Existence and Uniqueness of Linear Time Invariant System and its Application

M.Sc. Thesis

By

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DISCIPLINE OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

December 2018

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A THESIS

Submitted in partial fulfillment of
the requirements for the award of
the degree
of
Master of Science

by

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INDIAN INSTITUTE OF TECHNOLOGY INDORE

December 2018

INDIAN INSTITUTE OF TECHNOLOGY INDORE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **Existance and Uniqueness of Linear Time Invariant System and its Applications** in the partial fulfillment of the requirements for the award of the degree of **Master Of Science** and submitted in the **Discipline Of Mathematics, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from **June 2016 to December 2018** under the supervision of **Dr. Sk. Safique Ahmad, Associate Professor, Discipline of Mathematics, IIT Indore.**

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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This is to certify that the above statement made by the candidate is correct to the best of my/our knowledge.

Signature of the Supervisor of
M.Sc. thesis (with date)
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Acknowledgements

I owe my gratitude to the Discipline of Mathematics, IIT Indore for allowing me to undertake the M.Sc. research project work.
I wish to thank Dr. Md. Aquil Khan, Associate Professor, Head-Discipline of Mathematics, IIT Indore for the academic support and facilities provided to carry out various researches with ease.
I am very much indebted to my thesis supervisor Dr. Sk. Safique Ahmad, Associate Professor, Discipline of Mathematics, IIT Indore for his valuable guidance and supervision. I express my deepest sense of gratitude and obligation for his restless effort in every possible way for my project.
I would like to thank Prince Kanhya, Nitin Bisht, Research Scholar, IIT Indore and Dr. Sudhananda Maharana, Postdoctoral Fellow, IIT Indore who always helped me throughout the research work. I am also grateful to Dr. Vijay Kumar Sohani, Assistant Professor, Discipline of Mathematics and Gyan Swaroop Nag, Research Scholars, IIT Indore for their valuable suggestions.
I would like to thank my PSPC members Dr. Niraj Shukla, Assistant Professor, Discipline of Mathematics, Dr. Shanmugam Dhinakaran, Associate Professor, Discipline of Mechanical Engineering. and Dr. Santanu Manna, Visiting Assistant Professor, Discipline of Mathematics, IIT Indore for their valuable comments and suggestions at various stages of my research work.
A big thanks to all my classmates for supporting me.

Place: IIT Indore
Date: 07-12-2018

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Abstract

In this thesis, structural decomposition of linear periodic continuous time system is addressed. Decomposition of a state of a periodic system into controllable and uncontrollable parts is achieved by a continuously differentiable and periodic coordinate transformation with the same period of the system. Also, a counter-example has been examined for the conjecture. Hence we get a condition for the existence of such a coordinate transformation. This is a survey of the existing work done by I. Jikuya and I. Hodaka [4]. **The main highlight of this thesis is that we have proved existence and uniqueness theorem of Linear Time Variant system and extended these results for n^{th} order.**

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Notations

Symbol	**Interpretation**
LTV	Linear Time Variant
LTI	Linear Time Invariant
KCD	Kalman Canonical Decomposition
$x(t) \in \mathbb{R}^n$	State vector
$v(t) \in \mathbb{R}^q$	Output vector
$u(t) \in \mathbb{R}^p$	Input (or Control) vector
$P(t) \in \mathbb{R}^{n\times n}$	State matrix
$Q(t) \in \mathbb{R}^{n\times m}$	Input matrix
$R(t) \in \mathbb{R}^{q\times n}$	Output matrix
$S(t) \in \mathbb{R}^{q\times p}$	Feedthrough matrix
ρ	Rank

——— * ———

Chapter 1

Introduction

In this chapter we have discussed the Linear Time Invariant (LTI) and Linear Time Variant (LTV) systems which have been studied and developed by Rudolf Emil Kalman, an Electrical Engineer who has combined both the discrete-time and continuous-time case, the theory and design of linear systems with respect to quadratic criteria. In particular, in the current chapter we have studied the existence and uniqueness of the LTV system and furthermore, the results have been extended upto n^{th} order.

The transition matrix basically depends on a fundamental matrix of the LTV system. More precisely, we use the technique of transition matrix as per the methods and properties given by Chui and Chen in [3] which help us to determine the complete solution of an LTV system. We state reachability gramian matrix and controllability gramian matrix, respectively which will be used later on in Chapter 4 to find the reachability and controllability of an LTV system.

The controllability criteria for the LTV system as mentioned in [9, Theorem 3.3.1] has been retraced in Chapter 3. Furthermore, an LTV system can be decompose into two parts, namely, controllable part and uncontrollable part by the help of [10, Theorem 7]. A conjecture has also been discussed for periodic LTV system and a counterexample has also been given for that conjecture which stated in [4]. Finally, we have discussed about certain necessary and sufficient condition which validates the conjecture under consideration [4, Theorem 1].

1.1 Linear Time invariant system

Let us consider the standard form of the linear state equation as follows:

$$\left.\begin{aligned} x'(t) &= Px(t) + Qu(t) \\ v(t) &= Rx(t) + Su(t) \end{aligned}\right\}, \tag{1.1.1}$$

where $x(t) \in \mathbb{R}^n$ is a state vector, $v(t) \in \mathbb{R}^q$ is a output vector, $u(t) \in \mathbb{R}^p$ is a input (or control) vector, $P \in \mathbb{R}^{n\times n}$ is a state matrix, $Q \in \mathbb{R}^{n\times p}$ is a input matrix, $R \in \mathbb{R}^{q\times n}$ is a output matrix, $S \in \mathbb{R}^{q\times p}$ is a feedthrough matrix.

Definition 1.1.1. [3] (*Controllability*)

1. The system (1.1.1) is said to be *controllable* if the rank of $\mathcal{C} = \left[\begin{array}{cc} Q & PQ \dots P^{n-1}Q \end{array}\right]$ is full rank.

2. The system (1.1.1) is said to be *completely controllable* if all states are controllable.

Example 1.1.2. Determine the *controllability* for

$$x' = Px + Qu, \quad where \quad P = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Solution: To determine the *controllability* we need to check the rank of $\mathcal{C} = \left[\begin{array}{cc} Q & PQ \end{array}\right]$. First we find that

$$PQ = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -2b \end{bmatrix}.$$

Now

$$\mathcal{C} = \left[\begin{array}{cc} Q & PQ \end{array}\right] = \begin{bmatrix} a & -a \\ b & -2b \end{bmatrix},$$

and it follows that

$$\det(\mathcal{C}) = -ab, \begin{cases} \rho(\mathcal{C}) = 2, & if \quad a \neq 0 \quad and \quad b \neq 0 \\ \rho(\mathcal{C}) < 2, & if \quad either \quad a = 0 \quad or \quad b = 0. \end{cases}$$

The system is *controllable* if $a \neq 0$ and $b \neq 0$.

Definition 1.1.3. [3](*Observability*)

1. The system (1.1.1) is said to be observable if the rank of $\mathcal{O} = \begin{bmatrix} R^T & P^T R^T \dots (P^T)^{n-1} R^T \end{bmatrix}$ is full rank.

2. If all the possible initial states of the system (1.1.1) are observed then the system is said to be completely observable.

Example 1.1.4. Determine the observability for

$$x = Px + Qu, \quad v = Rx, \quad where \quad P = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} a & b \end{bmatrix}.$$

Solution: To determine the observability we need to check the rank of $\mathcal{O} = \begin{bmatrix} R^T & P^T R^T \end{bmatrix}$. Here we can see that

$$R^T = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad P^T R^T = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -2b \end{bmatrix}.$$

Thus, we obtain that

$$\mathcal{O} = \begin{bmatrix} R^T & P^T R^T \end{bmatrix} = \begin{bmatrix} a & -a \\ b & -2b \end{bmatrix},$$

which gives

$$\det(\mathcal{O}) = -ab, \begin{cases} \rho(\mathcal{O}) = 2, & if \;\; a \neq 0 \;\; and \;\; b \neq 0 \\ \rho(\mathcal{C}) < 2, & if \;\; either \;\; a = 0 \;\; or \;\; b = 0. \end{cases}$$

The system is observable if $a \neq 0$ and $b \neq 0$.

Definition 1.1.5. [2] *(T- periodicity)* A function f is periodic if the function values repeat at regular interval of the independent variable, The regular interval is referred to as the period. Function f is said to have T- periodicity if

$$f(t + T) = f(t)$$

for any value of t in the domain of f.

Definition 1.1.6. [3] *(Fundamental Matrix)* A matrix valued function ϕ is said to be a fundamental matrix of $x'(t) = P(t)x(t)$ if $\phi'(t) = P(t)\phi(t)$ and ϕ is non singular matrix for all $t \in \mathbb{R}$.

Remark 1.1.7. [5]*(Kalman Decomposition)* The Kalamn decomposition is just the combination of the controllable/uncontrollable and the observable/unobservable decomposition. Every state-space equation can be transformed into a canonical form that splits the states into

- *Controllable* and observable states
- *Controllable* but unobservable states
- *Uncontrollable* but observable states
- *Uncontrollable* and unobservable states

1.2 Linear Time Variant (LTV) system

Let us consider the standard form of the linear state equation as given in [3]:

$$\left.\begin{aligned} x'(t) &= P(t)x(t) + Q(t)u(t) \\ v(t) &= R(t)x(t) + S(t)u(t) \end{aligned}\right\}, \tag{1.2.1}$$

where $x(t) \in \mathbb{R}^n$ is a state vector, $v(t) \in \mathbb{R}^q$ is an output vector, $u(t) \in \mathbb{R}^p$ is an input (or control) vector, $P(t) \in \mathbb{R}^{n\times n}$ is a state matrix, $Q(t) \in \mathbb{R}^{n\times p}$ is an input matrix, $R(t) \in \mathbb{R}^{q\times n}$ is an output matrix, $S(t) \in \mathbb{R}^{q\times p}$ is a feedthrough matrix (If the system model does not have a direct feedthrough, then $S(t)$ is treated as zero matrix).

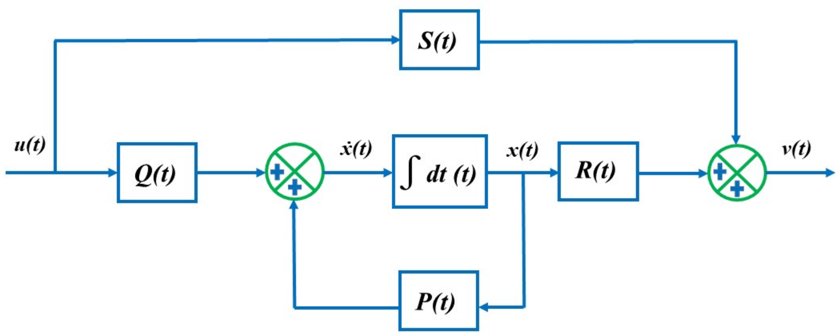

Figure 1.1: LTI System

Define, $x'(t) : \mathbb{R} \to \mathbb{R}^n$

$$where\ x'(t) = \begin{bmatrix} x'_1(t) \\ \vdots \\ x'_n(t) \end{bmatrix} \quad for \quad t \in \mathbb{R}, \quad \text{and} \quad x'(t) \in \mathbb{R}^n$$

In this general formulation, all matrices are allowed to be time variant, however, in the common linear time invariant case, matrices will be time invariant. The time variable t can be continuous (e.g. $t \in \mathbb{R}$). Depending on the assumptions taken, the state-space model representation can assume the following forms given in the Table 1.1.

