DEPENDENCE AND UNCERTAINTY: A COPULA-BASED FRAMEWORK

Ph.D. Thesis

By

SWAROOP GEORGY ZACHARIAH

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

APRIL 2025

DEPENDENCE AND UNCERTAINTY: A COPULA-BASED FRAMEWORK

A THESIS

Submitted in partial fulfillment of the

requirements for the award of the degree

of

DOCTOR OF PHILOSOPHY

by

Swaroop Georgy Zachariah

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

APRIL 2025

INDIAN INSTITUTE OF TECHNOLOGY INDORE

I hereby certify that the work which is being presented in the thesis entitled **DEPENDENCE AND UNCERTAINTY: A COPULA-BASED FRAMEWORK** in the partial fulfilment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DEPARTMENT OF MATHEMATICS, Indian Institute of Technology Indore,** is an authentic record of my own work carried out during the time period from December 2020 to April 2025 under the supervision of Dr. Mohd. Arshad, Assistant Professor, Department of Mathematics, Indian Institute of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

02/07/2025
Signature of the student with date

(Swaroop Georgy Zachariah)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

02-07-2025
Signature of Thesis Supervisor with date

(Dr. Mohd. Arshad)

Swaroop Georgy Zachariah has successfully given his Ph.D. Oral Examination held on 02/07/2025.

02-07-2025
Signature of Thesis Supervisor with date

(Dr. Mohd. Arshad)

ACKNOWLEDGEMENTS

The Ph.D. journey is never an easy one—it is filled with many ups and downs. There were countless moments that nearly broke me, but it was my unwavering faith in the Almighty Lord that gave me the strength and confidence to persevere and complete this journey. I am always deeply grateful to my Almighty Lord for His countless blessings and boundless mercy throughout this path.

I am deeply grateful to my grandparents for their constant prayers and moral support. During the difficult times of the COVID-19 pandemic, their financial support enabled me to pay my academic fees, for which I will always remain thankful. I would also like to express my heartfelt gratitude to my father, who has been a constant pillar of strength and a source of confidence during challenging times. My mother's prayers and unwavering moral support gave me hope when it was most needed. A special thanks to my brother, Thejus Geogy Zachariah, who has been the backbone of my confidence. I am sincerely grateful for the efforts and support of my family, who ensured that I never gave up on my dream.

I would like to express my sincere gratitude to my supervisor, Dr. Mohd. Arshad, for his guidance and support throughout my research. He gave me the freedom to explore topics aligned with my interests, while continuously encouraging me to step out of my comfort zone and take on challenging research proble. His support, motivation, and fatherly care ensured that I was shielded from the mental stress and frustrations that often accompany a research career.

I am also thankful to my collaborator, Dr. Ashok Kumar Pathak, for introducing me to the field of copula theory. His valuable insights and guidance helped me understand the contemporary trends in this area, which significantly influenced the direction and depth of my Ph.D. work. I am especially grateful to Dr. Alam Ali for his generous help and guidance with the implementation of copula models in R. His technical support played an important role in the empirical part of my thesis.

I would like to express my sincere gratitude to IIT Indore for providing financial support through the GATE Teaching Assistantship (TA) fellowship, which enabled me to pursue my doctoral research.

I would like to express my sincere gratitude to my PSPC members, Dr. Vijay Kumar Sohani and Dr. Vinay Kumar Gupta, for their constant support, insightful suggestions, and continuous motivation throughout my research journey. Their encouragement played a vital role in shaping my academic progress. I am also deeply thankful to our Head of the Mathematics Department, Dr. Sanjeev Singh, for his unwavering support and guidance. My sincere appreciation extends to the former Head of the Mathematics Department, Prof. Niraj Kumar Shukla, and the former DPGC of Mathematics, Prof. Sk. Safique Ahmad, for their encouragement and valuable inputs. A special note of thanks goes to Prof. V. Antony Vijesh, whose consistent encouragement towards pursuing statistical research has always been a source of inspiration. His thoughtful advice and positive motivation have given me energy and confidence in my work. I am immensely grateful to all the teaching and non-teaching staff members of the Mathematics Department for their constant support, care, and kindness throughout this academic journey. I would like to express my sincere gratitude to all the faculty members who taught me during my Ph.D. coursework. Their teachings have been immensely helpful and laid a strong foundation for my research.

I would also like to extend my heartfelt thanks to all members of our research group, Statistical Modelling & Simulation Research Group, for their academic and emotional support. A special mention goes to Dr. Mojammel Haque Sarkar for his continuous guidance and encouragement. The numerous late-night discussions with him were instrumental in shaping Chapters 6 and 7 of my thesis and provided me with much-needed support during the writing phase. I am thankful to Mushir Akthar for his moral support and his valuable suggestions during paper drafting. His insightful discussions on machine learning have left a meaningful impact on my research contributions. My sincere thanks to Shubham Alte, who has been like a brother to me. His unwavering support, especially during critical times like my open seminar, is truly admirable. His encouragement helped boost my confidence, particularly in my presentation skills. His selfless acts and dedication will always be remembered with deep appreciation. I would also like to thank Abishek Varshney for bringing positivity and consistent moral support into my life. Furthermore, I extend my gratitude to the MSc students of our group, Rahul Choudhary, Saurabh Maurya, Harsha Jain, and Shivani Sharma, for their continuous encouragement and help. I would like to acknowledge the alumni members of our research group Ishita Agrawal, Leela Krishna, Akshay Tak, and Chetanya Sharma for their valuable support and suggestions.

I am grateful to Dhivya Prabhu K for his invaluable support. Whenever a mathematical challenge arose during my research, he was always there like a guardian angel to guide me. His exceptional mathematical insight not only helped me overcome complex problems but also enhanced my own mathematical skills. I believe he will one day be known as the "Future Ramanujan". More than a friend, he has been a philosopher and a constant source of motivation. His advice has contributed not only to my academic development but also to my personal growth. His role in my Ph.D. journey has been crucial and irreplaceable.

I would also like to express my heartfelt gratitude to Snehalata Sahoo for always being a caring and supportive presence. Her help during the tough phases of my Ph.D. journey will always be remembered with gratitude. My sincere thanks to Diksha, who consistently looked after me whenever I was emotionally down. Her care and support were deeply meaningful. I also express my gratitude to Debdeep Roy, who always brought positivity and brotherly affection. He played a significant role in helping me understand chaos theory, which was essential for Chapter 5 of my thesis, and also supported me with MATLAB code implementations. I am also thankful to Rajni for her assistance in understanding MATLAB codes, bifurcation theory, and chaos. Our discussions were fruitful and are reflected in Chapter 5 of my work. I would like to express my sincere appreciation to Aditya Sharma for his constant encouragement and support. I would like to acknowldege Soham Ghosh for the support and encouragement.

I extend my heartfelt thanks to all the research scholar colleagues from Leelavati Lab, Optimal Lab, and AMG Lab. A special mention to Shreyas S. R. for always motivating me with his thoughtful suggestions and encouraging attitude. I would like to acknowledge the following Ph.D. scholars for their valuable help in rectifying mathematical analysis issues in my research work: Sajin Vincent A. W., Pinki Khatun, Navneet, Archit Agarwal, Meghali, Sheetal Sanjay Wankhede, Md. Sajid, and Dr. Pranav Kumar

I would like to express my sincere gratitude to Dr. Rohan Chacko Jacob, Dr. Renil Thomas, and my mother, Susan George, for their valuable help in clarifying medical terminologies and concepts. Their guidance has been instrumental, and its impact is clearly reflected in Chapter 7 of my thesis.

I would like to acknowledge Nimisha Ann Jose, Aneena M. S., and Ankitha C. S. for their support and encouragement throughout my research.

I gratefully acknowledge Dr. K. K. Jose, M. C. Jose, and all the teachers at the Department of Statistics, K. E. College, as well as the Department of Mathematics, S. B. College, for laying a strong foundation in statistics and mathematics during my undergraduate and postgraduate studies. Their dedication and teaching played a vital role in shaping my academic path.

I would also like to extend my sincere thanks to Dr. Abhijit Datta Banik, Associate Professor, IIT Bhubaneswar, for his valuable support and motivation.

I am deeply thankful to Jibin V. Sunny, Ankit Meena, and Kunal Motghare for their help in learning Monte Carlo simulation techniques and for guiding me in understanding the intersection of statistics and astronomy during the Astrostatistics course.

I would like to acknowledge Kamaljeet Singh and Ronald Scaria for their valuable support during my PhD coursework. Their assistance made a meaningful difference in my academic progress.

I would like to express my sincere gratitude to my flatmates Vijay Jain, Shivukumar Rakkasagi, Himanshu Kaushik, Deb Kumar Rath, and Deepak for being such wonderful companions. Their unwavering support and encouragement have played a significant role in helping me through the challenges of my PhD journey.

I would like to express my sincere gratitude to the Malayalee-Tamil community of IIT Indore for their unwavering support, love, and care during my research journey. I am especially thankful to Dr. Mohammed Ameen for his continuous encouragement and care, which proved invaluable throughout my PhD. I would also like to thank Jithu Jose Athalathil and Leon Noble for their constant support. My heartfelt thanks go to Navanit A. V. — our discussions were instrumental in identifying financial crises, a key component reflected in Chapter 5. I am also deeply grateful to Jerry Jones David for helping me understand the basic concepts of entropy and its connection to thermodynamics; our discussions were particularly insightful for Chapters 4 and 5.

I would like to express my sincere gratitude to Jyothi Justin and Justy Joseph for their support during the PhD coursework and for guiding me in understanding how to conduct effective research. Iam grateful to Aneesh Rajendran, Jerin Rajan, Sharath Madhavan K, Aswin T, Vaibhav Sharma, and Bhawani Singh for their continuous support and encouragement throughout my academic journey.

I would also like to extend my heartfelt appreciation to the following individuals for their companionship and care throughout my PhD journey: Tinto T. D., Nasmi S. Anand, Keerthi K., Aromal P., Jibin Jose, Parvathy Thankachy P., Dizna James, Thomas George P., Anugrah Mathew Prasad, Anoop K. R., Harikrishnan R., Gokul M Pillai, Renveer AR, Safwana Shirin K. M., Ardra T. M., Hiba K. P., Jayashree Parthiban and Ramamoorthy Velayutham.

I express my deepest gratitude to Abra Mathew and Abin Philip for their love, support, motivation, and genuine care that always uplifted my spirit.

Finally, I would like to express my heartfelt gratitude to Anlin Kurian, Varun Sam, Dr. Alestin Mawrie, and Susan Sharma for their unwavering support, prayers, and for instilling in me a deep faith in God. Their spiritual encouragement served as a powerful source of strength during the most challenging moments of my PhD journey.

"Humble yourselves, therefore, under God's mighty hand, that he may lift you up in due time. Cast all your anxiety on him because he cares for you."

— 1 Peter 5:6–7

Dedication

To the cherished memory of

***my grandmother, Lizy Zachariah**,*

whom I never had the privilege to meet,

yet whose statistical insight runs in my blood.

Her legacy continues through every step of this journey.

And to

***my dear friend, Aashish Chandra P**,*

whose presence brought light and laughter.

Your absence is deeply felt, and your memory dearly held.

This thesis is dedicated to both of you, with love and gratitude.

SYNOPSIS

In the era of machine learning and artificial intelligence, multivariate statistical analysis has become an inevitable tool due to the increasing complexity and dimensionality of data arising from diverse domains such as engineering, medicine, finance, and environmental science. Unlike univariate techniques, multivariate analysis provides a comprehensive framework to model, interpret, and infer the relationships among multiple random variables simultaneously.

Two crucial aspects of multivariate analysis are (i) the marginal behaviour of each component and (ii) the dependence structure among the variables. One of the main challenges in multivariate statistical analysis lies in the flexible and accurate representation of this dependence structure. Classical approaches, such as the multivariate normal distribution, often rely on strict assumptions such as linearity, which seldom occurs in reality. These limitations are particularly evident in the presence of non-linear relationships, asymmetries, or tail dependencies.

To address these shortcomings, the theory of copulas has emerged as a powerful tool, following the groundbreaking work of Sklar (1959). Sklar (1959) established the fundamental result that for any p-dimensional joint cumulative distribution function (CDF) H with marginal CDFs $F_1, \ldots, F_p$, there exists a copula C such that

$$H(x_1, \ldots, x_p) = C(F_1(x_1), \ldots, F_p(x_p)), \quad \forall (x_1, \ldots, x_p) \in \overline{\mathbb{R}}^p, \tag{0.1}$$

where $\overline{\mathbb{R}} = [-\infty, \infty]$ denotes the set of extended real number line. When the marginals are absolutely continuous, the copula C is unique; otherwise, it is uniquely determined by the range of the marginal distributions. Conversely, given any univariate CDFs $F_1, \ldots, F_p$ and any copula C, the function H defined in Eq. (0.1) is a valid joint CDF with marginals $F_1, \ldots, F_p$.

In a probabilistic sense, a copula is a multivariate distribution function with standard uniform marginals, capturing the dependence structure independently of the marginal behaviours. This separation allows for the construction of a wide class of multivariate distributions by combining arbitrary marginals with a suitable dependence model.

Mathematically, a function $C : \mathbb{I}^p \to \mathbb{I}$, where $\mathbb{I} = [0, 1]$, is called a p-dimensional copula if it satisfies the following properties:

(i) For every $\mathbf{u} \in \mathbb{I}^p$, if any component of $\mathbf{u}$ is zero, then

$$C(\mathbf{u}) = 0.$$

(ii) For every $\mathbf{u} \in \mathbb{I}^p$, and for a fixed $i \in \{1, 2, \ldots, p\}$, if all components u_j satisfy $u_j = 1$ except u_i (i.e., $u_j = 1$ for all $j \neq i, j = 1, 2, \ldots, p$), then

$$C(\mathbf{u}) = u_i.$$

(iii) For any two vectors $\mathbf{u}_1 = (u_{1,1}, u_{1,2}, \ldots, u_{1,p})$ and $\mathbf{u}_2 = (u_{2,1}, u_{2,2}, \ldots, u_{2,p})$ in $\mathbb{I}^p$, if $u_{1,i} \leq u_{2,i}$ for all $i \in \{1, 2, \ldots, p\}$, then

$$\Delta_{u_{1,p}}^{u_{2,p}} \Delta_{u_{1,p-1}}^{u_{2,p-1}} \cdots \Delta_{u_{1,1}}^{u_{2,1}} C(\mathbf{u}) \geq 0,$$

where Δ denotes the first-order difference operator. For more details, see Nelsen (2006), Trivedi et al. (2007) and Durante and Sempi (2016), Hofert et al. (2018).

Although introduced in the 1950s, copulas gained widespread recognition only in the early 2000s. While implicitly used earlier, their explicit application became prominent in the 21st century. The Gaussian copula, popular in finance, revealed critical limitations during the 2008 Global Financial Crisis, notably underestimating tail dependence and portfolio risks (Salmon, 2009; MacKenzie and Spears, 2012). This motivates the development of more flexible copula families to model nonlinear and tail dependencies. Today, copulas are essential tools in finance, insurance, medicine, engineering, and agriculture to model complex dependence structures.

In statistical analysis, modelling systems that exhibit randomness inherently involve uncertainty. As uncertainty increases, predictability decreases, and thus, information can be viewed as a measure of uncertainty reduction. To quantify uncertainty, Shannon (1948) introduced a fundamental concept known as *entropy*, initially defined for discrete random variables. The Shannon entropy of a discrete random variable X with probability mass function (PMF) $p_j = P(X = x_j)$, $j = 1, 2, \ldots, k$, is given by

$$\mathcal{H}(X) = -\sum_{j=1}^{k} p_j \log p_j.$$

Shannon entropy has found widespread applications in machine learning, information theory, reliability theory, physics, chemistry, finance, and the study of complex systems. Numerous generalizations and extensions of Shannon entropy have been proposed in the literature; see, for example, Rényi (1961), Varma (1966), Tsallis (1988), Rao et al. (2004), Ubriaco (2009), and Xiong et al. (2019).

In the context of multivariate analysis, uncertainty arises from two main sources: the marginal distributions of the variables and their underlying dependence structure. Quantifying the dependence structure is essential, as it reveals complex relationships among the components of a multivariate random vector. Copula functions offer a powerful framework for modelling this dependence structure independently of the marginal distributions, making copula-based information measures particularly important for modern statistical modelling.

This thesis focuses on two key aspects of multivariate statistical analysis: the construction of new, flexible copula families and the development of new copula-based information measures aimed at quantifying the uncertainty embedded in complex dependence structures. Copula misspecification can introduce substantial bias and lead to misleading inference. Therefore, *goodness-of-fit tests for copulas* become an essential component of model validation. Overall, this thesis contributes to the growing body of research by constructing new, flexible copulas tailored to capture complex dependencies and by introducing copula-based information measures for more accurate quantification of uncertainty in multivariate dependent datasets.

Motivation and Research Objectives

The primary motivation of this thesis stems from the need to develop flexible copula models and associated information-theoretic tools for analyzing multivariate data exhibiting complex dependence structures. The key research objectives of the thesis are outlined below:

1. A wide variety of copula families are available in the literature. Among them, the Farlie–Gumbel–Morgenstern (FGM) copula, introduced by Eyraud (1936), Morgenstern (1956), Gumbel (1960), and Farlie (1960), is well-known due to its simple mathematical structure and ability to model both positive and negative dependence. However, a major limitation of the FGM copula is its narrow dependence range. For example, Spearman's rank correlation coefficient, an important dependence measure bounded between -1 and 1—is limited to the interval $[-1/3, 1/3]$ under the FGM family. To address this, various generalizations have been proposed in the literature, primarily by introducing additional parameters. While such extensions improve the dependence range, they often result in complex formulations and computational challenges in parameter estimation. One of the principal objectives of this thesis is to propose a new FGM-type copula that is mathematically simple, involves fewer parameters, and possesses a broader dependence range across various dependence measures.
2. The second objective focuses on the construction of flexible copulas. Existing construction methods are typically confined to specific domains and often fail to model data with intricate or tail-heavy dependence structures adequately. For instance, the widely used FGM copula lacks tail dependence. This thesis aims to develop a new method for constructing copula families with an emphasis on improving tail and overall dependence characteristics.
3. Copula-based entropy is a measure of the uncertainty associated with the dependence structure among random variables. Ma and Sun (2011) showed that the mutual information (MI) of a multivariate random vector is equivalent to the negative of its copula entropy (CE), defined as

$$\zeta(c) = -\int_{\mathbb{I}^p} c(\mathbf{u}) \log c(\mathbf{u})\, d\mathbf{u}, \tag{0.2}$$

where $c(\mathbf{u})$ is the copula density. MI is a fundamental information-theoretic quantity with wide-ranging applications. However, this approach becomes inapplicable when the underlying copula is not absolutely continuous. To address this, Sunoj and Nair (2025) introduced cumulative copula entropy by replacing the density function with the copula function itself, but their work is mainly restricted to the bivariate case. Other copula-based information measures, such as the information generating function, inaccuracy measures, and Kullback–Leibler divergence, are discussed only to a limited extent in the literature. This thesis aims to develop new copula-based information measures using Shannon entropy and examine their applications in multivariate statistical analysis.

4. In thermodynamics and statistical physics, when a system is in a non-equilibrium state or exhibits strong interdependence among its components, non-additive entropies provide a more appropriate framework for uncertainty quantification. Tsallis (1988) introduced a non-additive entropy, now known as Tsallis entropy. Motivated by this, another objective of this thesis is to develop a class of copula-based information measures derived from Tsallis entropy.
5. Multivariate analysis plays a vital role in lifetime data analysis, particularly in the context of reliability engineering, where component lifetimes may be interdependent. Copula models are highly effective in such scenarios. In addition to the joint survival function, three primary reliability functions are frequently employed: the joint density function, the bivariate hazard rate function, and the bivariate mean residual life (BMRL) function. While the hazard rate function provides the instantaneous failure rate, the BMRL function offers insights into the expected remaining lifetime of components that have survived up to a given time. Kulkarni and Rattihalli (2002) proposed a nonparametric estimator for BMRL, which, however, suffers from discontinuity and cannot be evaluated beyond the largest observed failure times, resulting in significant bias at the extremes. This thesis aims to address these limitations by developing a smooth, continuous nonparametric estimator for the BMRL function.
6. The mean inactivity time function (MITF) has important applications in medical research, forensic science, and reliability theory, particularly in contexts where the exact time of failure or infection is of interest. Nair and Asha (2008) extended this concept to the bivariate case, resulting in the bivariate mean inactivity time function (BMITF). While parametric estimation methods for BMITF exist, they rely heavily on knowledge of the underlying distribution, a condition that is rarely met in practice. To the best of our knowledge, there is no nonparametric estimator for BMITF available in the existing literature. Therefore, the final objective of this thesis is to develop a new nonparametric estimator for the BMITF.

Outline of the Dissertation

The dissertation is structured into four thematic parts, comprising a total of seven chapters. The first part provides a comprehensive introduction, a review of the literature, and the motivation behind the study. The second part focuses on the construction of new copulas, discussed in Chapters 2 and 3. The third part is dedicated to the development of copula-based information measures, covered in Chapters 4 and 5. The final part presents applications in multivariate lifetime data analysis, with a focus on proposing nonparametric estimators for two important functions in reliability theory such as the bivariate mean residual life function and the bivariate mean inactivity time function, discussed in Chapters 6 and 7. A brief outline of each chapter is provided below.

Chapter 1 presents an introduction to the motivation behind this research, along with a comprehensive overview of the development of copula theory, from the foundational work of Sklar (1959) to recent advancements. It includes an extensive literature review covering univariate to multivariate information measures, as well as copula-based information measures. The chapter concludes with a discussion on bivariate reliability concepts, which are crucial in bivariate lifetime data analysis.

Chapter 2 introduces a new bivariate symmetric copula that captures both positive and negative dependence. The proposed copula is mathematically simple, exhibits a wider dependence range than the FGM copula and its extensions, and does not possess tail dependence. The maximum attainable value of Spearman's Rho is approximately $[-0.5866, 0.5866]$, significantly improving over the $[-1/3, 1/3]$ range of the FGM family. A bivariate Rayleigh distribution is then constructed using this copula, and its statistical properties are studied. The utility of the model is demonstrated through the analysis of a real dataset.

Chapter 3 presents a method for constructing a new class of copulas based on the probability generating function (PGF) of positive-integer-valued random variables. Several known copulas are shown to be special cases of this new family. Dependence measures, tail properties, and random generation algorithms are discussed. Concavity properties, including Schur and quasi-concavity, are examined. Two generalized FGM-type copulas derived from the geometric and discrete Mittag-Leffler PGFs are introduced, achieving improved ranges of Spearman's Rho up to $[-0.33, 0.4751]$ and $[-0.33, 0.9573]$, respectively. Real data applications are provided to illustrate practical utility.

Chapter 4 introduces the multivariate cumulative copula entropy (CCE) and explores its theoretical properties, including bounds and convergence. A cumulative copula information-generating function is defined and evaluated for several well-known copula families. A fractional generalization of CCE is also proposed. A nonparametric estimator for CCE is developed using the empirical beta copula. Furthermore, a new copula-based divergence measure is introduced via Kullback–Leibler divergence, and a corresponding goodness-of-fit test is formulated. The practical effectiveness of the measure is demonstrated through a copula selection procedure applied to real datasets.

Chapter 5 extends the framework from Chapter 4 by incorporating Tsallis entropy, a non-additive entropy measure that allows greater flexibility in modelling uncertainty. The cumulative copula Tsallis entropy is introduced, and its properties and bounds are derived. A nonparametric version is developed and validated using simulated data from coupled periodic and chaotic maps. The chapter also extends Kerridge's inaccuracy measure and KL divergence to the cumulative copula framework. Building on the relationship between KL divergence and mutual information, a cumulative mutual information (CMI) measure is proposed. A statistical test based on CMI is developed to assess mutual independence among random variables. Finally, the use of CMI as an economic indicator is demonstrated through analysis of real bivariate financial time series data.

Chapter 6 proposes a smooth nonparametric estimator for the bivariate mean residual life (BMRL) function and establishes its consistency. The proposed estimator addresses the limitations of the existing estimator by Kulkarni and Rattihalli (2002), particularly near the extremes of the data range. Simulation studies compare the performance of both estimators. The chapter concludes with the analysis of a bivariate warranty dataset, where the BMRL function is computed under four proposed warranty policies, highlighting the practical value of the estimator in warranty policy formulation.

Chapter 7 introduces a novel nonparametric estimator for the bivariate mean inactivity time function (BMITF), a concept with applications in medical and forensic studies. The asymptotic properties of the estimator, including bias, consistency, and asymptotic normality, are established. The estimator's performance is assessed through extensive

simulation studies across various bivariate models. Finally, a real dataset related to pink eye disease is analyzed demonstrating the practical applicability of the method.

LIST OF PUBLICATIONS

List of Published/Communicated Research Papers from the Thesis

1. Swaroop Georgy Zachariah, Mohd Arshad, and Ashok Kumar Pathak, "A new class of copulas having dependence range larger than FGM-type copulas," *Statistics & Probability Letters*, vol. 206, 2024, Art. no. 109988. DOI: 10.1016/j.spl.2023.109988.
2. Swaroop Georgy Zachariah, Mohd Arshad, and Ashok Kumar Pathak, "A new family of copulas based on probability generating functions," *Mathematica Slovaca*, vol. 74, no. 4, 2024, pp. 1039–1060. DOI: 10.1515/ms-2024-0076.
3. Mohd Arshad, Swaroop Georgy Zachariah, and Ashok Kumar Pathak, "Multivariate Information Measures: A Copula-Based Approach." Under Review.
4. Swaroop Georgy Zachariah, Mohd Arshad, and Ashok Kumar Pathak, "Dependence and Uncertainty: Information Measures using Tsallis Entropy." Under Review.
5. Swaroop Georgy Zachariah, Mohd Arshad, and Mojammel Haque Sarkar, "Smooth Estimation of Bivariate Mean Residual Life Function with Application in Bivariate Warranty Data." Under Review.
6. Swaroop Georgy Zachariah, Mohd Arshad, and Mojammel Haque Sarkar, "Nonparametric Estimation of a Bivariate Mean Inactivity Time Function with Application in Pink Eye Disease Data." Under Review.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

NOTATION

$\mathbb{R}$	Set of all real numbers
$\mathbb{R}^p$	p-dimensional real space
$\mathbb{R}_+$	Set of non-negative real numbers, i.e., $[0, \infty)$
$\mathbb{R}_+^p$	Non-negative orthant in p dimensions, i.e., $[0, \infty)^p$
$\bar{\mathbb{R}}$	Extended real number system, i.e., $[-\infty, \infty]$
$\mathbb{I}$	Unit interval, i.e., $[0, 1]$
$\mathbb{I}^p$	Unit hypercube in p dimensions, i.e., $[0, 1]^p$
$U(0, 1)$	Standard uniform distribution
$C(\cdot)$	Copula function
$c(\cdot)$	Copula density function
$\mathbf{u}$	A vector in $\mathbb{I}^p$
$W(\cdot)$	Fréchet–Hoeffding lower bound copula
$M(\cdot)$	Fréchet–Hoeffding upper bound copula
$\Pi(\cdot)$	Product copula
$\mathbf{I}(\cdot)$	Indicator function
log	Natural logarithm
$\mathbb{E}(\cdot)$	Expectation of a random variable
$\Gamma(p)$	Gamma function: $\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\, dt$ for $p > 0$
$\beta(q_1, q_2)$	Beta function: $\beta(q_1, q_2) = \int_0^1 t^{q_1-1}(1-t)^{q_2-1}\, dt$ for $q_1, q_2 > 0$
$\sim$	Concordance ordering
$\overset{\text{PLOD}}{\prec}$	Positive lower orthant dependent ordering
a.s.	Almost surely

ACRONYMS

AIC	Akaike Information Criterion
BIC	Bayesian Information Criterion
BMITF	Bivariate Mean Inactivity Time Function
BMRL	Bivariate Mean Residual Life
CCE	Cumulative Copula Entropy
CCIGF	Cumulative Copula Information Generating Function
CCKL	Cumulative Copula Kullback–Leibler Divergence
CCTD	Cumulative Copula Tsallis Divergence
CCTE	Cumulative Tsallis Entropy
CCTI	Cumulative Copula Tsallis Inaccuracy
CE	Cumulative Entropy
CDF	Cumulative Distribution Function
CMI	Cumulative Mutual Information
CRE	Cumulative Residual Entropy
CRKL	Cumulative Residual Kullback–Leibler Divergence
CRTE	Cumulative Residual Tsallis Entropy
CVM	Cramér–von Mises Statistic
DE	Differential Entropy
DF	Distribution Function
FCCE	Fractional Cumulative Copula Entropy
IFM	Inference Functions for Margins
KL	Kullback–Leibler
KS	Kolmogorov–Smirnov
LCSD	Left-Corner Set Decreasing

LL	Log-Likelihood
LTD	Left-Tail Decreasing
MI	Mutual Information
MITF	Mean Inactivity Time Function
MLE	Maximum Likelihood Estimate
MPL	Maximum Pseudo-Likelihood Method
MRL	Mean Residual Life
NLOD	Negatively Lower Orthant Dependent
NQD	Negative Quadrant Dependent
NUOD	Negatively Upper Orthant Dependent
PDF	Probability Density Function
PGF	Probability Generating Function
PMF	Probability Mass Function
PLOD	Positive Lower Orthant Dependent
PQD	Positive Quadrant Dependent
PUOD	Positively Upper Orthant Dependent
RTI	Right Tail Increasing
RV	Random Variable
SI	Stochastic Increasing
TP_2	Total Positivity of Order 2
TRKL	Tsallis Residual Kullback–Leibler Divergence
WAM	Weighted Arithmetic Mean
WGM	Weighted Geometric Mean

Chapter 1

Introduction and Review of the Literature

This chapter outlines various definitions and properties of copula theory and presents a literature review, tracing the development of copulas from the foundational work of Sklar (1959) to recent advancements that highlight their significance in multivariate dependence modelling. This chapter also discusses an overview of various information measures used to quantify uncertainty in stochastic systems, extending from univariate to multivariate settings. The chapter concludes with a discussion on the foundational concepts of bivariate reliability theory, which are crucial in the analysis of bivariate lifetime data.

1.1 Introduction

Many real-world phenomena involve complex relationships, and understanding the dependencies between variables and the uncertainty in those dependencies is crucial across diverse fields like finance, engineering, insurance, healthcare, and agriculture. In these disciplines, multivariate data are commonly encountered, and probability distributions play a key role in modelling such data. While the assumption of independence is often made for simplicity, it is rarely valid in real-world scenarios.

For instance, studying cancer progression requires considering multiple covariates, such as age, immune response, and tumour size, which are inherently dependent. Moreover, these covariates may follow different probability distributions; for example, tumour size may follow a log-normal distribution, while age may follow a geometric distribution. Hence, a flexible probabilistic framework is needed to accommodate different marginal distributions while accurately modelling their joint probability law.

The multivariate Gaussian distribution is one of the most commonly used multivariate distributions, with its marginals being univariate Gaussian distribution. However, it has significant limitations for practical applications. In multivariate lifetime data, for example, we often require multivariate extensions of various lifetime distributions, where marginal distributions may differ. Moreover, the multivariate Gaussian distribution primarily models linear dependencies and exhibits weak tail dependence, making it unsuitable for scenarios with strong tail dependencies.

In literature, some attempts are made to construct multivariate distributions (see, for example, Marshall and Olkin (1967), Clayton (1978), Olkin and Liu (2003) and Mirhosseini et al. (2015). However, these distributions are constructed based on specific properties and lack the flexibility to model a wide variety of multivariate datasets. To address these limitations, copulas, introduced by Sklar (1959), provide a powerful tool for constructing joint probability distributions. Copulas allow for modelling dependence structures separately from marginal distributions, offering flexibility in capturing nonlinear relationships and strong tail dependencies. They facilitate scale-free dependence modelling, making them particularly well-suited for applications requiring robust multivariate analysis.

Although copulas were introduced in the late 1950s, their widespread utility didn't occur until the early 2000s. While some researchers implicitly used copulas in multivariate analysis, their explicit application and recognition only became prominent in the 21st century. The Gaussian copula, derived from the multivariate normal distribution, is a common choice, particularly in finance. However, its limitations became strikingly clear during the 2008 Global Financial Crisis. The misapplication of the Gaussian copula in risk modelling led to a significant underestimation of dependencies between assets in diversified portfolios. Critically, it failed to adequately capture increased tail dependence, resulting in a severe underestimation of risk exposure (see Salmon (2009) and MacKenzie and Spears (2012)). This gained significant attention among many researchers to develop more flexible copula families capable of capturing nonlinear dependencies and strong tail dependence, addressing the shortcomings of the Gaussian copula. Currently, copulas are widely used not just in finance but also in diverse fields like insurance, medicine, engineering, and agriculture, wherever the modelling of complex dependencies is crucial. The following section discusses a brief literature review of copulas, tracing their development from Sklar's foundational work to more recent advancements. This review will highlight how copulas have become a powerful tool for dependence modelling.

1.2 Modelling Dependence with Copulas: A Literature Review

The word *copula* is a Latin word meaning "link" or "connection". As its meaning suggests, in probability and statistics, copulas are functions that link multivariate distributions to their respective univariate marginal distributions. The theory of copulas can be viewed as a multivariate extension of the well-known probability integral transformation theorem, as stated below.

Theorem 1.2.1. *Let X be an absolutely continuous random variable with cumulative distribution function (CDF) $F(\cdot)$. Then, the transformed random variable $U = F(X)$ follows a standard uniform distribution, i.e., $U \sim U(0,1)$.*

Now, we take a look at the history behind the motivation of copula theory. In the late 1950s, A. Sklar and B. Schweizer actively worked on probabilistic metric spaces. They submitted their work Schweizer and Sklar (1958) to M.Fréchet, who accepted it but posed a fundamental question: Is there a way to determine the relationship between a multivariate distribution and its univariate marginals? Sklar (1959) addressed this question by extending the probability integral transformation theorem, leading to what is now known as Sklar's Theorem, the foundation of copula theory. The theorem is formally stated as follows.

