RIPPLES IN FABRIC OF SPACE-TIME AND COSMIC STABILITY

M.Sc. Thesis

by

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RIPPLES IN FABRIC OF SPACE-TIME AND COSMIC STABILITY

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Submitted in partial fulfillment of the requirements for the award of the degree of

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by

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"RIPPLES IN FABRIC OF SPACE-TIME AND COSMIC STABILITY"** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF PHYSICS**, **Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2023 to May 2024 under the supervision of **Dr. Manavendra N. Mahato**, Associate Professor, Department of Physics, IIT Indore. The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

Ajoy Dawn

13/05/2024

Signature of the student with date

(**Ajoy Dawn**)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

13/05/2024

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Ajoy Dawn has successfully given his M.Sc. Oral Examination held on 15th May, 2024.

Signature of supervisor of M.Sc Thesis

Date: 22/5/2024

Signature of Convener,DPGC

Date: 22/5/2024

Dedicated to my family

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Abstract

This study focuses on investigating gravitational wave solutions and conducts stability analysis within the framework of EMY Black hole metric. It includes the non-abelian gauge fields and Maxwell's electromagnetic fields into our gravitational background. Our stability analysis look for the effect of these fields on the system. We also have conducted stability analysis across various metrics, including FLRW, Kantowski-Sachs, Reissner-Nordström. Utilizing the Regge-Wheeler gauge, we explore the behavior of gravitational perturbations or ripples through matter or vacuum.

Conventions

In this work certain mathematical conventions are used and understanding of General theory of Relativity is assumed.

- Metric signature is mostly positive, i.e. $(-, +, +, +)$.

- Universal Gravitational constant (G) and the speed of light (c) are expressed in natural units, i.e. G=c=1.

- Spatial 3-vectors are denoted by Latin indices, such as x, whereas space-time 4-vectors of the kind (t,x) are denoted by Greek indices, such as x. Unless indicated otherwise, the Einstein summation convention will be used.

- Conformal and cosmological time will be represented by η and t respectively.

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CHAPTER 1

Introduction

1.1 Gravitational Waves

General Theory of Relativity is one of the best theory of Gravity which we have till today. It passes all the test that have been done so far with a great accuracy. Our old understanding of gravity comes from Newton's framework, where gravity was described as an attractive force between two objects . However, his theory could not explain the reason behind this attractive force. Einstein with his ingenious mind, redefined this concept entirely with his General Theory of Relativity. Einstein proposed that gravity extends beyond being just a force and is result of massive objects deforming the fabric of space-time itself. This curvature is then perceived as gravitational attraction. This profound insight, often summarized as "matter tells space-time how to curve, and curved space-time tells matter how to move", fundamentally altered our understanding of gravity.

Exceptionally notable prediction that General Relativity made, was the existence of gravitational waves, which was absent in Newtonian Gravity. According to Einstein's theory, when massive objects accelerate or undergo asymmetric motion or even collide, they distort the fabric of space-time itself, That produces ripples in spacetime, which is analogous to the concentric waves spreading across a pool when a stone is tossed in. But these waves are quite interesting because when they pass by, they can distort passing of time as well as distort objects shapes. These gravitational waves circulate at light speed, and also carry information about the dynamic events that generated them, like the collision of massive black holes or the merging of neutron stars etc. In 2015, LIGO made the historic detection of gravitational waves, confirming Einstein's prediction and exploring a brand-new avenue into understanding of the universe. This discovery assured the profound impact of Einstein's modification of gravity.

1.2 Einstein's Equation

Einstein's field theory in General Relativity predicts how metric responds to energy and momentum and is given by ,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \tag{1.1}$$

Here, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, $T_{\mu\nu}$ denotes the energy-momentum tensor ,$R_{\mu\nu}$ signifies Ricci tensor and R denotes the Ricci scalar.

Einstein's equations solution, in a linearized regime is the best way to comprehend the nature of gravitational waves.

1.3 Linearized Gravity

In linearized gravity we treat our metric to be perturbed from a flat metric $\eta_{\mu\nu}$, It is an adequate approximation to general relativity.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}; \qquad ||h_{\mu\nu}|| << 1 \tag{1.2}$$

Here, $h_{\mu\nu}$ is the perturbed part and $\eta_{\mu\nu}$ is defined as diag(-1,1,1,1).

The linearized expression of the components of Christoffel symbols are given by :

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}(h_{\nu\lambda,\mu} + h_{\lambda\mu,\nu} - h_{\mu\nu,\lambda}) \tag{1.3}$$

For the Riemann tensor only the derivative of Γ's will contribute,

$$R^{\mu}_{\nu\rho\sigma} = \Gamma^{\mu}_{\nu\sigma,\rho} - \Gamma^{\mu}_{\nu\rho,\sigma} = \frac{1}{2}(h^{\mu}_{\sigma,\rho\nu} + \partial^{\mu}h_{\nu\rho,\sigma} - \partial^{\mu}h_{\nu\sigma,\rho} - h^{\mu}_{\rho,\sigma\nu}) \tag{1.4}$$

And Ricci tensor is given by

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = \frac{1}{2}(h^{\rho}_{\mu,\rho\nu} + \partial^{\rho}h_{\nu\rho,\mu} - \Box h_{\mu\nu} - h_{,\mu\nu}) \tag{1.5}$$

Here, $h = h^{\mu}_{\mu}$ is the trace of the perturbation metric.

Ricci scalar is

$$R = R^{\mu}_{\mu} = \eta^{\mu\nu}R_{\mu\nu} = \partial^{\mu}h^{\rho}_{\mu,\rho} \tag{1.6}$$

Thus from (1.1) we get,

$$h^{\rho}_{\mu,\rho\nu} + \partial^{\rho}h_{\mu\rho,\nu} - \Box h_{\mu\nu} - h_{,\mu\nu} - \eta_{\mu\nu}\partial^{\mu}h^{\rho}_{\sigma;\rho} + \eta_{\mu\nu}\Box h = 16\pi T_{\mu\nu} \tag{1.7}$$

Now, instead of working with the perturbation metric $h_{\mu\nu}$, we consider the trace reversed perturbation ($\bar{h}^{\mu}_{\mu} = -h$) for simplification of equations,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{1.8}$$

Thus (1.7) becomes

$$\bar{h}^{\rho}_{\mu,\rho\nu} + \partial^{\rho}\bar{h}_{\nu\rho,\mu} - \Box\bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial^{\mu}\bar{h}^{\rho}_{\sigma,\rho=} = 16\pi T_{\mu\nu} \tag{1.9}$$

To further simplify this, we use Lorentz gauge condition .

$$\partial^{\nu}\bar{h}_{\mu\nu} = 0 \tag{1.10}$$

Thus (1.9) simply reduces to :

$$\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{1.11}$$

in vacuum, it is

$$\Box\bar{h}_{\mu\nu} = 0 \tag{1.12}$$

This is a wave equation and has a set of homogeneous solutions that are superposition of plane waves:

$$\bar{h}_{\mu\nu}(\vec{x}, t) = Re\int d^3k A1_{\mu\nu}(k, \omega)e^{i(\vec{k}.\vec{x}-\omega t)} \tag{1.13}$$

Here, $\omega = |k|$ and $A1_{\mu\nu}(k)$ is the polarization tensor which contains the information about polarization of the wave.