System type	**State-space model**
Continuous time-invariant	$x'(t) = Px(t) + Qu(t)$ $v(t) = Rx(t) + Su(t)$
Continuous time-variant	$x'(t) = P(t)x(t) + Q(t)u(t)$ $v(t) = R(t)x(t) + S(t)u(t)$

Table 1.1

The advantage of linear time variant (LTV) system

- First advantage is that one can analyze the output of the system by decomposing the input.
- LTV system is an ideal system, which can also be satisfied with principle of superposition.

1.3 The existence and uniqueness theorem

Consider the initial value problem(IVP)

$$y' = f(x, y), \quad y(x_0) = y_0. \tag{1.3.1}$$

Theorem 1.3.1. *[1]***(Existence theorem)** *Suppose that $f(x, y)$ is continuous function in some region*

$$R = \{(x, y) : |x - x_0| \le a, |y - y_0| \le b\} \quad (a, b > 0).$$

Then the IVP (1.3.1) has atleast one solution $y = y(x)$ defined in the interval $|x - x_0| \leq \alpha$ where $\alpha = min\left\{a, \frac{b}{M}\right\}$. and there exists $M > 0$ such that $|f(x, y)| \leq M, \forall (x, y) \in R$.

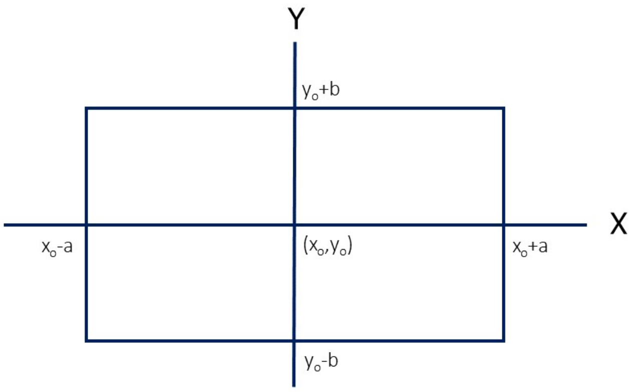

Figure 1.2: Rectangle R

Theorem 1.3.2. *[1]***(Uniquness Theorem)** *Suppose that f and $\frac{\partial f}{\partial y}$ are continuous functions in R (defined in the existence theorem). Then the IVP (1.3.1) has atmost one solution $y = y(x)$ defined in the interval $|x - x_0| \leq \alpha$ where*

$$\alpha = min\left\{a, \frac{b}{M}\right\}.$$

Combining with existence theorem, the IVP (1.3.1) has unique solution $y = y(x)$ defined in the interval $|x - x_0| \leq \alpha$ and there exists $M > 0$ such that $|f(x, y)| \leq M, \forall (x, y) \in R$.

Remark 1.3.3. [1] Condition (b) can be replaced by a weaker condition which is known as Lipschitz condition. Thus, instead of continuity of $\frac{\partial f}{\partial y}$, we require

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|, \quad \forall\ (x, y) \in R.$$

If $\frac{\partial f}{\partial y}$ exists and is bounded, then it necessarily saitisfies Lipschitz condition. On the other hand, a function $f(x, y)$ may be Lipschitz continuous but $\frac{\partial f}{\partial y}$ may not exists.

Example 1.3.4. [1] Test the existence and uniqueness of the solution of the IVP $y' = y^{1/2}$, $y(1) = 0$. In the suitable rectangle R. If solution is not unique, then find all solutions.

[1]http://home.iitk.ac.in/ sghorai/TEACHING/MTH203/ode5.pdf

Here $f(x,y) = y^{1/2}$, $x_0 = 1$, and $y_0 = 0$. Now $f(x,y)$ is continuous and bounded in R. Hence atleast one solution exists in some rectangle containing $(1,0)$. Let us now test the Lipschitz condition. We have

$$|f(x,y_1) - f(x,y_2)| = |y_2^{1/2} - y_1^{1/2}| = \left|\frac{(y_2^{1/2} + y_1^{1/2})(y_2^{1/2} - y_1^{1/2})}{(y_2^{1/2} - y_1^{1/2})}\right| = \left|\frac{y_2 - y_1}{y_2^{1/2} - y_1^{1/2}}\right|,$$

or

$$\frac{|f(x,y_1) - f(x,y_2)|}{|y_2 - y_1|} = \frac{1}{y_2^{1/2} - y_1^{1/2}},$$

This equation can be made as large as large as possible by choosing y_1 and y_2 sufficiently small, that is, a finite value for the Lipschitz constant L cannot be determined.

We have

$$y_1^{1/2} + y_2^{1/2} < 2y^{1/2}, \quad \text{if} \quad y = max\{y_1, y_2\},$$

and

$$\left|\frac{1}{y_1^{1/2} + y_2^{1/2}}\right| > \frac{1}{2y^{1/2}} > L, \quad \text{if} \quad y^{1/2} < \frac{1}{2L}.$$

In the neighborhood of $y = 0$, this criterion is satisfied for every $L > 0$.

Therefore, the IVP does not pass a unique solution. Infact, there are two solutions of the IVP which are

$$y = \frac{(x-1)^2}{4} \quad \textit{and} \quad y^2 = 0.$$

It can be easily verified for $y \neq 0$. From the differential equation $\frac{dy}{y^{1/2}} = dx$, on integrating, we have

$$2y^{1/2} = x + c \quad \textit{or} \quad y = \frac{(x+c)^2}{4}.$$

The condition $y(1) = 0$ is satisfied both above equation

$$\left[\frac{(c+1)^2}{4}\right] = 0 \quad \textit{or} \quad c = -1.$$

Hence, the solution are $y = 0$ and $y = \frac{(x-1)^2}{4}$.

Example 1.3.5. [1] Let $f : \mathbb{R} \to \mathbb{R}$ be function defined by $f(x,y) = x^2|y|$, $|x| \leq 1$, $|y| \leq 1$. Then f is Lipschitz continous in y but $\frac{\partial f}{\partial y}$ does not exist at $(x,0)$.

Example 1.3.6. [1] Consider the IVP

$$y'(x) = \frac{e^{y(x)^2 - 1}}{1 - x^2 y(x)^2}, \qquad y(-2) = 1. \tag{1.3.2}$$

We want to find an interval on which a solution surely exists. Let us define

$$F(x,y) = \frac{e^{y(x)^2-1}}{1-x^2y(x)^2}.$$

Thus we need to pick a rectangle R which is centered at $(-2,1)$. In this rectangle we need to have good control on F and $\frac{\partial f}{\partial y}$ and so we certainly have to choose R so small that it contains no points at which the denominator $1-x^2y^2$ vanishes. The exact choice of the rectangle is up to you but the properties of F and $\frac{\partial f}{\partial y}$, as required in the theorem, must be satisfied. Lets pick a, b small in the definition of R, say, lets choose $a = \frac{1}{2}$ and $b = \frac{1}{4}$ so that we work in the rectangle

$$R = \left\{ (x,y) : \frac{-5}{2} \leq x \leq \frac{-3}{2}, \frac{3}{4} \leq y \leq \frac{5}{4} \right\}.$$

Notice that for (x,y) in R we have

$$x^2 \geq \frac{9}{4}, y^2 \geq \frac{9}{16},$$

and

$$x^2y^2 \geq \frac{81}{64},$$

so

$$|1-x^2y^2| \geq \frac{81}{64} - 1.$$

Then $|(1-x^2y^2)^{-1}| \geq 4$ and $e^{y^2-1} \leq e^{\frac{25}{16}-1} = e^{\frac{9}{16}} < 3$, which implies

$$|F(x,y)| = |\frac{e^{y^2-1}}{1-x^2y^2}| \leq 3.4 = 12, \quad \text{for } (x,y) \text{ in } R.$$

Thus an appropriate choice for M in (a) is $M = 12$.

To verify (b), we compute

$$\frac{\partial F}{\partial y}(x,y) = \frac{2ye^{y^2-1}}{1-x^2y^2} + \frac{e^{y^2-1}}{(1-x^2y^2)^2}2yx^2.$$

Observe that $|2y| \leq \frac{5}{2}$ and $|2yx^2| \leq (\frac{5}{2})^3$ in R and using the above bounds, we can estimate for all (x,y) in R

$$\begin{aligned} \left|\frac{\partial F}{\partial y}(x,y)\right| &\leq \left|\frac{2ye^{y^2-1}}{1-x^2y^2}\right| + \left|\frac{e^{y^2-1}}{(1-x^2y^2)^2}2yx^2\right| \\ &\leq \frac{5}{2}.12 + 3.4.(\frac{5}{2})^3 = 780. \end{aligned}$$

Thus we see that condition is also satisfied, with $K = 780$.

Theorem 1.3.7. *[10] A sequence of function* $\{f_n\}, n = 1, 2, 3\ldots$ *converges uniformly on* $E \subseteq \mathbb{R}$ *to a function* f *if for every* $\epsilon > 0$ *there is an integer* N *such that* $n \leq N$ *implies*

$$|f_n(x) - F(x)| \leq \epsilon$$

for all $x \in E$.

Theorem 1.3.8. *[1] Let* $\{z_n(t)\}$ *be a sequence converges uniformly to* $z(t)$ *in* $[a, b]$*, and let* $f(t, z(t))$ *be a continuous function in the domain* D *such that for all* n *and* $t \in [a, b]$ *the point* $(t, z_n(t)) \in D$. *Then*

$$\lim_{n\to\infty} \int_a^b f(s, z_n(s))ds = \int_a^b \lim_{n\to\infty} f(s, z_n(s))ds = \int_a^b f(s, z(s))ds.$$

Theorem 1.3.9. *[1]***(Weierstrass' M- Test)** *Let* $\{z_n(t)\}$ *be a sequence of functions with* $|z_n(t)| \leq N_n$ *for all* $t \in [a, b]$ *with* $\sum_{n=0}^{\infty} N_n < \infty$. *Then* $\sum_{n=0}^{\infty} z_n(t)$ *converges uniformly in* $[a, b]$ *to a unique function* $z(t)$.

1.4 Existence and uniqueness of the solution of LTV system

Consider LTV system be given by

$$x'(t) = P(t)x(t) + Q(t)u(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1.4.1}$$

where $P(t)$ and $Q(t)$ are continuous on some interval $[a, b]$ containing t_0. It is evident that any solution of LTV system is also a solution of integral equation

$$x(t) = x_0 + \int_{t_0}^{t} [P(s)x(s) + Q(s)u(s)]ds, \tag{1.4.2}$$

and vice versa. Now, we shall solve the integral equation (1.4.2) by using the method of successive approximation. For this, let $x_0(t)$ be any continuous function which we assume to be the initial approximation of the unknown solution of (1.4.2), then we define $x_1(t)$ as

$$x_1(t) = x_0 + \int_{t_0}^{t} [P(s)x_0(s) + Q(s)u(s)]ds.$$

We take this $x_1(t)$ as our next approximation and substitute this for $x(t)$ on right side of equation (1.4.2) and call it $x_2(t)$ as

$$x_2(t) = x_0 + \int_{t_0}^{t} [P(s)x_1(s) + Q(s)u(s)]ds,$$

$$\vdots$$

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} [P(s)x_n(s) + Q(s)u(s)]ds. \tag{1.4.3}$$

Next we prove that $\{x_n(t)\}$ is a sequence converges uniformly to a continuous function $x(t)$ in some interval I containing t_0.

