ON THE SOLUTIONS AND PERTURBATION ANALYSIS OF GENERALIZED REDUCED BIQUATERNION MATRIX EQUATIONS WITH APPLICATIONS

Ph.D. Thesis

By

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APRIL 2025

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A THESIS

Submitted in partial fulfillment of the
requirements for the award of the degree

of

DOCTOR OF PHILOSOPHY

by

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INDIAN INSTITUTE OF TECHNOLOGY INDORE

APRIL 2025

INDIAN INSTITUTE OF TECHNOLOGY INDORE

I hereby certify that the work which is being presented in the thesis entitled **ON THE SOLUTIONS AND PERTURBATION ANALYSIS OF GENERALIZED REDUCED BIQUATERNION MATRIX EQUATIONS WITH APPLICATIONS** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DEPARTMENT OF MATHEMATICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2019 to December 2024 under the supervision of **Dr. Sk. Safique Ahmad**, Professor, Department of Mathematics, Indian Institute of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Neha
21/04/2025

Signature of the student with date
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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

21. 04. 2025

Signature of Thesis Supervisor with date
(Dr. Sk. Safique Ahmad)

Neha Bhadala has successfully given her Ph.D. Oral Examination held on **April 21, 2025**.

21. 04. 2025

Signature of Thesis Supervisor with date
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ACKNOWLEDGEMENTS

The journey of my Ph.D. at IIT Indore has been a remarkable and transformative experience. These years have been filled with moments of learning, introspection, and growth. While this journey has been deeply personal, it was made possible through the encouragement and contributions of many individuals, to whom I owe my deepest gratitude.

First and foremost, I would like to express my deepest gratitude to my supervisor, **Professor Sk. Safique Ahmad**, for his invaluable guidance, encouragement, and support throughout my doctoral journey. His profound knowledge and insightful suggestions have been instrumental in shaping the direction of my research. I deeply appreciate the numerous discussions on research topics and his guidance in enhancing my reading and writing abilities.

I also extend my heartfelt thanks to **Professor P. S. Stanimirović** (University of Niš, Serbia) for his collaboration on one of my research papers. His expertise and constructive inputs have greatly enriched the quality of our joint work.

I am profoundly grateful to **Professor Ivan Slapničar** (University of Split, Croatia) for his many fruitful discussions and invaluable insights. These discussions significantly deepened my understanding of the subject and inspired me to explore new dimensions in my research.

I am sincerely thankful to the **Director of IIT Indore** and the **Head of the Mathematics Department** for ensuring an environment conducive to research. My gratitude also extends to my **PSPC members**, **Dr. Niraj Kumar Shukla** and **Dr. Vijay Kumar Sohani**, for their constructive feedback and valuable suggestions.

During my time at IIT Indore, I had the privilege of meeting some amazing people. My heartfelt thanks go to **Sudipta, Sahil, Himanshi, Amaditya, Sajin, Prabhu, and Meghali**, among others, for their help, companionship, and making my stay at IIT Indore a memorable experience.

Special thanks to my friend **Navneet** for always being there for me, through both good times and tough moments. His support and friendship have been a great strength and have made this journey more special.

I am deeply thankful to my friend **Shreyas** for not only supporting me in this journey but also for his exceptional dedication to reading my manuscript and revisions. His insightful feedback and thoughtful suggestions have been a tremendous help throughout this process. Beyond his academic contributions, Shreyas has been an enduring source of motivation and encouragement, making this journey not only more productive but also deeply fulfilling.

I also want to thank my research group members, **Prince Bhaiya, Gyan Bhaiya, and Pinki**, for their unwavering support throughout this journey.

On a personal note, I owe everything to my family. **My parents** have been my unwavering support system, guiding me with love, wisdom, and unconditional care. They have been my strongest cheerleaders, standing by me through every challenge and celebrating every success, big or small. Words cannot capture the depth of my gratitude to them. I am equally thankful to my brother, **Siddharth**, for always being there for me, offering encouragement and understanding when I needed it the most.

Finally, I extend my thanks to everyone who, in some way, contributed to this journey. Each one of you has played a part in making this milestone possible, and for that, I am eternally grateful.

(Neha Bhadala)

DEDICATION

Dedicated to
My Parents & Brother

ABSTRACT

This thesis establishes comprehensive frameworks for addressing generalized reduced biquaternion matrix equations (RBMEs), exploring their solutions, applications, and sensitivity to perturbations. Firstly, the thesis focuses on structured least squares solutions for generalized RBMEs. To this end, it introduces the concept of reduced biquaternion L-structures, which accommodate linear relationships between matrix entries. A comprehensive framework is established for deriving L-structure least squares solutions to RBMEs, with particular attention to specialized structures such as Toeplitz, Hankel, symmetric Toeplitz, and circulant matrices. The developed techniques are further extended to applications like color image restoration and solving partially described inverse eigenvalue problems (PDIEPs) and generalized PDIEPs.

Next, the thesis investigates generalized inverses of RB matrices, such as the $\{2\}$-inverse and $\{1,2\}$-inverse, under prescribed conditions on row and/or column spaces. By solving RBMEs, conditions for the existence of these generalized inverses are derived, and their efficient representations are established.

Following this, the reduced biquaternion equality constrained least squares (RBLSE) problem is studied. Algebraic techniques are developed to compute real and complex solutions to the RBLSE problem. An upper bound for the relative forward error is derived to ensure the reliability and accuracy of the solutions. This analysis is particularly relevant for applications requiring robust solutions in the presence of data perturbations.

Expanding beyond least squares, the thesis also introduces the reduced biquaternion mixed least squares and total least squares (RBMTLS) method to solve overdetermined systems $AX \approx B$. This method is tailored to scenarios where errors exist in both matrix B and specific columns of matrix A. Two special cases—the reduced biquaternion total least squares (RBTLS) method, addressing errors in both A and B, and the reduced biquaternion least squares (RBLS) method, which assumes errors only in B—are also explored. For these methods, conditions for the existence and uniqueness of solutions are derived, along with explicit formulas for their relative normwise condition numbers. These condition numbers quantify the sensitivity of solutions to perturbations in input data.

Finally, perturbation analysis is performed for the RBMTLS, RBTLS, and RBLS methods, providing explicit bounds for the relative forward error. This ensures the reliability of the proposed solutions in practical applications.

LIST OF PUBLICATIONS

List of Published/Communicated Research Papers from the Thesis

1. **S. S. Ahmad and N. Bhadala**, L-structure least squares solutions of generalized reduced biquaternion matrix equations with applications, *Linear and Multilinear Algebra*, 1-29, 2024.
   DOI: https://doi.org/10.1080/03081087.2024.2437658.
2. **N. Bhadala, S. S. Ahmad, and P. S. Stanimirović**, Outer inverse of reduced biquaternion matrices, *Numerical Algorithms*, 1-28, 2024.
   DOI: https://doi.org/10.1007/s11075-024-01977-8.
3. **S. S. Ahmad and N. Bhadala**, On solutions of reduced biquaternion equality constrained least squares problem and their relative forward error bound, ***Under Review***.
4. **S. S. Ahmad and N. Bhadala**, Algebraic technique for mixed least squares and total least squares problem in the reduced biquaternion algebra, *Linear and Multilinear Algebra*, 1–21, 2024.
   DOI: https://doi.org/10.1080/03081087.2024.2410205.

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NOTATION

$\mathbb{Z}$	The set of integers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{Q}$	The set of quaternion numbers
$\mathbb{Q}_{\mathbb{R}}$	The set of all reduced biquaternion numbers
$\mathbb{R}^{m\times n}$	The set of real matrices of size $m\times n$
$\mathbb{C}^{m\times n}$	The set of complex matrices of size $m\times n$
$\mathbb{Q}^{m\times n}$	The set of quaternion matrices of size $m\times n$
$\mathbb{Q}_{\mathbb{R}}^{m\times n}$	The set of reduced biquaternion matrices of size $m\times n$
$\mathbb{F}$	Can be either $\mathbb{R}$ or $\mathbb{C}$
$\mathbb{K}$	Can be any of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Q}_{\mathbb{R}}$
$A\otimes B$	Kronecker product of matrices A and B
A^T	The transpose of $A\in\mathbb{K}^{m\times n}$
A^H	The conjugate transpose of $A\in\mathbb{K}^{m\times n}$
$\overline{A}$	The conjugate of $A\in\mathbb{F}^{m\times n}$
A^{-1}	The inverse of $A\in\mathbb{K}^{n\times n}$
$A^{\dagger}$	The pseudoinverse of $A\in\mathbb{K}^{m\times n}$
$\det(A)$	Determinant of $A\in\mathbb{K}^{n\times n}$
$\operatorname{rank}(A)$	Rank of matrix $A\in\mathbb{F}^{m\times n}$
I_n	The identity matrix of order n
0_n	The zero matrix of order n
$\lVert x\rVert_2=\sqrt{\sum_{i=1}^{n}\lvert x_i\rvert^2}$	The 2-norm on $\mathbb{C}^n$
$\lVert A\rVert_F=\sqrt{\operatorname{tr}(A^HA)}$	The Frobenius norm of $A\in\mathbb{C}^{m\times n}$

$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2$	The spectral norm of $A \in \mathbb{C}^{m\times n}$
e_i	The vector in $\mathbb{C}^n$ having 1 at the i-th position and 0 elsewhere
$\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$	The imaginary units
$\Re(A)$	For $A \in \mathbb{K}^{m\times n}$, the real part of A
$\Im(A)$	For $A \in \mathbb{K}^{m\times n}$, the imaginary part of A
$\mathcal{C}(A)$	For $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, the complex part of A
$\mathbb{TR}^{n\times n}$	The set of all $n\times n$ real Toeplitz matrices
$\mathbb{STR}^{n\times n}$	The set of all $n\times n$ real symmetric Toeplitz matrices
$\mathbb{HR}^{n\times n}$	The set of all $n\times n$ real Hankel matrices
$\mathbb{CR}^{n\times n}$	The set of all $n\times n$ real circulant matrices
$\mathbb{HC}^{n\times n}$	The set of all $n\times n$ complex Hankel matrices
$\mathbb{TQ}_{\mathbb{R}}^{n\times n}$	The set of all $n\times n$ reduced biquaternion Toeplitz matrices
$\mathbb{STQ}_{\mathbb{R}}^{n\times n}$	The set of all $n\times n$ reduced biquaternion symmetric Toeplitz matrices
$\mathbb{HQ}_{\mathbb{R}}^{n\times n}$	The set of all $n\times n$ reduced biquaternion Hankel matrices
$\mathbb{CQ}_{\mathbb{R}}^{n\times n}$	The set of all $n\times n$ reduced biquaternion circulant matrices
$\mathrm{diag}(\alpha_1,\alpha_2,\ldots,\alpha_n)$	For a diagonal matrix $A=(a_{ij})\in\mathbb{Q}_{\mathbb{R}}^{n\times n}$, we denote it as $\mathrm{diag}(\alpha_1,\alpha_2,\ldots,\alpha_n)$, where $a_{ij}=0$ whenever $i\neq j$ and $a_{ii}=\alpha_i$ for $i=1,\ldots,n$
$\mathrm{vec}(A)$	For matrix $A=(a_{ij})\in\mathbb{Q}_{\mathbb{R}}^{m\times n}$, $\mathrm{vec}(A)=[a_1,a_2,\ldots,a_n]^T$, where $a_j=[a_{1j},a_{2j},\ldots,a_{mj}]$ for $j=1,2,\ldots,n$
$[A,B]$	For $A\in\mathbb{K}^{m\times n_1}$ and $B\in\mathbb{K}^{m\times n_2}$, the notation $[A,B]$ represents the concatenated matrix $\begin{bmatrix}A & B\end{bmatrix}\in\mathbb{K}^{m\times(n_1+n_2)}$
$\Pi_{(d,n)}$	The vec-permutation matrix of size $dn\times dn$, defined as $\Pi_{(d,n)}=\sum_{i=1}^{d}\sum_{j=1}^{n}E_{ij}\otimes E_{ij}^T$, where $E_{ij}\in\mathbb{R}^{d\times n}$ has a one in position (i,j) and zeros elsewhere
$\mathrm{diag}(A,B)$	A block diagonal matrix formed by placing the matrices A and B along the main diagonal, with zero matrices in the off-diagonal blocks

$[A]_{I,J}$	For $A \in \mathbb{K}^{m\times n}$, represents the $k \times k$ minor of A whose rows are determined by the indices in the set I, and columns correspond to the indices in the set J. Here, I is a subset consisting of k elements from $\{1,\ldots,m\}$, and J is a subset of $\{1,\ldots,n\}$ consisting of k elements
$\mathcal{C}(A)$	For $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, the submodule generated by the columns of matrix A
$\mathcal{R}(A)$	For $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, the submodule generated by the rows of matrix A
$A(:,j)$	For any matrix $A \in \mathbb{F}^{m\times n}$ and $j \in \{1,2,\ldots,n\}$, this returns the j^{th} column of matrix A
$A(:,i:j)$	For any $i,j \in \{1,2,\ldots,n\}$ and $i < j$, this returns the submatrix of A consisting of all rows and columns from i to j
$\texttt{rand}(m,n)$	Returns an $m \times n$ matrix of uniformly distributed random numbers
$\texttt{randn}(m,n)$	Creates an $m \times n$ codistributed matrix of normally distributed random numbers, where every element lies between 0 and 1
$\texttt{reshape}(x,m,n)$	Reshapes the column vector x into a corresponding $m \times n$ matrix
$\texttt{ones}(m,n)$	Returns an $m \times n$ matrix with all entries equal to one
$\texttt{toeplitz}(u,v)$	For row vectors u and v of size n, returns a Toeplitz matrix with u as its first column and v as its first row
$\texttt{toeplitz}(u)$	Returns a symmetric Toeplitz matrix with u as its first column and first row
$\texttt{hankel}(u,v)$	Creates a Hankel matrix with u as its first column and v as its last row

ACRONYMS

RB	Reduced biquaternion
SVD	Singular value decomposition
RBME	Reduced biquaternion matrix equation
PDIEP	Partially described inverse eigenvalue problem
RBGI	Generalized inverse of reduced biquaternion matrix
LSE	Equality constrained least squares
RBLSE	Reduced biquaternion equality constrained least squares
LS	Least squares
TLS	Total least squares
MTLS	Mixed least squares and total least squares
RBLS	Reduced biquaternion least squares
RBTLS	Reduced biquaternion total least squares
RBMTLS	Reduced biquaternion mixed least squares and total least squares

CHAPTER 1

INTRODUCTION

The study of hypercomplex numbers, such as quaternions and reduced biquaternions, has significantly contributed to advancements in fields such as signal processing, image processing, and control theory [27, 41, 50, 60, 66, 88]. Among these number systems, quaternions have been widely recognized for their ability to represent multi-dimensional data in a compact form. Introduced by William Rowan Hamilton in 1843, quaternions form a four-dimensional hypercomplex number system consisting of one real component and three imaginary components. Quaternions have been applied effectively in control systems, computer graphics, signal, and image processing [1, 9, 23, 40, 51, 52, 59].

Despite their versatility, quaternions present a significant challenge: non-commutative multiplication, where the order of multiplication matters. This non-commutative property complicates many operations, leading to increased computational complexity and more intricate algorithm designs [25, 56, 57]. To address this limitation, reduced biquaternions (RBs), also known as commutative quaternions, were introduced [61]. Like quaternions, RBs are four-dimensional hypercomplex numbers, but their multiplication is commutative, making them more suitable for real-time applications where computational efficiency is crucial, particularly in signal and image processing. As a result, RBs have been applied to reduced biquaternion discrete Fourier transforms, convolutions, and correlation operations in signal and image processing [31, 55, 56]. Researchers have also explored the potential of RBs in theoretical physics, such as their connection to Maxwell's equations [8, 32], further broadening the applicability of RBs.

This growing interest in RBs has led to the development of several algorithms for matrix computations involving RBs. For instance, algorithms for eigenvalue and eigenvector computations, as well as singular value decompositions of RB matrices, are detailed in [57]. Zhang *et al.* explored the singular value decomposition and generalized inverses of RB matrices [85], and further investigated the diagonalization process in [86], where they established the necessary and sufficient conditions for diagonalization and introduced two numerical methods to facilitate this task. In [19], authors discussed the LU decomposition of RB matrices. These studies underscore the increasing relevance of RB matrices in both theoretical and applied contexts.

Color image processing is one area where RBs have demonstrated significant potential. A color image can be represented as an RB matrix, efficiently capturing the relationships between the red, green, and blue (RGB) color channels without losing spatial arrangement [57]. A color image I is expressed as

$$I = R\boldsymbol{i} + G\boldsymbol{j} + B\boldsymbol{k},$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ represent the basis imaginary units of RBs, and R, G, and B represent the red, green, and blue channels, respectively. This RB-based representation preserves inter-channel relationships and offers an advantage over conventional methods that treat each color channel separately.

In image restoration, RB matrix equations are particularly useful for modeling and correcting image degradation. The linear discrete model for image restoration is given by

$$g = Kf + n,$$

where g is the observed (degraded) image, f is the true image, K is the blurring matrix, and n is additive noise [39]. Solving this equation involves finding the purely imaginary least squares solution of the reduced biquaternion matrix equation $Kf = g$, thereby restoring the original image.

Additionally, eigenvalue problems involving RB matrices have been explored in the literature. For example, in [33], Guo *et al.* studied the eigenvalue problem of RB matrices by solving the matrix equation $AX = XB$. If $A \in \mathbb{Q}_{\mathbb{R}}^{n\times n}$ and there exists a non-zero RB vector α such that $A\alpha = \alpha\lambda$, where $\lambda \in \mathbb{Q}_{\mathbb{R}}$, then λ is called an eigenvalue of A, and α is its corresponding eigenvector. Such eigenvalue problems have proven effective in applications like color face recognition [33, 57].

Despite the advancements of RBs in various applications, the available literature on reduced biquaternion matrix equations (RBMEs) remains relatively sparse, particularly regarding structured solutions. For instance, in [18], the authors explored unstructured solutions to matrix equations of the form $XF - AX = BY$ and $XF - A\widetilde{X} = BY$ over commutative quaternions. Additionally, least squares solutions for matrix equations such as $AX = B$ and $AXC = B$ over commutative quaternions were investigated in [81]. The results in [67] further examine real representation methods for finding least squares solutions to the RB matrix equation $AXC = B$. Moreover, [75] discusses unstructured solutions to matrix equations over the commutative quaternion ring.

While significant progress has been made in finding unstructured solutions to the RBMEs, structured least squares solutions remain largely unexplored. For instance, [71] discusses least squares Toeplitz and bi-Hermitian solutions for the equation $X + AXB = C$, while [80] addresses Hermitian solutions for RBMEs of the form $(AXB, CXD) = (E, F)$. In this thesis, we extend the study to structured solutions for generalized RBMEs, covering L-structure solutions that encompass known structures such as Toeplitz, symmetric Toeplitz, Hankel, circulant, and lower triangular matrices. With our developed comprehensive framework, we can address the problem discussed in [71] in a more effective way.

Further studies have addressed equality constrained least squares problems, as seen in [83], where techniques for solving the reduced biquaternion equality constrained least squares (RBLSE) problem were developed. However, special solutions for the RBLSE problem, along with their detailed perturbation analysis, remain unexplored in the existing literature.

In numerical analysis, the concept of relative forward error is essential for evaluating the accuracy and stability of solutions to mathematical problems, particularly for the RBLSE problem in this case. This measure becomes especially critical when solutions are computed in the presence of data perturbations. While computing the solution of the RBLSE problem, inaccuracies arising from machine precision limits, floating-point arithmetic, or data input errors can introduce deviations between the computed solution and the true solution. The relative forward error effectively quantifies these discrepancies, helping to gauge the sensitivity of the solution to small data perturbations. By identifying the extent of these errors, researchers can evaluate the robustness and reliability of the solution methods employed. Furthermore, understanding these discrepancies is crucial in practical applications, where RB algebra is applied in domains such as robotics, image processing, and control systems. Therefore, conducting a detailed study of the perturbation analysis for the RBLSE problem is crucial to ensure solution reliability and accuracy—an aspect that has yet not been addressed in current research.

Recent research has also focused on the total least squares (TLS) method for finding an approximate solution to the matrix equation $AX \approx B$ in commutative quaternionic theory. For instance, [82] explored solutions to the TLS problem, while [84] examined special solutions in the commutative quaternionic theory. These studies provide valuable insights into TLS methods, but they leave open areas for further exploration, particularly concerning the reduced biquaternion mixed least squares and total least squares (RBMTLS) problem and its associated perturbation analysis.

Both least squares (LS) and TLS methods find approximate solutions to the linear system $AX \approx B$ by making certain assumptions about the input data that may not be valid across all practical applications. In scenarios where these assumptions do not hold, the RBMTLS method offers a more flexible and accurate approach to obtaining solutions. This flexibility arises from the RBMTLS method's capacity to account for errors in the matrix B and only a few columns of matrix A, which is not inherently addressed in standard LS and TLS methods.

This thesis addresses these research gaps by developing a comprehensive framework for solving RBMEs. The key contributions of this thesis are as follows:

Chapter 1 is introductory in nature and provides the history of RBs, the fundamental properties of RBs and RB matrices, background ideas, and prerequisites for the remaining chapters.

Chapter 2 focuses on least squares structured solutions for generalized RBMEs. In this chapter, we develop a comprehensive framework that accommodates various matrix structures, allowing for any set of linear relationships between matrix entries. This class of matrices is referred to as the *reduced biquaternion L-structure*, defined as follows:

Reduced Biquaternion L-structure: Let Ω be a submodule of $\mathbb{Q}_{\mathbb{R}}^{mn}$. The subset of RB matrices of order $m \times n$, given by

$$L(m,n) = \{X \in \mathbb{Q}_{\mathbb{R}}^{m\times n} \mid \operatorname{vec}(X) \in \Omega\}$$

is called the *reduced biquaternion L-structure.*

For example, consider the following matrices:

$$X_1 = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{11} & x_{12} \\ x_{31} & x_{21} & x_{11} \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_{11} + 2\boldsymbol{i} & x_{12} + 3 & x_{11} \\ x_{21} + (2 + 3\boldsymbol{i} + \boldsymbol{j}) & x_{21} & x_{21} \\ x_{11} & x_{12} & x_{11} \end{bmatrix},$$

where $x_{11}, x_{12}, x_{13}, x_{21}, x_{31} \in \mathbb{Q}_{\mathbb{R}}$. Clearly, X_1 is an L-structure matrix, but X_2 is not.

Applications of least squares solutions in color image restoration and solving inverse eigenvalue problems are explored. Both the partially described inverse eigenvalue problems (PDIEPs) and the generalized PDIEPs are addressed:

- **PDIEP:** Given vectors $\{u_1, u_2, \dots, u_k\} \subset \mathbb{F}^n$ $(k \leq n)$, values $\{\lambda_1, \lambda_2, \dots, \lambda_k\} \subset \mathbb{F}$, and a set $\mathcal{L}$ of structured matrices, find a matrix $M \in \mathcal{L}$ such that $Mu_i = \lambda_i u_i$ for $i = 1, 2, \dots, k$.
- **Generalized PDIEP:** Given vectors $\{u_1, u_2, \dots, u_k\} \subset \mathbb{F}^n$ $(k \leq n)$ and values $\{\lambda_1, \lambda_2, \dots, \lambda_k\} \subset \mathbb{F}$, find matrices $M, N \in \mathcal{L}$ such that $Mu_i = \lambda_i N u_i$ for $i = 1, \dots, k$.

Chapter 3 focuses on computing the $\{2\}$-inverse and $\{1,2\}$-inverse of RB matrices with predefined conditions on the row and/or column space. Conditions for the existence and effective representations of these generalized inverses are established by solving RBME of the form $(AXB, CXD) = (E, F)$. This chapter builds on the framework developed in Chapter 2, applying it to find unstructured solutions of the RBME. The results and algorithms presented here demonstrate the versatility of the techniques introduced earlier.

Chapter 4 addresses the RBLSE problem. The goal is to solve the system

$$\min_X \|AX - B\|_F, \quad \text{subject to} \quad CX = D.$$

where the constraints on X do not fall under the L-structure framework discussed in Chapter 2. In this chapter, algebraic techniques are developed to find both real and complex solutions to the RBLSE problem. An upper bound is also established for the relative forward error associated with these solutions:

$$\text{Relative Forward Error} = \frac{\|X_{\text{computed}} - X_{\text{exact}}\|}{\|X_{\text{exact}}\|}.$$

Minimizing this error ensures the accuracy of our solutions in practical applications.

Chapter 5 explores solutions to the RBME

$$AX \approx B,$$

a specific case of the generalized RBMEs studied in Chapters 2 and 3. In earlier chapters, a least squares framework was developed under the assumption that errors are only present in matrix B. However, in practical applications, matrix A may also contain errors, or these errors may be limited to only a few columns of A.

This chapter extends the framework to address these scenarios by introducing the RBMTLS method. The RBMTLS method is particularly well-suited for cases where errors occur in both matrix B and specific columns of matrix A. Two special cases of this method are also discussed:

- **Reduced biquaternion total least squares (RBTLS):** Suitable when both A and B contain errors.
- **Reduced biquaternion least squares (RBLS):** Suitable when only B contains errors, as detailed in earlier chapters.

This chapter explores the conditions for the existence and uniqueness of real solutions to the RBMTLS, RBTLS, and RBLS problems. Explicit formulas are derived for the relative normwise condition number, which quantifies the sensitivity of the solutions to small perturbations in the input data. Additionally, upper bounds for the relative forward error are determined for each method, ensuring the reliability and accuracy of the solutions.

Chapter 6 provides a summary of the thesis and outlines potential directions for future research.

By bridging the gaps in current research and providing new theoretical insights, this thesis advances the field of RBMEs and their applications across various domains. Furthermore, to ensure a comprehensive understanding, the fundamental properties of both RBs and RB matrices are reviewed.

1.1. Reduced Biquaternions

The concept of the reduced biquaternion was first introduced by Segre in 1892. A reduced biquaternion is a four-dimensional hyper-complex number system that extends the complex number system by incorporating one real component and three imaginary components. The set of reduced biquaternions is defined as:

$$\mathbb{Q}_{\mathbb{R}} = \{ r = r_0 + r_1 \boldsymbol{i} + r_2 \boldsymbol{j} + r_3 \boldsymbol{k} \mid r_0, r_1, r_2, r_3 \in \mathbb{R} \}, \tag{1.1.1}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are imaginary units. These units satisfy the following algebraic relationships

$$\boldsymbol{i}^2 = \boldsymbol{k}^2 = -1, \quad \text{and} \quad \boldsymbol{j}^2 = 1,$$

with the multiplication rules

$$\boldsymbol{ij} = \boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = \boldsymbol{kj} = \boldsymbol{i}, \quad \text{and} \quad \boldsymbol{ki} = \boldsymbol{ik} = -\boldsymbol{j}.$$

Given two reduced biquaternions $a = a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \in \mathbb{Q}_\mathbb{R}$ and $b = b_0 + b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k} \in \mathbb{Q}_\mathbb{R}$, we have

$$a = b \iff a_i = b_i, \quad \text{for} \quad i = 0, 1, 2, 3. \tag{1.1.2}$$

The operations of addition and multiplication are defined as follows:
For addition:

$$a + b = (a_0 + b_0) + (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j} + (a_3 + b_3)\boldsymbol{k},$$

and for multiplication:

$$\begin{aligned} ab = (a_0b_0 - a_1b_1 + a_2b_2 - a_3b_3) + (a_0b_1 + a_1b_0 + a_2b_3 + a_3b_2)\boldsymbol{i} \\ + (a_0b_2 - a_1b_3 + a_2b_0 - a_3b_1)\boldsymbol{j} + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)\boldsymbol{k}. \end{aligned}$$

These operations extend the familiar algebraic rules of complex numbers into a higher-dimensional system, enabling the representation of more complex relationships. The commutative properties of the multiplication operation make reduced biquaternions especially useful in applications where simplicity and computational efficiency are required.

Real number and complex numbers can be thought of as reduced biquaternions in a natural way. For a reduced biquaternion $r = r_0 + r_1\boldsymbol{i} + r_2\boldsymbol{j} + r_3\boldsymbol{k} \in \mathbb{Q}_\mathbb{R}$, the real, complex, and imaginary parts are defined as

$$\mathfrak{R}(r) = r_0, \quad \mathcal{C}(r) = r_0 + r_1\boldsymbol{i}, \quad \text{and} \quad \mathfrak{I}(r) = r_1\boldsymbol{i} + r_2\boldsymbol{j} + r_3\boldsymbol{k}.$$

Unlike the quaternion algebra, reduced biquaternion r has three types of conjugates [33]:

$$\bar{r}^{(1)} = r_0 - r_1\boldsymbol{i} + r_2\boldsymbol{j} - r_3\boldsymbol{k}, \quad \bar{r}^{(2)} = r_0 + r_1\boldsymbol{i} - r_2\boldsymbol{j} - r_3\boldsymbol{k}, \quad \bar{r}^{(3)} = r_0 - r_1\boldsymbol{i} - r_2\boldsymbol{j} + r_3\boldsymbol{k}.$$

$\bar{r}^{(1)}$ is also denoted by r^H. The norm of r is defined in [6, 33] as

$$\|r\| = \sqrt[4]{r\bar{r}^{(1)}\bar{r}^{(2)}\bar{r}^{(3)}} = \sqrt[4]{[(r_0 + r_2)^2 + (r_1 + r_3)^2][(r_0 - r_2)^2 + (r_1 - r_3)^2]}.$$

r is said to be nonsingular if there exists a reduced biquaternion p such that $rp = pr = 1$, written as $r^{-1} = p$, and

$$r^{-1} = \frac{\bar{r}^{(1)}\bar{r}^{(2)}\bar{r}^{(3)}}{\|r\|^4},$$

which is different from the inverse of quaternions [33].

Any reduced biquaternion number $r = r_0 + r_1\boldsymbol{i} + r_2\boldsymbol{j} + r_3\boldsymbol{k}$ can also be uniquely expressed as

$$r = (r_0 + r_1\boldsymbol{i}) + (r_2 + r_3\boldsymbol{i})\boldsymbol{j} = a_1 + a_2\boldsymbol{j},$$

where $a_1 = r_0 + r_1\boldsymbol{i}$ and $a_2 = r_2 + r_3\boldsymbol{i}$ are complex numbers.

The set $\mathbb{Q}_\mathbb{R}$ forms a commutative ring with identity, where addition and multiplication are defined in the usual way.

Zero Divisors of $\mathbb{Q}_\mathbb{R}$: Let $0 \neq p \in \mathbb{Q}_\mathbb{R}$, and if there exists another element $0 \neq q \in \mathbb{Q}_\mathbb{R}$ such that $pq = 0$, then p is referred to as a zero divisor of $\mathbb{Q}_\mathbb{R}$.

Let e_1 and e_2 be two special numbers defined as

$$e_1 = \frac{1+\boldsymbol{j}}{2} \quad \text{and} \quad e_2 = \frac{1-\boldsymbol{j}}{2}.$$

We have

$$e_1e_2 = 0, \quad e_1^n = e_1^{n-1} = \ldots = e_1^2 = e_1, \quad e_2^n = e_2^{n-1} = \ldots = e_2^2 = e_2.$$

Therefore, e_1 and e_2 are both idempotent elements $(e_1^2 = e_1,\ e_2^2 = e_2)$ and divisors of zero. Any RB of the form c_1e_1 or c_2e_2 (where c_1 and c_2 are any complex numbers) is also a divisor of zero.

Thus, for RBs, there are infinite solutions for the variable x in the following equation:

$$ux = 0, \quad \text{if} \quad u = c_1e_1 \text{ or } c_2e_2.$$

Hence, $\mathbb{Q}_\mathbb{R}$ does not form a complete division algebra.

Unit in $\mathbb{Q}_\mathbb{R}$: An element r in $\mathbb{Q}_\mathbb{R}$ is defined as a unit if there exists an element $s \in \mathbb{Q}_\mathbb{R}$ such that

$$rs = sr = 1,$$

where 1 represents the multiplicative identity in $\mathbb{Q}_\mathbb{R}$. In this context, s is referred to as the inverse of r. Units are thus sometimes called invertible elements in $\mathbb{Q}_\mathbb{R}$.

It is important to note that not every nonzero element in $\mathbb{Q}_\mathbb{R}$ is a unit, unlike in $\mathbb{R}$ and $\mathbb{C}$. For instance, there is no solution for the variable x in the following equation:

$$ux = 1, \quad \text{if} \quad u = c_1e_1 \text{ or } c_2e_2.$$

Since $\mathbb{Q}_\mathbb{R}$ forms a commutative ring, we can extend the concept of vector spaces to that of a module, where scalars are taken from a ring rather than a field. We begin by presenting the definition of a module [5].

Definition 1.1.1. *A module over a ring $\mathbb{Q}_{\mathbb{R}}$ is an abelian group $(M, +)$ together with a scalar multiplication operation $\mathbb{Q}_{\mathbb{R}} \times M \to M$, defined as $(r, x) \mapsto r \cdot x$, such that for all $r, s \in \mathbb{Q}_{\mathbb{R}}$ and $x, y \in M$, the following properties hold:*

(1) *Distributivity over module addition: $r \cdot (x + y) = r \cdot x + r \cdot y$,*
(2) *Distributivity over ring addition: $(r + s) \cdot x = r \cdot x + s \cdot x$,*
(3) *Associativity: $(rs) \cdot x = r \cdot (s \cdot x)$,*
(4) *Identity: $1 \cdot x = x$ (1 is multiplicative identity in $\mathbb{Q}_{\mathbb{R}}$).*

Then we say that M is a $\mathbb{Q}_{\mathbb{R}}$-module.

Next, we present some definitions concerning $\mathbb{Q}_{\mathbb{R}}$-module bases [5].

Definition 1.1.2. *Let M be a $\mathbb{Q}_{\mathbb{R}}$-module, and let $\Gamma = \{m_\alpha \mid \alpha \in \Delta\}$ be a subset of M.*

(1) *Γ is an $\mathbb{Q}_{\mathbb{R}}$-module basis of M if every $m \in M$ can be written as a finite linear combination of the elements of Γ.*
(2) *A finite subset $\{m_{\alpha_1}, \dots, m_{\alpha_n}\}$ of distinct elements of Γ is said to be linearly independent over $\mathbb{Q}_{\mathbb{R}}$ if, whenever for some $x_1, \dots, x_n \in \mathbb{Q}_{\mathbb{R}}$, we have*

$$x_1 m_{\alpha_1} + \cdots + x_n m_{\alpha_n} = 0 \implies x_1 = \cdots = x_n = 0.$$

(3) *Γ is linearly independent over $\mathbb{Q}_{\mathbb{R}}$ if every finite subset of distinct elements from Γ is linearly independent over $\mathbb{Q}_{\mathbb{R}}$.*
(4) *Γ is a free $\mathbb{Q}_{\mathbb{R}}$-module basis of M if Γ is a $\mathbb{Q}_{\mathbb{R}}$-module basis of M and Γ is linearly independent over $\mathbb{Q}_{\mathbb{R}}$.*
(5) *M is a free $\mathbb{Q}_{\mathbb{R}}$-module if M has a free $\mathbb{Q}_{\mathbb{R}}$-module basis.*

A $\mathbb{Q}_{\mathbb{R}}$-module basis of M is often referred to as the basis of M.

Remark 1.1.3. *The basis of a vector space differs from the basis of a module in the following ways:*

- *If Γ is a basis of M, this does not imply that every element of M can be written uniquely as a linear combination of elements from Γ.*
- *If Γ is a free $\mathbb{Q}_{\mathbb{R}}$-module basis of M, then every nonzero element of M can be written uniquely as a linear combination of elements from Γ.*

Definition 1.1.4. *A $\mathbb{Q}_{\mathbb{R}}$-module M is said to be finitely generated if M has a finite basis $\Gamma = \{m_1, \ldots, m_n\}$.*

The elements in a basis Γ are called generators of M. The rank of a free $\mathbb{Q}_{\mathbb{R}}$-module is the number of generators in a free basis of the module. Clearly, $\mathbb{Q}_{\mathbb{R}}$ itself is a $\mathbb{Q}_{\mathbb{R}}$-module with $\{1\}$ being a free $\mathbb{Q}_{\mathbb{R}}$-module basis of $\mathbb{Q}_{\mathbb{R}}$.

Definition 1.1.5. *Let M be a module over a ring $\mathbb{Q}_{\mathbb{R}}$. A subset $N \subseteq M$ is called a submodule if:*

(1) *N is closed under addition, i.e., $x + y \in N$ for all $x, y \in N$,*
(2) *N is closed under scalar multiplication, i.e., $r \cdot x \in N$ for all $r \in \mathbb{Q}_{\mathbb{R}}$ and $x \in N$,*
(3) *N contains the zero element of M.*

In simpler terms, a submodule is a subset of a module that retains the structure of a module over $\mathbb{Q}_{\mathbb{R}}$.

Now, we introduce the definition of the Moore-Penrose generalized inverse for an RB element, along with its algebraic representation, as detailed in [85].

Definition 1.1.6. *Let $r \in \mathbb{Q}_{\mathbb{R}}$. An RB element, denoted by $r^{\dagger}$, is called the Moore-Penrose generalized inverse of r if $x = r^{\dagger}$ satisfies the following four equations:*

$$(1)\, rxr = r, \quad (2)\, xrx = x, \quad (3)\, (rx)^H = rx, \quad (4)\, (xr)^H = xr.$$

Theorem 1.1.7. *Let $r = r_0 + r_1\boldsymbol{i} + r_2\boldsymbol{j} + r_3\boldsymbol{k} = b_1 + b_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}$, where $b_1 = r_0 + r_1\boldsymbol{i}$ and $b_2 = r_2 + r_3\boldsymbol{i}$. Then, the Moore-Penrose generalized inverse of r is given by*

$$r^{\dagger} = \begin{cases} \dfrac{r_0 + r_1\boldsymbol{i} - r_2\boldsymbol{j} - r_3\boldsymbol{k}}{(r_0 + r_1\boldsymbol{i})^2 - (r_2 + r_3\boldsymbol{i})^2}, & b_1^2 \neq b_2^2, \\ \dfrac{(r_0 + r_2) - (r_1 + r_3)\boldsymbol{i} + (r_0 + r_2)\boldsymbol{j} - (r_1 + r_3)\boldsymbol{k}}{2(r_0 + r_2)^2 + 2(r_1 + r_3)^2}, & b_1 = b_2 \neq 0, \\ \dfrac{(r_0 - r_2) - (r_1 - r_3)\boldsymbol{i} - (r_0 - r_2)\boldsymbol{j} + (r_1 - r_3)\boldsymbol{k}}{2(r_0 - r_2)^2 + 2(r_1 - r_3)^2}, & b_1 = -b_2 \neq 0, \\ 0, & b_1^2 = b_2^2 = 0. \end{cases}$$

1.2. Reduced Biquaternion Matrices

In this section, we introduce the foundational concepts and operations related to RB matrices, where each entry in the matrix is an RB number. Any matrix $A \in \mathbb{Q}_{\mathbb{R}}^{m \times n}$ can be uniquely expressed as

$$A = A_0 + A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k},$$

where $A_0, A_1, A_2, A_3 \in \mathbb{R}^{m\times n}$. Alternatively, A can also be written as

$$A = B_1 + B_2\boldsymbol{j},$$

where $B_1 = A_0 + A_1\boldsymbol{i}$ and $B_2 = A_2 + A_3\boldsymbol{i}$ are complex matrices of size $m \times n$.

The Frobenius norm for $A = (a_{ij}) \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ is defined in [80] as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}, \tag{1.2.1}$$

where, for any RB number $r = r_0 + r_1\boldsymbol{i} + r_2\boldsymbol{j} + r_3\boldsymbol{k}$, we define $|r|$ as

$$|r| = \sqrt{r_0^2 + r_1^2 + r_2^2 + r_3^2}.$$

For a matrix $A = A_0 + A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, its transpose is given by

$$A^T = A_0^T + A_1^T\boldsymbol{i} + A_2^T\boldsymbol{j} + A_3^T\boldsymbol{k},$$

and the ik-conjugate and the ik-conjugate transpose are defined as in [85]:

$$\widetilde{A} = A_0 - A_1\boldsymbol{i} + A_2\boldsymbol{j} - A_3\boldsymbol{k} \quad \text{and} \quad A^H = A_0^T - A_1^T\boldsymbol{i} + A_2^T\boldsymbol{j} - A_3^T\boldsymbol{k}.$$

RB Unitary Matrix: A matrix $A \in \mathbb{Q}_{\mathbb{R}}^{n\times n}$ is called an RB unitary matrix if it satisfies $AA^H = A^HA = I_n$ [85].

The collection of all $m\times n$ RB matrices, denoted by $\mathbb{Q}_{\mathbb{R}}^{m\times n}$, forms a $\mathbb{Q}_{\mathbb{R}}$-module, defined by the following operations:

- **Matrix Addition:** For $A, B \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, the (i,j)-th element of $A + B$ is

$$(A + B)_{ij} = A_{ij} + B_{ij}.$$

- **Scalar Multiplication:** For a scalar $r \in \mathbb{Q}_{\mathbb{R}}$ and a matrix $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, the (i,j)-th element of rA is

$$(rA)_{ij} = r(A)_{ij}.$$

For each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, let E_{ij} denote the $m \times n$ matrix whose entries are defined as follows:

$$(E_{ij})_{pq} = \begin{cases} 1 & \text{if } (p,q) = (i,j), \\ 0 & \text{if } (p,q) \neq (i,j). \end{cases}$$

The set of matrices $\Gamma = \{E_{ij} \mid 1 \le i \le m, 1 \le j \le n\}$ forms a free $\mathbb{Q}_\mathbb{R}$-module basis of $\mathbb{Q}_\mathbb{R}^{m\times n}$. Thus, $\mathbb{Q}_\mathbb{R}^{m\times n}$ is a finitely generated, free $\mathbb{Q}_\mathbb{R}$-module of rank mn.

Similarly, the set $\mathbb{Q}_\mathbb{R}^{m} = \mathbb{Q}_\mathbb{R}^{m\times 1}$, consisting of all column vectors of size m, is a free $\mathbb{Q}_\mathbb{R}$-module of rank m. Likewise, $\mathbb{Q}_\mathbb{R}^{1\times n}$, the set of all row vectors of size n, is a free $\mathbb{Q}_\mathbb{R}$-module of rank n.

We now define the determinant, minor, adjoint, and inverse of an RB matrix, along with a discussion of their properties [5, 48].

Determinant of an RB Matrix: Let $A = (a_{ij}) \in \mathbb{Q}_\mathbb{R}^{n\times n}$. Define

$$\det : \mathbb{Q}_\mathbb{R}^{n\times n} \to \mathbb{Q}_\mathbb{R}$$

by

$$\det(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

where the summation extends over all permutations σ of $\{1, 2, \ldots, n\}$, i.e., over all σ in the symmetric group S_n on n letters. The symbol $\operatorname{sgn}(\sigma)$ represents the sign of the permutation σ, which is $(+)$ for even permutations and $(-)$ for odd permutation. Recall that a permutation is even (odd) if it can be written as an even (odd) product of transpositions. The map $\det : \mathbb{Q}_\mathbb{R}^{n\times n} \to \mathbb{Q}_\mathbb{R}$ is called the determinant or determinant map, and $\det(A)$ is called the determinant of A.

For $A, B \in \mathbb{Q}_\mathbb{R}^{n\times n}$, the determinant of an RB matrix satisfies the following properties:

- $\det(AB) = \det(A)\det(B)$.
- $\det(A^T) = \det(A)$.

Minor of an RB Matrix: Let $A \in \mathbb{Q}_\mathbb{R}^{m\times n}$ and suppose $1 \le k \le \min\{m, n\}$. A $k \times k$ minor of A refers to the determinant of a $k \times k$ submatrix of A.

In particular, for $A \in \mathbb{Q}_\mathbb{R}^{m\times n}$, we will use the notation $[A]_{I,J}$ to denote the $k \times k$ minor of A, where the rows are indexed by the elements of the set I and the columns are indexed by the elements of the set J. Here, I is a subset of $\{1, \ldots, m\}$ with exactly k elements, and J is a subset of $\{1, \ldots, n\}$ with exactly k elements. This notation will be consistently used throughout the thesis to specify minors of RB matrices.

Definition 1.2.1. *For a matrix $A \in \mathbb{Q}_\mathbb{R}^{n\times n}$, the notion of cofactors is defined as follows:*

(1) *For any* $i, j = 1, 2, \ldots, n$*, let* $M_{ij}(A)$ *denote the* $(n-1) \times (n-1)$ *minor of* A*, obtained by deleting the* i*-th row and* j*-th column of* A*.*

(2) *The element* $(-1)^{i+j} M_{ij}(A)$ *is called the* i, j*-th cofactor of* A*. We denote this cofactor as* $cof_{ij}(A)$*.*

Adjoint of an RB Matrix: The adjoint of a matrix $A \in \mathbb{Q}_{\mathbb{R}}^{n \times n}$, denoted by $\mathrm{adj}(A)$, is defined by

$$(\mathrm{adj}(A))_{ij} = \mathrm{cof}_{ji}(A) \quad \text{for all} \quad i, j = 1, 2, \ldots, n.$$

For $A \in \mathbb{Q}_{\mathbb{R}}^{n \times n}$, we have

$$A\, \mathrm{adj}(A) = \mathrm{adj}(A)\, A = \det(A) I_n.$$

Inverse of an RB Matrix: A matrix $A \in \mathbb{Q}_{\mathbb{R}}^{n \times n}$ is an invertible matrix if there exists a matrix $B \in \mathbb{Q}_{\mathbb{R}}^{n \times n}$ such that

$$AB = BA = I_n.$$

The matrix B is called the inverse of A and is denoted by A^{-1}.

Proposition 1.2.2. *Let* $A \in \mathbb{Q}_{\mathbb{R}}^{n \times n}$*. Then* A *is invertible if and only if* $det(A)$ *is a unit in* $\mathbb{Q}_{\mathbb{R}}$*.*

Remark 1.2.3. *It is important to note that, in the context of reduced biquaternion algebra, a nonzero determinant* $\det(A)$ *does not necessarily imply invertibility of* A*, which differs from matrices over a field.*

In the case when $\det(A)$ is a unit in $\mathbb{Q}_{\mathbb{R}}$, A^{-1} is given by

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A).$$

Non-singular RB Matrix: A matrix $A \in \mathbb{Q}_{\mathbb{R}}^{n \times n}$ that is invertible is said to be a non-singular matrix.

Next, we present some examples to illustrate the concept of the determinant and the invertibility of an RB matrix.

Example 1.2.4. *Consider the matrix*

$$A = \begin{bmatrix} 1 + 2\boldsymbol{j} & 3 + 4\boldsymbol{j} \\ 5 + 6\boldsymbol{j} & 7 + 8\boldsymbol{j} \end{bmatrix}.$$

We compute the determinant as

$$det(A) = -16 - 16\boldsymbol{j}.$$

Since $det(A)$ is not a unit in $\mathbb{Q}_{\mathbb{R}}$, the matrix A is not invertible.

Example 1.2.5. *Consider the matrix*

$$A = \begin{bmatrix} 1-\boldsymbol{j} & \boldsymbol{i} \\ 2\boldsymbol{k} & 2 \end{bmatrix}.$$

The determinant is

$$det(A) = 2.$$

Here, $det(A)$ is a unit in $\mathbb{Q}_{\mathbb{R}}$, making A invertible. The inverse is given by

$$B = \frac{1}{2}\begin{bmatrix} 2 & -\boldsymbol{i} \\ -2\boldsymbol{k} & 1-\boldsymbol{j} \end{bmatrix}.$$

Verifying, we have $AB = BA = I_2$, thus confirming that $A^{-1} = B$.

Next, we define the SVD of an RB matrix and explore its utility in defining the Moore-Penrose generalized inverse, as well as various other generalized inverses of RB matrices. Additionally, we provide an explicit expression for the minimum norm solution of the RBME $Ax = b$, which is derived using the SVD approach [57, 85].

Theorem 1.2.6. *Let $A = B_1 + B_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, where $B_1, B_2 \in \mathbb{C}^{m\times n}$. Suppose the SVD of $B_1 - B_2$ and $B_1 + B_2$ are given by*

$$B_1 - B_2 = \hat{U}_1\hat{\Sigma}_1\hat{V}_1^H = \hat{U}_1\begin{bmatrix} \grave{\Sigma}_s & 0 \\ 0 & 0 \end{bmatrix}\hat{V}_1^H,$$

$$B_1 + B_2 = \hat{U}_2\hat{\Sigma}_2\hat{V}_2^H = \hat{U}_2\begin{bmatrix} \acute{\Sigma}_t & 0 \\ 0 & 0 \end{bmatrix}\hat{V}_2^H,$$

where $\hat{U}_1, \hat{U}_2 \in \mathbb{C}^{m\times m}$ and $\hat{V}_1, \hat{V}_2 \in \mathbb{C}^{n\times n}$ are unitary matrices, $\grave{\Sigma}_s = diag(\tau_1, \tau_2, \ldots, \tau_s)$, $\acute{\Sigma}_t = diag(\gamma_1, \gamma_2, \ldots, \gamma_t)$, with $s = rank(B_1 - B_2)$, $t = rank(B_1 + B_2)$, and $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_s > 0$, $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_t > 0$ being the singular values of $B_1 - B_2$ and $B_1 + B_2$, respectively.

Then there exist two RB unitary matrices $U \in \mathbb{Q}_{\mathbb{R}}^{m\times m}$ and $V \in \mathbb{Q}_{\mathbb{R}}^{n\times n}$ such that

$$A = U\begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}V^H, \tag{1.2.2}$$

where $\Sigma_r = diag(\sigma_1, \sigma_2, \ldots, \sigma_r)$ *with*

$$\sigma_l = \frac{\tau_l + \gamma_l}{2} + \frac{\gamma_l - \tau_l}{2}\boldsymbol{j}, \quad l = 1, 2, \ldots, r, \tag{1.2.3}$$

where $r = \max\{s, t\}$*, with* $\tau_r = 0$ *if* $r > s$ *and* $\gamma_r = 0$ *if* $r > t$*. We have* $|\sigma_1| \geq |\sigma_2| \geq \cdots \geq |\sigma_r| > 0$*, and* σ_r *is referred to as the singular values of the RB matrix* A*.*

Remark 1.2.7. *The singular values of an RB matrix differ from those in real, complex, or quaternion matrices. In particular, rather than being nonzero real numbers, the singular values have the form* $a + b\boldsymbol{j}$*, where* $a, b \in \mathbb{R}$*. Additionally, for RB matrices, the parameter* r *does not necessarily represent the rank of the matrix as it does in standard matrix theory.*

Definition 1.2.8. *Let* $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$*. A matrix* X *is called the Moore-Penrose generalized inverse of* A*, denoted by* $A^{\dagger}$*, if* X *satisfies the following Penrose conditions:*

(1) $AXA = A$,

(2) $XAX = X$,

(3) $(AX)^H = AX$,

(4) $(XA)^H = XA$.

The matrix $A^{\dagger}$ *is unique.*

We further define additional generalized inverses of RB matrices that satisfy a subset of the Penrose conditions. For any subset $\delta \subseteq \{1, 2, 3, 4\}$, where condition (i) corresponds to $i \in \delta$, the set of RB matrices that meet the specified conditions in δ is denoted by $A\{\delta\}$. Any RB matrix in $A\{\delta\}$ is referred to as the δ-inverse of A and is denoted by $A^{(\delta)}$.

Theorem 1.2.9. *Let* $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ *and suppose its SVD is*

$$A = U \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} V^H,$$

as in Theorem 1.2.6. Then the Moore-Penrose generalized inverse of the matrix A *is given by*

$$A^{\dagger} = V \begin{bmatrix} \Sigma_r^{\dagger} & 0 \\ 0 & 0 \end{bmatrix} U^H,$$

where $\Sigma_r^{\dagger} = diag(\sigma_1^{\dagger}, \sigma_2^{\dagger}, \ldots, \sigma_r^{\dagger})$*.*

Theorem 1.2.10. *Let $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ and suppose its SVD is*

$$A = U \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} V^H,$$

as in Theorem 1.2.6. Then

(1) *$G \in A\{1\}$ if and only if*

$$G = V \begin{bmatrix} \Sigma_r^{\dagger} & K \\ L & M \end{bmatrix} U^H,$$

where $K \in \mathbb{Q}_{\mathbb{R}}^{r\times(m-r)}$, $L \in \mathbb{Q}_{\mathbb{R}}^{(n-r)\times r}$, and $M \in \mathbb{Q}_{\mathbb{R}}^{(n-r)\times(m-r)}$.

(2) *$G \in A\{1,3\}$ if and only if*

$$G = V \begin{bmatrix} \Sigma_r^{\dagger} & 0 \\ L & M \end{bmatrix} U^H,$$

where $L \in \mathbb{Q}_{\mathbb{R}}^{(n-r)\times r}$ and $M \in \mathbb{Q}_{\mathbb{R}}^{(n-r)\times(m-r)}$.

(3) *$G \in A\{1,4\}$ if and only if*

$$G = V \begin{bmatrix} \Sigma_r^{\dagger} & K \\ 0 & M \end{bmatrix} U^H,$$

where $K \in \mathbb{Q}_{\mathbb{R}}^{r\times(m-r)}$ and $M \in \mathbb{Q}_{\mathbb{R}}^{(n-r)\times(m-r)}$.

Theorem 1.2.11. *Let $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ and $b \in \mathbb{Q}_{\mathbb{R}}^{m\times 1}$. Then the general solutions x and the minimum norm solution x_{LS} of*

$$\min_x \|Ax - b\|_F$$

are given by

$$\begin{aligned} x &= A^{\dagger}b + \left(I_n - A^{\dagger}A\right)z, \\ x_{LS} &= A^{\dagger}b, \end{aligned}$$

respectively, where $z \in \mathbb{Q}_{\mathbb{R}}^{n\times 1}$ is any RB vector.