Theorem 1.2.2 (Sklar's Theorem). *Let H be the joint CDF of a p-dimensional random vector with marginal CDFs $F_1, F_2, \ldots, F_p$. Then, there exists a function, called a copula, C, such that*

$$H(x_1, x_2, \ldots, x_p) = C(F_1(x_1), F_2(x_2), \ldots, F_p(x_p)), \quad \textit{for every } (x_1, x_2, \ldots, x_p) \in \bar{\mathbb{R}}, \tag{1.1}$$

where $\bar{\mathbb{R}}$ is the extended real line $[-\infty, \infty]$. If the marginal CDFs F_i for $i = 1, 2, \ldots, p$ are absolutely continuous, then the copula C is uniquely determined. Otherwise, it is uniquely *determined on the set* $\text{range}(F_1) \times \text{range}(F_2) \times \cdots \times \text{range}(F_p)$*, where* $\text{range}(F)$ *denotes the range of* F*. Conversely, if* $F_1, F_2, \ldots, F_p$ *are univariate CDFs and* C *is a copula function, then the function* H *defined in Eq. (1.1) is a valid joint CDF corresponding to a* p*-dimensional random vector with marginal CDFs* $F_1, F_2, \ldots, F_p$*.*

Sklar's theorem can be proved in various approaches. Among these, the probabilistic approach is an extension of the proof of the probability integral transformation. For more details, we refer to the book of Durante and Sempi (2016).

Let H be the joint CDF of a random vector of dimension p, where each marginal CDF is given by $F_1, F_2, \ldots, F_p$. Then, the underlying copula can be obtained as

$$C(u_1, u_2, \ldots, u_p) = H\left(F_1^{[-1]}(u_1), F_2^{[-1]}(u_2), \ldots, F_p^{[-1]}(u_p)\right), \tag{1.2}$$

where $F_i^{[-1]}$ denotes the quasi-inverse of F_i and is defined by

$$F_i^{[-1]}(u) = \sup\{x : F_i(x) \leq u\} = \inf\{x : F_i(x) \geq u\}, \ i = 1, 2, \ldots, p.$$

Now, we proceed to define the copula function. In a probabilistic sense, a copula function is simply the joint CDF of a p-dimensional random vector, where each marginal component follows a standard uniform distribution. That is,

$$C(\mathbf{u}) = P(U_1 \leq u_1, U_2 \leq u_2, \ldots, U_p \leq u_p),$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_p)$ and

$$P(U_j \leq u_j) = u_j, \quad \text{for } j \in \{1, 2, \ldots, p\} \text{ and } 0 \leq u_j \leq 1.$$

Now, we present the formal definition of a p-dimensional copula. For convenience, let us denote $\mathbb{I} = [0, 1]$.

Definition 1.2.3. *A function* $C : \mathbb{I}^p \to \mathbb{I}$ *is called a* p***-dimensional copula*** *if it satisfies the following conditions:*

(i) For every $\mathbf{u} \in \mathbb{I}^p$*, if any component of* $\mathbf{u}$ *is zero, then* $C(\mathbf{u}) = 0$*.*

(ii) For every $\mathbf{u} \in \mathbb{I}^p$*, and for a fixed* $i \in \{1, 2, \ldots, p\}$*, if all components* u_j *satisfy* $u_j = 1$ *except* u_i *(i.e.,* $u_j = 1$ *for all* $j \neq i, j = 1, 2, \ldots, p$*), then*

$$C(\mathbf{u}) = u_i.$$

(iii) For any two vectors $\mathbf{u}_1 = (u_{1,1}, u_{1,2}, \ldots, u_{1,p})$ *and* $\mathbf{u}_2 = (u_{2,1}, u_{2,2}, \ldots, u_{2,p})$ *in* $\mathbb{I}^p$*, if* $u_{1,i} \leq u_{2,i}$ *for every* $i \in \{1, 2, \ldots, p\}$*, then*

$$\Delta_{u_{1,p}}^{u_{2,p}} \Delta_{u_{1,p-1}}^{u_{2,p-1}} \cdots \Delta_{u_{1,1}}^{u_{2,1}} C(\mathbf{u}) \geq 0,$$

where Δ *represents the first-order difference operator.*

Conditions (i) and (ii) are known as the **boundary conditions**, while condition (iii) is referred to as the ***p*-increasing property** of a copula. For the special case of $p = 2$, the **2-increasing property** simplifies to

$$\Delta_{u_1}^{u_2} \Delta_{v_1}^{v_2} C(u, v) = C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0,$$

for all $u_1 \leq u_2$ and $v_1 \leq v_2$ with $u_1, u_2, v_1, v_2 \in \mathbb{I}$. For more details one can refer to the books of Nelsen (2006), Trivedi et al. (2007), Durante and Sempi (2016), Hofert et al. (2018) and the survey paper of Schweizer (1991). Kim et al. (2011) and Segers et al. (2017) showed that if a function $f : \mathbb{I}^p \to \mathbb{I}$ is infinitely differentiable on $\mathbb{I}^p$, then f is p-increasing if and only if

$$\frac{\partial^p f(\mathbf{u})}{\partial u_1 \, \partial u_2 \, \ldots \, \partial u_p} \geq 0$$

for every $\mathbf{u} \in \mathbb{I}^p$.

Like Jordan's decomposition theorem for distribution functions, any p-dimensional copula C can be decomposed as

$$C(\mathbf{u}) = \mathcal{A}(\mathbf{u}) + \mathcal{S}(\mathbf{u}),$$

where

$$\mathcal{A}(\mathbf{u}) = \int_0^{u_1} \int_0^{u_2} \cdots \int_0^{u_p} \frac{\partial^p C(v_1, v_2, \ldots, v_p)}{\partial v_1 \, \partial v_2 \, \ldots \, \partial v_p} \, dv_1 \, dv_2 \, \ldots \, dv_p$$

is the absolutely continuous component of C, and

$$\mathcal{S}(\mathbf{u}) = C(\mathbf{u}) - \mathcal{A}(\mathbf{u})$$

is the singular component of C. A copula C is said to be absolutely continuous if $\mathcal{S}(\mathbf{u}) = 0$, that is, $C(\mathbf{u}) = \mathcal{A}(\mathbf{u})$ almost everywhere in $\mathbb{I}^p$. Similarly, C is said to be singular if $\mathcal{A}(\mathbf{u}) = 0$, that is, almost everywhere in $\mathbb{I}^p$.

Definition 1.2.4. *Let C be a p-dimensional copula. If C is absolutely continuous, then the copula density is defined as*

$$c(\mathbf{u}) = \frac{\partial^p C(u_1, u_2, \ldots, u_p)}{\partial u_1\, \partial u_2\, \ldots\, \partial u_p}, \tag{1.3}$$

for every $\mathbf{u} \in \mathbb{I}^p$.

It is worth noting that for every p-dimensional copula C, there exist many cases where the copula density may not exist, at least for some values in $\mathbb{I}^p$. For example, consider the copula

$$C(u_1, u_2) = \min\{u_1, u_2\},$$

which does not possess a copula density at the point $u_1 = u_2$.

Theorem 1.2.5. *Let $\mathbf{X} = (X_1, X_2, \ldots, X_p)$ be a p-dimensional random vector with joint CDF $H(\mathbf{x})$. Suppose that each marginal CDF $F_i(x_i)$ of X_i, for $i = 1, 2, \ldots, p$, is continuous and has a density function $f_i(x_i)$. If H is absolutely continuous, then the joint density function $h(\mathbf{x})$ of $\mathbf{X}$ can be expressed as*

$$h(\mathbf{x}) = c\left(F_1(x_1), F_2(x_2), \ldots, F_p(x_p)\right) \prod_{i=1}^{p} f_i(x_i), \tag{1.4}$$

where c is the copula density corresponding to the random vector $\mathbf{X}$.

From the above theorem, we can infer that the joint density function of a random vector can be decomposed into two components: the marginal densities and the copula density function, which is independent of the marginal distributions. This decomposition highlights the importance of copulas in dependence modelling. Similar to Sklar's theorem for joint distribution functions, an alternative version exists for joint survival functions. The theorem is stated below.

Theorem 1.2.6 (Sklar's Theorem for Survival Functions). *Let $\mathbf{X} = (X_1, X_2, \ldots, X_p)$ be a p-dimensional random vector with joint survival function $\bar{H}$, where*

$$\bar{H}(x_1, x_2, \ldots, x_p) = P(X_1 > x_1, X_2 > x_2, \ldots, X_p > x_p).$$

*For each component of $\mathbf{X}$, the marginal survival function is defined by $\bar{F}_i(x_j) = P(X_i > x_i)$, $i = 1, 2, \ldots, p$. Then, there exists a function, called the **survival copula** $\hat{C}$, such that*

$$\bar{H}(x_1, x_2, \ldots, x_p) = \hat{C}(\bar{F}_1(x_1), \bar{F}_2(x_2), \ldots, \bar{F}_p(x_p)), \quad \text{for all } (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p. \tag{1.5}$$

If the marginal survival functions $\bar{F}_i$ *for* $i = 1, 2, \ldots, p$ *are absolutely continuous, then the survival copula* $\hat{C}$ *is uniquely determined. Otherwise, it is uniquely determined on the set* $\text{range}(\bar{F}_1) \times \text{range}(\bar{F}_2) \times \cdots \times \text{range}(\bar{F}_p)$. *Conversely, if* $\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_p$ *are univariate survival functions and* $\hat{C}$ *is a survival copula, then the function* $\bar{H}$ *defined in Eq. (1.5) is a valid joint survival function corresponding to a p-dimensional random vector with marginal survival functions* $\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_p$.

Remark 1.2.7. *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_p)$ *be a p-dimensional random vector whose dependence structure is captured by the copula* C. *Let* $\hat{C}$ *be the corresponding survival copula. Then,* C *and* $\hat{C}$ *satisfy the relation*

$$\hat{C}(\mathbf{u}) = \sum_{N \subseteq \{1,\ldots,p\}} (-1)^{|N|} C\Big((1-u_1)^{\mathbf{I}(1\in N)}, \ldots, (1-u_p)^{\mathbf{I}(p\in N)}\Big), \tag{1.6}$$

for every $\mathbf{u} \in \mathbb{I}^p$, *where the summation extends over all* 2^p *subsets of* $\{1, \ldots, p\}$, $|N|$ *denotes the number of elements in* N, *and* $\mathbf{I}(n \in N)$ *is the indicator function of* $n \in N$. *For* $p = 2$, *Eq. (1.6) simplifies to*

$$\hat{C}(u_1, u_2) = u_1 + u_2 - 1 + C(1-u_1, 1-u_2). \tag{1.7}$$

Remark 1.2.8. *Let* $\mathbf{U} = (U_1, U_2, \ldots, U_p)$ *be a p-dimensional random vector where each component follows a standard uniform distribution. Let*

$$\bar{C}(\mathbf{u}) = P(U_1 > u_1, U_2 > u_2, \ldots, U_p > u_p)$$

denote the joint survival function. Then, for every $\mathbf{u} = (u_1, u_2, \ldots, u_p) \in \mathbb{I}^p$, *the survival copula satisfies*

$$\bar{C}(\mathbf{u}) = \hat{C}(1-u_1, 1-u_2, \ldots, 1-u_p). \tag{1.8}$$

1.2.1 Properties

In this subsection, we present some important properties of copulas, which are discussed in the following theorems. For proofs and further details, we refer to the books of Nelsen (2006), Durante and Sempi (2016), and Hofert et al. (2018).

Theorem 1.2.9 (**Bounds**)**.** *For any p-dimensional copula* C, *the following inequality holds:*

$$W(\mathbf{u}) \leq C(\mathbf{u}) \leq M(\mathbf{u}), \tag{1.9}$$

for every $\mathbf{u} \in \mathbb{I}^p$*, where*

$$W(\mathbf{u}) = \max\{u_1 + u_2 + \cdots + u_p - p + 1, 0\},$$
$$M(\mathbf{u}) = \min\{u_1, u_2, \ldots, u_p\}.$$

It is important to note that $M(\mathbf{u})$ is a valid copula for any p, whereas $W(\mathbf{u})$ is a valid copula only when $p = 2$. These bounds are known as the *Fréchet–Hoeffding bounds*, where $W(\mathbf{u})$ is called the *Fréchet–Hoeffding lower bound*, and $M(\mathbf{u})$ is called the *Fréchet–Hoeffding upper bound*, sometimes also referred to as the *minimum copula*.

Theorem 1.2.10 (**Uniform Continuity**). *Let C be a p-dimensional copula. Then, for every two vectors $\mathbf{u_1}, \mathbf{u_2} \in \mathbb{I}^p$, we have*

$$|C(\mathbf{u_1}) - C(\mathbf{u_2})| \leq \sum_{i=1}^{p} |u_{2,i} - u_{1,i}|.$$

That is, C is uniformly continuous on $\mathbb{I}^p$.

Theorem 1.2.11 (**Convergence**). *Let $\{C_n : n \in \mathbb{N}\}$ be a sequence of copulas of the same dimension. If C_n converges pointwise to a function C as $n \to \infty$, then C is a valid copula. Furthermore, if C_n converges to C pointwise as $n \to \infty$, then C_n also converges to C uniformly.*

Theorem 1.2.12 (**Rank Invariance Property**). *Let $\mathbf{X} = (X_1, X_2, \ldots, X_p)$ be a continuous p-dimensional random vector. For every $j = 1, 2, \ldots, p$, let h_j be a strictly increasing function defined on the range of X_j. Then, the copula corresponding to $\mathbf{X}$ is identical to the copula corresponding to the transformed random vector $\mathbf{Y} = (h_1(X_1), h_2(X_2), \ldots, h_p(X_p))$.*

Theorem 1.2.13 (**Existence of Partial Derivatives**). *Let C be a p-dimensional copula. Then, for almost every $\mathbf{u} \in \mathbb{I}^p$, the partial derivative $\frac{\partial C(\mathbf{u})}{\partial u_i}$ exists and satisfies $\frac{\partial C(\mathbf{u})}{\partial u_i} \in \mathbb{I}$, for every $i = 1, 2, \ldots, p$.*

Theorem 1.2.14 (Schweizer (1991)). *Let $\mathbf{X} = (X_1, X_2)$ be a bivariate continuous random vector with joint CDF H and copula C. Then*

- *(a) If X_1 is almost surely an increasing function of X_2, then $C(u_1, u_2) = M(u_1, u_2) = \min\{u_1, u_2\}$.*
- *(b) If X_1 is almost surely a decreasing function of X_2, then $C(u_1, u_2) = W(u_1, u_2) = \max\{u_1 + u_2 - 1, 0\}$.*
- *(c) If X_1 and X_2 are independently distributed, then $C(u_1, u_2) = u_1 u_2$.*

1.2.2 Compendium of Copulas

Here, we present some well-known bivariate and multivariate copulas from the literature.

Example 1.2.1 (Product Copula). *Let $X_1, X_2, \ldots, X_p$ be p independent random variables. The copula corresponding to the independent case is called the* product copula*, denoted by $\Pi(\mathbf{u})$, and is given by*

$$\Pi(\mathbf{u}) = \prod_{i=1}^{p} u_i = u_1 u_2 \ldots u_p. \tag{1.10}$$

Example 1.2.2 (Gaussian Copula). *Let $\mathbf{X} = (X_1, X_2, \ldots, X_p)$ be a p-dimensional random vector following a multivariate Gaussian distribution with zero mean vector and correlation matrix $\boldsymbol{\rho}$, i.e., $\mathbf{X} \sim N_p(\mathbf{0}, \boldsymbol{\rho})$. The correlation matrix $\boldsymbol{\rho} = [\rho_{i,j}]$ satisfies $|\rho_{i,j}| < 1$ for every $i \neq j$ and $\rho_{i,j} = 1$ if $i = j$. The copula associated with the multivariate Gaussian random vector is called the* Gaussian copula *and is given by*

$$C(\mathbf{u}) = \Phi_{\boldsymbol{\rho}}\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_p)\right), \tag{1.11}$$

where $\Phi_{\boldsymbol{\rho}}$ is the joint CDF of a standard multivariate Gaussian distribution with correlation matrix $\boldsymbol{\rho}$, and Φ^{-1} is the inverse CDF of the standard Gaussian distribution. Note that the Gaussian copula, also referred to as the normal copula*, allows for both positive and negative dependence between variables. However, a key limitation of the Gaussian copula is that it is useful only when the data exhibit linear dependence.*

Example 1.2.3 (FGM Copula). *Eyraud (1936), Morgenstern (1956), Gumbel (1960), and Farlie (1960) introduced a bivariate family of distributions. The copula corresponding to this bivariate distribution was later rediscovered and is now popularly known as the* FGM (Farlie-Gumbel-Morgenstern) copula*, given by*

$$C(u_1, u_2) = u_1 u_2 + \delta(u_1 - u_1^2)(u_2 - u_2^2), \tag{1.12}$$

where $\delta \in [-1, 1]$ is the copula parameter. The FGM copula is widely used for modelling bivariate data exhibiting both positive and negative dependence. However, it only captures weak dependence, as it is limited to low values of Spearman's Rho and Kendall's Tau correlation coefficients. Due to this limitation, many researchers have proposed FGM-type copulas to improve the correlation coefficient. For references, see Huang and Kotz (1999), Bairamov and Kotz (2002), Bekrizadeh et al. (2015), and Chesneau (2022). A multivariate extension of the FGM copula is also available in the literature; see Nelsen (2006) and Nadarajah et al. (2017) for further details.

Example 1.2.4 (Marshall-Olkin Copula). *Marshall and Olkin (1967) proposed a bivariate exponential distribution, and the copula corresponding to this distribution is known as the Marshall-Olkin copula. It is defined as*

$$C(u_1, u_2) = \begin{cases} u_1^{1-\alpha_1} u_2, & \text{if } u_1^{\alpha_1} \geq u_2^{\alpha_2}, \\ u_1 u_2^{1-\alpha_2}, & \text{if } u_1^{\alpha_1} < u_2^{\alpha_2}, \end{cases}$$

where $\alpha_1, \alpha_2 \in [0, 1]$*. This copula is widely used in bivariate shock modelling.*

In literature, a wide variety of copulas available. The review article of Nadarajah et al. (2017) discussed most of the existing copulas in the literature. One of the popular family of copulas is called **Archimedean copula**, which was discussed by Genest and Mackay (1986). The copula is constructed via a function called Archimedean generator. Archimedean generator is a function $\varphi : \mathbb{I} \to (0, \infty)$ is convex strictly decreasing continuous function with $\varphi(1) = 0$ and $\varphi(0) \leq \infty$. Using Archimedean generator φ, one can construct a family of the bivariate copula, called Archimedean copula, as

$$C(u_1, u_2) = \varphi^{(-1)} \left[\varphi(u_1) + \varphi(u_2) \right], \ \forall u_1, u_2 \in \mathbb{I}, \tag{1.13}$$

where the pseudo-inverse function, $\varphi^{(-1)}$, defined by

$$\varphi^{(-1)}(s) = \begin{cases} \varphi^{-1}(s), & \text{if } 0 \leq s \leq \varphi(0), \\ 0, & \text{if } \varphi(0) \leq s \leq \infty. \end{cases} \tag{1.14}$$

If the generator satisfies $\varphi(0) = \infty$, the Archimedean copula is called a *strict Archimedean copula*; otherwise, it is referred to as a *non-strict Archimedean copula*. The Archimedean copula can also be extended to higher dimensions in a similar manner (Nelsen, 2006, p. 151). A total of 22 bivariate copulas belonging to this family are reported in the book of Nelsen (2006). Some of the most popular copulas in this family are discussed below.

Example 1.2.5 (Clayton Copula). *If the generator is given by*

$$\varphi(z) = \frac{z^{-\delta} - 1}{\delta}, \quad \delta \in (-1, \infty) \setminus \{0\},$$

then the corresponding copula is the ***Clayton copula****, given by*

$$C(u_1, u_2) = \left(\max\left\{u_1^{-\delta} + u_2^{-\delta} - 1, 0\right\}\right)^{-1/\delta}.$$

This copula was introduced by Clayton (1978).

Example 1.2.6 (Gumbel-Hougaard Copula). *If the generator is given by*

$$\varphi(z) = (-\log z)^{\delta}, \quad \delta \geq 1,$$

then the corresponding copula is the ***Gumbel-Hougaard copula****, given by*

$$C(u_1, u_2) = \exp\left\{-\left((-\ln u_1)^{\delta} + (-\ln u_2)^{\delta}\right)^{1/\delta}\right\}.$$

This copula was introduced by Gumbel (1960) and Hougaard (1984).

Example 1.2.7 (Frank Copula). *If the generator is given by*

$$\varphi(z) = -\log\left(\frac{e^{-\delta z} - 1}{e^{-\delta} - 1}\right), \quad \delta \in \mathbb{R},$$

then the corresponding copula is the ***Frank copula****, given by*

$$C(u_1, u_2) = -\frac{1}{\delta}\ln\left(1 + \frac{(e^{-\delta u_1} - 1)(e^{-\delta u_2} - 1)}{e^{-\delta} - 1}\right), \quad \delta \in \mathbb{R}.$$

This copula was introduced by Frank (1979).

Now, we will discuss a characterization theorem for Archimedean copulas. For more details, see Drouet Mari and Kotz (2001).

Theorem 1.2.15 (Characterization of Archimedean Copula). *A bivariate copula C is Archimedean if and only if there exists a function $\delta : (0,1) \to (0,\infty)$ satisfying*

$$\frac{C_{u_1}(u_1, u_2)}{C_{u_2}(u_1, u_2)} = \frac{\delta(u_1)}{\delta(u_2)}, \tag{1.15}$$

for every $u_1, u_2 \in \mathbb{I}$, where $C_{u_i}(u_1, u_2) = \frac{\partial C(u_1,u_2)}{\partial u_i}$, for $i = 1, 2$. The generator of C (up to a constant) is given by $\varphi(z) = \int_z^1 \delta(t)\, dt$.

1.2.3 Random Number Generation

We now discuss how to generate random numbers from a given bivariate copula. The following algorithm illustrates the procedure for generating random numbers from a given copula C.

Step 1: Generate two independent random numbers, u_1 and v, from the uniform distribution on $\mathbb{I}$.

Step 2: Determine u_2 as the solution to the equation $\frac{\partial C(u_1,u_2)}{\partial u_1} = v$.

Step 3: The generated sample from the copula is then given by (u_1, u_2).

This algorithm is called the conditional distribution method. To generate a random sample from a bivariate distribution with marginal CDFs F_1 and F_2, we apply the inverse transform method. Specifically, after obtaining (u_1, u_2) from the above algorithm, we compute

$$x_i = F_i^{-1}(u_i), \quad i = 1, 2.$$

The resulting pair (x_1, x_2) follows the desired bivariate distribution with the specified marginals and copula structure. The multivariate extension is discussed in Hofert et al. (2018).

1.2.4 Dependence Measures

Copula functions have been widely used for modelling dependence between random variables since they allow the separation of the dependence effect from the effects of the marginal distributions. In literature, there are various measures of dependence are available to measure the dependence structure captured by the copula. Some of the important dependence measures were discussed in this subsection.

1.2.4.1 Measures of Association

Karl Pearson's correlation coefficient is one of the most commonly used measures for quantifying the dependence between two random variables. However, it fundamentally assumes a linear relationship between the variables, an assumption that often does not hold in real-world data. To address this limitation, rank-based correlation measures such as Kendall's Tau and Spearman's Tau have gained prominence, particularly in the context of copula-based dependence modelling (see Nelsen (2006), Hofert et al. (2018)). These measures are invariant under strictly monotonic transformations and depend only on the ranks of the data, not their actual values.

Both Kendall's Tau and Spearman's rho range from -1 to 1, where negative values indicate negative dependence and positive values indicate positive dependence. Beyond these two, the literature also offers several other rank-based dependence measures, including Gini's Gamma coefficient, Spearman's Footrule coefficient (see Nelsen (2006)), and Blest's measure of rank correlation (see Genest and Plante (2003)).

Let $C(u_1, u_2)$ be the copula function associated with the random variables X_1 and X_2. Several important measures of dependence can be expressed directly in terms of the copula function, as discussed below.

- **Spearman's Rho**

$$\rho_c = 12 \int_0^1 \int_0^1 C(u_1, u_2)\, du_1\, du_2 - 3. \tag{1.16}$$

- **Kendall's Tau**

$$\begin{aligned}\tau_c &= 4 \int_0^1 \int_0^1 C(u_1, u_2)\, dC(u_1, u_2) - 1 \\ &= 1 - 4 \int_0^1 \int_0^1 \frac{\partial}{\partial u_1} C(u_1, u_2) \frac{\partial}{\partial u_2} C(u_1, u_2)\, du_1\, du_2.\end{aligned} \tag{1.17}$$

- **Gini's Gamma Coefficient**

$$\gamma_c = 4 \left\{ \int_0^1 C(u, 1-u)\, du - \int_0^1 (u - C(u,u))\, du \right\}. \tag{1.18}$$

- **Spearman's Footrule**

$$\phi_c = 6 \int_0^1 C(u,u)\, du - 2. \tag{1.19}$$

- **Blest's Measure of Rank Correlation**

$$\eta_c = 24 \int_0^1 \int_0^1 (1 - u_1)\, C(u_1, u_2)\, du_1\, du_2 - 2. \tag{1.20}$$

Theorem 1.2.16 (Nelsen (2006)). *Let C be a bivariate Archimedean copula with generator function $\varphi(s)$. Then, Kendall's Tau for C is given by*

$$\tau_C = 1 + 4 \int_0^1 \frac{\varphi(s)}{\varphi'(s)} ds. \tag{1.21}$$

Blomqvist's medial correlation coefficient, proposed by Blomqvist (1950), is another measure of association based on the median of two random variables. This measure is robust to outliers and skewed data. If C is a bivariate copula, then Blomqvist's medial correlation coefficient, denoted by β_C, is defined as

$$\beta_C = 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1. \tag{1.22}$$

Like Kendall's Tau and Spearman's Rho, Blomqvist's Beta has a range of $[-1, 1]$, with similar interpretations for positive and negative dependence.

It is worth noting that these rank-based correlations can be extended to higher dimensions. Nelsen (2002) discussed the multivariate extension of Spearman's Rho and Kendall's Tau. More recently, Bedő and Ong (2016) proposed two asymmetric versions of Spearman's Rho that generalize the bivariate case given in Eq. (1.16). Additionally, Úbeda-Flores (2005) introduced multivariate versions of Spearman's Footrule and Blomqvist's medial correlation coefficient, while Behboodian et al. (2007) provided the expression for Gini's Gamma coefficient in higher dimensions.

1.2.4.2 Quadrant Dependence

Two random variables X_1 and X_2 are said to be positively quadrant dependent (PQD) if

$$P(X_1 \leq x_1)P(X_2 \leq x_2) \leq P(X_1 \leq x_1, X_2 \leq x_2), \quad \forall x_1, x_2 \in \mathbb{R}. \tag{1.23}$$

Equivalently,

$$P(X_1 > x_1)P(X_2 > x_2) \geq P(X_1 > x_1, X_2 > x_2), \quad \forall x_1, x_2 \in \mathbb{R}. \tag{1.24}$$

Negative quadrant dependence (NQD) is defined by reversing the inequalities in Eqs. (1.23) and (1.24). Quadrant dependence is an important concept in reliability studies, particularly for modelling the failure time of two dependent components in a system (see Barlow and Proschan (1975), Lai and Balakrishnan (2009)).

The concept of PQD can also be defined in terms of copulas. A bivariate copula C exhibits the PQD property if

$$\Pi(u_1, u_2) \leq C(u_1, u_2), \quad \forall u_1, u_2 \in \mathbb{I}, \tag{1.25}$$

where $\Pi(u_1, u_2) = u_1 u_2$ is the product copula.

Conversely, if

$$\Pi(u_1, u_2) \geq C(u_1, u_2), \quad \forall u_1, u_2 \in \mathbb{I}, \tag{1.26}$$

then the copula is said to have the NQD property.

It is important to note that quadrant dependence for the random variables X_1 and X_2 with underlying copula C is equivalent to the corresponding copula-based quadrant dependence.

The notion of quadrant dependence can be extended to higher dimensions, where it is referred to as orthant dependence. However, the equivalent conditions defined in Eqs. (1.23) and (1.24) do not hold in the same way in higher dimensions. Similar to quadrant dependence, orthant dependence can also be expressed in terms of copulas.

Let C be a p-dimensional copula, and let $\bar{C}$ be the corresponding joint survival function of C, as defined in Eq. (1.6), associated with a multivariate random vector $\mathbf{X}$. Then, $\mathbf{X}$ is said to be positively lower orthant dependent (PLOD) if

$$\Pi(\mathbf{u}) \leq C(\mathbf{u}), \quad \forall \mathbf{u} \in \mathbb{I}^d. \tag{1.27}$$

Similarly, $\mathbf{X}$ is said to be positively upper orthant dependent (PUOD) if

$$\prod_{i=1}^{d}(1-u_i) \leq \bar{C}(\mathbf{u}), \quad \forall \mathbf{u} \in \mathbb{I}^d. \tag{1.28}$$

If the inequalities in Eqs. (1.27) and (1.28) are reversed, then $\mathbf{X}$ is said to be negatively lower orthant dependent (NLOD) and negatively upper orthant dependent (NUOD), respectively.

1.2.4.3 Total Positivity of Order 2 (TP_2) Property

A bivariate function $G(x, y)$ is said to be totally positive of order 2 (TP_2) if

$$G(x_1, y_1)G(x_2, y_2) \geq G(x_1, y_2)G(x_2, y_1), \quad \text{for every } x_1 < x_2,\ y_1 < y_2 \text{ in } \mathbb{R}.$$

Holland and Wang (1987) provided a sufficient condition: if the first-order partial derivatives of $G(x, y)$ exist almost everywhere in $\mathbb{R}^2$ and are continuous, then $G(x, y)$ satisfies the TP_2 property if

$$\zeta_G(x, y) = \frac{\partial^2 \ln G(x, y)}{\partial x \partial y} \geq 0, \quad \forall (x, y) \in \mathbb{R}^2.$$

The TP_2 property is one of the strongest dependence measures for a bivariate copula, as it implies positive quadrant dependence (PQD), left-tail decreasing (LTD), and left-corner set decreasing (LCSD) dependency properties.

Moreover, if a copula density $c(u_1, u_2)$ satisfies the TP_2 property, then the associated copula $C(u_1, u_2)$ exhibits stochastic increasing (SI) and right-tail increasing (RTI) properties in addition to LTD, LCSD, and PQD. A copula density with the TP_2 property is also referred to as positive likelihood ratio dependent (PLRD). For more details, see Nelsen (2006), Drouet Mari and Kotz (2001), Karlin (1968), and Joe (1997).

The notion of TP_2 for higher dimensions can be defined similarly and is discussed in Nelsen (2006).

1.2.4.4 Tail Dependence Coefficients

The tail dependence coefficients measure the level of dependence between random variables in the upper-right and lower-left quadrants of $\mathbb{I}^2$. In terms of copulas, the upper and lower tail dependence coefficients, denoted by λ_U^C and λ_L^C, respectively, are given by

$$\lambda_L^C = \lim_{u \to 0^+} \frac{C(u,u)}{u}, \quad \lambda_U^C = 2 - \lim_{u \to 1^-} \frac{1 - C(u,u)}{1-u}. \tag{1.29}$$

It is known that $0 \leq \lambda_L^C \leq 1$ and $0 \leq \lambda_U^C \leq 1$. If $\lambda_L^C \in (0,1]$, the copula $C(u_1, u_2)$ exhibits lower tail dependence; if $\lambda_L^C = 0$, the copula has no lower tail dependence. A similar interpretation holds for λ_U^C (see Nelsen (2006), p. 214).

Tail dependence coefficients play a crucial role in financial risk management, particularly in modelling extreme co-movements of asset returns (see Cherubini et al. (2004), Salmon (2009), and MacKenzie and Spears (2012)).

Recently, Pettere et al. (2018) generalized the concept of tail dependence to higher dimensions. The following theorem characterizes the tail dependence of Archimedean copulas using their generator function φ.

Theorem 1.2.17 (Nelsen (2006))**.** *Let C be a bivariate Archimedean copula with generator function φ. Then*

$$\lambda_L^C = \lim_{u \to \infty} \frac{\varphi^{(-1)}(2u)}{\varphi^{(-1)}(u)},$$

$$\lambda_U^C = 2 - \lim_{u \to 0^+} \frac{1 - \varphi^{(-1)}(2u)}{1 - \varphi^{(-1)}(u)},$$

where $\varphi^{(-1)}(\cdot)$ denotes the pseudo-inverse of φ, as defined in Eq. (1.14).

1.2.5 Weighted Arithmetic and Geometric Mean of Copulas

Let $C_1, C_2, \ldots, C_n$ be n copulas of dimension p. The weighted arithmetic mean (WAM) of these copulas is defined as

$$C^{\Sigma}(\mathbf{u}) = \sum_{i=1}^{n} \alpha_i C_i(\mathbf{u}), \tag{1.30}$$

for every $\mathbf{u} \in \mathbb{I}^d$, where the weights α_i satisfy $\alpha_i \in \mathbb{I}$ and $\sum_{i=1}^n \alpha_i = 1$.

It is straightforward to verify that the C^{Σ} is a valid copula. In a similar way, we can define weighted geometric mean (WGM) as

$$C^{\Pi}(\mathbf{u}) = \prod_{i=1}^{n} C_i(\mathbf{u})^{\alpha_i}, \quad \text{where } \alpha_i \in \mathbb{I} \text{ and } \sum_{i=1}^{n} \alpha_i = 1. \tag{1.31}$$

Cuadras (2009) proved that the WGM of two bivariate copulas may or may not be a copula. Recently, Diaz and Cuadras (2022)) showed that the WGM of two extended Gumbel-Barnett copulas is also a copula. Zhang et al. (2013) proved that the WGM of two bivariate copulas C_1 and C_2 is a copula if C_1 and C_2 has TP_2 property.

1.2.6 Convexity and Concavity Properties of Copulas

In this subsection, we discuss a certain family of copulas that exhibit convexity and concavity properties.

Definition 1.2.18. *A bivariate copula C is said to be convex if*

$$C(\alpha s_1 + (1-\alpha)t_1, \alpha s_2 + (1-\alpha)t_2) \leq \alpha C(s_1, s_2) + (1-\alpha)C(t_1, t_2)$$

and is said to be concave if

$$C(\alpha s_1 + (1-\alpha)t_1, \alpha s_2 + (1-\alpha)t_2) \geq \alpha C(s_1, s_2) + (1-\alpha)C(t_1, t_2)$$

for every $s_1, s_2, t_1, t_2 \in \mathbb{I}$ and $\alpha \in [0, 1]$.