Since $P(t)x(t) + Q(t)u(t)$ is continuous in closed rectangle Ⓢ : $|t - t_0| \leq a$, $\|x(t) - x(t_0)\| \leq b$ and hence there exist a $M > 0$ such that $\|P(t)x_0(t) + Q(t)u(t)\| \leq M$ for all $(t, x) \in$ Ⓢ. Thus, the initial guess $x_0(t)$ is continuous everywhere as $x_0(t) = x_0$.

First we show the successive approximations $\{x_n(t)\}$ are continuous function and converge to unique solution $x(t)$ in $J_h : |t - t_0| \leq h = min\{a, \frac{b}{M}\}$ for all $x \in J_h$. Since $x_0(t)$ is continuous for all $x : |t - t_0| \leq a$, the function $F_0(x) = P(t)x_0(t) + Q(t)u(t)$ is continuous in J_h, and hence $x_1(t)$ is continuous in J_h. Also,

$$\|x_1(t) - x_0\| \leq \left\| \int_{t_0}^{t} [P(s)x_0(s) + Q(s)u(s)]ds \right\| \leq Mh \leq b.$$

Assuming that the assertion is true for $x_{n-1}(t)$, then it is sufficient to prove that it is also true for $x_n(t)$. For this, since $x_{n-1}(t)$ is continuous in J_h, it follows that $F_{n-1}(x) = P(t)x_{n-1}(t) + Q(t)u(t)$ is also continuous in J_h, that is,

$$\|x_n(t) - x_0\| \leq \left\| \int_{t_0}^{t} [P(s)x_{n-1}(s) + Q(s)u(s)]ds \right\| \leq Mh \leq b.$$

Next we show that the sequence $\{x_n(t)\}$ converges uniformly in J_h. Since $x_1(t)$ and $x_0(t)$ are continuous in J_h, there exists a constant $N > 0$, such that $\|x_1(t) - x_0(t)\| \leq N$. We need to show that for all $x \in J_h$ the following inequality hold:

$$\|x_n(t) - x_{n-1}(t)\| \leq \frac{N(L_0|t - t_0|)^{(n-1)}}{(n-1)!},\ n = 1, 2, \ldots,$$

where $\|P(t)\| \leq L_0$.

For $n = 1$, the inequality is obvious. Furthermore, if it is true for $n = K$, then (1.4.3) yields

$$\|x_{k+1}(t) - x_k(t)\| \ \leq \ \left\| \int_{t_0}^{t} [P(s)x_k(s) + Q(s)u(s) - P(s)x_{k-1}(s) - Q(s)u(s)]ds \right\|$$

$$
\begin{aligned}
&\leq L_0 \left\| \int_{t_0}^{t} \|x_k(s) - x_{k-1}(s)\| ds \right\| \\
&\leq L_0 \left\| \int_{t_0}^{t} \frac{N(L_0|t-t_0|)^{(k-1)}}{(k-1)!} \right\| = N\frac{(L_0|t-t_0|)^k}{k!}.
\end{aligned}
$$

Thus the inequality is true for all n.

Next, since

$$
N\sum_{n=1}^{\infty} \frac{(L_0|t-t_0|)^{(n-1)}}{(n-1)!} \leq N\sum_{n=0}^{\infty} \frac{(L_0 h)^m}{m!} = Ne^{L_0 h} < \infty,
$$

from Theorem 1.3.9 it follows that the series $x_0(t) + \sum_{n=1}^{\infty}(x_n(t) - x_{n-1}(t))$ converges absolutely and uniformly in the interval J_h and hence its partial sum $x_1(t)$, $x_1(t)$, ... converge to a continuous function in this interval, i.e., $x(t) = \lim_{n\to\infty} x_n(t)$.

Then using Theorem 1.3.8 we may pass to the limit in both side of (1.4.3), to obtain

$$
\begin{aligned}
x(t) = \lim_{n\to\infty} x_{n+1}(t) &= x_0 + \lim_{n\to\infty} \int_{t_0}^{t} [P(s)x_n(s) + Q(s)u(s)]ds \\
&= x_0 + \int_{t_0}^{t} [P(s)x(s) + Q(s)u(s)]ds,
\end{aligned}
$$

so that $x(t)$ is desired solution.

Now we show the uniqueness: Let $x(t)$ and $y(t)$ be two solutions. Then

$$
x(t) = x_0 + \int_{t_0}^{t} [P(s)x(s) + Q(s)u(s)]ds,
$$

and

$$
y(t) = x_0 + \int_{t_0}^{t} [P(s)y(s) + Q(s)u(s)]ds.
$$

Since

$$
\begin{aligned}
\|x(t) - y(t)\| &\leq \int_{t_0}^{t} \|P(s)\|\|x(s) - y(s)\|ds, \\
&\leq L_0 \int_{t_0}^{t} \|x(s) - y(s)\|ds, \\
&\leq L_0 M \int_{t_0}^{t} ds = L_0 M|t - t_0|.
\end{aligned}
$$

Since we can choose t arbitrary therefore we can take $|t - t_0|$ as small as possible. Thus,

$$\|x(t) - y(t)\| \leq \epsilon, \text{ for all } \epsilon > 0,$$

$$\Rightarrow \|x(t) - y(t)\| = 0,$$

$$\Rightarrow x(t) = y(t).$$

Thus, $x(t) = y(t)$, and hence it is unique.

1.5 Existence and uniqueness of the solution of second order LTV system

Let us consider the second order LTV system given by the equation

$$x''(t) = P_1(t)x'(t) + P_2(t)x(t) + Q_1(t)u_1(t),$$

where $u_1(t) \in \mathbb{R}^p$ is an input (or control) vector, $P_1(t), P_2(t) \in \mathbb{R}^{n\times n}$ are state matrices, $Q_1(t) \in \mathbb{R}^{n\times p}$ is an input matrix.
On re-writing the above equation, we obtain

$$\begin{bmatrix} I & O \\ O & I \end{bmatrix} \begin{bmatrix} x''(t) \\ x'(t) \end{bmatrix} = \begin{bmatrix} P_1(t) & P_2(t) \\ I & O \end{bmatrix} \begin{bmatrix} x'(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} Q_1(t) & O \\ O & O \end{bmatrix} \begin{bmatrix} u_1(t) \\ O \end{bmatrix}. \tag{1.5.1}$$

Now, if we assume $y(t) = \begin{bmatrix} x'(t) \\ x(t) \end{bmatrix}$, then equation (1.5.1) is reduced to the following first order LTV system

$$y'(t) = P(t)y(t) + Q(t)u(t),$$

where, $P(t) = \begin{bmatrix} P_1(t) & P_2(t) \\ I & O \end{bmatrix}$, $Q(t) = \begin{bmatrix} Q_1(t) & O \\ O & O \end{bmatrix}$ and $u(t) = \begin{bmatrix} u_1(t) \\ O \end{bmatrix}$.

The existence and uniqueness of solution of second order LTV system will be deduced by the same procedure which we used to prove the existence and uniqueness of first order LTV system.

1.6 Existence and uniqueness of the solution of third order LTV system

Let us consider the third order LTV system given by the equation

$$x'''(t) = P_1(t)x''(t) + P_2(t)x'(t) + P_3(t)x(t) + Q_1(t)u_1(t),$$

where $u_1(t) \in \mathbb{R}^p$ is an input (or control) vector, $P_1(t), P_2(t), P_3(t) \in \mathbb{R}^{n\times n}$ are state matrices, $Q_1(t) \in \mathbb{R}^{n\times p}$ is an input matrix.
On re-writing the above equation, we obtain

$$\begin{bmatrix} I & O & O \\ O & I & O \\ O & O & I \end{bmatrix} \begin{bmatrix} x'''(t) \\ x''(t) \\ x'(t) \end{bmatrix} = \begin{bmatrix} P_1(t) & P_2(t) & P_3(t) \\ I & O & O \\ O & I & O \end{bmatrix} \begin{bmatrix} x''(t) \\ x'(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} Q_1(t) & O & O \\ O & O & O \\ O & O & O \end{bmatrix} \begin{bmatrix} u_1(t) \\ O \\ O \end{bmatrix}. \tag{1.6.1}$$

Now, if we assume $y(t) = \begin{bmatrix} x''(t) \\ x'(t) \\ x(t) \end{bmatrix}$, then equation (1.6.1) is reduced to the following first order LTV system

$$y'(t) = P(t)y(t) + Q(t)u(t),$$

where $P(t) = \begin{bmatrix} P_1(t) & P_2(t) & P_3(t) \\ I & O & O \\ O & I & O \end{bmatrix}$, $Q(t) = \begin{bmatrix} Q_1(t) & O & O \\ O & O & O \\ O & O & O \end{bmatrix}$ and $u(t) = \begin{bmatrix} u_1(t) \\ O \\ O \end{bmatrix}$.

The existence and uniqueness of solution of third order LTV system will be deduced by the same procedure which we used to prove the existence and uniqueness of first order LTV system.

1.7 Existence and uniqueness of the solution of n^{th} order LTV system

Let us consider the n^{th} order LTV system given by the equation

$$x^{(n)}(t) = \sum_{i=1}^{n} P_i(t)x^{(n-i)}(t) + Q_1(t)u_1(t),$$

where $u_1(t) \in \mathbb{R}^p$ is an input (or control) vector, $P_i(t) \in \mathbb{R}^{n\times n}$ are state matrix for all $i = 1, \ldots, n$, $Q_1(t) \in \mathbb{R}^{n\times p}$ is an input matrix.
On re-writing the above equation, we obtain

$$\begin{bmatrix} I & \ldots & O \\ \vdots & \ddots & \vdots \\ O & \ldots & I \end{bmatrix} \begin{bmatrix} x^{(n)}(t) \\ \vdots \\ x'(t) \end{bmatrix} = \begin{bmatrix} P_1(t) & \ldots & P_n(t) \\ I & \ddots & \vdots \\ O & I & O \end{bmatrix} \begin{bmatrix} x^{(n-1)}(t) \\ \vdots \\ x(t) \end{bmatrix} + \begin{bmatrix} Q_1(t) & \ldots & O \\ \vdots & \ddots & \vdots \\ O & \ldots & O \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ O \end{bmatrix}. \tag{1.7.1}$$

Now, if we assume $y(t) = \begin{bmatrix} x^{(n-1)}(t) \\ \vdots \\ x(t) \end{bmatrix}$, then equation (1.7.1) is reduced to first order LTV system

$$y'(t) = P(t)y(t) + Q(t)u(t),$$

where $P(t) = \begin{bmatrix} P_1(t) & \dots & P_n(t) \\ I & \ddots & \vdots \\ O & I & O \end{bmatrix}$, $Q(t) = \begin{bmatrix} Q_1(t) & \dots & O \\ \vdots & \ddots & \vdots \\ O & \dots & O \end{bmatrix}$ and $u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ O \end{bmatrix}$.