Now to further simplify things we use a special kind of coordinate system, called "Transverse-Traceless coordinate". It gives extra restriction on $A1_{\mu\nu}$

$$k_\mu A1^{\mu\nu} = 0 \tag{1.14}$$

Another one comes from the fact that gravitational wave propagate at speed of light.

$$k_\mu k^\mu = 0 \tag{1.15}$$

with $k^\mu = (\omega, \vec{k})$.

Now if we align our coordinate axis such that the wave propagates in z-direction, then (1.14) and (1.15) gives:

$$A1_{0z} = A1_{xz} = A1_{yz} = A1_{zz} = 0 \tag{1.16}$$

Thus the polarization tensor can be written as:

$$A1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A1_{xx} & A1_{xy} & 0 \\ 0 & A1_{xy} & -A1_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In transverse traceless gauge, the polarization tensor suggests that we have only two degrees of freedom, which are intrinsically two modes of gravitational wave . Plus

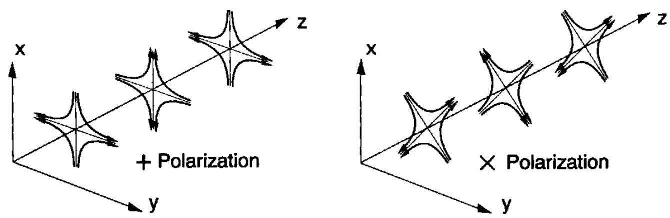

Figure 1.1: Force lines for entirely plus and cross polarization GW (Source: Google)

and cross polarization.

CHAPTER 2

Literature Review

2.1 Stability of a Schwarzchild Singularity

Karl Schwarzschild first discovered the solution of Einstein's field equation for a metric centered on a definite spherically symmetrical centre of mass in spherical polar coordinates (t, r, θ, ϕ):

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (2.1)$$

This metric explains the Schwarzschild Black hole as a non-rotating spherically symmetric black hole. In the beginning, these were not recognized as black holes, but rather as mathematical singularities. It was not known that such objects were physical or if they would be steady out there in the wild.As a result, well known physicist John Wheeler started the quest of stability for the Schwarzschild metric.

The equations are linear as in many stability problems in physics, and it's possible to dissect the disturbance into correct modes and determine their frequency, real or imaginary. It is found that Schwarzschild singularity is mainly stable. As a result [1], we conclude that very small deviations from equilibrium will oscillate around the equilibrium rather than growing with time.

2.2 First-order perturbations in polar coordinates

2.2.1 General equation

Stability is determined by introducing a small perturbation into the metric and then computing the variation of the Einstein's field equation. The space-time metric is denoted by $g_{\mu\nu}$, and a small perturbation in it is denoted by $h_{\mu\nu}$. Hence, if the new metric is $g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, then $R'_{\mu\nu} = R_{\mu\nu} + \delta R_{\mu\nu}$. From Eisenhart's calculation, we have:

$$\delta R_{\mu\nu} = \delta\Gamma^{\beta}_{\mu\nu;\beta} + \delta\Gamma^{\beta}_{\mu\beta;\nu} \tag{2.2}$$

Here the semicolons refer to the covariant derivative and we have used:

$$\delta\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\nu}(h_{\beta\nu;\gamma} + h\gamma\nu;\beta - h_{\beta\gamma;\nu}) \tag{2.3}$$

Imposing $\delta R_{\mu\nu} = 0$ implies that the perturbed metric also lacks any distributed mass or energy (vacuum).

2.2.2 Spherical Harmonics Analysis

This analysis begins with variable separation in polar coordinates. The primary move is to express the metric disturbances as the combination of spherical harmonics modes,Y^M_L, where L denotes angular momentum and M refers to the projection of angular momentum on z-axis. Rotation on two dimensional manifold $x^0 = T = constant$ and $x^1 = r = constant$ are used to analyse angular momentum. We can use partial and covariant derivatives to contract spherical harmonics and create quantities that transform like scalars, vector, and tensor, and then separate these structures into even parity $(-1)^L$ and odd parity $(-1)^{L+1}$, as determined by symmetry mirrored across the origin.

Scalar functions can be formed as:

$$\phi_L^M = constY_L^M(x_2, x_3) = constY_L^M(\theta, \Phi), \qquad parity(-1)^L \tag{2.4}$$

Vectors as:

$$\psi_L^M, \mu = const\frac{\partial}{\partial x^\mu}Y_L^M(\theta, \Phi), \qquad parity(-1)^L \tag{2.5}$$

$$\phi_L^M, \mu = const\epsilon_\mu^\nu \frac{\partial}{\partial x_\nu}Y_L^M(\theta, \Phi), \qquad parity(-1)^L + 1 \tag{2.6}$$

Tensors:

$$\psi_{L\ \mu\nu}^M = constY_{L\ \ ;\mu\nu}^M, \qquad parity(-1)^L \tag{2.7}$$

$$\phi_{L\ \mu\nu}^M = const\gamma_{\mu\nu}Y_L^M, \qquad parity(-1)^{L+1} \tag{2.8}$$

$$\chi_{L\ \mu\nu}^M = \frac{1}{2}const[\epsilon_\mu^\lambda \psi_{L\ \lambda\nu}^M + \epsilon_\nu^\lambda \psi_{L\ \lambda\mu}^M] \tag{2.9}$$

In these equations ϵ_μ^ν represents the quantities $\epsilon_2^2 = \epsilon_3^3 = 0$; $\epsilon_2^3 = \frac{-1}{\sin\theta)}$, $\epsilon_3^2 = \sin\theta$ and $\gamma_{\mu\nu} = \frac{g_{\mu\nu}}{r^2}$ represent the quantities $\gamma_{22} = 1$, $\gamma_{23} = 0 = \gamma_{32}$; $\gamma_{33} = \sin^2\theta$.