1.3. Preliminaries

This section introduces fundamental definitions and key results that will be utilized throughout this thesis. These foundational concepts, originating from the complex domain, are essential tools for solving problems in the RB domain, which is the primary focus of this work. The results presented here are primarily drawn from references [3, 30, 70] and will serve as the basis for the subsequent chapters.

We begin by discussing the definition and basic properties of the Kronecker product.

Definition 1.3.1. *Let $A = (a_{ij}) \in \mathbb{C}^{m\times n}$ and $B = (b_{ij}) \in \mathbb{C}^{p\times q}$. The Kronecker product of A and B, denoted by $A \otimes B$, is defined as follows:*

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{mp\times nq}.$$

Example 1.3.2. *Let* $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 5 & 7 \\ 3 & 2 & 4 \end{bmatrix}$. *Then*

$$A \otimes B = \begin{bmatrix} 1 & 5 & 7 & 3 & 15 & 21 \\ 3 & 2 & 4 & 9 & 6 & 12 \\ 2 & 10 & 14 & 4 & 20 & 28 \\ 6 & 4 & 8 & 12 & 8 & 16 \\ 3 & 15 & 21 & 1 & 5 & 7 \\ 9 & 6 & 12 & 3 & 2 & 4 \end{bmatrix}.$$

Some properties of the Kronecker product are as follows:

- For any A and B, we have

$$(A \otimes B)^T = A^T \otimes B^T \quad \text{and} \quad (A \otimes B)^H = A^H \otimes B^H.$$

- Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{r\times s}$, $C \in \mathbb{C}^{n\times t}$, and $D \in \mathbb{C}^{s\times q}$. Then

$$(A \otimes B)(C \otimes D) = (AC \otimes BD) \in \mathbb{C}^{mr\times tq}.$$

Pseudoinverse: The pseudoinverse of a matrix $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger \in \mathbb{C}^{n\times m}$, satisfies the following four properties, commonly referred to as the Moore-Penrose conditions:

(1) $AA^\dagger A = A$,
(2) $A^\dagger AA^\dagger = A^\dagger$,
(3) $(AA^\dagger)^H = AA^\dagger$,
(4) $(A^\dagger A)^H = A^\dagger A$.

The pseudoinverse has several notable properties:

- $A^\dagger$ always exists and is unique.
- If the matrix A is invertible, then $A^\dagger = A^{-1}$.
- The following identities hold: $(A^\dagger)^\dagger = A$, $(A^T)^\dagger = (A^\dagger)^T$, $\left(\overline{A}\right)^\dagger = \overline{A^\dagger}$, and $(A^H)^\dagger = (A^\dagger)^H$.
- For a scalar $\alpha \neq 0$, we have $(\alpha A)^\dagger = \alpha^{-1}A^\dagger$.

Remark 1.3.3. *The following properties hold for the pseudoinverse $A^\dagger$ under specific rank conditions of A:*

- *If A has full column rank, then A^HA is invertible, and the pseudoinverse of A is given by*
$$A^\dagger = \left(A^HA\right)^{-1}A^H.$$
- *If A has full row rank, then AA^H is invertible, and the pseudoinverse of A can be expressed as*
$$A^\dagger = A^H\left(AA^H\right)^{-1}.$$

Lemma 1.3.4. *Consider the complex matrix equation $AX = B$. The following results hold:*

(1) *The matrix equation has a solution X if and only if $AA^\dagger B = B$. In this case, the general solution is given by*
$$X = A^\dagger B + \left(I - A^\dagger A\right)Y,$$
where Y is an arbitrary matrix of suitable size. Furthermore, if the consistency condition is satisfied, then the matrix equation has a unique solution if and only if A is of full column rank. In this case, the unique solution is
$$X = A^\dagger B.$$

(2) *The least squares solutions of the matrix equation can be expressed as*
$$X = A^\dagger B + \left(I - A^\dagger A\right)Y,$$
where Y is an arbitrary matrix of suitable size. The least squares solution with the least norm is
$$X = A^\dagger B.$$

Singular Value Decomposition (SVD): For any nonzero matrix $A \in \mathbb{C}^{m\times n}$ with rank r, the matrix A can be decomposed as

$$A = U\Sigma V^H, \tag{1.3.1}$$

where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, and Σ is a diagonal matrix of the form $\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_r) \in \mathbb{R}^{m\times n}$. The entries $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are known as the singular values of A. This factorization is called the SVD of A.

Remark 1.3.5. *Let $A \in \mathbb{C}^{m\times n}$ be a matrix of rank r with SVD given by $A = U\Sigma V^H$, as in (1.3.1). The SVD of A provides a straightforward way to compute the pseudoinverse $A^\dagger$. Specifically, the pseudoinverse can be expressed as*

$$A^\dagger = V\Sigma^\dagger U^H,$$

where $\Sigma^\dagger = diag(1/\sigma_1, \dots, 1/\sigma_r) \in \mathbb{R}^{n\times m}$.

The SVD of matrix A provides a foundation for approximating A by matrices of lower rank.

Eckart-Young-Mirsky Matrix Approximation Theorem: Let the SVD of $A \in \mathbb{R}^{m\times n}$ be given by

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T,$$

where $r = \mathrm{rank}(A)$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are the singular values, and $U = [u_1, u_2, \dots, u_m] \in \mathbb{R}^{m\times m}$ and $V = [v_1, v_2, \dots, v_n] \in \mathbb{R}^{n\times n}$ are orthonormal matrices. If $k < r$, define

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T.$$

Then A_k is the best rank-k approximation to A in the Frobenius norm. Specifically, we have

$$\min_{\mathrm{rank}(B)=k} \|A - B\|_F = \|A - A_k\|_F = \sqrt{\sum_{i=k+1}^{r} \sigma_i^2}.$$

Inner Product: Let V be a vector space over a field $\mathbb{F}$. An inner product, denoted by $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$, is a function that satisfies the following properties:

(1) $\langle u, u \rangle \geq 0$ for all $u \in V$,
(2) $\langle u, u \rangle = 0$ if and only if $u = 0$,
(3) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$,
(4) $\langle cu, v \rangle = c\langle u, v \rangle$ for all $c \in \mathbb{F}$ and $u, v \in V$,

(5) $\langle u, v\rangle = \overline{\langle v, u\rangle}$.

For vectors $u, v \in \mathbb{C}^n$, the inner product is defined as $\langle u, v\rangle = v^H u$.

Norm: For a vector space V over a field $\mathbb{F}$, a function $\|\cdot\| : V(\mathbb{F}) \to \mathbb{R}$ is called a norm if it satisfies the following conditions:

(1) $\|v\| \geq 0$ for all $v \in V$,
(2) $\|v\| = 0$ if and only if $v = 0$,
(3) $\|\alpha v\| = |\alpha|\|v\|$ for all $v \in V$ and $\alpha \in \mathbb{F}$,
(4) $\|v + u\| \leq \|v\| + \|u\|$ for all $v, u \in V$ (triangle inequality).

For $x = (x_j) \in \mathbb{C}^n$, the function $\|x\| = \sqrt{\sum_{j=1}^{n} |x_j|^2}$ defines a norm on $\mathbb{C}^n$, commonly referred to as the 2-norm. Similarly, for a matrix $A = (a_{ij}) \in \mathbb{C}^{m\times n}$, the Frobenius norm of A is defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2}.$$

This norm can also be expressed in terms of the trace of $A^H A$ as

$$\|A\|_F = \sqrt{\mathrm{tr}(A^H A)},$$

where $\mathrm{tr}(\cdot)$ denotes the trace.

Unitary Matrix: A matrix $U \in \mathbb{C}^{n\times n}$ is called unitary if it satisfies the property

$$U^H U = I_n = UU^H.$$

Unitary matrices preserve norms, and the Frobenius norm exhibits the following property:

$$\|UAV\|_F = \|A\|_F,$$

for any matrix $A \in \mathbb{C}^{m\times n}$ and unitary matrices U and V.

Orthogonal Complement of S: Let S be a subset of $\mathbb{C}^n$. The orthogonal complement of S, denoted $S^\perp$, is defined as the set of vectors in $\mathbb{C}^n$ that are orthogonal to every vector in S. That is,

$$S^\perp = \{x \in \mathbb{C}^n \mid \langle x, y\rangle = 0 \text{ for all } y \in S\}.$$

The set $S^\perp$ is nonempty, as it includes at least the zero vector.

Consider $A \in \mathbb{C}^{m\times n}$, which can be viewed as a linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^m$. For a vector $x \in \mathbb{C}^n$, the transformation maps x to $Ax \in \mathbb{C}^m$. Two fundamental subspaces associated with this transformation are:

- The null space of A, denoted $\mathcal{N}(A)$, is a subspace of $\mathbb{C}^n$ defined by
$$\mathcal{N}(A) = \{x \in \mathbb{C}^n \mid Ax = 0\}.$$
- The range of A, denoted $\mathcal{R}(A)$, is a subspace of $\mathbb{C}^m$ defined by
$$\mathcal{R}(A) = \{Ax \mid x \in \mathbb{C}^n\}.$$

It is known that
$$\mathcal{R}(A^H)^\perp = \mathcal{N}(A).$$

QR Decomposition: Let $A \in \mathbb{C}^{n\times n}$ be a nonsingular matrix. There exist unique matrices $Q, R \in \mathbb{C}^{n\times n}$ such that Q is unitary, R is upper triangular with real, positive entries on the main diagonal, and $A = QR$.

Theorem 1.3.6. *For any matrix $A \in \mathbb{C}^{m\times n}$ with $m > n$, there exist matrices $Q \in \mathbb{C}^{m\times m}$ and $R \in \mathbb{C}^{m\times n}$ such that Q is unitary and*
$$R = \begin{bmatrix} \widehat{R} \\ 0 \end{bmatrix},$$
where $\widehat{R} \in \mathbb{C}^{n\times n}$ is upper triangular, and $A = QR$.

Theorem 1.3.7. *If $A = QR$ is the QR factorization of a matrix $A \in \mathbb{C}^{m\times n}$ with full column rank, and if*
$$A = [a_1, \ldots, a_n], \quad Q = [q_1, \ldots, q_m]$$
are the column partitionings of A and Q, respectively, then for $Q_1 = Q(1:m, 1:n)$, $Q_2 = Q(1:m, n+1:m)$, and $R_1 = R(1:n, 1:n)$, we have
$$\mathcal{R}(A) = \mathcal{R}(Q_1),$$
$$\mathcal{R}(A)^\perp = \mathcal{R}(Q_2).$$

CHAPTER 2

L-STRUCTURE LEAST SQUARES SOLUTIONS OF GENERALIZED REDUCED BIQUATERNION MATRIX EQUATIONS

This chapter presents a comprehensive framework for computing structure-constrained least squares solutions to generalized reduced biquaternion matrix equations (RBMEs). It investigates three main types of matrix equations: a linear matrix equation involving multiple unknown L-structures, a linear matrix equation with a single unknown L-structure, and general coupled linear matrix equations with one unknown L-structure. The proposed method leverages the complex representation of reduced biquaternion matrices to derive these solutions.

The versatility of the framework is demonstrated through the derivation of least squares purely imaginary solutions for the RBME $AX = E$, with applications to color image restoration. Furthermore, the framework is utilized to obtain structure-constrained solutions for complex and real matrix equations, broadening its applicability to various inverse problems. Specific attention is given to partially described inverse eigenvalue problems (PDIEPs) and generalized PDIEPs. The chapter concludes with illustrative numerical examples to validate the effectiveness of the proposed approach.

2.1. Introduction

In matrix theory, linear matrix equations play a crucial role due to their wide range of applications in control theory, inverse problems, and linear optimal control [15, 26, 35]. Owing to their widespread application in various fields, one encounters the problem of finding approximate solutions for linear matrix equations. There are many different forms of matrix equations. Some simple examples of these are:

$$AX = B, \quad AXB + CX^TD = E, \quad AXB + CYD = E.$$

A great deal of research has been carried out on real and complex matrix equations, which have applications across a range of scientific and engineering disciplines [11, 24, 37, 46]. Quaternion matrix equations, in particular, have been studied extensively due to their significance in areas like image and signal processing [43, 77, 78, 90]. However, a notable limitation of quaternions is their non-commutative multiplication, which restricts their applicability in certain contexts.

To address this issue, reduced biquaternions have emerged as a powerful alternative. Reduced biquaternions allow for commutative multiplication, simplifying many operations, especially in image and digital signal processing. For example, [56] demonstrated that analyzing complex symmetric multichannel systems and symmetric lattice filter systems using reduced biquaternions significantly reduces computational complexity. Additionally, reduced biquaternions have been shown to provide a more efficient and straightforward method for color-sensitive edge detection between two colors compared to traditional quaternions.

Further illustrating their utility, [57] demonstrated that reconstructing original color images using reduced biquaternion matrices requires only three-fourths of the computational complexity needed for quaternion matrices. Given these advantages, solving matrix equations that arise from commutative quaternion theory has become increasingly important in various practical fields.

Recent studies have focused on RBMEs. For example, Zhang *et al.* [81] investigated the least squares solutions for matrix equations such as $AXC = B$ and $AX = B$. The authors in [85] discussed the SVD and generalized inverse of reduced biquaternion matrices and used these tools to find the least squares solution of the RBME $Ax = b$. Similarly, [82] studied the total least squares solutions of the RBME $Ax = b$, while [83] explored the equality constrained least squares solutions of the RBME $AX = B$.

Most of the existing literature focuses on unstructured least squares solutions for RBMEs. Structured least squares solutions, however, have been relatively less explored. One of the few notable studies is [71], which addresses least squares Toeplitz and bi-Hermitian solutions for $X + AXB = C$. Furthermore, Yuan *et al.* [80] examined the Hermitian solutions of the RBME $(AXB, CXD) = (E, G)$. In this chapter, we extend this research direction by exploring least squares structured solutions for generalized RBMEs, specifically considering matrices whose entries adhere to specific linear constraints, referred to as reduced biquaternion L-structures.

Surprisingly, the least squares Toeplitz, symmetric Toeplitz, Hankel, and circulant solutions of the generalized RBMEs have not been discussed in the literature despite their significance in scientific computing, inverse problems, image restoration, and signal processing [7, 54, 89]. Given the above context, this chapter addresses least squares L-structure solutions for generalized RBMEs, with particular attention to reduced biquaternion Toeplitz, symmetric Toeplitz, Hankel, circulant, purely imaginary, complex, and real solution. The matrix equations considered are:

$$\sum_{l=1}^{r} A_l X_l B_l = E, \tag{2.1.1}$$

$$\sum_{l=1}^{r} A_l X B_l + \sum_{p=1}^{q} C_p X^T D_p = E, \tag{2.1.2}$$

$$(A_1 X B_1, A_2 X B_2, \ldots, A_r X B_r) = (E_1, E_2, \ldots, E_r). \tag{2.1.3}$$

In addition to deriving solutions for these RBMEs, the chapter also explores their applications, such as color image restoration and inverse eigenvalue problems. Several applications of the inverse eigenvalue problem, which involve reconstructing matrices from prescribed spectral data, deal with structured matrices. When the spectral data contain only partial information about the eigenpairs, this kind of inverse problem is called a PDIEP. In both PDIEP and generalized PDIEP, two pivotal questions arise: the theory of solvability and the numerical solution methodology (see textbook [12] and references therein). Regarding solvability, a major challenge has been identifying the necessary or sufficient conditions for a PDIEP or a generalized PDIEP to be solvable. On the other hand, numerical solution methods focus on developing procedures to construct matrices in a numerically stable manner when the given spectral data are feasible. In this chapter, we successfully develop a numerical solution methodology for both PDIEP and generalized PDIEP by employing our proposed framework. Our primary focus is on two structures: Hankel and symmetric Toeplitz matrices.

In summary, the main applications discussed in this chapter include:

- The application of the least squares purely imaginary reduced biquaternion solution of the matrix equation $AX = E$ to color image restoration.

- The use of the framework to determine structure-constrained solutions for complex and real matrix equations, which are a special case of RBMEs. This enables tackling various inverse eigenvalue problems, including PDIEP.
- Solutions for generalized PDIEP for symmetric Toeplitz and Hankel structures.

The chapter is structured as follows. Section 2.2 introduces preliminary results. In Section 2.3, we define reduced biquaternion L-structures and examine their properties. Section 2.4 outlines the general framework for solving RBMEs, with a specific focus on equations involving multiple unknown L-structures in Subsection 2.4.1. Section 2.5 applies the framework to practical cases, and Section 2.6 provides numerical examples to validate the results.

2.2. Preliminaries

To ensure this chapter is self-contained, we summarize key concepts and results that will be used in the following sections. For any reduced biquaternion matrix $Z = Z_1 + Z_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, where $Z_1, Z_2 \in \mathbb{C}^{m\times n}$, we represent it using the complex matrix form

$$\Psi_Z = [Z_1, Z_2] \in \mathbb{C}^{m\times 2n}.$$

Similarly, for any reduced biquaternion $r = r_1 + r_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}$, where r_1 and r_2 are complex numbers, we use the vector form

$$\Psi_r = [r_1, r_2] \in \mathbb{C}^{1\times 2}.$$

Lemma 2.2.1. *For any reduced biquaternion matrix $Z = Z_1 + Z_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, its Frobenius norm is given by*

$$\|Z\|_F = \|\Psi_Z\|_F = \sqrt{\|Z_1\|_F^2 + \|Z_2\|_F^2} = \sqrt{\|\Re(Z_1)\|_F^2 + \|\Im(Z_1)\|_F^2 + \|\Re(Z_2)\|_F^2 + \|\Im(Z_2)\|_F^2}.$$

Proof. The proof follows directly from the definition of the Frobenius norm of a reduced biquaternion matrix in (1.2.1). ■

The complex representation $h(Z)$ of a reduced biquaternion matrix $Z = Z_1 + Z_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, as defined in [80], is given by:

$$h(Z) = \begin{bmatrix} Z_1 & Z_2 \\ Z_2 & Z_1 \end{bmatrix}.$$

For matrices $Y \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ and $Z \in \mathbb{Q}_{\mathbb{R}}^{n\times p}$, the following property holds:

$$h(YZ) = h(Y)h(Z). \tag{2.2.1}$$

Lemma 2.2.2. *Let $\alpha \in \mathbb{R}$, $q = q_1 + q_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}$, and matrices $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $Y = Y_1 + Y_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, and $Z = Z_1 + Z_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times s}$. Then, the following properties hold:*

(1) $\Psi_{\alpha X} = \alpha\Psi_X$.
(2) $\Psi_{X+Y} = \Psi_X + \Psi_Y$.
(3) $\Psi_{qX} = \Psi_q h(X)$.
(4) $\Psi_{YZ} = \Psi_Y h(Z)$.

Proof. For part (1), we have $\alpha X = \alpha X_1 + \alpha X_2\boldsymbol{j}$, so

$$\Psi_{\alpha X} = [\alpha X_1, \alpha X_2] = \alpha[X_1, X_2] = \alpha\Psi_X.$$

For part (2), since $X + Y = (X_1 + Y_1) + (X_2 + Y_2)\boldsymbol{j}$, we get

$$\Psi_{X+Y} = [X_1 + Y_1, X_2 + Y_2] = \Psi_X + \Psi_Y.$$

For part (3), we have $qX = (q_1X_1 + q_2X_2) + (q_1X_2 + q_2X_1)\boldsymbol{j}$, which gives

$$\Psi_{qX} = [q_1X_1 + q_2X_2, q_1X_2 + q_2X_1] = \Psi_q h(X).$$

Finally, for part (4), since $YZ = (Y_1Z_1 + Y_2Z_2) + (Y_1Z_2 + Y_2Z_1)\boldsymbol{j}$, we get

$$\Psi_{YZ} = [Y_1Z_1 + Y_2Z_2, Y_1Z_2 + Y_2Z_1] = \Psi_Y h(Z). \quad \blacksquare$$

For any matrix $Z = Z_1 + Z_2\boldsymbol{j}$, the vector operator $\mathrm{vec}(Z)$ is defined as

$$\mathrm{vec}(Z) = \mathrm{vec}(Z_1) + \mathrm{vec}(Z_2)\boldsymbol{j}.$$

For the matrix Ψ_Z, the vector operator $\mathrm{vec}(\Psi_Z)$ is expressed as

$$\mathrm{vec}(\Psi_Z) = \begin{bmatrix} \mathrm{vec}(Z_1) \\ \mathrm{vec}(Z_2) \end{bmatrix}. \tag{2.2.2}$$

Now, let $Z = Z_1 + Z_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, and define $\overrightarrow{Z} = [\Re(Z_1), \Im(Z_1), \Re(Z_2), \Im(Z_2)] \in \mathbb{R}^{m\times 4n}$. The vectorization of $\overrightarrow{Z}$ is given by

$$\mathrm{vec}(\overrightarrow{Z}) = \begin{bmatrix} \mathrm{vec}(\Re(Z_1)) \\ \mathrm{vec}(\Im(Z_1)) \\ \mathrm{vec}(\Re(Z_2)) \\ \mathrm{vec}(\Im(Z_2)) \end{bmatrix}.$$

This establishes the following relationship:

$$\|Z\|_F = \|\Psi_Z\|_F = \|\mathrm{vec}(\Psi_Z)\|_F = \left\|\mathrm{vec}(\overrightarrow{Z})\right\|_F. \tag{2.2.3}$$

2.3. Reduced Biquaternion L-structure Matrices

This section aims to define the concept of reduced biquaternion L-structure and explore some specific examples of this class of matrices. A reduced biquaternion L-structure refers to the set of all reduced biquaternion matrices of a given order whose entries adhere to specific linear constraints. A notable example of this class includes unstructured matrices, where no linear restrictions are placed on the matrix entries. The subsequent definition offers a formalized explanation of this concept.

Definition 2.3.1. *Let Ω be a subspace of $\mathbb{Q}_{\mathbb{R}}^{mn}$. The subset of reduced biquaternion matrices of order $m \times n$ given by*

$$L(m,n) = \{X \in \mathbb{Q}_{\mathbb{R}}^{m\times n} | \mathrm{vec}(X) \in \Omega\} \tag{2.3.1}$$

is known as the reduced biquaternion L-structure.

Remark 2.3.2. $\mathbb{Q}_{\mathbb{R}}$ *and* $\mathbb{Q}_{\mathbb{R}}^{n}$ *are vector spaces over* $\mathbb{R}$ *with dimensions* 4 *and* $4n$*, respectively.*

To better comprehend the above definition, let us consider the following examples.

Example 2.3.3. *Let*

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and define

$$\Omega_1 = \{v \in \mathbb{Q}_{\mathbb{R}}^{9\times 1} \mid Av = 0\}.$$

Clearly, Ω_1 is a subspace of $\mathbb{Q}_{\mathbb{R}}^{9\times 1}$. The corresponding reduced biquaternion L-structure is given by

$$L(3,3) = \{X \in \mathbb{Q}_{\mathbb{R}}^{3\times 3} \mid \mathrm{vec}(X) \in \Omega_1\}.$$

The set $L(3,3)$ represents the class of diagonal matrices of size 3×3. In this case, six linear restrictions are imposed on the entries of the matrix $X = (x_{ij}) \in \mathbb{Q}_{\mathbb{R}}^{3\times 3}$, such that $x_{ij} = 0$ for $i \neq j$. Thus, the collection of all reduced biquaternion diagonal matrices of a given order belongs to the class of reduced biquaternion L-structures.

It is evident that the collection of all purely imaginary reduced biquaternion vectors of order n is a subspace of $\mathbb{Q}_{\mathbb{R}}^{n\times 1}$. Hence, the collection of all purely imaginary reduced biquaternion matrices forms a reduced biquaternion L-structure.

Example 2.3.4. *Let*

$$\Omega_2 = \{v \in \mathbb{Q}_{\mathbb{R}}^{16\times 1} \mid \mathfrak{R}(v) = 0\}.$$

Clearly, Ω_2 is a subspace of $\mathbb{Q}_{\mathbb{R}}^{16\times 1}$. The corresponding reduced biquaternion L-structure is given by

$$L(4,4) = \{X \in \mathbb{Q}_{\mathbb{R}}^{4\times 4} \mid \mathrm{vec}(X) \in \Omega_2\}.$$

The set $L(4,4)$ represents the collection of all purely imaginary reduced biquaternion matrices of size 4×4. Thus, the collection of all purely imaginary reduced biquaternion matrices of a given order forms an L-structure.

In the same way, the collection of all real reduced biquaternion matrices of a given order forms a reduced biquaternion L-structure. Other reduced biquaternion L-structure examples include the set of all reduced biquaternion Toeplitz, symmetric Toeplitz, Hankel, circulant, lower triangular, and upper triangular matrices of a given order. These classes of matrices consider only equality relationships between the matrix entries. Here is an example of a reduced biquaternion L-structure with some linear relationships between the matrix entries.

Example 2.3.5. *Let*

$$B = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix},$$

and define

$$\Omega_3 = \{v \in \mathbb{Q}_{\mathbb{R}}^{9\times 1} \mid Bv = 0\}.$$

Clearly, Ω_3 is a subspace of $\mathbb{Q}_{\mathbb{R}}^{9\times 1}$. The corresponding reduced biquaternion L-structure is given by

$$L(3,3) = \{X \in \mathbb{Q}_{\mathbb{R}}^{3\times 3} \mid \mathrm{vec}(X) \in \Omega_3\}.$$

This set $L(3,3)$ represents the collection of all reduced biquaternion matrices $X = (x_{ij}) \in \mathbb{Q}_{\mathbb{R}}^{3\times 3}$ with the following linear restrictions imposed on the entries of X:

$$x_{11} + x_{31} = x_{21}, \quad x_{12} + x_{22} = x_{32}, \quad x_{13} + x_{33} = x_{23}.$$

The remaining section focuses on some reduced biquaternion L-structure matrices that frequently appear in practical applications. Our primary focus lies on *reduced biquaternion Toeplitz, symmetric Toeplitz, Hankel, circulant, real, complex, and purely imaginary matrices.* To commence our exploration, we initially examine the vec-structure of some real structured matrices.

Definition 2.3.6. *A matrix $X \in \mathbb{R}^{n\times n}$ is Toeplitz if it has the following form:*

$$X = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & \cdots & x_{n-1} \\ x_{-1} & x_0 & x_1 & \ddots & & \vdots \\ x_{-2} & x_{-1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & x_1 & x_2 \\ \vdots & & \ddots & x_{-1} & x_0 & x_1 \\ x_{-n+1} & \cdots & \cdots & x_{-2} & x_{-1} & x_0 \end{bmatrix}.$$

For $X \in \mathbb{R}^{n\times n}$, $\mathrm{vec}_T(X)$ is defined as

$$\mathrm{vec}_T(X) := [x_{-n+1}, x_{-n+2}, \ldots, x_{-1}, x_0, x_1, \ldots, x_{n-1}]^T \in \mathbb{R}^{2n-1}. \tag{2.3.2}$$

Definition 2.3.7. *A matrix $X \in \mathbb{R}^{n\times n}$ is symmetric Toeplitz if it has the following form:*

$$X = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & \cdots & x_{n-1} \\ x_1 & x_0 & x_1 & \ddots & & \vdots \\ x_2 & x_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & x_1 & x_2 \\ \vdots & & \ddots & x_1 & x_0 & x_1 \\ x_{n-1} & \cdots & \cdots & x_2 & x_1 & x_0 \end{bmatrix}.$$

For $X \in \mathbb{R}^{n\times n}$, $\mathrm{vec}_{ST}(X)$ is defined as

$$\mathrm{vec}_{ST}(X) := [x_0, x_1, x_2, \ldots, x_{n-1}]^T \in \mathbb{R}^n. \tag{2.3.3}$$

Definition 2.3.8. *A matrix $X \in \mathbb{R}^{n\times n}$ is Hankel if it has the following form:*

$$X = \begin{bmatrix} x_{n-1} & \cdots & \cdots & x_2 & x_1 & x_0 \\ \vdots & & \iddots & x_1 & x_0 & x_{-1} \\ \vdots & \iddots & \iddots & \iddots & x_{-1} & x_{-2} \\ x_2 & x_1 & \iddots & \iddots & \iddots & \vdots \\ x_1 & x_0 & x_{-1} & \iddots & & \vdots \\ x_0 & x_{-1} & x_{-2} & \cdots & \cdots & x_{-n+1} \end{bmatrix}.$$

For $X \in \mathbb{R}^{n\times n}$, $\mathrm{vec}_H(X)$ is defined as

$$\mathrm{vec}_H(X) := [x_{n-1}, x_{n-2}, \dots, x_1, x_0, x_{-1}, \dots, x_{-n+1}]^T \in \mathbb{R}^{2n-1}. \tag{2.3.4}$$

Definition 2.3.9. *A matrix $X \in \mathbb{R}^{n\times n}$ is circulant if it has the following form:*

$$X = \begin{bmatrix} x_0 & x_{n-1} & \cdots & x_2 & x_1 \\ x_1 & x_0 & x_{n-1} & & x_2 \\ \vdots & x_1 & x_0 & \ddots & \vdots \\ x_{n-2} & & \ddots & \ddots & x_{n-1} \\ x_{n-1} & x_{n-2} & \cdots & x_1 & x_0 \end{bmatrix}.$$

For $X \in \mathbb{R}^{n\times n}$, $\mathrm{vec}_C(X)$ is defined as

$$\mathrm{vec}_C(X) := [x_0, x_1, x_2, \dots, x_{n-1}]^T \in \mathbb{R}^n. \tag{2.3.5}$$

In the following four lemmas, we describe the structure of some particular classes of real matrix sets.

Lemma 2.3.10. *If $X \in \mathbb{R}^{n\times n}$, then $X \in \mathbb{T}\mathbb{R}^{n\times n} \Leftrightarrow \mathrm{vec}(X) = K_T \mathrm{vec}_T(X)$, where $\mathrm{vec}_T(X)$ is of the form (2.3.2), and the matrix $K_T \in \mathbb{R}^{n^2\times(2n-1)}$ is represented as*

$$K_T = \begin{bmatrix} e_n & e_{n-1} & e_{n-2} & \cdots & e_2 & e_1 & 0 & \cdots & 0 & 0 \\ 0 & e_n & e_{n-1} & \cdots & e_3 & e_2 & e_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e_n & e_{n-1} & \cdots & e_2 & e_1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & e_n & e_{n-1} & \cdots & e_2 & e_1 \end{bmatrix}.$$

Proof. Consider the Toeplitz matrix X as defined in Definition 2.3.6. Let u_i for $i = 1, 2, \dots, n$ denote the i^{th} column of the matrix X. Then, we can express the vectorization of X as

$$\mathrm{vec}(X) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

We now calculate each column u_i. For u_1, we have

$$\begin{aligned} u_1 &= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \mathrm{vec}_T(X) \\ &= \begin{bmatrix} e_n & e_{n-1} & e_{n-2} & \cdots & e_2 & e_1 & 0 & \cdots & 0 & 0 \end{bmatrix} \mathrm{vec}_T(X). \end{aligned}$$

Next, for u_2, we get

$$\begin{aligned} u_2 &= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \mathrm{vec}_T(X) \\ &= \begin{bmatrix} 0 & e_n & e_{n-1} & \cdots & e_3 & e_2 & e_1 & \cdots & 0 & 0 \end{bmatrix} \mathrm{vec}_T(X). \end{aligned}$$

Finally, for u_n, we have

$$\begin{aligned} u_n &= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \mathrm{vec}_T(X) \\ &= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & e_n & e_{n-1} & \cdots & e_2 & e_1 \end{bmatrix} \mathrm{vec}_T(X). \end{aligned}$$

By substituting the computed values of u_1, u_2, …, u_n into the vectorization of X, we conclude that

$$\mathrm{vec}(X) = K_T \mathrm{vec}_T(X). \quad \blacksquare$$

Lemma 2.3.11. *If $X \in \mathbb{R}^{n\times n}$, then $X \in \mathbb{STR}^{n\times n} \Leftrightarrow \mathrm{vec}(X) = K_{ST}\mathrm{vec}_{ST}(X)$, where $\mathrm{vec}_{ST}(X)$ is of the form (2.3.3). When n is even, let $n = 2l$. In this case, the matrix $K_{ST} \in \mathbb{R}^{n^2\times n}$ is represented as*

$$K_{ST} = \begin{bmatrix} e_1 & e_2 & e_3 & \cdots & e_l & e_{l+1} & \cdots & e_{n-1} & e_n \\ e_2 & e_1 + e_3 & e_4 & \cdots & e_{l+1} & e_{l+2} & \cdots & e_n & 0 \\ e_3 & e_2 + e_4 & e_1 + e_5 & \cdots & e_{l+2} & e_{l+3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ e_l & e_{l-1} + e_{l+1} & e_{l-2} + e_{l+2} & \cdots & e_1 + e_{n-1} & e_n & \cdots & 0 & 0 \\ e_{l+1} & e_l + e_{l+2} & e_{l-1} + e_{l+3} & \cdots & e_2 + e_n & e_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & & \ddots & \vdots & \vdots \\ e_{n-1} & e_{n-2} + e_n & e_{n-3} & \cdots & \cdots & \cdots & \cdots & e_1 & 0 \\ e_n & e_{n-1} & e_{n-2} & \cdots & \cdots & \cdots & \cdots & e_2 & e_1 \end{bmatrix}.$$

When n is odd, let $n = 2l - 1$. In this case, the matrix $K_{ST} \in \mathbb{R}^{n^2\times n}$ is represented as

$$K_{ST} = \begin{bmatrix} e_1 & e_2 & e_3 & \cdots & e_l & e_{l+1} & \cdots & e_{n-1} & e_n \\ e_2 & e_1 + e_3 & e_4 & \cdots & e_{l+1} & e_{l+2} & \cdots & e_n & 0 \\ e_3 & e_2 + e_4 & e_1 + e_5 & \cdots & e_{l+2} & e_{l+3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ e_l & e_{l-1} + e_{l+1} & e_{l-2} + e_{l+2} & \cdots & e_1 + e_n & 0 & \cdots & 0 & 0 \\ e_{l+1} & e_l + e_{l+2} & e_{l-1} + e_{l+3} & \cdots & e_2 & e_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & & \ddots & \vdots & \vdots \\ e_{n-1} & e_{n-2} + e_n & e_{n-3} & \cdots & \cdots & \cdots & \cdots & e_1 & 0 \\ e_n & e_{n-1} & e_{n-2} & \cdots & \cdots & \cdots & \cdots & e_2 & e_1 \end{bmatrix}.$$

Proof. The proof is similar to the proof method used in Lemma 2.3.10. ∎

To gain a deeper understanding of the above lemma, let's explore specific cases for $n = 4$ and $n = 7$. In these cases, the matrix K_{ST} takes the following forms:

For $n = 4$, we have

$$K_{ST} = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ e_2 & e_1 + e_3 & e_4 & 0 \\ e_3 & e_2 + e_4 & e_1 & 0 \\ e_4 & e_3 & e_2 & e_1 \end{bmatrix}.$$

For $n = 7$, the matrix K_{ST} takes the following form:

$$K_{ST} = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ e_2 & e_1 + e_3 & e_4 & e_5 & e_6 & e_7 & 0 \\ e_3 & e_2 + e_4 & e_1 + e_5 & e_6 & e_7 & 0 & 0 \\ e_4 & e_3 + e_5 & e_2 + e_6 & e_1 + e_7 & 0 & 0 & 0 \\ e_5 & e_4 + e_6 & e_3 + e_7 & e_2 & e_1 & 0 & 0 \\ e_6 & e_5 + e_7 & e_4 & e_3 & e_2 & e_1 & 0 \\ e_7 & e_6 & e_5 & e_4 & e_3 & e_2 & e_1 \end{bmatrix}.$$

Lemma 2.3.12. *If $X \in \mathbb{R}^{n\times n}$, then $X \in \mathbb{HR}^{n\times n} \Leftrightarrow \mathrm{vec}(X) = K_H \mathrm{vec}_H(X)$, where $\mathrm{vec}_H(X)$ is of the form (2.3.4), and the matrix $K_H \in \mathbb{R}^{n^2\times(2n-1)}$ is represented as*

$$K_H = \begin{bmatrix} e_1 & e_2 & e_3 & \cdots & e_{n-1} & e_n & 0 & 0 & \cdots & 0 \\ 0 & e_1 & e_2 & \cdots & e_{n-2} & e_{n-1} & e_n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e_1 & e_2 & \cdots & e_{n-1} & e_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & e_1 & e_2 & \cdots & e_{n-1} & e_n \end{bmatrix}.$$

Proof. The proof is similar to the proof method used in Lemma 2.3.10. ■

Lemma 2.3.13. *If $X \in \mathbb{R}^{n\times n}$, then $X \in \mathbb{CR}^{n\times n} \Leftrightarrow \mathrm{vec}(X) = K_C \mathrm{vec}_C(X)$, where $\mathrm{vec}_C(X)$ is of the form (2.3.5), and the matrix $K_C \in \mathbb{R}^{n^2\times n}$ is represented as*

$$K_C = \begin{bmatrix} e_1 & e_2 & \cdots & e_{n-1} & e_n \\ e_2 & e_3 & \cdots & e_n & e_1 \\ e_3 & e_4 & \cdots & e_1 & e_2 \\ \vdots & \vdots & & \vdots & \vdots \\ e_n & e_1 & \cdots & e_{n-2} & e_{n-1} \end{bmatrix}.$$

Proof. The proof is similar to the proof method used in Lemma 2.3.10. ■

In the following lemmas, we present the vec-structure of reduced biquaternion L-structure matrices based on the vec-structure of real structure matrices.

Lemma 2.3.14. *If $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times n}$, then*

(1) $X \in \mathbb{TQ}_{\mathbb{R}}^{n\times n} \Leftrightarrow \mathrm{vec}(\overrightarrow{X}) = M_T \mathrm{vec}_T(\overrightarrow{X})$, *where*

$$M_T = \begin{bmatrix} K_T & 0 & 0 & 0 \\ 0 & K_T & 0 & 0 \\ 0 & 0 & K_T & 0 \\ 0 & 0 & 0 & K_T \end{bmatrix}, \quad \mathrm{vec}_T(\overrightarrow{X}) = \begin{bmatrix} \mathrm{vec}_T(\Re(X_1)) \\ \mathrm{vec}_T(\Im(X_1)) \\ \mathrm{vec}_T(\Re(X_2)) \\ \mathrm{vec}_T(\Im(X_2)) \end{bmatrix}.$$

(2) $X \in \mathbb{STQ}_{\mathbb{R}}^{n\times n} \Leftrightarrow \mathrm{vec}(\overrightarrow{X}) = M_{ST} \mathrm{vec}_{ST}(\overrightarrow{X})$, *where*

$$M_{ST} = \begin{bmatrix} K_{ST} & 0 & 0 & 0 \\ 0 & K_{ST} & 0 & 0 \\ 0 & 0 & K_{ST} & 0 \\ 0 & 0 & 0 & K_{ST} \end{bmatrix}, \quad \mathrm{vec}_{ST}(\overrightarrow{X}) = \begin{bmatrix} \mathrm{vec}_{ST}(\Re(X_1)) \\ \mathrm{vec}_{ST}(\Im(X_1)) \\ \mathrm{vec}_{ST}(\Re(X_2)) \\ \mathrm{vec}_{ST}(\Im(X_2)) \end{bmatrix}.$$

(3) $X \in \mathbb{HQ}_{\mathbb{R}}^{n\times n} \Leftrightarrow \mathrm{vec}(\overrightarrow{X}) = M_H \mathrm{vec}_H(\overrightarrow{X})$, *where*

$$M_H = \begin{bmatrix} K_H & 0 & 0 & 0 \\ 0 & K_H & 0 & 0 \\ 0 & 0 & K_H & 0 \\ 0 & 0 & 0 & K_H \end{bmatrix}, \quad \mathrm{vec}_H(\overrightarrow{X}) = \begin{bmatrix} \mathrm{vec}_H(\Re(X_1)) \\ \mathrm{vec}_H(\Im(X_1)) \\ \mathrm{vec}_H(\Re(X_2)) \\ \mathrm{vec}_H(\Im(X_2)) \end{bmatrix}.$$

(4) $X \in \mathbb{CQ}_{\mathbb{R}}^{n\times n} \Leftrightarrow \mathrm{vec}(\overrightarrow{X}) = M_C \mathrm{vec}_C(\overrightarrow{X})$, *where*

$$M_C = \begin{bmatrix} K_C & 0 & 0 & 0 \\ 0 & K_C & 0 & 0 \\ 0 & 0 & K_C & 0 \\ 0 & 0 & 0 & K_C \end{bmatrix}, \quad \mathrm{vec}_C(\overrightarrow{X}) = \begin{bmatrix} \mathrm{vec}_C(\Re(X_1)) \\ \mathrm{vec}_C(\Im(X_1)) \\ \mathrm{vec}_C(\Re(X_2)) \\ \mathrm{vec}_C(\Im(X_2)) \end{bmatrix}.$$

Proof. We will prove the first part of the statement, as the remaining parts follow using a similar argument.

It is known that $X \in \mathbb{TQ}_{\mathbb{R}}^{n\times n} \Leftrightarrow \Re(X_1), \Im(X_1), \Re(X_2), \Im(X_2) \in \mathbb{TR}^{n\times n}$. Utilizing this fact along with Lemma 2.3.10, we can write

$$\mathrm{vec}(\overrightarrow{X}) = \begin{bmatrix} \mathrm{vec}(\Re(X_1)) \\ \mathrm{vec}(\Im(X_1)) \\ \mathrm{vec}(\Re(X_2)) \\ \mathrm{vec}(\Im(X_2)) \end{bmatrix} = \begin{bmatrix} K_T\mathrm{vec}_T(\Re(X_1)) \\ K_T\mathrm{vec}_T(\Im(X_1)) \\ K_T\mathrm{vec}_T(\Re(X_2)) \\ K_T\mathrm{vec}_T(\Im(X_2)) \end{bmatrix}.$$

Thus, we can conclude

$$\mathrm{vec}(\overrightarrow{X}) = M_T \mathrm{vec}_T(\overrightarrow{X}). \quad \blacksquare$$

Lemma 2.3.15. *If* $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, *then*

(1) $X \in \mathbb{I}\mathbb{Q}_{\mathbb{R}}^{m\times n} \iff \mathrm{vec}(\overrightarrow{X}) = M_I \mathrm{vec}_I(\overrightarrow{X})$, *where*

$$M_I = \begin{bmatrix} 0 & 0 & 0 \\ I_{mn} & 0 & 0 \\ 0 & I_{mn} & 0 \\ 0 & 0 & I_{mn} \end{bmatrix}, \quad \mathrm{vec}_I(\overrightarrow{X}) = \begin{bmatrix} \mathrm{vec}(\Im(X_1)) \\ \mathrm{vec}(\Re(X_2)) \\ \mathrm{vec}(\Im(X_2)) \end{bmatrix}.$$

(2) $X \in \mathbb{C}^{m\times n} \iff \mathrm{vec}(\overrightarrow{X}) = M_c \mathrm{vec}_c(\overrightarrow{X})$, *where*

$$M_c = \begin{bmatrix} I_{mn} & 0 \\ 0 & I_{mn} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathrm{vec}_c(\overrightarrow{X}) = \begin{bmatrix} \mathrm{vec}(\Re(X_1)) \\ \mathrm{vec}(\Im(X_1)) \end{bmatrix}.$$

(3) $X \in \mathbb{R}^{m\times n} \iff \mathrm{vec}(\overrightarrow{X}) = M_R \mathrm{vec}_R(\overrightarrow{X})$, *where*

$$M_R = \begin{bmatrix} I_{mn} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathrm{vec}_R(\overrightarrow{X}) = \mathrm{vec}(\Re(X_1)).$$

Proof. The proof follows from the fact that $X \in \mathbb{I}\mathbb{Q}_{\mathbb{R}}^{m\times n} \iff \Re(X_1) = 0$, while $X \in \mathbb{C}^{m\times n} \iff \Re(X_2) = 0$ and $\Im(X_2) = 0$. Additionally, $X \in \mathbb{R}^{m\times n} \iff \Im(X_1) = 0$, $\Re(X_2) = 0$, and $\Im(X_2) = 0$. ■

Up to this point, we have explored the representation of a reduced biquaternion L-structure matrix using a corresponding real structure matrix for a specific class of matrix sets. Based on the preceding discussion regarding reduced biquaternion L-structure matrices, the findings can be summarized as follows:

For $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, we have $\overrightarrow{X} = [\Re(X_1), \Im(X_1), \Re(X_2), \Im(X_2)] \in \mathbb{R}^{m\times 4n}$. Let $\mathcal{G}$ be a subspace of $\mathbb{R}^{4mn}$ and M_L be the basis matrix for $\mathcal{G}$. The subset of real matrices of order $m \times 4n$ given by

$$L^R(m, 4n) = \{\overrightarrow{X} \in \mathbb{R}^{m\times 4n} \mid \mathrm{vec}(\overrightarrow{X}) \in \mathcal{G}\} \tag{2.3.6}$$

is called as a real L-structure.

Remark 2.3.16. *M_L represents the basis matrix of the subspace $\mathcal{G}$. For simplicity, we will refer to M_L as the basis matrix of $L^R(m,4n)$ throughout this chapter.*

Thus, we have the following Lemma.

Lemma 2.3.17. *Let M_L be the basis matrix of $L^R(m,4n)$. Then $X \in L(m,n) \Leftrightarrow \mathrm{vec}(\overrightarrow{X}) = M_L \mathrm{vec}_L(\overrightarrow{X})$, where $\mathrm{vec}_L(\overrightarrow{X})$ corresponds to the representation of $\overrightarrow{X}$ according to the basis matrix M_L.*

Proof. The proof follows from the generalization of Lemmas 2.3.10 and 2.3.14 to any L-structure matrix X. ■

Now that we have described the reduced biquaternion L-structure, we turn our attention to solving a RBME. Our approach for addressing the RBME involves transforming it into a complex matrix equation. For $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times s}$, and $B \in \mathbb{C}^{s\times t}$, we have

$$\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}(X). \tag{2.3.7}$$

In the context of reduced biquaternion algebra, we investigate $\mathrm{vec}(\Psi_{AXB})$ rather than $\mathrm{vec}(AXB)$.

Lemma 2.3.18. *Let $A = A_1 + A_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times s}$, and $B = B_1 + B_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{s\times t}$. Then*

$$\mathrm{vec}(\Psi_{AXB}) = \left(h(B)^T \otimes A_1 + h(B\boldsymbol{j})^T \otimes A_2\right)\mathrm{vec}(\Psi_X).$$

Proof. Using (2.2.1) and Lemma 2.2.2, we have

$$\Psi_{AXB} = \Psi_A h(XB) = \Psi_A h(X)h(B)$$

which can be expanded as

$$\Psi_{AXB} = [A_1X_1B_1 + A_2X_2B_1 + A_1X_2B_2 + A_2X_1B_2, A_1X_1B_2 + A_2X_2B_2 + A_1X_2B_1 + A_2X_1B_1].$$

Now, from (2.2.2) and (2.3.7), we get

$$\mathrm{vec}(\Psi_{AXB}) = \begin{bmatrix} (B_1^T \otimes A_1)\mathrm{vec}(X_1) + (B_1^T \otimes A_2)\mathrm{vec}(X_2) \\ +(B_2^T \otimes A_1)\mathrm{vec}(X_2) + (B_2^T \otimes A_2)\mathrm{vec}(X_1) \\ (B_2^T \otimes A_1)\mathrm{vec}(X_1) + (B_2^T \otimes A_2)\mathrm{vec}(X_2) \\ +(B_1^T \otimes A_1)\mathrm{vec}(X_2) + (B_1^T \otimes A_2)\mathrm{vec}(X_1) \end{bmatrix}$$

$$= \left(\begin{bmatrix} B_1^T & B_2^T \\ B_2^T & B_1^T \end{bmatrix} \otimes A_1 + \begin{bmatrix} B_2^T & B_1^T \\ B_1^T & B_2^T \end{bmatrix} \otimes A_2\right)\begin{bmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{bmatrix}.$$

Finally, we arrive at

$$\mathrm{vec}(\Psi_{AXB}) = \left(h(B)^T \otimes A_1 + h(B\boldsymbol{j})^T \otimes A_2\right)\mathrm{vec}(\Psi_X). \quad \blacksquare$$

Set

$$\mathcal{W}_{ns} = \begin{bmatrix} I_{ns} & \boldsymbol{i}I_{ns} & 0 & 0 \\ 0 & 0 & I_{ns} & \boldsymbol{i}I_{ns} \end{bmatrix}, \quad \mathcal{S}_{ns} = \begin{bmatrix} Q_{ns} & 0 \\ 0 & Q_{ns} \end{bmatrix}, \tag{2.3.8}$$

where Q_{ns} is the commutation matrix, a row permutation of the identity matrix I_{ns}.

We have examined $\mathrm{vec}(\Psi_{AXB})$ within reduced biquaternion algebra. The following lemma outlines $\mathrm{vec}(\Psi_{AXB})$ when X possesses an L-structure in reduced biquaternion algebra.

Lemma 2.3.19. *Let $A = A_1 + A_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $X = X_1 + X_2\boldsymbol{j} \in L(n,s)$, and $B = B_1 + B_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{s\times t}$. Then*

$$\mathrm{vec}(\Psi_{AXB}) = \left(h(B)^T \otimes A_1 + h(B\boldsymbol{j})^T \otimes A_2\right)\mathcal{W}_{ns}M_L\mathrm{vec}_L(\overrightarrow{X}),$$
$$\mathrm{vec}(\Psi_{AX^TB}) = \left(h(B)^T \otimes A_1 + h(B\boldsymbol{j})^T \otimes A_2\right)\mathcal{S}_{ns}\mathcal{W}_{ns}M_L\mathrm{vec}_L(\overrightarrow{X}),$$

where M_L represents the basis matrix of $L^R(n,4s)$, and $\mathcal{W}_{ns}$ and $\mathcal{S}_{ns}$ are defined as in (2.3.8).

Proof. Using (2.2.2), (2.3.8), and Lemmas 2.3.17 and 2.3.18, we obtain

$$\begin{aligned}
\mathrm{vec}(\Psi_{AXB}) &= \left(h(B)^T \otimes A_1 + h(B\boldsymbol{j})^T \otimes A_2\right)\mathrm{vec}(\Psi_X) \\
&= \left(h(B)^T \otimes A_1 + h(B\boldsymbol{j})^T \otimes A_2\right)\begin{bmatrix}\mathrm{vec}(X_1) \\ \mathrm{vec}(X_2)\end{bmatrix} \\
&= \left(h(B)^T \otimes A_1 + h(B\boldsymbol{j})^T \otimes A_2\right)\begin{bmatrix}\mathrm{vec}(\Re(X_1)) + \boldsymbol{i}\mathrm{vec}(\Im(X_1)) \\ \mathrm{vec}(\Re(X_2)) + \boldsymbol{i}\mathrm{vec}(\Im(X_2))\end{bmatrix} \\
&= \left(h(B)^T \otimes A_1 + h(B\boldsymbol{j})^T \otimes A_2\right)\mathcal{W}_{ns}\mathrm{vec}(\overrightarrow{X}) \\
&= \left(h(B)^T \otimes A_1 + h(B\boldsymbol{j})^T \otimes A_2\right)\mathcal{W}_{ns}\mathcal{M}_L\mathrm{vec}_L(\overrightarrow{X}).
\end{aligned}$$

Next, we have

$$\mathrm{vec}(\Psi_{X^T}) = \begin{bmatrix}\mathrm{vec}(X_1^T) \\ \mathrm{vec}(X_2^T)\end{bmatrix} = \begin{bmatrix}\mathcal{Q}_{ns}\mathrm{vec}(X_1) \\ \mathcal{Q}_{ns}\mathrm{vec}(X_2)\end{bmatrix} = \mathcal{S}_{ns}\begin{bmatrix}\mathrm{vec}(X_1) \\ \mathrm{vec}(X_2)\end{bmatrix} = \mathcal{S}_{ns}\mathcal{W}_{ns}\mathcal{M}_L\mathrm{vec}_L(\overrightarrow{X}).$$

The proof follows from simple calculations. $\blacksquare$

2.4. General Framework for Solving Constrained RBMEs

The purpose of this section is to demonstrate how we can solve constrained generalized linear matrix equations over commutative quaternions. As part of our approach, the constrained RBME is reduced to the following unconstrained real matrix system:

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} x = e, \tag{2.4.1}$$

where Q_1, Q_2 are real matrices of appropriate dimension, x and e are real matrices or vectors of appropriate size. From [13, Theorem 2] the generalized inverse of a partitioned matrix $[U, V]$ is given by

$$[U, V]^{\dagger} = \begin{bmatrix} U^{\dagger} - U^{\dagger} V H \\ H \end{bmatrix},$$

where

$$H = R^{\dagger} + \left(I - R^{\dagger} R\right) Z V^T U^{\dagger T} U^{\dagger} \left(I - V R^{\dagger}\right), \quad R = \left(I - U U^{\dagger}\right) V,$$
$$Z = \left(I + \left(I - R^{\dagger} R\right) V^T U^{\dagger T} U^{\dagger} V \left(I - R^{\dagger} R\right)\right)^{-1}.$$

We have

$$[U, V]^{T\dagger} = [U, V]^{\dagger T} = \begin{bmatrix} U^{\dagger} - U^{\dagger} V H \\ H \end{bmatrix}^T = \left[U^{T\dagger} - H^T V^T U^{T\dagger}, H^T\right].$$

By substituting $U = Q_1^T$ and $V = Q_2^T$, we get

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^{\dagger} = \left[Q_1^{\dagger} - H^T Q_2 Q_1^{\dagger}, H^T\right], \quad \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^{\dagger} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = Q_1^{\dagger} Q_1 + R R^{\dagger}, \tag{2.4.2}$$

where

$$\begin{aligned} H &= R^{\dagger} + \left(I - R^{\dagger} R\right) Z Q_2 Q_1^{\dagger} Q_1^{\dagger T} \left(I - Q_2^T R^{\dagger}\right), \quad R = \left(I - Q_1^{\dagger} Q_1\right) Q_2^T, \\ Z &= \left(I + \left(I - R^{\dagger} R\right) Q_2 Q_1^{\dagger} Q_1^{\dagger T} Q_2^T \left(I - R^{\dagger} R\right)\right)^{-1}. \end{aligned} \tag{2.4.3}$$

Using the results mentioned above, we deduce the following lemma that is helpful in developing the main results.

