangle \tag{3.2}$$

Using 3.2, we can expand the inner product in 3.1

$$A^\dagger \sigma^{1/2} \mathcal{N}^\dagger(B_0)\sigma^{1/2} = [\mathcal{P}^\dagger_{\sigma,\mathcal{N}}(A)]^\dagger \mathcal{N}(\sigma)^{1/2} B_0 \mathcal{N}(\sigma)^{1/2} \tag{3.3}$$

Now, we apply $B_0 \to \mathcal{N}(\sigma)^{-1/2} B_0 \mathcal{N}(\sigma)^{-1/2}$, we get

$$[\mathcal{P}^\dagger_{\sigma,\mathcal{N}}(A)]^\dagger B_0 = A^\dagger \sigma^{1/2} \mathcal{N}^\dagger(\mathcal{N}(\sigma)^{-1/2} B_0 \mathcal{N}(\sigma)^{-1/2})\sigma^{1/2} \tag{3.4}$$

In the left hand side $[\mathcal{P}^\dagger_{\sigma,\mathcal{N}}(A)]^\dagger B_0$ can be written in the following way

$$[\mathcal{P}^\dagger_{\sigma,\mathcal{N}}(A)]^\dagger B_0 = [\mathcal{P}^\dagger_{\sigma,\mathcal{N}} \cdot A]^\dagger \cdot B_0 = A^\dagger \cdot [\mathcal{P}^\dagger_{\sigma,\mathcal{N}}]^\dagger \cdot B_0 = A^\dagger \cdot \mathcal{P}_{\sigma,\mathcal{N}} \cdot B_0 = A^\dagger \mathcal{P}_{\sigma,\mathcal{N}}(B_0) \tag{3.5}$$

In the above relation, $[\mathcal{P}^\dagger_{\sigma,\mathcal{N}}(A)]^\dagger B_0$ belongs to $\mathcal{M}_0$ but $A^\dagger \mathcal{P}_{\sigma,\mathcal{N}}(B_0)$ belongs to $\mathcal{M}$

$$[\mathcal{P}^\dagger_{\sigma,\mathcal{N}}(A)]^\dagger B_0 \in \mathcal{M}_0 \text{ but } A^\dagger \mathcal{P}_{\sigma,\mathcal{N}}(B_0) \in \mathcal{M} \tag{3.6}$$

So, 3.4 becomes

$$A^\dagger \mathcal{P}_{\sigma,\mathcal{N}}(B_0) = A^\dagger \sigma^{1/2} \mathcal{N}^\dagger(\mathcal{N}(\sigma)^{-1/2} B_0 \mathcal{N}(\sigma)^{-1/2})\sigma^{1/2} \tag{3.7}$$

In last 3.7, all quantities in both sides belongs to $\mathcal{M}$ and we can compare both sides and see $\mathcal{P}_{\sigma,\mathcal{N}}(\cdot)$ has the form

$$\mathcal{P}_{\sigma,\mathcal{N}}(\cdot) = \sigma^{1/2} \mathcal{N}^\dagger(\mathcal{N}(\sigma)^{-1/2} (\cdot) \mathcal{N}(\sigma)^{-1/2})\sigma^{1/2} \tag{3.8}$$

Here $\mathcal{P}_{\sigma,\mathcal{N}}$ is Petz recovery map. Here we can see in the form of Petz map $\mathcal{P}_{\sigma,\mathcal{N}}(\cdot)$ there is only one operation that is $\mathcal{N}^\dagger(*)$. $\mathcal{P}_{\sigma,\mathcal{N}}(\cdot)$ and $\mathcal{N}^\dagger(*)$ both maps $\mathcal{M}_0$ to $\mathcal{M}$.

$$\mathcal{P}_{\sigma,\mathcal{N}} : \mathcal{M}_0 \to \mathcal{M}$$

$$\mathcal{N}^\dagger : \mathcal{M}_0 \to \mathcal{M}$$

This implies Petz map form is correct and works as exact recovery channel. The map $\mathcal{P}_{\sigma,\mathcal{N}}$ is unique if $\mathcal{N}(\sigma)$ is a faithful operators. If σ is on a finite-dimensional Hilbert space and $\mathcal{N}$ is a quantum channel with finite-dimensional inputs and outputs, then the Petz map takes the following explicit form,

$$\mathcal{P}_{\sigma,\mathcal{N}}(\cdot) \equiv \sigma^{1/2} \mathcal{N}^\dagger\big(\mathcal{N}(\sigma)^{-1/2} (\cdot) \mathcal{N}(\sigma)^{-1/2}\big)\sigma^{1/2} \tag{3.9}$$

3.2 Approximate recovery channel

Now consider two states ρ and σ as an input states belongs to the set of density operators on Hilbert space $\mathcal{H}$

$$\rho, \sigma \in S(\mathcal{H}) \tag{3.10}$$

we apply channel $\mathcal{N} : S(\mathcal{H}) \to S(\mathcal{H}_0)$ on input states and get $\mathcal{N}(\rho)$ and $\mathcal{N}(\sigma)$ respectively.

$$\mathcal{N}(\rho), \mathcal{N}(\sigma) \in S(\mathcal{H}_0) \tag{3.11}$$

For special case we get Petz recovery channel $\mathcal{P}$ which can recover the action of channel $\mathcal{N}$,

$$(\mathcal{P} \circ \mathcal{N})(\rho) = \rho \text{ and } (\mathcal{P} \circ \mathcal{N})(\sigma) = \sigma \tag{3.12}$$

Now the special case is the relative entropy of two input states and the output states is equal.

$$D(\rho||\sigma) = D(\mathcal{N}[\rho]||\mathcal{N}[\sigma]) \tag{3.13}$$

where $D(\rho||\sigma) := Tr(\rho \log \rho) - Tr(\rho \log \sigma)$ is the relative entropy between ρ, σ. This is called the saturation of monotonicity of relative entropy.

If equality does not holds or we can say it failure to saturate then exact reversal map $\mathcal{P}$ cannot exist but there is still the possibility of an approximate reversal map which would behave well in cases of near saturation. Indeed, an approximate version of the recovery channel was developed by *Junge et al.* [15] , who show that, for any $\rho, \sigma \in S(\mathcal{H})$ and any quantum chennel $\mathcal{N}$, there exists a recovery channel $\mathcal{R}_{\sigma,\mathcal{N}}$ such that

$$D(\rho||\sigma) - D(\mathcal{N}[\rho]||\mathcal{N}[\sigma]) \geq -2 \log F(\rho, \mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho)) \tag{3.14}$$

where $F(\rho, \sigma) \equiv ||\sqrt{\rho}\sqrt{\sigma}||_1$ and Schatten p-norm $||L||_p = (Tr(|L|^p))^{1/p}$. The inequality says that the fidelity between the recovered state and the original is controlled by the saturation gap in $D(\rho||\sigma) - D(\mathcal{N}[\rho]||\mathcal{N}[\sigma])$, with perfect fidelity in the case of saturation. Junge *et al.* [15] gave a concrete expression for the channel $\mathcal{R}_{\sigma,\mathcal{N}}$, called *approximate Petz recovery channel* or *twirled Petz map* and given by

$$\begin{aligned} \mathcal{R}_{\sigma,\mathcal{N}}(\cdot) &:= \int_{\mathbb{R}} dt \beta_0(t) \sigma^{-\frac{it}{2}} \mathcal{P}_{\sigma,\mathcal{N}}[\mathcal{N}(\sigma)^{\frac{it}{2}}(\cdot)\mathcal{N}(\sigma)^{\frac{-it}{2}}]\sigma^{\frac{it}{2}} \\ &= \int_{\mathbb{R}} dt \beta_0(t) \sigma^{\frac{1-it}{2}} \mathcal{N}^\dagger[\mathcal{N}(\sigma)^{\frac{-1+it}{2}}(\cdot)\mathcal{N}(\sigma)^{\frac{-1-it}{2}}]\sigma^{\frac{1+it}{2}} \end{aligned} \tag{3.15}$$

where $\mathcal{P}_{\sigma,\mathcal{N}}$ is so-called *Petz map* of 3.9 and β_0 is the probability density $\beta_0(t) := \frac{\pi}{2}(cosh(\pi t) + 1)^{-1}$. Derivation of $\mathcal{R}_{\sigma,\mathcal{N}}$ can be found in Appendix A.1.