Because our metric is spherically symmetric ,$h_{\mu\nu}^{odd}$ and $h_{\mu\nu}^{even}$ can be constructed and examined independently because even and odd parity do not combine in Einstein equations.

$$h_{\mu\nu}^{odd} = \begin{bmatrix} 0 & 0 & -h_0(T,r)(\partial_\Phi/\sin\theta)Y_L^M & h_0(T,r)(\sin\theta\partial_\theta)Y_L^M \\ 0 & 0 & -h_1(T,r)(\partial_\Phi/\sin\theta)Y_L^M & h_1(T,r)(\sin\theta\partial_\theta)Y_L^M \\ sym & sym & h_2(T,r)(\partial_\Phi\partial_\theta/\sin\theta - (\cos\theta/\sin^2\theta)\partial_\Phi)Y_L^M & sym \\ sym & sym & \frac{1}{2}h_2(T,r)(\partial_\Phi\partial_\phi/\sin\theta + \cos\theta\partial_\theta - \sin\theta\partial_\theta^2)Y_L^M & -h_2(T,r)(\sin\theta\partial_\Phi\partial_\theta - \cos\theta\partial_\Phi)Y_L^M \end{bmatrix}$$

Similar equation can be found for even-mode perturbation. The angular components and radial-temporal components have now been separated in perturbation matrix. Now, because our background metric maintains radial symmetry and is time independent, the dispersion relation $\omega = kc$ can be imposed and thus $h_{\mu\nu}$ must have the dependence

$e^{-i\omega t}$. Furthermore,L and M being constants of motion, M=0 can be chosen, keeping the constant radial dependence. This will make Φ vanish entirely from calculations.

2.2.3 Gauge or Coordinate Transformation

We look at an infinitesimal coordinate transformation:

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}; \;\; \xi^{\alpha} << x^{\alpha} \tag{2.10}$$

where ξ^{α} are transformed as a vector. In new frame:

$$g'_{\mu\nu} = h'_{\mu\nu} = g_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu} + h_{\mu\nu} \tag{2.11}$$

where

$$h'_{\mu\nu} = h_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu} \tag{2.12}$$

This significant condition can now be used to simplify the perturbation and make it unique.

2.2.4 Imposing Regge-Wheeler Gauge

h_0,h_1,h_2 are three undetermined functions of r, in the matrix. The Regge-Wheeler gauge is used to simplify these to only one radial wave equation. The transformation, known as the Regge-Wheeler gauge, was developed by Tullio Regge and John A. Wheeler in 1957 and for odd wave it is written as:

$$\xi^0 = 0, \;\; \xi^1 = 0, \;\; \xi^{\mu} = \Lambda(T, r)\epsilon^{\mu\nu}(\partial/\partial x^{\nu})Y_L^M(\theta, \Phi), \;\; (\mu, \nu = 2, 3) \tag{2.13}$$

where Λ is used to reduce the h_2 factor. The general form for an odd-parity wave with angular momentum L and projection M=0:

$$h_{\mu\nu} = e^{-ikT}(\sin\theta\partial_\theta)P_L(\cos\theta)\begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ sym & sym & 0 & 0 \end{bmatrix}$$

Similar transformation and the canonical form for an even-mode wave can be constructed where the seven undetermined functions to a single radial wave equation can be reduced.

2.2.5 Radial Wave Equations

In Einstein's field equations, we substitute the above matrix component for the linear order perturbation:

$$\delta\Gamma^{\beta}_{\mu\nu;\beta} - \delta\Gamma^{\beta}_{\mu\beta;\nu} = 0 \tag{2.14}$$

and for odd wave the second term is identically zero. Now variation of Ricci tensor can be analyzed under the above mentioned simplified expression of perturbation:

$$(1-\frac{2m}{r})^{-1}kh_0 + \frac{d}{dr}(1-2m/r)h_1 = 0, \;\; for \;\; \delta R_{23} = 0 \tag{2.15}$$

$$(1-\frac{2m}{r})^{-1}k(\frac{dh_0}{dr} - kh_1 - \frac{2h_0}{r}) + (L-1)(L+2)\frac{h_1}{r^2} = 0, \;\; for \;\; \delta R_{13} = 0 \tag{2.16}$$

$$\frac{d}{dr}(kh_1 - \frac{dh_0}{dr}) + \frac{2kh_1}{r} = r^{-2}(1-\frac{2m}{r})^{-1}(\frac{4mh_0}{r} - L(L+1)h_0), \;\; for \;\; \delta R_{03} = 0 \tag{2.17}$$

Defining new quantity:

$$Q = (1-\frac{2m}{r})\frac{h_1}{r}, \tag{2.18}$$

we eliminate h_0 and solve for the second order wave equation of Q:

$$\frac{d^2Q}{dr^{*2}} + k^2_{eff}(r)Q = 0 \tag{2.19}$$

where,

$$dr^* = e^{(\frac{\lambda}{2}-\frac{\nu}{2})}dr \tag{2.20}$$

and

$$k_{eff}^2 = k^2 - L(L+1)\frac{e^\nu}{r^2} + \frac{6me^\nu}{r^3} \tag{2.21}$$

Here, the new metric is defined in the terms of λ and ν:

$$ds^2 = -e^\nu dT^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\Phi^2) \tag{2.22}$$

Hence,

$$e^\nu = e^{-\lambda} = 1 - \frac{2m}{r} \tag{2.23}$$

2.2.6 Dynamic Modes

For the dynamic variation to be physical($k \neq 0$),solutions of (2.19) should have a regular behaviour at both points of singularity, i.e. at $r = 2m$ and at spatial infinity. the general solution behaves as :

$$Q \sim c_1 e^{i\delta}(\frac{r}{2m} - 1)^{2ikm} + c_1 e^{-i\delta}(\frac{r}{2m} - 1)^{-2ikm} \;\; for \;\; r \to 2m \tag{2.24}$$

$$Q \sim c_2 \sin(kr + \eta) \;\; for \;\; r \to \infty \tag{2.25}$$

Now if we look at the wave equations in these separate limits, we can see three possibilities:

- The first case involves having a high frequency,$k >> \frac{1}{2m}$, which remains constants through radius.For $r >> 2m$, these waves travel into the black hole.

- The second case involves having a low frequency near $r = 2m$, i.e. $k << \frac{1}{2m}$ that decreases as $r \to \infty$. These waves are caught up by the effective potential as they propagate out of the black hole.

- The third case involves having a high frequency for $r >> 2m$ which decays as $r \to 2m$. These waves travel towards the black hole before reflecting off of curved space-time.