The existence and uniqueness of solution of n^{th} order LTV system will be deduced by the same procedure which we used to proof the existence and uniqueness of first order LTV system.

1.8 Complete solution of LTV systems by using fundamental matrix

Let us consider LTV in state space given by

$$\left.\begin{aligned} x'(t) &= P(t)x(t) + Q(t)u(t) \\ v(t) &= R(t)x(t) + S(t)u(t) \end{aligned}\right\}. \tag{1.8.1}$$

Let $J(t) \in \mathbb{R}^{n\times n}$ be nonsingular for all t and define

$$x(t) = J(t)z(t).$$

Now substituting in Equation (1.8.1), we get the following:

$$x'(t) = J'(t)z(t) + J(t)z'(t) = P(t)J(t)z(t) + Q(t)u(t),$$

where

$$J(t)z'(t) = \left[P(t)J(t) - J'(t)\right] z(t) + Q(t)u(t).$$

Equivalent LTV system is

$$z'(t) = J(t)^{-1} \left[P(t)J(t) - J'(t)\right] x(t) + J(t)^{-1}Q(t)u(t),$$

and

$$y(t) = R(t)J(t)z(t) + S(t)u(t).$$

Now we define a fundamental matrix $J(t)$ as

$$J'(t) = P(t)J(t), \quad det(J(t)) \neq 0,$$

where $P(t) \in \mathbb{R}^{n\times n}$.

If $J(t)$ is a fundamental matrix then

$$z(t) = z(t_0) + \int_{t_0}^{t} J(\tau)^{-1}Q(\tau)u(\tau)d\tau,$$

and

$$\begin{aligned} x(t) &= J(t)z(t_0) + \int_{t_0}^{t} J(t)J(\tau)^{-1}Q(\tau)u(\tau)d\tau, \\ &= J(t)J(t_0)^{-1}x(t_0) + \int_{t_0}^{t} J(t)J(\tau)^{-1}Q(\tau)u(\tau)d\tau, \\ &= \psi(t,t_0)x(t_0) + \int_{t_0}^{t} \psi(t,\tau)Q(\tau)u(\tau)d\tau, \end{aligned} \tag{1.8.2}$$

where $\psi(t,\tau)$ is called the state transition matrix and is defined as

$$\psi(t,\tau) = J(t)J^{-1}(\tau). \tag{1.8.3}$$

Complete solution of LTV

$$\begin{aligned} y(t) &= R(t)x(t) + S(t)u(t) \\ &= R(t)\psi(t,t_0)x(t_0) + \int_{t_0}^{t} R(t)\psi(t,\tau)Q(\tau)u(\tau)d\tau + S(t)u(t). \end{aligned}$$

Remark 1.8.1. [3] Properties of state transition matrix:

Property 1:

$$\psi(\tau,\tau) = I. \tag{1.8.4}$$

Proof. $\psi(\tau,\tau) = J(\tau)J(\tau)^{-1} = I.$ □

Property 2:

$$\psi^{-1}(\tau,\tau) = \phi(\tau,\tau). \tag{1.8.5}$$

Proof. $\psi^{-1}(\tau_1,\tau_2) = [J(\tau_1)J^{-1}(\tau_2)]^{-1} = J(\tau_2)J^{-1}(\tau_1) = \psi(\tau_2,\tau_1).$ □

Property 3:

$$\psi(\tau_1,\tau_2) = \psi(\tau_1,\tau_0)\psi(\tau_0,\tau_2). \tag{1.8.6}$$

Proof. $\psi(\tau_1,\tau_2) = J(\tau_1)J^{-1}(\tau_0)J(\tau_0)J^{-1}(\tau_2) = \psi(\tau_1,\tau_0)\psi(\tau_0,\tau_2).$ □

Property 4:

$$\frac{d}{d\tau_1}\psi(\tau_1, \tau_2) = P\psi(\tau_1, \tau_2). \tag{1.8.7}$$

Proof. $\frac{d}{d\tau_1}\psi(\tau_1, \tau_2) = \frac{d}{d\tau_1}J(\tau_1)J(\tau_2)^{-1} = \dot{J}(\tau_1)J(\tau_2)^{-1} = P(\tau_1)J(\tau_1)J(\tau_2)^{-1} = P(\tau_1)\psi(\tau_1, \tau_2)$. □

Theorem 1.8.2. [9, Theorem 3.3.1] *The solution of*

$$x'(t) = P(t)x(t) + Q(t)u(t), \tag{1.8.8}$$

subject to the initial condition $x(t_0) = x_0$, is

$$x(f) = \psi(t_1, t_0)\left[x_0 + \int_{t_0}^{t_1} \psi(t_1, \tau)Q(\tau)u(\tau)d\tau\right], \tag{1.8.9}$$

where ψ is defined in (1.8.3).

Proof. The proof follows from the Equation (1.8.2). □

Example 1.8.3. [2] LTV system $x'(t) = P(t)x(t)$; where

$$P(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix},$$

is equivalent to the following $x_1' = 0$ and $x_2' = tx_1$, then integrating both side, we obtain

$$\begin{cases} x_1 = x_1(t_0), \\ x_2 = x_2(t_0) + \frac{1}{2}(t^2 - t_0^2)x_1(t_0). \end{cases}$$

Fundamental matrix for $t_0 = 0$ is given by

$$P(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{21}(t) & P_{22}(t) \end{bmatrix},$$

$$P'(t) = A(t)P(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}\begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{21}(t) & P_{22}(t) \end{bmatrix},$$

and

$$\begin{bmatrix} P_{11}'(t) & P_{12}'(t) \\ P_{21}'(t) & P_{22}'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ tP_{11}(t) & tP_{12}(t) \end{bmatrix}.$$

Since

$$P'_{11}(t) = 0 \Rightarrow P_{11}(t) = \text{Constant} = C_1,$$
$$P'_{12}(t) = 0 \Rightarrow P_{12}(t) = \text{Constant} = C_2,$$
$$P'_{21}(t) = tP_{11}(t) \Rightarrow P_{21}(t) = C_1\frac{t^2}{2} + C_3,$$
$$P'_{22}(t) = tP_{12}(t) \Rightarrow P_{22}(t) = C_2\frac{t^2}{2} + C_4,$$

we obtain

$$P(t) = \begin{bmatrix} C_1 & C_2 \\ C_1\frac{t^2}{2} + C_3 & C_2\frac{t^2}{2} + C_4 \end{bmatrix}.$$

Now by setting, $C_2 = C_3 = 0$, and $C_1 = C_4 = 1$, we get

$$P(t) = \begin{bmatrix} 1 & 0 \\ \frac{t^2}{2} & 1 \end{bmatrix}.$$

State transition matrix is

$$\begin{aligned} \psi(t, t_0) &= P(t)P(t_0)^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ \frac{t^2}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{t_0^2}{2} & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ \frac{t^2}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{t_0^2}{2} & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -\frac{(t^2-t_0^2)}{2} & 1 \end{bmatrix}. \end{aligned}$$

Remark 1.8.4. P(t) is not unique.

A counter example:

Let us choose

$$P(t) = \begin{bmatrix} 1 & 0 \\ 1 & \frac{t^2}{2} \end{bmatrix}.$$

Then state transition matrix is given by,

$$\psi(t, t_0) = \begin{bmatrix} 1 & 0 \\ 1 & \frac{t^2}{2} \end{bmatrix} \begin{bmatrix} \frac{t_0^2}{2} & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{(t^2-t_0^2)}{2} & 1 \end{bmatrix}.$$

Observe that choosing another $P(t)$ does not affect state transition matrix.

[2]http://control.ucsd.edu/mauricio/courses/mae280a/lecture8.pdf

Remark 1.8.5. Next we show that state transition matrix is unique. Let us choose

$$P(t) = \begin{bmatrix} C_1 & C_2 \\ C_1\frac{t^2}{2} + C_3 & C_2\frac{t^2}{2} + C_4 \end{bmatrix},$$

for which,

$$det(P(t)) = C_1C_2\frac{t^2}{2} + C_1C_4 - C_1C_2\frac{t^2}{2} - C_2C_3 = C_1C_4 - C_2C_3,$$

and

$$P^{-1}(t_0) = \frac{1}{C_1C_4 - C_2C_3}\begin{bmatrix} C_2\frac{t_0^2}{2} + C_4 & -C_2 \\ -C_1\frac{t_0^2}{2} - C_3 & C_1 \end{bmatrix}.$$

Thus, we obtain that

$$\begin{aligned} P(t)P^{-1}(t_0) &= \frac{1}{C_1C_4 - C_2C_3}\begin{bmatrix} C_1 & C_2 \\ C_1\frac{t^2}{2} + C_3 & C_2\frac{t^2}{2} + C_4 \end{bmatrix}\begin{bmatrix} C_2\frac{t_0^2}{2} + C_4 & -C_2 \\ -C_1\frac{t_0^2}{2} - C_3 & C_1 \end{bmatrix} \\ &= \frac{1}{C_1C_4 - C_2C_3}\begin{bmatrix} C_1C_2\frac{t_0^2}{2} + C_1C_4 - C_1C_2\frac{t_0^2}{2} - C_2C_3 & -C_1C_2 + C_1C_2 \\ C_1C_4\frac{t^2}{2} + C_2C_3\frac{t_0^2}{2} - C_2C_3\frac{t^2}{2} - C_1C_4\frac{t_0^2}{2} & -C_2C_3 + C_1C_4 \end{bmatrix} \\ &= \frac{1}{C_1C_4 - C_2C_3}\begin{bmatrix} C_1C_4 - C_2C_3 & 0 \\ C_1C_4\frac{(t^2-t_0^2)}{2} - C_2C_3\frac{(t^2-t_0^2)}{2} - & -C_2C_3 + C_1C_4 \end{bmatrix} \\ &= \frac{1}{C_1C_4 - C_2C_3}\begin{bmatrix} C_1C_4 - C_2C_3 & 0 \\ \frac{(t^2-t_0^2)}{2}(C_1C_4 - C_2C_3) & -C_2C_3 + C_1C_4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ \frac{(t^2-t_0^2)}{2} & 1 \end{bmatrix}. \end{aligned}$$

Hence it is unique.

1.9 Controllability of linear systems

Consider a continuous LTV system

$$\begin{aligned} x'(t) &= P(t)x(t) + Q(t)u(t), \\ x(t) &= \psi(t,t_0)x(t_0) + \int_{t_0}^{t}\psi(t,\tau)Q(\tau)u(\tau)d\tau. \end{aligned}$$

The **reachability map** on $[t_0,t]$ is defined to be

$$L_{r,[t_0,t]}(u(.)) = \int_{t_0}^{t}\psi(t,\tau)Q(\tau)u(\tau)d\tau.$$

Thus, it is **controllable** on $[t_0, t]$ if and only if $L_{r,[t_0,t]}(u(.))$ is onto.

$L_{r,[t_0,t]}(u(.))$ determines the set of states that can be reached from the origin at $\tau = t$. The study of the range space of the linear map

$$L_{r,[t_0,t]} : \{u(.)\} \longrightarrow \mathbb{R}^n,$$

is principal to the study of controllability.