Durante et al. (2006) proved that the only bivariate copula that is convex is the Fréchet–Hoeffding lower bound copula, given by

$$W(u_1, u_2) = \max\{u_1 + u_2 - 1, 0\},$$

and the only bivariate copula that is concave is the Fréchet–Hoeffding upper bound copula, given by

$$M(u_1, u_2) = \min\{u_1, u_2\}.$$

Since the convexity and concavity conditions are too restrictive, it is necessary to introduce weaker versions of these properties. The weaker versions of concavity for bivariate copulas are called *Schur-concavity* and *quasi-concavity*. Schur-concavity is introduced in the context of majorization ordering, whereas quasi-concavity arises in optimization theory. Schur-concavity of a bivariate copula is defined as follows.

Definition 1.2.19. *A bivariate copula C is said to be Schur-concave if*

$$C(u_1, u_2) \leq C(\alpha u_1 + (1-\alpha)u_2, (1-\alpha)u_1 + \alpha u_2),$$

for every $s, t \in \mathbb{I}$ and $\alpha \in [0, 1]$.

If the inequality is reversed, the copula is said to be *Schur-convex.* Durante and Sempi (2003) showed that the Fréchet–Hoeffding lower bound copula W is the only Schur-convex copula. Moreover, Durante and Sempi (2003) provided an equivalent condition for Schur-concavity, stated below.

Theorem 1.2.20 (Durante and Sempi (2003))**.** *Let C be a continuously differentiable bivariate copula. Then C is Schur-concave if and only if C is symmetric and*

$$\frac{\partial C(u_1, u_2)}{\partial u_1} \leq \frac{\partial C(u_1, u_2)}{\partial u_2}, \quad \textit{whenever } u_2 \leq u_1, \quad u_1, u_2 \in \mathbb{I}.$$

Theorem 1.2.21 (Durante and Sempi (2003))**.** *A bivariate copula C is Schur-concave if and only if its corresponding survival copula $\hat{C}$ is Schur-concave.*

Theorem 1.2.22 (Durante and Sempi (2003))**.** *Every Archimedean copula is Schur-concave.*

Definition 1.2.23. *A bivariate copula C is said to be quasi-concave if*

$$\min\{C(s_1, s_2), C(t_1, t_2)\} \leq C(\alpha s_1 + (1-\alpha)t_1, \alpha s_2 + (1-\alpha)t_2),$$

for every $s_1, s_2, t_1, t_2, \alpha \in \mathbb{I}$.

If the inequality is reversed, the copula is said to be *quasi-convex.* Similar to the case of Schur-convex copulas, Alvoni et al. (2007) and Alvoni and Papini (2007) proved that $W = \max\{u_1, u_2 - 1, 0\}$ is the only quasi-convex copula. Moreover, Alvoni and Papini (2007) also showed that if a bivariate copula C is quasi-concave, then it is Schur-concave if and only if it is symmetric.

1.2.7 Statistical Inference for Copulas

So far, we have discussed the fundamental definitions, properties, and key theorems of copula functions. Now, we turn our attention to their inferential aspects. Two crucial aspects must be considered in statistical inference for copulas:

1. **Estimation of Copula Function** Given a multivariate dataset, how can we estimate the copula function? Either assuming a known copula function and estimating the copula parameters or a nonparametric estimator for the copula function?
2. **Goodness-of-Fit Testing:** Once a copula model is chosen, how can we assess its adequacy in representing the dependence structure of the data? This includes goodness-of-fit tests, which play a critical role in multivariate data analysis. A special case of this problem is testing for mutual independence among random variables, i.e., verifying whether the underlying copula is the product copula.

In this subsection, we address these questions and review the relevant literature on copula estimation and goodness-of-fit testing. We begin by discussing estimation methods, which are broadly classified into three categories: parametric, semi-parametric, and nonparametric approaches. Below, we provide a brief review of these methods.

1.2.7.1 Parametric Methods for Estimating Copula Parameters

Let $\mathbf{X}_i = (X_{i,1}, X_{i,2}, \ldots, X_{i,k})$, for $i = 1, 2, \ldots, n$, be a random sample of size n drawn from a p-dimensional random vector $\mathbf{X}$. Assume that the dependence structure among the components of $\mathbf{X}$ is characterized by the absolutely continuous parametric copula $C(\cdot; \delta)$ and copula density $c(\cdot; \delta)$, where δ is the copula parameter. Additionally, let the marginal CDF and PDF of each component be denoted by $F_i(\cdot; \theta_i)$ and $f_i(\cdot; \theta_i)$, respectively, where $\theta_i \in \Theta$ for $i = 1, 2, \ldots, p$ represents the marginal parameters.

The goal is to estimate the parameters $(\theta_1, \theta_2, \ldots, \theta_p, \delta)$ from the given data. Using Theorem 1.2.5, the log-likelihood function can be expressed as

$$LL(\theta_1, \theta_2, \ldots, \theta_p, \delta) = \sum_{i=1}^{n} \log\left(c\left(F_1(X_{i,1}; \theta_1), \ldots, F_p(X_{i,p}; \theta_p); \delta\right)\right) + \sum_{i=1}^{n}\sum_{j=1}^{p} \log\left(f_i(X_{i,j}; \theta_j)\right). \tag{1.32}$$

Applying the standard maximum likelihood estimation (MLE) procedure, the unknown parameters are estimated by maximizing the log-likelihood function. The advantage of this method is that it provides estimates for both the copula parameters and the marginal parameters. However, it has two main drawbacks:

1. If the parametric assumptions on the marginal distributions are incorrect, the copula parameter estimates may be biased.

2. When the number of marginal parameters or components is large, estimating all parameters simultaneously becomes computationally challenging.

To improve computational efficiency, the *Inference Functions for Margins (IFM)* method is often used. This two-stage estimation procedure is outlined as follows:

1. First, estimate the marginal parameters by maximizing the likelihood function
$$\sum_{i=1}^{n} \log f_i(X_{i,j}; \theta_j)$$
for each $j = 1, 2, \dots, p$.

2. Let $\hat{\theta}_j$ be the MLE obtained from the above step. Then, estimate the copula parameter δ by substituting these estimated marginal parameters into the CDF and maximizing the likelihood function
$$\sum_{i=1}^{n} \log \left(c \left(F_1(X_{i,1}; \hat{\theta}_1), \dots, F_p(X_{i,p}; \hat{\theta}_p); \delta \right) \right).$$

Compared to the full MLE procedure, IFM is computationally more efficient. Although it may have slightly lower efficiency, its performance is generally comparable (see Kim et al. (2007) and Hofert et al. (2018)). However, similar to standard MLE, the IFM method also suffers from bias if the marginal CDFs are misspecified.

In addition to classical MLE, Bayesian methods for copula parameter estimation have also been explored in the literature. Some notable references on Bayesian estimation of copulas include Shemyakin and Kniazev (2017), Ning and Shephard (2018), and Henderson et al. (2021).

1.2.7.2 Semi-parametric Methods for Estimating Copula Parameters

As discussed earlier, if any of the marginal CDFs are misspecified in the parametric approach, the copula parameter estimates may suffer from bias. To address this issue, a semi-parametric approach estimates the marginal CDFs nonparametrically using the empirical CDF given by
$$\hat{F}_{n,j}(x) = \frac{1}{n+1} \sum_{i=1}^{n} \mathbf{I}(X_{i,j} \leq x),$$
for each $j = 1, 2, \dots, p$ and $\mathbf{I}(\cdot)$ is the usual indicator function. These nonparametric estimates are then substituted into the likelihood function, and the copula parameter δ is estimated by maximizing the pseudo-likelihood function
$$\sum_{i=1}^{n} \log \left(c \left(\hat{F}_{n,1}(X_{i,1}), \dots, \hat{F}_{n,p}(X_{i,p}); \delta \right) \right).$$

This method is referred to as the *maximum pseudo-likelihood method.*

Another widely used semi-parametric approach is the *method of moments*. In this method, copula parameters are estimated by equating sample measures of dependence, such as Spearman's rank correlation or Kendall's Tau, to their corresponding population values and solving for the parameters. Several authors discussed this approach for copula parameter estimation (see Genest (1987), Genest and Rivest (1993), Genest and Favre (2007), Yuan (2018), and Hofert et al. (2018)).

1.2.7.3 Nonparametric Estimation for Copula Functions

If the underlying copula is not specified correctly, subsequent data analysis may yield incorrect results and misleading conclusions. To overcome the limitations of the parametric approach, the literature discusses various nonparametric estimators for copula functions. A common choice is the empirical copula, defined as follows.

Definition 1.2.24 (Empirical Copula). *Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be a random sample of size n from a p-variate distribution, where $\mathbf{X}_i = (X_{i,1}, X_{i,2}, \ldots, X_{i,p})^T$. Let $R_{i,j}$ be the rank of the j^{th} component in the i^{th} observation $\mathbf{X}_i$. Then, the empirical copula is defined as*

$$\hat{C}_n(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{p} \mathbf{I}\left(\frac{R_{i,j}}{n+1} \leq u_j \right), \tag{1.33}$$

where $\mathbf{I}(\cdot)$ is the usual indicator function.

It is important to note that the empirical copula is not a valid copula, as it may lack the p-increasing property. However, it is asymptotically a valid copula. The concept of empirical copulas was first introduced by Deheuvels (1979). The asymptotic validity of the empirical copula is guaranteed by the following theorem.

Theorem 1.2.25 (Glivenko-Cantelli Theorem for Empirical Copulas). *Let $\hat{C}_n$ be the empirical copula, defined in Eq. (1.33), based on a random sample of size n from a multivariate population with underlying copula C. Then, as $n \to \infty$,*

$$\sup_{\mathbf{u} \in \mathbb{I}^p} \left| \hat{C}_n(\mathbf{u}) - C(\mathbf{u}) \right| \to 0, \quad a.s. \tag{1.34}$$

For further details, see Deheuvels (1979), Kiefer (1961), Shorack and Wellner (2009), and Janssen et al. (2012). The following theorem discusses the weak convergence of the empirical copula process. For more details, see Fermanian et al. (2004), Tsukahara (2005), and Kojadinovic and Holmes (2009).

Theorem 1.2.26 (Weak Convergence of the Empirical Copula Process). *Let C be a p-dimensional copula. Let $L_\infty(\mathbb{I}^p)$ denote the Banach space of real-valued bounded functions defined on $\mathbb{I}^p$, equipped with the supremum norm. If C has continuous partial derivatives for every $\mathbf{u} \in \mathbb{I}^d$, then the empirical process*

$$\mathbb{Z}_n(\mathbf{u}) = \sqrt{n}\,(C_n(\mathbf{u}) - C(\mathbf{u}))$$

converges weakly in $L_\infty(\mathbb{I}^p)$ to the tight, centered Gaussian process

$$\mathbb{Z}(\mathbf{u}) = \Gamma(\mathbf{u}) - \sum_{i=1}^{d} \partial_i C(\mathbf{u})\Gamma(\mathbf{u}_i),$$

where $\partial_i C(\mathbf{u})$ is the i-th partial derivative of C, $\mathbf{u}_i = (1, \dots, 1, u_i, 1, \dots, 1)$ with u_i in the i-th position, and $\Gamma(\mathbf{u})$ is a tight, centered Gaussian process on $\mathbb{I}^d$ with covariance function

$$\Sigma(\mathbf{u}, \mathbf{v}) = C(\mathbf{u} \wedge \mathbf{v}) - C(\mathbf{u})C(\mathbf{v}),$$

where $\mathbf{u} \wedge \mathbf{v} = \Big(\min(u_1, v_1), \dots, \min(u_p, v_p)\Big)$.

Since the empirical copula is not a valid copula and can be computationally challenging for various operations, a better estimator is needed to overcome its limitations. To address these issues, Sancetta and Satchell (2004) introduced Bernstein copulas using Bernstein polynomials.

Recall that the Bernstein polynomial approximation for any continuous bounded function g defined on $\mathbb{I}^p$ is given by

$$g^B(\mathbf{x}) = \sum_{\alpha_1=0}^{k_1} \cdots \sum_{\alpha_p=0}^{k_p} g\left(\frac{\alpha_1}{k_1}, \dots, \frac{\alpha_p}{k_p}\right) \prod_{j=1}^{p} \binom{k_j}{\alpha_j} x_j^{\alpha_j}(1 - x_j)^{k_j - \alpha_j},$$

where $\mathbf{k} = (k_1, k_2, \dots, k_p) \in \mathbb{N}^p$. It is important to note that as $k_i \to \infty$ for $i = 1, 2, \dots, p$, we have

$$g^B(\mathbf{x}) \to g(\mathbf{x}) \quad \text{for every } \mathbf{x} \in \mathbb{I}^p$$

(see (DeVore and Lorentz, 1993, p. 6)).

Using this approach, Sancetta and Satchell (2004) proposed the Bernstein copula, defined by

$$C^B(\mathbf{u}) = \sum_{\alpha_1=0}^{k_1} \cdots \sum_{\alpha_p=0}^{k_p} C\left(\frac{\alpha_1}{k_1}, \ldots, \frac{\alpha_p}{k_p}\right) \prod_{j=1}^{p} \binom{k_j}{\alpha_j} u_j^{\alpha_j} (1-u_j)^{k_j-\alpha_j}, \tag{1.35}$$

where C is some valid p-dimensional copula and $(k_1, k_2, \ldots, k_p) \in \mathbb{N}^p$. Furthermore, Sancetta and Satchell (2004) suggested a nonparametric estimator of the copula function by replacing C in Eq. (1.35) with the empirical copula $\hat{C}_n$, which is referred to as the empirical Bernstein copula. Segers et al. (2017) showed that the empirical Bernstein copula is a valid copula if and only if the sample size n is divisible by k_j for $j = 1, 2, \ldots, p$.

In particular, if $k_1 = k_2 = \cdots = k_p = n$, then the empirical Bernstein copula reduces to the empirical beta copula, which is defined as follows.

Definition 1.2.27 (**Empirical Beta Copula**). *Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be a random sample of size n from a p-variate distribution, where $\mathbf{X}_i = (X_{i,1}, X_{i,2}, \ldots, X_{i,p})^T$. Let $R_{i,j}$ be the rank of the j^{th} component in the i^{th} observation $\mathbf{X}_i$. Then, the empirical copula is defined as*

$$\hat{C}_n^{\beta}(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{p} \sum_{y=R_{i,j}}^{n} \binom{n}{y} u_j^y (1-u_j)^{n-y}. \tag{1.36}$$

The empirical beta copula provides a better estimate compared to the empirical copula in terms of bias and variance (see Segers et al. (2017), Kojadinovic and Yi (2024)) and, in many cases, even outperforms the empirical Bernstein copula. The analogues of Theorem 1.2.25 and Theorem 1.2.26 are discussed in Janssen et al. (2012) and Segers et al. (2017).

Apart from the empirical Bernstein copula, checkerboard copulas and sparse copulas have also been studied in the literature as methods to smooth the empirical copula and provide better approximations. For more details, we refer to the book of Durante and Sempi (2016). Now, we extend our discussion to the next aspect of inference, namely, goodness-of-fit tests for copulas.

1.2.7.4 Goodness-of-Fit Test for Copulas

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be a random sample of size n from a p-variate distribution with underlying copula $C \in C_\Theta$, where $C_\Theta = \{C_\theta : \theta \in \Theta\}$ is a family of copula functions. We aim to test the hypothesis

$$H_0 : C \in C_\Theta.$$

In the literature, most of the goodness-of-fit tests are based on the empirical copula process, defined as

$$\mathbb{C}_n = \sqrt{n}(\hat{C}_n - C).$$

Two commonly used test procedures are the Cramér-von Mises (CVM) statistic and the Kolmogorov–Smirnov (KS) statistic. The CVM statistic is given by

$$S_n = \int_{\mathbb{I}^p} \mathbb{C}_n(\mathbf{u})^2 \, d\mathbf{u},$$

while the KS statistic is defined as

$$T_n = \sup_{\mathbf{u}\in\mathbb{I}^p} |\mathbb{C}_n(\mathbf{u})|.$$

Another useful approach for characterizing dependence is based on Kendall's distribution, which is defined as the distribution of the random variable $W = C(\mathbf{U})$, where $\mathbf{U} = (U_1, U_2, \dots, U_p)$ is a random vector with joint CDF given by the copula function C. For further details, see Genest and Rivest (1993) and Wang and Wells (2000).

Barbe et al. (1996) showed that the empirical distribution function

$$\hat{K}_n(w) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(\hat{W}_i \leq w), \quad w \in \mathbb{I},$$

is a consistent estimator of the true Kendall distribution function $K(w) = P(W \leq w)$. This reformulates the goodness-of-fit test for copulas as

$$H_0' : K \in K_\Theta = \{K_\theta : \theta \in \Theta\}.$$

The empirical process associated with this approach is given by

$$\mathbb{K}_n = \sqrt{n}(\hat{K}_n - K).$$

Using this, the test statistics analogous to CVM and KS are defined as

$$S_n^K = \int_0^1 \mathbb{K}_n(w)^2 \, dw,$$

$$T_n^K = \sup_{w\in\mathbb{I}} |\mathbb{K}_n(w)|.$$

All the aforementioned test statistics do not have closed-form expressions for their limiting distributions under the null hypothesis. As a result, Monte Carlo methods are employed to compute approximate p-values. The validity of these approaches is discussed in Genest and Rémillard (2008).

Apart from these, several bootstrap-based test procedures exist for goodness-of-fit testing of copulas. For details on existing goodness-of-fit tests, one can refer to Panchenko (2005), Genest et al. (2009), Kojadinovic et al. (2011).

A special case of the goodness-of-fit test focuses on empirical copula process-based tests for independence, where the underlying copula is the product copula. The foundational idea was introduced by Deheuvels (1979). The Cramér–von Mises and Kolmogorov–Smirnov functionals are widely used for testing mutual independence among random variables. Moreover, recent advancements using the Möbius decomposition of the empirical copula process have been shown to improve the power of tests based on the Cramér–von Mises statistic. For further details, we refer to Genest and Rémillard (2004), Genest et al. (2006), Kojadinovic and Holmes (2009), Belalia et al. (2017), Herwartz and Maxand (2020), and Nasri and Remillard (2024). We conclude this section by highlighting several applications of copulas in the literature, showcasing their potential in modelling multivariate data analysis across various disciplines.

1.2.8 Applications

Copula theory has numerous real-world applications, particularly in constructing joint distributions using Sklar's theorem by incorporating appropriate marginal distributions. Several studies have focused on developing bivariate distributions using copulas. For instance, Achcar et al. (2015) used the Farlie-Gumbel-Morgenstern (FGM) copula to construct a bivariate generalized exponential distribution. Abd Elaal and Jarwan (2017) derived a bivariate generalized exponential distribution from Plackett and FGM copula functions, while Kundu and Gupta (2017) studied the bivariate Birnbaum-Saunders distribution using the Gaussian copula. Additionally, El-Sherpieny et al. (2018) explored a bivariate Weibull distribution based on the FGM copula, Mondal and Kundu (2020) proposed a bivariate inverse Weibull distribution using the Marshall-Olkin copula, and Almetwally and Muhammed (2020) developed a bivariate Fréchet distribution using the FGM copula.

Beyond distributional modelling, copulas have broad applications in finance, engineering, insurance, and medicine. In finance, Joe (1997) explored copula-based time series models, while Chen and Fan (2006) studied semiparametric estimation methods for such models, establishing their $\sqrt{n}$-consistency and asymptotic properties. Simard and Rémillard (2015) introduced a copula-based forecasting method for multivariate time series.

In insurance, Cossette et al. (2013) analyzed a portfolio of dependent risks using the FGM copula, assuming mixed Erlang-distributed marginals. Sarabia et al. (2018) derived explicit formulas for the probability density function of collective risk in multivariate mixed exponential distributions with Archimedean copula dependence. Marri and Moutanabbir (2022) addressed risk aggregation and capital allocation in dependent risk scenarios, modelling dependence using a mixed Bernstein copula. Recently, Blier-Wong et al. (2023) introduced novel representations based on symmetric multivariate Bernoulli distributions and order statistics, offering new insights into risk aggregation using FGM copulas.

For the last two decades, considerable efforts have been made to develop bivariate reliability models using copulas. Georges et al. (2001) studied the use of survival copulas in multivariate lifetime modelling. Kaishev et al. (2007) considered the problem for modelling the joint reliability function in a competing risk model using copula-based approach. Zhang and Lam (2016) developed efficient point estimators using copula approach for engineering applications. Gupta (2016) studied the reliability properties of the FGM family of bivariate distributions such as hazard rate components, hazard rate of the series system and the regression mean residual life of a parallel system. Emura et al. (2017) developed a joint frailty-copula model to study tumour progression and death, introducing dependency structures within the joint frailty framework. Nair et al. (2018) considered the bivariate survival copulas for modelling lifetime data. The authors provided the analogues of reliability function that were expressed in terms of survival copula. Yongjin et al. (2018) studied the reliability of a parallel system with dependent components and a cold standby where the dependency was expressed in terms of copula functions. Recently, Sreelakshmi (2018) introduced the notions of copula-based bivariate reliability concepts using the dependence structure and provided some characterization results based on bivariate hazard rate and bivariate mean residual life functions. Jia et al. (2018) formulated the efficiency of reliability and safety analysis of safety-critical series and parallel systems with dependent units using copula functions. Ebaid et al. (2020) proposed an FGM-type copula function and apply that copula function to model the stress-strength reliability with dependent stress and strength variables.

Environmental applications also benefit from copula theory. For example, Latif and Mustafa (2020) used vine copula constructions to analyze flood characteristics, such as peak, volume, and duration, in the Kelantan River Basin, Malaysia. Similarly, Das et al. (2020) utilized a copula-based approach to study drought characteristics in relation to climate indices across the Himalayan states in India.

These examples illustrate the versatility of copula-based methods in capturing dependencies across diverse fields, reinforcing their importance in multivariate data analysis.

1.3 Univariate and Multivariate Information Measures: A Literature Review

In multivariate data analysis, it is essential to consider not only the dependence structure among the random variables in a multivariate random vector but also the information content, i.e., the uncertainty associated with the dependence structure. Various information measures have been discussed in the literature, ranging from univariate to multivariate settings. However, the multivariate framework has been relatively less explored.

This section provides a brief review of the literature on different information measures, from univariate to multivariate contexts. We begin our discussion with Shannon entropy, a fundamental measure for quantifying the uncertainty associated with a discrete random variable.

1.3.1 Shannon Entropy

The concept of entropy was first introduced by Clausius (1850) in the context of the second law of thermodynamics. Later, Boltzmann (1872) provided a statistical definition of entropy by linking it to statistical mechanics. Shannon (1948) laid the mathematical foundation of entropy within the framework of communication theory, and it is now widely known as Shannon entropy.

In a probabilistic sense, Shannon entropy quantifies the uncertainty associated with a discrete random variable. Let X be a discrete random variable with probability mass function (PMF) $p_i = P(X = x_i)$ for $i = 1, 2, \ldots, n$. The Shannon entropy of X is defined as

$$\mathcal{H}(X) = -\sum_{i=1}^{n} p_i \log p_i.$$

Now, consider two discrete random variables X and Y with joint probability mass function $p_{i,j} = P(X = x_i, Y = y_j)$, for $i, j = 1, 2, \ldots, n$. The joint Shannon entropy, denoted by $\mathcal{H}(X, Y)$, is defined as

$$\mathcal{H}(X,Y) = -\sum_{i=1}^{n}\sum_{j=1}^{n} p_{i,j} \log p_{i,j}.$$

The conditional entropy of Y given X, denoted by $\mathcal{H}(Y \mid X)$, is defined as

$$\mathcal{H}(Y \mid X) = -\sum_{i=1}^{n}\sum_{j=1}^{n} p_{i,j} \log p_{j|i},$$

where $p_{j|i} = P(Y = y_j \mid X = x_i)$ is the conditional probability of $Y = y_j$ given $X = x_i$.

Shannon entropy satisfies several fundamental properties, including:

1. **Non-negativity:** $\mathcal{H}(X)$ is always non-negative and equals zero if X is deterministic.
2. **Symmetry:** The joint entropy satisfies $\mathcal{H}(X,Y) = \mathcal{H}(Y,X)$.
3. **Chain Rule:** $\mathcal{H}(X,Y) = \mathcal{H}(X) + \mathcal{H}(Y|X)$.
4. **Accumulation:** $\mathcal{H}(X,Y) \geq \mathcal{H}(X)$.

Note that when X and Y are independent, the joint entropy satisfies $\mathcal{H}(X,Y) = \mathcal{H}(X) + \mathcal{H}(Y)$, which is known as the **additive rule**.

Shannon entropy has applications in various fields, including machine learning, reliability theory, physics, chemistry, finance, and complex systems. For a comprehensive discussion on information theory, we refer to Cover (1999), Ash (2012) and Nair et al. (2022). Additionally, the role of Shannon entropy in thermodynamics is explored in detail in the book of Ben-Naim (2008).

The continuous counterpart of Shannon entropy is known as differential entropy (DE), which is defined for an absolutely continuous random variable with probability density function (PDF) $f(\cdot)$ as

$$\mathcal{D}(X) = -\int_{-\infty}^{\infty} f(x) \log f(x) dx. \tag{1.37}$$

However, Rao et al. (2004) pointed out certain limitations of DE. The main limitations are discussed below:

1. **Inconsistency:** DE can take negative values for certain distributions. For example, let X be a uniform random variable over $(0, \theta)$, $\theta > 0$. If $\theta < 1$, the differential

entropy is negative; if $\theta = 1$, the entropy is zero; and if $\theta > 1$, the entropy becomes positive. This inconsistency makes DE difficult to interpret in some cases.

2. **Defined only for absolutely continuous distributions:** DE is defined only for distributions that have a density function. However, many real-world distributions are of mixed type, incorporating both continuous and discrete components. DE is not applicable in such cases.
3. **Challenges in empirical estimation:** Approximating DE using empirical distribution functions is computationally challenging, making it difficult to estimate from observed data.

Considering these limitations, Rao et al. (2004) proposed an alternative measure called *cumulative residual entropy* (CRE). Let $\bar{F}(x)$ be the survival function of a non-negative random variable X. Then, CRE is defined as

$$\mathcal{CR}(X) = -\int_0^\infty \bar{F}(x) \log \bar{F}(x) dx.$$

More generally, if X is any random variable (not necessarily non-negative), CRE can be defined as

$$\mathcal{CR}(X) = -\int_0^\infty P(|X| > x) \log \left(P(|X| > x)\right) dx.$$

This formulation extends the concept of differential entropy while ensuring its applicability to both discrete and continuous random variables. An important advantage of CRE is that it can be estimated directly from sample data, and various asymptotic properties can be easily established.

On a similar line, Di Crescenzo and Longobardi (2009) introduced the *cumulative entropy* (CE) by replacing the probability density function (PDF) $f(x)$ in Eq. (1.37) with the cumulative distribution function (CDF) $F(x)$. This measure quantifies the uncertainty in a system's inactivity time.

1.3.2 Tsallis Entropy

In the context of thermodynamics, when a system is out of equilibrium or its component states exhibit strong interdependence, non-additive entropy provides a more appropriate measure for quantifying the uncertainty involved in the system. Tsallis (1988) proposed a non-additive entropy, commonly known as Tsallis entropy. It is defined for an absolutely continuous random variable X with probability density function $f(\cdot)$ as

$$\mathcal{T}_\alpha(X) = -\int_{-\infty}^{\infty} f(x)\log_{[\alpha]}(f(x))\,dx, \quad \alpha \in \mathcal{A},$$

where $\mathcal{A} = (0,1) \cup (1,\infty)$ and the generalized logarithmic function is given by

$$\log_{[\alpha]}(r) = \frac{r^{\alpha-1}-1}{\alpha-1}, \quad r \geq 0,$$

for every $\alpha \in \mathcal{A}$. It is to be noted that $\lim\limits_{\alpha\to 1} \log_{[\alpha]}(r) = \log(x)$. Consequently, $\log_{[\alpha]}(\cdot)$ can be interpreted as a fractional generalization of the standard natural logarithm function. As a result, Tsallis entropy reduces to Shannon entropy when $\alpha \to 1$. Recently, Rajesh and Sunoj (2019) generalized the CRE and proposed cumulative residual Tsallis entropy (CRTE), which is given by

$$\mathcal{TR}_\alpha(X) = -\int_{0}^{\infty} \bar{F}(x)\log_{[\alpha]}(\bar{F}(x))\,dx, \quad \alpha \in \mathcal{A}.$$

Similarly, Calì et al. (2017) proposed the cumulative Tsallis entropy (CTE), which generalizes the CE introduced by Di Crescenzo and Longobardi (2009). Various applications of Tsallis entropy and its variants have been discussed in the literature. For more details, we recommend readers to refer to Cartwright (2014), De Albuquerque et al. (2004), Sparavigna (2015), Singh et al. (2017), Mohamed et al. (2022), Toomaj and Atabay (2022), and the references therein. Apart from Tsallis entropy, various generalizations of Shannon entropy have been proposed in the literature. For more details, we refer to Rényi (1961), Varma (1966), Di Crescenzo and Longobardi (2006), Mathai and Haubold (2007) and Psarrakos and Toomaj (2017).

1.3.3 Fractional Order Entropy

Inspired by the concepts of fractional calculus, fractional variants of various information measures have been proposed, extending several entropy measures existing in the literature. The properties of fractional calculus allow these measures to capture long-range dependencies and non-local effects in complex random systems (see Kayid and Shrahili (2022) and Lopes and Machado (2020)).

One of the pioneering work on fractional order entropy by Ubriaco (2009) introduced a generalization of Shannon entropy using fractional calculus. The fractional version of

Shannon entropy is defined as

$$\mathcal{H}_r(X) = \sum_{i=1}^{n} p_i \left(-\ln p_i\right)^r, \quad r \in \mathbb{I}. \tag{1.38}$$

For $r = 1$, $\mathcal{H}_r(X)$ reduces to the standard Shannon entropy.

Xiong et al. (2019) extended this idea to generalizing CRE, which is defined as

$$\mathcal{CR}_r(X) = \int_0^{\infty} \bar{F}(x) \left[-\log \bar{F}(x)\right]^r dx, \quad r \in \mathbb{I}.$$

They demonstrated the application of fractional entropy in measuring uncertainty in financial datasets, showing that fractional entropy provides deeper insights compared to its classical counterpart. The fractional version of CE was further explored by Kayid and Shrahili (2022). For additional work in this direction, see Jumarie (2012), Karci (2016), Lopes and Machado (2020), Di Crescenzo et al. (2021), Foroghi et al. (2023) and Saha and Kayal (2023).

1.3.4 Divergence Measure and Mutual Information

Kullback and Leibler (1951) introduced an information measure to quantify the divergence between two probability distributions, widely known as Kullback-Leibler (KL) divergence, sometimes referred to as relative entropy. Let X_1 and X_2 be two continuous random variables with probability density functions (PDFs) $f_1(x)$ and $f_2(x)$, respectively. The KL divergence between X_1 and X_2 is defined as

$$KL(f_1 \| f_2) = \int_{-\infty}^{\infty} f_1(x) \log\left(\frac{f_1(x)}{f_2(x)}\right) dx. \tag{1.39}$$

Minimizing the KL divergence between an assumed distribution and an empirical distribution is equivalent to maximizing the likelihood of the sample (see (Murphy, 2022, p. 208)).

Motivated by the work of Rao et al. (2004), Baratpour and Rad (2012) proposed an alternative measure using the survival functions of non-negative random variables, called the *cumulative residual KL divergence* (CRKL), defined as

$$CRKL(\bar{F}_1 \| \bar{F}_2) = \int_0^{\infty} \bar{F}_1(x) \log\left(\frac{\bar{F}_1(x)}{\bar{F}_2(x)}\right) dx - \mathbb{E}(X_1) + \mathbb{E}(X_2).$$

Baratpour and Rad (2012) also discussed the application of CRKL divergence in goodness-of-fit testing for the exponential distribution. Similarly, Park and Kim (2014) proposed an alternative KL divergence measure based on the cumulative distribution function (CDF) of the random variables. Furthermore, Mao et al. (2020) extended the CRKL divergence using Tsallis entropy, defining the *Tsallis residual KL divergence* (TRKL) as

$$TRKL(\bar{F}_1\|\bar{F}_2) = \int_0^\infty \bar{F}_1(x)\log_{[\alpha]}\left(\frac{\bar{F}_1(x)}{\bar{F}_2(x)}\right)dx - \mathbb{E}(X_1) + \mathbb{E}(X_2), \quad \alpha \in \mathcal{A}.$$

This extension was applied in the financial sector to measure divergence in financial time series data. Recently, Mehrali and Asadi (2021) explored the application of cumulative KL divergence in estimation problems.

Another important measure in information sciences is *mutual information* (MI), which quantifies the amount of information one random variable contains about another. Let (X_1, X_2) be a bivariate continuous random vector with joint PDF $f(x_1, x_2)$ and marginal PDFs $f_1(x_1)$ and $f_2(x_2)$. The mutual information is defined as

$$\begin{aligned}\mathrm{MI}(X_1, X_2) =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2)\log\left(\frac{f(x_1, x_2)}{f_1(x_1)f_2(x_2)}\right)dx_1dx_2\\ =& \mathcal{D}(X_1) - \mathcal{D}(X_1|X_2).\end{aligned}$$

Thus, MI quantifies the reduction in uncertainty of one random variable given the knowledge of the other.

Joe (1987) extended this concept to higher dimensions for a multivariate random vector $\mathbf{X}$, defining the multivariate mutual information as

$$\mathrm{MI}(\mathbf{X}) = \int_{\mathbb{R}^p} f(\mathbf{x})\log\left(\frac{f(\mathbf{x})}{\prod_{i=1}^p f_i(x_i)}\right)d\mathbf{x}. \tag{1.40}$$

1.3.5 Inaccuracy Measures

Apart from entropy, several other information measures exist in information theory. One such measure is the *inaccuracy measure* proposed by Kerridge (1961). In Eq. (1.39), the KL divergence between two continuous random variables can be rewritten as

$$KL(f_1\|f_2) = -\mathcal{D}(X_1) + IN(f_1\|f_2),$$

where

$$IN(f_1|f_2) = -\int_{-\infty}^{\infty} f_1(x)\log f_2(x)\,dx$$

is the inaccuracy measure introduced by Kerridge (1961). The inaccuracy measure can be interpreted as follows: If $f_1(x)$ is the true PDF of the data but, due to experimental error, the experimenter assumes $f_2(x)$ as the PDF instead, then the average uncertainty in this incorrect assumption is quantified by $IN(f_1\|f_2)$. The cumulative version of the inaccuracy measure was introduced by Kumar and Taneja (2015), and recently, Raju et al. (2024) generalized it using Tsallis entropy. For more details, we refer the book of Nair et al. (2022).