Lemma 2.4.1. *Consider the real matrix system of the form* $\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} x = e$. *We have the following results:*

(1) *The matrix equation has a solution x if and only if* $\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^{\dagger} e = e$. *In this case, the general solution is*

$$x = \left[Q_1^{\dagger} - H^T Q_2 Q_1^{\dagger}, H^T\right] e + \left(I - Q_1^{\dagger} Q_1 - R R^{\dagger}\right) y,$$

where y is an arbitrary matrix or vector of suitable size. Furthermore, if the consistency condition is satisfied, then the matrix equation has a unique solution if and only if matrix $\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$ *is of full column rank. In this case, the unique solution is*

$$x = \left[Q_1^{\dagger} - H^T Q_2 Q_1^{\dagger}, H^T\right] e.$$

(2) *The least squares solutions of the matrix equation can be expressed as*

$$x = \left[Q_1^{\dagger} - H^T Q_2 Q_1^{\dagger}, H^T\right] e + \left(I - Q_1^{\dagger} Q_1 - R R^{\dagger}\right) y,$$

where y is an arbitrary matrix or vector of suitable size, and the least squares solution with the least norm is

$$x = \left[Q_1^{\dagger} - H^T Q_2 Q_1^{\dagger}, H^T\right] e.$$

The following lemma will be used for the development of main results.

Lemma 2.4.2. *Consider the matrix equation $AX = B$, where $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{R}^{n\times d}$, and $B \in \mathbb{C}^{m\times d}$. The matrix equation $AX = B$ is equivalent to the following linear system:*

$$\begin{bmatrix} \mathfrak{R}(A) \\ \mathfrak{I}(A) \end{bmatrix} X = \begin{bmatrix} \mathfrak{R}(B) \\ \mathfrak{I}(B) \end{bmatrix}.$$

In the following subsection, we aim to find Q_1, Q_2, and e for each of the three constrained RBMEs and solve them.

Remark 2.4.3. *It is important to emphasize that the values of Q_1, Q_2, and e vary depending on the specific matrix equation we are attempting to solve.*

2.4.1. Linear Matrix Equation in Several Unknown L-structures

The class of matrix equation (2.1.1) encompasses many important matrix equations. Some simple examples are $AXB + CYD = E$, $AX + YB = E$. We now introduce a general framework for finding the least squares solutions of RBME of the form (2.1.1). The problem can be formally stated as follows:

Problem 2.4.4. *Let* $A_l = A_{l1} + A_{l2}\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n_l}$*,* $B_l \in \mathbb{Q}_{\mathbb{R}}^{s_l\times t}$*, and* $E = E_1 + E_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times t}$ *for* $l = 1, 2, \dots, r$*. Let*

$$\mathcal{N}_{LE} = \left\{ [X_1, X_2, \dots, X_r] \,|\, X_l \in L_l(n_l, s_l), \left\| \sum_{l=1}^{r} A_l X_l B_l - E \right\|_F = \min_{\widetilde{X}_l \in L_l(n_l,s_l)} \left\| \sum_{l=1}^{r} A_l \widetilde{X}_l B_l - E \right\|_F \right\}.$$

Then find $[X_{1E}, X_{2E}, \dots, X_{rE}] \in \mathcal{N}_{LE}$ *such that*

$$\|[X_{1E}, X_{2E}, \dots, X_{rE}]\|_F = \min_{[X_1, X_2, \dots, X_r] \in \mathcal{N}_{LE}} \left(\|X_1\|_F^2 + \|X_2\|_F^2 + \cdots + \|X_r\|_F^2 \right)^{\frac{1}{2}}.$$

To solve Problem 2.4.4, we employ the following notations: for $l = 1, 2, \dots, r$, let M_{L_l} be the basis matrix of $L_l^R(n_l, 4s_l)$, and

$$S_l := \left(h(B_l)^T \otimes A_{l1} + h(B_l\boldsymbol{j})^T \otimes A_{l2} \right) \mathcal{W}_{n_l s_l} M_{L_l}, \tag{2.4.4}$$

$$x := \begin{bmatrix} \mathrm{vec}_{L_1}(\overrightarrow{X_1}) \\ \mathrm{vec}_{L_2}(\overrightarrow{X_2}) \\ \vdots \\ \mathrm{vec}_{L_r}(\overrightarrow{X_r}) \end{bmatrix}. \tag{2.4.5}$$

Additionally, Q_1, Q_2, and e (as in (2.4.1)) are in the following form:

$$\begin{aligned} Q_1 &:= [\Re(S_1), \Re(S_2), \dots, \Re(S_r)], \quad Q_2 := [\Im(S_1), \Im(S_2), \dots, \Im(S_r)], \\ \text{and} \quad e &:= \begin{bmatrix} \mathrm{vec}(\Re(\Psi_E)) \\ \mathrm{vec}(\Im(\Psi_E)) \end{bmatrix}. \end{aligned} \tag{2.4.6}$$

In case of inconsistency in matrix equation (2.1.1), we provide the least squares solutions. The following result provides the solution to Problem 2.4.4.

Theorem 2.4.5. *Let* $A_l \in \mathbb{Q}_{\mathbb{R}}^{m\times n_l}$*,* $B_l \in \mathbb{Q}_{\mathbb{R}}^{s_l\times t}$*, and* $E \in \mathbb{Q}_{\mathbb{R}}^{m\times t}$ *for* $l = 1, 2, \dots, r$*. Let* Q_1, Q_2*, and* e *be of the form* (2.4.6) *and* $\mathcal{T} = \mathrm{diag}(M_{L_1}, M_{L_2}, \dots, M_{L_r})$*. Then*

$$\mathcal{N}_{LE} = \left\{ [X_1, X_2, \dots, X_r] \left| \begin{bmatrix} \mathrm{vec}(\overrightarrow{X_1}) \\ \mathrm{vec}(\overrightarrow{X_2}) \\ \vdots \\ \mathrm{vec}(\overrightarrow{X_r}) \end{bmatrix} = \mathcal{T}\left[Q_1^\dagger - H^T Q_2 Q_1^\dagger, H^T \right] e + \mathcal{T}\left(I - Q_1^\dagger Q_1 - RR^\dagger \right) y \right. \right\}, \tag{2.4.7}$$

where y is any vector of suitable size. The unique solution $[X_{1E}, X_{2E}, \ldots, X_{rE}] \in \mathcal{N}_{LE}$ to Problem 2.4.4 satisfies

$$\begin{bmatrix} \mathrm{vec}(\overrightarrow{X_{1E}}) \\ \mathrm{vec}(\overrightarrow{X_{2E}}) \\ \vdots \\ \mathrm{vec}(\overrightarrow{X_{rE}}) \end{bmatrix} = \mathcal{T}\left[Q_1^{\dagger} - H^T Q_2 Q_1^{\dagger}, H^T\right] e. \tag{2.4.8}$$

Proof. By using (2.2.3) and Lemma 2.2.2, we get

$$\left\| \sum_{l=1}^{r} A_l X_l B_l - E \right\|_F^2 = \left\| \sum_{l=1}^{r} \Psi_{A_l X_l B_l} - \Psi_E \right\|_F^2 = \left\| \sum_{l=1}^{r} \mathrm{vec}\left(\Psi_{A_l X_l B_l}\right) - \mathrm{vec}\left(\Psi_E\right) \right\|_F^2 .$$

Using Lemma 2.3.19, we have

$$\mathrm{vec}\left(\Psi_{A_l X_l B_l}\right) = \left(h(B_l)^T \otimes A_{l1} + h(B_l \boldsymbol{j})^T \otimes A_{l2}\right) \mathcal{W}_{n_l s_l} M_{L_l} \mathrm{vec}_{L_l}(\overrightarrow{X_l}).$$

Now, using (2.4.4), we get

$$\begin{aligned} \sum_{l=1}^{r} \mathrm{vec}\left(\Psi_{A_l X_l B_l}\right) &= \sum_{l=1}^{r} \left(h(B_l)^T \otimes A_{l1} + h(B_l \boldsymbol{j})^T \otimes A_{l2}\right) \mathcal{W}_{n_l s_l} M_{L_l} \mathrm{vec}_{L_l}(\overrightarrow{X_l}) \\ &= \sum_{l=1}^{r} S_l \mathrm{vec}_{L_l}(\overrightarrow{X_l}). \end{aligned}$$

Using (2.4.5) and Lemma 2.4.2, we have

$$\begin{aligned} \left\| \sum_{l=1}^{r} A_l X_l B_l - E \right\|_F^2 &= \left\| \sum_{l=1}^{r} S_l \mathrm{vec}_{L_l}(\overrightarrow{X_l}) - \mathrm{vec}\left(\Psi_E\right) \right\|_F^2 \\ &= \left\| [S_1, S_2, \ldots, S_r] \begin{bmatrix} \mathrm{vec}_{L_1}(\overrightarrow{X_1}) \\ \mathrm{vec}_{L_2}(\overrightarrow{X_2}) \\ \vdots \\ \mathrm{vec}_{L_r}(\overrightarrow{X_r}) \end{bmatrix} - \mathrm{vec}\left(\Psi_E\right) \right\|_F^2 \\ &= \left\| \begin{bmatrix} \mathfrak{R}(S_1) & \mathfrak{R}(S_2) & \cdots & \mathfrak{R}(S_r) \\ \mathfrak{I}(S_1) & \mathfrak{I}(S_2) & \cdots & \mathfrak{I}(S_r) \end{bmatrix} x - \begin{bmatrix} \mathrm{vec}\left(\mathfrak{R}(\Psi_E)\right) \\ \mathrm{vec}\left(\mathfrak{I}(\Psi_E)\right) \end{bmatrix} \right\|_F^2 . \end{aligned}$$

Using (2.4.6), this simplifies to

$$\left\| \sum_{l=1}^{r} A_l X_l B_l - E \right\|_F^2 = \left\| \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} x - e \right\|_F^2 .$$

Hence, Problem 2.4.4 can be solved by finding the least squares solutions of the following unconstrained real matrix system:

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} x = e.$$

By Lemma 2.4.1, the least squares solutions of the above real matrix system is:

$$x = \left[Q_1^\dagger - H^T Q_2 Q_1^\dagger, H^T\right] e + \left(I - Q_1^\dagger Q_1 - RR^\dagger\right) y,$$

where y is any vector of suitable size, and the least squares solution with the least norm is

$$\left[Q_1^\dagger - H^T Q_2 Q_1^\dagger, H^T\right] e.$$

Using Lemma 2.3.17, we have

$$\begin{bmatrix} \mathrm{vec}(\overrightarrow{X_1}) \\ \mathrm{vec}(\overrightarrow{X_2}) \\ \vdots \\ \mathrm{vec}(\overrightarrow{X_r}) \end{bmatrix} = \mathcal{T} x.$$

Thus, we can obtain (2.4.7) and (2.4.8). ∎

The following theorem presents the consistency condition for obtaining the solution $X_l \in L_l(n_l, s_l)$ for the RBME of the form (2.1.1) and a general formulation for the solution.

Theorem 2.4.6. *Consider the RBME of the form (2.1.1) and let* $\mathcal{T} = \mathrm{diag}\left(M_{L_1}, M_{L_2}, \ldots, M_{L_r}\right)$. *Then the matrix equation (2.1.1) has an L-structure solution* $X_l \in L_l(n_l, s_l)$, *for* $l = 1, 2, \ldots, r$, *if and only if*

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^{\dagger} e = e, \tag{2.4.9}$$

where Q_1, Q_2, *and* e *are in the form of (2.4.6). In this case, the general solution* $X_l \in L_l(n_l, s_l)$ *satisfies*

$$\begin{bmatrix} \mathrm{vec}(\overrightarrow{X_1}) \\ \mathrm{vec}(\overrightarrow{X_2}) \\ \vdots \\ \mathrm{vec}(\overrightarrow{X_r}) \end{bmatrix} = \mathcal{T}\left[Q_1^\dagger - H^T Q_2 Q_1^\dagger, H^T\right] e + \mathcal{T}\left(I - Q_1^\dagger Q_1 - RR^\dagger\right) y,$$

where y is any vector of suitable size. Further, if the consistency condition holds, then the RBME of the form (2.1.1) has a unique solution $X_l \in L_l(n_l, s_l)$ if and only if

$$\operatorname{rank}\left(\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}\right) = \dim\left(\begin{bmatrix} \operatorname{vec}_{L_1}(\overrightarrow{X_1}) \\ \operatorname{vec}_{L_2}(\overrightarrow{X_2}) \\ \vdots \\ \operatorname{vec}_{L_r}(\overrightarrow{X_r}) \end{bmatrix}\right).$$

In this case, the unique solution $X_l \in L_l(n_l, s_l)$ satisfies

$$\begin{bmatrix} \operatorname{vec}(\overrightarrow{X_1}) \\ \operatorname{vec}(\overrightarrow{X_2}) \\ \vdots \\ \operatorname{vec}(\overrightarrow{X_r}) \end{bmatrix} = \mathcal{T}\left[Q_1^{\dagger} - H^T Q_2 Q_1^{\dagger}, H^T\right] e.$$

Proof. The proof follows using Lemma 2.4.1 and from the fact that

$$\sum_{l=1}^{r} A_l X_l B_l = E \Leftrightarrow \begin{bmatrix} \Re(S_1) & \Re(S_2) & \cdots & \Re(S_r) \\ \Im(S_1) & \Im(S_2) & \cdots & \Im(S_r) \end{bmatrix} \begin{bmatrix} \operatorname{vec}_{L_1}(\overrightarrow{X_1}) \\ \operatorname{vec}_{L_2}(\overrightarrow{X_2}) \\ \vdots \\ \operatorname{vec}_{L_r}(\overrightarrow{X_r}) \end{bmatrix} = \begin{bmatrix} \operatorname{vec}(\Re(\Psi_E)) \\ \operatorname{vec}(\Im(\Psi_E)) \end{bmatrix}. \quad \blacksquare$$

Remark 2.4.7. *The problem of finding the least squares real or purely imaginary solutions to the RBME $AX = E$ is a particular case of Problem 2.4.4. To solve this, we simply need to find the least squares solution of the matrix equation $\Psi_{AX} = \Psi_E$. This method is computationally less expensive compared to solving the least squares problem for* $\operatorname{vec}(\Psi_{AX}) = \operatorname{vec}(\Psi_E)$*, as the latter involves matrices of much larger dimensions due to the Kronecker product.*

Specifically, we have

$$\begin{aligned} \Psi_{AX} &= \Psi_A h(X) \\ &= [A_1, A_2] \begin{bmatrix} X_1 & X_2 \\ X_2 & X_1 \end{bmatrix} \\ &= [A_1X_1 + A_2X_2, A_1X_2 + A_2X_1] \\ &= [A_1\Re(X_1) + \boldsymbol{i}A_1\Im(X_1) + A_2\Re(X_2) + \boldsymbol{i}A_2\Im(X_2), \\ &\quad A_1\Re(X_2) + \boldsymbol{i}A_1\Im(X_2) + A_2\Re(X_1) + \boldsymbol{i}A_2\Im(X_1)]. \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
\|AX - E\|_F &= \|\Psi_{AX} - \Psi_E\|_F \\
&= \left\| \begin{bmatrix} A_1\mathfrak{R}(X_1) + \boldsymbol{i}A_1\mathfrak{I}(X_1) + A_2\mathfrak{R}(X_2) + \boldsymbol{i}A_2\mathfrak{I}(X_2) \\ A_1\mathfrak{R}(X_2) + \boldsymbol{i}A_1\mathfrak{I}(X_2) + A_2\mathfrak{R}(X_1) + \boldsymbol{i}A_2\mathfrak{I}(X_1) \end{bmatrix} - \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \right\|_F \\
&= \left\| \begin{bmatrix} A_1 & \boldsymbol{i}A_1 & A_2 & \boldsymbol{i}A_2 \\ A_2 & \boldsymbol{i}A_2 & A_1 & \boldsymbol{i}A_1 \end{bmatrix} \begin{bmatrix} \mathfrak{R}(X_1) \\ \mathfrak{I}(X_1) \\ \mathfrak{R}(X_2) \\ \mathfrak{I}(X_2) \end{bmatrix} - \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \right\|_F .
\end{aligned}
$$

For a purely imaginary solution, we set $\mathfrak{R}(X_1) = 0$. For a real solution, we impose the conditions $\mathfrak{I}(X_1) = 0$, $\mathfrak{R}(X_2) = 0$, and $\mathfrak{I}(X_2) = 0$. Similarly, any RBME of the form (2.1.1), where B_l for $l = 1, 2, \ldots, r$ are identity matrices and $X_l = X$ for all $l = 1, 2, \ldots, r$, can be solved using the same method.

The remaining subsection focuses on addressing the least squares problem associated with matrix equations (2.1.2) and (2.1.3). This involves finding the least squares solutions for the following unconstrained real matrix system:

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \mathrm{vec}_L(\overrightarrow{X}) = e. \tag{2.4.10}$$

Let M_L be the basis matrix of $L^R(n, 4s)$. Using Lemma 2.3.17, we get $\mathrm{vec}(\overrightarrow{X})$ from $\mathrm{vec}_L(\overrightarrow{X})$ in the following way:

$$\mathrm{vec}(\overrightarrow{X}) = M_L \mathrm{vec}_L(\overrightarrow{X}).$$

The methodology for solving RBMEs of the form (2.1.2) and (2.1.3) remains the same as outlined in Subsection 2.4.1. Therefore, our focus here is solely on presenting the values for Q_1, Q_2, and e, while intentionally omitting the detailed results.

Linear Matrix Equation in One Unknown L-structure

Consider the matrix equation (2.1.2) and let $A_l = A_{l1} + A_{l2}\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}, B_l \in \mathbb{Q}_{\mathbb{R}}^{s\times t}$, $C_p = C_{p1} + C_{p2}\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times s}$, $D_p \in \mathbb{Q}_{\mathbb{R}}^{n\times t}$, $E = E_1 + E_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times t}$ for $l = 1, 2, \ldots, r$ and $p = 1, 2, \ldots, q$. Let

$$S := \left(\sum_{l=1}^{r} \left(h(B_l)^T \otimes A_{l1} + h(B_l\boldsymbol{j})^T \otimes A_{l2} \right) \right) \mathcal{W}_{ns} M_L,$$

$$N := \left(\sum_{p=1}^{q} \left(h(D_p)^T \otimes C_{p1} + h(D_p\boldsymbol{j})^T \otimes C_{p2} \right) \right) \mathcal{S}_{ns} \mathcal{W}_{ns} M_L.$$

Q_1, Q_2, and e (as in (2.4.10)) for solving RBME of the form (2.1.2) are in the following form:

$$Q_1 := \mathfrak{R}(S) + \mathfrak{R}(N), \quad Q_2 := \mathfrak{I}(S) + \mathfrak{I}(N), \quad \text{and} \quad e := \begin{bmatrix} \operatorname{vec}(\mathfrak{R}(\Psi_E)) \\ \operatorname{vec}(\mathfrak{I}(\Psi_E)) \end{bmatrix}.$$

Generalized Coupled Linear Matrix Equations in One Unknown L-structure

Consider the matrix equation (2.1.3) and let $A_l = A_{l1} + A_{l2}\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}, B_l \in \mathbb{Q}_{\mathbb{R}}^{s\times t}$, and $E_l = E_{l1} + E_{l2}\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times t}$ for $l = 1, 2, \ldots, r$. Let

$$T := \begin{bmatrix} h(B_1)^T \otimes A_{11} + h(B_1\boldsymbol{j})^T \otimes A_{12} \\ h(B_2)^T \otimes A_{21} + h(B_2\boldsymbol{j})^T \otimes A_{22} \\ \vdots \\ h(B_r)^T \otimes A_{r1} + h(B_r\boldsymbol{j})^T \otimes A_{r2} \end{bmatrix} \mathcal{W}_{ns} M_L, \quad z := \begin{bmatrix} \operatorname{vec}(\Psi_{E_1}) \\ \operatorname{vec}(\Psi_{E_2}) \\ \vdots \\ \operatorname{vec}(\Psi_{E_r}) \end{bmatrix}.$$

Q_1, Q_2, and e (as in (2.4.10)) for solving RBME of the form (2.1.3) are in the following form:

$$Q_1 := \mathfrak{R}(T), \quad Q_2 := \mathfrak{I}(T), \quad \text{and} \quad e := \begin{bmatrix} \mathfrak{R}(z) \\ \mathfrak{I}(z) \end{bmatrix}.$$

2.5. Solutions of Matrix Equation $\mathbf{AXB + CYD = E}$

We now apply the framework developed in Section 2.4 to specific cases, exploring how the theory can be utilized in various applications. These include the least squares purely imaginary solution of the RBME $AX = E$ and its application to the image restoration problem, L-structure solutions for complex matrix equations, L-structure solutions for real matrix equations, solving PDIEP, and the generalized PDIEP.

2.5.1. The Least Squares Solutions of $\mathbf{AX = E}$ for $\mathbf{X \in \mathbb{IQ}_{\mathbb{R}}^{n\times s}}$

Our discussion in this subsection focuses on the least squares purely imaginary reduced biquaternion solutions to the following RBME

$$AX = E. \tag{2.5.1}$$

Problem 2.5.1. *Let $A = A_1 + A_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ and $E = E_1 + E_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times s}$. Define*

$$\mathcal{I}_{LE} = \left\{ X \mid X = X_1 + X_2\boldsymbol{j} \in \mathbb{IQ}_{\mathbb{R}}^{n\times s}, \|AX - E\|_F = \min_{\widetilde{X} \in \mathbb{IQ}_{\mathbb{R}}^{n\times s}} \left\|A\widetilde{X} - E\right\|_F \right\}.$$

Find $X_I = X_{I1} + X_{I2}\boldsymbol{j} \in \mathcal{I}_{LE}$ *such that*

$$\|X_I\|_F = \min_{X \in \mathcal{I}_{LE}} \|X\|_F .$$

The following notations will be used to solve Problem 2.5.1. Define

$$V := \begin{bmatrix} \boldsymbol{i}A_1 & A_2 & \boldsymbol{i}A_2 \\ \boldsymbol{i}A_2 & A_1 & \boldsymbol{i}A_1 \end{bmatrix}. \tag{2.5.2}$$

Further Q_1, Q_2, x, and e (as in (2.4.1)) for this problem are given by:

$$Q_1 := \Re(V), \quad Q_2 := \Im(V), \quad x := \begin{bmatrix} \Im(X_1) \\ \Re(X_2) \\ \Im(X_2) \end{bmatrix}, \quad \text{and} \quad e := \begin{bmatrix} \Re(E_1) \\ \Re(E_2) \\ \Im(E_1) \\ \Im(E_2) \end{bmatrix}. \tag{2.5.3}$$

The following result provides the expression for the solution to Problem 2.5.1.

Theorem 2.5.2. *Let* $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ *and* $E \in \mathbb{Q}_{\mathbb{R}}^{m\times s}$. *Let* $Q_1, Q_2,$ *and* e *be as defined in* (2.5.3). *Then*

$$\mathcal{I}_{LE} = \left\{ X \mid \Re(X_1) = 0, \begin{bmatrix} \Im(X_1) \\ \Re(X_2) \\ \Im(X_2) \end{bmatrix} = \left[Q_1^\dagger - H^T Q_2 Q_1^\dagger, H^T\right] e + \left(I - Q_1^\dagger Q_1 - RR^\dagger\right) Y \right\},$$

where Y *is any matrix of suitable size. The unique solution* $X_I \in \mathcal{I}_{LE}$ *to Problem* 2.5.1 *satisfies*

$$\Re(X_{I1}) = 0, \quad \begin{bmatrix} \Im(X_{I1}) \\ \Re(X_{I2}) \\ \Im(X_{I2}) \end{bmatrix} = \left[Q_1^\dagger - H^T Q_2 Q_1^\dagger, H^T\right] e. \tag{2.5.4}$$

Proof. From Remark 2.4.7 and using the fact that $X \in \mathbb{IQ}_{\mathbb{R}}^{n\times s}$ if and only if $\Re(X_1) = 0$, we have

$$\|AX - E\|_F = \left\| \begin{bmatrix} \boldsymbol{i}A_1 & A_2 & \boldsymbol{i}A_2 \\ \boldsymbol{i}A_2 & A_1 & \boldsymbol{i}A_1 \end{bmatrix} \begin{bmatrix} \Im(X_1) \\ \Re(X_2) \\ \Im(X_2) \end{bmatrix} - \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \right\|_F = \left\| \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} x - e \right\|_F .$$

The rest of the proof follows the same approach as the proof of Theorem 2.4.5. ∎

The following theorem provides the condition for the matrix equation (2.5.1) to have a purely imaginary reduced biquaternion solution, along with an expression for this solution.

Theorem 2.5.3. *The RBME $AX = E$ has a purely imaginary reduced biquaternion solution $X \in \mathbb{IQ}_{\mathbb{R}}^{n\times s}$ if and only if*

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^{\dagger} e = e, \tag{2.5.5}$$

where Q_1, Q_2, and e are as defined in (2.5.3). In this case, the general solution, given by $X = \Im(X_1)\, \boldsymbol{i} + \Re(X_2)\, \boldsymbol{j} + \Im(X_2)\, \boldsymbol{k}$, satisfies

$$\begin{bmatrix} \Im(X_1) \\ \Re(X_2) \\ \Im(X_2) \end{bmatrix} = \left[Q_1^{\dagger} - H^T Q_2 Q_1^{\dagger}, H^T\right] e + \left(I - Q_1^{\dagger} Q_1 - RR^{\dagger}\right) Y,$$

where Y is any matrix of suitable size. Further, if the consistency condition is satisfied, then the RBME of the form (2.5.1) has a unique solution if and only if

$$\operatorname{rank}\left(\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}\right) = 3n.$$

In this case, the unique solution, given by $X = \Im(X_1)\, \boldsymbol{i} + \Re(X_2)\, \boldsymbol{j} + \Im(X_2)\, \boldsymbol{k}$, satisfies

$$\begin{bmatrix} \Im(X_1) \\ \Re(X_2) \\ \Im(X_2) \end{bmatrix} = \left[Q_1^{\dagger} - H^T Q_2 Q_1^{\dagger}, H^T\right] e.$$

Proof. Based on Remark 2.4.7 and the fact that $X \in \mathbb{IQ}_{\mathbb{R}}^{n\times s}$ if and only if $\Re(X_1) = 0$, we have

$$AX = E \iff \begin{bmatrix} \Re(V) \\ \Im(V) \end{bmatrix} x = e. \quad \blacksquare$$

Using the solution to Problem 2.5.1, we can restore color images. Each pixel in a color image is composed of three primary color components: red, green, and blue (RGB). These colors are interrelated, and their relationships must be preserved during the restoration process. In 2004, Pei *et al.* proposed that the red, green, and blue values of each pixel in a color image can be represented as a pure imaginary reduced biquaternion [56]. Thus, an $m \times n$ color image I can be represented as a pure imaginary reduced biquaternion matrix:

$$I = R\boldsymbol{i} + G\boldsymbol{j} + B\boldsymbol{k},$$

where R, G, and B are real matrices representing the red, green, and blue channels, respectively.

The linear discrete model of image restoration can be described by the matrix-vector equation [39]:

$$g = Kf + n,$$

where g is the observed (degraded) image, f is the true or ideal image, n is additive noise, and K is a matrix representing the blurring phenomena [34, 79]. Image restoration methods aim to construct an approximation of f based on g, K, and, in some cases, statistical information about the noise. In most cases, the noise n is unknown, and we seek to find the solution f_K such that:

$$\|n\| = \|Kf_K - g\| = \min_f \|Kf - g\|.$$

Since a color image can be represented as a pure imaginary reduced biquaternion matrix, the image restoration problem can be reformulated as finding the least squares purely imaginary reduced biquaternion solution to the matrix equation $Kf = g$.

2.5.2. Solutions of Matrix Equation $\mathbf{AXB + CYD = E}$ for $[\mathbf{X}, \mathbf{Y}] \in \mathbb{HC}^{\mathbf{n\times n}} \times \mathbb{HC}^{\mathbf{n\times n}}$

As a special case, we now discuss the Hankel solutions of the complex matrix equation

$$AXB + CYD = E, \tag{2.5.6}$$

where $A, C \in \mathbb{C}^{m\times n}$, $B, D \in \mathbb{C}^{n\times s}$, and $E \in \mathbb{C}^{m\times s}$. The following notations will be used to solve matrix equation (2.5.6). Define

$$W := \left(B^T \otimes A\right) [I_{n^2}, \boldsymbol{i} I_{n^2}] \begin{bmatrix} K_H & 0 \\ 0 & K_H \end{bmatrix}, \quad J := \left(D^T \otimes C\right) [I_{n^2}, \boldsymbol{i} I_{n^2}] \begin{bmatrix} K_H & 0 \\ 0 & K_H \end{bmatrix}. \tag{2.5.7}$$

Further Q_1, Q_2, x, and e (as in (2.4.1)) are given in the form:

$$\begin{gathered} Q_1 := [\Re(W), \Re(J)], \quad Q_2 := [\Im(W), \Im(J)], \\ x := \begin{bmatrix} \mathrm{vec}_H(\Re(X)) \\ \mathrm{vec}_H(\Im(X)) \\ \mathrm{vec}_H(\Re(Y)) \\ \mathrm{vec}_H(\Im(Y)) \end{bmatrix}, \quad \text{and} \quad e := \begin{bmatrix} \mathrm{vec}(\Re(E)) \\ \mathrm{vec}(\Im(E)) \end{bmatrix}. \end{gathered} \tag{2.5.8}$$

Using (2.3.7), (2.5.7), and Lemma 2.3.12, we have

$$\begin{aligned}
\mathrm{vec}(AXB) &= (B^T \otimes A)\mathrm{vec}(X) \\
&= (B^T \otimes A)\left(\mathrm{vec}(\Re(X)) + \boldsymbol{i}\,\mathrm{vec}(\Im(X))\right) \\
&= (B^T \otimes A)\left[I_{n^2}, \boldsymbol{i} I_{n^2}\right]\begin{bmatrix}\mathrm{vec}(\Re(X)) \\ \mathrm{vec}(\Im(X))\end{bmatrix} \\
&= (B^T \otimes A)\left[I_{n^2}, \boldsymbol{i} I_{n^2}\right]\begin{bmatrix}K_H & 0 \\ 0 & K_H\end{bmatrix}\begin{bmatrix}\mathrm{vec}_H(\Re(X)) \\ \mathrm{vec}_H(\Im(X))\end{bmatrix} \\
&= W\begin{bmatrix}\mathrm{vec}_H(\Re(X)) \\ \mathrm{vec}_H(\Im(X))\end{bmatrix}.
\end{aligned}$$

Similarly,

$$\mathrm{vec}(CYD) = (D^T \otimes C)\left[I_{n^2}, \boldsymbol{i} I_{n^2}\right]\begin{bmatrix}K_H & 0 \\ 0 & K_H\end{bmatrix}\begin{bmatrix}\mathrm{vec}_H(\Re(Y)) \\ \mathrm{vec}_H(\Im(Y))\end{bmatrix} = J\begin{bmatrix}\mathrm{vec}_H(\Re(Y)) \\ \mathrm{vec}_H(\Im(Y))\end{bmatrix}.$$

Using the expressions for $\mathrm{vec}(AXB)$ and $\mathrm{vec}(CYD)$ along with equation (2.5.8) and Lemma 2.4.2, we obtain

$$\begin{aligned}
AXB + CYD = E &\Leftrightarrow \mathrm{vec}(AXB) + \mathrm{vec}(CYD) = \mathrm{vec}(E) \\
&\Leftrightarrow W\begin{bmatrix}\mathrm{vec}_H(\Re(X)) \\ \mathrm{vec}_H(\Im(X))\end{bmatrix} + J\begin{bmatrix}\mathrm{vec}_H(\Re(Y)) \\ \mathrm{vec}_H(\Im(Y))\end{bmatrix} = \mathrm{vec}(E) \\
&\Leftrightarrow [W, J]\begin{bmatrix}\mathrm{vec}_H(\Re(X)) \\ \mathrm{vec}_H(\Im(X)) \\ \mathrm{vec}_H(\Re(Y)) \\ \mathrm{vec}_H(\Im(Y))\end{bmatrix} = \mathrm{vec}(E) \\
&\Leftrightarrow \begin{bmatrix}\Re(W) & \Re(J) \\ \Im(W) & \Im(J)\end{bmatrix} x = \begin{bmatrix}\mathrm{vec}(\Re(E)) \\ \mathrm{vec}(\Im(E))\end{bmatrix}.
\end{aligned}$$

Finally, using Q_1, Q_2, and e from (2.5.8), we rewrite the equation as

$$AXB + CYD = E \Leftrightarrow \begin{bmatrix}Q_1 \\ Q_2\end{bmatrix} x = e.$$

Hence, matrix equation $AXB + CYD = E$ for $[X, Y] \in \mathbb{HC}^{n\times n} \times \mathbb{HC}^{n\times n}$ can be solved by solving the following unconstrained real matrix system:

$$\begin{bmatrix}Q_1 \\ Q_2\end{bmatrix} x = e.$$

By Lemma 2.3.12, we have

$$\begin{bmatrix} \mathrm{vec}(\Re(X)) \\ \mathrm{vec}(\Im(X)) \\ \mathrm{vec}(\Re(Y)) \\ \mathrm{vec}(\Im(Y)) \end{bmatrix} = \begin{bmatrix} K_H & 0 & 0 & 0 \\ 0 & K_H & 0 & 0 \\ 0 & 0 & K_H & 0 \\ 0 & 0 & 0 & K_H \end{bmatrix} x.$$

2.5.3. Solutions of Matrix Equation $\mathbf{AXB + CYD = E}$ for $[\mathbf{X}, \mathbf{Y}] \in \mathbb{STR}^{\mathbf{n \times n}} \times \mathbb{STR}^{\mathbf{n \times n}}$

As a special case, we now discuss the symmetric Toeplitz solutions of the real matrix equation

$$AXB + CYD = E, \tag{2.5.9}$$

where $A, C \in \mathbb{R}^{m \times n}$, $B, D \in \mathbb{R}^{n \times s}$, and $E \in \mathbb{R}^{m \times s}$. Using (2.3.7) and Lemma 2.3.11, we have

$$\begin{aligned} AXB + CYD = E &\Leftrightarrow \mathrm{vec}(AXB) + \mathrm{vec}(CYD) = \mathrm{vec}(E) \\ &\Leftrightarrow \left(B^T \otimes A\right) \mathrm{vec}(X) + \left(D^T \otimes C\right) \mathrm{vec}(Y) = \mathrm{vec}(E) \\ &\Leftrightarrow \left(B^T \otimes A\right) K_{ST} \mathrm{vec}_{ST}(X) + \left(D^T \otimes C\right) K_{ST} \mathrm{vec}_{ST}(Y) = \mathrm{vec}(E) \\ &\Leftrightarrow \left[\left(B^T \otimes A\right) K_{ST}, \left(D^T \otimes C\right) K_{ST}\right] \begin{bmatrix} \mathrm{vec}_{ST}(X) \\ \mathrm{vec}_{ST}(Y) \end{bmatrix} = \mathrm{vec}(E). \end{aligned}$$

Hence, matrix equation $AXB + CYD = E$ for $[X, Y] \in \mathbb{STR}^{n \times n} \times \mathbb{STR}^{n \times n}$ can be solved by solving the following unconstrained real matrix system:

$$\widetilde{Q} x = \widetilde{e},$$

where $\widetilde{Q} = [(B^T \otimes A) K_{ST}, (D^T \otimes C) K_{ST}]$, $x = \begin{bmatrix} \mathrm{vec}_{ST}(X) \\ \mathrm{vec}_{ST}(Y) \end{bmatrix}$, and $\widetilde{e} = \mathrm{vec}(E)$. Using Lemma 2.3.11, we have

$$\begin{bmatrix} \mathrm{vec}(X) \\ \mathrm{vec}(Y) \end{bmatrix} = \begin{bmatrix} K_{ST} & 0 \\ 0 & K_{ST} \end{bmatrix} x.$$

2.5.4. PDIEP and Generalized PDIEP

In this subsection, we aim to demonstrate the application of our developed framework in solving a range of inverse problems. Here, we develop a numerical solution methodology for the inverse problems in which the spectral constraints involve only a few eigenpair information rather than the entire spectrum. Mathematically, the problem statement is as follows:

Problem 2.5.4 (PDIEP). *Given vectors* $\{u_1, u_2, \ldots, u_k\} \subset \mathbb{F}^n$ $(k \le n)$, *values* $\{\lambda_1, \lambda_2, \ldots, \lambda_k\} \subset \mathbb{F}$, *and a set* $\mathcal{L}$ *of structured matrices, find a matrix* $M \in \mathcal{L}$ *such that*

$$Mu_i = \lambda_i u_i, \quad i = 1, 2, \ldots, k.$$

To simplify the discussion, we will use the matrix pair (Λ, Φ) to describe partial eigenpair information, where

$$\begin{aligned} \Lambda &= \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{F}^{k\times k}, \\ \Phi &= [u_1, u_2, \ldots, u_k] \in \mathbb{F}^{n\times k}. \end{aligned} \tag{2.5.10}$$

PDIEP can be written as $M\Phi = \Phi\Lambda$. By using the transformations

$$\begin{aligned} &A = I_n, \quad X = M, \\ &B = \Phi, \quad \text{and} \quad E = \Phi\Lambda, \end{aligned}$$

we can find solution to PDIEP by solving matrix equation $AXB = E$ for $X \in \mathcal{L}$.

Next, we investigate generalized PDIEPs. In a nutshell, the problem is:

Problem 2.5.5 (Generalized PDIEP). *Given vectors* $\{u_1, u_2, \ldots, u_k\} \subset \mathbb{F}^n$ $(k \le n)$, *values* $\{\lambda_1, \lambda_2, \ldots, \lambda_k\} \subset \mathbb{F}$, *and a set* $\mathcal{L}$ *of structured matrices, find pair of matrices* $M, N \in \mathcal{L}$ *such that*

$$Mu_i = \lambda_i N u_i, \quad i = 1, 2, \ldots, k.$$

Generalized PDIEP can be written as $M\Phi = N\Phi\Lambda$, where Λ and Φ are as in (2.5.10). By using the transformations

$$\begin{aligned} &A = I_n, \quad X = M, \quad B = \Phi, \quad C = -I_n, \\ &Y = N, \quad D = \Phi\Lambda, \quad \text{and} \quad E = 0, \end{aligned}$$

we can find solution to Generalized PDIEP by solving matrix equation $AXB + CYD = E$ for $X, Y \in \mathcal{L}$.

Though the primary emphasis of this paper is on inverse problems having symmetric Toeplitz or Hankel structures, the overall approach can be extended to encompass any structures where any set of linear relationships among matrix entries is permissible.

2.6. Numerical Verification

In this section, we present numerical examples to validate our proposed results. All computations are performed on an Intel Core i7-9700 @3.00GHz with 16GB RAM using MATLAB R2021b. Eight numerical examples are provided, each highlighting a different aspect of our framework.

We begin by computing the error between the least squares Toeplitz solution of the RBME $AXB+CYD=E$ obtained using our method and the corresponding exact solution. Additionally, we evaluate the error between the least squares Hankel solution of the RBME $(AXB, CXD)=(E,F)$ computed using our approach and its exact counterpart.

Our method is further applied to solve an image restoration problem, demonstrating its practical effectiveness. We also investigate the PDIEP for a Hankel and symmetric Toeplitz matrix, followed by an analysis of the generalized PDIEP for Hankel and symmetric Toeplitz structures.

Finally, we compare our approach for computing least squares Toeplitz solutions of the RBME $X+AXB=C$ with the method presented in [71], highlighting key differences and improvements.

We now provide an example for finding the structure-constrained least squares solution to the RBME of the form (2.1.1).

Example 2.6.1. *Consider the following matrices:*

$$A=\texttt{rand}(4,5)+\texttt{rand}(4,5)\boldsymbol{j}, \quad B=\texttt{rand}(5,7)+\texttt{rand}(5,7)\boldsymbol{j},$$
$$C=\texttt{ones}(4,5)+\texttt{rand}(4,5)\boldsymbol{j}, \quad D=\texttt{rand}(5,7)+\texttt{ones}(5,7)\boldsymbol{j}.$$

Let the column and row vectors for the Toeplitz matrices be defined as:

$$c_1=[\boldsymbol{i},\ 2+\boldsymbol{i},\ 0,\ 1,\ \boldsymbol{i}], \quad r_1=[\boldsymbol{i},\ 0,\ 2\boldsymbol{i},\ 1,\ 1+\boldsymbol{i}],$$
$$c_2=[1,\ 3\boldsymbol{i},\ 2+3\boldsymbol{i},\ 1,\ 0], \quad r_2=[1,\ 0,\ 1,\ \boldsymbol{i},\ 2].$$

Define the reduced biquaternion matrix $\widetilde{X}=\widetilde{X}_1+\widetilde{X}_2\boldsymbol{j}$*, where* $\widetilde{X}_1=\texttt{toeplitz}(c_1,r_1)$ *and* $\widetilde{X}_2=\texttt{toeplitz}(c_2,r_2)$*.*

Similarly, let the column and row vectors for another Toeplitz matrix be:

$$c_3 = [2 + \boldsymbol{i},\, 4,\, \boldsymbol{i},\, 1 + 3\boldsymbol{i},\, 2\boldsymbol{i}], \quad r_3 = [2 + \boldsymbol{i},\, 7 + 6\boldsymbol{i},\, 3 + 2\boldsymbol{i},\, \boldsymbol{i},\, 1 + \boldsymbol{i}],$$
$$c_4 = [1 + 3\boldsymbol{i},\, 3\boldsymbol{i},\, 2 + 3\boldsymbol{i},\, 3,\, 5 + \boldsymbol{i}], \quad r_4 = [1 + 3\boldsymbol{i},\, 5,\, 1 + 6\boldsymbol{i},\, 3 + \boldsymbol{i},\, 2\boldsymbol{i}].$$

Define the reduced biquaternion matrix $\widetilde{Y} = \widetilde{Y}_1 + \widetilde{Y}_2\boldsymbol{j}$*, where* $\widetilde{Y}_1 = \texttt{toeplitz}(c_3, r_3)$ *and* $\widetilde{Y}_2 = \texttt{toeplitz}(c_4, r_4)$*. Let*

$$E = A\widetilde{X}B + C\widetilde{Y}D.$$

Thus, $\left[\widetilde{X}, \widetilde{Y}\right]$ *is the least squares Toeplitz solution with the least norm of the RBME* $AXB + CYD = E$*.*

Now, we take the matrices $A, B, C, D,$ *and* E *as input to compute the least squares Toeplitz solution with the least norm for the RBME* $AXB + CYD = E$*. We obtain the matrices* $X = X_1 + X_2\boldsymbol{j}$ *and* $Y = Y_1 + Y_2\boldsymbol{j}$*, where*

$$X_1 = \begin{bmatrix} 0+1\boldsymbol{i} & 0-0\boldsymbol{i} & 0+2\boldsymbol{i} & 1-0\boldsymbol{i} & 1+1\boldsymbol{i} \\ 2+1\boldsymbol{i} & 0+1\boldsymbol{i} & 0-0\boldsymbol{i} & 0+2\boldsymbol{i} & 1-0\boldsymbol{i} \\ 0-0\boldsymbol{i} & 2+1\boldsymbol{i} & 0+1\boldsymbol{i} & 0-0\boldsymbol{i} & 0+2\boldsymbol{i} \\ 1+0\boldsymbol{i} & 0-0\boldsymbol{i} & 2+1\boldsymbol{i} & 0+1\boldsymbol{i} & 0-0\boldsymbol{i} \\ 0+1\boldsymbol{i} & 1+0\boldsymbol{i} & 0-0\boldsymbol{i} & 2+1\boldsymbol{i} & 0+1\boldsymbol{i} \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1-0\boldsymbol{i} & 0+0\boldsymbol{i} & 1-0\boldsymbol{i} & 0+1\boldsymbol{i} & 2-0\boldsymbol{i} \\ 0+3\boldsymbol{i} & 1-0\boldsymbol{i} & 0+0\boldsymbol{i} & 1-0\boldsymbol{i} & 0+1\boldsymbol{i} \\ 2+3\boldsymbol{i} & 0+3\boldsymbol{i} & 1-0\boldsymbol{i} & 0+0\boldsymbol{i} & 1-0\boldsymbol{i} \\ 1+0\boldsymbol{i} & 2+3\boldsymbol{i} & 0+3\boldsymbol{i} & 1-0\boldsymbol{i} & 0+0\boldsymbol{i} \\ 0+0\boldsymbol{i} & 1+0\boldsymbol{i} & 2+3\boldsymbol{i} & 0+3\boldsymbol{i} & 1-0\boldsymbol{i} \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} 2+1\boldsymbol{i} & 7+6\boldsymbol{i} & 3+2\boldsymbol{i} & 0+1\boldsymbol{i} & 1+1\boldsymbol{i} \\ 4+0\boldsymbol{i} & 2+1\boldsymbol{i} & 7+6\boldsymbol{i} & 3+2\boldsymbol{i} & 0+1\boldsymbol{i} \\ 0+1\boldsymbol{i} & 4+0\boldsymbol{i} & 2+1\boldsymbol{i} & 7+6\boldsymbol{i} & 3+2\boldsymbol{i} \\ 1+3\boldsymbol{i} & 0+1\boldsymbol{i} & 4+0\boldsymbol{i} & 2+1\boldsymbol{i} & 7+6\boldsymbol{i} \\ 0+2\boldsymbol{i} & 1+3\boldsymbol{i} & 0+1\boldsymbol{i} & 4+0\boldsymbol{i} & 2+1\boldsymbol{i} \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 1+3\boldsymbol{i} & 5+0\boldsymbol{i} & 1+6\boldsymbol{i} & 3+1\boldsymbol{i} & 0+2\boldsymbol{i} \\ 0+3\boldsymbol{i} & 1+3\boldsymbol{i} & 5+0\boldsymbol{i} & 1+6\boldsymbol{i} & 3+1\boldsymbol{i} \\ 2+3\boldsymbol{i} & 0+3\boldsymbol{i} & 1+3\boldsymbol{i} & 5+0\boldsymbol{i} & 1+6\boldsymbol{i} \\ 3+0\boldsymbol{i} & 2+3\boldsymbol{i} & 0+3\boldsymbol{i} & 1+3\boldsymbol{i} & 5+0\boldsymbol{i} \\ 5+1\boldsymbol{i} & 3+0\boldsymbol{i} & 2+3\boldsymbol{i} & 0+3\boldsymbol{i} & 1+3\boldsymbol{i} \end{bmatrix}.$$

Clearly, X *and* Y *are reduced biquaternion Toeplitz matrices. The error is given by* $\epsilon = \left\| [X, Y] - \left[\widetilde{X}, \widetilde{Y}\right] \right\|_F = 1.7470 \times 10^{-13}$*.*

From Example 2.6.1, we find that the error ϵ is in the order of 10^{-13} and is negligible. This demonstrates the effectiveness of our method in determining the structure-constrained least squares solution to the RBME of the form (2.1.1).

To further illustrate, we now provide an example for finding the structure-constrained least squares solution to the RBME of the form (2.1.3).

Example 2.6.2. *Consider the following matrices:*

$$A = \texttt{ones}(4,5) + \texttt{rand}(4,5)\boldsymbol{j}, \quad B = \texttt{ones}(5,7) + \texttt{rand}(5,7)\boldsymbol{j},$$

$$C = \texttt{rand}(4,5) + \texttt{rand}(4,5)\boldsymbol{j}, \quad D = \texttt{ones}(5,7) + \texttt{rand}(5,7)\boldsymbol{j}.$$

Let the column and row vectors for the Hankel matrices be defined as:

$$c_1 = [3+\boldsymbol{i}, 2+4\boldsymbol{i}, 6+\boldsymbol{i}, 2+\boldsymbol{i}, 3\boldsymbol{i}], \quad r_1 = [3\boldsymbol{i}, 7, 3+2\boldsymbol{i}, 1+\boldsymbol{i}, 9+\boldsymbol{i}],$$

$$c_2 = [1+2\boldsymbol{i}, 5+3\boldsymbol{i}, 3\boldsymbol{i}, 1+7\boldsymbol{i}, 3], \quad r_2 = [3, 1+\boldsymbol{i}, 2+8\boldsymbol{i}, 2+\boldsymbol{i}, 2+2\boldsymbol{i}].$$

Define the reduced biquaternion matrix $\widetilde{X} = \widetilde{X}_1 + \widetilde{X}_2\boldsymbol{j}$, *where* $\widetilde{X}_1 = \texttt{hankel}(c_1, r_1)$ *and* $\widetilde{X}_2 = \texttt{hankel}(c_2, r_2)$. *Next, let*

$$E = A\widetilde{X}B, \quad F = C\widetilde{X}D.$$

Thus, $\widetilde{X}$ *is the least squares Hankel solution with the least norm of the RBME* $(AXB, CXD) = (E, F)$.

Now, we take the matrices A, B, C, D, E, *and* F *as input to compute the least squares Hankel solution with the least norm for the RBME* $(AXB, CXD) = (E, F)$. *We obtain the matrix* $X = X_1 + X_2\boldsymbol{j}$, *where*

$$X_1 = \begin{bmatrix} 3+1\boldsymbol{i} & 2+4\boldsymbol{i} & 6+1\boldsymbol{i} & 2+1\boldsymbol{i} & 0+3\boldsymbol{i} \\ 2+4\boldsymbol{i} & 6+1\boldsymbol{i} & 2+1\boldsymbol{i} & 0+3\boldsymbol{i} & 7+0\boldsymbol{i} \\ 6+1\boldsymbol{i} & 2+1\boldsymbol{i} & 0+3\boldsymbol{i} & 7+0\boldsymbol{i} & 3+2\boldsymbol{i} \\ 2+1\boldsymbol{i} & 0+3\boldsymbol{i} & 7+0\boldsymbol{i} & 3+2\boldsymbol{i} & 1+1\boldsymbol{i} \\ 0+3\boldsymbol{i} & 7+0\boldsymbol{i} & 3+2\boldsymbol{i} & 1+1\boldsymbol{i} & 9+1\boldsymbol{i} \end{bmatrix},$$

and

$$X_2 = \begin{bmatrix} 1+2\boldsymbol{i} & 5+3\boldsymbol{i} & 0+3\boldsymbol{i} & 1+7\boldsymbol{i} & 3+0\boldsymbol{i} \\ 5+3\boldsymbol{i} & 0+3\boldsymbol{i} & 1+7\boldsymbol{i} & 3+0\boldsymbol{i} & 1+1\boldsymbol{i} \\ 0+3\boldsymbol{i} & 1+7\boldsymbol{i} & 3+0\boldsymbol{i} & 1+1\boldsymbol{i} & 2+8\boldsymbol{i} \\ 1+7\boldsymbol{i} & 3+0\boldsymbol{i} & 1+1\boldsymbol{i} & 2+8\boldsymbol{i} & 2+1\boldsymbol{i} \\ 3+0\boldsymbol{i} & 1+1\boldsymbol{i} & 2+8\boldsymbol{i} & 2+1\boldsymbol{i} & 2+2\boldsymbol{i} \end{bmatrix}.$$

Clearly, X *is a reduced biquaternion Hankel matrix. The error is given by* $\epsilon = \left\|X - \widetilde{X}\right\|_F = 5.7042 \times 10^{-13}$, *which is negligibly small.*

From Example 2.6.2, we find that the error ϵ is in the order of 10^{-13} and is negligible. This demonstrates the effectiveness of our method in determining the structure-constrained least squares solution to the RBME of the form (2.1.3).

To illustrate the practical application of our proposed framework, we present the following example.

Example 2.6.3. *Figure 2.6.1(a) shows a* 480×500 *color image* I. *The reduced biquaternion matrix representation of* I *is given by* $F = R\,\boldsymbol{i} + G\,\boldsymbol{j} + B\,\boldsymbol{k}$, *where* R, G, *and* B *are the real matrices corresponding to the red, green, and blue channels, respectively. The matrix* F *represents the image matrix of the original image* I.

To simulate a distorted image, the red channel matrix R *is disturbed using the parameters* $len = 30$ *and* $theta = 60$, *with the MATLAB function* `fspecial('motion', len, theta)` *to generate the disturbed matrix* R_D. *The corresponding blurring matrix* $K = R_d R^{\dagger}$ *is then used to disturb the green and blue channel matrices, obtaining* $G_D = KG$ *and* $B_D = KB$. *Consequently, the disturbed image matrix becomes* $F_D = R_D\,\boldsymbol{i} + G_D\,\boldsymbol{j} + B_D\,\boldsymbol{k}$, *and the corresponding distorted image* I_D *is shown in Figure 2.6.1(b).*

(a) (b) (c)

Figure 2.6.1. (a) Original image I (b) Distorted image I_D (c) Restored image I_R

Now, we take matrices K and F_D as input to compute the least squares purely imaginary reduced biquaternion solution F' of the matrix equation $KX = F_D$. This is done by solving:

$$\|KF' - F_D\| = \min_X \|KX - F_D\|.$$

The solution F' is the image matrix corresponding to the recovered image after the restoration process, and the recovered image I_R is shown in Figure 2.6.1(c). The computed error is $\epsilon = \|F' - F\| = 3.1014 \times 10^{-7}$, indicating that the error is negligible, as it is of the order 10^{-7}. This demonstrates the effectiveness of Theorem 2.5.2 in solving Problem 2.5.1.