The analysis of such solutions space dependence reveals that the real value of frequency is uniquely determined by the demand of not going to infinity at large r. Because it drops off for a large r, the solution is adequate because it also drops off at Schwarzschild radius. Therefore we deduce that there are no unstable solutions for odd parity waves.

CHAPTER 3

Einstein-Maxwell-Yang-Mills Black Hole

3.1 Introduction

The Einstein-Maxwell-Yang-Mills (EMY) black hole is unification of general relativity, electromagnetism, and non-abelian gauge theory, offering a unique arena to explore fundamental interactions.

If we include only Maxwell's electromagnetic field with gravity, we get Reissner Nordström Black holes. The quest to integrate other forces, notably the strong force, led to the EMY black hole solution, combining electromagnetic and Yang-Mills fields within gravity's framework.

The action is given by,

$$S = \int \sqrt{-g} d^4x \left[R - \Lambda - F_{\mu\nu}F^{\mu\nu} - (F_{\mu\nu}^{(a)} F^{(a)\mu\nu})^p \right] \tag{3.1}$$

Here, g is the determinant of $g_{\mu\nu}$, R is the Ricci scalar, p is a real parameter that introduces non-linearity.

Addition to that $F_{\mu\nu}$ is the electromagnetic field strength and $F_{\mu\nu}^{(a)}$ is the gauge strength tensor. These are given in terms of A_μ, and $A_\mu^{(a)}$, respectively.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.2}$$

$$F_{\mu\nu}^{(a)} = \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)} + \frac{1}{2\sigma} C_{(b)(c)}^{(a)} A_\mu^{(b)} A_\nu^{(c)} \tag{3.3}$$

Here, Greek indices takes the values from 0 to 3, and 'a' denotes the internal gauge index, taking the value from 1 to 3. $C_{(b)(c)}^{(a)}$ is the structure constant of 3 parameter Lie group $\mathcal{G}$, $A_\mu^{(a)}$ are the SO(3) gauge group Yang-Mills potential, σ denotes arbitrary coupling constant. The form of $A^{(a)}$ and A is given by,

$$A^{(a)} = \frac{q_{YM}}{r^2}(x_i dx_j - x_j dx_i) \tag{3.4}$$

Here, $2 \leq j+1 \leq i \leq 3$ and $1 \leq a \leq 3$, and $r^2 = \sum_{i=1}^{3} x_i^2$. And Maxwell potential is given by,

$$A = \frac{Q}{r} dt \tag{3.5}$$

Here, Q is electric charge and q_{YM} is the YM charge.

Varying the action considering the metric, we get the field equation given below,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \tag{3.6}$$

Here, the energy-momentum tensor has two contribution, (i)matter content,and (ii)Yang-Mills contribution.

$$T_{\mu\nu} = T_{\mu\nu}^{M} + T_{\mu\nu}^{YM} \tag{3.7}$$

$$T_{\mu\nu}^{M} = F_{\mu\rho} F_\nu^\rho - \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} g_{\mu\nu} \tag{3.8}$$

$$T_{\mu\nu}^{YM} = -\frac{1}{2} g_{\alpha\mu} \left[\delta_\nu^\alpha \mathcal{F}_{YM}^q - 4q Tr(F_{\nu\lambda}^{(a)} F^{(a)\alpha\lambda}) \mathcal{F}_{YM}^{q-1} \right] \tag{3.9}$$

Varying the action considering the gauge potentials A and $A^{(a)}$, we get Maxwell and Yang-Mills equations respectively,

$$d(*F) = 0 \tag{3.10}$$

$$d(*F^{(a)} \mathcal{F}_{YM}^{q-1}) + \frac{1}{\sigma} C_{(b)(c)}^{(a)} \mathcal{F}_{YM}^{q-1} A^{(b)} \Lambda * F^{(c)} = 0 \tag{3.11}$$

Here * means Hodge dual and $\mathcal{F}_{YM}$ is the trace of the YM strength tensor.

3.2 General Metric considering only electric charge

First, we only consider the case where we only have electromagnetism as long range force. Our black hole is not rotating and consists of electric charge. So, we started with the most general form of metric possessing spherical symmetry, given by,

$$ds^2 = -A1(t,r)dt^2 + B1(t,r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.12}$$

Then from the Einstein's field equation and Maxwell's equations, we finally recover the Reissner-Nordström's metric given by,

$$ds^2 = -f1(r)dt^2 + f1(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.13}$$

here, f1(r) is given by,

$$f1(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \tag{3.14}$$

here, m signifies the mass of the black hole and Q reffers to the electric charge.

Along with if we introduce the YM charge then the metric gets slightly modified, f(r) gets modified,

$$f1(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} + \frac{Q_{YM}^2}{r^{4p-2}} \tag{3.15}$$

here, p is the real parameter which was in the action (3.1).

3.2.1 Metric-Perturbation and Regge-Wheeler Gauge

Axial mode

For odd(or axial) parity perturbations, we only have two non-trivial components of $h_{\mu\nu}$, after we choose Regge-Wheeler gauge and they are denoted by:

$$h_{t\phi} = h_0(t,r)\sin\theta\frac{\partial Y}{\partial\theta} \qquad and \qquad h_{r\phi} = h_1(t,r)\sin\theta\frac{\partial Y}{\partial\theta} \tag{3.16}$$

Here,Y denotes the spherical harmonics $Y_{lm}(\theta,\phi)$, where we have chosen m to be zero. Further we have the identity,

$$\frac{\partial^2 Y}{\partial\theta^2} = -l(l+1)Y - \cot\theta\frac{\partial Y}{\partial\theta} \tag{3.17}$$

For wave like solutions, $l \geq 2$.