1.9.1 Reachability Gramian

The matrix $\mathcal{W}_r = L_r L_r^T \in \mathbb{R}$ is called the *Reachability Gramian* for the interval that L_r is defined. For LTV system

$$x'(t) = P(t)x(t) + Q(t)u(t),$$

the *Reachability Gramian* on the time interval $[t_0, t]$ is defined to be:

$$\mathcal{W}_{r,[t_0,t]} = \int_{t_0}^{t} \psi(t,\tau)Q(\tau)Q(\tau)^T\psi(t,\tau)^T d\tau.$$

1.9.2 Controllability Gramian

Controllability map: $L_{c,[t_0,t]}$

The controllability (to zero) map on $[t_0, t]$ is the map between $u(.)$ to the initial state x_0, such that $x(t_1) = 0$,

$$L_{c,[t_0,t]}(u(.)) = -\int_{t_0}^{t} \psi(t_0,\tau)Q(\tau)u(\tau)d\tau.$$

The controllability-to-zero grammian on the time interval $[t_0, t]$ is defined to be:

$$\mathcal{W}_{c,[t_0,t]} = L_r L_r^* = \int_{t_0}^{t} \psi(t_0,\tau)Q(\tau)Q(\tau)^T\psi(t_0,\tau)^T d\tau,$$

where $L_r^* = -L_r^A$.

———— * ————

Chapter 2

State-Space Explanation

Although the history of linear system theory can be traced back to the last century, the so-called state-space approach was not available till the early 1960s. An important feature of this approach over the traditional frequency domain considerations is that both time-varying and time-invariant linear or nonlinear systems can be treated systematically. The purpose of this chapter is to make a survey on existing work of state-space concept [3].

2.1 Introduction

A typical model that applied mathematicians and system engineers consider as a *machine* with an "input-output" relation given with the two terminals (Fig. 2.1) [3].

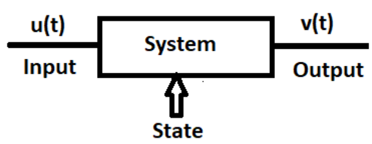

Figure 2.1: Input-output relation

This machine is also called a system which may represent certain biological, economical, or physical systems, or a mathematical description in terms of an algorithm, a system of integral or differential equations, etc. In many applications, a system is described by the totality of input-output relations (u, v) where u and v are functions or, when discretized, sequences, and may be either scalar or vector-valued.

First introduce some terms which we use frequently. The state of a system explains its past, present, and future situations. This is done by introducing a minimum number of variables which are called state variables that represent the present situation, using the past information, namely the initial state, and describe the future behavior of the system completely. The column vector of the state variables, in a given order, is called a *state vector*.

As an example, consider a system given by the differential equation

$$v'' + v = u.$$

In this situation, the totality of all input-output relations that determines the system is the set

$$S = \{(u, v) : v'' + v = u\},$$

and it is clear that the same input u gives rise to infinitely many outputs v. For example, $(1, \sin t + 1), (1, \cos t + 1)$, and even $(1, a\cos t + b\sin t + 1)$ for all arbitrary constants a and b, belong to S.

Let us return to the another example of the system described by the differential equation $v'' + v = u$ with a specified initial state. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be a state vector, where x_1 and x_2 are state variables satisfying the initial states $x_1(a) = b$ and $x_2(a) = c$. We can give a "state-space" description of this system by using a system of two equations:

$$\left.\begin{aligned} x' &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ v &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}\right\}, \tag{2.1.1}$$

where $\dot{x}$ denotes the derivative of the state vector x. Here, the first Equation (2.1.1) gives the input-state relation while the second equation describes the state-output relation.

The so-called state-space Equations (2.1.1) could be obtained by setting the state variables x_1 and x_2 to be v and v', respectively

$$x' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v \\ v' \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

and

$$v = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

Then by substituting the value of x' in the previous equation

$$\begin{aligned} \begin{bmatrix} v' \\ v'' \end{bmatrix} &= \begin{bmatrix} v' \\ -v \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix} \\ &= \begin{bmatrix} v' \\ -v + u \end{bmatrix}, \end{aligned}$$

which gives

$$v'' = -v + u,$$

that is,

$$v'' + v = u.$$

However, without the knowledge of such substitutions, it may not be immediately cleared that the input-output relation follows from the state-space Equation (2.1.1). To demonstrate this, we rewrite Equation (2.1.1) as

$$x' = Px + Qu,$$

and

$$v = Rx,$$

where P is a 2×2 matrix, Q is a 2×1 matrix, and R is a 1×2 matrix. Let $p(P)$ be the characteristic polynomial of P. In this example, $p(\lambda) = \lambda^2 + 1$, so that by Cayley-Hamilton Theorem, we have

$$p(P) = P^2 + I = O.$$

Hence, differentiating the second Equation in (2.1.1) twice (the number of times of differentiation will equal the degree of the characteristic polynomial of the square matrix A), and utilizing the first Equation (2.1.1) repeatedly, we have

$$\begin{aligned} &Rx = v, \\ &RPx = v' - RQu \\ &RP^2x = v'' - RQu' - RPQu. \end{aligned}$$

Therefore, the identity $p(P) = P^2 + I = 0$ can be used to eliminate x, yielding:

$$\Rightarrow (v'' - RQu' - u) + RPQv = RP^2x + Rx = R(P^2 + I)x = O,$$

or

$$\begin{aligned} v'' + v &= R(Qu' + PQu) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} u' + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \right], \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \left[\begin{bmatrix} 0 \\ u' \end{bmatrix} + \begin{bmatrix} u \\ 0 \end{bmatrix} \right], \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix}, \\ &= u. \end{aligned}$$

More generally, if the characteristic polynomial of 3×3 matrix A is

$$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3,$$

then

$$x' = Px + Qu,$$

and

$$v = Rx.$$

Now consider

$$Rx' = v'.$$

By substituting the value of x' in the above expression, we obtain

$$\begin{aligned} &R(Px + Qu) = v' \\ &\Rightarrow RPx = v' - RQu, \\ &\Rightarrow RPx' = v'' - RQu', \\ &\Rightarrow RP(Px + Qu) = v'' - RQu', \\ &\Rightarrow RP^2x = v'' - (RQu' + RPQu), \\ &\Rightarrow RP^2(Px + Qu) = v''' - (RQu'' + RPQu'), \\ &\Rightarrow RP^3x = v''' - (RQu'' + RPQu' + RP^2Qu). \end{aligned}$$

Now we get

$$p(P) = P^3 + a_1P^2 + a_2P + a_3 = 0,$$

$$\begin{aligned}
&v''' - (RQu'' + RPQu' + RP^2Qu) = RP^3x,\\
&\Rightarrow v''' - (RQu'' + RPQu' + RP^2Qu) + a_1v'' - a_1(RQu' + RPQu) + a_2v' - a_2RQu\\
&\qquad\qquad + a_3v = R(P^3 + a_1P^2 + a_2P + a_3I)x,\\
&\Rightarrow v''' + a_1v'' + a_2v' + a_3v = RQu'' + R(P + a_1)Qu' + R(P^3 + a_1P + a_2)Qu.
\end{aligned}$$

If matrix A is 4×4, then

$$p(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4,$$

$$\begin{aligned}
&RP^3x = v''' - (RQu'' + RPQu' + RP^2Qu),\\
&\Rightarrow RP^3x' = v'''' - (RQu''' + RPQu'' + RP^2Qu'),\\
&\Rightarrow RP^3(Px + Qu) = v'''' - (RQu''' + RPQu'' + RP^2Qu'),\\
&\Rightarrow RP^3Px + RP^3Qu = v'''' - (RQu''' + RPQu'' + RP^2Qu'),\\
&\Rightarrow CA^4x = v'''' - (CBu''' + CABu'' + CA^2Bu' + CA^3Bu),
\end{aligned}$$

$$p(A) = A^4 + a_1A^3 + a_2A^2 + a_3A + a_4 = 0,$$

$$\begin{aligned}
&\Rightarrow CA^4x + a_1CA^3x + a_2CA^2x + a_3CAx + a_4Cx,\\
&\quad = v'''' - (CBu''' + CABu'' + CA^2Bu' + CA^3Bu) + a_1v''' - a_1(CBu'' + CABu'\\
&\quad + CA^2Bu) + a_2v'' - a_2(CBu' + CABu) + a_3v' - a_3CBu' + a_4v = 0,\\
&\Rightarrow v'''' + a_1v''' + a_2v'' + a_3v' + a_4v = CBu''' + C(A - a_1)Bu'' + C(A^2 + a_1A + a_2)Bu'\\
&\qquad\qquad + C(A^3 + a_1A^2 + a_2A + a_3)Bu.
\end{aligned}$$

2.2 An example of input-output relations

In the previous section we have discussed the input-output relation of 3^{rd} order system. In this section we will discuss on the higher order input-output relation as given.

More generally, if the characteristic polynomial of an $n \times n$ matrix A in an input-state equation such as (2.1.1) is

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n,$$

then we proceed as discussed in the previous section we write

$$
\begin{aligned}
&Cx = v \\
&\Rightarrow CAx = v' - CBu, \\
&\Rightarrow CA^2x = v'' - CBu' - CABu, \\
&\ldots \\
&\Rightarrow CA^nx = v^{(n)} - CBu^{(n-1)} - CABu^{(n-1)} - \cdots - CA^{(n-1)}Bu,
\end{aligned}
$$

so that, by setting $a_0 = 1$, we have:

$v^n + a_1v^{n-1} + a_2v^{n-2} + \cdots + a_nv = CBu^{(n-1)} + C(A + a_1)Bu^{(n-2)} + C(A^2 + a_1A + a_2)Bu^{(n-3)} + C(A^3 + a_1A^2 + a_2A + a_3)Bu^{(n-4)} + \cdots + C(A^{n-1} + a_1A^{n-2} + a_2A^{n-32} + \cdots + a_{n-1})Bu,$

$$\sum_{k=0}^{n} a_k \left(v^{(n-k)} - C \sum_{j=0}^{n-k-1} A^j Bu^{(n-k-j-l)} \right) = Cp(A)x = 0.$$

That is, the input-output relation can be given by

$$\sum_{j=0}^{n} a_j v^{(n-j)} = C \sum_{k=0}^{n} a_k \sum_{j=0}^{n-k-1} A^j Bu^{(n-k-j-l)}, \tag{2.2.1}$$

with $a_0 = 1$.

A slightly more general form of (2.2.1) is given by $Lv = Mu$, where

$$
\begin{aligned}
L &= \sum_{j=0}^{n} a_j \frac{d^{n-j}}{dt^{n-j}}, && a_0 = 1, \\
M &= \sum_{k=0}^{m} b_k \frac{d^{n-k}}{dt^{n-k}}, && m \leq n.
\end{aligned}
\tag{2.2.2}
$$

We also remark in passing that even if it has such a description A, B and C are not unique.