Next, we will discuss Hankel PDIEPs [12, Problem 5.1]. Given a set of vectors $\{u_1, u_2, \ldots, u_k\} \subset \mathbb{C}^n$, where $k \geq 1$, and a set of numbers $\{\lambda_1, \lambda_2, \ldots, \lambda_k\} \subset \mathbb{C}$, our aim is to construct a Hankel matrix $M \in \mathbb{C}^{n\times n}$ satisfying $Mu_i = \lambda_i u_i$ for $i = 1, 2, \ldots, k$. Now, we will illustrate this problem with an example.

Example 2.6.4. *To establish test data, we first generate a Hankel matrix M. Define $M = \texttt{hankel}(c, r)$, where*

$$c = [1 + 2\boldsymbol{i},\ 2 - 4\boldsymbol{i},\ -1 + 3\boldsymbol{i},\ 4], \quad r = [4,\ 3 + 4\boldsymbol{i},\ 2\boldsymbol{i},\ 3].$$

Let (Λ, Φ) denote the eigenpairs of M, where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_4) \in \mathbb{C}^{4\times 4}$ and $\Phi = [u_1, u_2, u_3, u_4] \in \mathbb{C}^{4\times 4}$. The eigenvalues are

$$[\lambda_1, \lambda_2, \lambda_3, \lambda_4] = [-3.8029 + 7.9250\boldsymbol{i},\ -2.7826 - 3.5629\boldsymbol{i},\ 5.6954 - 1.0619\boldsymbol{i},\ 6.8900 + 5.6998\boldsymbol{i}],$$

and the matrix Φ of eigenvectors is given by

$$\Phi = \begin{bmatrix} 0.6240 + 0.0000\boldsymbol{i} & -0.4940 - 0.0377\boldsymbol{i} & -0.5395 - 0.2011\boldsymbol{i} & 0.1572 - 0.2047\boldsymbol{i} \\ -0.6145 - 0.0885\boldsymbol{i} & -0.5863 + 0.0219\boldsymbol{i} & 0.0172 - 0.1236\boldsymbol{i} & 0.4818 - 0.1113\boldsymbol{i} \\ 0.4246 + 0.0774\boldsymbol{i} & 0.1217 - 0.1368\boldsymbol{i} & 0.5855 + 0.0000\boldsymbol{i} & 0.6784 + 0.0000\boldsymbol{i} \\ -0.1893 + 0.0550\boldsymbol{i} & 0.6138 + 0.0000\boldsymbol{i} & -0.5259 - 0.1832\boldsymbol{i} & 0.4609 - 0.1275\boldsymbol{i} \end{bmatrix}.$$

***Case* 1.** *Reconstruction from one eigenpair ($k = 1$): Let the prescribed partial eigeninformation be given by*

$$\widetilde{\Lambda} = \lambda_3 \in \mathbb{C}, \quad \widetilde{\Phi} = u_3 \in \mathbb{C}^{4\times 1}.$$

Construct the Hankel matrix $\widetilde{M}$ such that $\widetilde{M}u_3 = \lambda_3 u_3$. Using the transformations $A = I_4$, $X = \widetilde{M}$, $B = \widetilde{\Phi}$, and $E = \widetilde{\Phi}\widetilde{\Lambda}$, we solve the matrix equation $AXB = E$ to obtain

$$\widetilde{M} = \begin{bmatrix} 1.6614 + 0.3115\boldsymbol{i} & 1.0564 + 0.6597\boldsymbol{i} & -1.8088 + 0.4921\boldsymbol{i} & 2.6736 - 0.4763\boldsymbol{i} \\ 1.0564 + 0.6597\boldsymbol{i} & -1.8088 + 0.4921\boldsymbol{i} & 2.6736 - 0.4763\boldsymbol{i} & 2.0823 - 0.5222\boldsymbol{i} \\ -1.8088 + 0.4921\boldsymbol{i} & 2.6736 - 0.4763\boldsymbol{i} & 2.0823 - 0.5222\boldsymbol{i} & -1.7415 + 0.7505\boldsymbol{i} \\ 2.6736 - 0.4763\boldsymbol{i} & 2.0823 - 0.5222\boldsymbol{i} & -1.7415 + 0.7505\boldsymbol{i} & 1.2459 + 0.2833\boldsymbol{i} \end{bmatrix}.$$

Thus, $\widetilde{M}$ is the desired Hankel matrix.

***Case* 2.** *Reconstruction from two eigenpairs ($k = 2$): Let the prescribed partial eigeninformation be given by*

$$\widetilde{\Lambda} = \mathrm{diag}(\lambda_2, \lambda_3) \in \mathbb{C}^{2\times 2}, \quad \widetilde{\Phi} = [u_2, u_3] \in \mathbb{C}^{4\times 2}.$$

*Construct the Hankel matrix $\widetilde{M}$ such that $\widetilde{M}u_i = \lambda_i u_i$ for $i = 2,3$. Using the same transformations as in **Case 1**, we solve the matrix equation $AXB = E$ and obtain*

$$\widetilde{M} = \begin{bmatrix} 1.0000 + 2.0000\boldsymbol{i} & 2.0000 - 4.0000\boldsymbol{i} & -1.0000 + 3.0000\boldsymbol{i} & 4.0000 + 0.0000\boldsymbol{i} \\ 2.0000 - 4.0000\boldsymbol{i} & -1.0000 + 3.0000\boldsymbol{i} & 4.0000 + 0.0000\boldsymbol{i} & 3.0000 + 4.0000\boldsymbol{i} \\ -1.0000 + 3.0000\boldsymbol{i} & 4.0000 + 0.0000\boldsymbol{i} & 3.0000 + 4.0000\boldsymbol{i} & 0.0000 + 2.0000\boldsymbol{i} \\ 4.0000 + 0.0000\boldsymbol{i} & 3.0000 + 4.0000\boldsymbol{i} & 0.0000 + 2.0000\boldsymbol{i} & 3.0000 + 0.0000\boldsymbol{i} \end{bmatrix}.$$

Thus, $\widetilde{M}$ is the desired Hankel matrix.

***Case* 1 (k = 1)**		***Case* 2 (k = 2)**	
Eigenpair	*Residual* $\left\lVert \widetilde{M}u_i - \lambda_i u_i \right\rVert_2$	*Eigenpairs*	*Residual* $\left\lVert \widetilde{M}u_i - \lambda_i u_i \right\rVert_2$
(λ_3, u_3)	2.7792×10^{-15}	(λ_2, u_2)	3.1349×10^{-14}
		(λ_3, u_3)	2.2761×10^{-14}

Table 2.6.1. Residual $\left\lVert \widetilde{M}u_i - \lambda_i u_i \right\rVert_2$ for Example 2.6.4.

From Table 2.6.1, we find that the residual $\left\lVert \widetilde{M}u_i - \lambda_i u_i \right\rVert_2$ for $i = 3$ in **Case 1** and for $i = 2,3$ in **Case 2** is on the order of 10^{-14}, which is negligible. This demonstrates the effectiveness of our method in solving the Hankel PDIEP.

Next, we illustrate the example for solving the symmetrix Toeplitz PDIEP.

Example 2.6.5. *To establish test data, we first generate a real symmetric Toeplitz matrix T. Define $T = \texttt{toeplitz}(c)$, where*

$$c = [5.30,\ 2.50,\ 4.60,\ -3.70,\ 2.80].$$

Let (Λ, Φ) denote the eigenpairs of T, where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_5) \in \mathbb{R}^{5\times5}$ and $\Phi = [u_1, u_2, u_3, u_4, u_5] \in \mathbb{R}^{5\times5}$. The eigenvalues are

$$[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5] = [-4.6650,\ -1.0842,\ 7.8650,\ 10.4951,\ 13.8891],$$

and the matrix Φ of eigenvectors is given by

$$\Phi = \begin{bmatrix} 0.4627 & 0.4077 & 0.5347 & -0.3460 & -0.4627 \\ -0.5347 & 0.2169 & 0.4627 & 0.6165 & -0.2699 \\ -0.0000 & -0.7573 & 0.0000 & -0.0193 & -0.6528 \\ 0.5347 & 0.2169 & -0.4627 & 0.6165 & -0.2699 \\ -0.4627 & 0.4077 & -0.5347 & 0.3460 & -0.4627 \end{bmatrix}.$$

Case 1. *Reconstruction from two eigenpairs* $(k = 2)$*: Let the prescribed partial eigen-information be given by*

$$\widetilde{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2) \in \mathbb{R}^{2\times 2}, \quad \widetilde{\Phi} = [u_1, u_2] \in \mathbb{R}^{5\times 2}.$$

We construct the symmetric Toeplitz matrix $\widetilde{T}$ *such that* $\widetilde{T}u_i = \lambda_i u_i$ *for* $i = 1, 2$*. Using the transformations* $A = I_5$*,* $X = \widetilde{T}$*,* $B = \widetilde{\Phi}$*, and* $E = \widetilde{\Phi}\widetilde{\Lambda}$*, we solve the matrix equation* $AXB = E$ *to obtain:*

$$\widetilde{T} = \begin{bmatrix} 5.30 & 2.50 & 4.60 & -3.70 & 2.80 \\ 2.50 & 5.30 & 2.50 & 4.60 & -3.70 \\ 4.60 & 2.50 & 5.30 & 2.50 & 4.60 \\ -3.70 & 4.60 & 2.50 & 5.30 & 2.50 \\ 2.80 & -3.70 & 4.60 & 2.50 & 5.30 \end{bmatrix}.$$

Thus, $\widetilde{T}$ *is the desired symmetric Toeplitz matrix.*

Case 2. *Reconstruction from two eigenpairs* $(k = 2)$*: Let the prescribed partial eigen-information be given by*

$$\widetilde{\Lambda} = diag(\lambda_1, \lambda_3) \in \mathbb{R}^{2\times 2}, \quad \widetilde{\Phi} = [u_1, u_3] \in \mathbb{R}^{5\times 2}.$$

We construct the symmetric Toeplitz matrix $\widetilde{T}$ *such that* $\widetilde{T}u_i = \lambda_i u_i$ *for* $i = 1, 3$*. Following the same approach as in* ***Case 1****, we solve the matrix equation* $AXB = E$ *and obtain:*

$$\widetilde{T} = \begin{bmatrix} 1.0667 & 3.1000 & 0.3667 & -3.1000 & -1.4333 \\ 3.1000 & 1.0667 & 3.1000 & 0.3667 & -3.1000 \\ 0.3667 & 3.1000 & 1.0667 & 3.1000 & 0.3667 \\ -3.1000 & 0.3667 & 3.1000 & 1.0667 & 3.1000 \\ -1.4333 & -3.1000 & 0.3667 & 3.1000 & 1.0667 \end{bmatrix}.$$

Thus, $\widetilde{T}$ *is the desired symmetric Toeplitz matrix.*

Case 1 $(k = 2)$		***Case 2*** $(k = 2)$	
Eigenpairs	*Residual* $\lVert\widetilde{T}u_i - \lambda_i u_i\rVert_2$	*Eigenpairs*	*Residual* $\lVert\widetilde{T}u_i - \lambda_i u_i\rVert_2$
(λ_1, u_1)	5.7430×10^{-15}	(λ_1, u_1)	2.2505×10^{-15}
(λ_2, u_2)	1.2200×10^{-14}	(λ_3, u_3)	6.1218×10^{-15}

Table 2.6.2. Residual $\lVert\widetilde{T}u_i - \lambda_i u_i\rVert_2$ for Example 2.6.5.

From Table 2.6.2, we observe that the residual $\left\|\widetilde{T}u_i - \lambda_i u_i\right\|_2$ for $i = 1, 2$ in **Case 1** and for $i = 1, 3$ in **Case 2** is on the order of 10^{-14}, which is negligible. This confirms the effectiveness of our method in solving the symmetric Toeplitz PDIEP.

Similar to PDIEP, one can solve the generalized PDIEP (Problem 2.5.5). We now illustrate the generalized PDIEP for Hankel and symmetric Toeplitz structure.

Example 2.6.6. *To establish test data, we first generate a linear matrix pencil $M - \lambda N$, where M and N are Hankel matrices. Specifically, we define $M = \texttt{hankel}(c_1, r_1)$ and $N = \texttt{hankel}(c_2, r_2)$, where*

$$c_1 = [4 + 2\boldsymbol{i},\ 2 - 4\boldsymbol{i},\ -1 + 3\boldsymbol{i},\ 4 + 3\boldsymbol{i}], \quad r_1 = [4 + 3\boldsymbol{i},\ 4\boldsymbol{i},\ 9 + 2\boldsymbol{i},\ 3 + \boldsymbol{i}],$$

$$c_2 = [3 + 2\boldsymbol{i},\ 6 - \boldsymbol{i},\ -5 + 2\boldsymbol{i},\ 4 + 7\boldsymbol{i}], \quad r_2 = [4 + 7\boldsymbol{i},\ 3 + 4\boldsymbol{i},\ 2 + 2\boldsymbol{i},\ 3 - 8\boldsymbol{i}].$$

Let (Λ, Φ) denote the eigenpairs of $M - \lambda N$, where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{C}^{4\times4}$ and $\Phi = [u_1, u_2, u_3, u_4] \in \mathbb{C}^{4\times4}$. The eigenvalues are

$$[\lambda_1, \lambda_2, \lambda_3, \lambda_4] = [-0.3953 + 0.6027\boldsymbol{i},\ 0.3708 - 0.7155\boldsymbol{i},\ 0.6743 - 0.3655\boldsymbol{i},\ 0.6761 + 0.1157\boldsymbol{i}],$$

and the matrix Φ of eigenvectors is given by

$$\Phi = \begin{bmatrix} -0.4881 + 0.1767\boldsymbol{i} & -0.4811 - 0.3552\boldsymbol{i} & -0.7739 + 0.1499\boldsymbol{i} & 0.7130 + 0.2870\boldsymbol{i} \\ 0.4383 + 0.4624\boldsymbol{i} & 0.4236 + 0.5764\boldsymbol{i} & -0.8976 + 0.1024\boldsymbol{i} & 0.1416 + 0.5177\boldsymbol{i} \\ 0.4194 - 0.5806\boldsymbol{i} & -0.1700 + 0.0352\boldsymbol{i} & -0.3007 + 0.3084\boldsymbol{i} & -0.3339 + 0.5007\boldsymbol{i} \\ -0.5678 - 0.0875\boldsymbol{i} & 0.3392 - 0.1123\boldsymbol{i} & 0.0061 + 0.1882\boldsymbol{i} & -0.3560 - 0.2370\boldsymbol{i} \end{bmatrix}.$$

***Case* 1.** *Reconstruction from one eigenpair ($k = 1$): Let the prescribed partial eigeninformation be given by*

$$\widetilde{\Lambda} = \lambda_1 \in \mathbb{C}, \quad \widetilde{\Phi} = u_1 \in \mathbb{C}^{4\times1}.$$

We construct the Hankel matrices $\widetilde{M}$ and $\widetilde{N}$ such that $\widetilde{M}u_i = \lambda_i \widetilde{N} u_i$ for $i = 1$. Using the transformations $A = I_4$, $X = \widetilde{M}$, $B = \widetilde{\Phi}$, $C = -I_4$, $Y = \widetilde{N}$, $D = \widetilde{\Phi}\widetilde{\Lambda}$, and $E = 0$, we solve the matrix equation $AXB + CYD = E$. The resulting matrices are

$$\widetilde{M} = \begin{bmatrix} 1.0472 + 0.3406\boldsymbol{i} & 1.1937 + 0.5288\boldsymbol{i} & 0.8984 + 0.8802\boldsymbol{i} & 1.0875 + 1.1282\boldsymbol{i} \\ 1.1937 + 0.5288\boldsymbol{i} & 0.8984 + 0.8802\boldsymbol{i} & 1.0875 + 1.1282\boldsymbol{i} & 0.7748 + 1.0806\boldsymbol{i} \\ 0.8984 + 0.8802\boldsymbol{i} & 1.0875 + 1.1282\boldsymbol{i} & 0.7748 + 1.0806\boldsymbol{i} & 0.6237 + 1.3399\boldsymbol{i} \\ 1.0875 + 1.1282\boldsymbol{i} & 0.7748 + 1.0806\boldsymbol{i} & 0.6237 + 1.3399\boldsymbol{i} & 0.3267 + 1.2860\boldsymbol{i} \end{bmatrix},$$

$$\widetilde{N} = \begin{bmatrix} 1.4161 + 0.7678\boldsymbol{i} & 1.3606 + 0.9305\boldsymbol{i} & 1.0320 + 0.8914\boldsymbol{i} & 0.9574 + 1.1034\boldsymbol{i} \\ 1.3606 + 0.9305\boldsymbol{i} & 1.0320 + 0.8914\boldsymbol{i} & 0.9574 + 1.1034\boldsymbol{i} & 0.8624 + 0.8961\boldsymbol{i} \\ 1.0320 + 0.8914\boldsymbol{i} & 0.9574 + 1.1034\boldsymbol{i} & 0.8624 + 0.8961\boldsymbol{i} & 0.6464 + 0.9076\boldsymbol{i} \\ 0.9574 + 1.1034\boldsymbol{i} & 0.8624 + 0.8961\boldsymbol{i} & 0.6464 + 0.9076\boldsymbol{i} & 0.5615 + 0.7072\boldsymbol{i} \end{bmatrix}.$$

Thus, $\widetilde{M} - \lambda\widetilde{N}$ is the desired Hankel matrix pencil.

***Case* 2.** *Reconstruction from two eigenpairs ($k = 2$): Let the prescribed partial eigeninformation be given by*

$$\widetilde{\Lambda} = \operatorname{diag}(\lambda_1, \lambda_3) \in \mathbb{C}^{2\times 2} \text{ and } \widetilde{\Phi} = [u_1, u_3] \in \mathbb{C}^{4\times 2}.$$

*Construct the Hankel matrices $\widetilde{M}$ and $\widetilde{N}$ such that $\widetilde{M}u_i = \lambda_i\widetilde{N}u_i$ for $i = 1, 3$. Following the same approach as in **Case 1**, we solve the matrix equation $AXB + CYD = E$ and obtain:*

$$\widetilde{M} = \begin{bmatrix} 0.2460 - 0.0000\boldsymbol{i} & -0.0696 - 0.0231\boldsymbol{i} & 0.1118 - 0.0226\boldsymbol{i} & -0.0519 + 0.0436\boldsymbol{i} \\ -0.0696 - 0.0231\boldsymbol{i} & 0.1118 - 0.0226\boldsymbol{i} & -0.0519 + 0.0436\boldsymbol{i} & 0.0299 + 0.1325\boldsymbol{i} \\ 0.1118 - 0.0226\boldsymbol{i} & -0.0519 + 0.0436\boldsymbol{i} & 0.0299 + 0.1325\boldsymbol{i} & 0.1243 - 0.0621\boldsymbol{i} \\ -0.0519 + 0.0436\boldsymbol{i} & 0.0299 + 0.1325\boldsymbol{i} & 0.1243 - 0.0621\boldsymbol{i} & 0.0711 + 0.0777\boldsymbol{i} \end{bmatrix},$$

$$\widetilde{N} = \begin{bmatrix} 0.1767 + 0.0416\boldsymbol{i} & 0.1067 - 0.0146\boldsymbol{i} & -0.0352 + 0.0850\boldsymbol{i} & 0.0696 - 0.0910\boldsymbol{i} \\ 0.1067 - 0.0146\boldsymbol{i} & -0.0352 + 0.0850\boldsymbol{i} & 0.0696 - 0.0910\boldsymbol{i} & -0.0943 + 0.1694\boldsymbol{i} \\ -0.0352 + 0.0850\boldsymbol{i} & 0.0696 - 0.0910\boldsymbol{i} & -0.0943 + 0.1694\boldsymbol{i} & -0.0396 + 0.0850\boldsymbol{i} \\ 0.0696 - 0.0910\boldsymbol{i} & -0.0943 + 0.1694\boldsymbol{i} & -0.0396 + 0.0850\boldsymbol{i} & -0.0269 + 0.0197\boldsymbol{i} \end{bmatrix}.$$

Thus, $\widetilde{M} - \lambda\widetilde{N}$ is the desired Hankel matrix pencil.

***Case* 1** ($k = 1$)		***Case* 2** ($k = 2$)	
Eigenpairs	*Residual* $\|\widetilde{M}u_i - \lambda_i\widetilde{N}u_i\|_2$	*Eigenpairs*	*Residual* $\|\widetilde{M}u_i - \lambda_i\widetilde{N}u_i\|_2$
(λ_1, u_1)	2.7626×10^{-15}	(λ_1, u_1)	1.0906×10^{-14}
		(λ_3, u_3)	2.7570×10^{-15}

Table 2.6.3. Residual $\|\widetilde{M}u_i - \lambda_i\widetilde{N}u_i\|_2$ for Example 2.6.6.

From Table 2.6.3, we find that the residual $\|\widetilde{M}u_i - \lambda_i\widetilde{N}u_i\|_2$, for $i = 1$ in **Case 1** and for $i = 1, 3$ in **Case 2**, is in the order of 10^{-14} and is negligible. This demonstrates the effectiveness of our method in solving the generalized PDIEP for Hankel structure.

Example 2.6.7. *To generate the test data, we first construct a linear matrix pencil $M - \lambda N$, where both M and N are symmetric Toeplitz matrices. Specifically, M and N are defined as $M = \texttt{toeplitz}(c_1)$ and $N = \texttt{toeplitz}(c_2)$, where the vectors c_1 and c_2 are given by*

$$c_1 = [7.80,\ 5.50,\ 3.70,\ -2.30,\ 8.90], \quad c_2 = [4.20,\ 1.20,\ -3.50,\ 3.90,\ 9.80].$$

Let (Λ, Φ) denote the eigenpairs of $M - \lambda N$, where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_5) \in \mathbb{C}^{5\times5}$ and $\Phi = [u_1, u_2, u_3, u_4, u_5] \in \mathbb{C}^{5\times5}$. The eigenvalues are

$$[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5] = [4.1157,\ -1.7144,\ 0.2371,\ -0.1060 + 1.1336\boldsymbol{i},\ -0.1060 - 1.1336\boldsymbol{i}],$$

and the matrix Φ of eigenvectors is given by

$$\Phi = \begin{bmatrix} -0.2481 & 0.3192 & -0.2773 & -0.2700 + 0.7300\boldsymbol{i} & -0.2700 - 0.7300\boldsymbol{i} \\ -0.4470 & -0.8953 & -0.4115 & 0.6140 + 0.1425\boldsymbol{i} & 0.6140 - 0.1425\boldsymbol{i} \\ -1.0000 & 1.0000 & 1.0000 & -0.0000 + 0.0000\boldsymbol{i} & -0.0000 + 0.0000\boldsymbol{i} \\ -0.4470 & -0.8953 & -0.4115 & -0.6140 - 0.1425\boldsymbol{i} & -0.6140 + 0.1425\boldsymbol{i} \\ -0.2481 & 0.3192 & -0.2773 & 0.2700 - 0.7300\boldsymbol{i} & 0.2700 + 0.7300\boldsymbol{i} \end{bmatrix}.$$

***Case* 1.** *Reconstruction from two eigenpairs ($k = 2$): Let the prescribed partial eigen-information be given by*

$$\widetilde{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_3) \in \mathbb{C}^{2\times2}, \quad \widetilde{\Phi} = [u_1, u_3] \in \mathbb{C}^{5\times2}.$$

Construct the symmetric Toeplitz matrices $\widetilde{M}$ and $\widetilde{N}$ such that $\widetilde{M}u_i = \lambda_i \widetilde{N} u_i$ for $i = 1, 3$. Using the transformations $A = I_5$, $X = \widetilde{M}$, $B = \widetilde{\Phi}$, $C = -I_5$, $Y = \widetilde{N}$, $D = \widetilde{\Phi}\widetilde{\Lambda}$, and $E = 0$, we solve the matrix equation $AXB + CYD = E$. The resulting matrices are

$$\widetilde{M} = \begin{bmatrix} 1.3921 & 1.0473 & 0.6772 & -0.2032 & 0.6735 \\ 1.0473 & 1.3921 & 1.0473 & 0.6772 & -0.2032 \\ 0.6772 & 1.0473 & 1.3921 & 1.0473 & 0.6772 \\ -0.2032 & 0.6772 & 1.0473 & 1.3921 & 1.0473 \\ 0.6735 & -0.2032 & 0.6772 & 1.0473 & 1.3921 \end{bmatrix},$$

$$\widetilde{N} = \begin{bmatrix} 0.6339 & 0.1905 & -0.3161 & 0.6055 & 0.7404 \\ 0.1905 & 0.6339 & 0.1905 & -0.3161 & 0.6055 \\ -0.3161 & 0.1905 & 0.6339 & 0.1905 & -0.3161 \\ 0.6055 & -0.3161 & 0.1905 & 0.6339 & 0.1905 \\ 0.7404 & 0.6055 & -0.3161 & 0.1905 & 0.6339 \end{bmatrix}.$$

Hence, $\widetilde{M} - \lambda\widetilde{N}$ is the desired symmetric Toeplitz matrix pencil.

***Case* 2.** *Reconstruction from three eigenpairs* ($k = 3$)*: Let the prescribed partial eigen-information be given by*

$$\widetilde{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^{3\times 3}, \quad \widetilde{\Phi} = [u_1, u_2, u_3] \in \mathbb{C}^{5\times 3}.$$

Construct the symmetric Toeplitz matrices $\widetilde{M}$ *and* $\widetilde{N}$ *such that* $\widetilde{M}u_i = \lambda_i \widetilde{N}u_i$ *for* $i = 1, 2, 3$. *Using the same transformations as in* ***Case* 1**, *we solve the matrix equation* $AXB + CYD = E$ *and obtain*

$$\widetilde{M} = \begin{bmatrix} 0.9214 & 0.6497 & 0.4371 & -0.2717 & 1.0513 \\ 0.6497 & 0.9214 & 0.6497 & 0.4371 & -0.2717 \\ 0.4371 & 0.6497 & 0.9214 & 0.6497 & 0.4371 \\ -0.2717 & 0.4371 & 0.6497 & 0.9214 & 0.6497 \\ 1.0513 & -0.2717 & 0.4371 & 0.6497 & 0.9214 \end{bmatrix},$$

$$\widetilde{N} = \begin{bmatrix} 0.4961 & 0.1417 & -0.4134 & 0.4607 & 1.1576 \\ 0.1417 & 0.4961 & 0.1417 & -0.4134 & 0.4607 \\ -0.4134 & 0.1417 & 0.4961 & 0.1417 & -0.4134 \\ 0.4607 & -0.4134 & 0.1417 & 0.4961 & 0.1417 \\ 1.1576 & 0.4607 & -0.4134 & 0.1417 & 0.4961 \end{bmatrix}.$$

Hence, $\widetilde{M} - \lambda\widetilde{N}$ *is the desired symmetric Toeplitz matrix pencil.*

***Case* 1 (k = 2)**		***Case* 2 (k = 3)**	
Eigenpairs	*Residual* $\lVert \widetilde{M}u_i - \lambda_i\widetilde{N}u_i \rVert_2$	*Eigenpairs*	*Residual* $\lVert \widetilde{M}u_i - \lambda_i\widetilde{N}u_i \rVert_2$
(λ_1, u_1)	3.3675×10^{-15}	(λ_1, u_1)	6.9900×10^{-15}
(λ_3, u_3)	2.3481×10^{-15}	(λ_2, u_2)	2.4962×10^{-15}
		(λ_3, u_3)	2.5686×10^{-15}

Table 2.6.4. Residual $\lVert \widetilde{M}u_i - \lambda_i\widetilde{N}u_i \rVert_2$ for Example 2.6.7.

From Table 2.6.4, we find that the residual $\lVert \widetilde{M}u_i - \lambda_i\widetilde{N}u_i \rVert_2$, for $i = 1, 3$ in **Case 1** and for $i = 1, 2, 3$ in **Case 2**, is in the order of 10^{-15} and is negligible. This demonstrates the effectiveness of our method in solving the generalized PDIEP for symmetric Toeplitz structure.

Next, we provide an example to compare our method for finding the least squares Toeplitz solutions of the matrix equation $X + AXB = C$, where $A, B, C \in \mathbb{Q}_{\mathbb{R}}^{n\times n}$, with the method reported in [71]. By setting $r = 2$, $q = 0$, $A_1 = B_1 = I_n$, $A_2 = A$, $B_2 = B$, and $E = C$ in (2.1.2), our proposed framework can solve the RBME reported in [71] more efficiently. We compare both the error and CPU time to demonstrate the accuracy and efficiency of our proposed method relative to that in [71].

Example 2.6.8. *Let*

$$A = \texttt{rand}(n) + \texttt{rand}(n)\boldsymbol{i} + \texttt{rand}(n)\boldsymbol{j} + \texttt{rand}(n)\boldsymbol{k},$$
$$B = \texttt{rand}(n) + \texttt{rand}(n)\boldsymbol{i} + \texttt{rand}(n)\boldsymbol{j} + \texttt{rand}(n)\boldsymbol{k}.$$

Define

$$\widetilde{X} = \texttt{toeplitz}(a_1, a_2) + \texttt{toeplitz}(b_1, b_2)\boldsymbol{i} + \texttt{toeplitz}(c_1, c_2)\boldsymbol{j} + \texttt{toeplitz}(d_1, d_2)\boldsymbol{k},$$

where $a_1 = b_1 = c_1 = d_1 = \texttt{randn}(n, 1)$ and $a_2 = b_2 = c_2 = d_2 = \texttt{randn}(1, n)$. Let

$$C = \widetilde{X} + A\widetilde{X}B.$$

Hence, $\widetilde{X}$ is the least squares Toeplitz solution with the least norm for $X + AXB = C$.

Next, we use matrices A, B, and C as input to calculate the least squares Toeplitz solution with the least norm for $X + AXB = C$. Let $\overline{X}$ be the solution obtained using our method, and $\widehat{X}$ be the solution obtained using Algorithm 2 from [71].

We compute the errors between the solution obtained by our framework and the actual solution, defined as $\epsilon_1 = \log_{10}(\|\overline{X} - \widetilde{X}\|_F)$, and the errors between the solution obtained by [71] and the actual solution, defined as $\epsilon_2 = \log_{10}(\|\widehat{X} - \widetilde{X}\|_F)$. From Table 2.6.5, we observe that the accuracy of both methods is high; ϵ_1 and ϵ_2 are comparable and consistently less than -11 for various matrix dimensions. This demonstrates that our method is as effective as the one proposed in [71].

Let t_1 and t_2 represent the CPU time consumed by our method and the method reported in [71], respectively, for computing $\overline{X}$ and $\widehat{X}$. As shown in Table 2.6.5, our method consistently requires less time compared to the method in [71] across various matrix dimensions, highlighting its superior efficiency. This is because our method employs only real and complex operations, which are more convenient and efficient. In contrast, the method in [71] involves reduced biquaternion operations, which are considerably more time-consuming.

	Error		CPU time	
n	ϵ_1	ϵ_2	$\mathbf{t_1}$	$\mathbf{t_2}$
5	−13.0918	−13.4447	0.0110	0.1379
10	−12.8390	−12.6899	0.0252	2.5883
15	−12.0256	−12.3873	0.0562	18.1414
20	−12.0689	−12.1151	0.1130	76.1936
25	−11.9322	−11.9998	0.2419	233.1581
30	−11.8615	−11.8008	0.5344	574.7308
35	−11.8275	−12.0263	1.1164	1.2119×10^3
40	−11.3621	−11.7243	2.3388	2.4118×10^3
45	−11.7765	−11.3134	4.2934	4.2807×10^3
50	−11.1779	−11.5833	7.2243	7.0931×10^3

Table 2.6.5. Comparison of error and CPU time for computing the Toeplitz solution of $X + AXB = C$ using our method and the method reported in [71] across various matrix dimensions.

Conclusion: In this chapter, we have explored various L-structure reduced biquaternion matrix sets, including reduced biquaternion Toeplitz, symmetric Toeplitz, Hankel, circulant, real, complex, and purely imaginary matrix sets. Furthermore, we have developed a generalized framework for finding the least squares L-structure solutions for three generalized RBMEs. Additionally, we have demonstrated how the proposed theory extends to several practical applications, such as image restoration problem, L-structure solutions for complex and real matrix equations, solution of PDIEP, and generalized PDIEP.

These contributions provide a foundation for further research in reduced biquaternion matrix theory, particularly in areas where the underlying L-structure plays a crucial role. The framework and methodologies discussed here open up new possibilities for solving advanced matrix equations in both theoretical and applied contexts.

CHAPTER 3

GENERALIZED INVERSE OF REDUCED BIQUATERNION MATRICES

This chapter focuses on computing the outer and $\{1,2\}$-generalized inverses of reduced biquaternion matrices (RBGI). The main results pertain to RBGIs that satisfy specific conditions related to column and/or row spaces. Conditions for the existence and effective representations of these generalized inverses are established. The existence condition is determined using the rank function and regularity, while the representation is achieved by solving RBME of the form $(AXB, CXD) = (E, F)$. Additionally, numerical algorithms based on these representations are presented, and their effectiveness is demonstrated through numerical examples.

3.1. Introduction

The concept of generalized inverse, first introduced by E.H. Moore in 1920 [53], was initially defined using matrix projectors. For several decades, little progress was made until the 1950s when interest in generalized inverses was renewed due to their applications in solving linear systems. In 1955, R. Penrose [58] advanced the field by demonstrating that Moore's inverse uniquely satisfies four matrix equations. However, for certain applications, matrices that satisfy fewer than all four equations are also of interest. The primary goal of constructing a generalized inverse is to extend the concept of an inverse matrix to a broader class of matrices, including those that are not invertible. Various types of generalized inverses have been introduced in the literature, particularly for solving both consistent and inconsistent systems of linear equations [3, 20, 68, 69].

This chapter focuses on generalized inverses with predefined conditions. The study of such generalized inverses for real and complex matrices is well-documented; see, for example, [10, 62, 64, 72, 73, 74]. In [63], the authors established determinantal representations of generalized inverses over the quaternion skew field using the theory of column and row determinants. Similarly, [4] explored generalized inverses with predefined column and/or row spaces, particularly for matrices over a commutative ring with identity.

In this chapter, we investigate the generalized inverses of RB matrices with predefined conditions on the column and/or row spaces. We begin by stating the definition of particular generalized inverses for RB matrices. Consider the following conditions:

$$(P_1)\ AXA = A, \quad (P_2)\ XAX = X. \tag{3.1.1}$$

The set of all RB matrices satisfying conditions defined by the set $\delta \subseteq \{1, 2\}$, where the condition (P_i) corresponds to $i \in \delta$, is denoted as $A\{\delta\}$. Any RB matrix in $A\{\delta\}$ is referred to as the δ-inverse of A and is denoted by $A^{(\delta)}$. The matrix $A^{(\delta)}$ is called the generalized inverse of matrix A, and $A^{(2)}$ is also referred to as the outer generalized inverse of A. A matrix $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ is said to be regular if there exists a matrix $X \in \mathbb{Q}_{\mathbb{R}}^{n\times m}$ satisfying $AXA = A$.

Additional important generalized inverses include outer inverses and $\{1, 2\}$-inverses with prescribed column and/or row spaces. An element $X \in A\{\delta\}$ satisfying $\mathcal{C}(X) = \mathcal{C}(S)$ (respectively $\mathcal{R}(X) = \mathcal{R}(T)$) is denoted by $A^{(\delta)}_{\mathcal{C}(S),*}$ (respectively $A^{(\delta)}_{*,\mathcal{R}(T)}$), where $\mathcal{C}(X)$ and $\mathcal{R}(X)$ denote the column space and row space of matrix X. If X satisfies both the requirements $\mathcal{C}(X) = \mathcal{C}(S)$ and $\mathcal{R}(X) = \mathcal{R}(T)$ it is denoted by $A^{(\delta)}_{\mathcal{C}(S),\mathcal{R}(T)}$.

In [85], Zhang *et al.* studied the generalized inverse problem of RB matrices using their singular value decomposition. Their work discussed the Moore-Penrose generalized inverse, $\{1\}$-inverse, and $\{1, 2\}$-inverse of RB matrices. Outer inverses with prescribed column and/or row spaces are explored in [4] for matrices over a commutative ring with identity. Notably, the set of all RB matrices forms a commutative ring with identity. In this chapter, we examine the existence and representation of generalized inverses of an RB matrix $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, including:

$$A^{(2)}_{\mathcal{C}(S),*}, \quad A^{(2)}_{*,\mathcal{R}(T)}, \quad A^{(2)}_{\mathcal{C}(S),\mathcal{R}(T)}, \quad A^{(1,2)}_{\mathcal{C}(S),*}, \quad A^{(1,2)}_{*,\mathcal{R}(T)}, \quad A^{(1,2)}_{\mathcal{C}(S),\mathcal{R}(T)}. \tag{3.1.2}$$

The existence conditions for these RBGIs are determined by the rank function, regularity, and the properties of the column and row spaces of RB matrices. The representation of RBGIs is derived by solving appropriate RBMEs. Solutions to these RBMEs are obtained by transforming them into equivalent systems of linear equations with complex matrix coefficients (CSoLE), utilizing tools like the Kronecker product and vectorization techniques.

This chapter is organized as follows: Section 3.2 introduces preliminary results. Section 3.3 presents the framework for solving RBMEs of the form $AXB = E$ and $(AXB, CXD) = (E, F)$. Section 3.4 investigates the existence and representation of outer and $\{1,2\}$-generalized inverses of RB matrices with specified column and/or row spaces. Finally, Section 3.5 provides numerical verification of the proposed results.

3.2. Preliminaries

To ensure this chapter is self-contained, we will briefly summarize key concepts and results that are essential for the discussions in the following sections. We start by presenting the definitions of the column space and row space of an RB matrix, which are provided in Definitions 3.2.1 and 3.2.2, respectively.

Definition 3.2.1. *Let $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ and $x = [x_1, \ldots, x_n]^T \in \mathbb{Q}_{\mathbb{R}}^{n}$. Suppose $\mathrm{Col}_j(A)$ represents jth column of A, for $j = 1, \ldots, n$. Then, the column space $\mathcal{C}(A)$ of A is defined as the span of its columns:*

$$\mathcal{C}(A) = \{x_1\mathrm{Col}_1(A) + \cdots + x_n\mathrm{Col}_n(A) \,|\, x_1, \ldots, x_n \in \mathbb{Q}_{\mathbb{R}}\} = \{Ax \,|\, x \in \mathbb{Q}_{\mathbb{R}}^{n}\}.$$

Definition 3.2.2. *Let $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ and $y = [y_1, \ldots, y_m] \in \mathbb{Q}_{\mathbb{R}}^{1\times m}$. Suppose $\mathrm{Row}_i(A)$ represents the ith row of A, for $i = 1, \ldots, m$. Then, the row space $\mathcal{R}(A)$of A is defined as the span of its rows:*

$$\mathcal{R}(A) = \{y_1\mathrm{Row}_1(A) + \cdots + y_m\mathrm{Row}_m(A) \,|\, y_1, \ldots, y_m \in \mathbb{Q}_{\mathbb{R}}\} = \{yA \,|\, y \in \mathbb{Q}_{\mathbb{R}}^{1\times m}\}.$$

Remark 3.2.3. (1) *The set $\mathbb{Q}_{\mathbb{R}}$ forms a commutative ring with identity, where addition and scalar multiplication are defined in the usual way.*

(2) *The set $\mathbb{Q}_{\mathbb{R}}^{m\times n}(\mathbb{Q}_{\mathbb{R}})$ is a $\mathbb{Q}_{\mathbb{R}}$-module with standard addition and scalar multiplication.*

(3) *For $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, the set $\mathcal{C}(A)$ is a $\mathbb{Q}_{\mathbb{R}}$-submodule of $\mathbb{Q}_{\mathbb{R}}^{m}$ generated by the columns of A.*

(4) *For $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, the set $\mathcal{R}(A)$ is a $\mathbb{Q}_{\mathbb{R}}$-submodule of $\mathbb{Q}_{\mathbb{R}}^{1\times n}$ generated by the rows of A.*

In the subsequent Lemma 3.2.4, we examine the properties of the column and row space of a matrix. This lemma will be utilized to present representations of the outer inverse and $\{1,2\}$-inverse that meet specific conditions related to the column space and/or row space.

Lemma 3.2.4. *Let $X \in \mathbb{Q}_{\mathbb{R}}^{n\times m}$, $S \in \mathbb{Q}_{\mathbb{R}}^{n\times k}$, and $T \in \mathbb{Q}_{\mathbb{R}}^{l\times m}$ be given RB matrices. Then*

(1) $\mathcal{C}(X) \subseteq \mathcal{C}(S)$ *if and only if there exists* $U \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$ *such that* $X = SU$.
(2) $\mathcal{R}(X) \subseteq \mathcal{R}(T)$ *if and only if there exists* $V \in \mathbb{Q}_{\mathbb{R}}^{n\times l}$ *such that* $X = VT$.

Proof.

(1) Let $Col_i(X)$ represent the ith column of matrix X for $i = 1,\dots,m$, and $Col_j(S)$ denote the jth column of matrix S for $j = 1,\dots,k$. Since $\mathcal{C}(X) \subseteq \mathcal{C}(S)$, for each $i = 1,\dots,m$, there exist scalars $[u_{1i},\dots,u_{ki}]^T$ such that

$$Col_i(X) = u_{1i}Col_1(S) + \cdots + u_{ki}Col_k(S).$$

Therefore, we can write

$$X = S\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ \vdots & \vdots & & \vdots \\ u_{k1} & u_{k2} & \cdots & u_{km} \end{bmatrix} = SU.$$

(2) Let $Row_i(X)$ represent the ith row of matrix X for $i = 1,\dots,n$, and $Row_j(T)$ denote the jth row of matrix T for $j = 1,\dots,l$. Since $\mathcal{R}(X) \subseteq \mathcal{R}(T)$, for each $i = 1,\dots,n$, there exist scalars $[v_{i1},\dots,v_{il}]$ such that

$$Row_i(X) = v_{i1}Row_1(T) + \cdots + v_{il}Row_l(T).$$

Therefore, we can write

$$X = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1l} \\ \vdots & \vdots & & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nl} \end{bmatrix} T = VT. \quad \blacksquare$$

For a matrix over a field, such as the fields of real or complex numbers, the concept of rank(A) is well defined as the dimension of the subspace generated by the columns of A. However, $\mathbb{Q}_{\mathbb{R}}$ does not constitute a field, since not every nonzero element in $\mathbb{Q}_{\mathbb{R}}$ has a multiplicative inverse $(e.g., 1 + \boldsymbol{j})$. Consequently, the standard properties of rank do not universally apply when dealing with matrices having entries from $\mathbb{Q}_{\mathbb{R}}$. In such instances, an alternative approach involving determinantal rank and a rank function is employed for matrices over $\mathbb{Q}_{\mathbb{R}}$.

Definition 3.2.5. *Given $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$; the determinantal rank of A, marked with $\rho(A)$, is the size of the largest submatrix of A with a nonzero determinant.*

For the definition of the determinant of a matrix $A \in \mathbb{Q}_{\mathbb{R}}^{n\times n}$, refer to [48, Page 18]. It is important to note that, in the context of reduced biquaternion algebra, a nonzero determinant $\det(A)$ does not necessarily imply non-singularity of A, which differs from matrices over a field. A matrix $A \in \mathbb{Q}_{\mathbb{R}}^{n\times n}$ is said to be nonsingular, or equivalently invertible, if its determinant $\det(A)$ is a unit in $\mathbb{Q}_{\mathbb{R}}$.

To study matrix properties in the RB domain, it is crucial to analyze minors and their behavior under matrix multiplication. Lemma 3.2.6 establishes a key relationship, expressing the minors of the product of two rectangular matrices in terms of the minors of the individual matrices. This result is instrumental in exploring determinantal rank properties.

Lemma 3.2.6. *Suppose $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $B \in \mathbb{Q}_{\mathbb{R}}^{n\times s}$ and $k \le \min\{m,n,s\}$. Then $k\times k$ minor $[AB]_{I,J}$ of matrix AB, where I is a subset of $\{1,2,\dots,m\}$ with k elements and J is a subset of $\{1,2,\dots,s\}$ with k elements, is equal to*

$$[AB]_{I,J} = \sum_{K} [A]_{I,K}[B]_{K,J},$$

where the sum performs over all subsets $K \subseteq \{1,\dots,n\}$ involving k elements.

Proof. Since $\mathbb{Q}_{\mathbb{R}}$ is a commutative ring with identity, the proof directly follows from Theorem I.5 in [48, Page 21]. ∎

In Lemma 3.2.7, we delve into properties of the determinantal rank of a matrix. These properties will be later utilized in this chapter to investigate the properties of the rank function of a matrix.

Lemma 3.2.7. *The subsequent statements hold for $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $B \in \mathbb{Q}_{\mathbb{R}}^{n\times s}$, and $C \in \mathbb{Q}_{\mathbb{R}}^{t\times m}$:*

(1) $\rho(AB) \le \min(\rho(A), \rho(B))$.
(2) *If B is a right invertible matrix and $n \le s$, then $\rho(AB) = \rho(A)$.*
(3) *If C is a left invertible matrix and $m \le t$, then $\rho(CA) = \rho(C)$.*

Proof.

(1) The proof is straightforward, follows from Lemma 3.2.6.
(2) Given that B is right invertible, there exists a matrix $P \in \mathbb{Q}_{\mathbb{R}}^{s\times n}$ such that $BP = I_n$. Therefore, we can express A as

$$A = AI_n = A(BP) = (AB)P.$$

Using part (1), we deduce that $\rho(A) \leq \rho(AB)$. Additionally, $\rho(AB) \leq \rho(A)$. Consequently, we conclude that

$$\rho(AB) = \rho(A).$$

(3) Given that C is left invertible, there exists a matrix $Q \in \mathbb{Q}_{\mathbb{R}}^{m\times t}$ such that $QC = I_m$. Therefore, we can express A as

$$A = I_m A = (QC)A = Q(CA).$$

Using part (1), we deduce that $\rho(A) \leq \rho(CA)$. Additionally, since $\rho(CA) \leq \rho(A)$, it follows that

$$\rho(CA) = \rho(A).$$

All parts of the proof are verified. ■

In [47, Definition 2.2], the author defined the rank function for a matrix over a commutative ring with identity. Since $\mathbb{Q}_{\mathbb{R}}$ is a commutative ring with identity, the corresponding definition of the rank function for a matrix over $\mathbb{Q}_{\mathbb{R}}$ is provided as follows.

Definition 3.2.8. *Let $\mathcal{E} = \{e \in \mathbb{Q}_{\mathbb{R}} \mid e^2 = e \text{ and } e \neq 0\}$ be the set involving all nonzero idempotent elements in $\mathbb{Q}_{\mathbb{R}}$. The rank function of a matrix $A \in \mathbb{Q}_{\mathbb{R}}$, denoted by $\mathscr{R}_A$, is an integer-valued function $\mathscr{R}_A : \mathcal{E} \to \mathbb{Z}$ defined by $\mathscr{R}_A(e) = \rho(eA)$ for all $e \in \mathcal{E}$.*

Remark 3.2.9. *For $\mathbb{Q}_{\mathbb{R}}$, the set $\mathcal{E}$ is equal to $\mathcal{E} = \left\{1, \frac{1}{2} + \frac{1}{2}\boldsymbol{j}, \frac{1}{2} - \frac{1}{2}\boldsymbol{j}\right\}$.*

Remark 3.2.10. *In the case of real and complex fields, the set $\mathcal{E}$ is defined by $\mathcal{E} = \{1\}$. Therefore, the rank function of a matrix in these fields corresponds to the usual rank of a matrix.*

Properties of the rank function are studied in the following Lemmas 3.2.11, 3.2.12, and 3.2.13. These lemmas will play a key role in characterizing the conditions for the existence of outer and $\{1, 2\}$-generalized inverse that meet specific criteria related to the column space and/or row space.

Lemma 3.2.11. *The subsequent characterization are valid for $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $B \in \mathbb{Q}_{\mathbb{R}}^{n\times s}$, and $C \in \mathbb{Q}_{\mathbb{R}}^{t\times m}$:*

(1) *If B is a right invertible matrix and $n \leq s$, then $\mathscr{R}_{AB} = \mathscr{R}_A$.*
(2) *If C is a left invertible matrix and $m \leq t$, then $\mathscr{R}_{CA} = \mathscr{R}_A$.*

Proof. The proof straightforwardly follows from Lemma 3.2.7. ■

As $\mathbb{Q}_{\mathbb{R}}$ forms a commutative ring with identity, Lemmas 3.2.12 and 3.2.13 that follow can be directly inferred from [47, Theorem 2.2] and utilizing Lemma 3.2.4.

Lemma 3.2.12. *Let $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ be regular and of determinantal rank r, and B be an $m \times p$ matrix. The following statements are equivalent for $T = \begin{bmatrix} A & B \end{bmatrix}$:*

(1) *$\rho(eA) = \rho(eT)$ for every idempotent $e \in \mathbb{Q}_{\mathbb{R}}$;*
(2) *$\mathscr{R}_A = \mathscr{R}_T$;*
(3) *The matrix equation $AX = B$ is consistent;*
(4) *$\mathcal{C}(B) \subseteq \mathcal{C}(A)$.*

Lemma 3.2.13. *Let $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ be regular and of determinantal rank r, and C be a matrix of size $q \times n$. The subsequent statements are equivalent for $S = \begin{bmatrix} A \\ C \end{bmatrix}$:*

(1) *$\rho(eA) = \rho(eS)$ for every idempotent $e \in \mathbb{Q}_{\mathbb{R}}$;*
(2) *$\mathscr{R}_A = \mathscr{R}_S$;*
(3) *Matrix equation $XA = C$ is solvable;*
(4) *$\mathcal{R}(C) \subseteq \mathcal{R}(A)$.*

3.3. Solutions to the RBME of the Form $(\mathbf{AXB}, \mathbf{CXD}) = (\mathbf{E}, \mathbf{F})$

The aim in this section is to derive existence conditions and solutions of the RBMEs $(AXB, CXD) = (E, F)$ and $AXB = E$. Let

$$\begin{aligned} &A = A_1 + A_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m_1\times n}, \quad X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times s}, \quad B = B_1 + B_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{s\times t_1}, \\ &C = C_1 + C_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m_2\times n}, \quad D = D_1 + D_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{s\times t_2}, \quad E = E_1 + E_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m_1\times t_1}, \\ &F = F_1 + F_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m_2\times t_2}. \end{aligned} \tag{3.3.1}$$

Before proceeding, we introduce the subsequent notations:

$$M = \begin{bmatrix} B_1^T \otimes A_1 + B_2^T \otimes A_2 & B_1^T \otimes A_2 + B_2^T \otimes A_1 \\ B_2^T \otimes A_1 + B_1^T \otimes A_2 & B_2^T \otimes A_2 + B_1^T \otimes A_1 \\ D_1^T \otimes C_1 + D_2^T \otimes C_2 & D_1^T \otimes C_2 + D_2^T \otimes C_1 \\ D_2^T \otimes C_1 + D_1^T \otimes C_2 & D_2^T \otimes C_2 + D_1^T \otimes C_1 \end{bmatrix} \quad \text{and} \quad e = \begin{bmatrix} \mathrm{vec}(E_1) \\ \mathrm{vec}(E_2) \\ \mathrm{vec}(F_1) \\ \mathrm{vec}(F_2) \end{bmatrix}. \tag{3.3.2}$$

Theorem 3.3.1. *Consider the RBME* $(AXB, CXD) = (E, F)$ *with coefficient matrices defined in* (3.3.1)*. Let* $M \in \mathbb{C}^{2(m_1t_1+m_2t_2)\times 2ns}$ *and* $e \in \mathbb{C}^{2(m_1t_1+m_2t_2)\times 1}$ *be as in* (3.3.2)*. Then* $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times s}$ *is a solution of the RBME* $(AXB, CXD) = (E, F)$ *if and only if* $MM^{\dagger}e = e$*. In addition, the generic solution* $X \in \mathbb{Q}_{\mathbb{R}}^{n\times s}$ *satisfies the following*

$$\begin{bmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{bmatrix} = M^{\dagger}e + \left(I_{2ns} - M^{\dagger}M\right)y, \tag{3.3.3}$$

where the vector $y \in \mathbb{C}^{2ns\times 1}$ *is arbitrary.*

Proof. Application of the operators vec and Ψ on the initial RBME leads to

$$(AXB, CXD) = (E, F) \Longleftrightarrow (\mathrm{vec}(\Psi_{AXB}), \mathrm{vec}(\Psi_{CXD})) = (\mathrm{vec}(\Psi_E), \mathrm{vec}(\Psi_F)).$$

Using (2.2.1) and Lemma 2.2.2 yields

$$\begin{aligned}
\Psi_{AXB} &= \Psi_A h(XB) \\
&= [A_1, A_2]\begin{bmatrix} X_1 & X_2 \\ X_2 & X_1 \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_2 & B_1 \end{bmatrix} \\
&= [A_1X_1B_1 + A_2X_2B_1 + A_1X_2B_2 + A_2X_1B_2, A_1X_1B_2 + A_2X_2B_2 + A_1X_2B_1 + A_2X_1B_1],
\end{aligned}$$

$$\Psi_{CXD} = [C_1X_1D_1 + C_2X_2D_1 + C_1X_2D_2 + C_2X_1D_2, C_1X_1D_2 + C_2X_2D_2 + C_1X_2D_1 + C_2X_1D_1].$$

Utilizing (2.2.2), we obtain the expression for $\mathrm{vec}(\Psi_{AXB})$ and $\mathrm{vec}(\Psi_{CXD})$ as follows

$$\begin{aligned}
\mathrm{vec}(\Psi_{AXB}) &= \begin{bmatrix} \mathrm{vec}(A_1X_1B_1 + A_2X_2B_1 + A_1X_2B_2 + A_2X_1B_2) \\ \mathrm{vec}(A_1X_1B_2 + A_2X_2B_2 + A_1X_2B_1 + A_2X_1B_1) \end{bmatrix} \\
&= \begin{bmatrix} (B_1^T \otimes A_1)\mathrm{vec}(X_1) + (B_1^T \otimes A_2)\mathrm{vec}(X_2) \\ + (B_2^T \otimes A_1)\mathrm{vec}(X_2) + (B_2^T \otimes A_2)\mathrm{vec}(X_1) \\ (B_2^T \otimes A_1)\mathrm{vec}(X_1) + (B_2^T \otimes A_2)\mathrm{vec}(X_2) \\ + (B_1^T \otimes A_1)\mathrm{vec}(X_2) + (B_1^T \otimes A_2)\mathrm{vec}(X_1) \end{bmatrix} \\
&= \begin{bmatrix} B_1^T \otimes A_1 + B_2^T \otimes A_2 & B_1^T \otimes A_2 + B_2^T \otimes A_1 \\ B_2^T \otimes A_1 + B_1^T \otimes A_2 & B_2^T \otimes A_2 + B_1^T \otimes A_1 \end{bmatrix}\begin{bmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{bmatrix}, \\
\mathrm{vec}(\Psi_{CXD}) &= \begin{bmatrix} D_1^T \otimes C_1 + D_2^T \otimes C_2 & D_1^T \otimes C_2 + D_2^T \otimes C_1 \\ D_2^T \otimes C_1 + D_1^T \otimes C_2 & D_2^T \otimes C_2 + D_1^T \otimes C_1 \end{bmatrix}\begin{bmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{bmatrix}.
\end{aligned}$$

Usage of (3.3.2) leads to

$$(AXB, CXD) = (E, F) \Leftrightarrow M\begin{bmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{bmatrix} = e.$$

Hence, the RBME $(AXB, CXD) = (E, F)$ is consistent if and only the above matrix equation, determined by (3.3.2), is solvable. Thus, $(AXB, CXD) = (E, F)$ is consistent if and only if $MM^{\dagger}e = e$ and the general solution X is determined by (3.3.3). ■

Remark 3.3.2. *The solution to the RBME $(AXB, CXD) = (E, F)$ can also be obtained using* [80, Theorem 3.1] *and* [75, Theorem 3.1]*. In* [80]*, the structured solution for the RBME is presented, while* [75] *focuses on the unstructured solution to a system of RBMEs. Both papers employ the complex representation of RB matrices. In their approach, they first convert the RBME into a complex linear system and then further transform it into a real linear system to find the solution. This additional transformation is essential when seeking structured solutions.*

In our method, to find the unstructured solution, we convert the RBME into a complex linear system and solve it without the additional transformation to a real linear system. This modification reduces computational overhead and improves overall efficiency in finding the unstructured solution to $(AXB, CXD) = (E, F)$.