1.3.6 Information-generating function

Recent research has focused on developing information-generating functions, which can generate a number of useful uncertainty and divergence measures. Golomb (1966) defined an information generating function by

$$SG_X(s) = \sum_{i=1}^{n} (p_i)^s \,,\ \ s \geq 1.$$

It may be observed that $SG_X(1) = 1$ and the first derivative of $SG_X(s)$ at $s = 1$ corresponds to the negative of the Shannon entropy. Guiasu and Reischer (1985) discussed the generating function for the relative entropy and showed that its first derivative at $s = 1$ gives a negative of the Kullback and Leibler divergence for two probability distributions. Fisher information generating function and associated results are reported in Papaioannou et al. (2007). The generating function and nonparametric estimator for the CRE are reported in a recent work of Smitha et al. (2023). Recently, Saha and Kayal (2024) defined the general weighted information and relative information generating functions and discussed its mathematical properties.

1.3.7 Multivariate Information Measures

The multivariate extension of univariate information measures has been widely studied in the literature. Nadarajah and Zografos (2005) derived expressions for the bivariate differential entropy of various bivariate distributions. Ebrahimi et al. (2007) developed information measures for the residual lifetime of a bivariate random vector.

Rajesh et al. (2009) introduced a vector-based bivariate residual entropy to quantify the uncertainty in the remaining lifetime of a bivariate random vector. Further, Rajesh et al. (2014) extended the bivariate version of dynamic cumulative residual entropy, initially proposed by Asadi and Zohrevand (2007). In addition, Kundu and Kundu (2017) generalized the cumulative entropy introduced by Di Crescenzo and Longobardi (2006) for bivariate random vectors and discussed its dynamic version. More recently, Raju et al.

(2020) proposed the bivariate cumulative residual Tsallis entropy (CRTE) and explored its properties and applications.

1.3.8 Copula-Based Information Measures

In this subsection, we review the literature on copula-based information measures. A natural question arises regarding the significance of copula-based dependence entropy. It is worth noting that copula-based entropy measures the uncertainty involved in the dependence structure among random variables. In multivariate data analysis, uncertainty associated with a multivariate random variable can be decomposed into two components: the uncertainty due to each marginal distribution and the uncertainty that arises from the dependence structure among the random variables. Note that the copula captures the dependence structure, making copula-based information measures relevant. The scope of copula-based information measures in multivariate data analysis was first discussed by Ma and Sun (2011). They showed that the MI of a multivariate random vector is equivalent to the negative of the copula entropy, which is defined as

$$\zeta\,(c) = -\int_{\mathbb{I}^p} c(\mathbf{u}) \log c(\mathbf{u})\, d\mathbf{u}, \tag{1.41}$$

where $c(\mathbf{u})$ is the copula density. Using the results of Ma and Sun (2011), the MI of a multivariate random vector $\mathbf{X}$ is independent of marginal distributions and depends only on the dependence structure, which is measured by the underlying copula density. Copula entropy has widespread applications across various fields, including science, engineering, hydrology, and finance (see Zhao and Lin (2011), Hao and Singh (2015), Singh and Zhang (2018)). However, when the underlying copula is not absolutely continuous, the copula density does not exist, making the copula entropy proposed by Ma and Sun (2011) inapplicable. Additionally, the copula entropy $\zeta(c)$ is always negative. Motivated by the works of Rao et al. (2004) and Di Crescenzo and Longobardi (2009), Sunoj and Nair (2025) replaced the copula density with the copula function and proposed the cumulative copula entropy (CCE) which is given by

$$\xi\,(C) = -\int_0^1 \int_0^1 C(u_1, u_2) \log C(u_1, u_2)\, du_1 du_2.$$

The copula-based inaccuracy measure was first proposed by Hosseini and Ahmadi (2019). Let C_1 and C_2 be two p-dimensional copulas. The copula-based inaccuracy measure is defined as

$$\mathcal{I}(C_1 \mid C_2) = -\int_{\mathbb{I}^p} C_1(\mathbf{u}) \log\left(C_2(\mathbf{u})\right) d\mathbf{u}.$$

The results were further extended to co-copulas, and the dual of a copula in Hosseini and Nooghabi (2021).

1.4 Bivariate Reliability Theory: Basic Concepts

In multivariate data analysis, it is common to encounter multivariate lifetime data, which requires specialized modelling approaches. Copulas offer a flexible framework for modelling the dependence structure among random variables and are instrumental in constructing multivariate lifetime models. For a detailed discussion on the role of copulas in bivariate reliability modelling, refer to Sreelakshmi (2018) and Nair et al. (2018). This section introduces key concepts in bivariate reliability theory.

In lifetime data analysis, in addition to density and survival functions, the hazard rate and mean residual life (MRL) function are essential tools to assess the reliability of a component or system. The hazard rate quantifies the instantaneous failure rate at a given time, while the MRL function represents the expected remaining lifetime given survival up to that time point.

The MRL function is useful for characterizing lifetime distributions. A reversed analogue of this function, referred to as the mean inactivity time (or reversed mean residual life) function, is applicable in scenarios involving left-censored data. The reversed hazard rate, defined as the ratio of the density function to the distribution function, is relevant when the exact time of failure is observed. This concept has been studied in the context of stochastic orderings by Keilson and Sumita (1982), Shaked and Shanthikumar (2007), and further examined by Block et al. (1998), Chandra and Roy (2001), and Finkelstein (2002).

Let X be a non-negative random variable. The mean inactivity time function is defined as

$$r(x) = \mathbb{E}(x - X \mid X \leq x), \quad x > 0,$$

and is sometimes referred to as the mean past lifetime. Various properties of this function are discussed in Nanda et al. (2003). This measure is particularly useful for analyzing left-censored data and has numerous applications in fields such as medicine, engineering, and forensic science (see Jayasinghe and Zeephongsekul (2013)). Now we will discuss the extension of the univariate reliability measures to higher dimensions. We here discuss for the bivariate case only and the multivariate extension is just a straightforward extension only.

Bivariate Hazard Rate Function

The concept of the bivariate hazard rate was introduced by Basu (1971). Let (X_1, X_2) be a non-negative bivariate random vector with joint probability density function $f(x_1, x_2)$ and joint survival function $\bar{F}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$. The bivariate hazard rate function proposed by Basu (1971) is defined as

$$\mu(x_1, x_2) = \frac{f(x_1, x_2)}{\bar{F}(x_1, x_2)}, \quad x_1 > 0, x_2 > 0.$$

Unlike the univariate case, the bivariate hazard rate function $\mu(x_1, x_2)$ introduced by Basu (1971) does not characterize the underlying joint distribution (see Yang and Nachlas (2001)). To overcome this limitation, Johnson and Kotz (1975) proposed the hazard gradient approach, defining the bivariate hazard rate function as

$$(h_1(x_1, x_2), h_2(x_1, x_2)) = \left(-\frac{\partial \ln \bar{F}(x_1, x_2)}{\partial x_1}, -\frac{\partial \ln \bar{F}(x_1, x_2)}{\partial x_2}\right). \tag{1.42}$$

Here, $h_1(x_1, x_2)$ denotes the conditional hazard rate of X_1 given $X_2 > x_2$, while $h_2(x_1, x_2)$ denotes that of X_2 given $X_1 > x_1$. Notably, the bivariate hazard rate function defined by Johnson and Kotz (1975) fully characterizes the joint distribution.

Bivariate Mean Residual Life Function

Arnold and Zahedi (1988) extended the univariate mean residual life function to the multivariate setting and studied its characterization properties. The bivariate MRL function is given by

$$(m_1(x_1, x_2), m_2(x_1, x_2)) = \left(\frac{\int_{x_1}^{\infty} \bar{F}(t, x_2)\, dt}{\bar{F}(x_1, x_2)}, \frac{\int_{x_2}^{\infty} \bar{F}(x_1, t)\, dt}{\bar{F}(x_1, x_2)}\right). \tag{1.43}$$

When the joint density satisfies the total positivity of order two (TP_2) property, it follows that the hazard component $h_1(x_1, x_2)$ is decreasing in x_2, and the MRL component $m_1(x_1, x_2)$ is increasing in x_2. For details, see Shaked (1975), Gupta and Akman (1995), and Gupta (2016).

Reversed Hazard Rate Components

A vector-based formulation of the reversed hazard rate was introduced by Roy (2002), analogous to the hazard gradient definition of Johnson and Kotz (1975). Let $F(x_1, x_2)$ denote the joint cumulative distribution function. The bivariate reversed hazard rate function is defined as

$$(l_1(x_1, x_2), l_2(x_1, x_2)) = \left(\frac{\partial \ln F(x_1, x_2)}{\partial x_1}, \frac{\partial \ln F(x_1, x_2)}{\partial x_2} \right).$$

Bivariate Mean Inactivity Time Function

Nair and Asha (2008) provided the definition of the bivariate mean inactivity time function and explored several of its properties. The function is defined as

$$(r_1(x_1, x_2), r_2(x_1, x_2)) = \left(\frac{\int_0^{x_1} F(t, x_2)\, dt}{F(x_1, x_2)}, \frac{\int_0^{x_2} F(x_1, t)\, dt}{F(x_1, x_2)} \right). \tag{1.44}$$

1.5 Outline of the Dissertation

In **Chapter 2**, we introduce a new bivariate symmetric copula exhibiting both positive and negative dependence. The proposed copula features a simple mathematical structure, a wider dependence range than the FGM copula and its generalizations. The maximum range of Spearman's Rho for the proposed copula is $[-0.5866, 0.5866]$, significantly improving the dependence range of the FGM copula. Using this copula, we construct a new bivariate Rayleigh distribution and study its statistical properties. A real dataset is analyzed to illustrate the practical relevance of the proposed bivariate distribution.

In **Chapter 3**, we propose a method for constructing a new class of copulas using the probability generating function (PGF) of a positive-integer-valued random variable. Several existing copulas in the literature emerge as special cases of the proposed family. We analyze dependence measures, tail dependence properties under PGF transformation, and provide an algorithm for generating random samples from the PGF copula. The bivariate concavity properties, including Schur concavity and quasi-concavity, are also examined. Two new generalized FGM copulas, derived using PGFs of geometric and discrete Mittag-Leffler distributions, improve Spearman's Rho to $(-0.3333, 0.4751)$ and $(-0.3333, 0.9573)$, respectively. Finally, we apply the proposed copulas to a real dataset to illustrate their practical utility.

In **Chapter 4**, we introduce multivariate cumulative copula entropy (CCE) and explore its theoretical properties, including bounds, stochastic orders, and convergence results. A cumulative copula information-generating function is defined and derived for several well-known families of multivariate copulas. Additionally, we propose a fractional generalization of multivariate CCE and investigate its characteristics. A nonparametric estimator of CCE, based on the empirical beta copula, is developed. Furthermore, we define a new copula-based divergence measure using the Kullback-Leibler (KL) divergence and introduce a goodness-of-fit test derived from this measure. The practical relevance of the proposed divergence measure is demonstrated through a copula selection procedure applied to real data.

In **Chapter 5**, we extend the framework of Chapter 4 by incorporating Tsallis entropy, a non-additive entropy that enhances flexibility in quantifying uncertainty. We introduce cumulative copula Tsallis entropy, derive its properties and bounds, and demonstrate its applicability through examples. A nonparametric version of the measure is developed and validated using coupled periodic and chaotic maps. Additionally, we extend Kerridge's inaccuracy measure and KL divergence to the cumulative copula framework. Using the relationship between KL divergence and mutual information, we propose a new cumulative mutual information (CMI) measure, which overcomes the limitations of density-based mutual information. We further introduce a test for assessing mutual independence among random variables based on the CMI measure. Finally, we illustrate the potential of the proposed CMI measure as an economic indicator through an analysis of real bivariate financial time series data.

In **Chapter 6** we propose a smooth nonparametric estimator for the bivariate mean residual life function. We establish the consistency of the proposed estimator and assess its finite-sample performance through extensive simulation studies, comparing it with existing methods. Furthermore, the practical relevance of the estimator is demonstrated via an application to a bivariate warranty dataset.

In **Chapter 7** we introduce a novel nonparametric estimator for the bivariate mean inactivity time function. The proposed estimator is shown to be asymptotically unbiased, consistent, and asymptotically normally distributed. Its performance is examined through simulation studies across various copula models. To illustrate real-world applicability, we analyze a dataset on pink eye disease, estimating the time since infection period for infections in the left and right eyes.

CHAPTER 2

Exponentiated FGM Copula

This chapter introduces a new FGM-type copula, called the Exponentiated FGM copula. It attains a maximum Spearman's Rho of $[-0.5866, 0.5866]$, exceeding the dependence range of the classical FGM copula and its generalizations. A new bivariate Rayleigh distribution is then constructed using this copula, and its key statistical properties are examined.

2.1 Introduction

Copula plays a significant role in the field of statistics, finance, engineering and medical sciences for modelling dependent data sets. If we have a family of copulas, we automatically have a collection of multivariate distributions with whatever marginal distributions we desire. This feature of the copula is useful in every branch of study where dependence modelling and simulation are an integral part. In literature, a wide variety of copulas are available; of them, Farlie-Gumbel-Morgenstern (FGM) copula received much attention

A part of this chapter has been published in the journal *Statistics & Probability Letters*, 206 (2024): 109988. DOI: 10.1016/j.spl.2023.109988

due to its simple mathematical structure and exhibited positive and negative dependence (see Eyraud (1936), Morgenstern (1956), Gumbel (1960) and Farlie (1960)).

In recent years, the FGM copula has been widely used across various disciplines due to its mathematical simplicity and flexibility in modelling both positive and negative dependence structures. For instance, Achcar et al. (2015) investigated a bivariate generalized exponential distribution using the FGM copula. Similarly, Abd Elaal and Jarwan (2017) examined inference procedures for the bivariate generalized exponential distribution based on the FGM copula. El-Sherpieny et al. (2018) constructed a bivariate Weibull distribution using the FGM copula, while Almetwally and Muhammed (2020) recently proposed a bivariate Fréchet distribution derived from the FGM copula.

In addition to the construction of new multivariate distributions, the FGM copula has found significant applications in lifetime modelling. For example, Louzada et al. (2012) applied the FGM copula to analyze Brazilian HIV data, and Gupta (2016) explored the reliability characteristics of bivariate distributions with an FGM copula structure. Shih and Emura (2019) studied generalized FGM copulas in the context of bivariate competing risks models. More recently, Ghalibaf (2022) analyzed the stress-strength reliability of the FGM bivariate family, highlighting its applications in medical sciences. Furthermore, Blier-Wong et al. (2023) discussed the relevance of FGM copulas in actuarial science. For a comprehensive discussion on the properties and applications of FGM copulas, we refer the reader to Sriboonchitta and Kreinovich (2018).

Despite its advantages, including ease of construction and interpretability, the FGM copula also exhibits certain limitations. One key issue lies in its restricted ability to model strong dependence. For instance, Spearman's Rho (ρ_c), a commonly used measure of dependence for copulas, lies within the interval $[-1, 1]$. However, in the case of the FGM copula, the range of values of Spearman's Rho is very low, i.e., $\rho_c \in [-0.33, 0.33]$ (see Farlie (1960)). So, the FGM copula is unsuitable for modelling data with a high dependence structure. Many researchers have attempted to propose an FGM-type copula for improving the correlation coefficient. Huang and Kotz (1999) proposed two extended FGM copulas, having $\rho_c \in [-0.33, 0.375]$ and $\rho_c \in [-0.33, 0.391]$ respectively. Bairamov and Kotz (2002) also extended FGM copula with $\rho_c \in [-0.48, 0.502]$. Pathak and Vellaisamy (2016a) proposed a new generalized FGM copula through order statistics with maximal range of ρ_C is $(-0.48, 0.53)$. Recently, Chesneau (2022) proposed a polynomial-sine copula exhibiting positive as well as negative dependence with $\rho_c \in [-0.4927, 0.4927]$. In most of these works, the parameters are added to improve Spearman's correlation coefficient range, resulting in a mathematically complex structure and computationally more expensive for estimating unknown parameters. To overcome these drawbacks, we propose a simple bivariate copula without adding any parameters to existing copulas. The proposed copula improves the dependence range of Spearman's correlation of various FGM-type copulas reported in the literature.

The main contributions of this chapter are summarised as follows:

- A new bivariate FGM-type copula with a simple and tractable mathematical structure is proposed.
- Closed-form expressions for various measures of association, including Spearman's Rho and Kendall's Tau, are derived and compared with those of existing FGM-type copulas in the literature.
- The dependence properties of the proposed copula, such as quadrant dependence, the TP_2 property, and tail dependence, are discussed.
- A new bivariate Rayleigh distribution is constructed using the proposed copula, and its statistical properties are explored. The applicability of the model is demonstrated through analysis of a real dataset.

The chapter is organized as follows. In Section 2.2, we introduce a new bivariate symmetric copula. Section 2.3 is dedicated to studying various dependency measures of the proposed copula. In Section 2.4, a new bivariate Rayleigh distribution is derived from the proposed copula, and expressions for the conditional distribution and product moments are obtained. Additionally, a real dataset is analyzed to illustrate the application of the proposed bivariate Rayleigh distribution.

2.2 New Bivariate Copula

In this section, we propose a new bivariate FGM-type copula, which is presented in the following proposition.

Proposition 2.2.1. *Let α be a real number and let $\delta^\star(\alpha)$ be defined as*

$$\delta^\star(\alpha) = \begin{cases} \frac{1}{\alpha^2}, & \text{if } \alpha \in (-\infty, 2] \setminus \{0\} \\ \\ \frac{1}{2\alpha} \exp\left\{1 - \frac{\alpha}{2}\right\}, & \text{if } \alpha > 2. \end{cases}$$

Then, the bivariate function

$$C(u, v; \delta, \alpha) = uv + \delta \left(1 - e^{\alpha(u-u^2)}\right)\left(1 - e^{\alpha(v-v^2)}\right), \quad (u, v) \in \mathbb{I}^2, \tag{2.1}$$

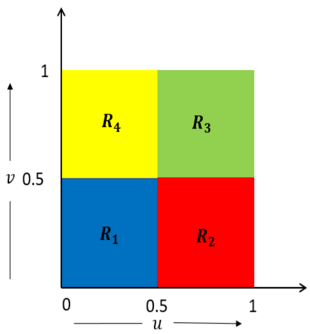
is a bivariate copula if $|\delta| \leq \delta^\star(\alpha)$.

Proof. The bivariate function, defined in Eq. (2.1), satisfied the boundary conditions of a bivariate copula. But we need to find the range of parameter δ for which this function is a valid bivariate copula, i.e., the function in Eq. (2.1) satisfy the 2-increasing property. Kim et al. (2011) proved that 2-increasing property in an absolutely continuous copula is equivalent to the condition that copula density $c(u, v; \delta, \alpha)$ is non-negative, i.e.,

$$c(u, v; \delta, \alpha) = \frac{\partial^2 C(u, v)}{\partial u \partial v} = 1 + \delta\alpha^2 g(u)g(v) \geq 0, \tag{2.2}$$

where $g(t) = (1 - 2t)\, e^{\alpha(t-t^2)}, t \in \mathbb{I}$. Clearly, non-negativity of copula density in Eq. (2.2) depends on the behaviour of function $g(\cdot)$ and the value of the parameter δ. For finding the feasible range of δ, we will divide the domain $\mathbb{I}^2$ of (u, v) into four quadrants as:

$$R_1 = \left\{(u, v) \in \mathbb{I}^2 : 0 \leq u \leq \frac{1}{2}, 0 \leq v \leq \frac{1}{2}\right\},$$
$$R_2 = \left\{(u, v) \in \mathbb{I}^2 : \frac{1}{2} < u \leq 1, 0 \leq v \leq \frac{1}{2}\right\},$$
$$R_3 = \left\{(u, v) \in \mathbb{I}^2 : \frac{1}{2} < u \leq 1, \frac{1}{2} < v \leq 1\right\},$$
$$R_4 = \left\{(u, v) \in \mathbb{I}^2 : 0 \leq u \leq \frac{1}{2}, \frac{1}{2} < v \leq 1\right\}.$$



Since the sign of $g(t)$ is positive if $t \in \left(0, \frac{1}{2}\right)$, and negative if $t \in \left(\frac{1}{2}, 1\right)$, it follows that the product $g(u)g(v)$ is positive on $R_1 \cup R_3$ and negative on $R_2 \cup R_4$. Thus, the copula density $c(u, v; \delta, \alpha)$ is non-negative if

$$\delta \geq \frac{-1}{\alpha^2 g(u)g(v)}, \quad (u, v) \in R_1 \cup R_3,$$

and

$$\delta \leq \frac{-1}{\alpha^2 g(u)g(v)}, \quad (u, v) \in R_2 \cup R_4.$$

Therefore, the copula density $c(u, v; \delta, \alpha)$ is non-negative if

$$\frac{-1}{\alpha^2 \sup\limits_{(u,v)\in R_1\cup R_3} \{g(u)g(v)\}} \leq \delta \leq \frac{-1}{\alpha^2 \inf\limits_{(u,v)\in R_2\cup R_4} \{g(u)g(v)\}}. \tag{2.3}$$

Since the behaviour of function $g(\cdot)$ depends on $\alpha \in \mathbb{R}$, we will consider the following three cases:

Case I: When $\alpha = 0$.

In this case, the copula density $c(u, v; \delta, 0) = 1$, which is non-negative, and hence the 2-increasing property holds for arbitrary value of δ. Moreover, the proposed copula in Eq . (2.1) reduces to the product copula. Note that the product copula is a well-known copula, which corresponds to the independence of two random variables.

Case II: When $\alpha \leq 2, \alpha \neq 0$.

In this case, $g(t)$ is a decreasing function on $\mathbb{I}$ with $g(0) = 1, g\left(\frac{1}{2}\right) = 0$ and $g(1) = -1$. It follows that $g(t)$ takes values in $[0, 1]$ for $t \in \left[0, \frac{1}{2}\right]$, and takes values in $[-1, 0)$ for $t \in \left(\frac{1}{2}, 1\right]$. Thus, the product function $g(u)g(v)$ is bounded above by 1 on $R_1 \cup R_3$, and the upper bound 1 is attended at $(u, v) \in \{(0, 0), (1, 1)\}$. Therefore, $\sup\limits_{(u,v)\in R_1\cup R_3} \{g(u)g(v)\} = 1$. Further, for $(u, v) \in R_2$, $g(u)$ takes values in $[-1, 0)$ and $g(v)$ takes values in $[0, 1]$. This implies that the product function $g(u)g(v)$ is bounded below by -1 on R_2, and the lower bound -1 is attended at $u = 1, v = 0$. Similarly, for $(u, v) \in R_4$, the product function $g(u)g(v)$ is bounded below by -1, which is attended at $u = 0, v = 1$. Therefore, $\inf\limits_{(u,v)\in R_2\cup R_4} \{g(u)g(v)\} = -1$. Now, using these values in inequality (2.3), we get the feasible range of the parameter δ as

$$\frac{-1}{\alpha^2} \leq \delta \leq \frac{1}{\alpha^2}. \tag{2.4}$$

Case III: When $\alpha > 2$.

Let $r_1 = \frac{1}{2} - \frac{1}{\sqrt{2\alpha}}$ and let $r_2 = \frac{1}{2} + \frac{1}{\sqrt{2\alpha}}$. Clearly, $0 < r_1 < \frac{1}{2} < r_2 < 1$. It can be observed that $g(t)$ increases on $t \in [0, r_1]$, decreases on $t \in (r_1, r_2)$, and increases on $t \in [r_2, 1]$. Also, $g(t)$ takes positive values on $t \in \left[0, \frac{1}{2}\right)$ and negative values on $t \in \left(\frac{1}{2}, 1\right]$, with $g\left(\frac{1}{2}\right) = 0$. Moreover, $g(t)$ is maximum at $t = r_1$ with maximum value $g(r_1) = \sqrt{\frac{2}{\alpha}} \exp\left\{\frac{\alpha}{4} - \frac{1}{2}\right\}$, and $g(t)$ is minimum at $t = r_2$ with minimum value $g(r_2) = -\sqrt{\frac{2}{\alpha}} \exp\left\{\frac{\alpha}{4} - \frac{1}{2}\right\}$. Since the functions $g(u)$ and $g(v)$ are positive on the quadrant R_1 and takes maximum at $u = r_1, v = r_1$, it follows that the product function $g(u)g(v)$

has maximum value $[g(r_1)]^2 = \frac{2}{\alpha}\exp\left\{\frac{\alpha}{2}-1\right\}$, on $(u,v)\in R_1$. Since the functions $g(u)$ and $g(v)$ are negative on R_3 and takes minimum at $u=r_2, v=r_2$, it follows that the product function $g(u)g(v)$ is positive and has maximum value $[g(r_2)]^2 = \frac{2}{\alpha}\exp\left\{\frac{\alpha}{2}-1\right\}$, on $(u,v)\in R_3$. Therefore, $\sup_{(u,v)\in R_1\cup R_3}\{g(u)g(v)\} = \frac{2}{\alpha}\exp\left\{\frac{\alpha}{2}-1\right\}$. Similarly, we have found that the infimum of the product function $g(u)g(v)$ on the quadrant $R_2\cup R_4$ is equal to $g(r_1)g(r_2) = -\frac{2}{\alpha}\exp\left\{\frac{\alpha}{2}-1\right\}$. Now, using these values in Eq. (2.3), we get the feasible range of δ as

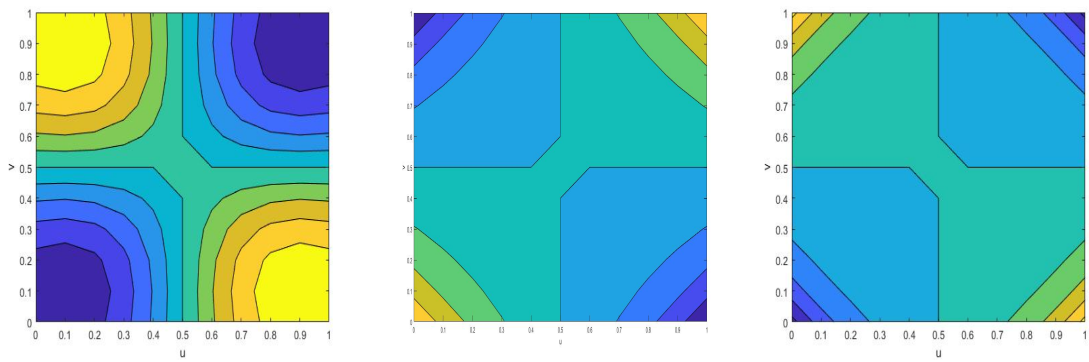
$$-\frac{1}{2\alpha}\exp\left\{1-\frac{\alpha}{2}\right\} \le \delta \le \frac{1}{2\alpha}\exp\left\{1-\frac{\alpha}{2}\right\}. \tag{2.5}$$

Now, the result follows from Eq. (2.4) and Eq. (2.5). □

Recall that our proposed copula reduced to the product copula when $\alpha = 0$. Also, the copula density $c(u,v) = \dfrac{\partial^2 C(u,v)}{\partial u \partial v}$ of the proposed copula is given by

$$c(u,v) = 1 + \alpha^2\delta\,(1-2u)\,(1-2v)\exp\left\{\alpha\left(u-u^2+v-v^2\right)\right\}, \quad (u,v)\in\mathbb{I}^2, \tag{2.6}$$

where $\alpha\in\mathbb{R}$ and $|\delta|\le\delta^\star(\alpha)$. We refer to the proposed copula as the **Exponentiated FGM** copula. The contour plots of the copula density are shown in Figure 2.1 for various choices of the parameters δ and α.



(A) $\alpha=-3, \delta=-0.1$ (B) $\alpha=2, \delta=0.2$ (C) $\alpha=3.8, \delta=-0.3$

FIGURE 2.1. Contour plots of copula density $c(u,v)$ for various values of α and δ.

2.3 Measures of Dependence

In this section, we derive several dependence measures for the Exponentiated FGM copula. We begin by focusing on various measures of association that quantify the strength and direction of dependence between the two variables. These measures quantify the degree of positive or negative dependence between the components of a bivariate distribution as captured by the copula. Each measure takes values in the range $[-1, 1]$, with negative values indicating negative dependence and positive values indicating positive dependence. The analytical expressions for these measures, expressed in terms of the copula function $C(u, v)$, are summarized in Table 2.1.

TABLE 2.1. Various measures of association in terms of copula function

Measure	**Expression**
Spearman's Rho ρ_c	$\rho_c = 12 \int_0^1 \int_0^1 C(u,v)\, du\, dv - 3$
Gini's Gamma γ_c	$\gamma_c = 4\left\{\int_0^1 C(u,1-u)\, du - \int_0^1 (u - C(u,u))\, du\right\}$
Kendall's Tau τ_c	$\tau_c = 4 \int_0^1 \int_0^1 \frac{\partial C(u,v)}{\partial u} \cdot \frac{\partial C(u,v)}{\partial v}\, du\, dv - 1$
Blest's Measure η_c	$\eta_c = 24 \int_0^1 \int_0^1 (1-u)\, C(u,v)\, du\, dv - 2$
Spearman's Footrule ϕ_c	$\phi_c = 6 \int_0^1 C(u,u)\, du - 2$

Now, we will provide the expressions of the various measures of association for the Exponentiated FGM copula function.

Proposition 2.3.1. *For the copula defined in Eq. (2.1), the Spearman's Rho ρ_C, and Gini's Gamma Coefficient γ_C are given by*

$$\rho_C = \begin{cases} 12\delta\left[1 - \sqrt{\frac{\pi}{|\alpha|}} e^{\alpha/4} erfi\left\{\frac{\sqrt{|\alpha|}}{2}\right\}\right]^2, & \text{if } \alpha < 0 \\ 12\delta\left[1 - \sqrt{\frac{\pi}{\alpha}} e^{\alpha/4} erf\left\{\frac{\sqrt{\alpha}}{2}\right\}\right]^2, & \text{if } \alpha > 0, \end{cases}$$

$$\gamma_C = \begin{cases} 8\delta\left[1 - 2\sqrt{\frac{\pi}{|\alpha|}} e^{\alpha/4} erfi\left\{\frac{\sqrt{|\alpha|}}{2}\right\} + \sqrt{\frac{\pi}{2|\alpha|}} e^{\alpha/2} erfi\left\{\sqrt{\frac{|\alpha|}{2}}\right\}\right], & \text{if } \alpha < 0 \\ 8\delta\left[1 - 2\sqrt{\frac{\pi}{\alpha}} e^{\alpha/4} erf\left\{\frac{\sqrt{\alpha}}{2}\right\} + \sqrt{\frac{\pi}{2\alpha}} e^{\alpha/2} erf\left\{\sqrt{\frac{\alpha}{2}}\right\}\right], & \text{if } \alpha > 0, \end{cases}$$

where $erf(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-z^2}dz$ *denotes the error function (see Abramowitz and Stegun (1972)) and* $erfi(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{z^2}dz$ *denotes the imaginary error function (see Marcinowski and Sadowski (2020)).*

Remark 2.3.2. *It can be verified that the expressions of other measures of dependence satisfied the following relation under the copula given in Eq. (2.1).*

$$\eta_c = \rho_c = \frac{3}{2}\tau_c, \quad \text{and} \quad \phi_c = \frac{3}{4}\gamma_c.$$

Table 2.2 presents the numerical values of Spearman's Rho and Gini's Gamma coefficient of the new copula for different values of the copula parameter α. Since the exponentiated FGM copula is symmetric, we have shown only the upper boundary values of the dependence measures in Table 2.2. The lower boundary value is the negative of upper boundary value. It is observed from the Table 2.2 that Spearman's Rho $\rho_c \in [-0.5866, 0.5866]$ when $\alpha = 3.8$, thereby extending the range of the dependence measure Spearman's Rho over the popular FGM copula and its various generalizations.

2.3.1 Comparison with FGM-type copulas

Here, we will compare the proposed copula in Eq. (2.1) with some FGM-type copulas, which are reported below. The various measures of dependence are reported in Table 2.3.