Chapter 4

Entanglement wedge reconstruction using Petz recovery channel

To tackle the entanglement wedge reconstruction problem using information-theoretic methods, we must first reinterpret our objective in terms of quantum information.

In the AdS/CFT correspondence, a bulk quantum gravity theory and a boundary conformal field theory are dual to one another. A true duality of theories should be represented by AdS/CFT if the "bulk" Hilbert space and the boundary Hilbert space H_{CFT} are isomorphic. However, a complete, non-perturbative, microscopic description of the entire Hilbert space from a purely bulk perspective, if one exists, remains unknown. Moreover, any such Hilbert space would be dominated by huge black holes. Usually, we limit our attention to a small subset of states having a smooth semiclassical bulk geometry, which we refer to as the "code subspace" H_{code}. Tiny bulk perturbations concerning the vacuum state may be included in this. As a consequence, we establish an isometry $J : \mathcal{H}_{code} \to \mathcal{H}_{CFT}$. One may consider an equivalent to be the quantum channel $\mathcal{J}(\cdot) = J(\cdot)J^{\dagger}$, which converts bulk density matrices to boundary density one. Rather than being a more generic isometry, it turns out that none of our results depend on $\mathcal{J}$ being a quantum channel.

The algebra of observables for the Hilbert space $\mathcal{H}_{code}$ is denoted by $\mathcal{B}(\mathcal{H}_{code})$, and $\mathcal{B}(\mathcal{H}_{CFT})$ for the algebra of observables on $\mathcal{H}_{CFT}$. When we consider the entanglement wedge a, we assume that it has an associated von Neumann subalgebra $\mathcal{M}_a \hookrightarrow \mathcal{B}(\mathcal{H}_{CFT})$, composed of bulk observables that act only on a, just as the boundary area A is associated with one.

The question of whether the channel $\mathcal{N} = Tr_{\bar{A}}[\mathcal{J}(\cdot)]$ forms an approximation error-correcting code for the algebra $\mathcal{M}_a$ can be reformulated as the question of entanglement wedge reconstruction. In this case, the restriction channel $Tr_{\bar{A}}[\cdot]$ merely projects the density matrix onto the algebra $\mathcal{M}_A$. Stated differently, the possibility of reconstructing an entanglement wedge is depends upon the existence of a decoding channel $\mathcal{D} : S(\mathcal{M}_A) \to S(\mathcal{M}_a)$

$$\mathcal{D} \circ \mathcal{N}(\rho) \approx \rho_a \tag{4.1}$$

for all states $\rho \in S(\mathcal{H}_{code})$; the restriction ρ_a is the projection of ρ onto $\mathcal{M}_a$.

Theorem 1. *Let $\mathcal{M}_a \hookrightarrow \mathcal{B}(\mathcal{H}_{code})$ be a von Neumann subalgebra acting on the code space $\mathcal{K}_{code}$ with dimention d_{code}, let $\mathcal{N}$ be a quantum channel, and suppose that there exists a channel $\mathcal{D}'$ such that*

$$||\mathcal{D}' \circ \mathcal{N}(\rho) - \rho_a||_1 < \delta. \tag{4.2}$$

Let

$$\mathcal{P}_{\tau,\mathcal{N}} := \frac{1}{d_{code}} \mathcal{N}^\dagger \left[\mathcal{N}(\tau)^{-1/2} (\cdot) \mathcal{N}(\tau)^{-1/2} \right] \tag{4.3}$$

be the Petz map with maximally mixed reference state τ. Then

$$||\mathcal{P}_{\tau,\mathcal{N}} \circ \mathcal{N}(\rho)|_a - \rho_a||_1 < d_{code}\sqrt{8\delta}. \tag{4.4}$$

Proof of **theorem 1** can be found in [16].

As we can see from **theorem 1**, if the error utilising the original decoding channel $\mathcal{D}'$ is non-perturbatively small, then the Petz map error will also be non-perturbatively small providing that the dimension of the code space does not expand superpolynomially in N. For most code spaces of interest, such as perturbations about the vacuum, for which the code space dimension will be $O(1)$, this aspect of the code space size is not a problem. The Petz map may always be relied upon, provided that we limit our analysis to perturbative excitations of quantum fields inside a specific gravitational backdrop. There is no need of universal recovery channel. As a result, entanglement wedge reconstruction is feasible with the Petz map, provided that the size of the code space that we expect to be able to reconstruct is manageable. Specifically, if the code space dimension does not grow superpolynomially in the limit of large N, the Petz map offers a strong recovery map.

No effort is made to assess the Petz map in specific situations. Even though the Petz map is much easier to record and examine than the twisted Petz map, there are still a lot of barriers to overcome. Let's talk about the difficulties at hand. We wish to explicitly evaluate

$$\mathcal{O}_A = \frac{1}{d_{code}} \tau_A^{-1/2} [J \phi J^\dagger]_A \tau_A^{-1/2}. \tag{4.5}$$

Projecting the global HKLL boundary reconstruction $\mathcal{O}^{HKLL}$ into the coding space yields the operator $J\phi_a J^\dagger$,

$$J\phi_a J^\dagger = P_{code}\mathcal{O}^{HKLL}P_{code}. \tag{4.6}$$

Thus, the main things is in identifying the operator's restriction to region A. To keep things simple, we'll assume that the CFT Hilbert space factorises as $\mathcal{H}_{CFT} \approxeq \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, and that $\mathcal{M}_A \approxeq \mathcal{B}(\mathcal{H}_A)$. In that case, the restriction map is merely a partial trace over $\mathcal{H}_{\bar{A}}$. As a matter of convention, this assumption is made for simplicity (although not with reality). One problem is that the operator $\mathcal{O}$ generated by the HKLL technique is not time-confined. We must recast $\mathcal{O}_A$ in terms of operators at time zero using the Heisenberg equations of motion in order to obtain the partial trace over region $\bar{A}$. These operators are typically very complex and challenging to analyse. The problem is essentially the standard problem to the evaluation of quantities that cannot be protected by symmetry on the boundary side of AdS/CFT. Strongly coupled quantum field theories are just challenging to work with; luckily, there is also a weakly coupled bulk. The entanglement wedge reconstruction for the extremely rare scenario of real free fields and low dimensional code space will be discussed in the upcoming chapter.

Chapter 5

Entanglement wedge reconstruction using approximate recovery channel

We suppose $\mathcal{H}_{code}$ is a code space Hilbert space with the set of density operators $S(\mathcal{H}_{code})$, while $\mathcal{H}_{CFT}$ represents the Hilbert space associated with the set of density operators $S(\mathcal{H}_{CFT})$. States in $S(\mathcal{H}_{code})$ are connected to states in $S(\mathcal{H}_{CFT})$ by the AdS/CFT correspondence. *Cotler et al.* frame a relationship between the code space and the CFT Hilbert space by an isometry $J : \mathcal{H}_{code} \longrightarrow \mathcal{H}_{CFT}$. We now bipartite the CFT into two sections A and $\bar{A}$, and the bulk into a and $\bar{a}$, where a is maintained solely on the entanglement wedge of A, as shown in Fig. 5.1. We factorize $\mathcal{H}_{CFT}$ and $\mathcal{H}_{code}$ into $\mathcal{H}_{CFT} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ and $\mathcal{H}_{code} = \mathcal{H}_a \otimes \mathcal{H}_{\bar{a}}$.