In accordance with Theorem 3.3.1, we now outline an algorithm designed to compute the solution X of RBME $(AXB, CXD) = (E, F)$.

Algorithm 3.3.1 Computation of Solution X to RBME $(\mathbf{AXB}, \mathbf{CXD}) = (\mathbf{E}, \mathbf{F})$

Input: $A = A_1 + A_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m_1\times n}$, $B = B_1 + B_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{s\times t_1}$, $C = C_1 + C_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m_2\times n}$, $D = D_1 + D_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{s\times t_2}$, $E = E_1 + E_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m_1\times t_1}$, $F = F_1 + F_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m_2\times t_2}$.

Output: $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times s}$.

Step 1: Matrix and Vector Computation: Compute M and e using equation (3.3.2).

Step 2: Consistency Check: Verify the consistency of the RBME $(AXB, CXD) = (E, F)$ by checking the condition $MM^{\dagger}e = e$. If this condition holds, proceed to the next step.

Step 3: Solution Calculation: Compute

$$x = \begin{bmatrix} \operatorname{vec}(X_1) \\ \operatorname{vec}(X_2) \end{bmatrix} = M^{\dagger}e + \left(I_{2ns} - M^{\dagger}M\right)y,$$

where $y \in \mathbb{C}^{2ns\times 1}$ is an arbitrary vector.

Step 4: Reshaping: Reshape x into the matrices X_1 and X_2 using the Matlab function `reshape`:

$$X_1 = \texttt{reshape}(x(1:ns,:), n, s), \quad X_2 = \texttt{reshape}(x(ns+1:2ns,:), n, s).$$

Next, we derive the explicit expression for existence condition and solution of the RBME $AXB = E$ with coefficient matrices defined in (3.3.1). Before proceeding we introduce certain notations

$$N = \begin{bmatrix} B_1^T \otimes A_1 + B_2^T \otimes A_2 & B_1^T \otimes A_2 + B_2^T \otimes A_1 \\ B_2^T \otimes A_1 + B_1^T \otimes A_2 & B_2^T \otimes A_2 + B_1^T \otimes A_1 \end{bmatrix} \quad \text{and} \quad f = \begin{bmatrix} \mathrm{vec}(E_1) \\ \mathrm{vec}(E_2) \end{bmatrix}. \tag{3.3.4}$$

Corollary 3.3.3. *Consider the RBME $AXB = E$ with coefficient matrices defined in (3.3.1). Let $N \in \mathbb{C}^{2m_1t_1 \times 2ns}$ and $f \in \mathbb{C}^{2m_1t_1 \times 1}$ be as in (3.3.4). Then $AXB = E$ has a solution $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n \times s}$ if and only if $NN^{\dagger}f = f$. In this case, the general solution $X \in \mathbb{Q}_{\mathbb{R}}^{n \times s}$ satisfies the following*

$$\begin{bmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{bmatrix} = N^{\dagger}f + \left(I_{2ns} - N^{\dagger}N\right)y, \tag{3.3.5}$$

where the vector $y \in \mathbb{C}^{2ns \times 1}$ is arbitrary.

Proof. The RBME $AXB = E$ can be considered as a specific instance of the RBME $(AXB, CXD) = (E, F)$. Therefore, the proof directly follows from the proof method of Theorem 3.3.1. ∎

Remark 3.3.4. *Algorithm 3.3.1 can also be used to solve the RBME $AXB = E$ under the particular settings $C = 0$, $D = 0$, and $F = 0$ of the algorithm.*

3.4. Generalized Inverse of RB Matrices

In this section, we will explore the properties of RBGIs $A^{(2)}_{\mathcal{C}(S),*}$, $A^{(2)}_{*,\mathcal{R}(T)}$, $A^{(2)}_{\mathcal{C}(S),\mathcal{R}(T)}$, $A^{(1,2)}_{\mathcal{C}(S),*}$, $A^{(1,2)}_{*,\mathcal{R}(T)}$, and $A^{(1,2)}_{\mathcal{C}(S),\mathcal{R}(T)}$. The formalized definitions for these generalized inverses are outlined as follows:

Definition 3.4.1. *An outer inverse of $A \in \mathbb{Q}_{\mathbb{R}}^{m \times n}$ with a predefined column space $\mathcal{C}(S)$, denoted by $A^{(2)}_{\mathcal{C}(S),*}$, is a matrix X that satisfies the following conditions: $XAX = X$ and $\mathcal{C}(X) = \mathcal{C}(S)$. In addition, if $A^{(2)}_{\mathcal{C}(S),*}$ also satisfies $AXA = A$, then it is referred to as a $\{1,2\}$-inverse of A with the predefined column space $\mathcal{C}(S)$, denoted as $A^{(1,2)}_{\mathcal{C}(S),*}$.*

Definition 3.4.2. *An outer inverse of $A \in \mathbb{Q}_{\mathbb{R}}^{m \times n}$ with a predefined row space $\mathcal{R}(T)$, denoted by $A^{(2)}_{*,\mathcal{R}(T)}$, is a matrix X that satisfies the following conditions: $XAX = X$ and $\mathcal{R}(X) = \mathcal{R}(T)$. In addition, if $A^{(2)}_{*,\mathcal{R}(T)}$ also satisfies $AXA = A$, then it is referred to as a $\{1,2\}$-inverse of A with the predefined row space $\mathcal{R}(T)$, denoted as $A^{(1,2)}_{*,\mathcal{R}(T)}$.*

Definition 3.4.3. *An outer inverse of* $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ *with a prescribed column space* $\mathcal{C}(S)$ *and row space* $\mathcal{R}(T)$*, denoted by* $A^{(2)}_{\mathcal{C}(S),\mathcal{R}(T)}$*, is a matrix* X *that satisfies the following conditions:* $XAX = X$*,* $\mathcal{C}(X) = \mathcal{C}(S)$*, and* $\mathcal{R}(X) = \mathcal{R}(T)$*. In addition, if* $A^{(2)}_{\mathcal{C}(S),\mathcal{R}(T)}$ *also satisfies* $AXA = A$*, then it is referred to as a* $\{1,2\}$*-inverse of* A *with the prescribed column space* $\mathcal{C}(S)$ *and row space* $\mathcal{R}(T)$*, denoted as* $A^{(1,2)}_{\mathcal{C}(S),\mathcal{R}(T)}$.

The notations $A\{2\}_{\mathcal{C}(S),*}$, $A\{1,2\}_{\mathcal{C}(S),*}$, $A\{2\}_{*,\mathcal{R}(T)}$, $A\{1,2\}_{*,\mathcal{R}(T)}$, $A\{2\}_{\mathcal{C}(S),\mathcal{R}(T)}$, and $A\{1,2\}_{\mathcal{C}(S),\mathcal{R}(T)}$ are used for the sets of generalized inverses $A^{(2)}_{\mathcal{C}(S),*}$, $A^{(1,2)}_{\mathcal{C}(S),*}$, $A^{(2)}_{*,\mathcal{R}(T)}$, $A^{(1,2)}_{*,\mathcal{R}(T)}$, $A^{(2)}_{\mathcal{C}(S),\mathcal{R}(T)}$, and $A^{(1,2)}_{\mathcal{C}(S),\mathcal{R}(T)}$, respectively.

Theorem 3.4.4 offers equivalent conditions for the existence and general representations of an outer inverse with a specified column space $\mathcal{C}(S)$.

Theorem 3.4.4. *Let the RB matrices* $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ *and* $S \in \mathbb{Q}_{\mathbb{R}}^{n\times k}$ *be given. Then*

(1) *The subsequent claims are equivalent to each other:*
- (i) *There exists* $X \in A\{2\}$ *satisfying* $\mathcal{C}(X) = \mathcal{C}(S)$*;*
- (ii) *There exists* $Y \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$ *such that* $SYAS = S$*;*
- (iii) AS *is regular and* $\mathcal{R}(AS) = \mathcal{R}(S)$*;*
- (iv) AS *is regular and* $\mathscr{R}_{AS} = \mathscr{R}_S$*;*
- (v) AS *is regular and* $S = S(AS)^{(1)}AS$ *for some* $(AS)^{(1)} \in AS\{1\}$.

(2) *If the statements in* (1) *hold, then*

$$A\{2\}_{\mathcal{C}(S),*} = \left\{SY \,\middle|\, Y \in \mathbb{Q}_{\mathbb{R}}^{k\times m}, SYAS = S\right\}$$
$$= \left\{S(AS)^{(1)} \,\middle|\, (AS)^{(1)} \in (AS)\{1\}\right\}.$$

Proof.

(1) $(i) \Rightarrow (ii)$. Let $X \in \mathbb{Q}_{\mathbb{R}}^{n\times m}$ satisfy $XAX = X$ and $\mathcal{C}(X) = \mathcal{C}(S)$. By Lemma 3.2.4, there exist matrices $Y \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$ and $W \in \mathbb{Q}_{\mathbb{R}}^{m\times k}$ such that $X = SY$ and $S = XW$. Therefore, we conclude that

$$S = XW = XAXW = XAS = SYAS.$$

$(ii) \Rightarrow (i)$. Let $Y \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$ such that $SYAS = S$. We will show that $X = SY$ is a $\{2\}$-inverse of A and satisfies $\mathcal{C}(X) = \mathcal{C}(S)$. First, since

$$XAX = SYASY = SY = X,$$

it follows that $X \in A\{2\}$. Additionally, since $X = SY$, it implies that

$$\mathcal{C}(X) = \mathcal{C}(SY) \subseteq \mathcal{C}(S).$$

Moreover, given that $S = SYAS = XAS$, we have

$$\mathcal{C}(S) = \mathcal{C}(XAS) \subseteq \mathcal{C}(X).$$

Thus, we conclude that $\mathcal{C}(X) = \mathcal{C}(S)$.

$(ii) \Rightarrow (iii)$. Let $Y \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$ such that $SYAS = S$. This implies that

$$AS = A(SYAS) = (AS)Y(AS),$$

which shows that $Y \in (AS)\{1\}$, meaning that AS is regular. By Definition 3.2.2, we conclude that

$$\mathcal{R}(AS) \subseteq \mathcal{R}(S).$$

Moreover, since $SYAS = S$, it follows that

$$\mathcal{R}(S) = \mathcal{R}(SYAS) \subseteq \mathcal{R}(AS).$$

Thus, we have $\mathcal{R}(AS) = \mathcal{R}(S)$.

$(iii) \Rightarrow (iv)$. Let AS be regular and assume that $\mathcal{R}(AS) = \mathcal{R}(S)$. Clearly, we have $\mathcal{R}(S) \subseteq \mathcal{R}(AS)$. Now, consider the matrix

$$\overline{S} = \begin{bmatrix} AS \\ S \end{bmatrix}.$$

By applying Lemma 3.2.13, we obtain $\mathscr{R}_{AS} = \mathscr{R}_{\overline{S}}$. Next, observe that

$$\overline{S} = \begin{bmatrix} AS \\ S \end{bmatrix} = \begin{bmatrix} A \\ I_n \end{bmatrix} S.$$

Notably, the matrix $\begin{bmatrix} A \\ I_n \end{bmatrix}$ is left invertible, since

$$\begin{bmatrix} 0 & I_n \end{bmatrix} \begin{bmatrix} A \\ I_n \end{bmatrix} = I_n.$$

Therefore, using Lemma 3.2.11, we conclude that $\mathscr{R}_{\overline{S}} = \mathscr{R}_S$. Consequently, we obtain $\mathscr{R}_{AS} = \mathscr{R}_S$.

$(iv) \Rightarrow (iii)$. Assume that AS is regular and $\mathscr{R}_{AS} = \mathscr{R}_S$. Consider the matrix

$$\overline{S} = \begin{bmatrix} AS \\ S \end{bmatrix}.$$

From the previous argument for $(iii) \Rightarrow (iv)$, we obtain that

$$\mathscr{R}_{\overline{S}} = \mathscr{R}_S.$$

Thus, it follows that

$$\mathscr{R}_{\overline{S}} = \mathscr{R}_{AS}.$$

By using Lemma 3.2.13, we have $\mathcal{R}(S) \subseteq \mathcal{R}(AS)$. Additionally, since $\mathcal{R}(AS) \subseteq \mathcal{R}(S)$, it follows that $\mathcal{R}(AS) = \mathcal{R}(S)$, as required.

$(iii) \Rightarrow (v)$. Let AS be regular, and suppose that $\mathcal{R}(AS) = \mathcal{R}(S)$. Since $\mathcal{R}(S) \subseteq \mathcal{R}(AS)$ is evident from the assumption, we can apply Lemma 3.2.4, which guarantees the existence of a matrix $Y \in \mathbb{Q}_{\mathbb{R}}^{n\times m}$ such that $S = YAS$.

Given that AS is regular, there exists an arbitrary $\{1\}$-inverse of AS, denoted by $(AS)^{(1)}$. Thus, we can express S as follows:

$$S = Y(AS) = YAS(AS)^{(1)}AS = S(AS)^{(1)}AS.$$

This establishes the necessary relationship for the result.

$(v) \Rightarrow (i)$. Let $S = S(AS)^{(1)}AS$ for some $(AS)^{(1)} \in (AS)\{1\}$. We aim to show that $X = S(AS)^{(1)}$ belongs to $A\{2\}$ and satisfies $\mathcal{C}(X) = \mathcal{C}(S)$. First, observe that

$$XAX = S(AS)^{(1)}AS(AS)^{(1)} = S(AS)^{(1)} = X.$$

This confirms that $X \in A\{2\}$. Next, since $X = S(AS)^{(1)}$, it follows that

$$\mathcal{C}(X) = \mathcal{C}(S(AS)^{(1)}) \subseteq \mathcal{C}(S).$$

Moreover, from $S = S(AS)^{(1)}AS = XAS$, we have

$$\mathcal{C}(S) = \mathcal{C}(XAS) \subseteq \mathcal{C}(X).$$

Consequently, we conclude that $\mathcal{C}(X) = \mathcal{C}(S)$.

(2) From the results in part (1), we derive the following chain of inclusions

$$\begin{aligned} A\{2\}_{\mathcal{C}(S),*} &\subseteq \{SY \mid Y \in \mathbb{Q}_{\mathbb{R}}^{k\times m},\ SYAS = S\} \\ &\subseteq \{S(AS)^{(1)} \mid (AS)^{(1)} \in (AS)\{1\}\} \\ &\subseteq A\{2\}_{\mathcal{C}(S),*}. \end{aligned}$$

The proof is complete. ∎

Corollary 3.4.5 reveals known results derived for complex matrices in [65].

Corollary 3.4.5. *Let $A \in \mathbb{C}^{m\times n}$ and $S \in \mathbb{C}^{n\times k}$ be fixed. Then $A^{(2)}_{\mathcal{C}(S),*}$ exists if and only if* $\operatorname{rank}(AS) = \operatorname{rank}(S)$. *In this case,* $A\{2\}_{\mathcal{C}(S),*} = \{S(AS)^{(1)} \,|\, (AS)^{(1)} \in (AS)\{1\}\}$.

Proof. Follows from Theorem 3.4.4, utilizing the fact that in the complex field, AS is regular, and the rank function corresponds to the usual rank of a matrix. ∎

Theorem 3.4.6 provides the existence conditions and representations of outer inverses with specified row space $\mathcal{R}(T)$.

Theorem 3.4.6. *Let the RB matrices $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ and $T \in \mathbb{Q}_{\mathbb{R}}^{l\times m}$ be given. Then*

(1) *The subsequent claims are equivalent to one another:*
   - (i) *There exists $X \in A\{2\}$ such that $\mathcal{R}(X) = \mathcal{R}(T)$;*
   - (ii) *There exists $Z \in \mathbb{Q}_{\mathbb{R}}^{n\times l}$ such that $TAZT = T$;*
   - (iii) *TA is regular and $\mathcal{C}(TA) = \mathcal{C}(T)$;*
   - (iv) *TA is regular and $\mathscr{R}_{TA} = \mathscr{R}_T$;*
   - (v) *TA is regular and $T = TA(TA)^{(1)}T$ for some $(TA)^{(1)} \in (TA)\{1\}$.*

(2) *If the statements in* (1) *are valid, then*

$$\begin{aligned} A\{2\}_{*,\mathcal{R}(T)} &= \left\{ZT \,\middle|\, Z \in \mathbb{Q}_{\mathbb{R}}^{n\times l}, TAZT = T\right\} \\ &= \left\{(TA)^{(1)}T \,\middle|\, (TA)^{(1)} \in (TA)\{1\}\right\}. \end{aligned}$$

Proof.

(1) $(i) \Rightarrow (ii)$. Let $X \in \mathbb{Q}_{\mathbb{R}}^{n\times m}$ be such that $XAX = X$ and $\mathcal{R}(X) = \mathcal{R}(T)$. According to Lemma 3.2.4, there exist matrices $Z \in \mathbb{Q}_{\mathbb{R}}^{n\times l}$ and $W \in \mathbb{Q}_{\mathbb{R}}^{l\times n}$ such that $X = ZT$ and $T = WX$. Therefore, we have

$$T = WX = WXAX = TAX = TAZT.$$

$(ii) \Rightarrow (i)$. Let $Z \in \mathbb{Q}_{\mathbb{R}}^{n\times l}$ be such that $TAZT = T$. We will demonstrate that $X = ZT$ is a $\{2\}$-inverse of A and satisfies $\mathcal{R}(X) = \mathcal{R}(T)$. First, since

$$XAX = ZTAZT = ZT = X,$$

it follows that $X \in A\{2\}$. Additionally, since $X = ZT$, it implies that

$$\mathcal{R}(X) = \mathcal{R}(ZT) \subseteq \mathcal{R}(T).$$

Moreover, given that $T = TAZT = TAX$, we have

$$\mathcal{R}(T) = \mathcal{R}(TAX) \subseteq \mathcal{R}(X).$$

Thus, we conclude that $\mathcal{R}(X) = \mathcal{R}(T)$.

$(ii) \Rightarrow (iii)$. Let $Z \in \mathbb{Q}_{\mathbb{R}}^{n\times l}$ such that $TAZT = T$. We have

$$TA = TAZTA = (TA)Z(TA),$$

which implies that $Z \in (TA)\{1\}$, meaning that TA is regular. According to Definition 3.2.1, we conclude that

$$\mathcal{C}(TA) \subseteq \mathcal{C}(T).$$

Moreover, since $TAZT = T$, we deduce

$$\mathcal{C}(T) = \mathcal{C}(TAZT) \subseteq \mathcal{C}(TA).$$

Thus, it follows that $\mathcal{C}(TA) = \mathcal{C}(T)$.

$(iii) \Rightarrow (iv)$. Let TA be regular and assume that $\mathcal{C}(TA) = \mathcal{C}(T)$. Clearly, we have $\mathcal{C}(T) \subseteq \mathcal{C}(TA)$. Define the matrix

$$\overline{T} = \begin{bmatrix} TA & T \end{bmatrix}.$$

By applying Lemma 3.2.12, we conclude that $\mathscr{R}_{TA} = \mathscr{R}_{\overline{T}}$. Now, observe that

$$\overline{T} = \begin{bmatrix} TA & T \end{bmatrix} = T\begin{bmatrix} A & I_m \end{bmatrix}.$$

The matrix $\begin{bmatrix} A & I_m \end{bmatrix}$ is right invertible since

$$\begin{bmatrix} A & I_m \end{bmatrix}\begin{bmatrix} 0 \\ I_m \end{bmatrix} = I_m.$$

As a result, using Lemma 3.2.11, we obtain $\mathscr{R}_{\overline{T}} = \mathscr{R}_T$. Thus, we can conclude that $\mathscr{R}_{TA} = \mathscr{R}_T$.

$(iv) \Rightarrow (iii)$. Let TA be regular and assume that $\mathscr{R}_{TA} = \mathscr{R}_T$. Define

$$\overline{T} = \begin{bmatrix} TA & T \end{bmatrix}.$$

Referring to the proof of $(iii) \Rightarrow (iv)$, we obtain that

$$\mathscr{R}_{\overline{T}} = \mathscr{R}_T.$$

Thus, it follows that

$$\mathscr{R}_{\overline{T}} = \mathscr{R}_{TA}.$$

By Lemma 3.2.12, we have $\mathcal{C}(T) \subseteq \mathcal{C}(TA)$. Furthermore, since $\mathcal{C}(TA) \subseteq \mathcal{C}(T)$, we conclude that $\mathcal{C}(TA) = \mathcal{C}(T)$.

$(iii) \Rightarrow (v)$. Let TA be regular and assume that $\mathcal{C}(TA) = \mathcal{C}(T)$. Since it is clear that $\mathcal{C}(T) \subseteq \mathcal{C}(TA)$, by Lemma 3.2.4, there exists a matrix $Y \in \mathbb{Q}_{\mathbb{R}}^{n\times m}$ such that $T = TAY$.

Given that TA is a regular matrix, there exist an arbitrary $\{1\}$-inverse of TA, denoted by $(TA)^{(1)}$. Thus, we can express T as follows:

$$T = (TA)Y = TA(TA)^{(1)}TAY = TA(TA)^{(1)}T.$$

This establishes the necessary relationship for the result.

$(v) \Rightarrow (i)$. Let $T = TA(TA)^{(1)}T$ for some $(TA)^{(1)} \in (TA)\{1\}$. We need to verify that $X = (TA)^{(1)}T$ is an element of $A\{2\}$ and satisfies $\mathcal{R}(X) = \mathcal{R}(T)$. First, observe that

$$XAX = (TA)^{(1)}TA(TA)^{(1)}T = (TA)^{(1)}T = X.$$

This confirms that $X \in A\{2\}$. Next, since $X = (TA)^{(1)}T$, it follows that

$$\mathcal{R}(X) = \mathcal{R}((TA)^{(1)}T) \subseteq \mathcal{R}(T).$$

Moreover, from $T = TA(TA)^{(1)}T = TAX$, we have

$$\mathcal{R}(T) = \mathcal{R}(TAX) \subseteq \mathcal{R}(X).$$

Consequently, we conclude that $\mathcal{R}(X) = \mathcal{R}(T)$.

(2) From the results in part (1), we derive the following chain of inclusions

$$\begin{aligned} A\{2\}_{*,\mathcal{R}(T)} &\subseteq \{ZT \mid Z \in \mathbb{Q}_{\mathbb{R}}^{n\times l},\ TAZT = T\} \\ &\subseteq \{(TA)^{(1)}T \mid (TA)^{(1)} \in (TA)\{1\}\} \\ &\subseteq A\{2\}_{*,\mathcal{R}(T)}. \end{aligned}$$

The proof is complete. ■

Corollary 3.4.7. *Let $A \in \mathbb{C}^{m\times n}$ and $T \in \mathbb{C}^{l\times m}$. In this case, $A^{(2)}_{*,\mathcal{R}(T)}$ exists if and only if* $\operatorname{rank}(TA) = \operatorname{rank}(T)$. *Moreover,* $A\{2\}_{*,\mathcal{R}(T)} = \{(TA)^{(1)}T \mid (TA)^{(1)} \in (TA)\{1\}\}$.

Proof. The proof follows from Theorem 3.4.6, using the fact that in the complex field, TA is regular, and the rank function corresponds to the usual rank of a matrix. ■

Now, Theorem 3.4.8 presents equivalent conditions for the existence and representation of an outer inverse with a prescribed column space $\mathcal{C}(S)$ and row space $\mathcal{R}(T)$.

Theorem 3.4.8. *Let the RB matrices $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $S \in \mathbb{Q}_{\mathbb{R}}^{n\times k}$, and $T \in \mathbb{Q}_{\mathbb{R}}^{l\times m}$ be given. Then*

(1) *The subsequent statements are equivalent:*
   (i) *There exists $X \in A\{2\}$ satisfying $\mathcal{C}(X) = \mathcal{C}(S)$ and $\mathcal{R}(X) = \mathcal{R}(T)$;*
   (ii) *There exists $Y \in \mathbb{Q}_{\mathbb{R}}^{k\times l}$ such that $SYTAS = S$ and $TASYT = T$;*
   (iii) *TAS is regular, $\mathcal{C}(TAS) = \mathcal{C}(T)$, and $\mathcal{R}(TAS) = \mathcal{R}(S)$;*
   (iv) *TAS is regular and $\mathscr{R}_{TAS} = \mathscr{R}_T = \mathscr{R}_S$;*
   (v) *TAS is regular, $S = S(TAS)^{(1)}TAS$, and $T = TAS(TAS)^{(1)}T$ for some $(TAS)^{(1)} \in (TAS)\{1\}$.*

(2) *If the statements in* (1) *are satisfied, then*

$$\begin{aligned} A^{(2)}_{\mathcal{C}(S),\mathcal{R}(T)} &= \left\{SYT \,\middle|\, Y \in \mathbb{Q}_{\mathbb{R}}^{k\times l},\ SYTAS = S, \text{ and } TASYT = T\right\} \\ &= \left\{S(TAS)^{(1)}T \,\middle|\, (TAS)^{(1)} \in (TAS)\{1\}\right\}. \end{aligned}$$

Proof.

(1) (i) $\Rightarrow$ (ii). Let $X \in \mathbb{Q}_{\mathbb{R}}^{n\times m}$ be such that $XAX = X$, $\mathcal{C}(X) = \mathcal{C}(S)$, and $\mathcal{R}(X) = \mathcal{R}(T)$. By Lemma 3.2.4, there exist matrices $Y_1 \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$, $Y_2 \in \mathbb{Q}_{\mathbb{R}}^{n\times l}$, $W \in \mathbb{Q}_{\mathbb{R}}^{m\times k}$, and $V \in \mathbb{Q}_{\mathbb{R}}^{l\times n}$ such that

$$X = SY_1 = Y_2T, \quad S = XW, \quad \text{and} \quad T = VX.$$

This implies

$$X = XAX = (SY_1)A(Y_2T) = S(Y_1AY_2)T.$$

Let $Y = Y_1AY_2$. Then, $Y \in \mathbb{Q}_{\mathbb{R}}^{k\times l}$ and $X = SYT$. Consequently, we obtain

$$S = XW = XAXW = XAS = SYTAS$$

and

$$T = VX = VXAX = TAX = TASYT.$$

(ii) $\Rightarrow$ (i). Let $Y \in \mathbb{Q}_{\mathbb{R}}^{k\times l}$ be such that $SYTAS = S$ and $TASYT = T$. We aim to show that $X = SYT$ is an element of $A\{2\}$, satisfying both $\mathcal{C}(X) = \mathcal{C}(S)$ and $\mathcal{R}(X) = \mathcal{R}(T)$. First, observe that

$$XAX = SYTASYT = SYT = X,$$

which implies that $X \in A\{2\}$. Furthermore, since $X = SYT$, it follows that

$$\mathcal{C}(X) = \mathcal{C}(SYT) \subseteq \mathcal{C}(S) \quad \text{and} \quad \mathcal{R}(X) = \mathcal{R}(SYT) \subseteq \mathcal{R}(T).$$

Additionally, from $S = SYTAS = XAS$ and $T = TASYT = TAX$, we deduce

$$\mathcal{C}(S) = \mathcal{C}(XAS) \subseteq \mathcal{C}(X) \quad \text{and} \quad \mathcal{R}(T) = \mathcal{R}(TAX) \subseteq \mathcal{R}(X).$$

Consequently, we conclude that $\mathcal{C}(X) = \mathcal{C}(S)$ and $\mathcal{R}(X) = \mathcal{R}(T)$.

$(ii) \Rightarrow (iii)$. Let $Y \in \mathbb{Q}_{\mathbb{R}}^{k\times l}$ be such that $SYTAS = S$ and $TASYT = T$. From this, it follows that

$$TAS = TA(SYTAS) = (TAS)Y(TAS),$$

which implies that $Y \in (TAS)\{1\}$, and thus TAS is regular. By Definitions 3.2.1 and 3.2.2, we conclude that

$$\mathcal{C}(TAS) \subseteq \mathcal{C}(T) \quad \text{and} \quad \mathcal{R}(TAS) \subseteq \mathcal{R}(S).$$

Furthermore, the conditions $SYTAS = S$ and $TASYT = T$ imply

$$\mathcal{C}(T) = \mathcal{C}(TASYT) \subseteq \mathcal{C}(TAS) \quad \text{and} \quad \mathcal{R}(S) = \mathcal{R}(SYTAS) \subseteq \mathcal{R}(TAS).$$

Thus, we conclude that

$$\mathcal{C}(TAS) = \mathcal{C}(T) \quad \text{and} \quad \mathcal{R}(TAS) = \mathcal{R}(S).$$

$(iii) \Rightarrow (iv)$. Let TAS be regular, with $\mathcal{C}(TAS) = \mathcal{C}(T)$ and $\mathcal{R}(TAS) = \mathcal{R}(S)$. It is clear that $\mathcal{C}(T) \subseteq \mathcal{C}(TAS)$. Define

$$\overline{T} = \begin{bmatrix} TAS & T \end{bmatrix}.$$

By applying Lemma 3.2.12, we conclude that

$$\mathscr{R}_{TAS} = \mathscr{R}_{\overline{T}}.$$

Now, expressing $\overline{T}$ as

$$\overline{T} = T\begin{bmatrix} AS & I_m \end{bmatrix},$$

it follows that the matrix $\begin{bmatrix} AS & I_m \end{bmatrix}$ is right invertible since

$$\begin{bmatrix} AS & I_m \end{bmatrix}\begin{bmatrix} 0 \\ I_m \end{bmatrix} = I_m.$$

Thus, by Lemma 3.2.11, we obtain

$$\mathscr{R}_{\overline{T}} = \mathscr{R}_T.$$

Consequently, we have

$$\mathscr{R}_{TAS} = \mathscr{R}_T.$$

Additionally, since $\mathcal{R}(S) \subseteq \mathcal{R}(TAS)$, define

$$\overline{S} = \begin{bmatrix} TAS \\ S \end{bmatrix}.$$

By Lemma 3.2.13, it follows that

$$\mathscr{R}_{TAS} = \mathscr{R}_{\overline{S}}.$$

Expressing $\overline{S}$ as

$$\overline{S} = \begin{bmatrix} TA \\ I_n \end{bmatrix} S,$$

where the matrix $\begin{bmatrix} TA \\ I_n \end{bmatrix}$ is left invertible because

$$\begin{bmatrix} 0 & I_n \end{bmatrix} \begin{bmatrix} TA \\ I_n \end{bmatrix} = I_n,$$

applying Lemma 3.2.11 gives

$$\mathscr{R}_{\overline{S}} = \mathscr{R}_S.$$

Thus, we conclude

$$\mathscr{R}_{TAS} = \mathscr{R}_S,$$

and therefore $\mathscr{R}_{TAS} = \mathscr{R}_T = \mathscr{R}_S$.

$(iv) \Rightarrow (iii)$. Let TAS be regular, with $\mathscr{R}_{TAS} = \mathscr{R}_T = \mathscr{R}_S$. Define

$$\overline{T} = \begin{bmatrix} TAS & T \end{bmatrix} \quad \text{and} \quad \overline{S} = \begin{bmatrix} TAS \\ S \end{bmatrix}.$$

From the proof of $(iii) \Rightarrow (iv)$, we can conclude

$$\mathscr{R}_{\overline{T}} = \mathscr{R}_T \quad \text{and} \quad \mathscr{R}_{\overline{S}} = \mathscr{R}_S.$$

Thus, we have

$$\mathscr{R}_{\overline{T}} = \mathscr{R}_{TAS} \quad \text{and} \quad \mathscr{R}_{\overline{S}} = \mathscr{R}_{TAS}.$$

By Lemma 3.2.12, it follows that

$$\mathcal{C}(T) \subseteq \mathcal{C}(TAS),$$

and by Lemma 3.2.13, we conclude that

$$\mathcal{R}(S) \subseteq \mathcal{R}(TAS).$$

Furthermore, we have

$$\mathcal{C}(TAS) \subseteq \mathcal{C}(T) \quad \text{and} \quad \mathcal{R}(TAS) \subseteq \mathcal{R}(S).$$

Consequently, it follows that $\mathcal{C}(TAS) = \mathcal{C}(T)$ and $\mathcal{R}(TAS) = \mathcal{R}(S)$.

$(iii) \Rightarrow (v)$. Let TAS be regular with the additional properties $\mathcal{C}(TAS) = \mathcal{C}(T)$ and $\mathcal{R}(TAS) = \mathcal{R}(S)$. It is evident that $\mathcal{C}(T) \subseteq \mathcal{C}(TAS)$ and $\mathcal{R}(S) \subseteq \mathcal{R}(TAS)$. By Lemma 3.2.4, there exist matrices $Y \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$ and $W \in \mathbb{Q}_{\mathbb{R}}^{n\times l}$ such that

$$T = TASY \quad \text{and} \quad S = WTAS.$$

Since TAS is a regular matrix, there exists an arbitrary $\{1\}$-inverse $(TAS)^{(1)}$ of TAS. Thus, we can express

$$T = (TAS)Y = TAS(TAS)^{(1)}TASY = TAS(TAS)^{(1)}T$$

and

$$S = W(TAS) = WTAS(TAS)^{(1)}TAS = S(TAS)^{(1)}TAS.$$

$(v) \Rightarrow (i)$. Let $S = S(TAS)^{(1)}TAS$ and $T = TAS(TAS)^{(1)}T$ for some $(TAS)^{(1)} \in (TAS)\{1\}$. We aim to show that $X = S(TAS)^{(1)}T$ belongs to $A\{2\}$ and satisfies $\mathcal{C}(X) = \mathcal{C}(S)$ and $\mathcal{R}(X) = \mathcal{R}(T)$. Indeed, since

$$XAX = S(TAS)^{(1)}TAS(TAS)^{(1)}T = S(TAS)^{(1)}T = X,$$

it follows that $X \in A\{2\}$. Moreover, $X = S(TAS)^{(1)}T$ implies that

$$\mathcal{C}(X) = \mathcal{C}(S(TAS)^{(1)}T) \subseteq \mathcal{C}(S) \quad \text{and} \quad \mathcal{R}(X) = \mathcal{R}(S(TAS)^{(1)}T) \subseteq \mathcal{R}(T).$$

Additionally, from the relationships

$$S = S(TAS)^{(1)}TAS = XAS \quad \text{and} \quad T = TAS(TAS)^{(1)}T = TAX,$$

we deduce that

$$\mathcal{C}(S) = \mathcal{C}(XAS) \subseteq \mathcal{C}(X) \quad \text{and} \quad \mathcal{R}(T) = \mathcal{R}(TAX) \subseteq \mathcal{R}(X).$$

Thus, we conclude that $\mathcal{C}(X) = \mathcal{C}(S)$ and $\mathcal{R}(X) = \mathcal{R}(T)$.

(2) From the proof of part (1), it follows

$$\begin{aligned} A\{2\}_{\mathcal{C}(S),\mathcal{R}(T)} &\subseteq \{SYT \,|\, Y \in \mathbb{Q}_{\mathbb{R}}^{k\times l},\ SYTAS = S \text{ and } TASYT = T\} \\ &\subseteq \{S(TAS)^{(1)}T \,|\, (TAS)^{(1)} \in (TAS)\{1\}\} \\ &\subseteq A\{2\}_{\mathcal{C}(S),\mathcal{R}(T)}. \end{aligned}$$

The proof is complete. ∎

Corollary 3.4.9 reveals known results derived for complex matrices.

Corollary 3.4.9. *If $A \in \mathbb{C}^{m\times n}$, $S \in \mathbb{C}^{n\times k}$, and $T \in \mathbb{C}^{l\times m}$ are given, then $A^{(2)}_{\mathcal{C}(S),\mathcal{R}(T)}$ exists if and only if* $\operatorname{rank}(TAS) = \operatorname{rank}(T) = \operatorname{rank}(S)$. *If these conditions are satisfied then*

$$A^{(2)}_{\mathcal{C}(S),\mathcal{R}(T)} = \left\{S(TAS)^{(1)}T \,\middle|\, (TAS)^{(1)} \in (TAS)\{1\}\right\}.$$

Proof. The proof follows from Theorem 3.4.8, using the fact that in the complex field, TAS is regular, and the rank function corresponds to the usual rank of a matrix. ∎

Next, we outline conditions for the existence and several representations of a $\{1,2\}$-inverse with a predefined column space $\mathcal{C}(S)$.

Theorem 3.4.10. *Let the RB matrices $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ and $S \in \mathbb{Q}_{\mathbb{R}}^{n\times k}$ be given. Then*

(1) *The subsequent claims are equivalent one another:*
   - (i) *There exists $X \in A\{1,2\}$ satisfying $\mathcal{C}(X) = \mathcal{C}(S)$;*
   - (ii) *There exists $U \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$ satisfying $SUAS = S$ and $ASUA = A$;*
   - (iii) *There exist $U, V \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$ such that $SUAS = S$ and $ASVA = A$;*
   - (iv) *AS is regular, $\mathcal{R}(AS) = \mathcal{R}(S)$, and $\mathcal{C}(AS) = \mathcal{C}(A)$;*
   - (v) *AS is regular and $\mathscr{R}_{AS} = \mathscr{R}_S = \mathscr{R}_A$;*
   - (vi) *AS is regular, $S = S(AS)^{(1)}AS$, and $A = AS(AS)^{(1)}A$ for some $(AS)^{(1)} \in AS\{1\}$.*

(2) *If the statements in* (1) *are satisfied, then*

$$\begin{aligned} A\{1,2\}_{\mathcal{C}(S),*} &= \left\{SU \,\middle|\, U \in \mathbb{Q}_{\mathbb{R}}^{k\times m},\ SUAS = S,\ \text{and } ASUA = A\right\} \\ &= \left\{S(AS)^{(1)} \,\middle|\, (AS)^{(1)} \in (AS)\{1\}\right\}. \end{aligned}$$

Proof.

(1) $(i) \Rightarrow (ii)$. Let $X \in \mathbb{Q}_{\mathbb{R}}^{n\times m}$ satisfy $X \in A\{1,2\}$ and $\mathcal{C}(X) = \mathcal{C}(S)$. According to Lemma 3.2.4, there exist matrices $U \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$ and $W \in \mathbb{Q}_{\mathbb{R}}^{m\times k}$ such that $X = SU$ and $S = XW$. As a result, we obtain the following

$$A = AXA = ASUA$$

and

$$S = XW = XAXW = XAS = SUAS.$$

$(ii) \Rightarrow (i)$. Let $U \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$ be such that $SUAS = S$ and $ASUA = A$. The objective is to show that $X = SU \in A\{1,2\}$ and that $\mathcal{C}(X) = \mathcal{C}(S)$. Indeed, since

$$AXA = ASUA = A,$$

by applying Theorem 3.4.4, part $(ii) \Rightarrow (i)$, it follows that $X \in A\{1,2\}$ and satisfies $\mathcal{C}(X) = \mathcal{C}(S)$.

$(ii) \Rightarrow (iii)$. The result follows directly from part (ii) of the theorem.

$(iii) \Rightarrow (ii)$. Let $U, V \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$ be such that $SUAS = S$ and $ASVA = A$. We can then proceed as follows

$$A = ASVA = A(SUAS)VA = ASU(ASVA) = ASUA.$$

$(ii) \Rightarrow (iv)$. Let $U \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$ be such that $SUAS = S$ and $ASUA = A$. By Theorem 3.4.4, from $(ii) \Rightarrow (iii)$, we know that AS is regular and satisfies $\mathcal{R}(AS) = \mathcal{R}(S)$. It is evident that $\mathcal{C}(AS) \subseteq \mathcal{C}(A)$. Moreover, since $A = ASUA$, we also have

$$\mathcal{C}(A) = \mathcal{C}(ASUA) \subseteq \mathcal{C}(AS).$$

Hence, we conclude that $\mathcal{C}(AS) = \mathcal{C}(A)$.

$(iv) \Rightarrow (v)$. Let AS be regular, with $\mathcal{R}(AS) = \mathcal{R}(S)$ and $\mathcal{C}(AS) = \mathcal{C}(A)$. It is evident that $\mathcal{C}(A) \subseteq \mathcal{C}(AS)$. Consider the matrix

$$\overline{S} = \begin{bmatrix} AS & A \end{bmatrix}.$$

Using Lemma 3.2.12, we obtain $\mathscr{R}_{AS} = \mathscr{R}_{\overline{S}}$. Now, we can express $\overline{S}$ as

$$\overline{S} = \begin{bmatrix} AS & A \end{bmatrix} = A\begin{bmatrix} S & I_n \end{bmatrix}.$$

Notably, the matrix $\begin{bmatrix} S & I_n \end{bmatrix}$ is right invertible, as

$$\begin{bmatrix} S & I_n \end{bmatrix}\begin{bmatrix} 0 \\ I_n \end{bmatrix} = I_n.$$

Consequently, applying Lemma 3.2.11, we conclude that $\mathscr{R}_{\overline{S}} = \mathscr{R}_A$. Thus, $\mathscr{R}_{AS} = \mathscr{R}_A$.

According to Theorem 3.4.4 $(iii) \Rightarrow (iv)$, we can assert that $\mathscr{R}_{AS} = \mathscr{R}_S$. Therefore, we have

$$\mathscr{R}_{AS} = \mathscr{R}_S = \mathscr{R}_A.$$

$(v) \Rightarrow (iv)$. Let AS be regular with $\mathscr{R}_{AS} = \mathscr{R}_S = \mathscr{R}_A$. Consider the matrix

$$\overline{S} = \begin{bmatrix} AS & A \end{bmatrix}.$$

Using the proof of $(iv) \Rightarrow (v)$, we deduce that $\mathscr{R}_{\overline{S}} = \mathscr{R}_A$. Thus, we have $\mathscr{R}_{\overline{S}} = \mathscr{R}_{AS}$. According to Lemma 3.2.12, it follows that $\mathcal{C}(A) \subseteq \mathcal{C}(AS)$. Furthermore, since $\mathcal{C}(AS) \subseteq \mathcal{C}(A)$, we conclude that $\mathcal{C}(AS) = \mathcal{C}(A)$.

Finally, by applying Theorem 3.4.4, part $(iv) \Rightarrow (iii)$, we can affirm that $\mathcal{R}(AS) = \mathcal{R}(S)$.

$(iv) \Rightarrow (vi)$. Let AS be regular, with $\mathcal{R}(AS) = \mathcal{R}(S)$ and $\mathcal{C}(AS) = \mathcal{C}(A)$. It follows that $\mathcal{C}(A) \subseteq \mathcal{C}(AS)$. By Lemma 3.2.4, there exists a matrix $W \in \mathbb{Q}_{\mathbb{R}}^{k \times n}$ such that $A = ASW$. Since AS is regular, let $(AS)^{(1)}$ be an arbitrary $\{1\}$-inverse of AS. Consequently, we have

$$A = (AS)W = AS(AS)^{(1)}ASW = AS(AS)^{(1)}A.$$

Applying Theorem 3.4.4, part $(iii) \Rightarrow (v)$, we can conclude that $S = S(AS)^{(1)}AS$.

$(vi) \Rightarrow (i)$. Let $S = S(AS)^{(1)}AS$ and $A = AS(AS)^{(1)}A$ for some $(AS)^{(1)} \in (AS)\{1\}$. We claim that $X = S(AS)^{(1)}$ is an element of $A\{1,2\}$ that satisfies $\mathcal{C}(X) = \mathcal{C}(S)$. To demonstrate this, we observe that

$$AXA = AS(AS)^{(1)}A = A.$$

By Theorem 3.4.4, part $(v) \Rightarrow (i)$, it is clear that $X \in A\{1,2\}$ and that $\mathcal{C}(X) = \mathcal{C}(S)$.

(2) From the proof of part (1), it follows

$$\begin{aligned} A\{1,2\}_{\mathcal{C}(S),*} &\subseteq \left\{ SU \,\middle|\, U \in \mathbb{Q}_{\mathbb{R}}^{k \times m}, SUAS = S, \text{ and } ASUA = A \right\} \\ &\subseteq \left\{ S(AS)^{(1)} \,\middle|\, (AS)^{(1)} \in (AS)\{1\} \right\} \\ &\subseteq A\{1,2\}_{\mathcal{C}(S),*}. \end{aligned}$$

The proof is complete. ∎

Corollary 3.4.11 reveals known results derived for complex matrices in [65].

Corollary 3.4.11. *Let $A \in \mathbb{C}^{m\times n}$ and $S \in \mathbb{C}^{n\times k}$. Then $A^{(1,2)}_{\mathcal{C}(S),*}$ exists if and only if* $\operatorname{rank}(AS) = \operatorname{rank}(S) = \operatorname{rank}(A)$. *In this case,*

$$A\{1,2\}_{\mathcal{C}(S),*} = \left\{S(AS)^{(1)} \,\middle|\, (AS)^{(1)} \in (AS)\{1\}\right\}.$$

Proof. The proof follows from Theorem 3.4.10, using the fact that in the complex field, AS is regular, and the rank function corresponds to the usual rank of a matrix. ∎

The following theorem offers equivalent conditions for the existence and representation of $\{1,2\}$-inverse with a specified row space $\mathcal{R}(T)$.

Theorem 3.4.12. *Let the RB matrices $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ and $T \in \mathbb{Q}_{\mathbb{R}}^{l\times m}$ be given. Then*

(1) *The subsequent claims are equivalent:*
   - (i) *There exists $X \in A\{1,2\}$ satisfying $\mathcal{R}(X) = \mathcal{R}(T)$;*
   - (ii) *There exists $U \in \mathbb{Q}_{\mathbb{R}}^{n\times l}$ such that $TAUT = T$ and $AUTA = A$;*
   - (iii) *There exist $U, V \in \mathbb{Q}_{\mathbb{R}}^{n\times l}$ such that $TAUT = T$ and $AVTA = A$;*
   - (iv) *TA is regular, $\mathcal{C}(TA) = \mathcal{C}(T)$, and $\mathcal{R}(TA) = \mathcal{R}(A)$;*
   - (v) *TA is regular and $\mathscr{R}_{TA} = \mathscr{R}_T = \mathscr{R}_A$;*
   - (vi) *TA is regular, $T = TA(TA)^{(1)}T$, and $A = A(TA)^{(1)}TA$ for some $(TA)^{(1)} \in (TA)\{1\}$.*

(2) *If the statements in* (1) *are valid, then*

$$\begin{aligned} A\{1,2\}_{*,\mathcal{R}(T)} &= \left\{UT \,\middle|\, U \in \mathbb{Q}_{\mathbb{R}}^{n\times l}, TAUT = T, \text{ and } AUTA = A\right\} \\ &= \left\{(TA)^{(1)}T \,\middle|\, (TA)^{(1)} \in (TA)\{1\}\right\}. \end{aligned}$$

Proof. The proof follows by employing Theorem 3.4.6 and follows a similar approach as outlined in Theorem 3.4.10. ∎

Corollary 3.4.13. *Let $A \in \mathbb{C}^{m\times n}$ and $T \in \mathbb{C}^{l\times m}$. Then $A^{(1,2)}_{*,\mathcal{R}(T)}$ exists if and only if* $\operatorname{rank}(TA) = \operatorname{rank}(T) = \operatorname{rank}(A)$. *In this case,*

$$A\{1,2\}_{*,\mathcal{R}(T)} = \left\{(TA)^{(1)}T \,\middle|\, (TA)^{(1)} \in (TA)\{1\}\right\}.$$

Proof. The proof follows from Theorem 3.4.12, using the fact that in the complex field, TA is regular, and the rank function corresponds to the usual rank of a matrix. ∎

In Theorem 3.4.14 we outline equivalent existence conditions for $\{1,2\}$-inverse with a prescribed column space $\mathcal{C}(S)$ and row space $\mathcal{R}(T)$.

Theorem 3.4.14. *Let the RB matrices* $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $S \in \mathbb{Q}_{\mathbb{R}}^{n\times k}$, *and* $T \in \mathbb{Q}_{\mathbb{R}}^{l\times m}$ *be given. Then*

(1) *The subsequent claims are equivalent to one another:*

(i) *There exists* $X \in A\{1,2\}$ *satisfying* $\mathcal{C}(X) = \mathcal{C}(S)$ *and* $\mathcal{R}(X) = \mathcal{R}(T)$*;*

(ii) *There exist* $U \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$ *and* $V \in \mathbb{Q}_{\mathbb{R}}^{n\times l}$ *such that* $SUAS = S$, $ASUA = A$, $TAVT = T$, *and* $AVTA = A$*;*

(iii) AS *and* TA *are regular,* $\mathcal{C}(AS) = \mathcal{C}(A)$, $\mathcal{R}(AS) = \mathcal{R}(S)$, $\mathcal{C}(TA) = \mathcal{C}(T)$, *and* $\mathcal{R}(TA) = \mathcal{R}(A)$*;*

(iv) AS *and* TA *are regular,* $\mathscr{R}_{AS} = \mathscr{R}_S = \mathscr{R}_A$, *and* $\mathscr{R}_{TA} = \mathscr{R}_T = \mathscr{R}_A$*;*

(v) AS *and* TA *are regular,* $S = S(AS)^{(1)}AS$, $T = TA(TA)^{(1)}T$, *and* $A = AS(AS)^{(1)}A = A(TA)^{(1)}TA$ *for some* $(AS)^{(1)} \in (AS)\{1\}$ *and* $(TA)^{(1)} \in (TA)\{1\}$.

(2) *If the statements in* (1) *are valid, then*

$$\begin{aligned}A\{1,2\}_{\mathcal{C}(S),\mathcal{R}(T)} &= \Big\{SUAVT \,\Big|\, U \in \mathbb{Q}_{\mathbb{R}}^{k\times m},\ V \in \mathbb{Q}_{\mathbb{R}}^{n\times l},\ SUAS = S,\\ &\qquad ASUA = A,\ TAVT = T,\ \text{and}\ AVTA = A\Big\}\\ &= \Big\{S(AS)^{(1)}A(TA)^{(1)}T \,\Big|\, (AS)^{(1)} \in (AS)\{1\}\ \text{and}\\ &\qquad (TA)^{(1)} \in (TA)\{1\}\Big\}.\end{aligned}$$

Proof.

(1) The proof follows by the application of Theorems 3.4.8, 3.4.10, and 3.4.12.

(2) Let $U \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$ and $V \in \mathbb{Q}_{\mathbb{R}}^{n\times l}$ be such that $SUAS = S$, $ASUA = A$, $TAVT = T$, and $AVTA = A$. We will verify that $X = SUAVT \in A\{1,2\}$ and that it satisfies $\mathcal{C}(X) = \mathcal{C}(S)$ and $\mathcal{R}(X) = \mathcal{R}(T)$. First, observe the following:

$$\begin{aligned}XAX &= (SUAVT)A(SUAVT) = SU(AVTA)SUAVT\\ &= (SUAS)UAVT = SUAVT = X,\end{aligned}$$

and

$$AXA = A(SUAVT)A = ASU(AVTA) = ASUA = A.$$

Hence, $X \in A\{1,2\}$. Furthermore, since $X = SUAVT$, it follows that

$$\mathcal{C}(X) = \mathcal{C}(SUAVT) \subseteq \mathcal{C}(S)$$

and

$$\mathcal{R}(X) = \mathcal{R}(SUAVT) \subseteq \mathcal{R}(T).$$

Additionally, from $S = SUAS = SU(AVTA)S = (SUAVT)AS = XAS$ and $T = TAVT = T(ASUA)VT = TA(SUAVT) = TAX$, we deduce that

$$\mathcal{C}(S) = \mathcal{C}(XAS) \subseteq \mathcal{C}(X) \quad \text{and} \quad \mathcal{R}(T) = \mathcal{R}(TAX) \subseteq \mathcal{R}(X).$$

Therefore, we conclude that $\mathcal{C}(X) = \mathcal{C}(S)$ and $\mathcal{R}(X) = \mathcal{R}(T)$.