1. The FGM copula (Eyraud (1936), Morgenstern (1956), Gumbel (1960), Farlie (1960)) is given by

$$C(u,v) = uv\left[1 + \theta(1-u)(1-v)\right], \; -1 \leq \theta \leq 1.$$

2. Huang and Kotz (1999) proposed two extended FGM copulas:

$$C(u,v) = uv\left[1 + \theta\left(1-u^k\right)\left(1-v^k\right)\right], \; k > 0, \; -\min\left\{1, \frac{1}{k^2}\right\} \leq \theta \leq \frac{1}{k}, \tag{2.7}$$

$$C(u,v) = uv\left[1 + \theta\left(1-u\right)^q\left(1-v\right)^q\right], \; q > 1, \; -1 \leq \theta \leq \left(\frac{q+1}{q-1}\right)^{q-1}. \tag{2.8}$$

3. Bairamov and Kotz (2002) proposed the following copula function:

$$C(u,v) = uv\left(1 + \theta\left[(1-u^k)(1-v^k)\right]^q\right), \; k > 0, q \geq 1,$$

$$-\min\left\{1, \frac{1}{k^2}\left(\frac{kq+1}{k(q-1)}\right)^{2(q-1)}\right\} \leq \theta \leq \frac{1}{k}\left(\frac{kq+1}{k(q-1)}\right)^{q-1}.$$

TABLE 2.2. Sperman's Rho and Gini's Gamma coefficient for various values of α

α	$\delta^{\star}(\alpha)$	ρ_{upper}	γ_{upper}	α	$\delta^{\star}(\alpha)$	ρ_{upper}	γ_{upper}
-3	0.1111	0.1899	0.1463	1.2	0.6944	0.4264	0.3473
-2.7	0.1372	0.2003	0.1547	1.5	0.4444	0.4544	0.3718
-2.4	0.1736	0.2113	0.1638	1.8	0.3086	0.4845	0.3984
-2.1	0.2268	0.2231	0.1736	2	0.25	0.506	0.4174
-1.8	0.3086	0.2358	0.1841	2.3	0.1871	0.5348	0.4434
-1.5	0.4444	0.2493	0.1954	2.6	0.1425	0.5561	0.4634
-1.2	0.6944	0.2638	0.2076	2.9	0.1099	0.571	0.4783
-0.9	1.2346	0.2794	0.2207	3.2	0.0858	0.5805	0.4888
-0.6	2.7778	0.2961	0.2349	3.5	0.0675	0.5855	0.4956
-0.3	11.1111	0.314	0.2502	3.8	0.0535	**0.5866**	0.4992
0	0	0	0	4.1	0.0427	0.5845	**0.5002**
0.3	11.1111	0.3541	0.2845	4.4	0.0342	0.5798	0.499
0.6	2.7778	0.3764	0.3038	4.7	0.0276	0.5729	0.4959
0.9	1.2346	0.4005	0.3247	5	0.0223	0.5643	0.4912

4. Bekrizadeh et al. (2015) has discussed the generalized FGM copula given by

$$\begin{aligned} C(u,v) =& uv\left[1+\theta(1-u^a)(1-v^a)\right]^n, \\ & -\min\left\{1,\frac{1}{na^2}\right\} \leq \theta \leq \frac{1}{na}, a>0, n\geq 1. \end{aligned}$$

5. Pathak and Vellaisamy (2016a) have extended the copula proposed by Bairamov and Kotz (2002) given by

$$C(u,v) = uv\left[\left(1+\theta\left[(1-u^k)(1-v^k)\right]^q\right)\right]^n,\ k>0, q\geq 1, n\in\mathbb{N},$$

$$-\min\left\{1,\frac{1}{n^2k^2}\left(\frac{nkq+1}{nk(q-1)}\right)^{2(q-1)}\right\} \leq \theta \leq \frac{1}{nk}\left(\frac{nkq+1}{nk(q-1)}\right)^{q-1}.$$

6. Chesneau (2022) proposed the polynomial-sine copula:

$$C(u,v) = uv + \frac{\theta}{\pi^2 ab}\left(\sin(\pi u)\right)^a\left(\sin(\pi u)\right)^b,\ \theta\in[-1,1], a,b\geq 1.$$

From Table 2.3, it is evident that the proposed copula exhibits a wider dependence range in comparison to the FGM-type copulas. With just two parameters, the proposed copula offers a broader range of dependencies, making it a more suitable choice for modelling bivariate datasets characterized by stronger dependence structures than the other FGM-type copulas.

In order to continue our discussion on dependence between random variables, there are some more dependence properties available in the literature. For example, quadrant dependence, totally positive of order 2 (TP_2), and tail dependence coefficient. For more detailed discussion on these properties, one can see Lehmann (1966), Barlow and Proschan (1975), Drouet Mari and Kotz (2001), Nelsen (2006) and Lai and Balakrishnan (2009). Now, we will discuss these dependence properties of Exponentiated FGM copula.

2.3.2 Quadrant Dependence

A copula $C(u, v)$ is said to be positively (negatively) quadrant dependent if

$$C(u, v) \geq (\leq)\ uv, \quad \forall (u, v) \in \mathbb{I}^2.$$

It can be verified that the quadrant dependence of the Exponentiated FGM copula defined in Eq.(2.1) depends only on the copula parameter δ. The proof is straightforward, so omitted. Thus, we have the following result.

Proposition 2.3.3. *The Exponentiated FGM copula has positive (negative) quadrant dependence if* $\delta \geq 0 \quad (\delta \leq 0)$*.*

2.3.3 Totally Positive of Order 2 (TP_2)

TP_2 property is one of the strongest notion of dependence. If a copula density $c(u, v)$ possesses the property TP_2, then the associated copula $C(u, v)$ has stochastic increasing properties (SI), right tail increasing properties (RTI), and positive quadrant dependence (PQD).

Holland and Wang (1987) proved that the function $g(x, y)$ has TP_2 property if

$$\zeta_g(x, y) = \frac{\partial^2 \ln g(x, y)}{\partial x \partial y} \geq 0, \quad \forall (x, y) \in \mathbb{R}^2.$$

Now, using the result of Holland and Wang (1987), we will prove the following result.

Table 2.3. Maximal dependence range values of some FGM-type copulas

Copula	**Dependency Measures**				
	τ_C	ρ_C	γ_C	ϕ_C	η_C
Farlie (1960)	$(-0.2222, 0.222)$	$(-0.3333, 0.3333)$	$(-0.26667, 0.26667)$	$(-0.2, 0.2)$	$(-0.3333, 0.3333)$
Huang and Kotz (1999) [Eq.(2.7)]	$(-0.2222, 0.25)$	$(-0.3333, 0.375)$	$(-0.26667, 0.3001)$	$(-0.2, 0.2287)$	$(-0.3333, 0.3542)$
Huang and Kotz (1999) [Eq. (2.8)]	$(-0.2222, 0.2608)$	$(-0.3333, 0.391)$	$(-0.2667, 0.31761)$	$(-0.2, 0.2401)$	$(-0.2667, 0.3176)$
Bairamov and Kotz (2002)	$(-0.32, 0.334)$	$(-0.48, 0.502)$	$(-0.4059, 0.4244)$	$(-0.3047, 0.3204)$	$(-0.4887, 0.4922)$
Bekrizadeh et al. (2015)	$(-0.2821, 0.3398)$	$(-0.4958, 0.4212)$	$(-0.3938, 0.3404)$	$(-0.2770, 0.2595)$	$(-0.5046, 0.41022)$
Pathak and Vellaisamy (2016a)	$(-0.3345, 0.3584)$	$(-0.48, 0.5308)$	$(-0.406, 0.4244)$	$(-0.3047, 0.3433)$	$(-0.4887, 0.5313)$
Chesneau (2022)	$(-0.3285, 0.3285)$	$(-0.4927, 0.4927)$	$(-0.4052, 0.4052)$	$(-0.3039, 0.3039)$	$(-0.4927, 0.4927)$
Exponentiated FGM Copula [Eq. (2.1)]	$(-0.3910, 0.3910)$	$(-0.5866, 0.5866)$	$(-0.5002, 0.5002)$	$(-0.3752, 0.3752)$	$(-0.5866, 0.5866)$

Proposition 2.3.4. *The copula density of Exponentiated FGM copula defined in Eq. (2.6) has TP_2 property if $\delta \geq 0$ and $\alpha \leq 2$.*

Proof. We have,

$$\begin{aligned}\zeta_c(u,v) =& \frac{\partial^2 \ln c(u,v)}{\partial x \partial y} \\ =& \frac{\delta\alpha^2\left[2-\alpha(1-2u)^2\right]\left[2-\alpha(1-2v)^2\right]\exp\left\{\alpha\left((u-u^2)+(v-v^2)\right)\right\}}{\left(1+\delta\alpha^2\left(1-2u\right)\left(1-2v\right)\exp\left\{\alpha\left((u-u^2)+(v-v^2)\right)\right\}\right)^2} \geq 0,\end{aligned}$$

for every $\delta \geq 0$ and $\alpha \leq 2$. □

2.3.4 Tail Dependence Coefficients

Tail dependence coefficients quantify the dependence in the joint lower and upper tails of the distribution. For a copula $C(u,v)$, the lower and upper tail dependence coefficients are defined as

$$\lambda_L^C = \lim_{u\to 0^+}\frac{C(u,u)}{u}, \quad \lambda_U^C = \lim_{u\to 1^-}\frac{1-2u+C(u,u)}{1-u}. \tag{2.9}$$

These coefficients lie in $[0,1]$. A copula has lower (upper) tail dependence if $\lambda_L^C > 0$ ($\lambda_U^C > 0$); otherwise, it exhibits tail independence. Now, we will prove the following result.

Proposition 2.3.5. *The Exponentiated FGM copula defined in Eq. (2.1) has no tail dependence.*

Proof. Using Eq. (2.9), we have

$$\begin{aligned}\lambda_L^C =& \lim_{u\to 0^+}\frac{u^2+\delta\left(1-e^{\alpha(u-u^2)}\right)^2}{u} \\ =& \delta\lim_{u\to 0^+}\frac{1}{u}\left(1-\sum_{n=0}^{\infty}\frac{\alpha^n u^n(1-u)^n}{n!}\right)^2 \\ =& \delta\lim_{u\to 0^+}\alpha^2 u(1-u)^2\left(\sum_{n=2}^{\infty}\frac{\alpha^{n-1}u^{n-1}(1-u)^{n-1}}{n!}\right)^2 \\ =& 0.\end{aligned}$$

$$\lambda_U^C = \lim_{u\to 1^-} \frac{1-2u+u^2+\delta\left(1-e^{\alpha(u-u^2)}\right)^2}{1-u}$$
$$= \lim_{u\to 1^-} \frac{(1-u)^2+\delta\left(1-\sum_{n=0}^{\infty}\frac{\alpha^n u^n(1-u)^n}{n!}\right)^2}{(1-u)}$$
$$= \delta \lim_{u\to 1^-} \alpha^2 u^2 (1-u)\left(\sum_{n=2}^{\infty}\frac{\alpha^{n-1}u^{n-1}(1-u)^{n-1}}{n!}\right)^2$$
$$= 0.$$

□

In the next section, we will develop a bivariate Rayleigh distribution as an application of the proposed Exponentiated FGM copula. We will study some statistical properties of new bivariate Rayleigh distribution, and a real data analysis involving the new distribution is also presented.

2.4 A New Bivariate Rayleigh distribution

Rayleigh distribution is one of the most popular models in medical sciences, engineering, particle physics and economics. A random variable X follows Rayleigh distribution with parameter λ, denoted by $Rayleigh(\lambda)$, if its cumulative distribution function (CDF) is given by $F(x;\lambda) = 1-e^{-x^2/2\lambda^2}$, $x>0, \lambda>0$, and corresponding probability density function (PDF) is given by $f(x;\lambda) = \frac{x}{\lambda^2}e^{-x^2/2\lambda^2}$, $x>0,\lambda>0$. Let X and Y be two random variables having $Rayleigh(\lambda_1)$ and $Rayleigh(\lambda_2)$ distributions respectively, and the dependence between X and Y is modelled by the Exponentiated FGM copula in Eq. (2.1). Then, the joint distribution function of X and Y is given by

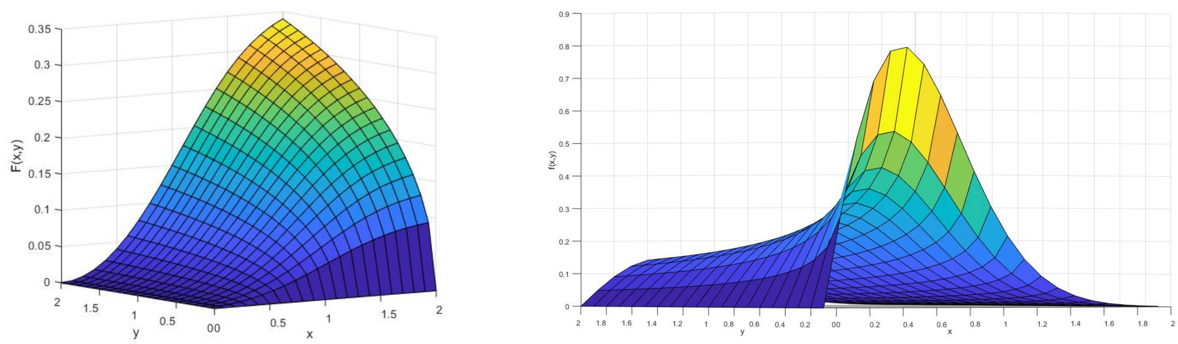
$$F(x,y;\Theta) = \left(1-e^{-x^2/2\lambda_1^2}-e^{-y^2/2\lambda_2^2}+e^{-\left(x^2/2\lambda_1^2+y^2/2\lambda_2^2\right)}\right)+ \delta\left(1-e^{\alpha\left(e^{-x^2/2\lambda_1^2}-e^{-x^2/\lambda_1^2}\right)}\right)\left(1-e^{\alpha\left(e^{-y^2/2\lambda_2^2}-e^{-y^2/\lambda_2^2}\right)}\right), \quad (2.10)$$

where $x>0, y>0, \lambda_1>0, \lambda_2>0, \alpha\in\mathbb{R}, |\delta|\le\delta^\star(\alpha)$ and $\Theta=(\lambda_1,\lambda_2,\alpha,\delta)$. A non-negative random vector (X,Y) is said to follow bivariate Rayleigh distribution with parameters $\lambda_1,\lambda_2,\alpha$ and δ, if its joint CDF is given by Eq. (2.10) and is denoted by

BRD$(\lambda_1, \lambda_2, \alpha, \delta)$. The corresponding joint density function is given by

$$f(x,y;\Theta) = \left(\frac{xy}{\lambda_1^2\lambda_2^2}e^{-\left(x^2/2\lambda_1^2+y^2/2\lambda_2^2\right)}\right)\left[1+\delta\alpha^2\left(2e^{-x^2/2\lambda_1^2}-1\right)\left(2e^{-y^2/2\lambda_2^2}-1\right)\right.$$
$$\left.\left(\exp\left\{\alpha\left(e^{-x^2/2\lambda_1^2}-e^{-x^2/\lambda_1^2}+e^{-y^2/2\lambda_2^2}-e^{-y^2/\lambda_2^2}\right)\right\}\right)\right], \qquad (2.11)$$

where $x > 0, y > 0, \lambda_1 > 0, \lambda_2 > 0, \alpha \in \mathbb{R}, |\delta| \leq \delta^\star(\alpha)$ and $\Theta = (\lambda_1, \lambda_2, \alpha, \delta)$. Surface plots of joint CDF (2.10) and joint density function (2.11) of the BRD family are shown in Figure 2.2. These figures are constructed using MATLAB R2021b. Now, we will provide



(A) (B)

FIGURE 2.2. Surface plots of $F(x,y)$ and $f(x,y)$ of the BRD distribution for $\lambda_1 = 3,\ \lambda_2 = 2,\ \delta = 0.5,\ \alpha = 3.8$.

expressions for conditional distribution and product moments.

Proposition 2.4.1. *Let $(X,Y) \sim BRD(\lambda_1, \lambda_2, \alpha, \delta)$. Then*

1. *the conditional density function of X given $Y = y$ is*

$$f(x|y) = \left(\frac{x}{\lambda_1^2}e^{-x^2/2\lambda_1^2}\right)\left[1+\delta\alpha^2\left(2e^{-x^2/2\lambda_1^2}-1\right)\left(2e^{-y^2/2\lambda_2^2}-1\right)\right.$$
$$\left.\left(e^{\alpha\left(e^{-x^2/2\lambda_1^2}-e^{-x^2/\lambda_1^2}+e^{-y^2/2\lambda_2^2}-e^{-y^2/\lambda_2^2}\right)}\right)\right],$$

2. *the conditional distribution function of X given $Y = y$ is*

$$F(x|y) = \left(1-e^{-x^2/2\lambda_1^2}\right)+\delta\alpha\left(e^{\alpha\left(e^{-x^2/2\lambda_1^2}-e^{-x^2/\lambda_1^2}\right)}-1\right)$$
$$\left(2e^{-y^2/2\lambda_2^2}-1\right)\left(e^{\alpha\left(e^{-y^2/2\lambda_2^2}-e^{-y^2/\lambda_2^2}\right)}\right),$$

where $x > 0, y > 0, \lambda_1 > 0, \lambda_2 > 0, \alpha \neq 0, |\delta| \leq \delta^\star(\alpha)$ and $\Theta = (\lambda_1, \lambda_2, \alpha, \delta)$.

Proposition 2.4.2. *Let* $(X, Y) \sim BRD(\lambda_1, \lambda_2, \alpha, \delta)$. *Then* (r, s)*-th order product moments can be expressed as*

$$
\begin{aligned}
E(X^r Y^s) = {} & \lambda_1^r \lambda_2^s 2^{(r+s)/2} \Gamma\left(1 + \frac{r}{2}\right) \Gamma\left(1 + \frac{s}{2}\right) \\
& + \delta\alpha^2 \left(\sum_{k=0}^{\infty} \alpha^k \sum_{t=0}^{k} \frac{(-1)^t}{\lambda_1^2} \binom{k}{t} \Gamma\left(\frac{r+2}{2}\right) \right. \\
& \times \left. \left(\left(\frac{2\lambda_1^2}{k+t+2} \right)^{(r+2)/2} - \frac{1}{2} \left(\frac{\lambda_1^2}{k+t+1} \right)^{(r+2)/2} \right) \right) \\
& \times \left(\sum_{k=0}^{\infty} \alpha^k \sum_{t=0}^{k} \frac{(-1)^t}{\lambda_2^2} \binom{k}{t} \Gamma\left(\frac{s+2}{2}\right) \right. \\
& \times \left. \left(\left(\frac{2\lambda_2^2}{k+t+2} \right)^{(s+2)/2} - \frac{1}{2} \left(\frac{\lambda_2^2}{k+t+1} \right)^{(s+2)/2} \right) \right).
\end{aligned}
$$

where $\Gamma(t)$ *denotes the well-known Gamma function.*

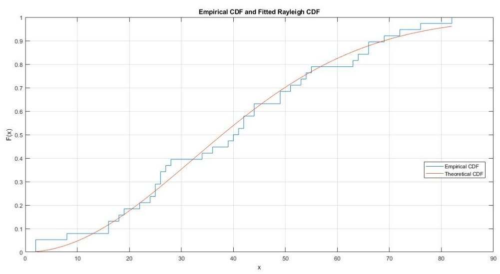
2.4.1 Real Data Application

We consider the UEFA Champions League data set from 2004 to 2006, reported in Meintanis (2007). In this data set, X and Y represent the time (in minutes) of the first goal scored by Team-A and Team-B, respectively. The table is presented in Table 7.5 in the Appendix section for reference. Before fitting the bivariate distribution, we first conduct exploratory data analysis. The basic descriptive statistics and measures of dependence, namely Pearson's correlation, Spearman's correlation, and Kendall's Tau, are presented in Table 2.4.

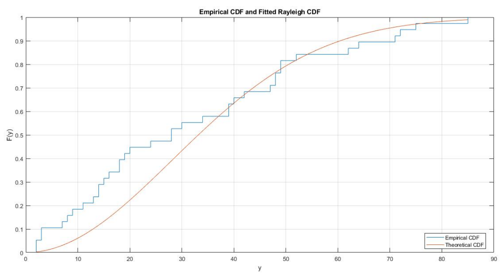
To check whether the marginal distributions of X and Y support the Rayleigh distribution, we perform Kolmogorov-Smirnov (KS) one-sample test. The results of the KS test suggest that X supports the Rayleigh distribution with parameter $\hat{\lambda_1} = 32.14599$ (p-value $= 0.934$ and statistic value of KS $= 0.088515$). Similarly, Y also supports Rayleigh distribution with parameter $\hat{\lambda_2} = 28.172255$ (p-value=0.07727 and KS statistic value=0.20968). These results can be verified graphically using the Figure 2.3. Now, we fit the proposed bivariate Rayleigh distribution, and the results are shown in Table 2.5. We compare the new BRD model with Marshall Olkin's bivariate exponential distribution (BMOED) by Meintanis (2007), bivariate generalized exponential distribution (BGED) by Mirhosseini et al. (2015), and bivariate generalized Rayleigh distribution (BGRD) proposed by Pathak

TABLE 2.4. Descriptive statistics and measures of dependence of the UEFA Champions League data.

Statistics	X		Y
Minimum	2.00		2.00
Maximum	82.00		85.00
1st Quantile	25.00		14.00
Mean	40.89		32.86
Median	41.00		28.00
3rd Quantile	54.00		48.00
Skewness	0.1712		0.5444
Kurtosis	2.1868		2.2825
Standard deviation	19.8641		22.5222
Pearson's correlation		0.4698	
Spearman's rho		0.4075	
Kendall's tau		0.3111	



(A) Fitted CDF plot of X



(B) Fitted CDF plot of Y

FIGURE 2.3. Fitted CDF plots of the UEFA Champions League Football data.

and Vellaisamy (2022). We use the log-likelihood (LL) function, Akaike Information Criteria (AIC) and Bayesian Information Criteria (BIC) as the comparison criteria. The formulas for AIC and BIC are given by

$$\mathrm{AIC} = 2k - 2\ln L, \quad \text{and} \quad \mathrm{BIC} = k\ln n - 2\ln L,$$

where k is the number parameters in the model, n is the sample size and L is the maximum value of the likelihood function. From Table 2.5, it is clear that bivariate Rayleigh distribution provides a better fit over BGED, BMOED and BGRD for the UEFA champions league data set.

TABLE 2.5. ML estimates, LL, AIC, and BIC values for the bivarite distributions using UEFA Champion's League data set.

Bivariate Distribution	ML Estimates	LL	AIC	BIC
BGED	$\hat{\alpha}_1 = 0.0244, \hat{\alpha}_2 = 0.0304, \hat{\theta} = 0.999$	-340.5234	687.0468	691.8795
BMOED	$\hat{\lambda}_1 = 0.012, \hat{\lambda}_2 = 0.014, \hat{\lambda}_3 = 0.022$	-339.006	684.012	688.8448
BGRD	$\hat{b}_1 = 0.000530, \hat{b}_2 = 0.000836, \hat{\theta} = 0.40331$	-331.879	664.589	672.6436
BRD	$\hat{\lambda}_1 = 33.39429, \hat{\lambda}_2 = 28.08949, \hat{\delta} = 10.39829, \hat{\alpha} = 0.2871858$	-327.256	664.512	668.9557

2.5 Conclusion and Future Direction

This chapter proposes a new bivariate symmetric copula exhibiting positive and negative dependence. The main features of the copula are: (i) it has a simple mathematical structure, (ii) it has a wider dependence range when compared to FGM copula and its generalizations, and (iii) there is no lower and upper tail dependence. Using the proposed copula, we developed a new bivariate Rayleigh distribution (BRD) and discussed some statistical properties. The proposed bivariate model provides a better fit for a real data set. Since we considered only the symmetric version of the bivariate copula, the asymmetric version is still an open problem for new researchers.

Chapter 3

A New Family of Copulas Based on Probability Generating Functions

This chapter introduces a new class of copulas constructed using the probability generating function of a positive-integer-valued random variable. Expressions for various dependence measures and concavity properties of the copula are examined, and an algorithm for generating random numbers from the proposed copula is presented.

3.1 Introduction

In the last few decades, copulas have received significant attention for modelling dependent data. However, many existing copulas in the literature are still not suitable for capturing complex dependence structures. For instance, the Gaussian copula is limited to modelling linear relationships and weak tail dependence. Therefore, it is necessary to develop methodologies for constructing new families of copulas that are flexible in modelling complex relationships.

A bivariate copula is a bivariate function $C : \mathbb{I}^2 \to \mathbb{I}$ satisfying the following conditions:

$$C(u, 0) = C(0, u) = 0; \; C(u, 1) = u = C(1, u), \;\; \forall u \in \mathbb{I}, \tag{3.1}$$

A part of this chapter has been published in the journal *Mathematica Slovaca*, 74(4) (2024): 1039–1060. DOI: 10.1515/ms-2024-0076

and

$$C(u_2, v_2) + C(u_1, v_1) \geq C(u_1, v_2) + C(u_2, v_1), \;\; u_1 < u_2, \;\; v_1 < v_2, \;\; u_1, u_2, v_1, v_2 \in \mathbb{I}. \quad (3.2)$$

As a consequence of the 2-increasing property, every bivariate copula C satisfies the inequality

$$C(u_2, v_2) \geq C(u_1, v_1), \quad (3.3)$$

for every $u_1 < u_2$ and $v_1 < v_2$ in $\mathbb{I}$.

From a practical point of view, proposing a new bivariate function that satisfies the copula conditions is often a cumbersome task. One of the most popular techniques for constructing a family of copulas is through the use of an **Archimedean generator**. An Archimedean generator is a convex, strictly decreasing, and continuous function $\varphi : \mathbb{I} \to (0, \infty)$, satisfying $\varphi(1) = 0$ and $\varphi(0) \leq \infty$. Using such a generator φ, one can define a family of bivariate copulas, known as *Archimedean copulas*, by

$$C(u, v) = \varphi^{(-1)} \left(\varphi(u) + \varphi(v)\right), \quad \forall u, v \in \mathbb{I},$$

where the pseudo-inverse $\varphi^{(-1)}$ is given by

$$\varphi^{(-1)}(s) = \begin{cases} \varphi^{-1}(s), & \text{if } 0 \leq s \leq \varphi(0), \\ 0, & \text{if } \varphi(0) \leq s < \infty. \end{cases}$$

For further details, one may refer to Drouet Mari and Kotz (2001), Nelsen (2006), Trivedi et al. (2007), and Chamizo et al. (2021).

Most of the existing construction methods impose restrictive conditions that are often difficult to verify or implement in practice. Therefore, a more feasible approach is to construct new copula families by modifying existing ones, aiming to enhance dependence properties relative to a baseline copula. This chapter is an attempt in that direction.

Various approaches are available in the literature for constructing new copulas from existing ones. For references, see p. 95 of Nelsen (2006), Kim et al. (2011) and Morillas (2005). Recently, Dolati et al. (2014) proposed a new class of copulas using the discrete Mittag-Leffler distribution's probability generating function (PGF). This motivates us to generalise the idea of Dolati et al. (2014) to the arbitrary PGF of a positive integer-valued random variable. This work is more general and generalizes several results of Dolati et al. (2014). The main contributions of this chapter are summarized as follows.

- A new class of copulas is proposed using the probability generating function of a positive-integer-valued random variable.
- An algorithm for generating random numbers from the PGF copula is presented.
- Expressions for various dependence measures of the proposed copula family are derived.
- A sufficient condition is established for the weighted geometric mean of the proposed copula family to be a valid copula.
- Bivariate concavity properties, such as Schur concavity and quasi-concavity, associated with the PGF copula are investigated.
- Two new generalized FGM copulas are introduced using the PGFs of geometric and discrete Mittag-Leffler distributions. The proposed copulas improve the Spearman's Rho of the classical FGM copula from -0.3333 to 0.4751 and 0.9573, respectively.

The chapter is organised as follows. In Section 3.2, we proposed a methodology for constructing a new family of copulas from existing copulas by using the probability generating functions as generators. In Section 3.3, we proposed an algorithm for generating random numbers from the PGF copula. In Section 3.4, various measures of stochastic dependence such as positive quadrant dependence, total positive of order 2 property, tail dependence coefficients and some measures of association such as Kendall's Tau, Spearman's Rho and Blomqvist's Beta coefficients are studied. Weighted geometric mean of a PGF copula is studied in Section 3.5. Schur concavity and quasi concavity properties associated with copula are discussed in Section 3.6. Finally, in Section 3.7, two generalized FGM copulas are proposed using PGFS of geometric and discrete Mittag-Leffler distributions.

3.2 New Class of Bivariate Copulas derived from Probability Generating Functions

Let N be a positive integer-valued random variable (RV) with probability mass function (PMF) given by $P_n = P(N = n), n \in \mathbb{N}$ (set of natural numbers), then PGF of N can be defined as

$$\gamma(t) = \mathbb{E}(t^N) = \sum_{n=1}^{\infty} t^n P_n, \quad t \in \mathbb{I}. \tag{3.4}$$

Note that $\gamma(t)$ exists for all values of $t \in \mathbb{I}$. If $P_1 = P(N=1) > 0$, and $P_0 = P(N=0) = 0$ (i.e., $\gamma(0) = 0$), then $\gamma(t)$ is a strictly increasing function on $\mathbb{I}$ and $\gamma^{-1}(t)$, inverse of γ, exists for every $t \in \mathbb{I}$. Throughout this chapter, we assume that the support of N maybe $\mathbb{N}$ or any finite or infinite subsets of $\mathbb{N}$ with $P_1 = P(N=1) > 0$ unless explicitly stated otherwise. Some examples of PGFs for positive integer-valued random variables are presented in Table 3.1. For further details on these PGFs, we refer to Harris (1948), Pillai and Jayakumar (1995) and Johnson et al. (2005).

Table 3.1. PGFs of some positive integer-valued random variables

Distribution	**PGF**	**Inverse PGF**
Zero-Truncated Poisson	$\gamma(t) = \dfrac{e^{\lambda t}-1}{e^{\lambda}-1},\ \lambda > 0$	$\gamma^{-1}(t) = \dfrac{1}{\lambda}\ln\left[1+t(e^{\lambda}-1)\right]$
Geometric	$\gamma(t) = \dfrac{pt}{1-(1-p)t},\ p \in (0,1]$	$\gamma^{-1}(t) = \dfrac{t}{(1-t)p+t}$
Logarithmic	$\gamma(t) = \dfrac{\ln(1-\mu t)}{\ln(1-\mu)},\ \mu \in (0,1)$	$\gamma^{-1}(t) = \dfrac{1-(1-\mu)^t}{\mu}$
Harris	$\gamma(t) = \left(\dfrac{\delta t^k}{1-(1-\delta)t^k}\right)^{1/k},\quad k>0,\ \delta \in (0,1)$	$\gamma^{-1}(t) = \left(\dfrac{t^k}{\delta+(1-\delta)t^k}\right)^{1/k}$
Discrete Mittag-Leffler	$\gamma(t) = 1-(1-t)^{\alpha},\ \alpha \in (0,1)$	$\gamma^{-1}(t) = 1-(1-t)^{1/\alpha}$

Let $(U_1, V_1), (U_2, V_2), \ldots$ be a sequence of pairwise independent and identically distributed random pairs from (U, V) with marginals are uniformily distributed over $\mathbb{I} = [0,1]$. Suppose that the joint distribution function (DF) of (U, V), in fact copula, is denoted by $C(u, v)$. Let N be a positive integer-valued RV independent of $(U_1, V_1), (U_2, V_2), \ldots$, having PMF, $P_n = P(N=n), n \in \mathbb{N}$ and PGF γ defined in Eq. (3.4). Define

$$X_N = \max\{U_1, U_2, \ldots, U_N\} \text{ and } Y_N = \max\{V_1, V_2, \ldots, V_N\}.$$

The joint DF of (X_N, Y_N) is given by

$$\begin{aligned} F(x,y) &= \sum_{n=1}^{\infty} P(X_N \le x, Y_N \le y | N=n) P_n \\ &= \sum_{n=1}^{\infty} P(U_1 \le x, \ldots, U_n \le x, V_1 \le y, \ldots, V_n \le y) P_n \\ &= \sum_{n=1}^{\infty} \left(P(U \le x, V \le y)\right)^n P_n \\ &= \sum_{n=1}^{\infty} \left(C(x,y)\right)^n P_n \\ &= \gamma\left(C(x,y)\right). \end{aligned}$$

Since $\lim_{y\to 1} F(x,y) = \gamma(C(x,1)) = \gamma(x)$ and $\lim_{x\to 1} F(x,y) = \gamma(C(1,y)) = \gamma(y)$, it follows that marginals of X and Y are $\gamma(x)$ and $\gamma(y)$. By Sklar's theorem, there exists a unique copula, denoted by $C^\gamma(u,v)$ satisfying

$$F(x,y) = \gamma(C(x,y)) = C^\gamma(\gamma(x), \gamma(y)).$$

Therefore, the underlying copula corresponds to the joint distribution of X and Y is given by

$$C^\gamma(u,v) = \gamma\left(C(\gamma^{-1}(u), \gamma^{-1}(v))\right). \tag{3.5}$$

We call $C^\gamma(u,v)$ as the PGF copula of $C(u,v)$ derived from γ. Following examples are some existing class of copulas which are in fact the sub-families of PGF copula.

Example 3.2.1. *Let $\Pi(u,v) = uv$ denote the product copula, which corresponds to independence of U and V. If $\gamma(t) = \frac{e^{\lambda t}-1}{e^{\lambda}-1}$ (PGF of Zero-truncated Poisson), then*

$$\Pi^\gamma(u,v) = \delta^{-1}\left(\exp\left\{\frac{1}{\ln(1+\delta)}(\ln[1+\delta u])(\ln[1+\delta v])\right\} - 1\right).$$

This copula is well-known Frank copula with dependency parameter $\delta = e^\lambda - 1$.

Example 3.2.2. *Let D be a degenerated RV with $P_d = P(D = d) = 1$ for some $d \in \mathbb{N}$, then $\gamma(t) = t^d$, for every $t \in \mathbb{I}$. For any bivariate copula $C(u,v)$, the PGF copula of $C(u,v)$ derived from the PGF of degenerated RV D is $C^\gamma(u,v) = \left[C(u^{1/d}, v^{1/d})\right]^d$ (Nelsen (2006)). Several authors used this copula to generalize the base copula $C(u,v)$. For example, if $C(u,v) = uv(1+\theta(1-u)(1-v))$; $\theta \in \mathbb{I}$, the well-known FGM copula, then $C^\gamma(u,v) = \left[(uv)^{1/d}\left(1+\theta(1-u^{1/d})(1-v^{1/d})\right)\right]^d$ and it was proposed by Bayramoglu and Bayramoglu (2014). Pathak and Vellaisamy (2016b) proposed the copula $C^\gamma(u,v) = [uv(1+\theta(1-u^\alpha)(1-v^\alpha))]^d$ by considering $C(u,v) = uv\left(1+\theta(1-u^{d\alpha})(1-v^{d\alpha})\right)$ as a baseline copula.*

Example 3.2.3. *Consider the PGF of geometric distribution $\gamma(t) = \frac{pt}{1-(1-p)t}$ and let $C(u,v)$, be any bivariate copula, then the corresponding PGF copula is*

$$C^\gamma(u,v) = \frac{pC\left(\frac{u}{(1-u)p+u}, \frac{v}{(1-v)p+v}\right)}{1-(1-p)C\left(\frac{u}{(1-u)p+u}, \frac{v}{(1-v)p+v}\right)}. \tag{3.6}$$

This copula was proposed by Marshall and Olkin (1997). If we replace $C(u,v)$ in Eq. (3.6) by product copula $\Pi(u,v)$. Then the corresponding PGF copula is

$$\Pi^{\gamma}(u,v)=\frac{uv}{1-(1-p)(1-u)(1-v)}.$$

This copula is popularly known as Ali-Mikhail-Haq (AMH) copula.