3.2.2 Perturbation equations

$$\begin{aligned} &h_0\left(6r^2\cot\theta Y + (2Q^2 - 4mr + r^2\cot^2\theta + r^2\csc^2\theta)Y' - r^2Y'''\right) \\ &-r\left(Q^2 + r(-2m+r)\right)Y'\left(rh_0'' - 2\dot{h}_1 - r\dot{h}_1'\right) = 0 \end{aligned} \tag{3.18}$$

$$\begin{aligned} &\frac{csc\theta h_1\left(6sin2\theta Y + (1+3cos2\theta)Y' - 2sin^2\theta Y'''\right)}{4r^2} \\ &-\frac{rsin\theta Y'\left(-2\dot{h}_0 + r(\dot{h}_0' - \ddot{h}_1)\right)}{2\left(Q^2 + r(-2m+r)\right)} = 0 \end{aligned} \tag{3.19}$$

$$\begin{aligned} &2\left(Q^4 + m(2m-r)r^2 + Q^2r(-3m+r)\right)h_1 \\ &-r\left((Q^2 + r(-2m+r))^2h_1' - r^4\dot{h}_0\right) = 0 \end{aligned} \tag{3.20}$$

Using Maxwell's equation we get the condition,

$$-2h_0 + r(h_0' - \dot{h}_1) = 0 \tag{3.21}$$

Using (3.21) in (3.20) we get,

$$\begin{aligned} &2\left[Q^2(Q^2 + r^2 - 2mr) - mQ^2r - mr^3 + 2m^2r^2\right]\left[\frac{2r}{Q^2 + r^2 - 2mr}h_0 - h_0'\right] \\ &-2r(Q^2 - r^2)h_0 - 2r^2\left[Q^2 + r(-2m+r)\right]h_0' \\ &+r\left[Q^2 + r(-2m+r)\right]^2h_0'' + r^5\ddot{h}_0 = 0 \end{aligned} \tag{3.22}$$

Solving the above equation for large r limit we get the equation,

$$\left(1 - \frac{4m}{r} + \frac{2Q^2 + 4m^2}{r^2}\right)h_0'' + \frac{2m}{r^2}h_0' + \ddot{h}_0 = 0 \tag{3.23}$$

By separation of variable we get the time dependent and radial dependent part. The time dependent part is exponentially decaying , while the radial part is oscillating. So overall the perturbation fades away after some time. Which indicates that it is basically stable under these perturbations.

$h_1(t,r)$ can also be found using the constraint given by Maxwell's equation and remaining field equation.

3.2.3 Solution

Graphical representation of the solution(Axial mode),

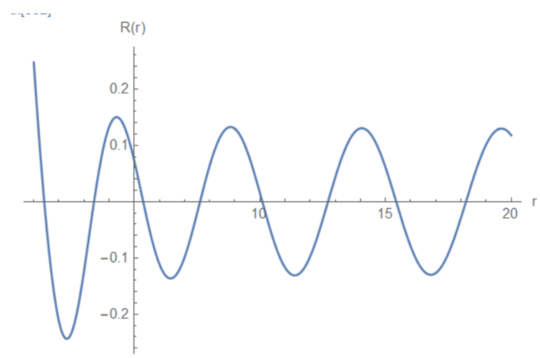

Figure 3.1: $h_0(r)$ for K=1,Q=1,and M =1.

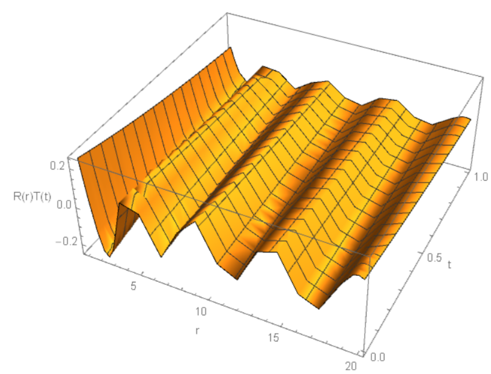

Figure 3.2: $h_0(t, r)$ for K=1,Q=1,and M =1.

3.2.4 Results

• We performed the stability analysis for axial mode or odd perturbations and found that it is stable under this perturbations. We also use the Maxwell's equations to get the constraint on $h_0(t,r)$ and $h_1(t,r)$.

• We also used Bianchi identity to find additional constraint, but the equation turns out to be identical with perturbation equations. So it satisfies the Bianchi identity also.

• If we take Q (electric charge) zero, we get the same equations Regge-Wheeler get in their stability analysis of non-rotating, uncharged black hole.

• It is to be noted that the system is not stable for all parameter space of m and Q. The system can become unstable for higher frequency of the perturbations, and there is a ratio of mass and charge, which can led system to be unstable.

3.3 General metric with electric charge and Yang-Mill's charge

Finally, we include our Yang-Mill's non abelian field into our system. Therefore our black hole is static, not rotating and consists both electric charge and Yang-Mill's charge. Metric of this system is as follows,

$$ds^2 = -A1(t,r)dt^2 + B1(t,r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.24}$$

Here, $A1(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} + \frac{Q_{YM}^2}{r^2}$, and $B1(r) = A1(r)^{-1}$. Here, we have used the value of p to be 1 to deal with only linearity. Thus the background is also time independent. So, we can use the time dependency of the perturbations as $e^{-i\omega t}$.

3.3.1 Field perturbation for axial mode

For this case, if we use Regge-Wheeler gauge condition, we get the trivial solution with all perturbations becoming zero. Thus we will be using the most general metric perturbation, addition to that we also have to perturb the electromagnetic field to get non zero perturbation dynamics. These perturbations are given by,

Metric perturbation

$$h_{\mu\nu} = e^{-i\omega t} \begin{bmatrix} 0 & 0 & 0 & h_0(r)\sin\theta Y'(\theta) \\ 0 & 0 & 0 & h_1(r)\sin\theta Y'(\theta) \\ 0 & 0 & 0 & \frac{h_2(r)}{2}\sin\theta(-\cot\theta Y'(\theta) + Y''(\theta)) \\ sym & sym & sym & 0 \end{bmatrix}$$

(* sym means they are related by even symmetry, $h_{\mu\nu} = h_{\nu\mu}$)

Electromagnetic perturbation

$$f_{\mu\nu} = e^{-i\omega t}\sin\theta \begin{bmatrix} 0 & 0 & 0 & f_{02}(r)Y'(\theta) \\ 0 & 0 & 0 & f_{12}(r)Y'(\theta) \\ 0 & 0 & 0 & f_{23}(r)Y(\theta) \\ sym & sym & sym & 0 \end{bmatrix}$$

(* sym means they are related by anti-symmetry, $f_{\mu\nu} = -f_{\nu\mu}$)

Therefore the whole metric is given as,

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu} \tag{3.25}$$

Here, $g^0_{\mu\nu}$ is given by (3.24).

And the electromagnetic field tensor is given by,

$$F_{\mu\nu} = F^0_{\mu\nu} + f_{\mu\nu} \tag{3.26}$$

Here $F^0_{\mu\nu}$ is given by,

$$F^0_{\mu\nu} = \begin{bmatrix} 0 & \frac{Q}{r^2} & 0 & 0 \\ -\frac{Q}{r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

These electromagnetic field perturbation can be derived from perturbation in terms of potential a_μ and they are as follows:

$a_0 = 0,\, a_1 = 0,\, a_2 = 0,\, a_3 = -\frac{f_{23}\sin\theta Y'(\theta)}{(l^2+l)}$.