Similarly, it can be verified that $X = S(AS)^{(1)}A(TA)^{(1)}T \in A\{1,2\}$, and that $\mathcal{C}(X) = \mathcal{C}(S)$ and $\mathcal{R}(X) = \mathcal{R}(T)$. ∎

Corollary 3.4.15. *For arbitrary $A \in \mathbb{C}^{m\times n}$, $S \in \mathbb{C}^{n\times k}$, and $T \in \mathbb{C}^{l\times m}$ the $\{1,2\}$-inverse $A^{(1,2)}_{\mathcal{C}(S),\mathcal{R}(T)}$ exists if and only if* $\operatorname{rank}(AS) = \operatorname{rank}(TA) = \operatorname{rank}(T) = \operatorname{rank}(S) = \operatorname{rank}(A)$*. In this case,*

$$A\{1,2\}_{\mathcal{C}(S),\mathcal{R}(T)} = \left\{ S(AS)^{(1)}A(TA)^{(1)}T \,\middle|\, (AS)^{(1)} \in (AS)\{1\} \text{ and } (TA)^{(1)} \in (TA)\{1\} \right\}.$$

Proof. The proof follows directly from Theorem 3.4.14, using the fact that in the complex field, AS and TA are regular, and the rank function corresponds to the usual rank of a matrix. ∎

3.5. Algorithms for Computing RBGIs and Numerical Verification

Building on the discussions from the preceding section, this section presents numerical algorithms for computing the outer inverse and $\{1,2\}$-inverse of an RB matrix $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, adhering to specific conditions regarding column and/or row space. Additionally, we provide examples to validate the efficiency of the proposed algorithms. Implementation and numerical experiments are carried out on an Intel Core $i7-9700@3.00GHz/16GB$ computer utilizing $MATLAB\ R2021b$ software.

Building on Theorem 3.4.4, we propose Algorithm 3.5.1 for computing the outer inverse of $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ with a specified column space $\mathcal{C}(S)$. To further evaluate the accuracy and efficiency of our proposed method, we present an example demonstrating the computation of the generalized inverse $A^{(2)}_{\mathcal{C}(S),*}$ for reduced biquaternion matrices.

Algorithm 3.5.1 Computation of $\mathbf{X} = \mathbf{A}^{(2)}_{\mathcal{C}(\mathbf{S}),*}$

Input: $A = A_1 + A_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $S = S_1 + S_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times k}$.

Output: $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times m}$.

Step 1: Consistency Check: Verify the consistency of the RBME $SYAS = S$ for $Y = Y_1 + Y_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$ using Remark 3.3.4. Proceed with the next steps only if the equation is consistent.

Step 2: Solve: Solve the RBME $SYAS = S$ using Remark 3.3.4.

Step 3: Compute: Determine $A^{(2)}_{\mathcal{C}(S),*} = SY$.

Example 3.5.1. *Consider $A = A_1 + A_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{3\times 2}$ and $S = S_1 + S_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{2\times 2}$, where*

$$A_1 = \begin{bmatrix} 1+2\boldsymbol{i} & 1+3\boldsymbol{i} \\ 1+4\boldsymbol{i} & 7+6\boldsymbol{i} \\ 4+9\boldsymbol{i} & 8+6\boldsymbol{i} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3+4\boldsymbol{i} & 4+5\boldsymbol{i} \\ 6+\boldsymbol{i} & 5\boldsymbol{i} \\ 5 & 2\boldsymbol{i} \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 3+4\boldsymbol{i} & 5+10\boldsymbol{i} \\ 3+2\boldsymbol{i} & 7+3\boldsymbol{i} \end{bmatrix}, \quad S_2 = \begin{bmatrix} 3+3\boldsymbol{i} & 2+4\boldsymbol{i} \\ 1+2\boldsymbol{i} & 4 \end{bmatrix}.$$

We apply Algorithm 3.5.1 to determine the outer inverse $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{2\times 3}$ of RB matrix A with specified column space $\mathcal{C}(S)$. Solving the matrix equation $SYAS = S$ for $Y = Y_1 + Y_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{2\times 3}$ gives

$$Y_1 = \begin{bmatrix} -0.0073+0.0123\boldsymbol{i} & 0.0194+0.0324\boldsymbol{i} & 0.0162-0.0155\boldsymbol{i} \\ 0.0026+0.0040\boldsymbol{i} & 0.0049-0.0064\boldsymbol{i} & -0.0129+0.0025\boldsymbol{i} \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} -0.0085-0.0182\boldsymbol{i} & -0.0448+0.0029\boldsymbol{i} & 0.0144-0.0133\boldsymbol{i} \\ 0.0061-0.0007\boldsymbol{i} & 0.0119-0.0138\boldsymbol{i} & -0.0064+0.0109\boldsymbol{i} \end{bmatrix}.$$

Then $X = SY$ is given by

$$X_1 = \begin{bmatrix} -0.0545-0.0035\boldsymbol{i} & -0.0468+0.0860\boldsymbol{i} & 0.0476-0.0989\boldsymbol{i} \\ 0.0117+0.0203\boldsymbol{i} & 0.0438-0.0361\boldsymbol{i} & -0.0029+0.0239\boldsymbol{i} \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 0.0147+0.0020\boldsymbol{i} & 0.0481+0.0416\boldsymbol{i} & 0.0148-0.0363\boldsymbol{i} \\ 0.0338-0.0443\boldsymbol{i} & -0.0413-0.0963\boldsymbol{i} & -0.0120+0.0730\boldsymbol{i} \end{bmatrix}.$$

Taking into account $X = SY$ and $S = SYAS = XAS$, we get $\mathcal{C}(X) = \mathcal{C}(S)$. Additionally, $\|XAX - X\|_F = 1.4550 \times 10^{-15}$. Thus, $X = A^{(2)}_{\mathcal{C}(S),*}$.

Example 3.5.2. *Let*

$$A = A_1 + A_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}, \quad S = S_1 + S_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times k},$$

where the complex components are generated as:

$$A_1 = A_2 = \texttt{rand}(m,n) + \texttt{rand}(m,n)\boldsymbol{i} \in \mathbb{C}^{m\times n},$$

$$S_1 = S_2 = \texttt{rand}(n,k) + \texttt{rand}(n,k)\boldsymbol{i} \in \mathbb{C}^{n\times k}.$$

The matrix dimensions are chosen based on a variable parameter t, such that:

$$m = t, \quad n = t+5, \quad k = t.$$

In this example, t varies from 5 to 70 in increments of 5.

Objective: *We compute $A^{(2)}_{\mathcal{C}(S),*}$ given by $X = SY$, where Y is the solution to the RBME:*

$$SYAS = S.$$

We apply three different methods to compute Y:

(a) *Our proposed Algorithm 3.3.1.*
(b) *The approach based on Theorem 3.1 from [80].*
(c) *The approach based on Theorem 3.1 from [75].*

Let X_1, X_2, and X_3 denote the computed solutions using these three methods, respectively.

Error and CPU Time Evaluation: *The error metrics are defined as:*

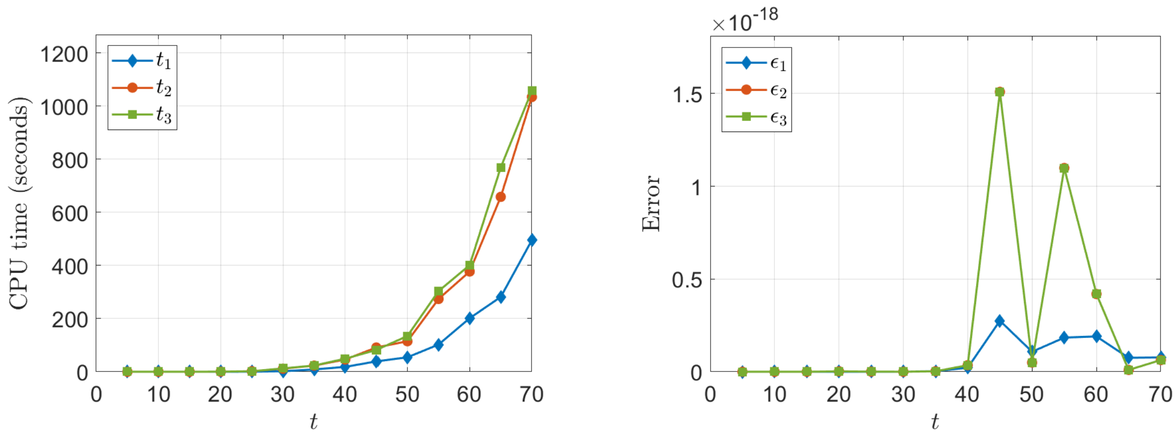
$$\epsilon_1 = \|X_1AX_1 - X_1\|_F, \quad \epsilon_2 = \|X_2AX_2 - X_2\|_F, \quad \epsilon_3 = \|X_3AX_3 - X_3\|_F.$$

We conduct 50 trials and compute the average CPU time for each method. Let t_1, t_2, and t_3 be the average CPU time for computing X_1, X_2, and X_3, respectively.

Results and Discussion:

- Figure 3.5.1(a) illustrates the average CPU times, demonstrating that our method is computationally more efficient than the other methods.
- Figure 3.5.1(b) presents a comparative analysis of the error values, verifying whether the computed solutions satisfy the {2}-inverse property. To confirm this, we compute the error $\|XAX - X\|_F$, ensuring that all computed solutions meet the required inverse conditions. The results demonstrate that the errors remain below 10^{-18}, validating the accuracy and correctness of the obtained solutions.

Following the results obtained in Theorem 3.4.6, we now propose Algorithm 3.5.2 for calculating the outer inverse of $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ with a specified row space $\mathcal{R}(T)$.



(a) CPU time comparison. (b) Error comparison.

Figure 3.5.1. CPU time and error comparison for computing $A^{(2)}_{\mathcal{C}(S),*}$ using different methods.

Algorithm 3.5.2 **Computation of $\mathbf{X} = \mathbf{A}^{(2)}_{*,\mathcal{R}(\mathbf{T})}$**

Input: $A = A_1 + A_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $T = T_1 + T_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{l\times m}$.

Output: $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times m}$.

Step 1: Consistency Check: Verify the consistency of the RBME $TAZT = T$ for $Z = Z_1 + Z_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times l}$ using Remark 3.3.4. Proceed with the next steps only if the equation is consistent.

Step 2: Solve: Solve the RBME $TAZT = T$ using Remark 3.3.4.

Step 3: Compute: Determine $A^{(2)}_{*,\mathcal{R}(T)} = ZT$.

Example 3.5.3. *Consider $A = A_1 + A_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{2\times 3}$ and $T = T_1 + T_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{2\times 2}$, where*

$$A_1 = \begin{bmatrix} 7+2\boldsymbol{i} & 5+8\boldsymbol{i} & 2+3\boldsymbol{i} \\ 9+3\boldsymbol{i} & 8+2\boldsymbol{i} & 3+3\boldsymbol{i} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 6+5\boldsymbol{i} & 1+3\boldsymbol{i} & 2+9\boldsymbol{i} \\ 8+\boldsymbol{i} & 7+2\boldsymbol{i} & 2+5\boldsymbol{i} \end{bmatrix},$$
$$T_1 = \begin{bmatrix} 5+7\boldsymbol{i} & 3+13\boldsymbol{i} \\ 5+3\boldsymbol{i} & 7+9\boldsymbol{i} \end{bmatrix}, \quad T_2 = \begin{bmatrix} 5+3\boldsymbol{i} & 8+7\boldsymbol{i} \\ 5+7\boldsymbol{i} & 4+4\boldsymbol{i} \end{bmatrix}.$$

We employ Algorithm 3.5.2 to determine the outer inverse $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{3\times 2}$ of RB matrix A with specified row space $\mathcal{R}(T)$. Solving the matrix equation $TAZT = T$ for

$Z = Z_1 + Z_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{3\times 2}$ gives

$$Z_1 = \begin{bmatrix} -0.0155 - 0.0036\boldsymbol{i} & -0.0066 - 0.0038\boldsymbol{i} \\ 0.0061 - 0.0149\boldsymbol{i} & -0.0034 + 0.0139\boldsymbol{i} \\ 0.0149 + 0.0036\boldsymbol{i} & -0.0058 - 0.0186\boldsymbol{i} \end{bmatrix},$$

$$Z_2 = \begin{bmatrix} 0.0005 + 0.0026\boldsymbol{i} & 0.0247 + 0.0061\boldsymbol{i} \\ 0.0099 - 0.0078\boldsymbol{i} & -0.0187 + 0.0107\boldsymbol{i} \\ -0.0005 + 0.0156\boldsymbol{i} & -0.0093 - 0.0069\boldsymbol{i} \end{bmatrix}.$$

Then $X = ZT = X_1 + X_2\boldsymbol{j}$ is given by

$$X_1 = \begin{bmatrix} 0.0027 + 0.0529\boldsymbol{i} & 0.0499 - 0.1501\boldsymbol{i} \\ -0.0184 - 0.0588\boldsymbol{i} & 0.0804 + 0.0770\boldsymbol{i} \\ 0.0282 - 0.0112\boldsymbol{i} & 0.0011 + 0.0784\boldsymbol{i} \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 0.0177 - 0.0083\boldsymbol{i} & -0.0226 + 0.1011\boldsymbol{i} \\ -0.0596 + 0.0174\boldsymbol{i} & -0.0109 - 0.0218\boldsymbol{i} \\ 0.0270 - 0.0587\boldsymbol{i} & -0.0626 - 0.0566\boldsymbol{i} \end{bmatrix}.$$

Taking into account $X = ZT$ and $T = TAZT = TAX$, we get $\mathcal{R}(X) = \mathcal{R}(T)$. Additionally, $\|XAX - X\|_F = 4.7851 \times 10^{-15}$. Thus, $X = A^{(2)}_{*,\mathcal{R}(T)}$.

By Theorem 3.4.8, we present Algorithm 3.5.3 for computing the outer inverse of $A \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ with a specified column space $\mathcal{C}(S)$ and row space $\mathcal{R}(T)$.

Algorithm 3.5.3 Computation of $\mathbf{X} = \mathbf{A}^{(2)}_{\mathcal{C}(\mathbf{S}),\mathcal{R}(\mathbf{T})}$

Input: $A = A_1 + A_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $S = S_1 + S_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times k}$, $T = T_1 + T_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{l\times m}$.

Output: $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times m}$.

Step 1: Consistency Check: Verify the consistency of the RBME $(SYTAS, TASYT) = (S, T)$ for $Y = Y_1 + Y_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{k\times l}$ using Step 2 of Algorithm 3.3.1. If the equation is consistent, proceed to the subsequent steps.

Step 2: Solve: Solve the RBME $(SYTAS, TASYT) = (S, T)$ using Steps 3 and 4 of Algorithm 3.3.1.

Step 3: Compute: Determine $A^{(2)}_{\mathcal{C}(S),\mathcal{R}(T)} = SYT$.

Example 3.5.4. *Consider* $A = A_1 + A_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{2\times 3}$, $S = S_1 + S_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{3\times 2}$, *and* $T = T_1 + T_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{3\times 2}$, *where*

$$A_1 = \begin{bmatrix} 3+2\boldsymbol{i} & 6+5\boldsymbol{i} & 4+7\boldsymbol{i} \\ 1+\boldsymbol{i} & 6+9\boldsymbol{i} & 3+8\boldsymbol{i} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1+5\boldsymbol{i} & 2+9\boldsymbol{i} & 4+2\boldsymbol{i} \\ 1+11\boldsymbol{i} & 5+5\boldsymbol{i} & 7 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 4+3\boldsymbol{i} & 5\boldsymbol{i} \\ 3+2\boldsymbol{i} & 4+7\boldsymbol{i} \\ 2 & 5+4\boldsymbol{i} \end{bmatrix}, \quad S_2 = \begin{bmatrix} 4+2\boldsymbol{i} & 2\boldsymbol{i} \\ 2 & 4\boldsymbol{i} \\ 3+4\boldsymbol{i} & 5+9\boldsymbol{i} \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 2 & 5\boldsymbol{i} \\ 1+\boldsymbol{i} & 2+5\boldsymbol{i} \\ 3+2\boldsymbol{i} & 2+3\boldsymbol{i} \end{bmatrix}, \quad T_2 = \begin{bmatrix} 3+4\boldsymbol{i} & 4+3\boldsymbol{i} \\ 3 & 3\boldsymbol{i} \\ 2+8\boldsymbol{i} & 9+4\boldsymbol{i} \end{bmatrix}.$$

We apply Algorithm 3.5.3 to compute the outer inverse $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{3\times 2}$ *of the RB matrix* A *with the specified column space* $\mathcal{C}(S)$ *and row space* $\mathcal{R}(T)$*. Solving the matrix system* $(SYTAS, TASYT) = (S, T)$ *for* $Y = Y_1 + Y_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{2\times 3}$ *yields*

$$Y_1 = \begin{bmatrix} 0.0617 - 0.0152\boldsymbol{i} & 0.0504 + 0.0463\boldsymbol{i} & -0.0247 + 0.0487\boldsymbol{i} \\ -0.0315 + 0.0125\boldsymbol{i} & -0.0300 - 0.0207\boldsymbol{i} & 0.0095 - 0.0266\boldsymbol{i} \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} -0.0613 + 0.0175\boldsymbol{i} & -0.0512 - 0.0423\boldsymbol{i} & 0.0253 - 0.0521\boldsymbol{i} \\ 0.0306 - 0.0136\boldsymbol{i} & 0.0292 + 0.0184\boldsymbol{i} & -0.0090 + 0.0285\boldsymbol{i} \end{bmatrix}.$$

By computing $X = SYT$*, we obtain*

$$X_1 = \begin{bmatrix} 0.0110 + 0.0880\boldsymbol{i} & -0.0075 - 0.0870\boldsymbol{i} \\ 0.0724 + 0.0735\boldsymbol{i} & -0.0635 - 0.0336\boldsymbol{i} \\ -0.1262 - 0.0219\boldsymbol{i} & 0.0797 - 0.0543\boldsymbol{i} \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 0.1023 + 0.0452\boldsymbol{i} & -0.0777 - 0.0513\boldsymbol{i} \\ -0.0243 - 0.1447\boldsymbol{i} & 0.0325 + 0.0789\boldsymbol{i} \\ 0.0429 - 0.0631\boldsymbol{i} & 0.0101 + 0.1059\boldsymbol{i} \end{bmatrix}.$$

Taking into account that $X = SYT$, $S = SYTAS = XAS$, *and* $T = TASYT = TAX$, *we conclude that* $\mathcal{C}(X) = \mathcal{C}(S)$ *and* $\mathcal{R}(X) = \mathcal{R}(T)$*. Additionally,* $\|XAX - X\|_F = 1.9577 \times 10^{-15}$, *confirming that* $X = A^{(2)}_{\mathcal{C}(S),\mathcal{R}(T)}$.

By Theorem 3.4.10, we introduce Algorithm 3.5.4 for computing $A^{(1,2)}_{\mathcal{C}(S),*}$. Furthermore, Algorithm 3.5.5, which is based on the results of Theorem 3.4.12, is designed to compute the $\{1,2\}$-inverse of A with a specified row space $\mathcal{R}(T)$. Finally, by Theorem 3.4.14, we present Algorithm 3.5.6 for computing $A^{(1,2)}_{\mathcal{C}(S),\mathcal{R}(T)}$.

Algorithm 3.5.4 **Computation of $\mathbf{X} = \mathbf{A}^{(1,2)}_{\mathcal{C}(\mathbf{S}),*}$**

Input: $A = A_1 + A_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $S = S_1 + S_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times k}$.

Output: $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times m}$.

Step 1: Consistency Check: Verify the consistency of the RBME $(SUAS, ASUA) = (S, A)$ for $U = U_1 + U_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$ using Step 2 of Algorithm 3.3.1. If the equation is consistent, proceed with the subsequent steps.

Step 2: Solve: Solve the RBME $(SUAS, ASUA) = (S, A)$ using Steps 3 and 4 of Algorithm 3.3.1.

Step 3: Compute: Calculate $A^{(2)}_{\mathcal{C}(S),*} = SU$.

Algorithm 3.5.5 **Computation of $\mathbf{X} = \mathbf{A}^{(1,2)}_{*,\mathcal{R}(\mathbf{T})}$**

Input: $A = A_1 + A_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $T = T_1 + T_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{l\times m}$.

Output: $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times m}$.

Step 1: Consistency Check: Verify the consistency of the RBME $(TAUT, AUTA) = (T, A)$ for $U = U_1 + U_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times l}$ using Step 2 of Algorithm 3.3.1. If the equation is consistent, proceed with the subsequent steps.

Step 2: Solve: Solve the RBME $(TAUT, AUTA) = (T, A)$ using Steps 3 and 4 of Algorithm 3.3.1.

Step 3: Compute: Calculate $A^{(2)}_{*,\mathcal{R}(T)} = UT$.

Algorithm 3.5.6 **Computation of $\mathbf{X} = \mathbf{A}^{(1,2)}_{\mathcal{C}(\mathbf{S}),\mathcal{R}(\mathbf{T})}$**

Input: $A = A_1 + A_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $S = S_1 + S_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times k}$, $T = T_1 + T_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{l\times m}$.

Output: $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times m}$.

Step 1: Consistency Check: Verify the consistency of the RBMEs $(SUAS, ASUA) = (S, A)$ for $U = U_1 + U_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{k\times m}$ and $(TAVT, AVTA) = (T, A)$ for $V = V_1 + V_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times l}$ using Step 2 of Algorithm 3.3.1. If both equations are consistent, proceed with the next steps.

Step 2: Solve: Solve the RBMEs $(SUAS, ASUA) = (S, A)$ and $(TAVT, AVTA) = (T, A)$ using Steps 3 and 4 of Algorithm 3.3.1.

Step 3: Compute: Calculate $A^{(1,2)}_{\mathcal{C}(S),\mathcal{R}(T)} = SUAVT$.

Example 3.5.5. *Consider* $A = A_1 + A_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{3\times 2}$, $S = S_1 + S_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{2\times 2}$, *and* $T = T_1 + T_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{2\times 3}$, *where*

$$A_1 = \begin{bmatrix} 2+3\boldsymbol{i} & 2\boldsymbol{i} \\ 3+4\boldsymbol{i} & 3 \\ 7 & 1+5\boldsymbol{i} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1+4\boldsymbol{i} \\ 4+3\boldsymbol{i} & 7\boldsymbol{i} \\ 5+2\boldsymbol{i} & 3+3\boldsymbol{i} \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 1+\boldsymbol{i} & 1 \\ \boldsymbol{i} & 2+3\boldsymbol{i} \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1+2\boldsymbol{i} & 2+3\boldsymbol{i} \\ 3\boldsymbol{i} & 3 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 1+3\boldsymbol{i} & 3 & 2+2\boldsymbol{i} \\ 1+\boldsymbol{i} & 3+4\boldsymbol{i} & 4\boldsymbol{i} \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1+5\boldsymbol{i} & 2+7\boldsymbol{i} & 9 \\ 3+3\boldsymbol{i} & \boldsymbol{i} & 5+3\boldsymbol{i} \end{bmatrix}.$$

Using Algorithm 3.5.6, we compute the $\{1,2\}$*-inverse* $X = X_1 + X_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{3\times 2}$ *of the RB matrix* A*, with specified column space* $\mathcal{C}(S)$ *and row space* $\mathcal{R}(T)$*. Solving the RBMEs* $(SUAS, ASUA) = (S, A)$ *for* $U = U_1 + U_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{2\times 3}$*, and* $(TAVT, AVTA) = (T, A)$ *for* $V = V_1 + V_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{2\times 2}$*, we obtain*

$$U_1 = \begin{bmatrix} 0.0275+0.0433\boldsymbol{i} & -0.0738-0.0132\boldsymbol{i} & -0.0059+0.0301\boldsymbol{i} \\ 0.0401-0.0116\boldsymbol{i} & 0.0012+0.0284\boldsymbol{i} & -0.0244-0.0144\boldsymbol{i} \end{bmatrix},$$

$$U_2 = \begin{bmatrix} -0.0600+0.0182\boldsymbol{i} & 0.0137-0.0489\boldsymbol{i} & 0.0693+0.0163\boldsymbol{i} \\ -0.0030-0.0130\boldsymbol{i} & 0.0098+0.0202\boldsymbol{i} & 0.0075-0.0360\boldsymbol{i} \end{bmatrix},$$

and

$$V_1 = \begin{bmatrix} -0.0074+0.0017\boldsymbol{i} & 0.0134-0.0519\boldsymbol{i} \\ -0.0277-0.0209\boldsymbol{i} & 0.0458+0.0059\boldsymbol{i} \end{bmatrix},$$

$$V_2 = \begin{bmatrix} 0.0179+0.0460\boldsymbol{i} & -0.0395+0.0018\boldsymbol{i} \\ -0.0195-0.0112\boldsymbol{i} & 0.0131+0.0103\boldsymbol{i} \end{bmatrix}.$$

Thus, $X = SUAVT$ *is determined as*

$$X_1 = \begin{bmatrix} -0.2830-0.0362\boldsymbol{i} & -0.0622+0.0810\boldsymbol{i} & 0.1481+0.3466\boldsymbol{i} \\ 0.1198-0.0907\boldsymbol{i} & 0.0598-0.0075\boldsymbol{i} & -0.1784+0.0762\boldsymbol{i} \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 0.0187-0.0886\boldsymbol{i} & -0.0466-0.0493\boldsymbol{i} & 0.0928-0.2341\boldsymbol{i} \\ 0.2134-0.0503\boldsymbol{i} & 0.0245-0.1400\boldsymbol{i} & -0.0955-0.0300\boldsymbol{i} \end{bmatrix}.$$

Using $X = SUAVT$*, along with* $S = SUAS = SU(AVTA)S = (SUAVT)AS = XAS$ *and* $T = TAVT = T(ASUA)VT = TA(SUAVT) = TAX$*, we confirm that* $\mathcal{C}(X) = \mathcal{C}(S)$ *and* $\mathcal{R}(X) = \mathcal{R}(T)$*. Additionally, we find* $\|AXA - A\|_F = 7.2786 \times 10^{-14}$ *and* $\|XAX - X\|_F = 2.4072 \times 10^{-15}$*. Therefore,* $X = A^{(1,2)}_{\mathcal{C}(S),\mathcal{R}(T)}$.

In the previous examples, we demonstrated the effectiveness of Algorithm 3.5.6 in computing $A^{(1,2)}_{\mathcal{C}(S),\mathcal{R}(T)}$. Now, we will focus on comparing the efficiency of different methods for computing $X = A^{(1,2)}_{\mathcal{C}(S),\mathcal{R}(T)}$.

Example 3.5.6. *Let*

$$A = A_1 + A_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}, \quad S = S_1 + S_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{n\times k}, \quad T = T_1 + T_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{l\times m},$$

where the complex components are generated as:

$$\begin{aligned}
A_1 = A_2 &= \texttt{rand}(m,n) + \texttt{rand}(m,n)\boldsymbol{i} \in \mathbb{C}^{m\times n},\\
S_1 = S_2 &= \texttt{rand}(n,k) + \texttt{rand}(n,k)\boldsymbol{i} \in \mathbb{C}^{n\times k},\\
T_1 = T_2 &= \texttt{rand}(l,m) + \texttt{rand}(l,m)\boldsymbol{i} \in \mathbb{C}^{l\times m}.
\end{aligned}$$

The matrix dimensions depend on a positive integer t as follows:

$$m = t, \quad n = t+5, \quad k = t, \quad l = t+10.$$

In this example, t varies from 2 to 50 in increments of 2.

Objective: *We compute $A^{(1,2)}_{\mathcal{C}(S),\mathcal{R}(T)}$ given by $X = SUAVT$, where U and V are solutions to the RBMEs:*

$$(SUAS, ASUA) = (S, A), \quad (TAVT, AVTA) = (T, A).$$

We apply three different methods to compute U and V:

(a) *Our proposed Algorithm 3.3.1.*
(b) *The approach based on Theorem 3.1 from [80].*
(c) *The approach based on Theorem 3.1 from [75].*

Let X_1, X_2, and X_3 be the computed $A^{(1,2)}_{\mathcal{C}(S),\mathcal{R}(T)}$ using our method, the approach based on [80], and the approach based on [75], respectively.

Error and CPU Time Evaluation: *To assess accuracy, we define the following error metrics:*

$$\begin{aligned}
&\epsilon_1^1 = \|AX_1A - A\|_F, \quad \epsilon_2^1 = \|AX_2A - A\|_F, \quad \epsilon_3^1 = \|AX_3A - A\|_F,\\
&\epsilon_1^2 = \|X_1AX_1 - X_1\|_F, \quad \epsilon_2^2 = \|X_2AX_2 - X_2\|_F, \quad \epsilon_3^2 = \|X_3AX_3 - X_3\|_F.
\end{aligned}$$

To ensure reliable results, we run each experiment for 50 *trials and compute the average CPU time for each method. Let* t_1*,* t_2*, and* t_3 *be the average CPU time for computing* X_1*,* X_2*, and* X_3*, respectively.*

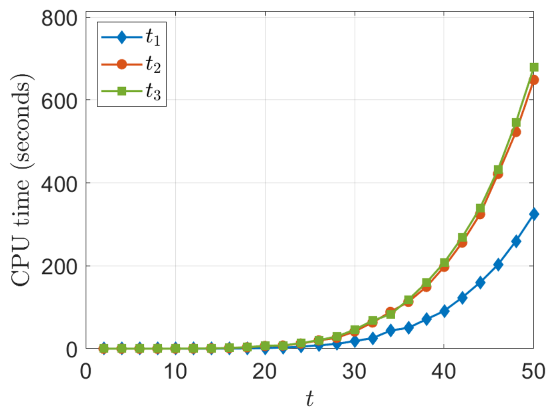
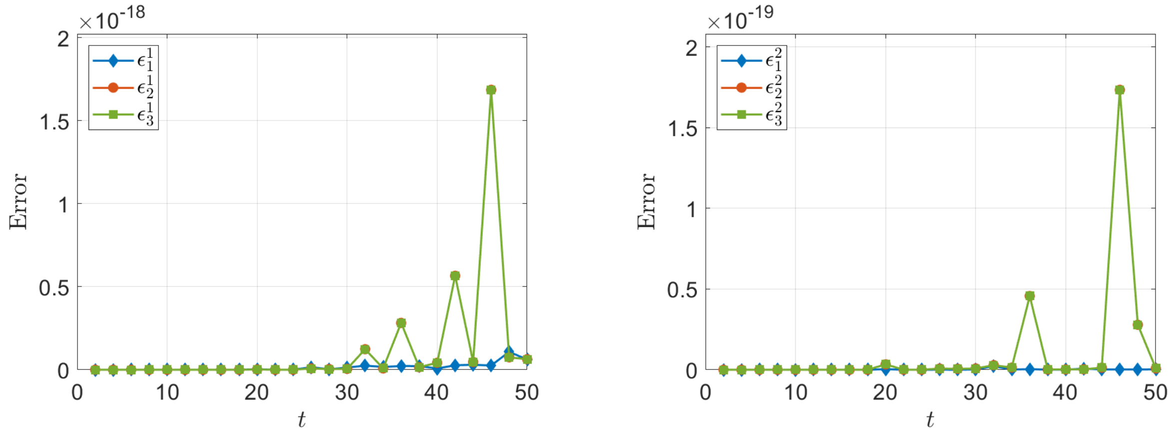


Figure 3.5.2. Comparison of CPU time for computing $A^{(1,2)}_{\mathcal{C}(S),\mathcal{R}(T)}$ using different methods.



(a) Error comparison for {1}-inverse. (b) Error comparison for {2}-inverse.

Figure 3.5.3. Error comparison for computing $A^{(1,2)}_{\mathcal{C}(S),\mathcal{R}(T)}$ using different methods.

Results and Discussion:

- Figure 3.5.2 illustrates the average CPU times for different methods. The results show that our method outperforms the other methods in terms of computational efficiency.
- Figure 3.5.3 presents a comparative analysis of the error values obtained using the three different methods. Specifically, we assess whether the computed solutions X

satisfy the conditions for being a $\{1, 2\}$-inverse of A. To verify this, we compute the errors $\|XAX - X\|_F$ for the $\{2\}$-inverse property and $\|AXA - A\|_F$ for the $\{1\}$-inverse property. The results indicate that in all cases, the errors remain below 10^{-18}, confirming the high accuracy and reliability of the proposed methods.

Conclusion: In this chapter, we have examined existence condition and representation of outer inverses and $\{1, 2\}$-inverses with predefined column and/or row space of RB matrices (RBGI). Some new relationships between computation of RBGIs and solution of RBMEs are established. The research in this chapter generalizes results obtained in [65]. We presented the transformation of necessary RBMEs into equivalent complex systems of linear equations. Some specificities of the basic terms, such as rank function, column, and row space of RB matrices, are also studied. Known results and algorithms about complex matrices are derived as particular cases.

One of promising possibility for future research is solving the corresponding RBMEs in the time-varying case, as well as calculating RBGIs using development of corresponding continuous-time recurrent neural networks, such as zeroing neural networks.

CHAPTER 4

ON SOLUTIONS OF REDUCED BIQUATERNION EQUALITY CONSTRAINED LEAST SQUARES PROBLEM

This chapter addresses the problem of solving the reduced biquaternion equality constrained least squares (RBLSE) problem. The main results focus on developing algebraic methods to derive both complex and real solutions for the RBLSE problem by exploiting the complex and real forms of reduced biquaternion matrices. In addition, a detailed perturbation analysis is conducted to evaluate the sensitivity of these solutions, and an upper bound for the relative forward error is established. Numerical examples are provided to demonstrate the effectiveness of the proposed methods and validate the accuracy of the derived upper bound for the relative forward errors.

4.1. Introduction

In many practical applications, determining the solution to a linear system, typically expressed as $AX \approx B$, is a common challenge. The least squares method is a well-established approach to address this problem. An extension of the least squares problem is the equality constrained least squares problem, which has been studied extensively in real and complex domains. Several valuable results for the real (or complex) equality constrained least squares problem have been obtained in the literature [2, 16, 17, 22, 44].

To represent multi-dimensional data in a compact form, quaternions and reduced biquaternions are frequently utilized, particularly in applications related to digital signal and image processing. When studying the theoretical and numerical aspects of these applications, one often encounters equality constrained least squares problems in the quaternion and reduced biquaternion domains.

The quaternion equality constrained least squares (QLSE) problem has garnered significant attention. For example, in [36], the authors solve the QLSE problem using the complex representation and generalized SVD of quaternion matrices. In [38], employing the complex representation of quaternion matrices, the relationship between the solutions of the QLSE problem and the complex equality constrained least squares (CLSE) problem is established, leading to a novel technique for finding solutions to the QLSE problem. Li *et al.* [42] proposed a real structure-preserving algorithm for solving the QLSE problem by transforming it into the corresponding quaternion weighted least squares problem. The work in [91] provides another approach, where the authors solve the QLSE problem using quaternion SVD and the real representation of quaternion matrices. In [87], a real structure-preserving algorithm for the minimal norm solution of the QLSE problem is proposed by leveraging quaternion QR decomposition and the real representation of quaternion matrices.

Despite extensive research on the equality constrained least squares problem in real, complex, and quaternion domains, the study of these problems in the reduced biquaternion domain remains sparse. Previous research, such as the work in [81], explored least squares solutions for matrix equations like $AX = B$ and $AXC = B$ over commutative quaternions. In [83], the authors discussed solution techniques for computing reduced biquaternion solutions to the RBLSE problem.

This chapter aims to advance the study of the RBLSE problem by developing methods for obtaining both complex and real solutions. Additionally, an upper bound for the relative forward error associated with these solutions is established, ensuring accuracy and reliability in solving RBLSE problems.

The remainder of this chapter is organized as follows: Section 4.2 introduces the preliminary concepts required for understanding the RBLSE problem. Section 4.3 outlines the method for finding the complex solution to the RBLSE problem. In Section 4.4, we discuss the technique for obtaining real solutions to the RBLSE problem. Finally, Section 4.5 presents the numerical validation of the proposed methods.

4.2. Preliminaries

To ensure this chapter is self-contained, we present key results relevant to the subsequent sections. In particular, we define the real and complex representations of an RB matrix M, denoted as M^R and M^C, respectively. Let M be an RB matrix expressed as:

$$M = M_0 + M_1\boldsymbol{i} + M_2\boldsymbol{j} + M_3\boldsymbol{k} = N_1 + N_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n},$$

where M_0, M_1, M_2, M_3 are real matrices, and $N_1 = M_0 + M_1\boldsymbol{i}$, $N_2 = M_2 + M_3\boldsymbol{i}$ are complex matrices.

The real and complex representations of M are defined as:

$$M^R = \begin{bmatrix} M_0 & -M_1 & M_2 & -M_3 \\ M_1 & M_0 & M_3 & M_2 \\ M_2 & -M_3 & M_0 & -M_1 \\ M_3 & M_2 & M_1 & M_0 \end{bmatrix}, \quad M^C = \begin{bmatrix} N_1 & N_2 \\ N_2 & N_1 \end{bmatrix}. \tag{4.2.1}$$

Let M_c^R denote the first block column of the matrix M^R, which is defined as:

$$M_c^R = \begin{bmatrix} M_0^T & M_1^T & M_2^T & M_3^T \end{bmatrix}^T. \tag{4.2.2}$$

Using M_c^R, the matrix M^R can be represented as:

$$M^R = \left[M_c^R, Q_m M_c^R, R_m M_c^R, S_m M_c^R\right], \tag{4.2.3}$$

where the matrices Q_m, R_m, and S_m are given by:

$$\begin{gathered} Q_m = \begin{bmatrix} 0 & -I_m & 0 & 0 \\ I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_m \\ 0 & 0 & I_m & 0 \end{bmatrix}, \quad R_m = \begin{bmatrix} 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_m \\ I_m & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \end{bmatrix}, \\ \text{and} \quad S_m = \begin{bmatrix} 0 & 0 & 0 & -I_m \\ 0 & 0 & I_m & 0 \\ 0 & -I_m & 0 & 0 \\ I_m & 0 & 0 & 0 \end{bmatrix}. \end{gathered} \tag{4.2.4}$$

The following lemma relates the Frobenius norm of a matrix M to its real representation.

Lemma 4.2.1. *Let $M \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, and let M_c^R and M^R be defined as in (4.2.2) and (4.2.3), respectively. Then, the Frobenius norm of M can be expressed as:*

$$\|M\|_F = \frac{1}{2}\left\|M^R\right\|_F = \left\|M_c^R\right\|_F.$$

Proof. The proof directly follows from the definition of the Frobenius norm for a reduced biquaternion matrix and a real matrix. ∎

The following result shows the relationship between the Frobenius norm of a real matrix and that of a block real matrix, where each block has entries of equal norm.

Lemma 4.2.2. *Let $P \in \mathbb{R}^{m\times n}$, $Q \in \mathbb{R}^{m\times d}$, $R \in \mathbb{R}^{m\times p}$, and $S \in \mathbb{R}^{m\times q}$. If $\|P\|_F = \|Q\|_F = \|R\|_F = \|S\|_F$, then we have*

$$\|P\|_F = \frac{1}{2}\|[P, Q, R, S]\|_F.$$

Proof. The proof directly follows from the definition of the Frobenius norm for a real matrix. ∎

Next, consider the complex representation. Let M_c^C represent the first block column of the matrix M^C, which is defined as:

$$M_c^C = \begin{bmatrix} N_1^T & N_2^T \end{bmatrix}^T. \tag{4.2.5}$$

Using M_c^C, the matrix M^C can be represented as:

$$M^C = \left[M_c^C, P_m M_c^C\right], \tag{4.2.6}$$

where the matrix P_m is defined as:

$$P_m = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}. \tag{4.2.7}$$

The following lemma relates the Frobenius norm of a matrix M to its complex representation.

Lemma 4.2.3. *Let $M \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, and let M_c^C and M^C be defined as in (4.2.5) and (4.2.6), respectively. Then, the Frobenius norm of M can be expressed as:*

$$\|M\|_F = \frac{1}{\sqrt{2}}\left\|M^C\right\|_F = \left\|M_c^C\right\|_F.$$

Proof. The proof directly follows from the definition of the Frobenius norm for a reduced biquaternion matrix and a complex matrix. ∎

We also provide the following result, which illustrates how the Frobenius norm of a complex matrix is related to a block complex matrix whose each entry has equal norm.

Lemma 4.2.4. *Let $P \in \mathbb{C}^{m\times n}$ and $Q \in \mathbb{C}^{m\times d}$. If $\|P\|_F = \|Q\|_F$, then we have*

$$\|P\|_F = \frac{1}{\sqrt{2}}\|[P, Q]\|_F.$$

Proof. The proof directly follows from the definition of the Frobenius norm for a complex matrix. ∎

Finally, we present results for both the real and complex representations of RB matrices.

Lemma 4.2.5. *For $\alpha \in \mathbb{R}$, $\beta \in \mathbb{C}$, $P, Q \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, and $R \in \mathbb{Q}_{\mathbb{R}}^{n\times t}$, the following properties hold:*

(1) $P = Q \iff P^C = Q^C \iff P^R = Q^R$.
(2) $(P+Q)^R = P^R + Q^R$, $(P+Q)^C = P^C + Q^C$.
(3) $(\alpha P)^R = \alpha P^R$, $(\beta P)^C = \beta P^C$.
(4) $(PR)^R = P^R R^R$, $(PR)^C = P^C R^C$.

4.3. Algebraic Method for Complex Solution of RBLSE Problem

This section focuses on an algebraic approach to derive the complex solution for the RBLSE problem. The method is based on analyzing the solution of the associated complex LSE problem. Suppose

$$A = M_1 + M_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}, \quad B = N_1 + N_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times d}, \tag{4.3.1}$$

$$C = R_1 + R_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{p\times n}, \quad D = S_1 + S_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{p\times d}. \tag{4.3.2}$$

We will limit our discussion to the scenario where $m \geq n + d$, and both matrices C and C_c^C have full row rank. With these assumptions, the RBLSE problem can be stated as follows:

$$\min_{X\in\mathbb{C}^{n\times d}} \|AX - B\|_F \quad \text{subject to} \quad CX = D. \tag{4.3.3}$$

To establish the connection between the RBLSE problem and its complex counterpart, consider the following complex LSE problem:

$$\min_{X\in\mathbb{C}^{n\times d}} \left\|A_c^C X - B_c^C\right\|_F \quad \text{subject to} \quad C_c^C X = D_c^C. \tag{4.3.4}$$

To find the complex solution of the RBLSE problem, we begin by computing the QR factorization of $(C_c^C)^H$, which is given by:

$$\left(C_c^C\right)^H = \widetilde{Q}\begin{bmatrix}\widetilde{R}\\ 0\end{bmatrix}, \tag{4.3.5}$$

where $\widetilde{Q} \in \mathbb{C}^{n\times n}$ is a unitary matrix and $\widetilde{R} \in \mathbb{C}^{2p\times 2p}$ is a nonsingular upper triangular matrix. Next, partition $A_c^C\widetilde{Q}$ as:

$$A_c^C\widetilde{Q} = \left[\widetilde{P}_1, \widetilde{P}_2\right], \tag{4.3.6}$$

where $\widetilde{P}_1 \in \mathbb{C}^{2m\times 2p}$ and $\widetilde{P}_2 \in \mathbb{C}^{2m\times(n-2p)}$. With these notations in place, we now present the main result of this section.

Theorem 4.3.1. *Consider the RBLSE problem defined in (4.3.3) and the complex LSE problem in (4.3.4), with notations specified in (4.3.5) and (4.3.6). For a matrix $X \in \mathbb{C}^{n\times d}$, X is a complex solution of the RBLSE problem (4.3.3) if and only if X solves the complex LSE problem (4.3.4). In this scenario, the unique solution X with minimum norm can be expressed as:*

$$X = \widetilde{Q}\begin{bmatrix} \left(\widetilde{R}^H\right)^{-1} D_c^C \\ \widetilde{P}_2^{\dagger}\left(B_c^C - \widetilde{P}_1\left(\widetilde{R}^H\right)^{-1} D_c^C\right)\end{bmatrix}. \tag{4.3.7}$$

Proof. If $X \in \mathbb{C}^{n\times d}$ is a solution of the complex LSE problem (4.3.4), then

$$\left\|A_c^C X - B_c^C\right\|_F = \min, \quad C_c^C X = D_c^C. \tag{4.3.8}$$

The Frobenius norm of a complex matrix remains invariant under unitary transformations. Since the matrix P_m in (4.2.7) is unitary, it follows that:

$$\left\|A_c^C X - B_c^C\right\|_F = \left\|P_m\left(A_c^C X - B_c^C\right)\right\|_F.$$

Using equations (4.2.1), (4.2.6), along with Lemmas 4.2.3, 4.2.4 and 4.2.5, we obtain

$$\begin{aligned}
\left\|A_c^C X - B_c^C\right\|_F &= \frac{1}{\sqrt{2}}\left\|\left[\left(A_c^C X - B_c^C\right), P_m\left(A_c^C X - B_c^C\right)\right]\right\|_F \\
&= \frac{1}{\sqrt{2}}\left\|\left[A_c^C X, P_m A_c^C X\right] - \left[B_c^C, P_m B_c^C\right]\right\|_F \\
&= \frac{1}{\sqrt{2}}\left\|\left[A_c^C, P_m A_c^C\right]\begin{bmatrix} X & 0 \\ 0 & X\end{bmatrix} - \left[B_c^C, P_m B_c^C\right]\right\|_F \\
&= \frac{1}{\sqrt{2}}\left\|A^C X^C - B^C\right\|_F \\
&= \frac{1}{\sqrt{2}}\left\|(AX - B)^C\right\|_F \\
&= \|AX - B\|_F.
\end{aligned}$$

From (4.3.8), we obtain

$$\left\|A_c^C X - B_c^C\right\|_F = \|AX - B\|_F = \min, \tag{4.3.9}$$

and

$$\left[C_c^C, P_p C_c^C\right]\begin{bmatrix} X & 0 \\ 0 & X\end{bmatrix} = \left[D_c^C, P_p D_c^C\right].$$

Using (4.2.6), we know that $C^C = [C_c^C, P_p C_c^C]$ and $D^C = [D_c^C, P_p D_c^C]$. Applying this, we get

$$\begin{aligned} C^C X^C &= D^C, \\ (CX)^C &= D^C, \\ CX &= D. \end{aligned} \tag{4.3.10}$$

By combining (4.3.9) and (4.3.10), we conclude that $X \in \mathbb{C}^{n\times d}$ is a complex solution to the RBLSE problem (4.3.3), and vice versa.

To find the expression for X, we solve the complex LSE problem (4.3.4). Set

$$\widetilde{Q}^H X = \begin{bmatrix} Y \\ Z \end{bmatrix},$$

where $Y \in \mathbb{C}^{2p\times d}$ and $Z \in \mathbb{C}^{(n-2p)\times d}$. Equation 4.3.4 can be rewritten as

$$\min_X \left\| A_c^C \widetilde{Q}\widetilde{Q}^H X - B_c^C \right\|_F \quad \text{subject to} \quad C_c^C \widetilde{Q}\widetilde{Q}^H X = D_c^C. \tag{4.3.11}$$

Utilizing (4.3.5), we have

$$C_c^C \widetilde{Q}\widetilde{Q}^H X = D_c^C \implies \begin{bmatrix} \widetilde{R}^H & 0 \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix} = D_c^C.$$

Since $\widetilde{R}^H$ is a nonsingular matrix, we get $Y = \left(\widetilde{R}^H\right)^{-1} D_c^C$. Using (4.3.6), equation (4.3.11) takes the form

$$\min_Z \left\| \widetilde{P}_2 Z - \left(B_c^C - \widetilde{P}_1 Y\right) \right\|_F.$$

The minimum norm solution of the above least squares problem is $Z = \widetilde{P}_2^{\dagger} \left(B_c^C - \widetilde{P}_1 Y\right)$. Thus, we can derive the desired expression for X. ∎

Next, we aim to examine how perturbations in A, B, C, and D affect the complex solution X_{CL} of the RBLSE problem (4.3.3). Let

$$\begin{aligned} \widehat{A} &= A + \Delta A, \quad \widehat{B} = B + \Delta B, \\ \widehat{C} &= C + \Delta C, \quad \widehat{D} = D + \Delta D, \end{aligned} \tag{4.3.12}$$

where ΔA, ΔB, ΔC, and ΔD represent the perturbations of the input data A, B, C, and D, respectively. We assume that the perturbations ΔA, ΔB, ΔC, and ΔD are small enough to guarantee that the perturbed matrices $\widehat{C}$ and $\widehat{C}_c^C$ retain full row rank. These perturbations are measured normwise by the smallest ϵ for which

$$\begin{aligned} \|\Delta A\|_F &\le \epsilon \|A\|_F, \quad \|\Delta B\|_F \le \epsilon \|B\|_F, \\ \|\Delta C\|_F &\le \epsilon \|C\|_F, \quad \|\Delta D\|_F \le \epsilon \|D\|_F. \end{aligned} \tag{4.3.13}$$

Let $\widehat{X}_{CL}$ be the complex solution to the perturbed RBLSE problem

$$\min_{X\in\mathbb{C}^{n\times d}} \|\widehat{A}X - \widehat{B}\|_F \quad \text{subject to} \quad \widehat{C}X = \widehat{D}, \tag{4.3.14}$$

and let $\Delta X_{CL} = \widehat{X}_{CL} - X_{CL}$.

Theorem 4.3.2. *Consider the RBLSE problem defined in (4.3.3) and the perturbed RBLSE problem described in (4.3.14). If the perturbations ΔA, ΔB, ΔC, and ΔD are sufficiently small, as described in (4.3.13), then we have*

$$\begin{aligned} \frac{\|\Delta X_{CL}\|_F}{\|X_{CL}\|_F} \le \epsilon \Bigg(& \mathcal{K}_A^C \left(\frac{\|D_c^C\|_F}{\|C_c^C\|_F \|X_{CL}\|_F} + 1 \right) + \mathcal{K}_B^C \left(\frac{\|B_c^C\|_F}{\|A_c^C\|_F \|X_{CL}\|_F} + 1 \right) \\ & + \left(\mathcal{K}_B^C\right)^2 \left(\frac{\|C_c^C\|_F}{\|A_c^C\|_F} \|A_c^C \mathcal{L}_c\|_2 + 1 \right) \frac{\|R_c\|_F}{\|A_c^C\|_F \|X_{CL}\|_F} \Bigg) + O(\epsilon^2) \equiv U_{CL}, \end{aligned} \tag{4.3.15}$$

where

$$\mathcal{K}_B^C = \|A_c^C\|_F \left\| \left(A_c^C P_c\right)^\dagger \right\|_2, \quad \mathcal{K}_A^C = \|C_c^C\|_F \|\mathcal{L}_c\|_2, \quad \mathcal{L}_c = \left(I_n - \left(A_c^C P_c\right)^\dagger A_c^C \right) \left(C_c^C\right)^\dagger,$$
$$P_c = I_n - \left(C_c^C\right)^\dagger C_c^C, \quad R_c = B_c^C - A_c^C X_{CL}.$$

Proof. The perturbed complex LSE problem corresponding to the perturbed RBLSE problem (4.3.14) is given by:

$$\min_{X\in\mathbb{C}^{n\times d}} \left\| \left(\widehat{A}\right)_c^C X - \left(\widehat{B}\right)_c^C \right\|_F \quad \text{subject to} \quad \left(\widehat{C}\right)_c^C X = \left(\widehat{D}\right)_c^C. \tag{4.3.16}$$

Using Theorem 4.3.1, we know that $\widehat{X}_{CL}$ is the solution to the perturbed complex LSE problem (4.3.16). From (4.3.12) and utilizing Lemma 4.2.5, we have

$$\begin{aligned} \left(\widehat{A}\right)_c^C &= A_c^C + (\Delta A)_c^C, \quad \left(\widehat{B}\right)_c^C = B_c^C + (\Delta B)_c^C, \\ \left(\widehat{C}\right)_c^C &= C_c^C + (\Delta C)_c^C, \quad \left(\widehat{D}\right)_c^C = D_c^C + (\Delta D)_c^C. \end{aligned}$$

Thus, the perturbed complex LSE problem (4.3.16) can be rewritten as:

$$\begin{aligned} &\min_{X\in\mathbb{C}^{n\times d}} \left\| \left(A_c^C + (\Delta A)_c^C\right) X - \left(B_c^C + (\Delta B)_c^C\right) \right\|_F \quad \text{subject to} \\ &\qquad \left(C_c^C + (\Delta C)_c^C\right) X = \left(D_c^C + (\Delta D)_c^C\right). \end{aligned} \tag{4.3.17}$$

Using (4.3.13) and Lemma 4.2.3, we can establish the following bounds for the perturbation:

$$\begin{aligned} \left\|(\Delta A)_c^C\right\|_F &\leq \epsilon \left\|A_c^C\right\|_F, \quad \left\|(\Delta B)_c^C\right\|_F \leq \epsilon \left\|B_c^C\right\|_F, \\ \left\|(\Delta C)_c^C\right\|_F &\leq \epsilon \left\|C_c^C\right\|_F, \quad \left\|(\Delta D)_c^C\right\|_F \leq \epsilon \left\|D_c^C\right\|_F. \end{aligned} \tag{4.3.18}$$

With the perturbed problem (4.3.17) and the bounds in (4.3.18), and using Theorem 4.3.1, the sensitivity analysis of the complex solution to the RBLSE problem (4.3.3) reduces to evaluating the sensitivity of the solution to the complex LSE problem (4.3.4). Consequently, the upper bound U_{CL} for the relative forward error of the complex solution to the RBLSE problem can be obtained from [14, Equation 4.11]. ∎

4.4. Algebraic Method for Real Solution of RBLSE Problem

This section focuses on an algebraic approach to derive the real solution for the RBLSE problem. The method is based on analyzing the solution of the associated real LSE problem. Suppose

$$A = A_0 + A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}, \quad B = B_0 + B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times d}, \tag{4.4.1}$$

$$C = C_0 + C_1\boldsymbol{i} + C_2\boldsymbol{j} + C_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{p\times n}, \quad D = D_0 + D_1\boldsymbol{i} + D_2\boldsymbol{j} + D_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{p\times d}. \tag{4.4.2}$$

We will limit our discussion to the scenario where $m \geq n + d$, and both matrices C and C_c^R have full row rank. With these assumptions, the RBLSE problem can be stated as follows:

$$\min_{X\in\mathbb{R}^{n\times d}} \|AX - B\|_F \quad \text{subject to} \quad CX = D. \tag{4.4.3}$$

To establish the connection between the RBLSE problem and its real counterpart, consider the following real LSE problem:

$$\min_{X\in\mathbb{R}^{n\times d}} \left\|A_c^R X - B_c^R\right\|_F \quad \text{subject to} \quad C_c^R X = D_c^R. \tag{4.4.4}$$

To find the real solution of the RBLSE problem, we first compute the QR factorization of $(C_c^R)^T$, given by:

$$\left(C_c^R\right)^T = \overline{Q}\begin{bmatrix}\overline{R}\\ 0\end{bmatrix}, \tag{4.4.5}$$

where $\overline{Q} \in \mathbb{R}^{n\times n}$ is an orthonormal matrix and $\overline{R} \in \mathbb{R}^{4p\times 4p}$ is a nonsingular upper triangular matrix. Next, partition $A_c^R\overline{Q}$ as:

$$A_c^R\overline{Q} = \left[\overline{P}_1, \overline{P}_2\right], \tag{4.4.6}$$

where $\overline{P}_1 \in \mathbb{R}^{4m\times 4p}$ and $\overline{P}_2 \in \mathbb{R}^{4m\times(n-4p)}$. With these notations, we now present the main result of this section.