We construct an example,

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ v' \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \\
v &= \begin{bmatrix} 0 & 1 \end{bmatrix} x,
\end{aligned}
$$

$$v'' + v \quad = \quad C(Bu' + ABu),$$

$$\begin{aligned}
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} u' + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \right], \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \left[\begin{bmatrix} u' \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix} \right], \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} u' \\ u \end{bmatrix}, \\
&= u.
\end{aligned}$$

Thus, A, B and C are not unique.

2.3 An example of state-space explanation

A more general state-space explanation of a system with input-output pairs (u, v) is given by

$$\left.\begin{aligned} x' &= Px + Qu \\ v &= Rx + Su \end{aligned}\right\}, \tag{2.3.1}$$

where P, Q, R, S are matrices with appropriate dimensions.

By eliminating the state vector x and its derivative with the help of the Cayley-Hamilton Theorem, it is not difficult to see that the input-output pair (u, v) given in (2.3.1) satisfies the relation $Lv = Mu$ given in (2.2.2) with appropriate choices of constants a_j and b_k.

To see the converse, that is, to show that the input-output relations in (2.2.2) have a state-space description as given in (2.3.1), we follow the standard technique of transforming an n^{th} order linear differential equation to a first order vector differential equation as was done in the simple example discussed earlier by choosing the matrix A to be

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix},$$

$$R = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Hence, by setting

$$Q = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix}^T,$$

and

$$S = \begin{bmatrix} \beta_0 \end{bmatrix},$$

we see that the variables of the vector $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$ in (2.3.1) satisfy the equations:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u,$$

$$v = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \beta_0 \end{bmatrix} u$$

$x_1' = x_2 + \beta_1 u$

$x_2' = x_3 + \beta_2 u$

$x_3' + a_3 x_1 + a_2 x_2 + a_1 x_3 = \beta_1 u$

$v = x_1 + \beta_0 u$

$x_1 = v - \beta_0 u$

$x_2 = x_1' - \beta_1 u = v' - \beta_0 u' - \beta_1 u$

$x_3 = x_2' - \beta_2 u = v'' - \beta_0 u'' - \beta_1 u' - \beta_2 u$

and must be satisfy the constraint

$x_3' + a_3 x_1 + a_2 x_2 + a_1 x_3 = \beta_3 u,$

$v''' - \beta_0 u''' - \beta_1 u'' - \beta_2 u' + a_3 v - a_3 \beta_0 u + a_2 v' - a_2 \beta_0 u' - a_2 \beta_1 u + a_1 v'' - a_1 \beta_0 u'' - a_1 \beta_1 u' - a_1 \beta_2 u = \beta_3 u, \ \ and$

$v''' + a_1 v'' + a_2 v' + a_3 v = (a_1\beta_3 + a_1\beta_2 + a_2\beta_1 + a_3\beta_0)u + (a_1\beta_2 + a_1\beta_1 + a_2\beta_0)u' + (a_0\beta_1 + a_1\beta_0)u'' + a_0\beta_0 u'''.$

More generalize,

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & & \vdots \\ -a_n & \cdots & & -a_2 & -a_1 \end{bmatrix}.$$

Of course there are other choices of A. But with this“so-called” standard choice, it is clear that the matrix C must be given by

$$R = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Hence, by setting

$$Q = \begin{bmatrix} \beta_1 & \cdots & \beta_n \end{bmatrix}^T,$$

and

$$S = \begin{bmatrix} \beta_0 \end{bmatrix},$$

we see that the variables of the vector $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$ in (2.3.1) satisfy the equations:

$x_1' = x_2 + \beta_1 u$

$x_2' = x_3 + \beta_2 u$

...

$x_{n-1}' = x_n + \beta_{n-1} u$

$x_n' + a_1 x_n + ... + a_n x_1 = \beta_n u$

$v = x_1 + \beta_0 u$

That is, the state variables are defined by

$x_l = v - \beta_0 u,$

$x_2 = x_1' - \beta_1 u = v' - (\beta_0 u' + \beta_1 u),$

$x_3 = x_2' - \beta_2 u = v'' - (\beta_0 u'' + \beta_1 u' + \beta_2 u),$

...

$x_n = x_{n-1}' - \beta_{n-1} u = v^{(n-1)} - (\beta_0 u^{(n-1)} + ... + \beta_{n-1} u),$

and must satisfy the constraint:

$x_n' + a_1 x_n + ... + a_n x_1 = \beta_n u,$

or equivalently,

$$\sum_{j=0}^{n} a_j v^{(n-j)} = \left(\sum_{i=0}^{n} a_i \beta_{n-i} \right) u + \left(\sum_{i=0}^{n-1} a_i \beta_{n-i-1} \right) u' + ... + (a_1 \beta_0 + a_0 \beta_1) + a_0 \beta_0 u^{(n)}.$$

Hence, the constants $\beta_0, ..., \beta_n$ are uniquely determined by the linear matrix equation

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_1 \\ 0 & \cdots & 0 & -a_0 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_{n-1} \\ \vdots \\ \beta_0 \end{bmatrix} = \begin{bmatrix} b_m \\ b_{m-1} \\ \vdots \\ b_{m-n} \end{bmatrix},$$

where $a_o = 1$ and $b_j = 0$ for $j < 0$. We remark that the highest derivative of u in **(1.6)** is n, and hence the order m of the differential operator M in (2.3.1) is not allowed to exceed n.

—————— * ——————

Chapter 3

Controllability

Let us consider the state equation of the form $x'(t) = A(t)x(t) + B(t)u(t)$. In this chapter, we shall only deal with the above type state equation and discuss the "*Controllability*" for these equations. Here, *Controllability* refer to that u which allows us to obtain a desire state x. The "*Controllability*" has been studied in two ways, that is, '*Controllability* to the origin' and 'Controllability from the origin', and furthermore, some conditions have been found for the *Controllability* by introducing the *Controllability Grammian* matrix $\mathcal{W}_c(t)$. At the end of this chapter, we find a sufficient condition for *Controllability* and drive the control $u(t)$ for a state x.

3.1 Definitions of Controllability

Controllability is a major issue that one should control for a state equation system. A linear controllable system may be defined as a system which can be steered to any desired state from the zero initial state. In this section, we find some suitable control function to transfer a linear system from an arbitrary given state to any desired state.

Definition 3.1.1. [9] The LTV system is ruled by

$$\left.\begin{aligned} x'(t) &= P(t)x(t) + Q(t)u(t) \\ v(t) &= R(t)x(t) + S(t)u(t) \end{aligned}\right\}, \tag{3.1.1}$$

is called as *completely controllable* if at any time t_0, any initial state $x(t_0) = x_0$, and any final state x_f there exists a final time $t_1 \geq t_0$, and a continuous control signal

$u(t)$, $t_0 \leq t \leq t_1$ such that the corresponding solution of

$$x'(t) = P(t)x(t) + Q(t)u(t),$$

at some time t_f is "the zero solution" $x(t_f) = x_f$.

From the expression Equation (1.8.2) for the solution of Equation (4.1.1) we have

$$x(f) = \psi(t_1, t_0)\left[x_0 + \int_{t_0}^{t_1} \psi(t_1, \tau)Q(\tau)u(\tau)d\tau\right].$$

Using the Property 2 Equation (1.8.2) of the state transition matrix and rearranging, we obtain

$$0 = \psi(t_1, t_0)\{[x_0 + \psi(t_0, t_1)x(f)] + \int_{t_0}^{t_1} \psi(t_1, \tau)Q(\tau)u(\tau)d\tau\}.$$

Both the equation and nonsingularity of ψ implies that if $u(t)$ transfer x_0 to x_f then $[x_0 + \psi(t_0, t_1)x(f)]$ is also transferred to the origin in the same time interval. As x_0 to x_f are choose to be arbitrary, hence in the definition the given final state can be chosen to be the null vector.

3.2 Criteria for Controllability

Theorem 3.2.1. [9, Theorem 3.3.1] *The LTV system*

$$x'(t) = P(t)x(t) + Q(t)u(t)$$

*is called completely controllable if and only if (***controllability Grammian** *matrix)*

$$\mathcal{W}_c(t_0, t_1) = \int_{t_0}^{t_1} \psi(t_0, \tau)Q(\tau)Q(\tau)^T\psi(t_0, \tau)^T d\tau \ , \tag{3.2.1}$$

where ψ is defined in Equation (1.3.2), is positive definite for any $t > 0$.

Here, the control

$$u(t) = -Q^T(t)\psi^T(t_0, t)\mathcal{W}_c^{-1}(t_0, t_1)[x_0 + \psi(t_0, t_1)x(f)], \tag{3.2.2}$$

defined on $t_0 \leq t \leq t_1$, transfers $x(t_0) = x_0$, to $x(t_f) = x_f$

Proof. Assuming $det(\mathcal{W}_c \neq 0)$, the control, (3.2.2), exists. By the Definition 3.1.1 we shall show that

$$x(t_f) = x_f.$$

Putting

$$u(t) = -Q^T(t)\psi^T(t_0,t)\mathcal{W}_c^{-1}(t_0,t_1)[x_0 + \psi(t_0,t_1)x(f)]$$

in equation **(1.9)** for $x(t_1)$, we get

$$x(f) = \psi(t_1,t_0)\left[x_0 + \int_{t_0}^{t_1} \psi(t_1,\tau)Q(\tau)u(\tau)d\tau\right],$$

Then,

$x(f) = \psi(t_1,t_0)[x_0+$

$\int_{t_0}^{t_1} \psi(t_1,\tau)Q(\tau)\{-Q^T(t)\psi^T(t_0,t)\mathcal{W}_c^{-1}(t_0,t_1)[x_0 + \psi(t_0,t_1)x(f)]\}d\tau],$

$x(f) = \psi(t_1,t_0)[x_0-$

$\int_{t_0}^{t_1} \psi(t_1,\tau)Q(\tau)Q^T(t)\psi^T(t_0,t)\mathcal{W}_c^{-1}(t_0,t_1)[x_0 + \psi(t_0,t_1)x_f]d\tau],$

$\psi(t_1,t_0)[x_0 - \int_{t_0}^{t_1} \psi(t_1,\tau)Q(\tau)Q^T(t)\psi^T(t_0,t)\mathcal{W}_c^{-1}(t_0,t_1)x_0 d\tau,$

$-\int_{t_0}^{t_1} \psi(t_0,\tau)Q(\tau)Q^T(\tau)\psi^T(t_0,t)\mathcal{W}_c^{-1}(t_0,t_1)\psi(t_0,t_1)x_f d\tau]$

$\psi(t_1,t_0)[x_0 - \int_{t_0}^{t_1} \psi(t_1,\tau)Q(\tau)Q^T(t)\psi^T(t_0,t)d\tau\mathcal{W}_c^{-1}(t_0,t_1)x_0,$

$-\int_{t_0}^{t_1} \psi(t_0,\tau)Q(\tau)Q^T(\tau)\psi^T(t_0,t)d\tau\mathcal{W}_c^{-1}(t_0,t_1)\psi(t_0,t_1)x_f]$

From (3.2.1)

$$\psi(t_1,t_0)[x_0 - \overbrace{\mathcal{W}_c(t_0,t_1)W_c^{-1}(t_0,t_1)}\,x_0 - \overbrace{\mathcal{W}_c(t_0,t_1)W_c^{-1}(t_0,t_1)}\,\psi(t_0,t_1)x_f].$$

We know that:

$$\mathcal{W}_c(t_0,t_1)\mathcal{W}_c^{-1}(t_0,t_1) = 1$$

Thus,

$$x(t_1) = \psi(t_1,t_0)[x_0 - Ix_0 - I\psi(t_0,t_1)x_f] = \psi(t_1,t_0)\psi(t_0,t_1)x_f].$$

From Equation (1.8.4), we get

$$\psi(t_1,t_0)\psi(t_0,t_1) = I,$$

which gives $x(t_f) = x_f$. Hence proved. $\square$

———— * ————

Chapter 4

Kalman Canonical Decomposition (KCD)

4.1 Introduction

For linear periodic system the Kalman canonical decomposition has been reconsidered here. The decomposition of a state into controllable and uncontrollable parts has been discussed in this chapter.