The fact that $f_{\mu\nu}$ is derived from a potential gives,

$$f_{12} = \frac{1}{(l^2+l)}\frac{\partial f_{23}}{\partial r} \tag{3.27}$$

$$f_{02} = \frac{1}{(l^2+l)}\frac{\partial f_{23}}{\partial t} \tag{3.28}$$

Above equations can be derived from the field equations,

$$f_{\mu\nu,\lambda} + f_{\lambda\mu,\nu} + f_{\nu\lambda,\mu} = 0 \tag{3.29}$$

We now have to substitute these perturbations in Einstein's field equations and Maxwell's equation.

3.3.2 Perturbation equations

Einstein's field equations

$$\begin{aligned}&8Q(q^2+Q^2-2mr+r^2)f_{12}-2(4q^2+2Q^2+r(-4m+l(l+1)r))h_0+\\&ir(4(q^2+Q^2-2mr+r^2)\omega h_1+(l^2+l-2)r\omega h_2+\\&2r(q^2+Q^2-2mr+r^2)(\omega h_1'-ih_0''))=0\end{aligned} \tag{3.30}$$

$$\begin{aligned}&-\frac{8Qr^4f_{02}}{q^2+Q^2-2mr+r^2}+4q^2h_1-4r^2h_1+2lr^2h_1-2l^2r^2h_1-\\&\frac{2r^5\omega(2ih_0+r\omega h_1-irh_0')}{q^2+Q^2-2mr+r^2}+(l^2+l-2)r(-2h_2+rh_2')=0\end{aligned} \tag{3.31}$$

$$\begin{aligned}&-2Q^2h_2+2r(q^2+Q^2-mr)h_1-2(q^2+Q^2-2mr)h_2\\&-2(q^2+Q^2-2mr+r^2)h_2-\frac{r^6\omega(2ih_0+\omega h_2)}{q^2+Q^2-2mr+r^2}\\&-r(2r(q^2+Q^2-2mr+r^2)h_1'-(3(q^2+Q^2)-5mr+2r^2)h_2'\\&+r(q^2+Q^2-2mr+r^2)h^{2\prime\prime})=0\end{aligned} \tag{3.32}$$

Maxwell's Equation

$$\begin{aligned}&\frac{i\omega f_{02}}{q^2+Q^2-2mr+r^2}+\\&\frac{-2(q^2+Q^2-mr)f_{12}-2Qh_0+r(-f_{23}+iQ\omega h_1+(q^2+Q^2-2mr+r^2)f_{12}')}{r^5}=0\end{aligned} \tag{3.33}$$

From (3.27) and (3.28), we rearrange the equations,

Einstein's field equations

$$\begin{aligned}
&\frac{8Q(q^2+Q^2-2mr+r^2)f'_{23}}{(l^2+l)} - 2(4q^2+2Q^2+r(-4m+L(l+1)r))h_0 \\
&+ir(4(q^2+Q^2-2mr+r^2)\omega h_1 + (l^2+l-2)r\omega h_2 \\
&+2r(q^2+Q^2-2mr+r^2)(\omega h'_1 - ih''_0)) = 0
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
&\frac{8iQr^4\omega f_{23}}{(l^2+l)(q^2+Q^2-2mr+r^2)} + 4q^2h_1 - 4r^2h_1 + 2lr^2h_1 + 2l^2r^2h_1 \\
&-\frac{2r^5\omega(2ih_0+r\omega h_1 - irh'_0)}{q^2+Q^2-2mr+r^2} + (l^2+l-2)r(-2h_2+rh'_2) = 0
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
&-2Q^2h_2 + 2r(q^2+Q^2-mr)h_1 - 2(q^2+Q^2-2mr)h_2 \\
&-2(q^2+Q^2-2mr+r^2)h_2 - \frac{r^6\omega(2ih_0+\omega h_2)}{q^2+Q^2-2mr+r^2} \\
&-r(2r(q^2+Q^2-2mr+r^2)h'_1 - (3(q^2+Q^2)-5mr+2r^2)h'_2 \\
&+r(q^2+Q^2-2mr+r^2)h^{2\prime\prime}) = 0
\end{aligned} \tag{3.36}$$

Maxwell's equation

$$\begin{aligned}
&\frac{\omega^2 f_{23}}{(l^2+l)(q^2+Q^2-2mr+r^2} + \frac{1}{r^5}(-2Qh_0 - \frac{2(q^2+Q^2-mr)f'_{23}}{(l^2+l)} \\
&+r(-f_{23}+iQ\omega h_1 + Qh'_0 + \frac{(q^2+Q^2-2mr+r^2)f''_{23}}{(l^2+l)})) = 0
\end{aligned} \tag{3.37}$$

3.3.3 Solution

Before jumping to the solution we want to do a variable substitution so that all the exterior region of the black hole (outside of outer event horizon) is smoothly mapped. If r_h is the radius of outer event horizon, then $x = \frac{r_h}{r}$. So, when r is near to r_h, x is very close to 1, and when r is very very large, x tends to zero. Substituting these in (3.34-3.37), and solving numerically, we can get the behaviour of these perturbations in terms of graphical solution.

For large r limit

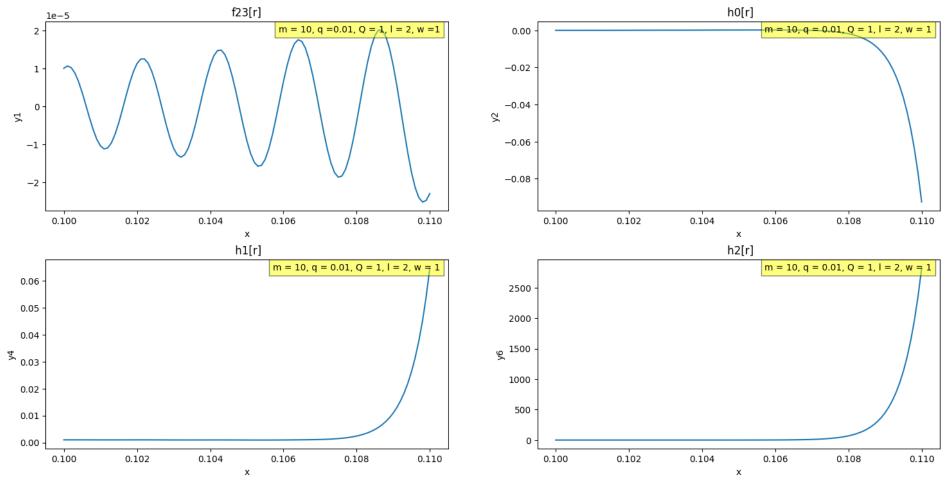

Figure 3.3: Behaviour of perturbations at large r limit

For r near horizon (outer most)