The challenge of entanglement wedge reconstruction can be described as generating a boundary observable $\mathcal{O}_A$ supported solely on A, such that, for every bulk operator ϕ_a on

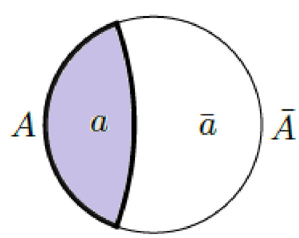

Figure 5.1: The boundary is divided into two parts: $\bar{A}$ and A, which are entangled with each other. Here, A part's causal wedge, indicated by a in the figure, and the entanglement wedge of A coincide. In such case, $\bar{a}$ is a's complement.

the entanglement wedge of A

$$|\langle \mathcal{O}_A \rangle_{J\rho J^\dagger} - \langle \phi_a \rangle_\rho| \leq \delta ||\phi_a||, \tag{5.1}$$

for all $\rho \in S(\mathcal{H}_{code})$ and for very small $\delta > 0$. If we write like the monotonicity relation, 3.13, for all $\rho, \sigma \in S(\mathcal{H}_{code})$,

$$|D(\rho_a||\sigma_a) - D((J\rho J^\dagger)_A||(J\sigma J^\dagger)_A)| \leq \epsilon, \tag{5.2}$$

where ϵ is for $1/N$, and the notation $(\cdot)_A := Tr_{\bar{A}}(\cdot)$ is a shorthand. ρ_a and σ_a are bulk states on entanglement wedge a. $(J\rho J^\dagger)_A$ and $(J\sigma J^\dagger)_A$ are the corresponding boundary states of ρ_a and σ_a respectively. The channel applied here to get boundary state on boundary sub-region A from bulk states is $\mathcal{N}(\cdot) = (J(\cdot)J^\dagger)_A = Tr_{\bar{A}}(J(\cdot)J^\dagger)$.

We can obtain a recovery channel that would reverse the partial trace over $\bar{A}$ as the respective entropies are roughly equal. However, there is a problem: Not only does $(J\rho J^\dagger)_A$ depend on the reduced state on the entanglement wedge, ρ_a, but it also depends on the state ρ, which is suppported on the entire bulk. In this form, the principle of recovery channels can not be used. To get rid of this obstacle, at first we will limit the recovery problem to particular code states which takes the form $\rho = \rho_a \otimes 1_{\bar{a}}$. Consequently, we obtain a quantum channel $\rho_a \to (J\rho J^\dagger)_A$, which maps the states on the boundary region to the states on the entanglement wedge.

With only a small amount of error added, we will see that the recovery channel $\mathcal{R}$ obtained for this channel truly functions for all code states ρ. The CFT states corresponding to any ρ and $(\rho_a \otimes 1_{\bar{a}})$ are almost similar on the boundary area A, according to 5.2.

Entanglement wedge reconstruction is possible because the adjoint of the recovery channel $\mathcal{R}^\dagger$ maps bulk operators ϕ_a on the entanglement wedge to boundary operators $\mathcal{O}_A$ on boundary sub-region A and satisfies 5.1.

We specify the local channel $\mathcal{N} : S(\mathcal{H}_a) \to S(\mathcal{H}_A)$ by

$$\mathcal{N}[\rho_a] := Tr_{\bar{A}}[J(\rho_a \otimes 1_{\bar{a}})J^\dagger] = (J(\rho_a \otimes 1_{\bar{a}})J^\dagger)_A \tag{5.3}$$

for all states $\rho_a \in S(\mathcal{H}_a)$. 3.14 can be used to obtain a recovery channel $\mathcal{R} = \mathcal{R}_{\sigma_a, \mathcal{N}}$ such that, for all $\rho_a \in S(\mathcal{H}_a)$,

$$D(\rho_a||\sigma_a) - D(\mathcal{N}[\rho_a]||\mathcal{N}[\sigma_a]) \geq -2 \log F(\rho_a, \mathcal{R}_{\sigma_a, \mathcal{N}} \circ \mathcal{N}(\rho_a)) \tag{5.4}$$

However, from 5.2, we have

$$|D(\rho_a||\sigma_a) - D(\mathcal{N}[\rho_a]||\mathcal{N}[\sigma_a])| \leq \epsilon, \tag{5.5}$$

and hence we may state that there will be good fidelity in the recovery channel $\mathcal{R}$. The reduced state supported by the entanglement wedge is recovered via the channel $\mathcal{R}$ for all code states ρ, not simply those with form $\rho = \rho_a \otimes 1_{\bar{a}}$, as may be shown. This shows that using one of the Fuchs-van de Graaf inequalities [17]

$$||\rho_a - \mathcal{R}[\mathcal{N}[\rho_a]]||_1 \leq 2\sqrt{\epsilon} := \delta_1 \tag{5.6}$$

for all $\rho_a \in S(\mathcal{H}_a)$.

$$\begin{aligned} ||\mathcal{N}[\rho_a] - (J\rho J^\dagger)_A||_1^2 &= ||(J(\rho_a \otimes 1_{\bar{a}})J^\dagger)_A - (J\rho J^\dagger)_A||_1^2 \\ &\leq (2\ln 2)D((J(\rho_a \otimes 1_{\bar{a}})J^\dagger)_A||(J\rho J^\dagger)_A) \\ &\leq (2\ln 2)\epsilon =: \delta_2^2, \end{aligned} \tag{5.7}$$

here we used Pinsker's inequality in first inequality and 5.2 in second inequality, with the one state ρ and the other state to $\rho_a \otimes 1_{\bar{a}}$. Therefore, we get that, for all $\rho \in S(\mathcal{H}_{code})$,

$$\begin{aligned} &||\rho_a - \mathcal{R}[(J\rho J^\dagger)_A]||_1 \\ \leq &||\rho_a - \mathcal{R}[\mathcal{N}[\rho_a]]||_1 + ||\mathcal{R}[\mathcal{N}[\rho_a]] - \mathcal{R}[(J\rho J^\dagger)_A]||_1 \\ \leq &||\rho_a - \mathcal{R}[\mathcal{N}[\rho_a]]||_1 + ||\mathcal{N}[\rho_a] - (J\rho J^\dagger)_A||_1 \\ \leq &\delta_1 + \delta_2 =: \delta \end{aligned} \tag{5.8}$$

As anticipated, we can observe that $\mathcal{R}$ accurately retrieves any bulk states that are supported on the entanglement wedge.

It is now evident that the entanglement wedge reconstruction problem can be solved by $\mathcal{R}^\dagger$, as expressed in 5.1. Define $\mathcal{O}_A = \mathcal{R}^\dagger[\phi_a]$, for any bulk operator ϕ_a on the entanglement wedge of A. Then, for every $\rho \in S(\mathcal{H}_{code})$, we obtain,

$$\begin{aligned} &|\langle \mathcal{O}_A \rangle_{J\rho J^\dagger} - \langle \phi_a \rangle_\rho| \\ =&|Tr\mathcal{R}^\dagger[\phi_a](J\rho J^\dagger)_A - Tr\phi_a\rho_a| \\ =&|Tr\phi_a\mathcal{R}(J\rho J^\dagger)_A - Tr\phi_a\rho_a| \\ =&|Tr\phi_a(\mathcal{R}(J\rho J^\dagger)_A - \rho_a)| \\ \leq&||\mathcal{R}(J\rho J^\dagger)_A - \rho_a||_1||\phi_a|| \leq \delta||\phi_a|| \end{aligned} \tag{5.9}$$

Here, we employ 5.8 for the second inequality and Hölder's inequality for the first. Therefore, $\mathcal{R}^\dagger$, which has an explicit form, can be used to achieve entanglement wedge reconstruction,

$$\mathcal{O}_A = \mathcal{R}^\dagger(\phi_a) = \frac{1}{d_{code}} \int dt \beta_0(t) e^{\frac{1-it}{2} H_A} \mathcal{N}(\phi_a) e^{\frac{1+it}{2} H_A}$$
$$= \frac{1}{d_{code}} \int_{\mathbb{R}} dt \beta_0(t) e^{\frac{1-it}{2} H_A} Tr_{\bar{A}}[J(\phi_a \otimes 1_{\bar{a}}) J^\dagger] e^{\frac{1+it}{2} H_A} \tag{5.10}$$

where d_{code} is the dimension of code space, $H_A = -\log(\mathcal{N}(\tau_{code})) = -\log(J\tau_{code}J^\dagger)_A$ and τ_{code} is maximally mixed code space state. Derivation of $\mathcal{R}^\dagger$ can be found in Appendix A.2.

5.1 Recovery channel for 2-dimensional code space

To keep things simple, we take the AdS_3 situation into consideration and utilise *Poincaré* patch coordinates,

$$ds^2 = \frac{l^2}{z^2}(-dt^2 + dx^2 + dz^2), \tag{5.11}$$

and we designate $Y = (t, x, z)$ for bulk coordinates. where $y = (t, x)$ for the border coordinates.