Take a LTV system

$$x'(t) = P(t)x(t) + Q(t)u(t), \tag{4.1.1}$$

$x(t) \in \mathbb{R}^n$ is a state vector, $u(t) \in \mathbb{R}^m$ is a input vector, $P(t) \in \mathbb{R}^{n\times n}$ is a state matrix, $Q(t) \in \mathbb{R}^{n\times m}$ is a input matrix, $P(t)$, $Q(t)$ are supposed to be continuous, $u(t)$ is supposed to be piecewise continuous.

Definition 4.1.1. [4] The controllable Gramian is defined by

$$\mathcal{W}(t,s) := \int_t^s \psi(t,\tau)Q(\tau)Q(\tau)^T\psi(t,\tau)^T d\tau, \tag{4.1.2}$$

Where, $\psi(t,\tau)$ is defined in 1.8.3.

Next we shall state [10, Theorem 7] that helps to decomposed the LTV system into controllable and uncontrollable part.

Theorem 4.1.2. [10, Theorem 7] *Consider the system Equation (4.1.1) with controllability matrix $\mathcal{C}(t, t+\mu(t))$ and suppose rank $\mathcal{C}(t, t+\mu(t)) = r_c < n$ for all t.*

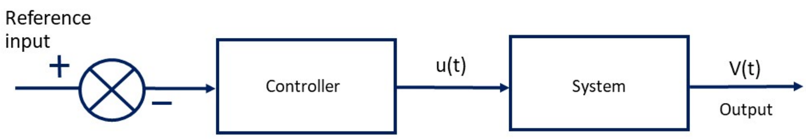

Figure 4.1: Controllable system

Then there exists a diffeomorphic coordinate transformation of the state of (4.1.1) *with respect to which* (4.1.1) *takes on the form*

$$\begin{aligned} x_1'(t) &= F_{11}(t)x_1(t) + F_{12}(t)x_2(t) + G_1(t)u(t), \\ x_2'(t) &= F_{22}(t)x_2(t), \\ y(t) &= H_1(t)x_1(t) + H_2(t)x_2(t), \end{aligned}$$

valid for all time, where $x_1(t)$ is an r_c- vector.

4.2 Conjecture of KCD over periodic system

Let $\zeta = B(t)x$, be a coordinate transformation, where $B(t) \in \mathbb{R}^{n\times n}$ is continuous differentiable and invertible for all $t \in \mathbb{R}$. Then system (4.1.1) is transformed to

$$\zeta' = F(t)\zeta + G(t)u, \tag{4.2.1}$$

where

$$F(t) := (B'(t) + B(t)P(t))B(t)^{-1}, \tag{4.2.2}$$

and

$$G(t) := B(t)Q(t). \tag{4.2.3}$$

Let $P(t)$ and $Q(t)$ be T-periodic. Then by Theorem 4.1.2, it gives possibility to construct $B(t)$ so that the system (4.2.1) can be decomposed to controllable and uncontrollable part, i.e., there exists a non negative integer $n_c \leq n$, such that

$$F(t) = \begin{bmatrix} F_{11}(t) & F_{12}(t) \\ 0 & F_{22}(t) \end{bmatrix}, G(t) = \begin{bmatrix} G_1(t) \\ 0 \end{bmatrix},$$

where $F_{11}(t) \in \mathbb{R}^{n_e\times n_e}$, $F_{12}(t) \in \mathbb{R}^{n_e\times(n-n_e)}$, $F_{22}(t) \in \mathbb{R}^{(n-n_e)\times(n-n_e)}$, $G(t) \in \mathbb{R}^{n_c\times m}$.

- F_{11}, G_1 is controllable.

  Conjecture 1. [4] Suppose that $P(t) \in \mathbb{R}^{n\times n}$ and $Q(t) \in \mathbb{R}^{n\times m}$ are continuous T-periodic. Then there exist a continuously differentiable and T-periodic matrix $B(t) \in \mathbb{R}^{n\times n}$ which is invertible for all $t \in \mathbb{R}$. $F(t)$ defined by Equation (4.2.2) and $G(t)$ defined by Equation (4.2.3) satisfy the following block structure

$$F(t) = \begin{bmatrix} F_{11}(t) & F_{12}(t) \\ 0 & F_{22}(t) \end{bmatrix}, \tag{4.2.4}$$

  and

$$G(t) = \begin{bmatrix} G_1(t) \\ 0 \end{bmatrix}. \tag{4.2.5}$$

- (F_{11}, G_1) is controllable.

4.3 A counter example

Next we study a counter example for Conjecture 1 given in [4]. Let $\widehat{A} \in \mathbb{R}^{2\times 2}$ be a constant matrix and $\widehat{B}(t) \in \mathbb{R}^{2\times 1}$ be continuous T-periodic matrix given by

$$\widehat{P} = \begin{bmatrix} 0 & \dfrac{\pi}{T} \\ -\dfrac{\pi}{T} & 0 \end{bmatrix}, \tag{4.3.1}$$

and

$$\widehat{Q}(t) = \begin{bmatrix} \sin\left(\dfrac{\pi t}{T}\right)\left(\cos\left(\dfrac{\pi t}{T}\right) + \sin\left(\dfrac{\pi t}{T}\right)\right) \\ \sin\left(\dfrac{\pi t}{T}\right)\left(\cos\left(\dfrac{\pi t}{T}\right) - \sin\left(\dfrac{\pi t}{T}\right)\right) \end{bmatrix}. \tag{4.3.2}$$

Then the controllability Gramain over $[t, t+2T]$ is given by

$$\widehat{\mathcal{W}}(t, t+nT) := \int_t^{t+2T} e^{\widehat{P}(\tau-t)}\widehat{Q}(\tau)\widehat{Q}^T(\tau)e^{\widehat{P}^T(\tau-t)}d\tau.$$

First we find,

$$\begin{aligned} e^{\widehat{P}(\tau-t)} &= \exp\left(\begin{bmatrix} 0 & \dfrac{\pi}{T} \\ -\dfrac{\pi}{T} & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} \cos\left(\frac{\pi}{T}\right) & \sin\left(\frac{\pi}{T}\right) \\ -\sin\left(\frac{\pi}{T}\right) & \cos\left(\frac{\pi}{T}\right) \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned}
\widehat{Q}(t)\widehat{Q}^T(t) &= \begin{bmatrix} \sin\left(\frac{\pi t}{T}\right)\left(\cos\left(\frac{\pi t}{T}\right)+\sin\left(\frac{\pi t}{T}\right)\right) \\ \sin\left(\frac{\pi t}{T}\right)\left(\cos\left(\frac{\pi t}{T}\right)-\sin\left(\frac{\pi t}{T}\right)\right) \end{bmatrix} \\
&\quad \times \begin{bmatrix} \sin\left(\frac{\pi t}{T}\right)\left(\cos\left(\frac{\pi t}{T}\right)+\sin\left(\frac{\pi t}{T}\right)\right) & \sin\left(\frac{\pi t}{T}\right)\left(\cos\left(\frac{\pi t}{T}\right)-\sin\left(\frac{\pi t}{T}\right)\right) \end{bmatrix} \\
&= \begin{bmatrix} \sin^2\left(\frac{\pi t}{T}\right)\left(\cos\left(\frac{\pi t}{T}\right)+\sin\left(\frac{\pi t}{T}\right)\right)^2 & \sin^2\left(\frac{\pi t}{T}\right)\left(\cos^2\left(\frac{\pi t}{T}\right)-\sin^2\left(\frac{\pi t}{T}\right)\right) \\ \sin^2\left(\frac{\pi t}{T}\right)\left(\cos^2\left(\frac{\pi t}{T}\right)-\sin^2\left(\frac{\pi t}{T}\right)\right) & \sin^2\left(\frac{\pi t}{T}\right)\left(\cos\left(\frac{\pi t}{T}\right)-\sin\left(\frac{\pi t}{T}\right)\right)^2 \end{bmatrix} \\
&= \begin{bmatrix} \sin^2\left(\frac{\pi t}{T}\right)\left(1+2\sin\left(\frac{2\pi t}{T}\right)\right) & \sin^2\left(\frac{\pi t}{T}\right)\cos\left(\frac{2\pi t}{T}\right) \\ \sin^2\left(\frac{\pi t}{T}\right)\cos\left(\frac{2\pi t}{T}\right) & \sin^2\left(\frac{\pi t}{T}\right)\left(1+2\sin\left(\frac{2\pi t}{T}\right)\right) \end{bmatrix}
\end{aligned}$$

and

$$\widehat{\mathcal{W}}(t,t+nT) := \begin{bmatrix} T\left(1+2\sin\left(\frac{2\pi t}{T}\right)\right) & T\cos\left(\frac{2\pi t}{T}\right) \\ T\cos\left(\frac{2\pi t}{T}\right) & T\left(1+2\sin\left(\frac{2\pi t}{T}\right)\right) \end{bmatrix},$$

so that

$$\mathrm{rank}\widehat{\mathcal{W}}(t,t+2T) = 1.$$

for all $t \in \mathbb{R}$, and therefore $(\widehat{P}, \widehat{Q})$ is uncontrollable.

Suppose that, followed by Conjecture 1, matrix $B(t) \in \mathbb{R}^{2\times 2}$ such that $\widehat{P}$ and $\widehat{Q}(t)$ are transformed to $F(t)$ and $G(t)$ of the form Equation (4.2.4) and Equation (4.2.5). By the T-periodicity of $B(t)$, the monodromy matrices in x-coordinate in ζ-coordinate are similar, and the characteristic multipliers are invariant with respect to a coordinate transformations $\zeta = B(t)x$. Here $e^{\widehat{P}T} = -I$, the characteristic multipliers in x-coordinate are -1 with multiplicity 2, therefore they are also $-1\,(<0)$ with multiplicity 2 in ζ-coordinate. On the contrary, it follows from Equation (4.2.4) that the characteristic multipliers in ζ-coordinate are given by

$$\exp\left(\int_0^T F_{11}(\tau)d\tau\right) \text{ and } \exp\left(\int_0^T F_{22}(\tau)d\tau\right) \quad (>0).$$

Therefore we have a contradiction, which proves that a pair $\widehat{P}$ and $\widehat{Q}(t)$ is a counterexample to the conjecture.

4.4 KCD over periodic system

It can be seen that *Conjecture 1* is not always satisfied for all linear periodic systems as it has already been discussed earlier.