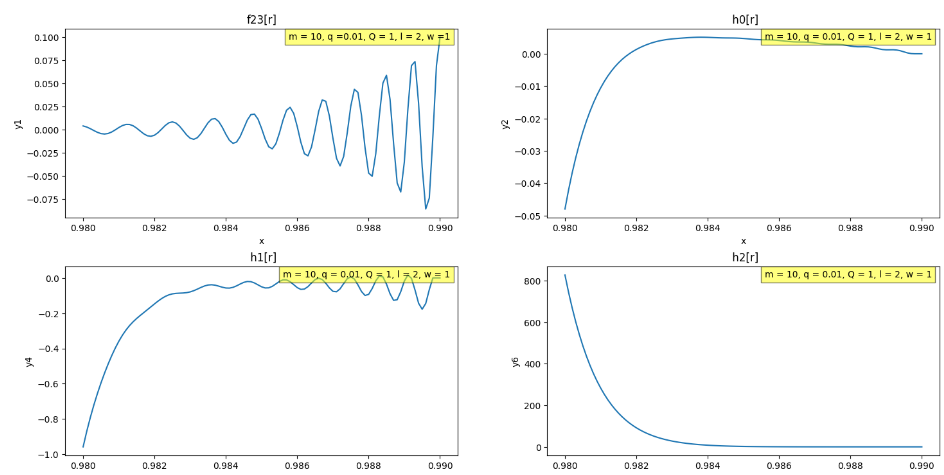

Figure 3.4: Behaviour of perturbations near outer most horizon

CHAPTER 4

Discussion

• We performed this harmonics analysis for axial mode or odd mode perturbations and found that the system is stable and these perturbations are well behaved at both limit.

• If we take q to be zero and $f_{23}(r)$ to be zero, we get back the old set of equations, for the case where we only consider electric charged black hole.

• The stability is dependent on the amount of electric charge and Yang-Mill's charges, we have in our system. Although we have not exactly pinpoint the limits of these charges which can trigger the system to become unstable but we have found that there is some critical mass to these charges ratio, after which system is unstable and these perturbations grows over time.

• We also used the Bianchi identity to find additional constraint, but the equation turns out to be identical with perturbation equations. So it satisfies the Bianchi identity also.

CHAPTER 5

Conclusion

From these graphs and the considered value of the parameter m, Q, l, q_{YM}, we can certainly say that these perturbations behave smoothly at both end, and for large r limit they decay very quickly. Which clearly meant that the system is stable under this perturbation and selected parameter values. This indicates that there is a possibility of detecting these kind of black hole and gravitational waves can be a tool for detecting them and looking for their characteristics.

CHAPTER 6

Scope for future work

1. **Nonlinear Stability Analysis**: Extend the stability analysis beyond linearized perturbations to investigate the nonlinear stability of EMY black holes.

2. **Incorporating Quantum Effects**: Explore the effects of quantum mechanics on the stability properties of EMY black holes. Quantum corrections, such as Hawking radiation and quantum fluctuations, can have significant implications for the stability and evolution of black hole solutions.

3. **Charged Black Hole Thermodynamics**: Investigate the thermodynamic properties of charged EMY black holes, including their entropy, temperature, and heat capacity.

4. **Black Hole Hair**: Investigate the existence and properties of ”black hole hair” in EMY black holes, referring to additional degrees of freedom beyond mass, charge, and angular momentum.

APPENDIX A

Energy Condition

Energy condition in General Relativity is a mathematical formulation that generalises the statement “a region of space cannot have a negative energy density”. For a perfect fluids($T^{\mu}_{\nu} = diag(-\rho, p, p, p)$), these can be stipulated in mathematical form as:

- **Null Energy Condition**: $\rho + p \geq 0$
- **Weak Energy Condition**: $\rho \geq 0, \rho + p \geq 0$
- **Dominant Energy Condition** : $\rho \geq |p|$
- **Strong Energy Condition**: $\rho + p \geq 0, \rho + 3p \geq 0$

APPENDIX B

FLRW Metric

B.1 Introduction

The Friedmann-Lemaître-Robertson-Walker (FLRW) metric is the simplest metric with great number of symmetry and it can describe our universe in extremely large cosmic scale, where the universe is approximately homogeneous and spherically symmetric. The FLRW metric provides a framework in which we can get an expanding or contracting or even static universe. Even though many believed at first that our universe is static, not expanding or contracting but Hubble's discovery of distant galaxy's redshift confirms that our universe indeed is expanding.

B.2 General Metric

The FLRW metric is grounded on the presumption of spatial homogeneity and isotropy . It also makes the assumption that the metric's spatial component is time-dependent

.The following is a generic metric that meets these criteria (in spherical coordinates):

$$ds^2 = -dt^2 + a^2(t)[\frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\Phi^2] \tag{B.1}$$

Here, $a(t)$ is the cosmic scale factor; the increasing scale factor implies that the universe is expanding. The curvature of the space, which can be elliptical, Euclidean or hyperbolic, is represented by the constant κ. It's values can be taken as +1, -1, or 0 for positive, negative and zero curvature respectively. r is unitless ,while $a(t)$ has length units.When $\kappa = 1$,$a(t)$ is the space's radius of curvature, which can also be sometimes written as R(t).

B.3 Einstein's Field Equations

For any form of the scale factor a(t), the FLRW metric is applicable; We will consider our spacetime to be filled with perfect fluids. It's evident that, if a fluid, isotropic in one frame results in a metric that's isotropic in another frame, those two frames will align. In other words the fluid will be stationary in co-moving coordinates. The four-velocity vector is then:

$$U^\mu = (1, 0, 0, 0) \tag{B.2}$$

and the energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \tag{B.3}$$

Here, ρ denotes the energy density of the fluid and p is the fluid pressure. Friedmann equation results after we put this in the Einstein field equation.

00 component of (1.1) will give:

$$\frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} - \frac{\Lambda}{3} = \frac{\rho}{3} \tag{B.4}$$

ij component will give:

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} - \Lambda = -p \tag{B.5}$$

These two equations are known as **friedmann equations**.