Example 3.2.4. *Consider the PGF of discrete Mittag-Leffler distribution $\gamma(t)=1-(1-t)^{\alpha},\alpha\in(0,1)$, proposed by Pillai and Jayakumar (1995). Let $C(u,v)$ be any bivariate copula. Then $C^{\gamma}(u,v)=1-\left(1-C\left(1-(1-u)^{1/\alpha},1-(1-v)^{1/\alpha}\right)\right)^{\alpha}$, which was proposed by Dolati et al. (2014).*

Example 3.2.5. *If γ is any arbitrary PGF of a positive integer-valued RV and $\Pi(u,v)=uv$, denotes the product copula, then $\Pi^{\gamma}(u,v)=\gamma\left(\gamma^{-1}(u)\gamma^{-1}(v)\right)$. This copula was proposed by Alhadlaq and Alzaid (2020).*

Proposition 3.2.1. *Let $Z_N=\min\{U_1,U_2,\ldots,U_N\}$ and $W_N=\min\{V_1,V_2,\ldots,V_N\}$, then survival copula of (Z,W) is $\hat{C}^{\gamma}(u,v)=\gamma\left(\hat{C}(\gamma^{-1}(u),\gamma^{-1}(v))\right)$, where $\hat{C}(u,v)$ is the survival copula of (U,V).*

Proof. The joint survival function $S(z,w)$ of (Z_N,W_N) is given by

$$\begin{aligned}S(z,w)=&\sum_{n=1}^{\infty}P(Z_N\geq z,W_N\geq w|N=n)P_n\\=&\sum_{n=1}^{\infty}\left(P(U\geq z,V\geq w)\right)^nP_n\\=&\sum_{n=1}^{\infty}\left(\hat{C}(1-z,1-w)\right)^nP_n\\=&\gamma\left(\hat{C}(1-z,1-w)\right).\end{aligned}$$

It follows from Sklar's theorem that there exists a survival copula $\hat{C}^{\gamma}(u,v)$ satisfying $S(z,w)=\hat{C}^{\gamma}\left(\gamma(1-z),\gamma(1-w)\right)$. Then, the survival PGF copula can be written as $\hat{C}^{\gamma}(u,v)=\gamma\left(\hat{C}(\gamma^{-1}(u),\gamma^{-1}(v))\right)$. □

Remark 3.2.2. *Consider the PGF of discrete Mittag-Leffler distribution $\gamma(t)=1-(1-t)^{\alpha}$, then $\hat{\Pi}^{\gamma}(u,v)=u+v-uv\left[u^{-1/\alpha}+v^{-1/\alpha}-1\right]^{\alpha},\alpha\in(0,1)$. A concrete study on this copula were reported by Mirhosseini et al. (2015), Pathak and Vellaisamy (2022), Pathak et al. (2023), and Arshad et al. (2023).*

Proposition 3.2.3. *The minimum copula is invariant under the PGF transformation.*

Proof. Since γ^{-1} is strictly increasing and continuous function, it follows that for $u, v \in \mathbb{I}$, $u \leq v$ if and only if $\gamma^{-1}(u) \leq \gamma^{-1}(v)$. Thus, for the Fréchet-Hoeffding upper bound copula, $M(u,v) = \min\{u,v\}$, we have $M^\gamma(u,v) = \gamma\left(M(\gamma^{-1}(u), \gamma^{-1}(v))\right) = \gamma\left(\min(\gamma^{-1}(u), \gamma^{-1}(v))\right) = \min\{u,v\} = M(u,v)$. □

Remark 3.2.4. *Consider the Fréchet-Hoeffding lower bound copula*

$$W(u,v) = \max\{u+v-1, 0\}$$

and γ be any PGF, then the PGF copula $W^\gamma(u,v)$ need not be equal to $W(u,v)$.

Proposition 3.2.5. *The copula C^γ defined in Eq.(3.5) is an Archimedean copula if and only if C is Archimedean. The generator φ (up to a constant) is given by*

$$\varphi(\gamma^{-1}(s)) = \int_{\gamma^{-1}(s)}^{1} \delta(t)dt, \tag{3.7}$$

where the function $\delta : (0,1) \to (0,\infty)$ satisfies the relation $\delta(u)C_v = \delta(v)C_u$, for every $u, v \in \mathbb{I}$ with $C_u = \dfrac{\partial C(u,v)}{\partial u}$ and $C_v = \dfrac{\partial C(u,v)}{\partial v}$.

Proof. A copula C is Archimedean if and only if there exists a function $\delta : (0,1) \to (0,\infty)$ satisfying

$$\frac{C_u}{C_v} = \frac{\delta(u)}{\delta(v)}, \tag{3.8}$$

for every $u, v \in \mathbb{I}$ and the generator of C (upto a constant) is given by $\varphi(s) = \int_s^1 \delta(t)dt$ (see Drouet Mari and Kotz (2001)). Since γ is strictly increasing and differentiable function on $\mathbb{I}$, then $\dfrac{d\gamma^{-1}(t)}{dt} = \dfrac{1}{\gamma'(\gamma^{-1}(t))}$, for every $t \in \mathbb{I}$. Therefore,

$$\frac{C_u^\gamma}{C_v^\gamma} = \frac{\delta\left(\gamma^{-1}(u)\right)\left(\gamma'(\gamma^{-1}(u))\right)^{-1}}{\delta\left(\gamma^{-1}(v)\right)\left(\gamma'(\gamma^{-1}(v))\right)^{-1}},$$

where $C_u^\gamma = \dfrac{\partial C^\gamma(u,v)}{\partial u}$ and $C_v^\gamma = \dfrac{\partial C^\gamma(u,v)}{\partial v}$. Thus, C^γ is Archimedean if and only if C is Archimedean and the generator of C^γ is nothing but $\varphi(\gamma^{-1}(s))$. □

Following examples are some new class of Archimedean-PGF family of copulas with generators.

Example 3.2.6. *Consider the Clayton copula,* $C(u,v) = \left[u^{-\eta} + v^{-\eta} - 1\right]^{-1/\eta}$*, where* $\eta \in (0,\infty)$*. The Archimedean generator* $\varphi(t)$ *corresponds to Clayton copula is* $\varphi(t) = \eta^{-1}\left(t^{-\eta} - 1\right), t \in \mathbb{I}$*. Then, PGF copula corresponds to Clayton copula derived from PGF of geometric RV with parameter* p*, is given by*

$$C^{\gamma}(u,v) = \frac{p\left(\left[\left(\frac{u}{(1-u)p+u}\right)^{-\eta} + \left(\frac{v}{(1-v)p+v}\right)^{-\eta} - 1\right]^{-1/\eta}\right)}{1-(1-p)\left(\left[\left(\frac{u}{(1-u)p+u}\right)^{-\eta} + \left(\frac{v}{(1-u)p+v}\right)^{-\eta} - 1\right]^{-1/\eta}\right)}$$

and the Archimedean generator of this Clayton-PGF copula is

$$\varphi(t) = \eta^{-1}\left(\left(\frac{p^2(1-2t) + t^2p(1-p) + t(1+p)}{(1-t)\,pt + t^2}\right)^{\eta} - 1\right).$$

Example 3.2.7. *The Gumbel-Barnett copula, a member of Archimedean family, is defined as* $C(u,v) = uv\exp\{-\phi \ln u \ln v\}$*,* $\phi \in (0,1]$*, with generator function* $\varphi(t) = \ln\left(2t^{-\phi} - 1\right), t \in \mathbb{I}$*. Then the PGF Gumbel-Barnett copula corresponds to the PGF of logarithmic distribution with parameter* $\mu \in (0,1)$ *is given by*

$$C^{\gamma}(u,v) = \frac{1}{\ln\theta}\left[\ln\left(1 - \left(\frac{(1-\theta^u)(1-\theta^v)}{\mu}\right)\exp\left\{-\phi\ln\left(\frac{1-\theta^u}{\mu}\right)\ln\left(\frac{1-\theta^v}{\mu}\right)\right\}\right)\right],$$

where $\theta = 1-\mu$ *and the generator function is* $\varphi(t) = \ln\left(2\left(\frac{1-(1-\mu)^t}{\mu}\right)^{-\phi} - 1\right)$*,* $t \in \mathbb{I}$*.*

Example 3.2.8. *If the Archimedean generator is* $\varphi(t) = \ln\left(\frac{1-\eta(1-t)}{t}\right), \eta \in [-1,1), t \in \mathbb{I}$*, coreesponds to AMH copula. The AMH copula can be defined as* $C(u,v) = \frac{uv}{1-\eta(1-u)(1-v)}$*. The discrete Mittag-Leffler AMH copula is*

$$C^{\gamma}(u,v) = 1 - \left(1 - \left[\frac{\left(1-(1-u)^{1/\alpha}\right)\left(1-(1-v)^{1/\alpha}\right)}{1-\eta\left(1-\left(1-(1-u)^{1/\alpha}\right)\right)\left(1-\left(1-(1-v)^{1/\alpha}\right)\right)}\right]\right)^{\alpha},$$

where $0 < \alpha < 1$ *and the corresponding generator is*

$$\varphi(t) = \ln\left(\frac{1-\eta\left(1-\left(1-(1-t)^{1/\alpha}\right)\right)}{\left(1-(1-t)^{1/\alpha}\right)}\right), t \in \mathbb{I}.$$

Proposition 3.2.6. *For every* $u, v \in \mathbb{I}$*, the inequality* $C^{\gamma}(u,v) \geq (C(u,v))^{\mu}$ *holds, where* $\mu = \mathbb{E}(N)$ *denotes the expected value of the RV* N*.*

Proof. Since $\ln(\cdot)$ is a concave function, it follows from the Jensen's inequality that $\mathbb{E}\left(\ln(t^N)\right) \leq \ln\left(\mathbb{E}(t^N)\right)$. It implies that

$$\mu\ln(t) \leq \ln\left(\gamma(t)\right). \tag{3.9}$$

Substituting $t = C\left(\gamma^{-1}(u), \gamma^{-1}(v)\right)$ in Eq. (3.9), we have

$$\mu \ln(C\left(\gamma^{-1}(u), \gamma^{-1}(v)\right)) \leq \ln(C^{\gamma}(u, v)).$$

It is obvious that $\mu > 0$ and using the fact that $C(u, v) \leq C\left(\gamma^{-1}(u), \gamma^{-1}(v)\right)$ (see Eq. 3.3), we have

$$\ln\left(\gamma(t)\right) \geq \mu \ln\left(C(u, v)\right). \tag{3.10}$$

The result follows by exponentiating both sides of the Eq (3.10). □

Definition 3.2.7. *(Nelsen (2006)) Let C be a bivariate copula, then*

1. *C is said to be symmetric if $C(u, v) = C(v, u)$, for every $u, v \in \mathbb{I}$;*
2. *C is said to be associative if $C\left(C(u, v), w\right) = C\left(u, C(v, w)\right)$, for every $u, v, w \in \mathbb{I}$.*

Proposition 3.2.8. *The PGF copula C^{γ} is associative if and only if C is associative.*

Proof. Assume C^{γ} is associative, it implies that

$$C^{\gamma}\left(C^{\gamma}(u, v), w\right) = C^{\gamma}\left(u, C^{\gamma}(v, w)\right),$$

for every $u, v, w \in \mathbb{I}$. Since γ is a bijective function, it follows that

$$C\left(C\left(\gamma^{-1}(u), \gamma^{-1}(v)\right), \gamma^{-1}(w)\right) = C\left(\gamma^{-1}(u), C\left(\gamma^{-1}(v), \gamma^{-1}(w)\right)\right).$$

Thus C is associative. In a similar argument, one can easily prove the converse part. □

Remark 3.2.9. *The PGF copula C^{γ} is symmetric if and only if C is symmetric. This can be proved similar to the proof of Proposition 3.2.8.*

Proposition 3.2.10. *Let $\{\gamma_n : n \in \mathbb{N}\}$ be a sequence of PGFs of positive integer-valued RVs converges uniformly to the PGF γ and C be any bivariate copula. Let C^{γ_n} and C^{γ} be the PGF copulas of C derived from the PGFs γ_n and γ respectively. Then C^{γ_n} converges uniformly to C^{γ}.*

Proof. Since γ is a PGF, it follows that γ^{-1} is continuous and strictly increasing on $\mathbb{I}$. Therefore for every $t \in \mathbb{I}$ and for a given $\zeta > 0$, there exists an $\eta > 0$ satisfying

$$\left|\gamma^{-1}\left(t - \frac{\eta}{2}\right) - \gamma^{-1}(t)\right| < \frac{\zeta}{2} \text{ and } \left|\gamma^{-1}\left(t + \frac{\eta}{2}\right) - \gamma^{-1}(t)\right| < \frac{\zeta}{2}. \tag{3.11}$$

Given γ_n converges uniformly to γ, then for a given $\eta > 0$, there exists some $n_0 \in \mathbb{N}$ such that

$$|\gamma_n(t) - \gamma(t)| < \frac{\eta}{2} \tag{3.12}$$

for every $t \in \mathbb{I}$ and for every $n \geq n_0$. It implies that

$$\left|\gamma_n\left(\gamma^{-1}\left(t - \frac{\eta}{2}\right)\right) - \left(t - \frac{\eta}{2}\right)\right| < \frac{\eta}{2} \text{ and } \left|\gamma_n\left(\gamma^{-1}\left(t + \frac{\eta}{2}\right)\right) - \left(t + \frac{\eta}{2}\right)\right| < \frac{\eta}{2},$$

for every $n \geq n_0$. It follows that

$$\gamma_n\left(\gamma^{-1}\left(t - \frac{\eta}{2}\right)\right) < t < \gamma_n\left(\gamma^{-1}\left(t + \frac{\eta}{2}\right)\right),$$

for every $n \geq n_0$. Since γ_n^{-1} is strictly increasing function, it implies that

$$\gamma^{-1}\left(t - \frac{\eta}{2}\right) < \gamma_n^{-1}(t) < \gamma^{-1}\left(t + \frac{\eta}{2}\right), \quad \text{for every } n \geq n_0.$$

Using Eq. (3.11), we have for every $n \geq n_0$,

$$\gamma_n^{-1}(t) \in \left(\gamma^{-1}\left(t - \frac{\eta}{2}\right), \gamma^{-1}\left(t + \frac{\eta}{2}\right)\right) \subset \left(\gamma^{-1}(t) - \frac{\zeta}{2}, \gamma^{-1}(t) + \frac{\zeta}{2}\right),$$

for every $n \geq n_0$. Therefore, we can conclude that

$$\left|\gamma_n^{-1}(t) - \gamma^{-1}(t)\right| < \frac{\zeta}{2}, \quad \text{for every } n \geq n_0 \text{ and for all } t \in \mathbb{I}.$$

In other words, we can say γ_n^{-1} converges uniformly to γ^{-1}. Since every bivariate copula satisfies Lipchitz condition (see Nelsen (2006), Theorem 2.24), we have

$$\begin{aligned} |C(\gamma_n^{-1}(u), \gamma_n^{-1}(v)) - C(\gamma^{-1}(u), \gamma^{-1}(v))| &< |\gamma_n^{-1}(u) - \gamma^{-1}(u)| + |\gamma_n^{-1}(v) - \gamma^{-1}(v)| \\ &< \frac{\zeta}{2} + \frac{\zeta}{2} = \zeta, \end{aligned}$$

for every $n \geq n_0$ and for all $u, v \in \mathbb{I}$. γ_n converges uniformly to γ, then $\{\gamma_n : n \in \mathbb{N}\}$ is an equi-continuous family of functions (see Rudin (1976), Theorem 7.24), i.e.,

$$|\gamma_n(t_2) - \gamma_n(t_1)| < \frac{\eta}{2} \tag{3.13}$$

for all $n \in \mathbb{N}$, whenever $|t_2 - t_1| < \zeta$ with $t_1, t_2 \in \mathbb{I}$. Substitute $t_1 = C\left(\gamma^{-1}(u), \gamma^{-1}(v)\right)$ and $t_2 = C\left(\gamma_n^{-1}(u), \gamma_n^{-1}(v)\right)$ in Eq. (3.13) and $t = C\left(\gamma^{-1}(u), \gamma^{-1}(v)\right)$ in Eq. (3.12), we have

$$\left|\gamma_n\left(C\left(\gamma_n^{-1}(u), \gamma_n^{-1}(v)\right)\right) - \gamma_n\left(C\left(\gamma^{-1}(u), \gamma^{-1}(v)\right)\right)\right| < \frac{\eta}{2},$$

and

$$\left|\gamma_n\left(C\left(\gamma^{-1}(u), \gamma^{-1}(v)\right)\right) - \gamma\left(C\left(\gamma^{-1}(u), \gamma^{-1}(v)\right)\right)\right| < \frac{\eta}{2},$$

for every $n \geq n_0$. Thus,

$$\begin{aligned}
|C^{\gamma_n}(u,v) - C^{\gamma}(u,v)| &= \left|\gamma_n\left(C\left(\gamma_n^{-1}(u), \gamma_n^{-1}(v)\right)\right) - \gamma\left(C\left(\gamma^{-1}(u), \gamma^{-1}(v)\right)\right)\right| \\
&\leq \left|\gamma_n\left(C\left(\gamma_n^{-1}(u), \gamma_n^{-1}(v)\right)\right) - \gamma_n\left(C\left(\gamma^{-1}(u), \gamma^{-1}(v)\right)\right)\right| \\
&\quad + \left|\gamma_n\left(C\left(\gamma^{-1}(u), \gamma^{-1}(v)\right)\right) - \gamma\left(C\left(\gamma^{-1}(u), \gamma^{-1}(v)\right)\right)\right| \\
&< \frac{\eta}{2} + \frac{\eta}{2} = \eta, \text{ for every } n \geq n_0 \text{ and for every } u, v \in \mathbb{I}.
\end{aligned}$$

Thus, C^{γ_n} converges uniformly to C^{γ}. □

Definition 3.2.11. *(Nelsen (2006)) Let C_1 and C_2 be any two bivariate copulas, then C_1 is said to be smaller than C_2, denoted by $C_1 \prec C_2$, if $C_1(u,v) \leq C_2(u,v)$ for all $u, v \in \mathbb{I}$.*

Proposition 3.2.12. *Let $C_1^{\gamma}(u,v)$ and $C_2^{\gamma}(u,v)$ be PGF copulas of $C_1(u,v)$ and $C_2(u,v)$ respectively derived from the PGF γ. Then, $C_1^{\gamma} \prec C_2^{\gamma}$ if and only if $C_1 \prec C_2$.*

The proof is straightforward, so omitted.

3.3 Random Number Generation

In this section, we present an algorithm for generating random numbers from the PGF copula. The algorithm is as follows:

Step 1: Generate a RV N from the distribution with PGF $\gamma(\cdot)$.

Step 2: Generate N independent random samples $(u_1, v_1), (u_2, v_2), \ldots, (u_N, v_N)$ from the baseline copula $C(u,v)$.

Step 3: Set $x = \max\{u_1, u_2, \ldots, u_N\}$ and $y = \max\{v_1, v_2, \ldots, v_N\}$.

Step 4: Set $u = \gamma(x)$ and $v = \gamma(y)$. Finally the desired sample is (u, v).

Following examples are new class of PGF copulas derived from well known copulas. We generate 1000 random numbers from the PGF copula and is depicted in scatterplot. We use R-software (version 3.6.3) for random number generation. We use `copula` package in R for generating random numbers from the baseline copula.

Example 3.3.1. *Consider the Gumbel-Hougaard copula*

$$C(u,v) = \exp\left\{-\left[(-\ln u)^{\phi} + (-\ln v)^{\phi}\right]^{1/\phi}\right\},$$

$\phi \geq 1$. *Then the Geometric-Gumbel-Hougaard copula derived from geometric PGF is given by*

$$C^{\gamma}(u,v) = \frac{p\left(\exp\left\{-\left[\left(-\ln\left(\frac{u}{(1-u)p+u}\right)\right)^{\phi} + \left(-\ln\left(\frac{v}{(1-v)p+v}\right)\right)^{\phi}\right]^{1/\phi}\right\}\right)}{1-(1-p)\left(\exp\left\{-\left[\left(-\ln\left(\frac{u}{(1-u)p+u}\right)\right)^{\phi} + \left(-\ln\left(\frac{v}{(1-v)p+v}\right)\right)^{\phi}\right]^{1/\phi}\right\}\right)},$$

for $\phi \geq 1$ *and* $p \in (0,1]$.

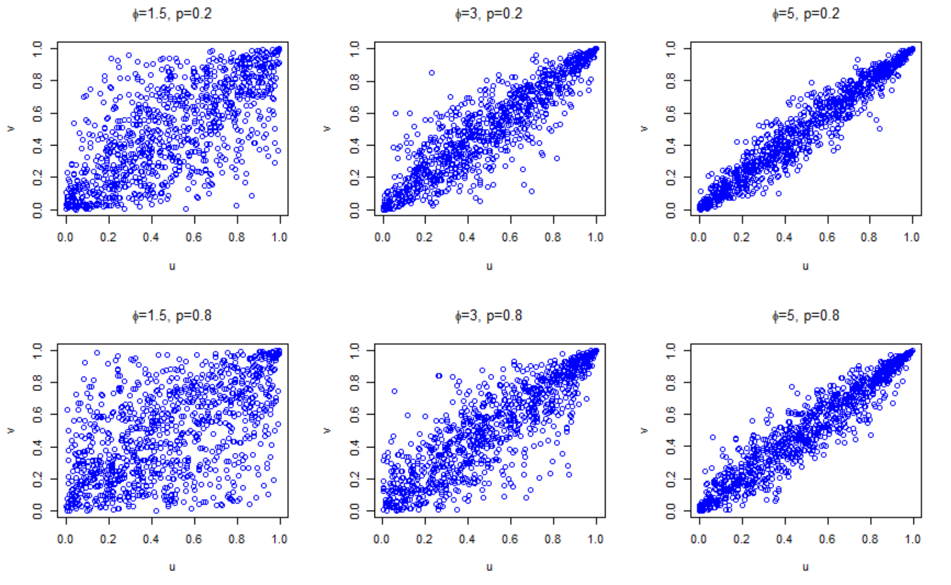


FIGURE 3.1. Random numbers from Geometric-Gumbel-Hougaard copula with different parameters

Example 3.3.2. *Marshall and Olkin (1967) proposed a bivariate copula, defined by*

$$C(u,v) = \begin{cases} u^{1-\alpha}v, & \text{if } u^{\alpha} \geq v^{\beta}, \\ uv^{1-\beta}, & \text{if } u^{\alpha} < v^{\beta} \end{cases}$$

for $0 \leq \alpha, \beta \leq 1$. *Then the Logarithmic-Marshall-Olkin copula derived from the PGF of logarithmic distribution is given by*

$$C^{\gamma}(u,v) = \begin{cases} (\ln\theta)^{-1}\ln\left(1-\mu^{\alpha-1}\left(1-\theta^{u}\right)^{1-\alpha}\left(1-\theta^{v}\right)\right), & \text{if } u^{\alpha} \geq v^{\beta}, \\ (\ln\theta)^{-1}\ln\left(1-\mu^{\beta-1}\left(1-\theta^{u}\right)\left(1-\theta^{v}\right)^{1-\beta}\right), & \text{if } u^{\alpha} < v^{\beta}, \end{cases}$$

for $\alpha, \beta \in [0,1], \mu \in (0,1)$ *and* $\theta = 1-\mu$.

In the following section, we will discuss some important dependence measures of the PGF copula.

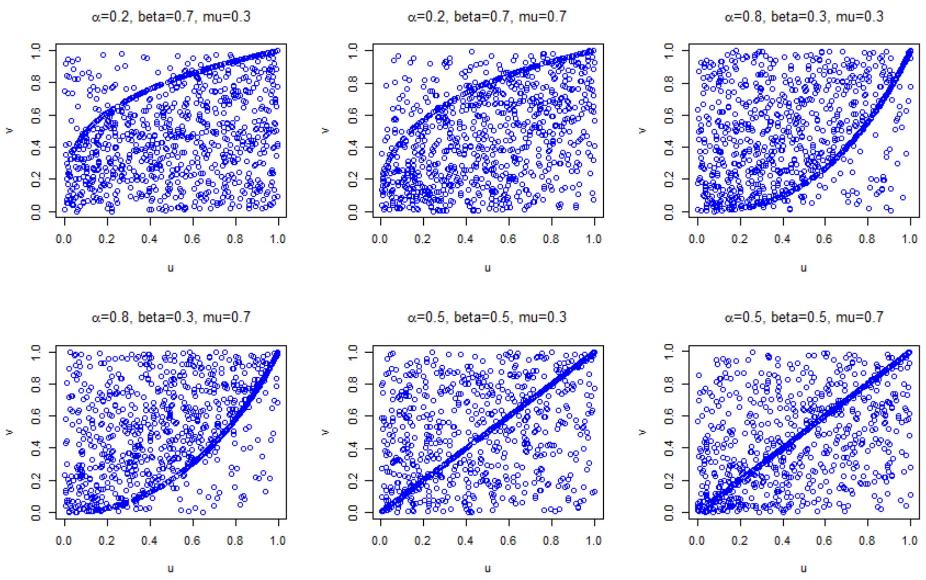


FIGURE 3.2. Random numbers from Logarithmic-Marshall-Olkin copula with different parameters

3.4 Stochastic Dependence

Copula functions are widely used in modelling dependent data sets. In literature, there are various measures of dependence are available to measure the dependence structure captured by the copula. Some of the important dependence measures were discussed here.

3.4.1 Measures of Association

Let $C(u, v)$ be a bivariate copula. Then the Kendall's Tau and Spearman's Rho in terms of copula, denoted by τ_C and ρ_C, can be defined as

$$\tau_C = 1 - 4\int_0^1\int_0^1 \frac{\partial}{\partial u}\{C(u,v)\}\frac{\partial}{\partial v}\{C(u,v)\}\,dudv, \tag{3.14}$$

and

$$\rho_C = 12\int_0^1\int_0^1 C(u,v)dudv - 3. \tag{3.15}$$

Using the expressions in Eq. (3.14) and Eq. (3.15), the Kendall's Tau and Spearman's Rho for the PGF copula in Eq.(3.5) can be defined as

$$\tau_{C^\gamma} = 1 - 4\int_0^1\int_0^1 \frac{\partial}{\partial u}\{\gamma(C(u,v))\}\frac{\partial}{\partial v}\{\gamma(C(u,v))\}\,dudv,$$

and

$$\rho_{C^\gamma} = 12 \int_0^1 \int_0^1 \gamma\left(C(u,v)\right) \gamma'(u)\gamma'(v) du dv - 3. \tag{3.16}$$

Remark 3.4.1. *Let C be an Archimedean bivariate copula with Archimedean generator $\varphi(s)$, then Kendall's Tau for C will be $\tau_C = 1 + 4\int_0^1 \frac{\varphi(s)}{\varphi'(s)} ds$. As a consequence of Proposition 3.2.5, Kendall's Tau for the PGF copula C^γ derived from the PGF γ is $\tau_{C^\gamma} = 1 + 4\int_0^1 \frac{\varphi(\gamma^{-1}(s))}{\varphi'(\gamma^{-1}(s))} ds$.*

Proposition 3.4.2. *Let $C^\gamma(u,v)$ be the PGF copula of a bivariate copula $C(u,v)$ derived from PGF γ. Then,*

$$\max\left\{1 - \left[\gamma'(1)\right]^2 (\tau_C - 1), -1\right\} \leq \tau_{C^\gamma} \leq 1 - \left[\gamma'(0)\right]^2 (\tau_C - 1). \tag{3.17}$$

Proof. Since $\gamma''(t) \geq 0$, for all $t \in \mathbb{I}$, it follows that γ' is an increasing function. Hence,

$$\gamma'(0) \leq \gamma'(t) \leq \gamma'(1), \text{ for all } t \in \mathbb{I}. \tag{3.18}$$

Substituting $t = C(u,v)$ into Eq. (3.18), we obtain

$$\left[\gamma'(0)\right]^2 \leq \left[\gamma'\left(C(u,v)\right)\right]^2 \leq \left[\gamma'(1)\right]^2.$$

Since $0 < \frac{\partial}{\partial u}C(u,v) < 1$ and $0 < \frac{\partial}{\partial v}C(u,v) < 1$, multiplying both sides of the above inequality by

$$4\frac{\partial}{\partial u}C(u,v)\frac{\partial}{\partial v}C(u,v),$$

and integrating over the unit square $\mathbb{I}^2$, and using Eq. (3.14), we obtain the inequality (3.17).

It is important to note that $\gamma'(0)$ always exists, whereas $\gamma'(1)$ may not exist in general. To provide a tight lower bound, we consider

$$\max\left\{1 - \left[\gamma'(1)\right]^2 (\tau_C - 1),\ -1\right\},$$

since the expression $\left[\gamma'(1)\right]^2 (\tau_C + 1) - 1$ can potentially be less than -1.

□

Blomqvist's medial correlation coefficient, proposed by Blomqvist (1950), is a measure of association based on the medians of the two RVs. If C is a bivariate copula, then Blomqvist's Medial correlation coefficient, denoted by β_C, can be defined as

$$\beta_C = 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1.$$

Like Kendall's Tau and Spearman's Rho, the range of Blomqvist's Beta is also $[-1, 1]$, and similar interpretation can be made for positive and negative values of β_C. The Blomqvist's Beta coefficient for the PGF copula is given by

$$\beta_{C^\gamma} = 4\gamma\left(C\left(\gamma^{-1}\left(\frac{1}{2}\right), \gamma^{-1}\left(\frac{1}{2}\right)\right)\right) - 1.$$

Following examples will provide the Blomqvist's Medial correlation coefficient of some well-known families of copulas.

Example 3.4.1. *The Blomqvist's Beta for product copula, $\Pi(u, v) = uv$, is zero. Then the product PGF copula derived from the PGF of Harris distribution $\gamma(t) = \left(\frac{\delta t^k}{1-(1-\delta)t^k}\right)^{1/k}$; $k > 0, \delta \in (0, 1)$ is given by*

$$\beta_{C^\gamma} = \left[\left(1 - 2^k\right)\left(\delta(1 - 2^k) - 2\right) + 1\right]^{-1/k}.$$

Example 3.4.2. *Consider the Galambos (1975) copula*

$$C(u, v) = uv\exp\left\{\left[(1 - u)^{-\theta} + (1 - v)^{-\theta}\right]^{-1/\theta}\right\},$$

where $\theta > 0$. Then $\beta_C = \exp\left\{2^{-\frac{1}{\theta}-1}\right\} - 1$. Hence, Blomqvist's Medial correlation coefficient for the PGF copula corresponds to the PGF $\gamma(t) = 1 - (1 - t)^\alpha$; $\alpha \in (0, 1)$ is given by

$$\beta_{C^\gamma} = 3 - 4\left(1 - \left[1 - 2^{-\frac{1}{\alpha}}\right]^2 \exp\left\{2^{-\frac{1}{\theta}-\frac{1}{\alpha}}\right\}\right)^\alpha.$$

Example 3.4.3. *Consider the Cuadras and Augé (1981) copula*

$$C(u, v) = (uv)^{1-\theta} (\min\{u, v\})^\theta; \theta \in [0, 1].$$

Then $\beta_C = 2^\theta - 1$. It follows that the Blomqvist's Beta of the PGF copula for the PGF $\frac{pt}{1-(1-p)t}$ is

$$\beta_{C^\gamma} = \frac{(1 + p)^{\theta-2}(3p + 1) - 1}{1 - (1 - p)(1 + p)^{\theta-2}}.$$

3.4.2 Positive Quadrant Dependence

A bivariate copula is said to have the positively quadrant dependent (PQD) property if $uv \leq C(u, v)$ for all $u, v \in [0, 1]$. Conversely, if $uv \geq C(u, v)$ for all $u, v \in [0, 1]$, the copula is said to exhibit negatively quadrant dependent (NQD) behavior. The following lemma will be useful in establishing the PQD property of the PGF copula.

Lemma 3.4.3. *Let γ be a PGF of some positive integer-valued RV N. Then*

$$\gamma(a)\gamma(b) \leq \gamma(ab), \quad \textit{for every } a, b \in \mathbb{I}.$$

Proof. Let Ω denote the support of N. First, we will prove that the result is true for the cardinality of Ω is finite. Without loss of generality, we assume $\Omega = \{1, 2, 3, \ldots, K\}$, where $K \in \mathbb{N}$. Since $a \in \mathbb{I}$, it follows that $(a^x - a^y) \geq 0$ for $x \geq y$ and $(a^x - a^y) \leq 0$ for $x < y$, $x, y \in \mathbb{N}$. This implies that for $a, b \in \mathbb{I}$, the product term $(a^x - a^y)(b^y - b^x) \leq 0$ for all $x, y \in \mathbb{I}$. Therefore,

$$\begin{aligned}
\frac{1}{2}\sum_{x=1}^{K}\sum_{y=1}^{K}(a^x - a^y)(b^y - b^x) P_y P_x =& \frac{1}{2}\sum_{x=1}^{K}\left[\sum_{y=1}^{K}\left[a^x b^y - (ab)^x - (ab)^y + a^y b^x\right] P_y\right] P_x \\
=& \frac{1}{2}\sum_{x=1}^{K}\left[a^x\gamma(b) - (ab)^x - \gamma(ab) + \gamma(a)b^x\right] P_x \\
=& \frac{1}{2}\left[2\gamma(a)\gamma(b) - 2\gamma(ab)\right] \\
=& \gamma(a)\gamma(b) - \gamma(ab).
\end{aligned}$$

Clearly, $\gamma(a)\gamma(b) - \gamma(ab) \leq 0$ for $a, b \in \mathbb{I}$. The result is also valid for infinite case by letting $K \to \infty$. □

Corollary 3.4.4. *Let γ^{-1} be the inverse of the PGF γ, then*

$$\gamma^{-1}(ab) \leq \gamma^{-1}(a)\gamma^{-1}(b), \quad \textit{for all } a, b \in \mathbb{I}.$$

Proposition 3.4.5. *If the bivariate copula C has bivariate PQD copula, then corresponding PGF copula C^γ generated from the PGF γ also has PQD property.*

Proof. Since C has PQD, then by definition $C(u, v) \geq uv$ for all $u, v \in \mathbb{I}$. It follows that

$$C\left(\gamma^{-1}(u), \gamma^{-1}(v)\right) \geq \gamma^{-1}(u)\gamma^{-1}(v).$$

Using Lemma 3.4.3, we obtain

$$\gamma\left(C\left(\gamma^{-1}(u), \gamma^{-1}(v)\right)\right) \geq \gamma\left(\gamma^{-1}(u)\gamma^{-1}(v)\right) \geq uv,$$

for all $u, v \in \mathbb{I}$. Hence, C^γ has positive quadrant dependence property. □

Remark 3.4.6. *If C has NQD, then it is not necessary that the corresponding PGF copula C^γ has NQD property. For instance, take $\gamma(t) = 1 - (1-t)^{0.8}, t \in \mathbb{I}$ and $C(u, v) = uv(1 - 0.7(1-u)(1-v))$, Farlie-Gumbel-Morgenstern (FGM) copula. It is well-known* *result that the given FGM copula has NQD (see Drouet Mari and Kotz (2001), p.119), but* $C^{\gamma}(0.1, 0.9) = 0.09101 \nleq 0.09$.