Theorem 4.4.1. [4, Theorem 1] *Consider the system given in Equation (4.1.1), where $P(t) \in \mathbb{R}^{n\times n}$ and $Q(t) \in \mathbb{R}^{n\times m}$ are supposed to be continuous T-periodic. Let $\widehat{n}_e$= rank $\mathcal{W}(t, t+nT)$ where $\mathcal{W}$ is defined by Equation (4.1.2). There exist a T-periodic matrix $B(t) \in \mathbb{R}^{n\times n}$ which is continuously differentiable and invertible for all $t \in \mathbb{R}$ such that*

- *$F(t)$ defined by Equation (4.2.2) has a block structure of the form Equation (4.2.4)*
- *$G(t)$ is defined in Equation (4.2.3) has a block structure of the form Equation (4.2.5)*
- *(F_{11}, G_1) is controllable*

if and only if there exists a T-periodic matrix $Q(t) \in \mathbb{R}^{n\times n}$ which is continuously differentiable and orthogonal for all $t \in \mathbb{R}$ and a T-periodic matrix $E(t) \in \mathbb{R}^{\widetilde{n}_e\times\widetilde{n}_e}$ which is continuously differentiable and positive definite symmetric for all $t \in \mathbb{R}$ such that the controllability Gramian is factored by

$$\mathcal{W}(t, t+nT) = Q(t)' \begin{bmatrix} E(t) & 0 \\ 0 & 0(t) \end{bmatrix} Q(t). \tag{4.4.3}$$

Moreover, if there exists such $B(t)$, n_c which is a size of $F_{11}(t)$ is given by $n_e = \widehat{n}_e$.

Proof. Let M be the controllability Gramian for (F, G)-pair, then we have

$$M(t, t+nT) = B(t)\mathcal{W}(t, t+nT)B(t)'.$$

Let $\widehat{M}$ be the controllability Gramian for (F_{11}, G_1)-pair. As $F_{11}(t)$ and $G_1(t)$ are T-periodic, $\widehat{M}(t, t+nT)$ is also T-periodic. By controllability of (F_{11}, G_1), $\widehat{M}(t, t+nT)$ is positive definite symmetric for all $t \in \mathbb{R}$. Observe that M and $\widehat{M}$ satisfy the equation

$$M(t, t+nT) = \begin{bmatrix} \widehat{M}(t, t+nT) & 0 \\ 0 & 0(t) \end{bmatrix}.$$

Thus $\mathcal{W}(t, t+nT)$ is factored by

$$\mathcal{W}(t, t+nT) = B(t)^{-1} \begin{bmatrix} \widehat{M}(t, t+nT) & 0 \\ 0 & 0(t) \end{bmatrix} (B(t)')^{-1}.$$

As $B(t)$ is invertible for all $t \in \mathbb{R}$, Gram-Schmidts process can be applied to column vectors of $B(t)^{-1}$ pointwise. There exist a T-periodic matrix $K(t)$ which is continuously differentiable and orthogonal for all $t \in \mathbb{R}$ and an upper triangular T-periodic matrix $L(t)$ whose diagonal entries are positive for all $t \in \mathbb{R}$ such that

$$B(t)^{-1} = K(t)'L(t).$$

Decompose $L(t)$ followed by the block structure of $M(t, t+nT)$ and denote an upper left part of $L(t)$ by $L_{11}(t)$, and define

$$E(t) := L_{11}(t)\widehat{M}(t, t+nT)L_{11}(t)',$$

then $E(t)$ is T-periodic and positive definite symmetric for all $t \in \mathbb{R}$. Hence the necessity part.

For sufficiency part, factor $K(t)$ followed by the factorization Equation (4.4.3)

$$K(t) = \begin{bmatrix} K_1(t) \\ K_2(t) \end{bmatrix}.$$

Set $B(t) = K(t)$ and consider $\zeta = K(t)x$. It follows from Equation (4.4.3) that the controllability Gramian over $[t, t+nT]$ in ζ-coordinate is $K(t)\mathcal{W}(t, t+nT)K(t)'$. With respect to the state transition map

$$Im \begin{bmatrix} E(s) & 0 \\ 0 & 0 \end{bmatrix} = \oplus(s,t) Im \begin{bmatrix} E(s) & 0 \\ 0 & 0 \end{bmatrix},$$

where $\oplus$ denotes the state transition matrix in the ζ-coordinate

$$\oplus(s,t) := K(s)\phi(s,t)K(t)^{-1},$$

the controllability subspace is invariant. Since $E(t)$ is positive definite symmetric for all $t \in \mathbb{R}$, a lower left part of $\oplus$ is identically 0

$$\oplus(s,t) = \begin{bmatrix} \oplus_{11}(s,t) & \oplus_{12}(s,t) \\ 0 & \oplus_{22}(s,t) \end{bmatrix}.$$

By the definition, $\oplus(s,t)$ is continuously differentiable and invertible for all $s, t \in \mathbb{R}$ which satisfies

$$\oplus(s+kT, t+kT) = \oplus(s,t), \tag{4.4.4}$$

for all $s, t \in \mathbb{R}$ and $k \in \mathbb{N}$. Define $F(t)$ by

$$F(s) := \frac{\partial \oplus (s,t)}{\partial s} \oplus (s,t)^{-1}, \tag{4.4.5}$$

then $F(t)$ has a block structure of the form Equation (4.2.4), and the size of $F_{11}(t)$ and $E(t)$ is equivalent. Note that the right hand side of Equation (4.4.5) is independent of t, therefore $F(t)$ is well-defined. It follows from

$$\begin{aligned} F(s,T) &= \frac{\partial \oplus (s+T, t+T)}{\partial s} \oplus (s+T, t+T)^{-1} \\ &= \frac{\partial \oplus (s,t)}{\partial s} \oplus (s,t)^{-1} \\ &= F(s) \end{aligned}$$

that $F(t)$ is continuous T-periodic, where we have used the equation $\oplus(s,t) = \oplus(s, t+T) \oplus (t+T, t)$ in the second identity and have used Equation (4.4.4) in the third identity. Denote the B-matrix in the ζ-coordinate by $G(t) := K(t)Q(t), G(t)$ is continuous T-periodic. It can be shown that $G(t)$ has a block structure of the form Equation (4.2.5). Indeed, multiplying Equation (4.4.3) by $K(t)$ from the left and $K(t)'$ from the right, it follows that

$$\begin{aligned} &K_2(t)\mathcal{W}(t, t+nT)K_2(t)' \\ &= \int_t^{t+nT} \oplus_{22}(t,\tau)K_2(\tau)Q(\tau)Q(\tau)'K_2(\tau)' \oplus (t,\tau)' d\tau \\ &= 0. \end{aligned}$$

Since the integrand is positive semidefinite symmetric and continuous, it is equivalent to 0 for all $\tau \in [t, t+nT]$. Moreover, since $\oplus_{22}(s,t)$ is invertible for all $s, t \in \mathbb{R}$ and $K(t)$ and $Q(t)$ is T -periodic, it follows that

$$K_2(t)Q(t) = 0,$$

for all $t \in \mathbb{R}$. Multiplying Equation (4.4.3) by $K(t)$ form the left and $K(t)'$ from the right, it follows that

$$\begin{aligned} &K_1(t)\mathcal{W}(t, t+nT)K_1(t)' \\ &= \int_t^{t+nT} \oplus_{11}(t,\tau)G_1(\tau)G_1(\tau)' \oplus_{11} (t,\tau)'G_1(t)' d\tau \\ &= E(t). \end{aligned}$$

therefore (F_{11}, G_1) is controllable. □

Example 4.4.2. [4] Let there exists a T-periodic coordinate transformation $B(t)$ which transforms $(\widehat{P}, \widehat{Q})$-pair into the block structure of the forms Equation (4.2.4) and Equation (4.2.5). Since the eigenvalues of $\widehat{\mathcal{W}}(t, t+2T)$ are given by 0 and $2T$, followed by Theorem 4.4.1, there exist a continuously differentiable, orthogonal and T-periodic matrix $K(t) \in \mathbb{R}^{2\times 2}$ such that

$$\widehat{\mathcal{W}}(t, t+nT) = K(t)' \begin{bmatrix} 2T & 0 \\ 0 & 0 \end{bmatrix} K(t), \tag{4.4.6}$$

which corresponds to Equation (4.4.3). We note that first column vector of $K(t)'$, which is denoted by $w(t)$, is a eigenvector of $\mathcal{W}(t, t+2T)$ for the eigenvalue $2T$. On the other hand,

$$v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} 1 + \sin(\frac{2\pi t}{T}) \\ \cos(\frac{2\pi t}{T}) \end{bmatrix}$$

is also a eigenvector of $\mathcal{W}(t, t+2T)$ for the eigenvalue $2T$. Since $v_1(t)$ and $v_2(t)$ has a common zero at $t = \frac{3T}{4}$, there exists a function $g(t) \in \mathbb{R}$ which has a singular point at $t = \frac{3T}{4}$ and satisfies

$$w(t) = v(t)g(t).$$

Then $h(t)$ satisfies the following on $[0, T]$.

(1) $h(t)$ is continuously differentiable except for $t = \frac{3T}{4}$ and T-periodic, therefore it follows that

$$h(0) = h(T) > 0,$$

or

$$h(0) = h(T) < 0.$$

(2) $h(t)$ has a 1^{st} order pole at $t = \frac{3T}{4}$, therefore it follows that

$$\lim_{t\to\frac{3T}{4}-} h(t) = -\infty, \quad \text{and} \quad \lim_{t\to\frac{3T}{4}+} h(t) = +\infty$$

or

$$\lim_{t\to\frac{3T}{4}-} h(t) = +\infty, \quad \text{and} \quad \lim_{t\to\frac{3T}{4}+} h(t) = -\infty.$$

We note that $v_1(t)$ has a 2^{nd} order zero at $t = \frac{3T}{4}$ and $v_2(t)$ has a 1^{st} order zero at $t = \frac{3T}{4}$. Therefore $h(t)$ has a 1^{st} order pole at $t = \frac{3T}{4}$.

Those properties are not simultaneously satisfied for each cases. Hence there is no T-periodic coordinate transformation $B(t)$ which transforms $(\widehat{P}, \widehat{Q})$-pair into the block triangular structure of Equation (4.2.4) and Equation (4.2.5), as shown in Theorem 4.4.1.

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Chapter 5

Conclusion and Future Plan

In this study, we have reviewed the work of [2]. The Kalman canonical decomposition for the linear periodic system is well-studied in the literature. In this thesis, we studied the existence and uniqueness of the first order LTV system and furthermore, we have generalized our results for the higher order. Also, we reconsider the problem of transforming a linear periodic system into a Kalman canonical decomposition, and studied the problem through a continuously differentiable coordinate transformation. It is a well-known conjecture in this direction that it is always possible to construct such a transformation with the same period of the system. Our focus in this direction is the existing conjecture, and nevertheless, it can be seen from a counterexample to it. Furthermore, a necessary and sufficient condition can be found in the literature for the existence of such a transformation.

Future plan

My future direction is to study the controllability and uncontrollability of the system. To be more precise, My aim to study how much perturbation is required to change a controllable system to uncontrollable and vice-versa.

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