Theorem 4.4.1. *Consider the RBLSE problem defined in (4.4.3) and the real LSE problem in (4.4.4), with notations specified in (4.4.5) and (4.4.6). For a matrix $X \in \mathbb{R}^{n\times d}$, X is a real solution of the RBLSE problem (4.4.3) if and only if X solves the real LSE problem (4.4.4). In this scenario, the unique solution with minimum norm X can be expressed as:*

$$X = \overline{Q}\begin{bmatrix} \left(\overline{R}^T\right)^{-1} D_c^R \\ \overline{P}_2^{\dagger}\left(B_c^R - \overline{P}_1\left(\overline{R}^T\right)^{-1} D_c^R\right)\end{bmatrix}. \tag{4.4.7}$$

Proof. If $X \in \mathbb{R}^{n\times d}$ is a solution of the real LSE problem (4.4.4), then

$$\left\|A_c^R X - B_c^R\right\|_F = \min, \quad C_c^R X = D_c^R. \tag{4.4.8}$$

The Frobenius norm of a real matrix remains invariant under orthogonal transformations. Since the matrices Q_m, R_m, and S_m in (4.2.4) are orthogonal, it follows that:

$$\left\|A_c^R X - B_c^R\right\|_F = \left\|Q_m\left(A_c^R X - B_c^R\right)\right\|_F = \left\|R_m\left(A_c^R X - B_c^R\right)\right\|_F = \left\|S_m\left(A_c^R X - B_c^R\right)\right\|_F.$$

Using equations (4.2.1), (4.2.3), along with Lemmas 4.2.1, 4.2.2 and 4.2.5, we obtain

$$\begin{aligned}
\left\|A_c^R X - B_c^R\right\|_F &= \frac{1}{2}\left\|\left[\left(A_c^R X - B_c^R\right), Q_m\left(A_c^R X - B_c^R\right), R_m\left(A_c^R X - B_c^R\right), S_m\left(A_c^R X - B_c^R\right)\right]\right\|_F \\
&= \frac{1}{2}\left\|\left[A_c^R X, Q_m A_c^R X, R_m A_c^R X, S_m A_c^R X\right] - \left[B_c^R, Q_m B_c^R, R_m B_c^R, S_m B_c^R\right]\right\|_F \\
&= \frac{1}{2}\left\|\left[A_c^R, Q_m A_c^R, R_m A_c^R, S_m A_c^R\right]\begin{bmatrix} X & 0 & 0 & 0 \\ 0 & X & 0 & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & X\end{bmatrix}\right. \\
&\qquad \left. - \left[B_c^R, Q_m B_c^R, R_m B_c^R, S_m B_c^R\right]\right\|_F \\
&= \frac{1}{2}\left\|A^R X^R - B^R\right\|_F \\
&= \frac{1}{2}\left\|(AX - B)^R\right\|_F \\
&= \|AX - B\|_F.
\end{aligned} \tag{4.4.9}$$

From (4.4.8), we obtain

$$\left\|A_c^R X - B_c^R\right\|_F = \|AX - B\|_F = \min, \tag{4.4.10}$$

and

$$\left[C_c^R, Q_p C_c^R, R_p C_c^R, S_p C_c^R\right]\begin{bmatrix} X & 0 & 0 & 0 \\ 0 & X & 0 & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & X \end{bmatrix} = \left[D_c^R, Q_p D_c^R, R_p D_c^R, S_p D_c^R\right].$$

Using (4.2.3), we get $C^R = [C_c^R, Q_p C_c^R, R_p C_c^R, S_p C_c^R]$, and $D^R = [D_c^R, Q_p D_c^R, R_p D_c^R, S_p D_c^R]$. Applying this, we obtain

$$\begin{aligned} C^R X^R &= D^R, \\ (CX)^R &= D^R, \\ CX &= D. \end{aligned} \tag{4.4.11}$$

By combining (4.4.10) and (4.4.11), we conclude that $X \in \mathbb{R}^{n\times d}$ is a real solution to the RBLSE problem (4.4.3), and vice versa. The expression for X can be obtained by following the proof method of Theorem 4.3.1. ■

Next, we aim to examine how perturbations in A, B, C, and D affect the real solution X_{RL} of the RBLSE problem (4.4.3). Let

$$\begin{aligned} \widehat{A} &= A + \Delta A, \quad \widehat{B} = B + \Delta B, \\ \widehat{C} &= C + \Delta C, \quad \widehat{D} = D + \Delta D, \end{aligned} \tag{4.4.12}$$

where ΔA, ΔB, ΔC, and ΔD represent the perturbations of the input data A, B, C, and D, respectively. We assume that the perturbations ΔA, ΔB, ΔC, and ΔD are small enough to ensure that the perturbed matrices $\widehat{C}$ and $\widehat{C}_c^R$ retain full row rank. These perturbations are measured normwise by the smallest ϵ for which

$$\begin{aligned} \|\Delta A\|_F &\leq \epsilon \|A\|_F, \quad \|\Delta B\|_F \leq \epsilon \|B\|_F, \\ \|\Delta C\|_F &\leq \epsilon \|C\|_F, \quad \|\Delta D\|_F \leq \epsilon \|D\|_F. \end{aligned} \tag{4.4.13}$$

Let $\widehat{X}_{RL}$ be the real solution to the perturbed RBLSE problem

$$\min_{X\in\mathbb{R}^{n\times d}} \left\|\widehat{A}X - \widehat{B}\right\|_F \quad \text{subject to} \quad \widehat{C}X = \widehat{D}, \tag{4.4.14}$$

and let $\Delta X_{RL} = \widehat{X}_{RL} - X_{RL}$.

Theorem 4.4.2. *Consider the RBLSE problem defined in (4.4.3) and the perturbed RBLSE problem described in (4.4.14). If the perturbations ΔA, ΔB, ΔC, and ΔD are sufficiently* *small, as described in* (4.4.13)*, then we have*

$$\begin{aligned} \frac{\|\Delta X_{RL}\|_F}{\|X_{RL}\|_F} \le \epsilon \Bigg(& \mathcal{K}_A^R \left(\frac{\|D_c^R\|_F}{\|C_c^R\|_F \|X_{RL}\|_F} + 1 \right) + \mathcal{K}_B^R \left(\frac{\|B_c^R\|_F}{\|A_c^R\|_F \|X_{RL}\|_F} + 1 \right) \\ & + \left(\mathcal{K}_B^R\right)^2 \left(\frac{\|C_c^R\|_F}{\|A_c^R\|_F} \left\|A_c^R \mathcal{L}_r\right\|_2 + 1 \right) \frac{\|R_r\|_F}{\|A_c^R\|_F \|X_{RL}\|_F} \Bigg) + O(\epsilon^2) \equiv U_{RL}, \end{aligned} \tag{4.4.15}$$

where

$$\mathcal{K}_B^R = \left\|A_c^R\right\|_F \left\|\left(A_c^R P_r\right)^\dagger\right\|_2, \quad \mathcal{K}_A^R = \left\|C_c^R\right\|_F \|\mathcal{L}_r\|_2, \quad \mathcal{L}_r = \left(I_n - \left(A_c^R P_r\right)^\dagger A_c^R\right)\left(C_c^R\right)^\dagger,$$
$$P_r = I_n - \left(C_c^R\right)^\dagger C_c^R, \quad R_r = B_c^R - A_c^R X_{RL}.$$

Proof. The perturbed real LSE problem corresponding to the perturbed RBLSE problem (4.4.14) is given by:

$$\min_{X \in \mathbb{R}^{n \times d}} \left\| \left(\widehat{A}\right)_c^R X - \left(\widehat{B}\right)_c^R \right\|_F \quad \text{subject to} \quad \left(\widehat{C}\right)_c^R X = \left(\widehat{D}\right)_c^R. \tag{4.4.16}$$

Using Theorem 4.4.1, we know that $\widehat{X}_{RL}$ is the solution to the perturbed real LSE problem (4.4.16). From (4.4.12) and utilizing Lemma 4.2.5, we have

$$\left(\widehat{A}\right)_c^R = A_c^R + (\Delta A)_c^R, \quad \left(\widehat{B}\right)_c^R = B_c^R + (\Delta B)_c^R,$$
$$\left(\widehat{C}\right)_c^R = C_c^R + (\Delta C)_c^R, \quad \left(\widehat{D}\right)_c^R = D_c^R + (\Delta D)_c^R.$$

Thus, the perturbed real LSE problem (4.4.16) can be rewritten as:

$$\begin{aligned} & \min_{X \in \mathbb{R}^{n \times d}} \left\| \left(A_c^R + (\Delta A)_c^R\right) X - \left(B_c^R + (\Delta B)_c^R\right) \right\|_F \quad \text{subject to} \\ & \qquad \left(C_c^R + (\Delta C)_c^R\right) X = \left(D_c^R + (\Delta D)_c^R\right). \end{aligned} \tag{4.4.17}$$

Using (4.4.13) and Lemma 4.2.1, we can establish the following bounds for the perturbation:

$$\begin{aligned} \left\|(\Delta A)_c^R\right\|_F \le \epsilon \left\|A_c^R\right\|_F, \quad \left\|(\Delta B)_c^R\right\|_F \le \epsilon \left\|B_c^R\right\|_F, \\ \left\|(\Delta C)_c^R\right\|_F \le \epsilon \left\|C_c^R\right\|_F, \quad \left\|(\Delta D)_c^R\right\|_F \le \epsilon \left\|D_c^R\right\|_F. \end{aligned} \tag{4.4.18}$$

With the perturbed problem (4.4.17) and the bounds in (4.4.18), and using Theorem 4.4.1, the sensitivity analysis of the real solution to the RBLSE problem (4.4.3) reduces to evaluating the sensitivity of the solution to the real LSE problem (4.4.4). Consequently, the upper bound U_{RL} for the relative forward error of the real solution to the RBLSE problem can be obtained from [14, Equation 4.11]. ■

4.5. Numerical Verification

Building on the previous discussions, this section presents numerical algorithms designed to find special solutions to the RBLSE problem. We also include numerical examples to validate these algorithms. Additionally, we assess the upper bound for the relative forward error of both complex and real solutions to the RBLSE problem.

All computations were performed using a computer equipped with an Intel Core $i7-9700$ processor at 3.00 GHz and 16 GB of RAM, running MATLAB $R2021b$ software.

Building on Theorem 4.3.1, we now outline Algorithm 4.5.1, which is designed to compute the complex solution to the RBLSE problem (4.3.3).

Algorithm 4.5.1 Complex Solution to the RBLSE Problem (4.3.3)

Input: $A = M_1 + M_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $B = N_1 + N_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times d}$, $C = R_1 + R_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{p\times n}$, $D = S_1 + S_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{p\times d}$. Assume $m \geq n + d$ and that both matrices C and C_c^C have full row rank.

Output: X (the complex solution to the RBLSE problem).

Step 1: QR Factorization: Find the QR factorization of $(C_c^C)^H$ as described in (4.3.5).

Step 2: Matrix Partitioning: Partition the matrix $A_c^C\widetilde{Q}$ as shown in (4.3.6).

Step 3: Solution Computation: Compute the complex solution X to the RBLSE problem (4.3.3) using the formula given in (4.3.7).

Building on Theorem 4.4.1, we now outline Algorithm 4.5.2, which is designed to compute the real solution to the RBLSE problem (4.4.3).

Algorithm 4.5.2 Real Solution to the RBLSE Problem (4.4.3)

Input: $A = A_0 + A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $B = B_0 + B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times d}$, $C = C_0 + C_1\boldsymbol{i} + C_2\boldsymbol{j} + C_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{p\times n}$, $D = D_0 + D_1\boldsymbol{i} + D_2\boldsymbol{j} + D_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{p\times d}$. Assume $m \geq n + d$ and that both matrices C and C_c^R have full row rank.

Output: X (the real solution to the RBLSE problem).

Step 1: QR Factorization: Find the QR factorization of $(C_c^R)^T$ as described in (4.4.5).

Step 2: Matrix Partitioning: Partition the matrix $A_c^R\overline{Q}$ as shown in (4.4.6).

Step 3: Solution Computation: Compute the real solution X to the RBLSE problem (4.4.3) using the formula given in (4.4.7).

Next, to thoroughly assess the performance of the proposed algorithms, we present a comprehensive step-by-step flop count analysis. This detailed examination systematically quantifies the computational cost at each stage, offering valuable insights into the overall efficiency of the algorithm. By explicitly evaluating the number of floating-point operations required, we provide a deeper understanding of the computational complexity involved. In particular, we conduct an in-depth analysis of the efficiency of the complex solution algorithm. A meticulous step-by-step flop count is outlined, highlighting the additional computations necessitated by the complex structure.

Step	**Description**	**Flop Count**
1	QR decomposition of $(C_c^C)^H$	$O(32np^2)$
2	Compute $A_c^C\widetilde{Q}$	$O(16mn^2 - 4mn)$
3	Partition $A_c^C\widetilde{Q}$ into $\widetilde{P}_1$ and $\widetilde{P}_2$	$O(1)$
4	Compute $\widetilde{P}_2^\dagger$	$O\big(48m(n-2p)^2 + 40(n-2p)^3\big)$
5	Solve $\left(\widetilde{R}^H\right)^{-1} D_c^C$	$O(16p^2d + 10pd)$
6	Compute $\widetilde{P}_1\left(\widetilde{R}^H\right)^{-1} D_c^C$	$O(32mpd - 4md)$
7	Compute $B_c^C - \widetilde{P}_1\left(\widetilde{R}^H\right)^{-1} D_c^C$	$O(4md)$
8	Compute $\widetilde{P}_2^\dagger(B_c^C - \widetilde{P}_1\left(\widetilde{R}^H\right)^{-1} D_c^C)$	$O(16m(n-2p)d)$
9	Compute X	$O(8n^2d)$
	Total Flop Count	$O\big(32np^2 + 16mn^2 - 4mn + 48m(n-2p)^2 + 40(n-2p)^3 + 16p^2d + 10pd + 32mpd - 4md + 4md + 16m(n-2p)d + 8n^2d\big)$

Table 4.5.1. Flop count for the computational steps to find the complex solution of the RBLSE problem.

To analyze the efficiency of the real solution algorithm, a detailed step-by-step flop count is provided below.

Step	Description	Flop Count
1	QR decomposition of $(C_c^R)^T$	$O(32np^2)$
2	Compute $A_c^R\overline{Q}$	$O(8mn^2 - 4mn)$
3	Partition $A_c^R\overline{Q}$ into $\overline{P}_1$ and $\overline{P}_2$	$O(1)$
4	Compute $\overline{P}_2^\dagger$	$O\big(24m(n-4p)^2 + 10(n-4p)^3\big)$
5	Solve $\left(\overline{R}^T\right)^{-1} D_c^R$	$O(16p^2d)$
6	Compute $\overline{P}_1\left(\overline{R}^T\right)^{-1} D_c^R$	$O(32mpd - 4md)$
7	Compute $B_c^R - \overline{P}_1\left(\overline{R}^T\right)^{-1} D_c^R$	$O(4md)$
8	Compute $\overline{P}_2^\dagger(B_c^R - \overline{P}_1\left(\overline{R}^T\right)^{-1} D_c^R)$	$O(8m(n-4p)d)$
9	Compute X	$O(2n^2d)$
	Total Flop Count	$O\big(32np^2 + 8mn^2 - 4mn + 24m(n-4p)^2 + 10(n-4p)^3 + 16p^2d + 32mpd - 4md + 4md + 8m(n-4p)d + 2n^2d\big)$

Table 4.5.2. Flop count for the computational steps to find the real solution of the RBLSE problem.

Next, we provide examples to evaluate the effectiveness of the proposed algorithms.

Example 4.5.1. *Let* $A = M_1 + M_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $B = N_1 + N_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times d}$, $C = R_1 + R_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{p\times n}$, *and* $D = S_1 + S_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{p\times d}$. *Let*

$$M_i = \texttt{rand}(m,n) + \texttt{rand}(m,n)\boldsymbol{i} \in \mathbb{C}^{m\times n}, \quad \text{for } i = 1,2,$$

$$N_i = \texttt{rand}(m,d) + \texttt{rand}(m,d)\boldsymbol{i} \in \mathbb{C}^{m\times d}, \quad \text{for } i = 1,2,$$

$$R_i = \texttt{rand}(p,n) + \texttt{rand}(p,n)\boldsymbol{i} \in \mathbb{C}^{p\times n}, \quad \text{for } i = 1,2,$$

$$S_i = \texttt{rand}(p,d) + \texttt{rand}(p,d)\boldsymbol{i} \in \mathbb{C}^{p\times d}, \quad \text{for } i = 1,2.$$

Take $m = 40t$, $n = 6t$, $p = 2t$, *and* $d = 3$. *Here,* t *is an arbitrary number. We apply Algorithm 4.5.1 to determine the complex solution of the RBLSE problem. Let* X_{CL} *be the* *complex solution of the RBLSE problem*

$$\min_{X,R_c} \|R_c\|_F \quad \textit{subject to} \quad AX = B + R_c, \quad CX = D.$$

Let $\epsilon_1 = \|AX_{CL} - (B + R_c)\|_F$ *and* $\epsilon_2 = \|CX_{CL} - D\|_F$*. In Table 4.5.3, we compute* ϵ_1 *and* ϵ_2 *for different values of* t*.*

t	ϵ_1	ϵ_2
1	8.5131×10^{-16}	2.0907×10^{-15}
3	7.0956×10^{-16}	2.5624×10^{-15}
5	1.0499×10^{-15}	3.7683×10^{-15}
7	1.2804×10^{-15}	4.5681×10^{-15}
9	1.4232×10^{-15}	8.1546×10^{-15}

Table 4.5.3. Computational accuracy of Algorithm 4.5.1 for computing the complex solution of the RBLSE problem (4.3.3)

Table 4.5.3 shows that the errors ϵ_1 and ϵ_2 across different values of t are consistently below 10^{-15}. This indicates that Algorithm 4.5.1 is highly effective in determining the complex solution for the RBLSE problem (4.3.3).

Example 4.5.2. *In Example 4.5.1, we introduce random perturbations* ΔA*,* ΔB*,* ΔC*, and* ΔD *to the input matrices* A*,* B*,* C*, and* D*, respectively, to examine how these small perturbations affect the complex solution* X_{CL} *of the RBLSE problem* (4.3.3)*. These perturbations are measured normwise by the smallest* ϵ*, as in* (4.3.13)*. We define the exact relative forward errors for these solutions as* $\frac{\|\widehat{X}_{CL}-X_{CL}\|_F}{\|X_{CL}\|_F}$*. Table 4.5.4 presents the calculated exact relative forward errors and the corresponding upper bound* U_{CL} (*calculated using equation* (4.3.15)) *for these solutions across different values of* t *and* ϵ*.*

Table 4.5.4 shows that the exact relative forward errors of the complex solution to the RBLSE problem (4.3.3) are consistently lower than their respective upper bounds across various values of t and ϵ. This verifies the reliability of the derived upper bound U_{CL} for the relative forward error.

t	ϵ	$\frac{\|\widehat{X}_{CL}-X_{CL}\|_F}{\|X_{CL}\|_F}$	U_{CL}
1	7.0103×10^{-13}	1.3055×10^{-12}	2.7846×10^{-11}
	7.0580×10^{-10}	1.3349×10^{-9}	2.8036×10^{-8}
	8.9885×10^{-7}	1.81513×10^{-6}	3.5704×10^{-6}
5	7.1245×10^{-11}	1.1503×10^{-10}	9.5002×10^{-9}
	8.9399×10^{-9}	2.0884×10^{-8}	1.1921×10^{-6}
	1.0173×10^{-8}	1.5357×10^{-8}	1.3565×10^{-6}
9	7.1497×10^{-12}	1.2430×10^{-11}	1.3370×10^{-9}
	6.6856×10^{-10}	1.2308×10^{-9}	1.2502×10^{-7}
	1.0212×10^{-7}	2.5587×10^{-7}	1.9096×10^{-5}

Table 4.5.4. Comparison of relative forward errors and their upper bounds for the complex solution of a perturbed RBLSE problem (4.3.3)

Example 4.5.3. *Let* $A = A_0 + A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $B = B_0 + B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times d}$, $C = C_0 + C_1\boldsymbol{i} + C_2\boldsymbol{j} + C_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{p\times n}$, *and* $D = D_0 + D_1\boldsymbol{i} + D_2\boldsymbol{j} + D_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{p\times d}$. *Let*

$$\begin{aligned} A_i &= \texttt{randn}(m,n) \in \mathbb{R}^{m\times n}, \quad \textit{for } i = 0:3,\\ B_i &= \texttt{randn}(m,d) \in \mathbb{R}^{m\times d}, \quad \textit{for } i = 0:3,\\ C_i &= \texttt{randn}(p,n) \in \mathbb{R}^{p\times n}, \quad \textit{for } i = 0:3,\\ D_i &= \texttt{randn}(p,d) \in \mathbb{R}^{p\times d}, \quad \textit{for } i = 0:3. \end{aligned}$$

Take $m = 30t$, $n = 10t$, $p = 2t$, *and* $d = 2$. *Here,* t *is an arbitrary number. We apply Algorithm 4.5.2 to determine the real solution of the RBLSE problem. Let* X_{RL} *be the real solution of the RBLSE problem*

$$\min_{X,R_r} \|R_r\|_F \quad \textit{subject to} \quad AX = B + R_r, \quad CX = D.$$

Let $\epsilon_1 = \|AX_{RL} - (B + R_r)\|_F$ *and* $\epsilon_2 = \|CX_{RL} - D\|_F$. *In Table 4.5.5, we compute* ϵ_1 *and* ϵ_2 *for different values of* t.

Table 4.5.5 shows that the errors ϵ_1 and ϵ_2 across different values of t are consistently below 10^{-14}. This indicates that Algorithm 4.5.2 is highly effective in determining the real solution for the RBLSE problem (4.4.3).

t	ϵ_1	ϵ_2
1	2.9063×10^{-15}	3.0851×10^{-15}
3	2.5053×10^{-15}	5.5184×10^{-15}
5	3.7915×10^{-15}	1.0949×10^{-14}
7	3.9204×10^{-15}	1.3185×10^{-14}
9	5.2026×10^{-15}	1.7247×10^{-14}

Table 4.5.5. Computational accuracy of Algorithm 4.5.2 for computing the real solution of the RBLSE problem (4.4.3)

t	ϵ	$\frac{\|\widehat{X}_{RL}-X_{RL}\|_F}{\|X_{RL}\|_F}$	U_{RL}
1	7.3638×10^{-12}	2.3814×10^{-11}	8.0339×10^{-10}
	7.3677×10^{-10}	2.1476×10^{-9}	8.0381×10^{-8}
	7.9614×10^{-8}	7.6845×10^{-7}	8.6858×10^{-6}
3	7.1742×10^{-14}	2.6216×10^{-13}	9.0524×10^{-12}
	8.1702×10^{-11}	1.5855×10^{-10}	1.0309×10^{-8}
	7.1758×10^{-9}	2.6438×10^{-8}	9.0544×10^{-7}
5	7.2567×10^{-11}	2.4717×10^{-10}	1.0430×10^{-8}
	7.2735×10^{-9}	2.5130×10^{-8}	1.0454×10^{-6}
	8.7538×10^{-7}	1.8374×10^{-6}	1.2581×10^{-4}

Table 4.5.6. Comparison of relative forward errors and their upper bounds for the real solution of a perturbed RBLSE problem (4.4.3)

Example 4.5.4. *In Example 4.5.3, we introduce random perturbations ΔA, ΔB, ΔC, and ΔD to the input matrices A, B, C, and D, respectively, to examine how these small perturbations affect the real solution X_{RL} of the RBLSE problem (4.4.3). These perturbations are measured normwise by the smallest ϵ, as in (4.4.13). We define the exact relative forward errors for these solutions as $\frac{\|\widehat{X}_{RL}-X_{RL}\|_F}{\|X_{RL}\|_F}$. Table 4.5.6 presents the computed exact relative forward errors and the corresponding upper bound U_{RL} (calculated using equation (4.4.15)) for these solutions across different values of t and ϵ.*

Table 4.5.6 shows that the exact relative forward errors of the real solution to the RBLSE problem (4.4.3) are consistently lower than their respective upper bounds across various values of t and ϵ. This verifies the reliability of the derived upper bound U_{RL} for the relative forward error.

Example 4.5.5. *Let* $A = M_1 + M_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times 10}$, $C = R_1 + R_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{3\times 10}$, *and* $X_0 = \texttt{rand}(10,3) + \texttt{rand}(10,3)\boldsymbol{i} \in \mathbb{C}^{10\times 3}$. *Let*

$$M_i = \texttt{rand}(m,10) + \texttt{rand}(m,10)\boldsymbol{i} \in \mathbb{C}^{m\times 10}, \quad \textit{for } i = 1,2$$

$$R_i = \texttt{rand}(3,10) + \texttt{rand}(3,10)\boldsymbol{i} \in \mathbb{C}^{3\times 10}, \quad \textit{for } i = 1,2.$$

Take $B = AX_0$ *and* $D = CX_0$. *Thus,* X_0 *is the complex solution to the RBLSE problem (4.3.3). To measure the performance of the proposed technique for the RBLSE problem (4.3.3), Algorithm 4.5.1 is employed to determine the complex solution* X. *Let the error be* $\epsilon = \|X - X_0\|_F$. *Here,* m *is a variable parameter. We evaluate the errors* ϵ *for various values of* m. *The relationship between the errors* ϵ *and* m *is presented in Table 4.5.7.*

m	$\epsilon = \|X - X_0\|_F$
100	1.3154×10^{-14}
200	6.2150×10^{-15}
300	6.5603×10^{-15}
400	6.1485×10^{-15}
500	3.2441×10^{-15}

Table 4.5.7. Computational accuracy of Algorithm 4.5.1 for computing the complex solution of the RBLSE problem (4.3.3)

Table 4.5.7 shows that the error ϵ between the complex solution derived from Algorithm 4.5.1 and the corresponding true solution to the RBLSE problem (4.3.3) remains consistently below 10^{-14} for various values of m. This indicates the high accuracy of Algorithm 4.5.1 in computing the complex solution to the RBLSE problem (4.3.3).

Example 4.5.6. *Let* $A = A_0 + A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times 50}$, $C = C_0 + C_1\boldsymbol{i} + C_2\boldsymbol{j} + C_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{10\times 50}$, *and* $X_0 = \texttt{randn}(50,30) \in \mathbb{R}^{50\times 30}$. *Let*

$$A_i = \texttt{randn}(m,50) \in \mathbb{R}^{m\times 50}, \quad \textit{for } i = 0:3$$

$$C_i = \texttt{randn}(10,50) \in \mathbb{R}^{10\times 50}, \quad \textit{for } i = 0:3.$$

Take $B = AX_0$ and $D = CX_0$. Clearly, X_0 is the real solution to the RBLSE problem (4.4.3). To measure the performance of the proposed technique for the RBLSE problem (4.4.3), Algorithm 4.5.2 is employed to determine the real solution X. Define the error as $\epsilon = \|X - X_0\|_F$. Here, m is a variable parameter. We evaluate the errors ϵ for various values of m. The relationship between the errors ϵ and m is presented in Table 4.5.8.

m	$\epsilon = \|X - X_0\|_F$
1000	3.8948×10^{-14}
2000	4.4732×10^{-14}
3000	4.1257×10^{-14}
4000	3.7531×10^{-14}
5000	4.6532×10^{-14}

Table 4.5.8. Computational accuracies of Algorithm 4.5.2 for computing the real solution of the RBLSE problem (4.4.3)

Table 4.5.8 shows that the error ϵ between the real solution derived from Algorithm 4.5.2 and the corresponding true solution to the RBLSE problem (4.4.3) remains consistently below 10^{-14} for various values of m. This indicates the high accuracy of Algorithm 4.5.2 in computing the real solution to the RBLSE problem (4.4.3).

The following example evaluates the efficiency of our method in computing both real and complex solutions of the RBLSE problem. To assess its performance and scalability, we apply our approach to large matrix sizes and compare the real and complex solutions in terms of computational accuracy and CPU runtime, demonstrating the effectiveness of our method for large-scale problems.

Example 4.5.7. *To evaluate the accuracy and performance of the proposed methods for solving the RBLSE problem, we use random data matrices of varying sizes. The problem matrices are defined as follows:*

$$A = A_0 + A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k} = M_1 + M_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}, \quad B = B_0 + B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k} = N_1 + N_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{m\times d},$$
$$C = C_0 + C_1\boldsymbol{i} + C_2\boldsymbol{j} + C_3\boldsymbol{k} = R_1 + R_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{p\times n}, \quad D = D_0 + D_1\boldsymbol{i} + D_2\boldsymbol{j} + D_3\boldsymbol{k} = S_1 + S_2\boldsymbol{j} \in \mathbb{Q}_{\mathbb{R}}^{p\times d}.$$

The real matrix components A_i, B_i, C_i, and D_i are generated as:

$$A_i = \texttt{rand}(m, n) \in \mathbb{R}^{m\times n}, \quad i = 0, 1, 2, 3,$$
$$B_i = \texttt{rand}(m, d) \in \mathbb{R}^{m\times d}, \quad i = 0, 1, 2, 3,$$
$$C_i = \texttt{rand}(p, n) \in \mathbb{R}^{p\times n}, \quad i = 0, 1, 2, 3,$$
$$D_i = \texttt{rand}(p, d) \in \mathbb{R}^{p\times d}, \quad i = 0, 1, 2, 3.$$

The matrix dimensions depend on a positive integer t defined as:

$$m = 30t, \quad n = 10t, \quad p = 2t, \quad d = 2.$$

In this example, t is varied from 10 to 400 in increments of 10.

Objective: *We aim to compute the real and complex solutions of the RBLSE problem and compare their accuracy and computational efficiency.*

Computational Methods:

(a) ***Real Solution*** X_{RL}*: The real solution of the RBLSE problem is computed as:*

$$\min_{X, R_r} \|R_r\|_F \quad \text{subject to} \quad AX = B + R_r, \quad CX = D.$$

(b) ***Complex Solution*** X_{CL}*: The complex solution of the RBLSE problem is computed as:*

$$\min_{X, R_c} \|R_c\|_F \quad \text{subject to} \quad AX = B + R_c, \quad CX = D.$$

Error and CPU Time Evaluation: *To assess accuracy, we define the following error metrics:*

$$\epsilon_{r1} = \|AX_{RL} - (B + R_r)\|_F, \quad \epsilon_{r2} = \|CX_{RL} - D\|_F.$$

$$\epsilon_{c1} = \|AX_{CL} - (B + R_c)\|_F, \quad \epsilon_{c2} = \|CX_{CL} - D\|_F.$$

To ensure reliable results, we run each experiment for 50 trials and compute the average CPU time. Let:

- t_r *be the average CPU time for computing the real solution* X_{RL}.
- t_c *be the average CPU time for computing the complex solution* X_{CL}.

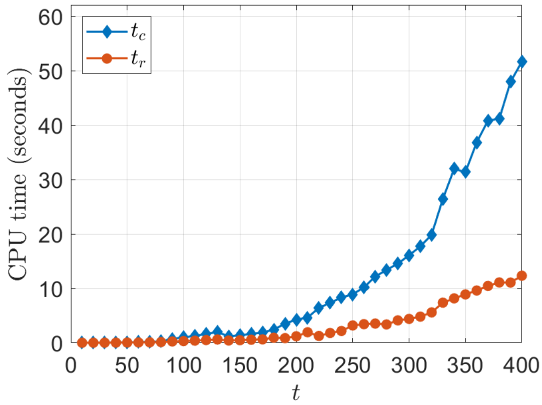
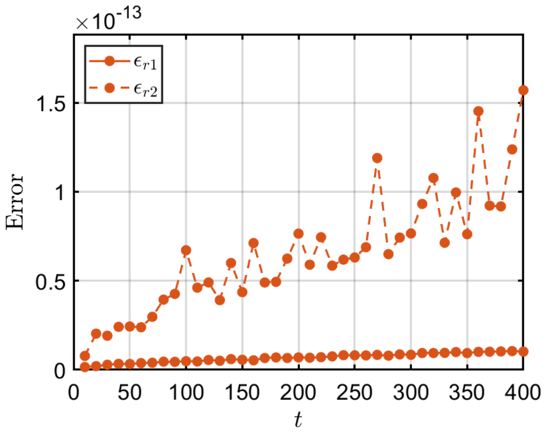
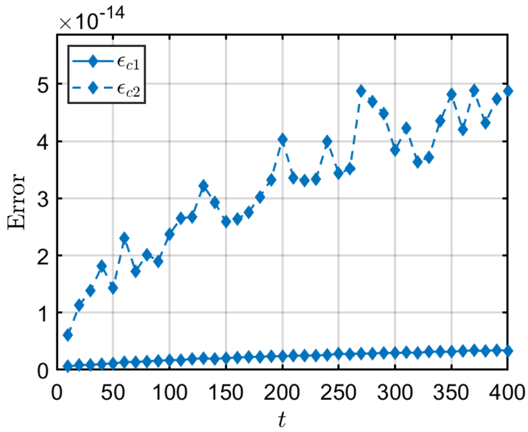


Figure 4.5.1. Comparison of CPU time for computing real and complex solution of the RBLSE problem.



(a) Error for computing real solution.



(b) Error for computing complex solution.

Figure 4.5.2. Accuracy of our method for computing the real and complex solutions of the RBLSE problem.

Results and Discussion:

- Figure 4.5.1 compares the average CPU times to compute real and complex solutions of the RBLSE problem. The results indicate that the algorithm for computing the real solution takes significantly less time than the one for computing the complex solution. This is consistent with the theoretical complexity of the methods.
- Figure 4.5.2 presents the error comparison for both real and complex solutions. Both solutions exhibit accuracy with error values below 10^{-13}, confirming the accuracy of the methods.

Conclusion: In this chapter, we have developed an algebraic method for solving the RBLSE problem by transforming it into equivalent complex and real LSE problems. This transformation is achieved by utilizing the complex and real representations of reduced biquaternion matrices, which facilitates efficient computation of both the complex and real solutions to the RBLSE problem. Furthermore, we have derived the upper bound for the relative forward error associated with these solutions, thereby demonstrating the accuracy of our proposed method in effectively solving the RBLSE problem.

CHAPTER 5

ALGEBRAIC TECHNIQUE FOR REDUCED BIQUATERNION MIXED LEAST SQUARES AND TOTAL LEAST SQUARES PROBLEM

This chapter introduces the reduced biquaternion mixed least squares and total least squares (RBMTLS) method for solving the overdetermined system $AX \approx B$ within the reduced biquaternion algebra. The main results focus on leveraging the real representations of RB matrices to derive conditions under which a real RBMTLS solution exists and to provide an explicit formula for this solution. The RBMTLS method also encompasses two important special cases: the reduced biquaternion total least squares (RBTLS) method and the reduced biquaternion least squares (RBLS) method. Furthermore, this chapter demonstrates the application of the RBMTLS method in finding the best approximate solution to $AX \approx B$ over the complex field. Additionally, a perturbation analysis of the real RBMTLS, RBTLS, and RBLS solutions is conducted to evaluate their stability and sensitivity to input variations. Numerical examples are provided to validate the theoretical results and illustrate the effectiveness of the proposed methods.

5.1. Introduction

The formulation of a solution procedure for many application problems often entails finding the best approximate solution to an inconsistent linear system. In this chapter, we explore how to compute the best approximate solutions to an overdetermined linear system

$$AX \approx B, \tag{5.1.1}$$

where $A = A_0 + A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ $(m > n)$, $B = B_0 + B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times d}$, and $m \geq n + d$ that arises within the framework of commutative quaternionic theory. Our primary focus is on addressing inconsistent systems. This chapter investigates several methods for solving the linear approximation problem (5.1.1), among which the least squares (LS) approach is a widely used method to find the best approximate solution.

The multidimensional RBLS problem is formulated as:

$$\min_{X,\widehat{G}} \|\widehat{G}\|_F \quad \text{subject to} \quad AX = B + \widehat{G}. \tag{5.1.2}$$

Once the minimizing $\widehat{G}$ is found, then any X that solves the corrected system in (5.1.2) is referred to as the RBLS solution.

However, the RBLS method assumes that all errors are contained in matrix B, with matrix A being error-free. In practice, matrix A may also be corrupted by noise, particularly in real-world applications. The RBLS method fails to account for errors in matrix A, potentially leading to suboptimal results. To address this issue, the total least squares (TLS) approach was introduced, which handles errors in both A and B.

The multidimensional RBTLS problem is formulated as:

$$\min_{X,\widehat{E},\widehat{G}} \|[\widehat{E},\widehat{G}]\|_F \quad \text{subject to} \quad (A+\widehat{E})X = B + \widehat{G}. \tag{5.1.3}$$

Once the minimizing $[\widehat{E},\widehat{G}]$ is found, then any X that solves the corrected system in (5.1.3) is called the RBTLS solution.

The TLS method is extensively applied in areas such as system theory, signal processing, and computer algebra. However, in some applications, the errors may be confined to the observation matrix B and only a few columns of matrix A, while other columns of A remain free from errors. Perturbing these accurately known columns using the RBTLS method can reduce the accuracy of the estimated parameter X. To handle such situations, the RBMTLS method is proposed.

While the LS, TLS, and MTLS techniques have been well-studied in the context of real matrices [28, 29, 76], only the LS method has been examined within the RB domain. For example, Zhang *et al.* [81] investigated the least squares solutions to reduced biquaternion matrix equations $AXC = B$ and $AX = B$. To the best of our knowledge, the RBMTLS solution techniques have not yet been explored in the RB domain. Notably, the RBMTLS method encompasses both the RBLS and RBTLS methods, making it more widely applicable.

In this chapter, we focus on finding real solutions to the linear approximation problem (5.1.1) in the RB domain. The key contributions of this chapter are summarized as follows:

- We present the RBMTLS method for obtaining the best approximate solution to the multidimensional overdetermined linear system $AX \approx B$. Additionally, we investigate the existence conditions for a unique real RBMTLS solution and derive an explicit expression for the solution.
- We propose the RBTLS and RBLS solution techniques as special cases of the RBMTLS problem. Specifically, when all columns of matrix A are contaminated with noise, the RBMTLS method reduces to the RBTLS method, and we derive the conditions for the existence of a unique real RBTLS solution. Similarly, when matrix A is error-free, the RBMTLS method reduces to the RBLS problem, and we use our developed technique to find real RBLS solutions.
- The developed solution methods are also applied to solve the complex matrix equation $AX \approx B$ as a special case of the reduced biquaternion matrix equation.
- We establish upper bounds for the relative forward errors of the real RBMTLS, RBTLS, and RBLS solutions using their relative normwise condition numbers.

The chapter is organized as follows: In Section 5.2, we provide preliminary results. Section 5.3 presents the solution techniques for RBMTLS, RBTLS, and RBLS problems. In Section 5.4, we conduct a perturbation analysis for the real RBMTLS, RBTLS, and RBLS solutions. Finally, Section 5.5 provides numerical verification of the developed results.

5.2. Preliminaries

To ensure this chapter is self-contained, we present key results relevant to the subsequent sections. Let $a = a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}$ and $b = b_0 + b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}$. Then, the equality property of RB numbers states:

$$a = b \iff a_0 = b_0,\ a_1 = b_1,\ a_2 = b_2, \text{ and } a_3 = b_3.$$

We now establish two essential lemmas that will be utilized in the subsequent analysis.

Lemma 5.2.1. *Let $A = A_0 + A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $m > n$. Then matrix A has full column rank if and only if matrix $A_c^R = [A_0^T, A_1^T, A_2^T, A_3^T]^T \in \mathbb{R}^{4m\times n}$ has full column rank.*

Proof. Let $A = (a_{ij})$, where $a_{ij} = a_{ij0} + a_{ij1}\boldsymbol{i} + a_{ij2}\boldsymbol{j} + a_{ij3}\boldsymbol{k}$. Let $v_j \in \mathbb{Q}_{\mathbb{R}}^m$ denote the j^{th} column of matrix A. The proof follows from the fact that the set of vectors $\{v_1, v_2, \dots, v_n\}$

is linearly independent if the vector equation $x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0$ has only the trivial solution $x_1 = x_2 = \cdots = x_n = 0$, and by the equality property of RB numbers. $\blacksquare$

Using the real representation in (4.2.1) and utlizing (1.2.1), we derive the following lemma, which establishes key relationships between the Frobenius norms of block RB matrices and their real representations.

Lemma 5.2.2. *Let $A = A_0 + A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ and $B = B_0 + B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times d}$. Denote $A_c^R = [A_0^T, A_1^T, A_2^T, A_3^T]^T$ and $B_c^R = [B_0^T, B_1^T, B_2^T, B_3^T]^T$. Then, the following properties hold.*

(1) $\|[A, B]\|_F = \frac{1}{2}\|[A, B]^R\|_F$.
(2) $\|[A, B]^R\|_F = \|[A^R, B^R]\|_F$.
(3) $\|[A^R, B^R]\|_F = 2\|[A_c^R, B_c^R]\|_F$.

To support the main findings of this chapter, we recall some well-known results. Specifically, we rephrase the Eckart-Young-Mirsky matrix approximation theorem [21] to align it with our analysis.

Lemma 5.2.3. *Let the SVD of $A \in \mathbb{R}^{m\times n}$ be given by $A = \bar{U}\bar{\Sigma}\bar{V}^T$ with $r = \operatorname{rank}(A)$ and $k < r$. Let*

$$\bar{U} = \begin{array}{c} \begin{array}{cc} k & m-k \end{array} \\ \begin{bmatrix} \bar{U}_1 & \bar{U}_2 \end{bmatrix} m \end{array}, \quad \bar{\Sigma} = \begin{array}{cc} & \begin{array}{cc} k & n-k \end{array} \\ & \begin{bmatrix} \bar{\Sigma}_1 & 0 \\ 0 & \bar{\Sigma}_2 \end{bmatrix} \begin{array}{l} k \\ m-k \end{array} \end{array}, \quad \text{and} \quad \bar{V} = \begin{array}{cc} & \begin{array}{cc} k & n-k \end{array} \\ & \begin{bmatrix} \bar{V}_{11} & \bar{V}_{12} \\ \bar{V}_{21} & \bar{V}_{22} \end{bmatrix} \begin{array}{l} k \\ n-k \end{array} \end{array},$$

where $\bar{U} \in \mathbb{R}^{m\times m}$ and $\bar{V} \in \mathbb{R}^{n\times n}$ are orthornormal matrices. Denote the diagonal matrices as $\bar{\Sigma}_1 = \operatorname{diag}(\bar{\sigma}_1, \ldots, \bar{\sigma}_k)$ and $\bar{\Sigma}_2 = \operatorname{diag}(\bar{\sigma}_{k+1}, \ldots, \bar{\sigma}_r)$. If

$$A_k = [\bar{U}_1, \bar{U}_2]\begin{bmatrix} \bar{\Sigma}_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{V}_{11} & \bar{V}_{12} \\ \bar{V}_{21} & \bar{V}_{22} \end{bmatrix}^T = [\bar{U}_1\bar{\Sigma}_1\bar{V}_{11}^T, \bar{U}_1\bar{\Sigma}_1\bar{V}_{21}^T],$$

then

$$\min_{\operatorname{rank}(B)=k} \|A - B\|_F = \|A - A_k\|_F = \sqrt{\sum_{i=k+1}^{r} \bar{\sigma}_i^2}.$$

In this lemma, A_k represents the best rank-k approximation of matrix A with respect to the Frobenius norm.

5.3. An Algebraic Technique for RBMTLS Problem

In this section, we derive an algebraic solution technique for the RBMTLS problem by exploring the solution of the corresponding real MTLS problem. Suppose

$$A = A_0 + A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times n} \quad \text{and} \quad B = B_0 + B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times d}. \tag{5.3.1}$$

Let the first n_1 columns of matrix A be known exactly, and the remaining n_2 columns be contaminated by noise, where $n_1 + n_2 = n$. Partition A and X as

$$A = [A_a, A_b] \quad \text{and} \quad X = [X_a^T, X_b^T]^T, \tag{5.3.2}$$

where $A_a = A_{a0} + A_{a1}\boldsymbol{i} + A_{a2}\boldsymbol{j} + A_{a3}\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times n_1}$, $A_b = A_{b0} + A_{b1}\boldsymbol{i} + A_{b2}\boldsymbol{j} + A_{b3}\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times n_2}$, and partitioning of X is conformal with A_a and A_b. For this analysis, we confine ourselves to the case when $m \geq n + d$ and A_a has full column rank.

The multidimensional RBMTLS problem can be formulated as:

$$\min_{X_a, X_b, \widehat{E}_b, \widehat{G}} \left\|[\widehat{E}_b, \widehat{G}]\right\|_F \quad \text{subject to} \quad A_a X_a + \left(A_b + \widehat{E}_b\right) X_b = B + \widehat{G}. \tag{5.3.3}$$

Once a minimizing $[\widehat{E}_b, \widehat{G}]$ is found, then any $X = [X_a^T, X_b^T]^T$ which solves the corrected system in (5.3.3) is called the RBMTLS solution.

Remark 5.3.1. *By varying n_1 from 0 to n, the above formulation can incorporate the RBTLS, RBMTLS, and RBLS problems:*

- *When $n_1 = 0$, the formulation reduces to the RBTLS problem.*
- *When $0 < n_1 < n$, it represents the RBMTLS problem.*
- *When $n_1 = n$, the formulation reduces to the RBLS problem.*

To connect the RBMTLS problem with its real counterpart, let

$$C_a = \begin{bmatrix} A_{a0}^T & A_{a1}^T & A_{a2}^T & A_{a3}^T \end{bmatrix}^T \in \mathbb{R}^{4m\times n_1}, \quad C_b = \begin{bmatrix} A_{b0}^T & A_{b1}^T & A_{b2}^T & A_{b3}^T \end{bmatrix}^T \in \mathbb{R}^{4m\times n_2},$$

and define $C = [C_a, C_b]$ and $D = \begin{bmatrix} B_0^T & B_1^T & B_2^T & B_3^T \end{bmatrix}^T \in \mathbb{R}^{4m\times d}$. Then, consider a multidimensional real MTLS problem

$$\min_{X_a, X_b, \widetilde{E}_b, \widetilde{G}} \left\|[\widetilde{E}_b, \widetilde{G}]\right\|_F \quad \text{subject to} \quad C_a X_a + \left(C_b + \widetilde{E}_b\right) X_b = D + \widetilde{G}. \tag{5.3.4}$$

Once a minimizing $[\widetilde{E}_b, \widetilde{G}]$ is found, then any $X = [X_a^T, X_b^T]^T$ which solves the corrected system in (5.3.4) is called the real MTLS solution.

In the forthcoming results on the RBMTLS solution, we will be using the following notations: Let

$$\widetilde{E}_b = [E_{b0}^T, E_{b1}^T, E_{b2}^T, E_{b3}^T]^T \in \mathbb{R}^{4m\times n_2} \quad \text{and} \quad \widetilde{G} = [G_0^T, G_1^T, G_2^T, G_3^T]^T \in \mathbb{R}^{4m\times d},$$

where $E_{bt} \in \mathbb{R}^{m\times n_2}$ and $G_t \in \mathbb{R}^{m\times d}$ for $t = 0,1,2,3$. The next theorem establishes the equivalence between the RBMTLS and real MTLS problems.

Theorem 5.3.2. *Consider the RBMTLS problem (5.3.3) and the real MTLS problem (5.3.4). Let $X = [X_a^T, X_b^T]^T$ be a real matrix. Then, X is an RBMTLS solution if and only if X is a real MTLS solution. In this case, if X represents a real MTLS solution, then there exist $\widetilde{E}_b$ and $\widetilde{G}$ such that*

$$\left\|[\widetilde{E}_b, \widetilde{G}]\right\|_F = \min, \quad C_aX_a + (C_b + \widetilde{E}_b)X_b = D + \widetilde{G}.$$

Let $\widehat{E}_b = E_{b0} + E_{b1}\boldsymbol{i} + E_{b2}\boldsymbol{j} + E_{b3}\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times n_2}$ and $\widehat{G} = G_0 + G_1\boldsymbol{i} + G_2\boldsymbol{j} + G_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times d}$. Then,

$$\left\|[\widehat{E}_b, \widehat{G}]\right\|_F = \min, \quad A_aX_a + (A_b + \widehat{E}_b)X_b = B + \widehat{G}.$$

Therefore, there exist $\widehat{E}_b$ and $\widehat{G}$ such that X is an RBMTLS solution.