3.4.3 Total Positive of Order 2 (TP_2) Property

A bivariate function $G(x, y)$ is said to be totally positive of order 2 (TP_2) if

$$G(x_1, y_1)G(x_2, y_2) \geq G(x_2, y_1)G(x_1, y_2), \text{ for every } x_1 < x_2 \text{ and } y_1 < y_2.$$

The TP_2 property is one of the strongest forms of dependence. If a bivariate copula possesses the TP_2 property, it implies positively quadrant dependence (PQD), left tail decreasing (LTD), and left corner set decreasing (LCSD) dependence.

Lemma 3.4.7. *Let* $p, q, r, s \in \mathbb{I}$ *with* $r \leq \min\{p, q\} \leq \max\{p, q\} \leq s$. *If* $pq \leq rs$, *then* $\gamma(p)\gamma(q) \leq \gamma(r)\gamma(s)$.

Proof. Without loss of generality, assume $\min\{p, q\} = p$ and $\max\{p, q\} = q$, then $pq \leq rs$ implies that $(pq)^x \leq (rs)^x$ for every $x > 0$. Then,

$$(pq)^x - (qr)^x \leq (rs)^x - (qr)^x \leq q^x \left[s^x - q^x\right].$$

It follows that

$$p^x - r^x \leq s^x - q^x,$$

for every $x > 0$ and $pq \leq rs$. Therefore, if $pq \leq rs$ then, $p^x + q^x \leq r^x + s^x$, for all $x > 0$. Let $\gamma(t), t \in \mathbb{I}$ be the PGF of the positive integer-valued RV N. Let Ω be the support of N. First we will show the result is true for the finite support of N, i.e. $\Omega = \{1, 2, 3, \ldots, K\}$, where K is a fixed natural number.

$$\begin{aligned}
\gamma(p)\gamma(q) = &\left(\sum_{x=1}^{K} p^x P_x\right)\left(\sum_{y=1}^{K} q^y P_y\right) \\
=&pqP_1\left(P_1 + [p+q]\,P_2 + \left[p^2+q^2\right]P_3 + \cdots + \left[p^{K-1}+q^{K-1}\right]P_K\right) + \\
&p^2q^2P_2\left(P_2 + [p+q]\,P_3 + \left[p^3+q^3\right]P_4 + \cdots \left[p^{K-1}+q^{K-1}\right]P_K\right) + \\
&\vdots \\
&+ p^{K-1}q^{K-1}P_{K-1}\left(P_{K-1} + [p+q]\,P_K\right) + p^Kq^KP_KP_K
\end{aligned}$$

$$
\begin{aligned}
\leq & rsP_1\left(P_1+[r+s]\,P_2+\left[r^2+s^2\right]P_3+\cdots+\left[r^{K-1}+s^{K-1}\right]P_K\right)+\\
& r^2s^2P_2\left(P_2+[r+s]\,P_3+\left[r^3+s^3\right]P_4+\cdots\left[r^{K-1}+s^{K-1}\right]P_K\right)+\\
& \vdots\\
& +r^{K-1}s^{K-1}P_{K-1}\left(P_{K-1}+[r+s]\,P_K\right)+r^Ks^KP_KP_K\\
= & \gamma(r)\gamma(s).
\end{aligned}
$$

Letting $K\to\infty$, the conclusion holds true even in the infinite case. □

Proposition 3.4.8. *Let C be a bivariate copula. If C has TP_2 property, then the corresponding PGF copula C^{γ} derived from PGF γ also has TP_2 property.*

Proof. If the bivariate copula C has TP_2 property, then

$$C(u_1,v_1)C(u_2,v_2)\geq C(u_1,v_2)C(u_2,v_1),$$

for every $u_1<u_2$ and $v_1<v_2$ with $u_1,u_2,v_1,v_2\in\mathbb{I}$. Substitute $p=C(u_1,v_2)$, $q=C(u_2,v_1)$, $r=C(u_1,v_1)$ and $s=C(u_2,v_2)$ in Lemma 3.4.7, the result immediately follows. □

3.4.4 Tail Dependence Coefficients

Tail dependence coefficients quantify the association between the tails of two RVs. For a bivariate copula C, the lower and upper tail dependence coefficients are defined as

$$\lambda_L^C=\lim_{u\to 0^+}\frac{C(u,u)}{u},\quad \lambda_U^C=2-\lim_{u\to 1^-}\frac{1-C(u,u)}{1-u}. \tag{3.19}$$

It holds that $0\leq\lambda_L^C,\lambda_U^C\leq 1$ (see (Nelsen, 2006, p. 214)). A value of zero indicates tail independence, while a positive value suggests tail dependence.

Proposition 3.4.9. *The lower tail dependence coefficient is invariant under the PGF transformation (i.e., $\lambda_L^{C^{\gamma}}=\lambda_L^C$). Similarly, if $\gamma'(1)$ exist, then the upper tail dependence coefficient is invariant under the PGF transformation.*

Proof. By definition,

$$\begin{aligned}
\lambda_L^{C^\gamma} &= \lim_{u\to 0^+} \frac{\gamma\left(C(\gamma^{-1}(u), \gamma^{-1}(u))\right)}{u} \\
&= \lim_{u\to 0^+} \frac{\gamma\left(C(u,u)\right)}{\gamma(u)} \\
&= \lim_{u\to 0^+} \frac{\gamma'\left(C(u,u)\right)}{\gamma'(u)} \lim_{u\to 0^+} \frac{d}{du}\{C(u,u)\} \\
&= \lim_{u\to 0^+} \frac{\gamma'\left(C(u,u)\right)}{\gamma'(u)} \lim_{u\to 0^+} \frac{C(u,u)}{u},
\end{aligned}$$

Now, the result follows from the fact that $\lim_{u\to 0^+} \frac{\gamma'\left(C(u,u)\right)}{\gamma'(u)} = 1$, as $\gamma'(0)$ is finite, and $\lim_{u\to 0^+} \frac{C(u,u)}{u} = \lambda_L^C$. Further, assume that $\gamma'(1)$ exists, we have

$$\begin{aligned}
\lambda_U^{C^\gamma} =& 2 - \lim_{u\to 1^-} \frac{1-\gamma\left(C(\gamma^{-1}(u), \gamma^{-1}(v)\right)}{1-u} \\
=& 2 - \lim_{u\to 1^-} \frac{1-\gamma\left(C(u,v)\right)}{1-\gamma(u)} \\
=& 2 - \lim_{u\to 1^-} \frac{\gamma'\left(C(u,v)\right)}{\gamma'(u)} \lim_{u\to 1^-} \frac{d}{du}\{C(u,u)\} \\
=& 2 - \lim_{u\to 1^-} \frac{\gamma'\left(C(u,v)\right)}{\gamma'(u)} \lim_{u\to 1^-} \frac{1-C(u,u)}{1-u} \\
=& \lambda_U^C,
\end{aligned}$$

where the last equality holds by using the fact that $\lim_{u\to 1^-} \frac{\gamma'\left(C(u,u)\right)}{\gamma'(u)} = 1$. □

Remark 3.4.10. *If $\gamma'(1)$ does not exist, then the upper tail dependence coefficient of the transformed copula C^γ may differ from that of the original copula C. For instance, consider the PGF $\gamma(t) = 1-(1-t)^\alpha$, with $0<\alpha<1$. In this case, as shown by Dolati et al. (2014), we have*

$$\lambda_U^{C^\gamma} = 2 - \left(2 - \lambda_U^C\right)^\alpha.$$

This example illustrates that even when the baseline copula exhibits no or weak upper tail dependence, an appropriate PGF transformation can yield a new copula with enhanced upper tail dependence.

Example 3.4.4. *Consider the Marshall-Olkin copula, defined by*

$$C(u,v) = \begin{cases} u^{1-p}v^q, & \text{if } u^p \geq v^q \\ u^p v^{1-q}, & \text{if } u^p \leq v^q, \end{cases}$$

where $p, q \in (0,1)$. *It is easy to show that* $\lambda_L^C = 0$ *and* $\lambda_U^C = \min\{p,q\}$. *The lower and upper tail coefficients are of Marshall-Olkin PGF copula derived from the PGF of discrete Mittag-Leffler distribution are* $\lambda_L^{C^\gamma} = 0$ *and* $\lambda_U^{C^\gamma} = 2 - (2 - \min\{p,q\})^\alpha$, *respectively.*

Example 3.4.5. *Consider the Polynomial-Sine copula, proposed by Chesneau (2022), defined by,*

$$C(u,v) = uv + \theta \sin(\pi u)\sin(\pi v),$$

where $\theta \in \left[-\frac{1}{\pi^2}, \frac{1}{\pi^2}\right]$. *Clearly,* $\lambda_L^C = 0$ *and* $\lambda_U^C = 0$. *The lower and upper tail dependence coefficients of Polynomial-Sine PGF copula derived from the PGF* $\gamma(t) = 1 - (1-t)^\alpha$ *are* $\lambda_L^{C^\gamma} = 0$ *and* $\lambda_U^{C^\gamma} = 2 - 2^\alpha$, *respectively.*

3.5 Weighted Geometric Mean

Let C_1 and C_2 be two bivariate copulas. The weighted geometric mean of C_1 and C_2 can be defined as

$$C(u,v) = [C_1(u,v)]^\theta [C_2(u,v)]^{1-\theta}, \;\; \theta \in \mathbb{I}. \tag{3.20}$$

Cuadras (2009) proved that weighted geometric mean of two bivariate copulas may or may not be a copula. Zhang et al. (2013) proved that the weighted geometric mean of two bivariate copulas C_1 and C_2 is a copula if C_1 and C_2 has TP_2 property. Now using Proposition 3.4.8 and the result of Zhang et al. (2013), we will state the following proposition without proof.

Proposition 3.5.1. *Let* $C_1^{\gamma_1}$ *and* $C_2^{\gamma_2}$ *be the PGF copulas of two bivariate copulas* C_1 *and* C_2 *derived from the PGFs* γ_1 *and* γ_2 *respectively. Then the weighted geometric mean of* $C_1^{\gamma_1}$ *and* $C_2^{\gamma_2}$

$$C = [C_1^{\gamma_1}]^\theta [C_2^{\gamma_2}]^{1-\theta}, \;\; \theta \in \mathbb{I},$$

is a copula if C_1 *and* C_2 *have* TP_2 *property.*

3.6 Concavity Property

In this section, we will discuss Schur-concavity and quasi-concavity property of a copula. Schur-concavity and quasi-concavity of a bivariate copula can be defined as follows.

Definition 3.6.1. *A bivariate copula C is said to be Schur-concave if*

$$C(s,t) \leq C(\alpha s + (1-\alpha)t, \alpha t + (1-\alpha)s),$$

for every $s, t, \alpha \in \mathbb{I}$.

Definition 3.6.2. *A bivariate copula C is said to be quasi-concave if*

$$\min\{C(s_1,s_2), C(t_1,t_2)\} \leq C(\alpha s_1 + (1-\alpha)t_1, \alpha s_2 + (1-\alpha)t_2),$$

for every $s_1, s_2, t_1, t_2, \alpha \in \mathbb{I}$.

Proposition 3.6.3. *Let C be a bivariate copula.*

1. *If C is Schur-concave, then the PGF copula C^γ is Schur-concave.*
2. *If C is quasi-concave then C^γ is quasi-concave.*

Proof.

1. First we prove that γ^{-1} is concave. Since $\gamma''(t) \geq 0$, for all $t \in \mathbb{I}$, (i.e. γ is convex). It follows that

$$\gamma(\alpha s + (1-\alpha)t) \leq \alpha\gamma(s) + (1-\alpha)\gamma(t),$$

for any $s, t, \alpha \in \mathbb{I}$. Let $p = \gamma^{-1}(\alpha s + (1-\alpha)t)$ and $q = \alpha\gamma^{-1}(s) + (1-\alpha)\gamma^{-1}(t)$. It is obvious that $p, q \in \mathbb{I}$. Now consider

$$\begin{aligned}\gamma(p) =& \alpha s + (1-\alpha)t \\ =& \alpha\gamma\left(\gamma^{-1}(s)\right) + (1-\alpha)\gamma\left(\gamma^{-1}(t)\right) \\ \geq& \gamma\left(\alpha\gamma^{-1}(s) + (1-\alpha)\gamma^{-1}(t)\right) \\ =& \gamma(q).\end{aligned}$$

Since γ is a bijective and strictly increasing function, it follows that $\gamma(p) \geq \gamma(q)$ if and only if $p \geq q$. Therefore,

$$\gamma^{-1}(\alpha s + (1-\alpha)t) \geq \alpha\gamma^{-1}(s) + (1-\alpha)\gamma^{-1}(t),$$

for every $s, t, \alpha \in \mathbb{I}$. Hence γ^{-1} is concave. Assume that C is Schur-concave. By definition and using inequality (3.3), we have

$$\begin{aligned} C\left(\gamma^{-1}(s), \gamma^{-1}(t)\right) \leq & C\left(\alpha\gamma^{-1}(s) + (1-\alpha)\gamma^{-1}(t), \alpha\gamma^{-1}(t) + (1-\alpha)\gamma^{-1}(s)\right) \\ \leq & C\left(\gamma^{-1}\left(\alpha s + (1-\alpha)t\right), \gamma^{-1}\left(\alpha t + (1-\alpha)s\right)\right). \end{aligned}$$

Since γ is strictly increasing, it follows that

$$C^{\gamma}(s,t) \leq C^{\gamma}(\alpha s + (1-\alpha)t, \alpha t + (1-\alpha)s),$$

for every $s, t, \alpha \in \mathbb{I}$. Hence C^{γ} is Schur-concave.

2. Assume that C is quasi-concave. Using inequality (3.3) and concavity property of γ^{-1}, for every $s_1, s_2, t_1, t_2, \alpha \in [0,1]$, we have

$$\begin{aligned} & \min\left\{C\left(\gamma^{-1}(s_1), \gamma^{-1}(s_2)\right), C\left(\gamma^{-1}(t_1), \gamma^{-1}(t_2)\right)\right\} \\ & \quad \leq C\left(\alpha\gamma^{-1}(s_1) + (1-\alpha)\gamma^{-1}(s_2), \alpha\gamma^{-1}(t_1) + (1-\alpha)\gamma^{-1}(t_2)\right) \\ & \quad \leq C\left(\gamma^{-1}\left(\alpha s_1 + (1-\alpha)t_1\right), \gamma^{-1}\left(\alpha s_2 + (1-\alpha)t_2\right)\right). \end{aligned}$$

It follows that

$$\min\left\{C^{\gamma}(s_1, s_2), C^{\gamma}(t_1, t_2)\right\} \leq C^{\gamma}(\alpha s_1 + (1-\alpha)t_1, \alpha s_2 + (1-\alpha)t_2).$$

□

3.7 A New Class of PGF-FGM Copulas

Farlie-Gumbel-Morgenstern (FGM) copula is one of the popular copula and widely used in the literature for constructing bivariate distributions. The FGM copula is defined as

$$C(u,v) = uv\left[1 + \theta(1-u)(1-v)\right], \ \theta \in [-1,1].$$

The FGM copula is commonly used to model datasets exhibiting both positive and negative dependence, facilitated by an appropriate choice of the parameter θ. However, a limitation arises from its relatively narrow dependence range, specifically with Spearman's Rho (ρ) confined within $[-0.33, 0.33]$. In response to this limitation, various researchers have endeavored to enhance the dependence range of the FGM copula. In this section, we will generalize the FGM copula by using PGF of some positive-integer valued RVs. We consider the PGF's of geometric and Discrete Mittag-Leffler distributions. We also compute the Spearman's Rho for both copulas for analyzing the dependence performance.

3.7.1 Geometric-FGM Copula

Using the PGF of Geometric distribution, the PGF copula is given by

$$C^{\gamma}(u,v) = \frac{p\left(\frac{u}{(1-u)p+u}\right)\left(\frac{v}{(1-v)p+v}\right)\left[1+\theta\left(\frac{(1-u)p}{(1-u)p+u}\right)\left(\frac{(1-v)p}{(1-v)p+v}\right)\right]}{1-(1-p)\left(\frac{u}{(1-u)p+u}\right)\left(\frac{v}{(1-v)p+v}\right)\left[1+\theta\left(\frac{(1-u)p}{(1-u)p+u}\right)\left(\frac{(1-v)p}{(1-v)p+v}\right)\right]}, \tag{3.21}$$

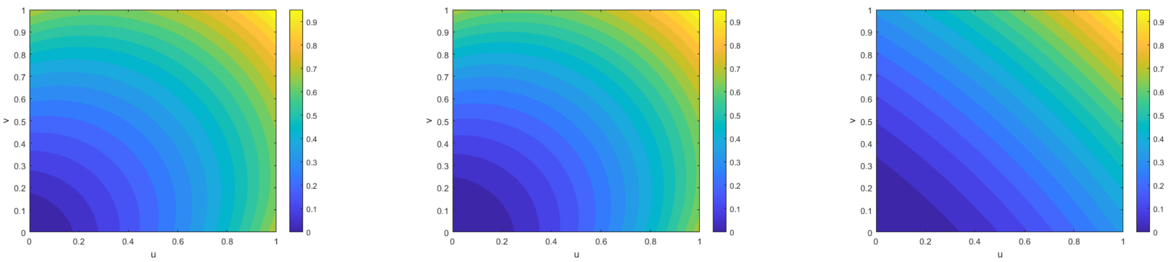
where $\theta \in [-1,1], p \in (0,1)$. It is easy to verify that C^{γ} reduces to FGM copula, when $p = 1$. Using Eq. (3.16), the Spearman's Rho for the Geometric-FGM copula defined in Eq. (3.21) is given by

$$12\int_0^1\int_0^1 \frac{p^3\left[uv\left(1+\theta(1-u)(1-v)\right)\right]}{\left(1-(1-p)\left[uv\left(1+\theta(1-u)(1-v)\right)\right]\right)\left(1-(1-p)u\right)^2\left(1-(1-p)v\right)^2}dudv - 3$$

Since the above integral is complicated in nature, so the explicit expression for the Spearman's Rho is difficult to obtain. Therefore, we use numerical integration technique to evaluate the above integral. We use `integral2()` function in MATLAB (R2023b) to compute the Spearman's Rho for various values of the parameters p and θ. The results are given in Table 3.2. It is observed from Table 3.2 that Geometric-FGM copula improved the dependence range of Spearman's Rho by $(-0.33333, 0.475118)$. Moveover, the contour plots of the Geometric-FGM copula for various values of the parameters are shown in Fig. 3.3.

TABLE 3.2. Spearman's $\rho_{C^{\gamma}}$ of the Geometric-FGM copula in Eq. (3.21) for various values of p and θ.

$\theta \rightarrow$ $\downarrow p$	**-1**	**-0.8**	**-0.6**	**-0.4**	**-0.2**	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**
0.1	0.343081	0.355567	0.368202	0.380989	0.393933	0.407037	0.420306	0.433744	0.447355	0.461145	**0.47512**
0.2	0.232094	0.254112	0.276415	0.299009	0.321903	0.345102	0.368616	0.392452	0.416619	0.441125	0.465981
0.3	0.13582	0.16584	0.196221	0.22697	0.258096	0.289608	0.321513	0.353822	0.386543	0.419686	0.453261
0.4	0.050158	0.087141	0.124511	0.162277	0.200445	0.239023	0.278018	0.31744	0.357296	0.397594	0.438345
0.5	-0.02728	0.015889	0.059433	0.103359	0.147674	0.192383	0.237492	0.283008	0.328937	0.375286	0.422061
0.6	-0.09806	-0.04933	-0.00026	0.049156	0.098913	0.149019	0.199477	0.250294	0.301472	0.353016	0.404931
0.7	-0.16332	-0.10952	-0.05545	-0.0011	0.053529	0.108438	0.163631	0.219111	0.27488	0.33094	0.387295
0.8	-0.22387	-0.16544	-0.10681	-0.04798	0.011042	0.070264	0.129685	0.189306	0.249129	0.309155	0.369386
0.9	-0.28037	-0.21767	-0.15485	-0.09194	-0.02892	0.034198	0.097422	0.16075	0.224183	0.287719	0.351361
1	**-0.33333**	-0.26667	-0.2	-0.13333	-0.06667	0	0.066667	0.133333	0.2	0.266667	0.333333



(A) $p = 0.1, \theta = 0.9$ (B) $p = 0.2, \theta = -0.8$ (C) $p = 0.7, \theta = 0.5$

FIGURE 3.3. Contour plots of the Geometric-FGM copula for various values of p and θ

3.7.2 Discrete Mittag-Leffler-FGM Copula

We further propose one more generalization of FGM copula using the PGF of discrete Mittag-Leffler distribution. The proposed copula is defined as

$$C^{\gamma}(u,v) = 1 - \left(1 - \left(1-(1-u)^{1/\alpha}\right)\left(1-(1-v)^{1/\alpha}\right)\left[1+\theta\left(1-u-v+uv\right)^{1/\alpha}\right]\right)^{\alpha}, \tag{3.22}$$

where $\theta \in [-1, 1]$ and $\alpha \in (0, 1)$. When $\alpha \to 1$, Discrete Mittag-Leffler-FGM copula reduces to FGM copula. The Spearman's Rho for the discrete Mittag-Leffler-FGM copula can be obtained by evaluating the following integral

$$12\int_0^1\int_0^1 1-\left(1 - \left(1-(1-u)^{1/\alpha}\right)\left(1-(1-v)^{1/\alpha}\right)\left[1+\theta\left(1-u-v+uv\right)^{1/\alpha}\right]\right)^{\alpha} du\,dv-3.$$

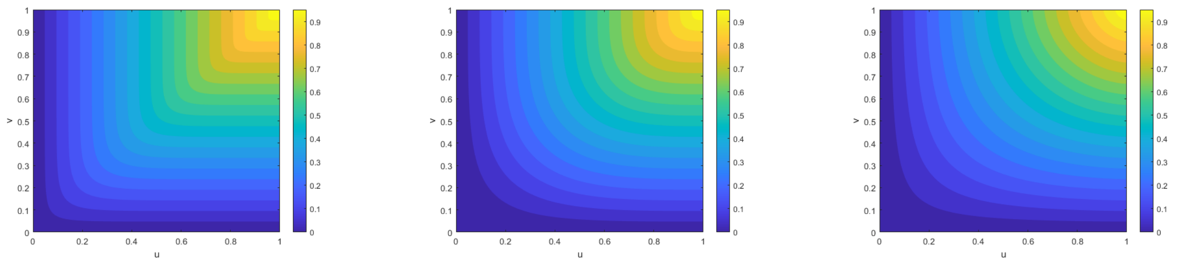
As the above integral is not in explicit form, we use numerical integration technique to compute the Spearman's Rho for various values of α and θ. The computed values are presented in Table 3.3. It is observed from Table 3.3 that the discrete Mittag-Leffler-FGM copula improved the range of Spearman's Rho by $(-0.33333, 0.95734)$. Further, the contour plots of the proposed copula for various values of the parameters are shown in Fig. 3.4.

3.7.3 Data Analysis

In this subsection, we analyze a real data set to illustrate the practical applicability of PGF copula. We use maximum likelihood (ML) estimation method for estimating the unknown parameters. The R software (version: 4.3.2) is used for numerical computations and data analysis purpose. Here, we consider UEFA Champion's League data set first reported in Meintanis (2007). In this data set, X and Y represents the time (in minutes) of

TABLE 3.3. Spearman's $\rho_{C^{\gamma}}$ of the Discrete Mittag-Leffler-FGM copula in Eq. (3.22) for various values of α and θ.

$\theta \rightarrow$ $\downarrow \alpha$	**-1**	**-0.8**	**-0.6**	**-0.4**	**-0.2**	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**
0.1	0.948074	0.948967	0.949868	0.950776	0.95169	0.952613	0.953542	0.954479	0.955425	0.956378	**0.957334**
0.2	0.831057	0.835698	0.840373	0.845084	0.84983	0.854614	0.859435	0.864294	0.773357	0.874133	0.879113
0.3	0.685005	0.695777	0.706624	0.717547	0.728547	0.739626	0.750787	0.762029	0.773357	0.78477	0.796273
0.4	0.528052	0.546391	0.564844	0.583411	0.602096	0.6209	0.639827	0.658878	0.678056	0.697364	0.716805
0.5	0.369742	0.396351	0.423099	0.449989	0.477024	0.504207	0.53154	0.559027	0.58667	0.614473	0.642439
0.6	0.215077	0.250169	0.28541	0.320803	0.356351	0.392056	0.427921	0.463949	0.500144	0.536509	0.573046
0.7	0.066602	0.110087	0.153713	0.19748	0.241391	0.285449	0.329656	0.374015	0.418529	0.463199	0.508029
0.8	-0.0745	-0.02289	0.028823	0.080653	0.132597	0.184657	0.236833	0.28913	0.341548	0.394089	0.446755
0.9	-0.20781	-0.14846	-0.08904	-0.02956	0.029981	0.089592	0.14927	0.209016	0.268831	0.328715	0.388671
1	**-0.33333**	-0.26667	-0.2	-0.13333	-0.06667	0	0.066667	0.133333	0.2	0.266667	0.333333



(A) $\alpha = 0.1, \theta = 0.9$ (B) $\alpha = 0.3, \theta = -0.8$ (C) $\alpha = 0.6, \theta = 0.5$

FIGURE 3.4. Contour plots of the discrete Mittag-Leffler-FGM copula for various values of α and θ

the first goal scored by any team and home team, respectively. We fit Weibull distribution for both marginals, i.e., X and Y follows Weibull distribution with parameters a_1, b_1 and a_2, b_2, with DFs

$$G_1(x) = 1 - e^{-(x/b_1)^{a_1}}; x > 0; a_1 > 0, b_1 > 0 \text{ and } G_2(y) = 1 - e^{-(y/b_2)^{a_2}}; y > 0; a_2 > 0, b_2 > 0.$$

We perform Kolmogorov-Smirnov (KS) one sample test for the goodness of fit test. KS test suggests that X supports Weibull distribution with parameter $\hat{a}_1 = 2.120, \hat{b}_1 = 45.938$ (p-value=0.8042, KS statistic value=0.10555). Similary Y also supports Weibull distribution with parameter $\hat{a}_2 = 1.421, \hat{b}_2 = 36.052$ (p-value=0.9602, KS statistic value=0.8042). We now fit the bivariate FGM distribution, bivariate Geometric-FGM and bivariate discrete Mittag-Leffler-FGM distribution with marginal distributions are Weibull distribution. Now we fitted these bivariate distributions for UEFA Champion's League data using maximum likelihood procedure. We use log-likelihood (LL), Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) for the purpose of model selection. Parameter estimates, LL, AIC, and BIC values are tabulated in Table 3.4. We also compare these bivariate models with the bivariate generalized exponential distribution by Mirhosseini et al. (2015), bivariate linear exponential distribution by Pathak and Vellaisamy (2022), bivariate Weibull-linear exponential by Arshad et al. (2023). From Table 3.4, it is clear that discrete Mittag-Leffler FGM Weibull is more appropriate for modelling UEFA champion's league data.

Table 3.4. ML estimates, LL, AIC, and BIC values for the bivarite distributions using UEFA champion's league data set.

Bivariate Distribution	ML Estimates	LL	AIC	BIC
Discrete Mittag-Leffler-FGM Weibull	$\hat{a}_1 = 2.0713,\ \hat{b}_1 = 45.5798,\ \hat{a}_2 = 1.4124,\ \hat{b}_2 = 36.2686,\ \hat{\theta} = 0.0546,\ \hat{\alpha} = 0.5539$	**-320.5838**	**653.8331**	**662.8331**
FGM Weibull	$\hat{a}_1 = 2.11,\ \hat{b}_1 = 46.233,\ \hat{a}_2 = 1.43,\ \hat{b}_2 = 36.206,\ \hat{\theta} = 0.977$	-323.3560	656.7120	664.7670
Bivariate Linear Exponential	$\hat{a}_1 = 0.00001,\ \hat{b}_1 = 0.00079,\ \hat{a}_2 = 0.00311,\ \hat{b}_2 = 0.00092,\ \hat{\theta} = 0.75905$	–323.7027	657.4054	665.4599
Geometric-FGM Weibull	$\hat{a}_1 = 2.037,\ \hat{b}_1 = 45.5518,\ \hat{a}_2 = 1.4158,\ \hat{b}_2 = 36.2230,\ \hat{\theta} = 0.01634\ \hat{p} = 0.1049$	-322.8890	657.7180	667.4435
Bivariate Weibull-Linear Exponential	$\hat{a} = 0.0017845,\ \hat{b} = 0.0007325,\ \hat{c} = 0.0055796,\ \hat{d} = 1.3609911,\ \hat{\theta} = 0.6679616$	–326.7092	663.4180	671.4731
Bivariate Generalized Exponential	$\hat{\alpha}_1 = 0.0244,\ \hat{\alpha}_2 = 0.0304,\ \hat{\theta} = 0.999$	-340.5234	687.0468	691.8795

3.8 Conclusion and Future Direction

A method to generalize the copula using PGF of a positive integer-valued RV is proposed. Several copulas in the literature are the sub-family of the proposed copula. An algorithm for generating random numbers from the PGF copula is discussed. Various dependence measures of the PGF copula are discussed. Further using this method, two generalized FGM copulas using PGFs of geometric and discrete Mittag-Leffler distribution are proposed. The improved dependence range of these two copulas are $(-0.33333, 0.475118)$ and $(-0.33333, 0.957339)$ respectively. A real data is analyzed to show the practical applicability of PGF copula. It is important to note that our focus on generalizing the FGM copula does not limit the applications of this approach; any bivariate copula can be utilized to propose a family of copulas, with improved dependence ranges. Moreover, while this study has concentrated on the bivariate case, the method can naturally be extended to the multivariate setting.

CHAPTER 4

Copula-Based Information Measures Using Shannon Entropy

This chapter introduces various copula-based information measures, such as entropy, the information generating function, and a Kullback–Leibler divergence, based on Shannon entropy. A consistent nonparametric estimator based on the empirical beta copula is also discussed. Finally, the application of the proposed copula-based divergence measure in goodness-of-fit testing and a copula selection criterion is demonstrated.

4.1 Introduction

Entropy is a fundamental concept in information theory with wide-ranging applications across disciplines such as statistical mechanics, machine learning, finance, insurance, physics, chemistry, and reliability. The formulation and generalization of various entropy measures have recently gained significant interest from both theoretical and applied perspectives. The origins of entropy date back to the seminal work of Shannon (1948), who introduced it to quantify uncertainty in information systems.

In Chapter 1, we reviewed the development of entropy, tracing its evolution from Shannon's foundational work to recent multivariate extensions.In multivariate data analysis, uncertainty can be decomposed into two components: (i) the uncertainty contributed by the marginal distributions of individual variables, and (ii) the uncertainty arising from the dependence structure among the variables. Copulas have emerged as essential tools for modelling and quantifying such dependence structures, making copula-based information measures highly relevant.

The role of copula-based information measures in multivariate data analysis was first explored by Ma and Sun (2011), who showed that the mutual information (MI) of a multivariate random vector is equal to the negative of the copula entropy, defined as

$$\zeta(c) = -\int_{\mathbb{I}^p} c(\mathbf{u}) \log c(\mathbf{u})\, d\mathbf{u}, \tag{4.1}$$

where $c(\mathbf{u})$ denotes the copula density. Consequently, MI of a multivariate random vector $\mathbf{X}$ is entirely determined by the dependence structure, captured by the copula, and is independent of the marginal distributions. Furthermore, when the marginals of $\mathbf{X}$ are identical, the differential entropy coincides with the copula entropy $\zeta(c)$.

Copula entropy has been successfully applied in various domains, including image processing, financial engineering, and hydrology (see Zhao and Lin (2011), Hao and Singh (2015), Singh and Zhang (2018)).

However, copula entropy has certain limitations. Notably, it is always non-positive, and its definition requires the existence of the copula density, which may not hold in many cases. These challenges motivate the development of the **multivariate cumulative copula entropy (CCE)**, extending the bivariate version proposed by Sunoj and Nair (2025). The proposed measure addresses the limitations of copula entropy and offers a more flexible and robust framework for quantifying the uncertainty associated with the dependence structure in multivariate data.

The primary objective of this chapter is to study copula-based multivariate information measures using Shannon entropy. The main contributions of the chapter are summarized below:

- A multivariate cumulative copula entropy (CCE) is proposed and its mathematical properties are discussed, including bounds, stochastic orders, and convergence results. It is shown that the CCE of the weighted arithmetic mean of copulas always exceeds the weighted arithmetic mean of the individual CCEs.
- A cumulative copula information-generating function (CCIGF) is introduced. It is showed that the first derivative of the CCIGF at $s = 1$ yields the negative of the CCE.

- A fractional extension of the CCE is proposed and its properties are explored.
- A nonparametric estimator of the proposed CCE is developed using the empirical beta copula, and its convergence behavior is examined.
- A Kullback-Leibler-based cumulative copula divergence is introduced, which is effective in copula selection problems. Its application is demonstrated using real medical data. In addition, a goodness-of-fit test based on the proposed divergence measure is discussed.

The chapter is organized as follows. In Section 4.2, we discuss the mathematical properties of the multivariate CCE and provide illustrative examples. Section 4.3 introduces the CCIGF and explores its key properties. In Section 4.4, we present a fractional extension of the CCE. Section 4.5 is dedicated to the development of a nonparametric estimator based on the empirical beta copula. In Section 4.6, we propose a new divergence measure between two copulas, based on the Kullback–Leibler divergence, and develop a corresponding goodness-of-fit test for copula models using the proposed divergence. Section 4.7 presents a Monte Carlo simulation study to evaluate the 95th percentile and the power of the proposed test across various copula models. Furthermore, a real dataset is analyzed to illustrate the copula selection criteria based on the proposed divergence measure. Finally, conclusions and future research directions are provided in Section 4.8.