First, let us rebuild a bulk operator $\phi(Y)$ for $Y \in a$ that is supported on the Rindler wedge A's boundary. The ground state is excited by $|\tilde{1}\rangle = \phi(Y)|\tilde{0}\rangle$, which we have taken to be normalised, and the vacuum state is represented by $|\tilde{0}\rangle$ in our notation. Our two-dimensional code space will be $\mathcal{H}_{code} = span|\tilde{0}\rangle, |\tilde{1}\rangle$ in this case. Reconstructing the impact of the operator $\phi(Y)$ on the code space within the boundary interval A is our goal. In this case, $\tau = \frac{1}{2}(|\tilde{0}\rangle\langle\tilde{0}| + |\tilde{1}\rangle\langle\tilde{1}|$, represents the maximally mixed code space state. Observe that since there are no degrees of freedom in $\bar{a}$, we will assume for simplicity's sake that $\mathcal{H}_a = \mathcal{H}_{code}$.

In the code space, any operator $\phi(Y)$ translates the vacuum state to the excited state and vice versa. It will behave as follows: $X := |\tilde{1}\rangle\langle\tilde{0}| + |\tilde{0}\rangle\langle\tilde{1}|$. We currently possess every tool, the operator, and the maximally mixed state, and we can write 5.10 as

$$\mathcal{R}^\dagger(X) = \frac{1}{2} \int_{\mathbb{R}} dt \beta_0(t) \mathcal{N}(\tau)^{\frac{-1+it}{2}} \mathcal{N}[\tilde{1}\rangle\langle\tilde{0}| + |\tilde{0}\rangle\langle\tilde{1}|] \mathcal{N}(\tau)^{-\frac{1+it}{2} H_A} \tag{5.12}$$

where we used $\mathcal{N}(x) = Tr_{\bar{A}}[JxJ^\dagger]$ as a shorthand.

If we try to evaluate 5.12, we need to compute the below term

$$\mathcal{N}[|\tilde{x}\rangle\langle\tilde{y}|] = Tr_{\bar{A}}[|x\rangle\langle y|], \tag{5.13}$$

where $x, y \in 0, 1$, and the states are mapped like $|x\rangle := J|\tilde{x}\rangle$. The vacuum AdS $|\tilde{0}\rangle$ is mapped via $J|\tilde{0}\rangle = |0\rangle$ to the CFT vacuum state. The operation $\mathcal{N}$ is a linear operation, so,

$$\mathcal{N}[\tilde{1}\rangle\langle\tilde{0}| + |\tilde{0}\rangle\langle\tilde{1}|] = \mathcal{N}[\tilde{1}\rangle\langle\tilde{0}|] + \mathcal{N}[|\tilde{0}\rangle\langle\tilde{1}|] \tag{5.14}$$

HKLL maps the excited state $|\tilde{1}\rangle$ to

$$J|\tilde{1}\rangle =: |1\rangle = \int_{y'\in D} dy' K_g(Y, y')\Phi(y')|0\rangle \tag{5.15}$$

where $\Phi(y)$ is a boundary operator, D is a boundary spacetime domain, and K_g is a bulk-to-boundary kernel (g stands for "global").

In $1/N$, $\Phi(y)$ behaves as a generalised free field to leading order. Unfortunately, the breakdown into Rindler modes is usually not obeyed by generalised free fields. Therefore, we expanded the boundary field in terms of Rindler modes a_l, b_l on A and $\bar{A}$, respectively, treating it as a real free field:

$$\Phi(y) = \sum_l f_{a,l}(y)a_l + f^*_{a,l}(y)a^\dagger_l + f_{b,l}(y)b_l + f^*_{b,l}(y)b^\dagger_l. \tag{5.16}$$

We indecate ground state density matrix $\rho_{A,0} = Tr_{\bar{A}}[|0\rangle\langle 0|]$ for A.

$$\begin{aligned}
\mathcal{N}[|\tilde{1}\rangle\langle\tilde{0}|] &= Tr_{\bar{A}}[|1\rangle\langle 0|] \\
&= Tr_{\bar{A}}[\int_{y'\in D} dy' K_g(Y, y')\Phi(y')|0\rangle\langle 0|] \\
&= \int_{y'\in D} dy' K_g(Y, y') Tr_{\bar{A}}\big[\big(\sum_l f_{a,l}(y)a_l + f^*_{a,l}(y)a^\dagger_l + f_{b,l}(y)b_l + f^*_{b,l}(y)b^\dagger_l\big)|0\rangle\langle 0|\big]
\end{aligned} \tag{5.17}$$

To perform trace out operation, we use "transpose trick", using which the operators on the complement of set A, denoted as $\bar{A}$, can be expressed in terms of operators on set A. :

$$\begin{aligned}
b_l|0\rangle &= \rho^{1/2}_{A,0} a^\dagger_l \rho^{-1/2}_{A,0}|0\rangle \\
b^\dagger_l|0\rangle &= \rho^{1/2}_{A,0} a_l \rho^{-1/2}_{A,0}|0\rangle,
\end{aligned} \tag{5.18}$$

we denote

$$\begin{aligned}
\hat{f}_{a,l} &= \int_{y\in D} dy K_g(Y, y) f_{a,l}(y) \\
\hat{f}_{b,l} &= \int_{y\in D} dy K_g(Y, y) f_{b,l}(y),
\end{aligned} \tag{5.19}$$

so, $\mathcal{N}[|\tilde{1}\rangle\langle\tilde{0}|]$ becomes,

$$\begin{aligned}\mathcal{N}[|\tilde{1}\rangle\langle\tilde{0}|] &= \int_{y'\in D} dy' K_g(Y,y') Tr_{\bar{A}}\big[\big(\sum f_{a,l}(y)a_l + f^*_{a,l}(y)a^\dagger_l + f_{b,l}(y)b_l + f^*_{b,l}(y)b^\dagger_l\big)|0\rangle\langle 0|\big] \\ &= \big(\sum \hat{f}_{a,l}(y)a_l + \hat{f}^*_{a,l}(y)a^\dagger_l + \hat{f}_{b,l}(y)\rho^{1/2}_{A,0}a^\dagger_l\rho^{-1/2}_{A,0} + \hat{f}^*_{b,l}(y)\rho^{1/2}_{A,0}a_l\rho^{-1/2}_{A,0}\big)|0\rangle\langle 0| \\ &= Q_A|0\rangle\langle 0|\end{aligned} \tag{5.20}$$

where we write Q_A as

$$Q_A = \sum \hat{f}_{a,l}(y)a_l + \hat{f}^*_{a,l}(y)a^\dagger_l + \hat{f}_{b,l}(y)\rho^{1/2}_{A,0}a^\dagger_l\rho^{-1/2}_{A,0} + \hat{f}^*_{b,l}(y)\rho^{1/2}_{A,0}a_l\rho^{-1/2}_{A,0} \tag{5.21}$$

In similar way, we can calculate for $\mathcal{N}[|\tilde{0}\rangle\langle\tilde{1}|]$, we will get,

$$\mathcal{N}[|\tilde{0}\rangle\langle\tilde{1}|] = |0\rangle\langle 0|Q^\dagger_A \tag{5.22}$$

and eventually, $\mathcal{N}(X)$ becomes

$$\begin{aligned}\mathcal{N}[\tilde{1}\rangle\langle\tilde{0}| + |\tilde{0}\rangle\langle\tilde{1}|] &= \mathcal{N}[\tilde{1}\rangle\langle\tilde{0}|] + \mathcal{N}[|\tilde{0}\rangle\langle\tilde{1}|] \\ &= Q_A|0\rangle\langle 0| + |0\rangle\langle 0|Q^\dagger_A\end{aligned} \tag{5.23}$$