B.4 Change of coordinate

We will substitute in FLRW metric:

$$\frac{dt}{a} = d\eta \tag{B.6}$$

for flat space-times($\kappa = 0$), FLRW metric can be written as:

$$ds^2 = a^2(\eta)\left(-d\eta^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\Phi^2\right) \tag{B.7}$$

Along with we define the 'conformal scale factor' as:

$$C(\eta) = a^2(\eta) = A + B\tanh\rho_0\eta; \quad A \geq B \tag{B.8}$$

Here, A, B and ρ_0 are some constants. The space-time then become Minkowskian in the distant past and future, because:

$$C(\eta) \to A \pm B, \quad \eta \to \pm\infty \tag{B.9}$$

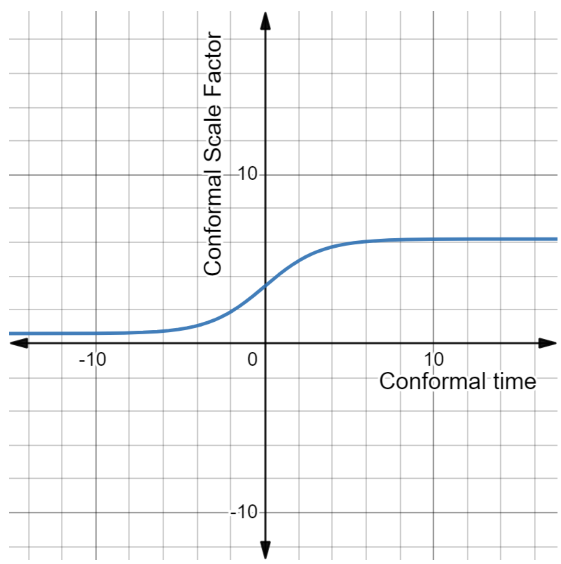

Figure B.1: $C(\eta) = A + Btanh(\rho_0\eta)$ represents a universe that is asymptotically static and expands smoothly.

Addition to that we define "conformal" Hubble constant,

$$H(\eta) := \frac{\partial_\eta a}{a} \tag{B.10}$$

Thus the background solution of friedmann equations gives the following relations:

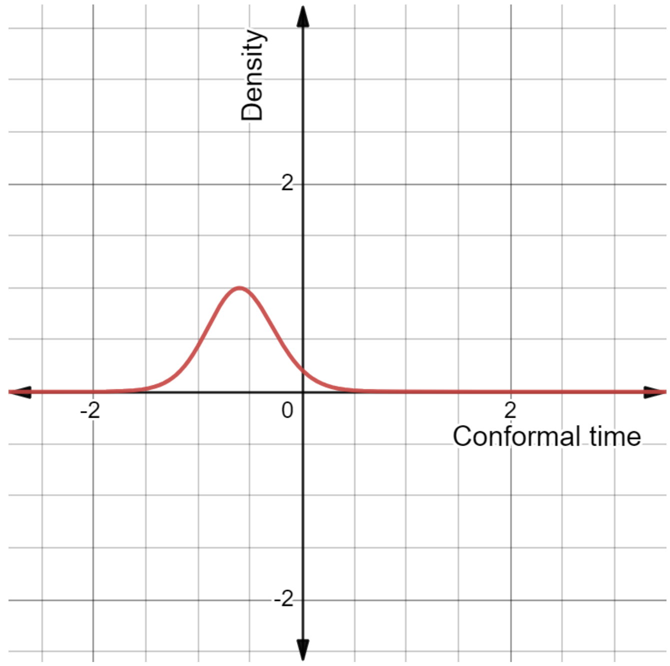

Figure B.2: Density variation .

$$\rho = \frac{3}{a^2} H^2 - \Lambda \tag{B.11}$$

$$p = \Lambda - \frac{H^2}{a^2} - \frac{2}{a^2} \frac{dH}{d\eta} \tag{B.12}$$

B.5 Interpretation of ρ and p

The solution of (3.12) and (3.13) can be easily done by calculating H and its derivative.

$$\rho = \frac{3B^2\rho_0^2}{4} \frac{sech^4(\rho_0\eta)}{(A + B\tanh(\rho_0\eta))^3} - \Lambda \tag{B.13}$$

$$p = \Lambda + \left[\frac{2B\rho_0^2 sech^2(\rho_0\eta)\tanh(\rho_0\eta)}{(A + B\tanh(\rho_0\eta))^2} + \frac{3B^2\rho_0^2 sech^4(\rho_0\eta)}{4(A + B\tanh(\rho_0\eta))^3}\right] \tag{B.14}$$

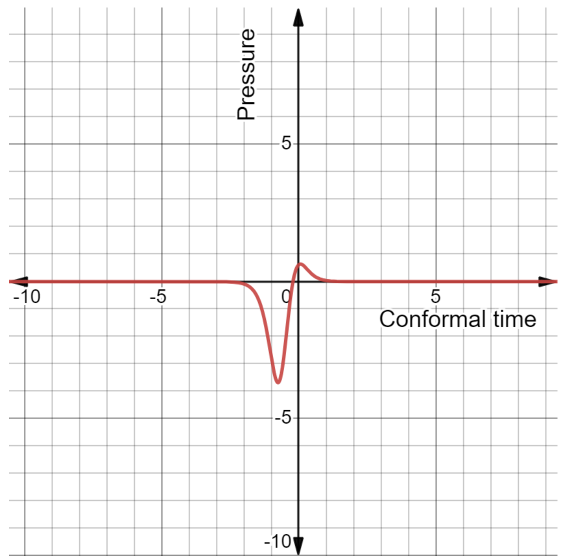

Figure B.3: Pressure variation.

B.6 Discussion

To get a physically valid background it must satisfy certain energy conditions, like Null Energy Condition, Weak Energy Condition, Dominant Energy Condition, and Strong Energy Condition. However, our analysis revealed that certain criteria of parameter space of A, B, and ρ_0 are not met. Because of it we were not able to further investigate and look for gravitational waves in this background.

It is shown in the graph, that the density of the fluid remains positive throughout all time. We can notice some peculiar behaviour of the fluid pressure. Its graph shifts from positive to negative values, a behavior that is very unusual and inconsistent with perfect fluid known to us. We are not able to identify the reason for this behaviour, particularly during the phase where pressure is too much negative. During this phase, the model fails to satisfy any of the energy conditions for any value of the parameters, thus loosing its viability as a valid feasible background.

The above plots were drawn, considering Λ equal to zero ($\Lambda = 0$). Similar plots can be drawn for other values of Λ, but for physically reliable results, Λ should be either zero or negative ($\Lambda \leq 0$). This ensures that the density of the fluid remains positive throughout the analysis.

B.7 Conclusion

While our investigation raises questions about the validity of the assumed cosmological scale factor for our background, but it's essential to understand that our findings don't prove that it's impossible. Further investigations and analysis are required to establish the precise implications of our results.

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