Proof. If $X = [X_a^T, X_b^T]^T \in \mathbb{R}^{n\times d}$ is a real MTLS solution, then there exist real matrices $\widetilde{E}_b \in \mathbb{R}^{4m\times n_2}$ and $\widetilde{G} \in \mathbb{R}^{4m\times d}$ such that

$$\left\|[\widetilde{E}_b, \widetilde{G}]\right\|_F = \min, \quad [C_a, C_b + \widetilde{E}_b]X = D + \widetilde{G}.$$

We have

$$\begin{aligned}\left[[C_a, C_b + \widetilde{E}_b], Q_m[C_a, C_b + \widetilde{E}_b], R_m[C_a, C_b + \widetilde{E}_b], S_m[C_a, C_b + \widetilde{E}_b]\right]\begin{bmatrix} X & 0 & 0 & 0\\ 0 & X & 0 & 0\\ 0 & 0 & X & 0\\ 0 & 0 & 0 & X\end{bmatrix} \\ = \left[(D+\widetilde{G}), Q_m(D+\widetilde{G}), R_m(D+\widetilde{G}), S_m(D+\widetilde{G})\right].\end{aligned} \tag{5.3.5}$$

Now,

$$[C_a, C_b + \widetilde{E}_b] = \begin{bmatrix} A_{a0} & A_{b0} + E_{b0}\\ A_{a1} & A_{b1} + E_{b1}\\ A_{a2} & A_{b2} + E_{b2}\\ A_{a3} & A_{b3} + E_{b3}\end{bmatrix}, \quad D + \widetilde{G} = \begin{bmatrix} B_0 + G_0\\ B_1 + G_1\\ B_2 + G_2\\ B_3 + G_3\end{bmatrix}. \tag{5.3.6}$$

Construct the following reduced biquaternion matrices

$$\begin{aligned}
\widehat{A} &:= [A_{a0}, A_{b0} + E_{b0}] + [A_{a1}, A_{b1} + E_{b1}]\boldsymbol{i} + [A_{a2}, A_{b2} + E_{b2}]\boldsymbol{j} + [A_{a3}, A_{b3} + E_{b3}]\boldsymbol{k},\\
\widehat{B} &:= (B_0 + G_0) + (B_1 + G_1)\boldsymbol{i} + (B_2 + G_2)\boldsymbol{j} + (B_3 + G_3)\boldsymbol{k},\\
\widehat{E}_b &:= E_{b0} + E_{b1}\boldsymbol{i} + E_{b2}\boldsymbol{j} + E_{b3}\boldsymbol{k},\\
\widehat{G} &:= G_0 + G_1\boldsymbol{i} + G_2\boldsymbol{j} + G_3\boldsymbol{k}.
\end{aligned}$$

Using (4.2.1), (4.2.2), and (5.3.6), we have

$$\widehat{A}_c^R = [C_a, C_b + \widetilde{E}_b], \quad \widehat{B}_c^R = D + \widetilde{G}, \quad \left(\widehat{E}_b\right)_c^R = \widetilde{E}_b, \quad \text{and} \quad \widehat{G}_c^R = \widetilde{G}.$$

Using (4.2.1), (4.2.3), and (5.3.6), we get

$$\begin{aligned}
\widehat{A}^R &= \left[[C_a, C_b + \widetilde{E}_b], Q_m[C_a, C_b + \widetilde{E}_b], R_m[C_a, C_b + \widetilde{E}_b], S_m[C_a, C_b + \widetilde{E}_b]\right],\\
\widehat{B}^R &= \left[(D + \widetilde{G}), Q_m(D + \widetilde{G}), R_m(D + \widetilde{G}), S_m(D + \widetilde{G})\right],\\
X^R &= \begin{bmatrix} X & 0 & 0 & 0\\ 0 & X & 0 & 0\\ 0 & 0 & X & 0\\ 0 & 0 & 0 & X \end{bmatrix}.
\end{aligned}$$

Therefore, equation 5.3.5 is equivalent to

$$\widehat{A}^R X^R = \widehat{B}^R, \tag{5.3.7}$$

$$(\widehat{A}X)^R = \widehat{B}^R,$$

$$\widehat{A}X = \widehat{B}. \tag{5.3.8}$$

Now,

$$\begin{aligned}
\widehat{A} &= [A_{a0}, A_{b0} + E_{b0}] + [A_{a1}, A_{b1} + E_{b1}]\boldsymbol{i} + [A_{a2}, A_{b2} + E_{b2}]\boldsymbol{j} + [A_{a3}, A_{b3} + E_{b3}]\boldsymbol{k}\\
&= [(A_{a0} + A_{a1}\boldsymbol{i} + A_{a2}\boldsymbol{j} + A_{a3}\boldsymbol{k}), (A_{b0} + A_{b1}\boldsymbol{i} + A_{b2}\boldsymbol{j} + A_{b3}\boldsymbol{k}) + (E_{b0} + E_{b1}\boldsymbol{i} + E_{b2}\boldsymbol{j} + E_{b3}\boldsymbol{k})]\\
&= [A_a, A_b + \widehat{E}_b].
\end{aligned} \tag{5.3.9}$$

and

$$\begin{aligned}
\widehat{B} &= (B_0 + G_0) + (B_1 + G_1)\boldsymbol{i} + (B_2 + G_2)\boldsymbol{j} + (B_3 + G_3)\boldsymbol{k}\\
&= (B_0 + B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k}) + (G_0 + G_1\boldsymbol{i} + G_2\boldsymbol{j} + G_3\boldsymbol{k}) = B + \widehat{G}.
\end{aligned} \tag{5.3.10}$$

Using (5.3.9) and (5.3.10), equation 5.3.8 is equivalent to

$$[A_a, A_b + \widehat{E}_b]X = B + \widehat{G},$$

$$[A_a, A_b + \widehat{E}_b]\begin{bmatrix} X_a \\ X_b \end{bmatrix} = B + \widehat{G},$$

$$A_aX_a + (A_b + \widehat{E}_b)X_b = B + \widehat{G}. \tag{5.3.11}$$

Using Lemma 5.2.2, we can verify that

$$\left\|[\widehat{E}_b, \widehat{G}]\right\|_F = \frac{1}{2}\left\|[\widehat{E}_b, \widehat{G}]^R\right\|_F = \frac{1}{2}\left\|[\widehat{E}_b^R, \widehat{G}^R]\right\|_F = \left\|[\widetilde{E}_b, \widetilde{G}]\right\|_F = \min. \tag{5.3.12}$$

Combining (5.3.11) and (5.3.12), we can conclude that there exist RB matrices $\widehat{E}_b \in \mathbb{Q}_{\mathbb{R}}^{m\times n_2}$ and $\widehat{G} \in \mathbb{Q}_{\mathbb{R}}^{m\times d}$ such that $X = [X_a^T, X_b^T]^T \in \mathbb{R}^{n\times d}$ is an RBMTLS solution, and vice versa. ■

Next, we derive an explicit expression for the real RBMTLS solution X. To begin, perform n_1 Householder transformations using a matrix $Q \in \mathbb{R}^{4m\times 4m}$ on the matrix $[C, D]$ such that

$$Q^T[C, D] = Q^T[C_a, C_b, D] = \begin{array}{cc} \begin{array}{ccc} n_1 & n_2 & d \end{array} & \\ \begin{bmatrix} R_{11} & R_{12} & R_{1d} \\ 0 & R_{22} & R_{2d} \end{bmatrix} & \begin{array}{l} n_1 \\ 4m-n_1 \end{array} \end{array}. \tag{5.3.13}$$

Partition Q as $Q = [Q_1, Q_2]$, where $Q_1 \in \mathbb{R}^{4m\times n_1}$ and $Q_2 \in \mathbb{R}^{4m\times(4m-n_1)}$. Next, compute the SVD of $[R_{22}, R_{2d}]$:

$$[R_{22}, R_{2d}] = U\Sigma V^T, \tag{5.3.14}$$

where U and V are real orthonormal matrices, $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_{n_2+d})$, and the singular values of $[R_{22}, R_{2d}]$ satisfy

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{n_2} > \sigma_{n_2+1} \geq \ldots \geq \sigma_{n_2+d} > 0. \tag{5.3.15}$$

Partition U, Σ, and V as

$$U = \begin{array}{cc} \begin{array}{cc} n_2 & 4m-n_1-n_2 \end{array} & \\ \begin{bmatrix} U_1, & U_2 \end{bmatrix} & 4m-n_1 \end{array}, \quad \Sigma = \begin{array}{cc} \begin{array}{cc} n_2 & d \end{array} & \\ \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} & \begin{array}{l} n_2 \\ 4m-n_1-n_2 \end{array} \end{array}, \quad \text{and} \quad V = \begin{array}{cc} \begin{array}{cc} n_2 & d \end{array} & \\ \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} & \begin{array}{l} n_2 \\ d \end{array} \end{array}. \tag{5.3.16}$$

In the following theorem, we present the conditions for the existence of a unique real RBMTLS solution, and in this case, provide an explicit expression for the real RBMTLS solution.

Theorem 5.3.3. *With the notations in (5.3.13) and (5.3.16), consider the RBMTLS problem (5.3.3). Let the SVD of $[R_{22}, R_{2d}]$ be as in (5.3.14), and let its singular values be as in (5.3.15). If $\sigma_{n_2} > \sigma_{n_2+1}$ and V_{22} is nonsingular, then the real RBMTLS solution exists and is unique. In this case, the real RBMTLS solution is given by*

$$X = \begin{bmatrix} R_{11}^{-1} R_{1d} \\ 0 \end{bmatrix} + \begin{bmatrix} R_{11}^{-1} R_{12} \\ -I_{n_2} \end{bmatrix} V_{12} V_{22}^{-1}. \tag{5.3.17}$$

Proof. Using Theorem 5.3.2, X represents an RBMTLS solution if and only if X is a real MTLS solution. Therefore, to find the RBMTLS solution, we find the real MTLS solution. Now, the real linear system corresponding to (5.1.1) is given by

$$[C_a, C_b] \begin{bmatrix} X_a \\ X_b \end{bmatrix} \approx D, \quad [C_a, C_b, D] \begin{bmatrix} X_a \\ X_b \\ -I_d \end{bmatrix} \approx 0.$$

To find the real MTLS solution, we modify the above system in such a way that it becomes compatible. We achieve this by perturbing matrices C_b and D while keeping matrix C_a exact, as in (5.3.4). By pre-multiplying both sides of the above system by Q^T and using (5.3.13), we get

$$[Q_1, Q_2]^T [C_a, C_b, D] \begin{bmatrix} X_a \\ X_b \\ -I_d \end{bmatrix} \approx 0, \quad \begin{bmatrix} R_{11} & R_{12} & R_{1d} \\ 0 & R_{22} & R_{2d} \end{bmatrix} \begin{bmatrix} X_a \\ X_b \\ -I_d \end{bmatrix} \approx 0.$$

Let

$$R := \begin{array}{c} \begin{array}{ccc} \scriptstyle n_1 & \scriptstyle n_2 & \scriptstyle d \end{array} \\ \left[\begin{array}{ccc} R_{11} & R_{12} & R_{1d} \\ 0 & R_{22} & R_{2d} \end{array} \right] \end{array} \begin{array}{l} \scriptstyle n_1 \\ \scriptstyle 4m-n_1 \end{array}.$$

To make the above system compatible, the matrix $[X_a^T, X_b^T, -I_d]^T$ should be in the null space of R. Therefore, by the rank-nullity theorem, the rank of the matrix R must be reduced to $n_1 + n_2$. We achieve this by modifying matrix R. To keep matrix C_a exact, we modify matrix R without perturbing the matrix R_{11}.

Now, matrix A_a has full column rank. In view of Lemma 5.2.1, the matrix C_a also has full column rank n_1, which implies that R_{11} is a nonsingular upper triangular matrix. As a result, modifying R_{12} and R_{1d} does not affect the rank of the matrix R. Consequently, we do not modify these matrices. Instead, we modify matrices R_{22} and R_{2d}.

Let

$$\widetilde{R} := \begin{bmatrix} R_{11} & R_{12} & R_{1d} \\ 0 & \widetilde{R}_{22} & \widetilde{R}_{2d} \end{bmatrix}$$

be the modified matrix such that the system $\widetilde{R}[X_a^T, X_b^T, -I_d]^T = 0$ is compatible. Now our aim is to find $\widetilde{R}_{22}$ and $\widetilde{R}_{2d}$. We first focus on the reduced real TLS problem $R_{22}X_b \approx R_{2d}$. We have

$$[R_{22}, R_{2d}] \begin{bmatrix} X_b \\ -I_d \end{bmatrix} \approx 0.$$

To find a solution to the reduced real TLS problem, the matrix $\left[X_b^T, -I_d\right]^T$ should be in the null space of $[R_{22}, R_{2d}]$. Therefore, by the rank-nullity theorem, the rank of the matrix $[R_{22}, R_{2d}]$ must be reduced to n_2. Let $[\widetilde{R}_{22}, \widetilde{R}_{2d}]$ denote the best rank n_2 approximation of $[R_{22}, R_{2d}]$. By Lemma 5.2.3, we have

$$[\widetilde{R}_{22}, \widetilde{R}_{2d}] = [U_1\Sigma_1 V_{11}^T, U_1\Sigma_1 V_{21}^T].$$

If $\sigma_{n_2} > \sigma_{n_2+1}$, then $[\widetilde{R}_{22}, \widetilde{R}_{2d}]$ represents the unique rank n_2 approximation of $[R_{22}, R_{2d}]$, and the columns of the matrix $\begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix}$ represent a basis for the null space of $[\widetilde{R}_{22}, \widetilde{R}_{2d}]$. We have

$$[\widetilde{R}_{22}, \widetilde{R}_{2d}] \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = 0.$$

If V_{22} is nonsingular, then we get

$$[\widetilde{R}_{22}, \widetilde{R}_{2d}] \begin{bmatrix} -V_{12}V_{22}^{-1} \\ -I_d \end{bmatrix} = 0.$$

Hence, the reduced real TLS solution is unique and is given by $X_b = -V_{12}V_{22}^{-1}$. Notice that the rank of the modified matrix $\widetilde{R}$ is $n_1 + n_2$.

After computing X_b, we calculate X_a. We have $\widetilde{R}[X_a^T, X_b^T, -I_d]^T = 0$. Since R_{11} is nonsingular, we obtain a unique solution $X_a = R_{11}^{-1}(R_{1d} - R_{12}X_b)$. ■

Remark 5.3.4. *The perturbation $\widetilde{E}_b$ to the matrix C_b is given by $\widetilde{E}_b = \widetilde{C}_b - C_b$, and the perturbation $\widetilde{G}$ to the matrix D is given by $\widetilde{G} = \widetilde{D} - D$. We have*

$$[C_a, \widetilde{C}_b, \widetilde{D}] = [Q_1, Q_2]\widetilde{R} = [Q_1, Q_2] \begin{bmatrix} R_{11} & R_{12} & R_{1d} \\ 0 & \widetilde{R}_{22} & \widetilde{R}_{2d} \end{bmatrix}.$$

We obtain the perturbed matrices $\widetilde{C}_b := Q_1R_{12} + Q_2\widetilde{R}_{22}$ and $\widetilde{D} := Q_1R_{1d} + Q_2\widetilde{R}_{2d}$, where $\widetilde{R}_{22} = U_1\Sigma_1V_{11}^T$ and $\widetilde{R}_{2d} = U_1\Sigma_1V_{21}^T$. Now, we can obtain $\widehat{E}_b$ from $\widetilde{E}_b$ and $\widehat{G}$ from $\widetilde{G}$ using Theorem 5.3.2.

Algebraic Technique for RBTLS Problem:

In the case where all columns of matrix A are contaminated by noise (i.e., $n_1 = 0$ and $n_2 = n$), the RBMTLS problem (5.3.3) simplifies to an RBTLS problem (5.1.3). In this scenario, we have $A_a = 0$ and $A_b = A$, as well as $C_a = 0$ and $C_b = C$. Let $C = [A_0^T, A_1^T, A_2^T, A_3^T]^T \in \mathbb{R}^{4m\times n}$. We now consider the corresponding multidimensional real TLS problem, which can be formulated as:

$$\min_{X,\widetilde{E},\widetilde{G}} \left\|[\widetilde{E},\widetilde{G}]\right\|_F \quad \text{subject to} \quad (C+\widetilde{E})X = D+\widetilde{G}. \tag{5.3.18}$$

Once a minimizing $[\widetilde{E},\widetilde{G}]$ is found, then any X which solves the corrected system in (5.3.18) is called the real TLS solution.

In the forthcoming results on the RBTLS solution, we will be using the following notations: Let

$$\widetilde{E} = [E_0^T, E_1^T, E_2^T, E_3^T]^T \in \mathbb{R}^{4m\times n} \quad \text{and} \quad \widetilde{G} = [G_0^T, G_1^T, G_2^T, G_3^T]^T \in \mathbb{R}^{4m\times d},$$

where $E_t \in \mathbb{R}^{m\times n}$ and $G_t \in \mathbb{R}^{m\times d}$ for $t = 0,1,2,3$. In the following corollary, we provide the solution technique for the RBTLS problem (5.1.3).

Corollary 5.3.5. *Consider the RBTLS problem (5.1.3) and the real TLS problem (5.3.18). Let X be a real matrix. Then, X is an RBTLS solution if and only if X is a real TLS solution. In this case, if X represents a real TLS solution, then there exist $\widetilde{E}$ and $\widetilde{G}$ such that*

$$\left\|[\widetilde{E},\widetilde{G}]\right\|_F = \min, \quad (C+\widetilde{E})X = D+\widetilde{G}.$$

Let $\widehat{E} = E_0 + E_1\boldsymbol{i} + E_2\boldsymbol{j} + E_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$ and $\widehat{G} = G_0 + G_1\boldsymbol{i} + G_2\boldsymbol{j} + G_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times d}$. Then,

$$\left\|[\widehat{E},\widehat{G}]\right\|_F = \min, \quad (A+\widehat{E})X = B+\widehat{G}.$$

Therefore, there exist $\widehat{E}$ and $\widehat{G}$ such that X is an RBTLS solution.

Proof. By taking $n_1 = 0$ and $n_2 = n$, the proof proceeds in a manner analogous to the proof of Theorem 5.3.2. ∎

We now derive an explicit expression for the real RBTLS solution X. By taking $n_1 = 0$ and $n_2 = n$, equations (5.3.13), (5.3.14), (5.3.15), and (5.3.16) simplify to

$$Q^T[C,D] = \begin{array}{cc} \begin{array}{cc} \;\; n & \quad d \end{array} & \\ \left[\begin{array}{cc} R_{22} & R_{2d} \end{array}\right] & 4m \end{array}. \tag{5.3.19}$$

Thus, the SVD of $[R_{22}, R_{2d}]$ is given by

$$[R_{22}, R_{2d}] = U\Sigma V^T, \tag{5.3.20}$$

where U and V are real orthonormal matrices, $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_{n+d})$, and the singular values of $[R_{22}, R_{2d}]$ satisfy

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > \sigma_{n+1} \geq \dots \geq \sigma_{n+d} > 0. \tag{5.3.21}$$

We partition U, Σ, and V as follows:

$$U = \begin{array}{cc} \begin{array}{cc} n & 4m-n \end{array} & \\ \begin{bmatrix} U_1 & U_2 \end{bmatrix} & 4m \end{array}, \quad \Sigma = \begin{array}{cc} \begin{array}{cc} n & d \end{array} & \\ \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} & \begin{array}{c} n \\ 4m-n \end{array} \end{array}, \quad \text{and} \quad V = \begin{array}{cc} \begin{array}{cc} n & d \end{array} & \\ \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} & \begin{array}{c} n \\ d \end{array} \end{array}. \tag{5.3.22}$$

In the following corollary, we present the conditions for the existence of a unique real RBTLS solution, and in this case, provide an explicit expression for the real RBTLS solution.

Corollary 5.3.6. *With the notations in (5.3.19) and (5.3.22), consider the RBTLS problem (5.1.3). Let the SVD of $[R_{22}, R_{2d}]$ be as in (5.3.20), and let its singular values be as in (5.3.21). If $\sigma_n > \sigma_{n+1}$ and V_{22} is nonsingular, then the real RBTLS solution exists and is unique. In this case, the real RBTLS solution is given by*

$$X = -V_{12}V_{22}^{-1}. \tag{5.3.23}$$

Proof. By taking $n_1 = 0$ and $n_2 = n$, the proof follows similar to the proof of Theorem 5.3.3. ∎

Remark 5.3.7. *The perturbation $\widetilde{E}$ to the matrix C is given by $\widetilde{E} = \widetilde{C} - C$, and the perturbation $\widetilde{G}$ to the matrix D is given by $\widetilde{G} = \widetilde{D} - D$. We have*

$$[\widetilde{C}, \widetilde{D}] = Q[\widetilde{R}_{22}, \widetilde{R}_{2d}].$$

We get the perturbed matrices $\widetilde{C} := Q\widetilde{R}_{22}$ and $\widetilde{D} := Q\widetilde{R}_{2d}$, where $\widetilde{R}_{22} = U_1\Sigma_1V_{11}^T$ and $\widetilde{R}_{2d} = U_1\Sigma_1V_{21}^T$. Now, we can obtain $\widehat{E}$ from $\widetilde{E}$ and $\widehat{G}$ from $\widetilde{G}$ using Corollary 5.3.5.

Algebraic Technique for RBLS Problem:

When all columns of matrix A are error-free, i.e., $n_1 = n$ and $n_2 = 0$, the RBMTLS problem (5.3.3) becomes an RBLS problem (5.1.2). In this scenario, we have $A_a = A$ and $A_b = 0$, also $C_a = C$ and $C_b = 0$. Let $C = [A_0^T, A_1^T, A_2^T, A_3^T]^T \in \mathbb{R}^{4m\times n}$. Consider a multidimensional real LS problem

$$\min_X \|CX - D\|_F. \tag{5.3.24}$$

In the following corollary, we provide the solution technique for RBLS problem (5.1.2).

Corollary 5.3.8. *Consider the RBLS problem (5.1.2) and the real LS problem (5.3.24). Let X be a real matrix. Then, X is an RBLS solution if and only if X is a real LS solution. In this case, the solution X is given by*

$$X = C^\dagger D + (I - C^\dagger C)Z, \tag{5.3.25}$$

where Z is an arbitrary matrix of suitable size and the least squares solution with the least norm is $X = C^\dagger D$.

Proof. By taking $n_1 = n$ and $n_2 = 0$ in Theorem 5.3.2, we get that X is an RBLS solution if and only if X is a real LS solution. Using Lemma 1.3.4, we get the desired expression for the solution X. ∎

The results developed in this section can also be applied to several other special cases. The following remarks are in order.

Remark 5.3.9. *When $d = 1$, our results also include single-right-hand-side RBMTLS, RBTLS, and RBLS problems.*

Remark 5.3.10. *Complex matrix equations are special cases of reduced biquaternion matrix equations. Hence, our developed solution techniques are well-suited for finding the best approximate solution to $AX \approx B$ over complex fields.*

We take the real representation of matrix $A = A_0 + A_1\boldsymbol{i} \in \mathbb{C}^{m\times n}$, where $A_t \in \mathbb{R}^{m\times n}$ for $t = 0, 1$, denoted by $\widetilde{A}^R$ as

$$\widetilde{A}^R = \begin{bmatrix} A_0 & -A_1 \\ A_1 & A_0 \end{bmatrix}.$$

Let $\widetilde{Q}_m = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix}$. Let $\widetilde{A}_c^R$ denotes the first block column of the block matrix $\widetilde{A}^R$ i.e. $\widetilde{A}_c^R = [A_0^T, A_1^T]^T$. We have $\widetilde{A}^R = [\widetilde{A}_c^R, \widetilde{Q}_m\widetilde{A}_c^R]$. By taking $\widetilde{A}^R$, $\widetilde{Q}_m$, and $\widetilde{A}_c^R$ as above, we can obtain results to solve the complex LS, TLS, and MTLS problems.

5.4. Perturbation Analysis of the RBMTLS Solution

Perturbation analysis is a crucial aspect of numerical analysis, focusing on how sensitive a solution is to small changes in the input data, which is quantified by the condition number. This section explores first-order perturbation bounds for real RBMTLS, RBTLS, and RBLS solutions using their relative normwise condition numbers.

For the reduced biquaternion linear approximation system $AX \approx B$, our objective is to analyze how perturbations in A and B affect the real RBMTLS solution X_M. Let

$$\widehat{A} = A + \Delta A = [\widehat{A}_a, \widehat{A}_b] \in \mathbb{Q}_{\mathbb{R}}^{m\times n} \quad \text{and} \quad \widehat{B} = B + \Delta B \in \mathbb{Q}_{\mathbb{R}}^{m\times d},$$

where $\Delta A = \Delta A_0 + \Delta A_1 \boldsymbol{i} + \Delta A_2 \boldsymbol{j} + \Delta A_3 \boldsymbol{k}$ and $\Delta B = \Delta B_0 + \Delta B_1 \boldsymbol{i} + \Delta B_2 \boldsymbol{j} + \Delta B_3 \boldsymbol{k}$ represent the perturbations of the input matrices A and B, respectively. Here, $\widehat{A}_a \in \mathbb{Q}_{\mathbb{R}}^{m\times n_1}$ and $\widehat{A}_b \in \mathbb{Q}_{\mathbb{R}}^{m\times n_2}$. Let $\widehat{X}_M$ denote the real RBMTLS solution to the perturbed reduced biquaternion system $\widehat{A}X \approx \widehat{B}$. When the norm $\|[\Delta A, \Delta B]\|_F$ is sufficiently small, the perturbation analysis of singular values guarantees the existence of a unique solution $\widehat{X}_M$. Let $\Delta X_M = \widehat{X}_M - X_M$ be the change in the solution.

Next, consider the perturbed matrices $\widehat{C}$ and $\widehat{D}$, where

$$\widehat{C} = C + \Delta C = [\widehat{C}_a, \widehat{C}_b] \in \mathbb{R}^{4m\times n} \quad \text{and} \quad \widehat{D} = D + \Delta D \in \mathbb{R}^{4m\times d}.$$

Here, the matrices $\widehat{C}_a \in \mathbb{R}^{4m\times n_1}$ and $\widehat{C}_b \in \mathbb{R}^{4m\times n_2}$ represent the partitioned columns of $\widehat{C}$. The perturbation matrices are given by

$$\Delta C = \begin{bmatrix} \Delta A_0 \\ \Delta A_1 \\ \Delta A_2 \\ \Delta A_3 \end{bmatrix} \quad \text{and} \quad \Delta D = \begin{bmatrix} \Delta B_0 \\ \Delta B_1 \\ \Delta B_2 \\ \Delta B_3 \end{bmatrix}.$$

The relative normwise condition number of the real RBMTLS solution X_M is defined as follows:

$$k^n_{RBMTLS}(X_M, A, B) = \lim_{\epsilon\to 0} \sup \left\{ \frac{\|\Delta X_M\|_F}{\epsilon \|X_M\|_F} \mid \|[\Delta A, \Delta B]\|_F \le \epsilon \|[A, B]\|_F \right\}.$$

Theorem 5.4.1. *Consider the RBMTLS problem* (5.3.3) *and the real MTLS problem* (5.3.4)*. Assume the conditions specified in Theorem* 5.3.3 *for the existence and uniqueness of the real RBMTLS solution X_M are satisfied. Let $C_a = U_a S_a V_a^T$ be the thin SVD of C_a,* *and denote* $\overline{C}_b = [C_b, D]$. *With the notations in* (5.3.15) *and* (5.3.16), *set*

$$\overline{\Sigma}_1 = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_{n_2}), \quad \overline{\Sigma}_2 = \mathrm{diag}(\sigma_{n_2+1}, \sigma_{n_2+2}, \ldots, \sigma_{n_2+d}),$$
$$\overline{C} = \begin{bmatrix} -C_a^{\dagger}\overline{C}_b \\ I_{n_2+d} \end{bmatrix}, \quad V_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix}, \quad V_2 = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix},$$
$$\overline{S}_a = \begin{bmatrix} S_a & 0 \\ 0 & \overline{\Sigma}_1 \end{bmatrix}, \quad \overline{V}_{11} = \begin{bmatrix} V_a & 0 \\ (C_a^{\dagger}C_b)^T V_a & V_{11} \end{bmatrix}.$$

Then, the relative normwise condition number of the real RBMTLS solution X_M *is expressed as*

$$k^n_{RBMTLS}(X_M, A, B) = \|HG\overline{Z}\|_2 \frac{\|[C, D]\|_F}{\|X_M\|_F}, \tag{5.4.1}$$

where

$$H = \Pi_{(d,n)} \left(\overline{V}_{11}^{-T} \otimes V_{22}^{-T}\right),$$
$$G = \left(\left(\overline{S}_a^2 \otimes I_d\right) - \begin{bmatrix} 0_{n_1} & 0 \\ 0 & I_{n_2} \end{bmatrix} \otimes (\overline{\Sigma}_2^T \overline{\Sigma}_2)\right)^{-1} \left[I_n \otimes \overline{\Sigma}_2^T, \ \overline{S}_a \otimes I_d\right],$$
$$\overline{Z} = \mathrm{diag}\left(\begin{bmatrix} I_{n_1} & 0 \\ -V_1^T (C_a^{\dagger}\overline{C}_b)^T V_a & I_{n_2} \end{bmatrix} \otimes I_d, \ I_n \otimes (\overline{C}V_2)^T\right).$$

Proof. Based on Theorem 5.3.2, X_M is a real RBMTLS solution of $AX \approx B$ if and only if X_M is a real MTLS solution of the corresponding real linear system $CX \approx D$. Using (1.2.1), we have $\|[\Delta A, \Delta B]\|_F = \|[\Delta C, \Delta D]\|_F$ and $\|[A, B]\|_F = \|[C, D]\|_F$. Consequently, we obtain

$$k^n_{RBMTLS}(X_M, A, B) = \lim_{\epsilon \to 0} \sup \left\{ \frac{\|\Delta X_M\|_F}{\epsilon \|X_M\|_F} \mid \|[\Delta C, \Delta D]\|_F \leq \epsilon \|[C, D]\|_F \right\},$$

which is same as the relative normwise condition number $(k^n_{MTLS}(X_M, C, D))$ of the MTLS solution X_M to $CX \approx D$.

Therefore, to study the perturbation analysis of the real RBMTLS solution, we only need to study the perturbation analysis of the real MTLS solution of the corresponding real linear system. Using [45, Theorem 3.3], we get the desired expression for $k^n_{MTLS}(X_M, C, D)$ and, therefore, for $k^n_{RBMTLS}(X_M, A, B)$. ∎

Let $\varepsilon_n = \frac{\|[\Delta A, \Delta B]\|_F}{\|[A,B]\|_F}$. Then, the upper bound U_M for the relative forward error of the real RBMTLS solution X_M is given by

$$\frac{\|\Delta X_M\|_F}{\|X_M\|_F} \le k^n_{RBMTLS}(X_M, A, B)\varepsilon_n \equiv U_M. \tag{5.4.2}$$

Perturbation Analysis of the RBTLS Solution:

Now, we will examine how perturbations in A and B affect the real RBTLS solution X_T of $AX \approx B$. For the RBTLS problem (i.e., $n_1 = 0$ and $n_2 = n$), let $\widehat{X}_T$ be the real RBTLS solution to the perturbed system $\widehat{A}X \approx \widehat{B}$. Let $\Delta X_T = \widehat{X}_T - X_T$. The relative normwise condition number of the real RBTLS solution X_T is defined as follows:

$$k^n_{RBTLS}(X_T, A, B) = \lim_{\epsilon\to 0}\sup\left\{\frac{\|\Delta X_T\|_F}{\epsilon\|X_T\|_F} \mid \|[\Delta A, \Delta B]\|_F \le \epsilon\|[A,B]\|_F\right\}.$$

Theorem 5.4.2. *Consider the RBTLS problem (5.1.3) and the real TLS problem (5.3.18). Assume the conditions specified in Corollary 5.3.6 for the existence and uniqueness of the real RBTLS solution X_T are satisfied. With the notations in (5.3.21) and (5.3.22), set*

$$\overline{\Sigma}_1 = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n), \quad \overline{\Sigma}_2 = \mathrm{diag}(\sigma_{n+1}, \sigma_{n+2}, \ldots, \sigma_{n+d}), \quad V_2 = \begin{bmatrix} V_{12} \\ V_{22}\end{bmatrix}.$$

Then, the relative normwise condition number of the real RBTLS solution X_T is expressed as

$$k^n_{RBTLS}(X_T, A, B) = \|HG\overline{Z}\|_2 \frac{\|[C,D]\|_F}{\|X_T\|_F}, \tag{5.4.3}$$

where

$$\begin{aligned}
H &= \Pi_{(d,n)}\left(V_{11}^{-T} \otimes V_{22}^{-T}\right), \\
G &= \left(\left(\overline{\Sigma}_1^2 \otimes I_d\right) - \left(I_n \otimes (\overline{\Sigma}_2^T\overline{\Sigma}_2)\right)\right)^{-1}\left[I_n \otimes \overline{\Sigma}_2^T,\ \overline{\Sigma}_1 \otimes I_d\right], \\
\overline{Z} &= \mathrm{diag}\left(I_n \otimes I_d,\ I_n \otimes V_2^T\right).
\end{aligned}$$

Proof. The proof follows by setting $n_1 = 0$ and $n_2 = n$ in Theorem 5.4.1. ■

The upper bound U_T for the relative forward error of the real RBTLS solution X_T is given by

$$\frac{\|\Delta X_T\|_F}{\|X_T\|_F} \le k^n_{RBTLS}(X_T, A, B)\varepsilon_n \equiv U_T. \tag{5.4.4}$$

Perturbation Analysis of the RBLS Solution:

Next, we examine how perturbations in A and B affect the real RBLS solution X_L of $AX \approx B$. For the RBLS problem (i.e., $n_1 = n$ and $n_2 = 0$), let $\widehat{X}_L$ be the real RBLS solution to the perturbed system $\widehat{A}X \approx \widehat{B}$. Let $\Delta X_L = \widehat{X}_L - X_L$. Additionally, let $\mathcal{S}$ be the set of perturbations in matrix A such that

$$\mathcal{S} = \left\{\Delta A \,\middle|\, \mathcal{R}(\Delta A) \subseteq \mathcal{R}(A),\ \mathcal{R}((\Delta A)^T) \subseteq \mathcal{R}(A^T)\right\}.$$

The relative normwise condition number of the real RBLS solution X_L is defined as follows:

$$k^n_{RBLS}(X_L, A, B) = \lim_{\epsilon \to 0} \sup \left\{ \frac{\|\Delta X_L\|_F}{\epsilon \|X_L\|_F} \mid \|[\Delta A, \Delta B]\|_F \leq \epsilon \|[A, B]\|_F, \Delta A \in \mathcal{S} \right\}.$$

Theorem 5.4.3. *Consider the RBLS problem* (5.1.2) *and the real LS problem* (5.3.24)*. Let A be rank deficient, then the relative normwise condition number of the real RBLS solution X_L is expressed as*

$$k^n_{RBLS}(X_L, A, B) = \frac{\|C^\dagger\|_2 \|[C, D]\|_F}{\|X_L\|_F} \sqrt{1 + \|X_L\|_2^2}. \tag{5.4.5}$$

Proof. The proof follows along similar lines as Theorem 5.4.1 and by applying [49, Theorem 3.1] to the corresponding real linear system $CX \approx D$. ■

The upper bound U_L for the relative forward error of the real RBLS solution X_L is given by

$$\frac{\|\Delta X_L\|_F}{\|X_L\|_F} \leq k^n_{RBLS}(X_L, A, B)\varepsilon_n \equiv U_L. \tag{5.4.6}$$

5.5. Numerical Verification

In this section, we present numerical algorithms for solving the RBMTLS, RBTLS, and RBLS problems and provide numerical examples to validate these algorithms. First, we illustrate the effectiveness of the RBMTLS method in solving the linear system $AX \approx B$, particularly when errors are present in all columns of matrix B and only a few columns of matrix A. Next, we examine the upper bounds for the relative forward errors associated with the real RBMTLS, RBTLS, and RBLS solutions.

Building on Theorem 5.3.3 and Corollary 5.3.6, we now outline algorithms to solve the RBMTLS problem (5.3.3) and the RBTLS problem (5.1.3), respectively.

Algorithm 5.5.1 For RBMTLS problem

Input: $A = A_0 + A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $B = B_0 + B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times d}$, where $m \geq n + d$. Let the first n_1 columns of matrix A be known exactly, and the remaining n_2 columns be contaminated by noise, where $n_1 + n_2 = n$. Partition $A = [A_a, A_b]$, where $A_a = A_{a0} + A_{a1}\boldsymbol{i} + A_{a2}\boldsymbol{j} + A_{a3}\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times n_1}$, $A_b = A_{b0} + A_{b1}\boldsymbol{i} + A_{b2}\boldsymbol{j} + A_{b3}\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times n_2}$, and A_a has full column rank.

Output: Perturbation $\widehat{E}_b$, $\widehat{G}$, and the solution X.

Step 1: Matrix Computation: Define $C = [C_a, C_b] \in \mathbb{R}^{4m\times n}$, $C_a = [A_{a0}^T, A_{a1}^T, A_{a2}^T, A_{a3}^T]^T$, $C_b = [A_{b0}^T, A_{b1}^T, A_{b2}^T, A_{b3}^T]^T$, and $D = [B_0^T, B_1^T, B_2^T, B_3^T]^T \in \mathbb{R}^{4m\times d}$.

Step 2: QR Decomposition: Find the orthogonal matrix $Q = [Q_1, Q_2] \in \mathbb{R}^{4m\times 4m}$, where $Q_1 \in \mathbb{R}^{4m\times n_1}$ and $Q_2 \in \mathbb{R}^{4m\times(4m-n_1)}$, that performs n_1 Householder transformations on the matrix $[C, D]$ as in (5.3.13).

Step 3: SVD Computation: Compute the SVD of the matrix $[R_{22}, R_{2d}]$. Let the SVD of $[R_{22}, R_{2d}]$ be as in (5.3.14).

Step 4: Solution Computation: If $\sigma_{n_2} > \sigma_{n_2+1}$ and V_{22} is nonsingular, compute the solution X to the RBMTLS problem using Theorem 5.3.3.

Step 5: Perturbation Computation: Compute the perturbations $\widehat{E}_b$ for matrix A_b and $\widehat{G}$ for matrix B using Theorem 5.3.2 and Remark 5.3.4.

Algorithm 5.5.2 For RBTLS problem

Input: $A = A_0 + A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $B = B_0 + B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times d}$, where $m \geq n + d$.

Output: Perturbations $\widehat{E}$, $\widehat{G}$, and the solution X.

Step 1: Matrix Computation: Set $C = [A_0^T, A_1^T, A_2^T, A_3^T]^T \in \mathbb{R}^{4m\times n}$ and $D = [B_0^T, B_1^T, B_2^T, B_3^T]^T \in \mathbb{R}^{4m\times d}$.

Step 2: Orthogonal Matrix: Find the orthogonal matrix $Q \in \mathbb{R}^{4m\times 4m}$ such that $Q^T[C, D] = [R_{22}, R_{2d}]$ as in (5.3.19).

Step 3: SVD Computation: Compute the SVD of the matrix $[R_{22}, R_{2d}]$ as described in (5.3.20).

Step 4: Solution Computation: If $\sigma_n > \sigma_{n+1}$ and V_{22} is nonsingular, compute the solution X to the RBTLS problem using Corollary 5.3.6.

Step 5: Perturbation Calculation: Compute the perturbations $\widehat{E}$ for matrix A and $\widehat{G}$ for matrix B using Corollary 5.3.5 and Remark 5.3.7.

Based on Corollary 5.3.8, we now describe algorithms for solving the RBLS problem (5.1.2).

Algorithm 5.5.3 For RBLS problem

Input: $A = A_0 + A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times n}$, $B = B_0 + B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times d}$, where $m \geq n + d$.

Output: X.

Step 1: Matrix Computation: Define $C = [A_0^T, A_1^T, A_2^T, A_3^T]^T \in \mathbb{R}^{4m\times n}$ and $D = [B_0^T, B_1^T, B_2^T, B_3^T]^T \in \mathbb{R}^{4m\times d}$.

Step 2: Solution Computation: Compute the solution X to the RBLS problem using Corollary 5.3.8.

We now present numerical examples. All calculations are performed on an Intel Core $i7 - 9700@3.00GHz/16GB$ computer using MATLAB $R2021b$ software.

Example 5.5.1. *Let* $F = F_0 + F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{m\times 50}$ $(m > 50)$, *where matrix components are defined as follows:*

$$F_0 = F_1 = F_2 = F_3 = \texttt{randn}(m, 50) \in \mathbb{R}^{m\times 50}.$$

Let $X_0 = \texttt{randn}(50, 35) \in \mathbb{R}^{50\times 35}$. *Set* $G = FX_0$, *which implies that the reduced biquaternion matrix equation* $FX = G$ *is consistent, and* X_0 *is its exact solution. We partition the matrix* F *as*

$$F = [F_a, F_b],$$

where $F_a \in \mathbb{Q}_{\mathbb{R}}^{m\times 20}$ *and* $F_b \in \mathbb{Q}_{\mathbb{R}}^{m\times 30}$.

To assess the effectiveness of the proposed solution techniques in finding the best approximate solution to an inconsistent linear system, we introduce errors into the entries of matrices F *and* G, *which makes the original system inconsistent.*

Let the error terms be denoted as $dA \in \mathbb{Q}_{\mathbb{R}}^{m\times 20}$, $dB \in \mathbb{Q}_{\mathbb{R}}^{m\times 30}$, *and* $dG \in \mathbb{Q}_{\mathbb{R}}^{m\times 35}$. *The modified matrices are then defined as*

$$A_a = F_a + dA, \quad A_b = F_b + dB, \quad \text{and} \quad B = G + dG.$$

Consequently, we have an overdetermined linear system:

$$AX \approx B,$$

where $A = [A_a, A_b] \in \mathbb{Q}_{\mathbb{R}}^{m\times 50}$ *and* $B \in \mathbb{Q}_{\mathbb{R}}^{m\times 35}$ *are known, and* $X \in \mathbb{R}^{50\times 35}$ *is unknown. Now we will consider three different cases. In the first case, errors are introduced in matrices* F_b

and G. In the second case, errors are introduced in matrices F_a, F_b, and G. Lastly, in the third case, errors are introduced only in matrix G.

***Case* 1**: *Take $R = \texttt{rand}(65, 65)$ and $E = 0.01\,(\texttt{rand}(m, 65)R)$. Let*

$$\begin{aligned}
dA &= 0,\\
dB &= E(:, 1:30) + E(:, 1:30)\boldsymbol{i} + E(:, 1:30)\boldsymbol{j} + E(:, 1:30)\boldsymbol{k},\\
dG &= E(:, 31:65) + E(:, 31:65)\boldsymbol{i} + E(:, 31:65)\boldsymbol{j} + E(:, 31:65)\boldsymbol{k}.
\end{aligned}$$

We define $A_a = F_a, A_b = F_b + dB$, and $B = G + dG$.

***Case* 2**: *Take $R = \texttt{rand}(85, 85)$ and $E = 0.01\,(\texttt{rand}(m, 85)R)$. Let*

$$\begin{aligned}
dA &= E(:, 1:20) + E(:, 1:20)\boldsymbol{i} + E(:, 1:20)\boldsymbol{j} + E(:, 1:20)\boldsymbol{k},\\
dB &= E(:, 21:50) + E(:, 21:50)\boldsymbol{i} + E(:, 21:50)\boldsymbol{j} + E(:, 21:50)\boldsymbol{k},\\
dG &= E(:, 51:85) + E(:, 51:85)\boldsymbol{i} + E(:, 51:85)\boldsymbol{j} + E(:, 51:85)\boldsymbol{k}.
\end{aligned}$$

We define $A_a = F_a + dA$, $A_b = F_b + dB$, and $B = G + dG$.

***Case* 3**: *Take $R = \texttt{rand}(35, 35)$ and $E = 0.01\,(\texttt{rand}(m, 35)R)$. Let*

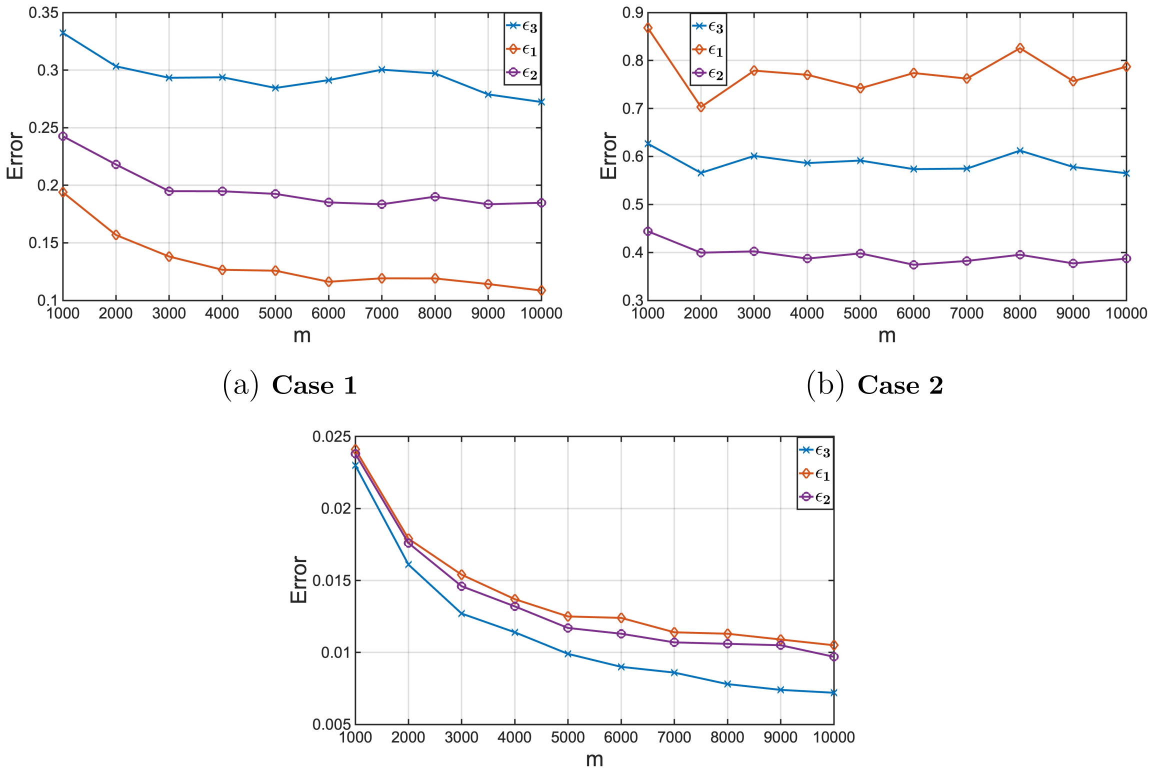
$$\begin{aligned}
dA &= 0,\\
dB &= 0,\\
dG &= E + E\boldsymbol{i} + E\boldsymbol{j} + E\boldsymbol{k}.
\end{aligned}$$

We define $A_a = F_a$, $A_b = F_b$, and $B = G + dG$.

In each of the three cases, due to the presence of errors in matrices A and B, an exact solution for the system $AX \approx B$ is not attainable, and thus, an approximate solution is required. In this example, we compute the RBMTLS solution (X_M), the RBTLS solution (X_T), and the RBLS solution (X_L) for the inconsistent system $AX \approx B$ across all three cases.

Note: *To achieve the highest possible accuracy in the estimated solution X, it is essential to eliminate any errors present in the entries of matrices A and B. In all three cases, if we remove all errors from these matrices, they reduce to matrices F and G, respectively. Therefore, X_0 represents the most accurate approximate solution for the system $AX \approx B$ in all cases.*

Next, we calculate X_M, X_T, and X_L using Algorithms 5.5.1, 5.5.2, and 5.5.3, respectively. Let the errors be denoted by $\epsilon_1 = \|X_M - X_0\|_F$, $\epsilon_2 = \|X_T - X_0\|_F$, and



(a) **Case 1** (b) **Case 2**

(c) **Case 3**

Figure 5.5.1. The errors from the three solution techniques for Cases 1, 2, and 3.

$\epsilon_3 = \|X_L - X_0\|_F$. In this example, m represents an arbitrary value. We compute the errors ϵ_1, ϵ_2, and ϵ_3 for various values of m. Since the input matrices are generated randomly, we calculate ϵ_1, ϵ_2, and ϵ_3 by averaging the results from solving this example twenty times for each value of m.

Figure 5.5.1 presents comparison for **Case 1** (5.5.1 (a)), **Case 2** (5.5.1 (b)), and **Case 3** (5.5.1 (c)), respectively, between ϵ_1, ϵ_2, and ϵ_3. These comparisons are obtained by taking different values of m. For all values of m, we observe that in **Case 1**, $\epsilon_1 < \epsilon_2 < \epsilon_3$, while in **Case 2**, $\epsilon_2 < \epsilon_3 < \epsilon_1$. Lastly, in **Case 3**, $\epsilon_3 < \epsilon_2 < \epsilon_1$.

We conclude Example 5.5.1 with the following remark:

Remark 5.5.2. (1) *If there is an error in matrix B along with a few columns of matrix A, then the RBMTLS solution technique offers the most accurate approximate solution to the overdetermined system $AX \approx B$.*

(2) *In cases where errors are present in both matrix A and matrix B, the RBTLS solution technique yields the most accurate approximate solution to the overdetermined system $AX \approx B$.*

(3) *When the error is solely present in matrix B, the RBLS solution technique provides the most accurate approximate solution to the overdetermined system $AX \approx B$.*

Example 5.5.3. *Consider the linear problem $AX \approx B$, where $A = A_0 + A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{500\times50}$ and $B = B_0 + B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k} \in \mathbb{Q}_{\mathbb{R}}^{500\times10}$. The matrix components are generated as follows:*

$$A_0 = A_1 = A_2 = A_3 = \texttt{rand}(500, 50), \quad B_0 = B_1 = B_2 = B_3 = \texttt{rand}(500, 10).$$

Let $n_1 = 20$. In this example, we introduce random perturbations ΔA and ΔB to the input matrices A and B, respectively. Our goal is to analyze how the real RBMTLS, RBTLS, and RBLS solutions to $AX \approx B$ are affected when A and B are subject to these small perturbations. The exact relative forward errors for the solutions are defined as follows:

$$\frac{\|\Delta X_M\|_F}{\|X_M\|_F} \text{ for RBMTLS}, \quad \frac{\|\Delta X_T\|_F}{\|X_T\|_F} \text{ for RBTLS}, \quad \frac{\|\Delta X_L\|_F}{\|X_L\|_F} \text{ for RBLS}.$$

Using equations (5.4.2), (5.4.4), and (5.4.6), we compute the upper bounds for these relative forward errors, denoted by U_M for RBMTLS, U_T for RBTLS, and U_L for RBLS.

	RBMTLS Method		**RBTLS Method**		**RBLS Method**	
$\mathbf{\|[\Delta A, \Delta B]\|_F}$	$\mathbf{\frac{\|\Delta X_M\|_F}{\|X_M\|_F}}$	$\mathbf{U_M}$	$\mathbf{\frac{\|\Delta X_T\|_F}{\|X_T\|_F}}$	$\mathbf{U_T}$	$\mathbf{\frac{\|\Delta X_L\|_F}{\|X_L\|_F}}$	$\mathbf{U_L}$
$1e-10$	3.5540e-11	2.7202e-09	1.5860e-10	8.5467e-09	6.6243e-12	2.1272e-11
$1e-09$	1.9327e-09	1.7719e-07	1.0217e-08	5.5672e-07	3.8986e-10	1.3856e-09
$1e-08$	7.5040e-09	3.8307e-07	2.1910e-08	1.2036e-06	8.8797e-10	2.9957e-09
$1e-07$	4.3153e-08	2.7253e-06	1.0701e-07	8.5629e-06	6.2069e-09	2.1312e-08
$1e-06$	6.6442e-07	2.8282e-05	1.8317e-06	8.8861e-05	7.1476e-08	2.2117e-07

Table 5.5.1. Comparison of relative forward errors and their upper bounds for a perturbed problem with different methods.

Table 5.5.1 presents a comparision of the exact relative forward errors of the real RBMTLS, RBTLS, and RBLS solutions with their corresponding upper bounds for varying random perturbations $\|[\Delta A, \Delta B]\|_F$. It is observed that the exact relative forward errors obtained using the three methods are consistently less than their respective upper bounds. This confirms the validity of the derived upper bounds for the relative forward error.

Conclusion: In this chapter, we have introduced a method to find the best approximate solution for an inconsistent linear system arising in commutative quantum theory. The algebraic solution technique presented focuses on addressing the RBMTLS problem. By transforming the RBMTLS problem into a real MTLS problem through the real representation of reduced biquaternion matrices, we deduced conditions for the existence of a unique real RBMTLS solution and derived explicit expressions for this solution.

Additionally, we proposed solution techniques for both the RBTLS and the RBLS problems. These techniques can be considered special cases of the RBMTLS solution method. Furthermore, the developed methods have been applied to solve the linear system $AX \approx B$ over the complex field, illustrating their versatility in handling complex matrix equations, which are special cases of reduced biquaternion matrix equations.

We also conducted a perturbation analysis of the real RBMTLS, RBTLS, and RBLS solutions, deriving upper bounds for the relative forward errors. Numerical examples were provided to verify the accuracy and efficiency of the proposed methods.

Future research could explore the mixed and componentwise condition numbers of the RBMTLS, RBTLS, and RBLS solutions. Additionally, tighter upper bound estimates for the relative forward error could be derived to further enhance the reliability of these solutions. The methods developed in this chapter have potential applications in digital signal processing and image analysis within the framework of commutative quaternionic theory.

CHAPTER 6

CONCLUSION AND SCOPE FOR FUTURE WORK

Conclusion

This thesis develops comprehensive frameworks for solving generalized RBMEs, focusing on their solutions, practical applications, and sensitivity to perturbations. Reduced biquaternions, a class of four-dimensional hypercomplex numbers, are explored for their computational advantages and unique properties. By formulating and solving RBMEs with different constraints, this thesis provides insights into their mathematical structure and practical implications. Below is a summary of the major contributions presented in each chapter:

In Chapter 1, the foundational concepts of RBs and RB matrices are introduced. This chapter covers basic definitions, properties, and historical developments essential for understanding the generalized RBMEs discussed in subsequent chapters.

In Chapter 2, a comprehensive framework for finding least squares structured solutions to generalized RBMEs is developed. The notion of reduced biquaternion L-structures is introduced, accommodating specific matrix constraints such as Toeplitz, symmetric Toeplitz, Hankel, and circulant structures. Applications in color image restoration and inverse eigenvalue problems, including PDIEP and generalized PDIEP, are also explored.

In Chapter 3, the focus shifts to computing $\{2\}$-inverse and $\{1,2\}$-inverse of RB matrices with predefined conditions on the row and/or column space. Conditions for existence and effective representations of these generalized inverses are established by solving RBME of the form $(AXB, CXD) = (E, F)$. The results build upon the framework in Chapter 2 to find the unstructured matrix solutions.

In Chapter 4, the RBLSE problem is addressed, where the system $AX \approx B$ is subject to additional constraints $CX = D$. Both real and complex solutions to the RBLSE problem are derived, along with an upper bound for the relative forward error. Minimizing this error ensures the accuracy of solutions in practical applications.

In Chapter 5, the RBMTLS method is introduced to solve the overdetermined system $AX \approx B$ within the reduced biquaternion algebra. Explicit conditions for the existence and uniqueness of real RBMTLS solutions are derived, and an expression for obtaining these solutions is presented. Special cases of RBMTLS, namely the RBTLS method and the RBLS method, are also covered. Perturbation analysis is conducted to evaluate the sensitivity of RBMTLS, RBTLS, and RBLS solutions to input variations. Relative normwise condition numbers and forward error bounds are derived to ensure reliability in practical applications.

Future Scope

To advance the findings of this thesis, it is vital to explore new research directions that extend the current work. The following are potential directions for future research based on the findings of this thesis:

- Investigate perturbation analysis of constrained solutions of generalized RBMEs to understand the sensitivity of the solutions under data perturbations.
- Explore the QR decomposition of RB matrices to uncover new theoretical insights and computational techniques for handling reduced biquaternion systems.
- Study the generalized inverse of quaternion matrices.

The methodologies developed in this thesis for solving overdetermined linear systems have potential applications in digital signal and image processing. Extending these methods to real-time and dynamic systems could significantly enhance their relevance in practical applications.

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