The recovery channel will be, see appendix in [1],

$$\begin{aligned}\mathcal{R}^\dagger[|\tilde{1}\rangle\langle\tilde{0}|] &= \int \beta_0(t)\rho^{it/2}_{A,0}\big(\rho^{-1/2}_{A,0}Q_A\rho^{1/2}_{A,0}\big)\rho^{-it/2}_{A,0} \\ &= \sum_l \hat{f}_{a,l}a_l e^{-\pi E_l + i\pi E_l t} + \hat{f}^*_{a,l}a^\dagger_l e^{\pi E_l - i\pi E_l t} + \hat{f}_{b,l}a^\dagger_l e^{-i\pi E_l t} + \hat{f}^*_{b,l}(y)a_l e^{i\pi E_l t}\end{aligned} \tag{5.24}$$

and $\mathcal{R}^\dagger(X)$ is

$$\begin{aligned}\mathcal{R}^\dagger[\tilde{1}\rangle\langle\tilde{0}| + |\tilde{0}\rangle\langle\tilde{1}|] &= \mathcal{R}^\dagger[\tilde{1}\rangle\langle\tilde{0}|] + \mathcal{R}^\dagger[|\tilde{0}\rangle\langle\tilde{1}|] \\ &= \int \beta_0(t)\big[\rho^{it/2}_{A,0}\big(\rho^{-1/2}_{A,0}Q_A\rho^{1/2}_{A,0}\big)\rho^{-it/2}_{A,0} + \rho^{it/2}_{A,0}\big(\rho^{1/2}_{A,0}Q^\dagger_A\rho^{-1/2}_{A,0}\big)\rho^{-it/2}_{A,0}\big].\end{aligned} \tag{5.25}$$

We are mainly interested on $\mathcal{N}(X)$ because in the next chapter we will see this term is actually $\delta\rho$ term in *Sarosi Ugajin* [2] formula for first order change in excited state modular Hamiltonian in [2].

Chapter 6

Excited state modular Hamiltonian

The modular Hamiltonian of a density matrix is defined as,

$$H = -\log \rho \tag{6.1}$$

where ρ is the density matrix and H is the corresponding modular Hamiltonian. If we excite a ground state density matrix ρ_0, we will get an excited state density matrix ρ_{ex}. The ground state density matrix and excited state density matrix will have corresponding ground state modular Hamiltonian H_0 and excited state modular Hamiltonian H_{ex}, respectively. In [2], *Sarosi, Ugajin* have given an expression of excited state modular Hamiltonian where we get a relation between the first order change of excited state modular Hamiltonian δH and the first order change of excited state density matrix $\delta\rho$. This relation is exactly matched with the explicit formula 5.10.

Let us have a ground state density matrix on boundary sub-region A,

$$\rho_{0,A} = Tr_{\bar{A}}|0\rangle\langle 0| \tag{6.2}$$

corresponding modular Hamiltonian is,

$$H_{0,A} = -\log \rho_{0,A} = -\log[Tr_{\bar{A}}|0\rangle\langle 0|] \tag{6.3}$$

and we get excited state density matrix by

$$\begin{aligned}
\rho_{ex,A} &= Tr_{\bar{A}}[e^{\epsilon Q}|0\rangle\langle 0|e^{\epsilon Q^\dagger}] \\
&= Tr_{\bar{A}}[|0\rangle\langle 0| + \epsilon Q|0\rangle\langle 0| + |0\rangle\langle 0|\epsilon Q^\dagger] + \mathcal{O}(\epsilon^2) \\
&= \rho_{0,A} + \epsilon Q_A|0\rangle\langle 0| + |0\rangle\langle 0|\epsilon Q_A^\dagger + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{6.4}$$

Corresponding excited state modular Hamiltonian is,

$$\begin{aligned} H_{ex,A} &= H_{0,A} + \epsilon \delta H_{1,A} + \mathcal{O}(\epsilon^2) \\ &= -\log(\rho_{0,A} + \epsilon Q_A|0\rangle\langle 0| + |0\rangle\langle 0|\epsilon Q_A^\dagger + \mathcal{O}(\epsilon^2)) \\ &= -\log(\rho_{0,A} + \epsilon\delta\rho_{1,A} + \mathcal{O}(\epsilon^2)), \end{aligned} \tag{6.5}$$

here, the first-order change in the reduced density matrix is

$$\delta\rho_{1,A} = Q_A|0\rangle\langle 0| + |0\rangle\langle 0|Q_A^\dagger \tag{6.6}$$

6.1 Connection with universal recovery channel

In [2], Sarosi, Ugajin gave the expression between first order δH and first order $\delta\rho$,

$$\delta H_{1,A} = -\frac{1}{2}\int_{\mathbb{R}} \frac{ds}{\cosh(s)+1}\rho_{0,A}^{-\frac{1}{2}-\frac{is}{2\pi}}(\delta\rho_{1,A})\rho_{0,A}^{-\frac{1}{2}+\frac{is}{2\pi}}. \tag{6.7}$$

We can derive 6.7 from the universal recovery channel that *Cotler et al.*[1] has given,

$$\mathcal{O}_A = \mathcal{R}^\dagger(\phi_a) = \frac{1}{d_{code}}\int_{\mathbb{R}} dt\beta_0(t)e^{\frac{1-it}{2}H_{0,A}}Tr_{\bar{A}}[J(\phi_a\otimes 1_{\bar{a}})J^\dagger]e^{\frac{1+it}{2}H_{0,A}} \tag{6.8}$$

and

$$\mathcal{O}_A = -\frac{1}{d_{code}}\frac{d}{d\epsilon}\Big|_{\epsilon=0} H_{0,A}\left[Tr_{\bar{A}}[\tau_{CFT}] + \epsilon\left(Tr_{\bar{A}}[J(\phi_a\otimes 1_{\bar{a}})J^\dagger]\right)\right] \tag{6.9}$$

where, $H_{0,A}[\rho] = -\log(\rho)$.

We consider $Tr_{\bar{A}}[\tau_{CFT}]$ as $\rho_{0,A}$ and $Tr_{\bar{A}}[J(\phi_a\otimes 1_{\bar{a}})J^\dagger]$ as $\delta\rho_{1,A}$. We will prove this statement later. So, 6.8 becomes

$$\mathcal{O}_A = \mathcal{R}^\dagger(\phi_a) = \frac{1}{d_{code}}\int_{\mathbb{R}} dt\beta_0(t)e^{\frac{1-it}{2}H_{0,A}}(\delta\rho_{1,A})e^{\frac{1+it}{2}H_{0,A}} \tag{6.10}$$

and 6.9 becomes

$$\begin{aligned} \mathcal{O}_A &= -\frac{1}{d_{code}}\frac{d}{d\epsilon}\Big|_{\epsilon=0} H_{0,A}\left[\rho_{0,A} + \epsilon\left(\delta\rho_{1,A}\right)\right] \\ &= \frac{1}{d_{code}}\frac{d}{d\epsilon}\Big|_{\epsilon=0}\log(\rho_{0,A} + \epsilon\delta\rho_{1,A}) \\ &= \frac{1}{d_{code}}\delta H_{1,A} \end{aligned} \tag{6.11}$$

From 6.10 and 6.11, we can write

$$\delta H_{1,A} = \int_{\mathbb{R}} dt \beta_0(t) e^{\frac{1-it}{2} H_{0,A}} (\delta\rho_{1,A}) e^{\frac{1+it}{2} H_{0,A}} \tag{6.12}$$

here $H_{0,A} = -\log(\rho_{0,A})$ and $\beta_0(t) = \frac{\pi}{2(\cosh(\pi t)+1)}$.

Now we do transform of $t = -s/\pi$ and $\delta H_{1,A}$ becomes

$$\delta H_{1,A} = -\frac{1}{2} \int_{\mathbb{R}} \frac{ds}{\cosh(s)+1} e^{(\frac{1}{2}+\frac{is}{2\pi})H_{0,A}} (\delta\rho_{1,A}) e^{(\frac{1}{2}-\frac{is}{2\pi})H_{0,A}}. \tag{6.13}$$

We know $H_{0,A} = -\log(\rho_{0,A})$, so we can replace $e^{-H_{0,A}}$ by $\rho_{0,A}$ and $\delta H_{1,A}$ becomes

$$\delta H_{1,A} = -\frac{1}{2} \int_{\mathbb{R}} \frac{ds}{\cosh(s)+1} \rho_{0,A}^{-\frac{1}{2}-\frac{is}{2\pi}} (\delta\rho_{1,A}) \rho_{0,A}^{-\frac{1}{2}+\frac{is}{2\pi}}. \tag{6.14}$$

Now, we are going to see how $Tr_{\bar{A}}[J(\phi_a \otimes 1_{\bar{a}})J^\dagger]$ can be $\delta\rho_{1,A}$. We have considered $Tr_{\bar{A}}[J(\phi_a \otimes 1_{\bar{a}})J^\dagger]$ as $\delta\rho_{1,A}$ and both quantities is boundary quantities. If we take $(\phi_a \otimes 1_{\bar{a}})$ bulk field as $(\phi_a \otimes 1_{\bar{a}}) = |\tilde{1}\rangle\langle\tilde{0}| + |\tilde{0}\rangle\langle\tilde{1}|$ for two dimensional code space, where $|\tilde{0}\rangle$ is bulk vacuum state and $|0\rangle$ is boundary vacuum state. Then $Tr_{\bar{A}}[J(\phi_a \otimes 1_{\bar{a}})J^\dagger]$ becomes

$$Tr_{\bar{A}}[J(\phi_a \otimes 1_{\bar{a}})J^\dagger] = Tr_{\bar{A}} \left(J|\tilde{1}\rangle\langle\tilde{0}|J^\dagger + J|\tilde{0}\rangle\langle\tilde{1}|J^\dagger \right). \tag{6.15}$$

Here, J and $J^\dagger$ are defined so that J and $J^\dagger$ applied on bulk state give boundary states.

$$J|\tilde{0}\rangle = |0\rangle \text{ and } \langle\tilde{0}|J^\dagger = \langle 0|. \tag{6.16}$$

So $Tr_{\bar{A}}[J(\phi_a \otimes 1_{\bar{a}})J^\dagger]$ becomes

$$Tr_{\bar{A}}[J(\phi_a \otimes 1_{\bar{a}})J^\dagger] = Tr_{\bar{A}}[|1\rangle\langle 0| + |0\rangle\langle 1|]. \tag{6.17}$$

Now let's say the excitation is happening due to an operator Q which excites a boundary vacuum state to a boundary excited state,

$$Q|0\rangle = |1\rangle \text{ and } \langle 0|Q^\dagger = \langle 1|. \tag{6.18}$$

So we can write

$$Tr_{\bar{A}}[|1\rangle\langle 0| + |0\rangle\langle 1|] = Tr_{\bar{A}}[Q|0\rangle\langle 0| + |0\rangle\langle 0|Q^\dagger]. \tag{6.19}$$

Here, $(Q|0\rangle\langle 0| + |0\rangle\langle 0|Q^\dagger)$ is a boundary state supported on the whole boundary. $Tr_{\bar{A}}(\cdot)$ operations remove or trace out the action of Q on $\bar{A}$ sub-region of the boundary and make

Q as Q_A which acts on boundary vacuum state and give boundary excited state confined in A sub-region of the boundary,

$$\begin{aligned} Tr_{\bar{A}}[J(\phi_a \otimes 1_{\bar{a}})J^\dagger] &= Tr_{\bar{A}}[Q|0\rangle\langle 0| + |0\rangle\langle 0|Q^\dagger] \\ &= Q_A|0\rangle\langle 0| + |0\rangle\langle 0|Q_A^\dagger. \end{aligned} \tag{6.20}$$

In our case of excited state modular Hamiltonian, from 6.6, $\delta\rho_{1,A}$ is

$$\delta\rho_{1,A} = Q_A|0\rangle\langle 0| + |0\rangle\langle 0|Q_A^\dagger, \tag{6.21}$$

so, we can say $Tr_{\bar{A}}[J(\phi_a \otimes 1_{\bar{a}})J^\dagger]$ is a first-order change in the excited state reduced density matrix in case of our excitation in 6.4.

Discussion

We have derived the theoretical framework behind the Petz recovery map and establishes its role in faithfully recovering quantum states after they undergo a quantum channel. It explains the uniqueness of the Petz map under certain conditions, emphasizing its significance in quantum information processing. Furthermore, the exploration of approximate recovery channels, as discussed by Junge et al., extends the utility of recovery maps beyond ideal scenarios, offering a practical approach to recovering quantum states in cases where exact reversal is not certainly possible. The concrete expression for the approximate Petz recovery channel, referred to as the twirled Petz map, provides a mathematical tool to quantify the fidelity of the recovered state with respect to the original.

The Petz recovery channel serves as a capable tool for entanglement wedge reconstruction, emphasizing its effectiveness in scenarios where perturbative excitations dominate and the code space dimension remains manageable. Even evaluating Petz map is quite challenging, particularly in dealing with non-localized operators and the complexities arising from the strong coupling regime.

By bridging the gap between bulk and boundary states through the isometric embedding $J : \mathcal{H}_{code} \longrightarrow \mathcal{H}_{CFT}$, *Cotler et al.*[1] have shown the groundwork for recovering boundary observables supported solely on region A from bulk operators residing in the entanglement wedge. Through the formulation of a recovery channel $\mathcal{R}$ and its adjoint $\mathcal{R}^\dagger$, *Cotler et al.* demonstrates the high-fidelity reconstruction of arbitrary bulk states within the entanglement wedge onto the boundary region A. In section 5.1, we have approached to calculate the reconstruction formula 5.10. This example is comparable to Rindler wedge reconstruction, except it is only strictly valid for true free fields. We have taken a bulk operator ϕ_a in AdS_3, which is supported on the entanglement wedge of a boundary sub-region A, and we have selected a two-dimensional code space defined by states $|\tilde{0}\rangle$ and $\phi_a|\tilde{0}\rangle$, where $|\tilde{0}\rangle$ is the ground state. We find expression for reconstructed boundary operator $\mathcal{O}_A = \mathcal{R}^\dagger(\phi_a)$.

We establishes a crucial link between the modular Hamiltonians for excited states and the boundary oparator in entanglement wadge reconstruction. The formula of first order change of excited state modular hamiltoian $\delta H_{1,A}$ is adjoint of appriximate recovery channel $\mathcal{R}^\dagger(\phi_a)$ with ground state density matrix is maximally mixed reference state and first order change of density matrix $\delta\rho_{1,A}$ is $\mathcal{N}(\phi_a)$. *Cotler et al.*[1] claimed $\mathcal{O}_A = \mathcal{R}^\dagger(\phi_a)$. So, we claim first order change of excited state modular hamiltoian $\delta H_{1,A}$ is reconstructed boundary operator $\mathcal{O}_A$ with some scailing factor of $1/d_{code}$.

$$\mathcal{O}_A = \frac{1}{d_{code}}\delta H_{1,A} \tag{6.22}$$

Our definition of excited state modular Hamiltonian becomes

$$\begin{aligned} &H_{ex,A} \\ =&H_{0,A} + \epsilon\delta H_{1,A} + O(\epsilon^2) \\ =&H_{0,A} + \epsilon d_{code}\mathcal{O}_A + O(\epsilon^2) = -log([|0\rangle\langle 0|]_A + \epsilon(Q_A|0\rangle\langle 0| + |0\rangle\langle 0|Q_A^\dagger) + O(\epsilon^2)) \end{aligned} \tag{6.23}$$

where

$$\mathcal{O}_A = \frac{1}{d_{code}}\int dt\beta_0(t)e^{\frac{1-it}{2}H_{0,A}}(Q_A|0\rangle\langle 0| + |0\rangle\langle 0|Q_A^\dagger)e^{\frac{1+it}{2}H_{0,A}}$$

From 6.23, one can find a new definition of ground state modular Hamiltonian $H_{0,A}$ other than $H_{0,A} = -log([|0\rangle\langle 0|]_A)$.

APPENDICES

Appendix A

A.1 Derivation of Junge et al. relation

Let $P(A)$ is set of non-negative trace-class operators on a Hilbert space A. Let $S(A)$ denote the set of density operators on A. For a $\sigma \in P(A)$, we define

$$S_\sigma(A) = \{\rho \in S(A) : Supp(\rho) \subseteq Supp(\sigma)\} \tag{A.1}$$

Theorem A.1.1: Let A and B be separable Hilbert spaces. $TPCP$ is Trace preserving compleately positive channels which map $A \to B$. For any $\sigma \in P(A)$, any $\rho \in S_\sigma(A)$ and any $\mathcal{N} \in TPCP(A, B)$, we have

$$D(\rho||\sigma) - D(\mathcal{N}[\rho]||\mathcal{N}[\sigma]) \geq -2\int_{\mathbb{R}} dt\beta_0(t)logF(\rho, \mathcal{R}^{\frac{t}{2}}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho)) \tag{A.2}$$

where $D(\rho||\sigma)$ is relative entropy defined as

$$\begin{aligned} D(\rho||\sigma) &= \sum_i \langle\phi_i|\rho(log\rho - log\sigma)|\phi_i\rangle \\ &= \sum_{i,j} |\langle\phi_i|\psi_j\rangle|^2[p(i)logp(i) - p(i)logq(j)] \end{aligned} \tag{A.3}$$

where $\rho = \sum_i p(i)|\phi_i\rangle\langle\phi_i|$ and $\sigma = \sum_j q(j)|\psi_j\rangle\langle\psi_j|$ are spectral decomposition of ρ and σ respectively. The *fidelity* of ρ and σ is defined by

$$F(\rho, \sigma) := ||\sqrt{\rho}\sqrt{\sigma}||_1 \text{ and } ||L||_p := (Tr|L|^p)^{\frac{1}{p}} \; p \in [1, \infty) \tag{A.4}$$

The recovery map is given by

$$\mathcal{R}^t_{\sigma,\mathcal{N}} : X \to \int_{\mathbb{R}} \beta_0(t)\sigma^{\frac{1-it}{2}}\mathcal{N}^\dagger[\mathcal{N}(\sigma)^{\frac{-1+it}{2}}(X)\mathcal{N}(\sigma)^{\frac{-1-it}{2}}]\sigma^{\frac{1+it}{2}} \tag{A.5}$$

and the *probability density function* $\beta_0(t) = \frac{\pi}{2}(cosh(\pi t) + 1)^{-1}$.

Proof of Theorem A.1.1: A *Rényi* generalization of a relative entropy difference is defined as

$$\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) = \frac{2\alpha}{\alpha - 1} log||(\mathcal{N}(\rho)^{\frac{1-\alpha}{2\alpha}} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \otimes id_E) U_{A\to BE} \sigma^{\frac{1-\alpha}{2\alpha}} \rho^{\frac{1}{2}}||_{2\alpha} \tag{A.6}$$

where $\alpha \in (0, 1) \cup (1, \infty)$ and $U_{A\to BE}$ is an isometric extension of the channel $\mathcal{N}$. $U_{A\to BE}$ is a linear isometry satisfying $Tr_E(U_{A\to BE}(\cdot)U^\dagger_{A\to BE}) = \mathcal{N}(\cdot)$ and $U^\dagger_{A\to BE}U_{A\to BE} = id_A$. All isometric extensions of a channel are related by an isometry acting on the environment system E, so that the definition in Eq.A.6 is invariant under any such choice. Adjoint of channel $\mathcal{N}^\dagger$ is given as $\mathcal{N}^\dagger(\cdot) = U^\dagger_{A\to BE}((\cdot) \otimes id_E)U_{A\to BE}$.

Lemma A.1.2: Let A and B be finite-dimensional Hilbert spaces. The following limit holds for $\sigma \in P(A)$, any $\rho \in S_\sigma(A)$ and any $\mathcal{N} \in TPCP(A, B)$

$$\lim_{\alpha\to 1} \tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) = D(\rho||\sigma) - D(\mathcal{N}[\rho]||\mathcal{N}[\sigma]) \tag{A.7}$$

For $\alpha = \frac{1}{2}$, observe that

$$\begin{aligned} \tilde{\Delta}_{\frac{1}{2}}(\rho, \sigma, \mathcal{N}) &= -2log||(\mathcal{N}(\rho)^{\frac{1}{2}} \mathcal{N}(\sigma)^{-\frac{1}{2}} \otimes id_E) U_{A\to BE} \sigma^{\frac{1}{2}} \rho^{\frac{1}{2}}||_1 \\ &= -2logF(\rho, \mathcal{P}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho)) \end{aligned} \tag{A.8}$$

where $\mathcal{P}_{\sigma,\mathcal{N}}$ denotes petz recovery map,

$$\mathcal{P}_{\sigma,\mathcal{N}}(\cdot) \equiv \sigma^{1/2} \mathcal{N}^\dagger\big(\mathcal{N}(\sigma)^{-1/2}(\cdot)\mathcal{N}(\sigma)^{-1/2}\big)\sigma^{1/2} \tag{A.9}$$

Lemma A.1.3: Let $\theta \in (0, 1)$ and define P_θ by

$$\frac{1}{P_\theta} = \frac{1-\theta}{P_0} + \frac{\theta}{P_1} \tag{A.10}$$

where $P_0, P_1 \in [1, \infty)$. Then the following bound holds

$$log(||G(\theta)||_{p(\theta)}) \leq \int_{\mathbb{R}} dt(\alpha_\theta(t) log(||G(it)||^{1-\theta}_{p_0}) + \beta_\theta(t) log(||G(1 + it)||^{\theta}_{p_1})) \tag{A.11}$$

where $\alpha_\theta(t)$ and $\beta_\theta(t)$ are defined by

$$\alpha_\theta(t) = \frac{sin(\pi\theta}{2(1-\theta)(cosh(\pi t) - cos(\pi\theta))} \text{ and } \beta_\theta(t) = \frac{sin(\pi\theta)}{2\theta(cosh(\pi t) + cos(\pi\theta))} \tag{A.12}$$

Remark A.1.4: Observe that $\alpha_\theta(t), \beta_\theta(t) \geq 0$ for all $t \in \mathbb{R}$ and we have

$$\int_{\mathbb{R}} dt\alpha_\theta(t) = \int_{\mathbb{R}} dt\beta_\theta(t) = 1 \tag{A.13}$$

So that $\alpha_\theta(t)$ and $\beta_\theta(t)$ can be interpreted as probability density functions. Furthermore, the following limit holds

$$\lim_{\theta \to 0} \beta_\theta(t) = \frac{\pi}{2(cosh(\pi t) + 1)} = \beta_0(t) \tag{A.14}$$

where β_0 is also a probability density function on $\mathbb{R}$. The operator valued-function $G(z)$ is defined as

$$G(z) = (\mathcal{N}(\rho)^{\frac{z}{2}} \mathcal{N}(\sigma)^{-\frac{z}{2}} \otimes id_E) U \sigma^{\frac{z}{2}} \rho^{\frac{1}{2}} \tag{A.15}$$

Here we abbreviate the isometric extension $U_{A \to BE}$ of the channel $\mathcal{N}$ as U in above equation. we fix $P_0 = 2$, $P_1 = 1$ and $\theta \in (\ 0, 1)$ which fix $P_\theta = \frac{2}{1+\theta}$. The operator valued function $G(z)$ satisfies inequality in **lemma A.1.3**.

For above choices,

$$||G(\theta)||_{\frac{2}{1+\theta}} = ||(\mathcal{N}(\rho)^{\frac{\theta}{2}} \mathcal{N}(\sigma)^{-\frac{\theta}{2}} \otimes id_E) U \sigma^{\frac{\theta}{2}} \rho^{\frac{1}{2}}||_{\frac{2}{1+\theta}} \tag{A.16}$$

and

$$\begin{aligned} &||G(it)||_2 = ||(\mathcal{N}(\rho)^{\frac{it}{2}} \mathcal{N}(\sigma)^{-\frac{it}{2}} \otimes id_E) U \sigma^{\frac{it}{2}} \rho^{\frac{1}{2}}||_2 \\ &||G(it)||_2 \leq ||\rho^{\frac{1}{2}}||_2 \\ &||G(it)||_2 \leq 1 \end{aligned} \tag{A.17}$$

as well as

$$\begin{aligned} ||G(1+it)||_1 &= ||(\mathcal{N}(\rho)^{\frac{1+it}{2}} \mathcal{N}(\sigma)^{-\frac{1+it}{2}} \otimes id_E) U \sigma^{\frac{1+it}{2}} \rho^{\frac{1}{2}}||_1 \\ &= ||(\mathcal{N}(\rho)^{\frac{1}{2}} \mathcal{N}(\rho)^{\frac{it}{2}} \mathcal{N}(\sigma)^{-\frac{1}{2}} \mathcal{N}(\sigma)^{-\frac{it}{2}} \otimes id_E) U \sigma^{\frac{1}{2}} \sigma^{\frac{it}{2}} \rho^{\frac{1}{2}}||_1 \\ &= F(\rho, \mathcal{R}^{\frac{t}{2}}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho)) \end{aligned} \tag{A.18}$$

Now from the inequality of **lemma A.1.3** and applying the fact $||G(it)||_2 \leq 1$, we conclude the following

$$log||(\mathcal{N}(\rho)^{\frac{\theta}{2}} \mathcal{N}(\sigma)^{-\frac{\theta}{2}} \otimes id_E) U \sigma^{\frac{\theta}{2}} \rho^{\frac{1}{2}}||_{\frac{2}{1+\theta}} \leq \int_{\mathbb{R}} dt\beta_\theta log(F(\rho, \mathcal{R}^{\frac{t}{2}}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho)))^\theta \tag{A.19}$$

which implies

$$-\frac{2}{\theta}log||(\mathcal{N}(\rho)^{\frac{\theta}{2}}\mathcal{N}(\sigma)^{-\frac{\theta}{2}}\otimes id_E)U\sigma^{\frac{\theta}{2}}\rho^{\frac{1}{2}}||_{\frac{2}{1+\theta}} \geq -2\int_{\mathbb{R}} dt\beta_\theta logF(\rho,\mathcal{R}^{\frac{t}{2}}_{\sigma,\mathcal{N}}\circ\mathcal{N}(\rho)). \quad (A.20)$$

Letting $\theta = \frac{1-\alpha}{\alpha}$ and from Eq.A.6 we get

$$\tilde{\Delta}_\alpha(\rho,\sigma,\mathcal{N}) \geq -2\int_{\mathbb{R}} dt\beta_\theta(t)logF(\rho,\mathcal{R}^{\frac{t}{2}}_{\sigma,\mathcal{N}}\circ\mathcal{N}(\rho)). \quad (A.21)$$

We can take limit $\alpha \to 1$ to get *Junge et al* relation,

$$\lim_{\alpha\to 1}\tilde{\Delta}_\alpha(\rho,\sigma,\mathcal{N}) = D(\rho||\sigma) - D(\mathcal{N}[\rho]||\mathcal{N}[\sigma]) \quad (A.22)$$

and

$$\lim_{\alpha\to 1}\tilde{\Delta}_\alpha(\rho,\sigma,\mathcal{N}) \geq -2\int_{\mathbb{R}} dt\beta_0(t)logF(\rho,\mathcal{R}^{\frac{t}{2}}_{\sigma,\mathcal{N}}\circ\mathcal{N}(\rho)) \quad (A.23)$$

So, from above two relations

$$D(\rho||\sigma) - D(\mathcal{N}[\rho]||\mathcal{N}[\sigma]) \geq -2\int_{\mathbb{R}} dt\beta_0(t)logF(\rho,\mathcal{R}^{\frac{t}{2}}_{\sigma,\mathcal{N}}\circ\mathcal{N}(\rho)) \quad (A.24)$$

Using the concavity of the logarithm and the fidelity, we can write,

$$D(\rho||\sigma) - D(\mathcal{N}[\rho]||\mathcal{N}[\sigma]) \geq -2logF(\rho,\mathcal{R}_{\sigma,\mathcal{N}}\circ\mathcal{N}(\rho)) \quad (A.25)$$

where

$$\begin{aligned}\mathcal{R}_{\sigma,\mathcal{N}}\circ\mathcal{N}(\rho) &= \int_{\mathbb{R}} dt\beta_0(t)\rho,\mathcal{R}^{\frac{t}{2}}_{\sigma,\mathcal{N}}\circ\mathcal{N}(\rho) \\ &= \int_{\mathbb{R}} dt\beta_0(t)\sigma^{\frac{1-it}{2}}\mathcal{N}^\dagger[\mathcal{N}(\sigma)^{\frac{-1+it}{2}}(\cdot)\mathcal{N}(\sigma)^{\frac{-1-it}{2}}]\sigma^{\frac{1+it}{2}}\end{aligned} \quad (A.26)$$

In Appendix A.2, we will see how we can get the expression of adjoint of recovery channel $\mathcal{R}^\dagger$.

A.2 Derivation of adjoint of approximate recovery channel

From *Junge et al.* relation approximate recovery channel $\mathcal{R}_{\sigma,\mathcal{N}}$ has the form

$$\mathcal{R}_{\sigma,\mathcal{N}}(\cdot) = \int_{\mathbb{R}} dt \beta_0(t) \sigma^{\frac{1-it}{2}} \mathcal{N}^{\dagger}[\mathcal{N}(\sigma)^{\frac{-1+it}{2}}(\cdot)\mathcal{N}(\sigma)^{\frac{-1-it}{2}}]\sigma^{\frac{1+it}{2}}. \tag{A.27}$$

We can break down above operation into three small operations. These three small operations are the followings,

1. $\mathcal{O}_1(\cdot) = \mathcal{N}(\sigma)^{\frac{-1+it}{2}}(\cdot)\mathcal{N}(\sigma)^{\frac{-1-it}{2}}$
2. $\mathcal{O}_2(\cdot) = \mathcal{N}^{\dagger}(\cdot)$
3. $\mathcal{O}_3(\cdot) = \sigma^{\frac{1-it}{2}}(\cdot)\sigma^{\frac{1+it}{2}}$,

so, our recovery channel can be written as,

$$\mathcal{R}_{\sigma,\mathcal{N}}(\cdot) = \int_{\mathbb{R}} dt \beta_0(t) \mathcal{O}_3(\mathcal{O}_2(\mathcal{O}_1(\cdot))) \tag{A.28}$$

Now adjoint of recovery channel can be found by operating the operators in reverse order like the following,

$$\mathcal{R}^{\dagger}_{\sigma,\mathcal{N}}(\cdot) = \int_{\mathbb{R}} dt \beta_0(t) \mathcal{O}_1(\mathcal{O}_2(\mathcal{O}_3(\cdot))). \tag{A.29}$$

So, adjoint of recovery channel becomes,

$$\mathcal{R}^{\dagger}_{\sigma,\mathcal{N}}(\cdot) = \int_{\mathbb{R}} dt \beta_0(t) \mathcal{N}(\sigma)^{\frac{-1+it}{2}} \mathcal{N}[\sigma^{\frac{1-it}{2}}(\cdot)\sigma^{\frac{1+it}{2}}]\mathcal{N}(\sigma)^{\frac{-1-it}{2}}. \tag{A.30}$$

Now, we write above channel in terms of modular Hamiltonian,

$$\mathcal{R}^{\dagger}_{\sigma,\mathcal{N}}(\cdot) = \int_{\mathbb{R}} dt \beta_0(t) e^{\frac{1-it}{2}H_A} \mathcal{N}[e^{\frac{-1+it}{2}H_a}(\cdot)e^{\frac{-1-it}{2}H_a}]e^{\frac{1+it}{2}H_A}. \tag{A.31}$$

where $H_a = -log\sigma_a$ and $H_A = -log(\mathcal{N}(\sigma_a))$. When σ_a is chosen to be maximally mixed state the $\mathcal{R}^{\dagger}$ takes quite simple form. When $\mathcal{R}^{\dagger}$ is applied on bulk operator defined on whole code space $(\phi_a \otimes 1_{\bar{a}})$, then recovery channel takes the explicit form,

$$\begin{aligned}\mathcal{R}^{\dagger}_{\sigma,\mathcal{N}}(\phi) &= \frac{1}{d_{code}} \int_{\mathbb{R}} dt \beta_0(t) e^{\frac{1-it}{2} H_A} \mathcal{N}[(\phi_a \otimes 1_{\bar{a}})] e^{\frac{1+it}{2} H_A} \\ &= \frac{1}{d_{code}} \int_{\mathbb{R}} dt \beta_0(t) e^{\frac{1-it}{2} H_A} Tr_{\bar{A}}[J(\phi_a \otimes 1_{\bar{a}}) J^{\dagger}] e^{\frac{1+it}{2} H_A}.\end{aligned} \tag{A.32}$$

where $H_A = -log[Tr_{\bar{A}}(J\tau J^{\dagger})]$ is the boundary modular Hamiltonian on subregion A associated with the maximally mixed state τ on code subspace. This is the explicit formula that *Cotler et al.* [1] has found in their paper.

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