4.2 Multivariate Cumulative Copula Entropy

In this section, we propose a p-dimensional CCE, which extends the bivariate CCE proposed by Sunoj and Nair (2025). Let $C(\mathbf{u})$ be a p-dimensional copula, then p-dimensional CCE is defined as

$$\xi\left(C\right) = -\int_{\mathbb{I}^p} C(\mathbf{u}) \log(C(\mathbf{u})) d\mathbf{u},$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_p)$. Since $f(x) = -x\log(x)$ is non-negative and bounded by e^{-1} on $\mathbb{I}$, it follows that $0 \leq \xi(C) \leq e^{-1}$. Now we consider some examples of the multivariate CCE of some well-known multivariate copulas.

Example 4.2.1. *Consider the product copula $\Pi(\mathbf{u}) = u_1 u_2 \ldots u_p$, which corresponds to the independence of random variables. Then, the p-dimensional CCE is given by*

$$\xi\left(\Pi\right) = \frac{p}{2^{p+1}},$$

which is a decreasing function of $p \geq 2$. *This implies that the uncertainty in a system of independent components decreases with an increase in a number of components.*

Example 4.2.2. *Consider the minimum copula* $M(\mathbf{u}) = \min\{u_1, u_2, \ldots, u_p\}$, *then*

$$\xi(M) = -\int_0^1 \int_0^1 \cdots \int_0^1 \min\{u_1, u_2, \ldots, u_p\} \log\left(\min\{u_1, u_2, \ldots, u_p\}\right) du_1 du_2 \ldots du_p. \tag{4.2}$$

We can solve the above integral using the concept of order statistics. Let $U_1, U_2, \ldots, U_p$ *be a random sample of sample size* p *from uniform distribution over* $\mathbb{I}$ *and let* $U_{(1)} = \min\{U_1, U_2, \ldots, U_p\}$. *The probability density function corresponds to* $U_{(1)}$ *is given by*

$$f_{U_{(1)}}(u) = \begin{cases} p(1-u)^{p-1} & \text{if } u \in \mathbb{I}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Now, the integral in Eq. (4.2) can be viewed as $\mathbb{E}(-U_{(1)} \log\left(U_{(1)}\right)$, *which is given by*

$$\begin{aligned} \xi(M) =& \mathbb{E}(-U_{(1)} \log\left(U_{(1)}\right) \\ =& -\int_0^1 u \log(u) p(1-u)^{p-1} du \\ =& -p \sum_{x=0}^{p-1} \binom{p-1}{x} (-1)^x \int_0^1 u^{x+1} \log(u) du \\ =& p \sum_{x=0}^{p-1} \binom{p-1}{x} \frac{(-1)^x}{(x+2)^2}. \end{aligned}$$

Example 4.2.3. *Consider the* $p-$*variate version of Cuadras-Augé copula, proposed by Cuadras (2009), is given by*

$$C(\mathbf{u}) = u_{(1)} \prod_{i=2}^{p} u_{(i)}^{\prod_{j=1}^{i-1}(1-\alpha_{ij})}, \tag{4.4}$$

where $u_{(1)} \leq u_{(2)} \cdots \leq u_{(p)}$ *and* $\alpha_{ij} \in \mathbb{I}$. *Let* $\theta_1 = 1$, $\theta_i = \prod_{j=1}^{i-1}(1-\alpha_{ij})$, *and* $k(i) = k(i-1) + \theta_i + 1$ *with* $k(1) = 2$, *for* $i = 2, 3, \ldots, p$. *The CCE corresponds to Cuadras-Augé copula is given by*

$$\begin{aligned} \xi(C) =& -\int_0^1 \int_0^1 \cdots \int_0^1 \prod_{i=1}^{p} u_{(i)}^{\theta_i} \log\left(\prod_{i=1}^{p} u_{(i)}^{\theta_i}\right) du_1 du_2 \ldots du_p \\ =& -p! \int_0^1 \int_0^{u_p} \int_0^{u_{p-1}} \cdots \int_0^{u_2} \prod_{i=1}^{p} u_i^{\theta_i} \log\left(\prod_{i=1}^{p} u_i^{\theta_i}\right) du_1 du_2 \ldots du_p \\ =& p! \sum_{j=1}^{p} \theta_j I_j, \end{aligned}$$

where for every $j = 1, 2, 3 \dots, p$,

$$
\begin{aligned}
I_j =& - \int_0^1 \int_0^{u_p} \int_0^{u_{p-1}} \cdots \int_0^{u_2} u_1 u_2^{\theta_2} u_3^{\theta_3} \dots u_p^{\theta_p} \log(u_j) du_1 du_2 \dots du_p \\
=& \frac{1}{\prod_{i=1}^{p} p(i)} \left(\sum_{i=j}^{p} \frac{1}{k(j)} \right).
\end{aligned}
$$

In literature, there exist several dependence measures for quantifying the dependence ability captured by the copula. One of the popular measures is Spearman's correlation. For bivariate case Spearman's Rho for the copula C is defined as

$$\rho_2(C) = 12 \int_0^1 \int_0^1 C(u_1, u_2) du_1 du_2 - 3.$$

Due to the lack of symmetry, the concordance measures in the multivariate case, Spearman's Rho can be defined in two ways,

$$\rho_p^-(C) = n(p) \left[2^p \int_{\mathbb{I}^p} C(\mathbf{u}) d\mathbf{u} - 1 \right] \tag{4.5}$$

and

$$\rho_p^+(C) = n(p) \left[2^p \int_{\mathbb{I}^p} \Pi(\mathbf{u}) dC(\mathbf{u}) - 1 \right],$$

where $n(p) = \dfrac{p+1}{2^p - p - 1}$ (for more details see Schmid et al. (2010) and Bedő and Ong (2016)). Using the multivariate version of Spearman's $\rho_p^-(C)$, we have the following theorem.

Theorem 4.2.1. *For every p-dimensional copula,*

$$\xi(C) \leq -\mathcal{B}_p(C) \log \left(\mathcal{B}_p(C) \right),$$

where $\mathcal{B}_p(C) = 2^{-p} \left[\dfrac{\rho_p^-(C)}{n(p)} + 1 \right]$ *and* $\rho_p^-(C)$ *is the multivariate version of Spearman's correlation defined in Eq. (4.5).*

Proof. Using log-sum inequality, we have

$$
\begin{aligned}
\int_{\mathbb{I}^p} C(\mathbf{u}) \log \left(C(\mathbf{u}) \right) d\mathbf{u} \geq & \left[\int_{\mathbb{I}^p} C(\mathbf{u}) d\mathbf{u} \right] \left[\log \left(\frac{\int_{\mathbb{I}^p} C(\mathbf{u}) d\mathbf{u}}{\int_{\mathbb{I}^p} d\mathbf{u}} \right) \right] \\
=& \mathcal{B}_p(C) \log \left(\mathcal{B}_p(C) \right).
\end{aligned}
$$

The theorem follows by multiplying both sides by -1. □

Definition 4.2.2. *Let $C_1(\mathbf{u}), C_2(\mathbf{u}), \ldots, C_m(\mathbf{u})$ be m copulas of same dimension. Then, the weighted arithmetic mean of m copulas is defined as*

$$C^{\Sigma}(\mathbf{u}) = \sum_{i=1}^{m} \alpha_i C_i(\mathbf{u}),$$

where $\alpha_i \in \mathbb{I},\ i = 1, 2, \ldots, m$ with $\sum_{i=1}^{m} \alpha_i = 1$.

Note that the weighted arithmetic mean of m copulas of the same dimension is always a valid copula.

Theorem 4.2.3. *The weighted arithmetic mean of the CCE of m copulas of the same dimension never exceeds the CCE of the weighted arithmetic mean of m copulas.*

Proof. Let $C_1, C_2, \ldots C_m$ be m copulas and $C^{\Sigma}(\mathbf{u}) = \sum_{i=1}^{m} \alpha_i C_i(\mathbf{u})$ be the arithmetic mean of m copulas, where $\alpha_i \in \mathbb{I},\ i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} \alpha_i = 1$. Since $f(x) = -x\log(x)$ is concave on $\mathbb{I}$, it follows that for every $\alpha_i \in \mathbb{I},\ i = 1, 2, \ldots, m$, with $\sum_{i=1}^{m} \alpha_i = 1$, we have

$$f\left(\sum_{i=1}^{m} \alpha_i x_i\right) \geq \sum_{i=1}^{m} \alpha_i f(x_i), \tag{4.6}$$

for every $x_i \in \mathbb{I}$. Substituting $x_i = C_i(\mathbf{u})$ and integrating over $\mathbb{I}^p$, the result immediately follows. □

Theorem 4.2.4. *Let $\{C_n : n \in \mathbb{N}\}$ be a sequence of copulas of the same dimension that converges point-wise to C, then $\xi(C_n)$ converges uniformly to $\xi(C)$.*

Proof. The sequence of copula $\{C_n : n \in \mathbb{N}\}$ converges point-wise to the copula C implies that C_n converges uniformly to C (see Theorem 1.7.6 of Durante and Sempi (2016)). It follows that for a given $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$|C_n(\mathbf{u}) - C(\mathbf{u})| < \delta, \quad \text{for every } n \geq n_0 \text{ and for every } \mathbf{u} \in \mathbb{I}^p. \tag{4.7}$$

Since $f(x) = -x\log(x)$ is uniformly continuous on $\mathbb{I}$, it implies that for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \varepsilon, \tag{4.8}$$

whenever $|x_1 - x_2| < \delta$. Substituting $x_1 = C_n(\mathbf{u})$ and $x_2 = C(\mathbf{u})$ in Eq.(4.8) and using Eq. (4.7), we obtain

$$\lim_{n\to\infty} -C_n(\mathbf{u})\log(C_n(\mathbf{u})) = -C(\mathbf{u})\log(C(\mathbf{u})).$$

Since $-C_n(\mathbf{u})\log(C_n(\mathbf{u}))$ is bounded on $\mathbb{I}^p$, using bounded convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty}\xi(C_n) &= \lim_{n\to\infty}\int_{\mathbb{I}^p} -C_n(\mathbf{u})\log(C_n(\mathbf{u}))\,d\mathbf{u}\\ &= \int_{\mathbb{I}^p}\lim_{n\to\infty} -C_n(\mathbf{u})\log(C_n(\mathbf{u}))\,d\mathbf{u}\\ &= \xi(C).\end{aligned}$$

□

Definition 4.2.5. *(Nelsen (2006)) Let $C_1(\mathbf{u})$ and $C_2(\mathbf{u})$ be two p-dimensional copulas. Then, C_1 is said to be less positive lower orthant dependent (PLOD) than C_2, denoted by $C_1 \overset{PLOD}{\prec} C_2$, if*

$$C_1(\mathbf{u}) \le C_2(\mathbf{u}) \quad \textit{for all } \mathbf{u}\in\mathbb{I}^p.$$

Next, we show that PLOD ordering does not necessarily imply the corresponding CCE ordering through a counterexample by considering

$$C_1(u_1,u_2) = \left(1+\left[\left(u_1^{-1}-1\right)^2+\left(u_2^{-1}-1\right)^2\right]^{0.5}\right)^{-1}$$

and $C_2(u_1,u_2)=\min\{u_1,u_2\}$. It is a well-known result that $C_1 \overset{\text{PLOD}}{\prec} C_2$. But, $\xi(C_1) = 0.2790$ and $\xi(C_2)=0.2777$.

4.3 Cumulative Copula Information Generating Function

In this section, we introduce a generating function for CCE and study its important properties. Let $C(\mathbf{u})$ be a p-dimensional copula, then we define the cumulative copula information generating function (CCIGF) as follows.

Definition 4.3.1. *Let $C(\mathbf{u})$ be a p-dimensional copula, then CCIGF, defined as*

$$\mathcal{G}_C(s) = \int_{\mathbb{I}^p}[C(\mathbf{u})]^s\,d\mathbf{u},\ \ s>0. \tag{4.9}$$

It is easy to show that the first derivative of $\mathcal{G}_C(s)$ at $s=1$ reduces to $-\xi(C)$. So, we call $\mathcal{G}_C(s)$ as a cumulative copula information generating function. Moreover, $\mathcal{G}_C(1)=\mathcal{B}_p(C)=2^{-p}\left[\dfrac{\rho_p^-(C)}{n(p)}+1\right]$.

The following are examples of of some well-known copulas available in the literature.

Example 4.3.1. *Consider the Marshall-Olkin copula defined by*

$$C(u_1,u_2)=u_1^{1-\alpha_1}u_2^{1-\alpha_2}\min\{u_1^{\alpha_1},u_2^{\alpha_2}\},\alpha_1,\alpha_2\in\mathbb{I}.$$

Then the CCIGF corresponds to Marshall-Olkin copula is given by

$$\begin{aligned}\mathcal{G}_C(s)=&\int_0^1\int_0^1\left(u_1^{1-\alpha_1}u_2^{1-\alpha_2}\min\{u_1^{\alpha_1},u_2^{\alpha_2}\}\right)^s du_1du_2\\ =&\int_0^1\int_0^{u_2^{\alpha_2/\alpha_1}}u_1^s u_2^{(1-\alpha_2)s}du_1du_2+\int_0^1\int_0^{u_1^{\alpha_1/\alpha_2}}u_2^s u_1^{(1-\alpha_1)s}du_2du_1\\ =&\frac{1}{(s+1)}\left[\frac{\alpha_1+\alpha_2}{(\alpha_1+\alpha_2)(s+1)-\alpha_1\alpha_2 s}\right].\end{aligned}$$

Example 4.3.2. *Consider the FGM copula* $C(u_1,u_2)=u_1u_2\left(1+\theta(1-u_1)(1-u_2)\right)$, *where* $\theta\in\mathbb{I}$. *The CCIGF of FGM copula is given by*

$$\begin{aligned}\mathcal{G}_C(s)=&\int_0^1\int_0^1\left(u_1u_2\left[1+\theta(1-u_1)(1-u_2)\right)\right]^s du_1du_2\\ =&\sum_{x=0}^{\infty}\binom{s+x-1}{x}\theta^x\int_0^1\int_0^1 u_1^s(1-u_1)^x u_2^s(1-u_2)^x du_1du_2\\ =&\sum_{x=0}^{\infty}\binom{s+x-1}{x}\theta^x\left[\beta(s+1,x+1)\right]^2.\end{aligned}$$

Example 4.3.3. *The CCIGF corresponds to the product copula is given by*

$$\begin{aligned}\mathcal{G}_\Pi(s)=&\int_0^1\int_0^1\cdots\int_0^1(u_1u_2\dots u_p)^s du_1du_2\dots u_p.\\ =&(s+1)^{-p}.\end{aligned}$$

Example 4.3.4. *The CCIGF of the Cuadras-Augé copula is given by*

$$\begin{aligned}\mathcal{G}_C(s)=&\int_0^1\int_0^1\cdots\int_0^1\prod_{i=1}^p u_{(i)}^{s\theta_i}\mathbf{1}(u_{(1)}\le u_{(2)}\cdots\le u_{(p)})\,du_1du_2\dots du_p\\ =&p!\int_0^1\int_0^{u_p}\int_0^{u_{k-1}}\cdots\int_0^{u_2}\prod_{i=1}^p u_i^{s\theta_i}du_1du_2\dots du_p\\ =&p!\prod_{i=1}^p[q(i)]^{-1},\end{aligned}$$

where $q(i)=\alpha_i s+1+q(i-1)$ *with* $q(1)=s+1$, $u_{(1)}\le u_{(2)}\cdots\le u_{(p)}$, *and* $\theta_1=1$, $\theta_i=\prod_{j=1}^{i-1}(1-\alpha_{ij}),\alpha_{ij}\in\mathbb{I}$, *for every* $i=2,3,\dots,p$.

Using the definition of multivariate Spearman's Rho defined in Eq. (4.5), we have the following theorem.

Theorem 4.3.2. *For any p-dimensional copula C, the following inequality holds.*

$$\mathcal{G}_C(s) \begin{cases} \geq [\mathcal{B}_p(C)]^s, & \textit{if} \;\; s > 1 \\ \leq [\mathcal{B}_p(C)]^s, & \textit{if} \; 0 \leq s \leq 1. \end{cases}$$

Proof. For every $s \in \mathbb{I}$ $(s > 1)$, $f(x) = x^s$ is concave (convex) on $x \in \mathbb{I}$. Using Jensens's inequality and using the definition of multivariate Spearman's Rho defined in Eq. (4.5), the theorem immediately follows. □

The following theorem discusses the ordering property of the CCIGF. The proof is straightforward, so it is omitted here.

Theorem 4.3.3. *Let C_1 and C_2 be two copula of same dimension, then if $C_1 \overset{PLOD}{\prec} C_2$, then $\mathcal{G}_{C_1}(s) \leq \mathcal{G}_{C_2}(s)$, for every $s > 0$.*

The following theorem is due to Theorem 4.3.3, which provides a tight bound for every CCIGF.

Theorem 4.3.4. *For any p-dimensional copula C,*

$$\left(\prod_{i=1}^{p}(s+i)\right)^{-1} \leq \mathcal{G}_C(s) \leq p\beta(s+1,p),$$

where $\beta(q_1, q_2)$ is the standard Beta function and $s > 0$.

Proof. From Eq. (4.9), we have

$$\begin{aligned} \mathcal{G}_C(s) &= \int_{\mathbb{I}^p} [C(\mathbf{u})]^s \, d\mathbf{u} \\ &\geq \int_{\mathbb{I}^p} [W(\mathbf{u})]^s \, d\mathbf{u} \\ &= \underbrace{\int_0^1 \int_0^1 \cdots \int_0^1}_{p \text{ times}} \left(\max\left\{ \sum_{i=1}^{p} u_i - p + 1, 0 \right\} \right)^s du_1 du_2 \ldots du_p \\ &= \left(\prod_{i=1}^{p}(s+i)\right)^{-1}. \end{aligned}$$

Similarly, using the upper bound of every copula, we have

$$\begin{aligned}\mathcal{G}_C(s) \leq& \int_{\mathbb{I}^p} [M(\mathbf{u})]^s \, d\mathbf{u}\\ =&\underbrace{\int_0^1 \int_0^1 \cdots \int_0^1}_{p \text{ times}} (\min\{u_1, u_2, \ldots, u_p\})^s \, du_1 du_2 \ldots du_p\\ =&p\int_0^1 u^s(1-u)^{p-1} du\\ =&p\beta(s+1, p).\end{aligned}$$

□

Definition 4.3.5. *Let $C_1, C_2, \ldots, C_m$ be m copulas of the same dimension. The weighted geometric mean of m copulas is defined as*

$$C^{\Pi}(\mathbf{u}) = C_1(\mathbf{u})^{\alpha_1} C_2(\mathbf{u})^{\alpha_2} \ldots C_m(\mathbf{u})^{\alpha_m},$$

where $\alpha_i \in \mathbb{I}$ for $i = 1, 2, \ldots, m$ with $\sum_{i=1}^{m} \alpha_i = 1$.

Remark 4.3.6. *The weighted geometric mean of copulas may not always be a valid copula. However, it can be a valid copula under certain conditions. For further details, one may refer to Cuadras (2009) and Zhang et al. (2013).*

Theorem 4.3.7. *The CCIGF of the weighted geometric mean of m copulas never exceeds the weighted geometric mean of m copulas.*

Proof. Let $C_1, C_2, \ldots, C_m$ be m copulas, and $C^{\Pi}(\mathbf{u}) = C_1(\mathbf{u})^{\alpha_1} C_2(\mathbf{u})^{\alpha_2} \ldots C_m(\mathbf{u})^{\alpha_m}$ be the weighted geometric mean (WGM) of m copulas, where $\alpha_i \in \mathbb{I}$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} \alpha_i = 1$. Let $\mathcal{G}_{C^{\Pi}}(s)$ and $\mathcal{G}_{C_i}(s); i = 1, 2, \ldots, m$ be CCIGF of $C^{\Pi}(\mathbf{u})$ and $C_i(\mathbf{u}); i = 1, 2, \ldots, m$. The CCIGF of the WGM of m copulas is given by

$$\begin{aligned}\mathcal{G}_C(s) =& \int_{\mathbb{I}^p} [C(\mathbf{u})]^s \, d\mathbf{u}\\ =& \int_{\mathbb{I}^p} [C_1(\mathbf{u})^{\alpha_1} C_2(\mathbf{u})^{\alpha_2} \ldots . C_m(\mathbf{u})^{\alpha_m}]^s \, d\mathbf{u}\end{aligned}$$

Using Hölder's inequality on m integrals (see Kufner et al. (1977) and Finner (1992)), we have

$$\mathcal{G}_{C^{\Pi}}(s) \leq \left(\int_{\mathbb{I}^p} [C_1(\mathbf{u})]^s \, d\mathbf{u}\right)^{\alpha_1} \left(\int_{\mathbb{I}^p} [C_2(\mathbf{u})]^s \, d\mathbf{u}\right)^{\alpha_2} \ldots \left(\int_{\mathbb{I}^p} [C_m(\mathbf{u})]^s \, d\mathbf{u}\right)^{\alpha_m} = \prod_{i=1}^{m} (\mathcal{G}_{C_i}(s))^{\alpha_i}.$$

□

The proofs of the following theorems are similar to the proofs given in Section 4.2, so we left them out here.

Theorem 4.3.8. *Let* $C_1, C_2, \dots C_m$ *be* m *copulas and* $C^{\Sigma}(\mathbf{u}) = \sum_{i=1}^{m} \alpha_i C_i(\mathbf{u})$ *be the arithmetic mean of* m *copulas, where* $\alpha_i \in \mathbb{I}$, $i = 1, 2, \dots, m$ *and* $\sum_{i=1}^{m} \alpha_i = 1$. *Let* $\mathcal{G}_{C^{\Sigma}}(s)$ *and* $\mathcal{G}_{C_i}(s); i = 1, 2, \dots, m$ *be the CCIGF of* C^{Σ} *and* $C_i; i = 1, 2, \dots, m$. *Then*

$$\mathcal{G}_{C^{\Sigma}}(s) \begin{cases} \leq \sum_{i=1}^{m} \alpha_i \mathcal{G}_{C_i}(s), & \text{if } s > 1 \\ \geq \sum_{i=1}^{m} \alpha_i \mathcal{G}_{C_i}(s), & \text{if } 0 \leq s \leq 1. \end{cases}$$

Theorem 4.3.9. *Let* $\{C_n : n \in \mathbb{N}\}$ *be a sequence of copulas of the same dimension that converges point-wise to* C, *then* $\mathcal{G}_{C_n}(s)$ *converges uniformly to* $\mathcal{G}_C(s)$, *for every* $s > 0$.

4.4 Fractional Multivariate Cumulative Copula Entropy

In this section, we generalize the concept of multivariate CCE using fractional calculus. Using the Riemann-Liouville fractional derivative, Ubriaco (2009) proposed the fractional Shannon entropy, which is given in Eq. (1.38). Xiong et al. (2019) and Kayid and Shrahili (2022) further extended the concept to propose the fractional version of cumulative residual entropy and cumulative entropy, respectively. To the best of our knowledge, no prior work has addressed the fractional version of copula entropy, even in the context of bivariate cases.

Definition 4.4.1. *Let* $C(\mathbf{u})$ *be a* p*-dimensional copula, then fractional cumulative copula entropy (FCCE) can be defined as*

$$\xi_{[r]}(C) = \int_{\mathbb{I}^p} C(\mathbf{u}) \left(-\log(C(\mathbf{u})\right)^r d\mathbf{u}, \; 0 \leq r \leq 1. \tag{4.10}$$

For $r = 1$, FCCE reduces to CCE. Following are examples of FCCE of some well-known bivariate and multivariate copulas.

Example 4.4.1. *The fractional CCE of Fréchet-Hoeffding lower bound copula is given by*

$$\begin{aligned}\xi_{[r]}(W) &= \int_0^1 \int_0^1 \max\{u_1 + u_2 - 1, 0\} \left[-\log\left(\max\{u_1 + u_2 - 1, 0\}\right)\right]^r du_1 du_2 \\ &= \int_0^1 \int_{1-u_2}^1 (u_1 + u_2 - 1) \left[-\log\left(u_1 + u_2 - 1\right)\right]^r du_1 du_2 \\ &= \int_0^1 \int_0^{u_2} u_1[-\log(u_1)]^r du_1 du_2.\end{aligned}$$

Using the transformation $t = -\log(u_1)$, *we get* $\xi_{[r]}(W) = \Gamma(r+1)\left(2^{-r-1} - 3^{-r-1}\right)$.

Example 4.4.2. *Consider the bivariate Cuadras-Augé copula given by*

$$C(u_1, u_2) = \left(\min\{u_1, u_2\}\right)^{1-\alpha} \left(u_1 u_2\right)^{\alpha}, \ \alpha \in \mathbb{I}^2.$$

Then the FCCE corresponds to Cuadras-Augé copula is

$$\begin{aligned}\xi_{[r]}(C) &= \int_0^1 \int_0^1 \left(\min\{u_1, u_2\}\right)^{1-\alpha} \left(u_1 u_2\right)^{\alpha} \left[-\log\left(\left(\min\{u_1, u_2\}\right)^{1-\alpha} \left(u_1 u_2\right)^{\alpha}\right)\right]^r du_1 du_2 \\ &= 2! \int_0^1 \int_0^{u_2} u_1 u_2^{\alpha} \left[-\log\left(u_1 u_2^{\alpha}\right)\right]^r du_1 du_2 \\ &= 2! \int_O^{\infty} \int_{e^{-\frac{t}{\alpha+1}}}^1 e^{-2t} t^r u_2^{-\alpha} du_2 dt \qquad (\textit{Using the transformation } t = -\log(u_1).) \\ &= \frac{2!\Gamma(r+1)}{1-\alpha} \left[\frac{1}{2^{r+1}} - \left(\frac{\alpha+1}{\alpha+3}\right)^{r+1}\right].\end{aligned}$$

Example 4.4.3. *The FCCE corresponds to the* $p-$*dimensional product copula is*

$$\begin{aligned}\xi_{[r]}(\Pi) &= \int_0^1 \int_0^1 \cdots \int_0^1 u_1 u_2 \ldots u_p \left[-\log\left(u_1 u_2 \ldots u_p\right)\right]^r du_1 du_2 \ldots du_p \\ &= \int_0^1 \int_0^1 \cdots \int_0^1 \int_{-\log(u_2 u_3 \ldots u_p)}^{\infty} \frac{e^{-2t} t^r}{u_2 u_3 \ldots u_p} du_1 du_2 \ldots du_p \\ &= \int_0^{\infty} \int_{e^{-t}}^1 \int_{e^{-t} u_2^{-1}}^{u_2^{-1}} \cdots \int_{e^{-t} u_2^{-1} \ldots u_p^{-1}}^{u_2^{-1} \ldots u_p^{-1}} \frac{e^{-2t} t^r}{u_2 u_3 \ldots u_p} du_p du_{p-1} \ldots dt \\ &= \frac{\Gamma(r+p)}{2^{r+p}}.\end{aligned}$$

Example 4.4.4. *The FCCE corresponds to the minimum copula is*

$$\begin{aligned}
\xi_{[r]}(M) =& \int_0^1 \int_0^1 \cdots \int_0^1 \min\{u_1, u_2, \ldots, u_p\} \left[-\log\left(\min\{u_1, u_2, \ldots, u_p\}\right)\right]^r du_1 du_2 \ldots du_p \\
=& p \int_0^1 u \left[-\log(u)\right]^r (1-u)^{p-1} \\
=& p \sum_{x=0}^{p-1} \binom{p-1}{x} (-1)^x \int_0^1 u^{x+1} \left[-\log(u)\right]^r du \\
=& p \sum_{x=0}^{p-1} \binom{p-1}{x} (-1)^x \int_0^\infty t^r e^{-(x+2)t} dt \\
=& p \sum_{x=0}^{p-1} \binom{p-1}{x} (-1)^x \frac{\Gamma(r+1)}{(x+2)^{r+2}}.
\end{aligned}$$

In the following theorem, we obtain an upper bound for FCCE in terms of CCE.

Theorem 4.4.2. *Let $C(\mathbf{u})$ be a p-dimensional copula, then $(\xi(C))^r \geq \xi_{[r]}(C)$, for every $0 \leq r \leq 1$.*

Proof. For fixed $r \in \mathbb{I}$, the function $f(x) = x^r$ is concave on $x \in \mathbb{I}$. Using Jensen's inequality on concave function, we have

$$\begin{aligned}
(\xi(C))^r =& \left(-\int_{\mathbb{I}^p} C(\mathbf{u}) \log(C(\mathbf{u})) d\mathbf{u}\right)^r \\
\geq& \int_{\mathbb{I}^p} \left(-C(\mathbf{u}) \log(C(\mathbf{u}))\right)^r d\mathbf{u} \\
\geq& \int_{\mathbb{I}^p} C(\mathbf{u}) \left(-\log(C(\mathbf{u}))\right)^r d\mathbf{u} \\
=& \xi_{[r]}(C).
\end{aligned}$$

□

The proofs of the following theorems are similar to the proofs given in Section 4.2, so we omitted them.

Theorem 4.4.3. *The weighted arithmetic mean of the FCCE of m copulas of the same dimension is always less than or equal to the FCCE of the weighted arithmetic mean of m copulas.*

Theorem 4.4.4. *Let $\{C_n; n \in \mathbb{N}\}$ be a sequence of copulas of the same dimension that converges point-wise to C. Then, $\lim_{n\to\infty} \xi_{[r]}(C_n) = \xi_{[r]}(C)$, for every $r \in \mathbb{I}$.*

4.5 Empirical Beta Cumulative Copula Entropy

In this section, we will develop a nonparametric estimator of CCE and its generating function using empirical beta copula. Sunoj and Nair (2025) proposed an empirical CCE for the bivariate case. However, computationally evaluating the empirical CCE is a time-consuming task for higher dimensional cases. Moreover, the empirical copula is not even a valid copula. Segers et al. (2017) proposed empirical beta copula given by

$$\hat{C}_N(\mathbf{u}) = \frac{1}{N}\sum_{i=1}^{N}\prod_{j=1}^{p}\sum_{y=R_{i,j}}^{N}\binom{N}{y}u_j^y(1-u_j)^{N-y},$$

where $R_{i,j}$ is the rank of the i^{th} observation of the j^{th} component $X_{i,j}$. It is to be noted that an empirical beta copula is a valid copula when there are no ties in the data. Moreover, the empirical beta copula is a particular case of empirical Bernstein copula, introduced by Sancetta and Satchell (2004), when all Bernstein polynomials have degrees equal to the sample size N. In case of ties, we need to break the ties at random, then the empirical beta copula will become a valid copula. Furthermore, the empirical beta copula provides a better estimate compared to the empirical copula in terms of bias and variance (see Segers et al. (2017), Kojadinovic and Yi (2024)). Using the definition of the empirical beta copula, we can define the empirical beta CCE, which is given below.

Definition 4.5.1. *Let $\mathbf{X}_j, j = 1, 2, \ldots, N$ be a random sample of size N from a continuous p-variate distribution, then the empirical beta CCE can be defined as*

$$\xi(\hat{C}_N) = -\int_{\mathbb{I}^p}\hat{C}_N(\mathbf{u})\log\left(\hat{C}_N(\mathbf{u})\right)d\mathbf{u}. \tag{4.11}$$

Analogous to empirical CCE defined in Eq. (4.11), we can define the fractional version of empirical beta CCE and empirical beta CCIGF.

Definition 4.5.2. *The fractional empirical beta CCE corresponds to the N random samples given by*

$$\xi_{[r]}(\hat{C}_N) = \int_{\mathbb{I}^p}\hat{C}_N(\mathbf{u})\left[-\log\left(\hat{C}_N(\mathbf{u})\right)\right]^r d\mathbf{u}, r \in \mathbb{I}.$$

Definition 4.5.3. *For every N random sample, the empirical beta can be defined as*

$$\mathcal{G}_{\hat{C}_N}(s) = \int_{\mathbb{I}^p}\left[\hat{C}_N(\mathbf{u})\right]^s, s > 0.$$

The following theorem asserts that the fractional empirical beta CCE and the empirical beta are always consistent estimators for the FCCE and its information-generating function.

Theorem 4.5.4. *The fractional empirical beta CCE and empirical beta converges to FCCE, and empirical beta CCIGF converges to CCIGF almost surely. That is*

1. $\xi_{[r]}(\hat{C}_N) \xrightarrow{a.s.} \xi_{[r]}(C)$, *as* $N \to \infty$ *and for every* $r \in \mathbb{I}$,
2. $\mathcal{G}_{\hat{C}_N}(s) \xrightarrow{a.s.} \mathcal{G}_C(s)$, *as* $N \to \infty$ *and for every* $s > 0$.

Proof. We prove only the first part of the theorem, second part is similar and are therefore omitted. Let $\mathbf{X}_j$, for $j = 1, 2, \ldots, N$, represent N random samples from a continuous p-variate distribution with underlying copula C. Let C_N be an empirical copula obtained from the sample. Then, the Glivenco-Cantelli theorem on empirical copula states that

$$\sup_{\mathbf{u}\in\mathbb{I}^p} |C_N(\mathbf{u}) - C(\mathbf{u})| \xrightarrow{a.s.} 0, \tag{4.12}$$

as $N \to \infty$. For more details, one could refer Kiefer (1961), Shorack and Wellner (2009), Janssen et al. (2012) and González-Barrios and Hoyos-Argüelles (2021). Segers et al. (2017) showed that for any $p-$dimensional copula C,

$$\sup_{\mathbf{u}\in\mathbb{I}^p} \left|C_N(\mathbf{u}) - \hat{C}_N(\mathbf{u})\right| \leq p\left[\left(\frac{\log N}{N}\right)^{1/2} + N^{-1/2} + N^{-1}\right] \xrightarrow{a.s.} 0, \tag{4.13}$$

as $N \to \infty$. Using Eq.(4.12) and Eq.(4.13), we have

$$\sup_{\mathbf{u}\in\mathbb{I}^p} \left|\hat{C}_N(\mathbf{u}) - C(\mathbf{u})\right| \xrightarrow{a.s.} 0,$$

as $N \to \infty$. Since we $f(x) = -x\log(x)$ is continuous on $\mathbb{I}$, it follows that

$$\lim_{N\to\infty} \sup_{\mathbf{u}\in\mathbb{I}^p} \left|-\hat{C}_N(\mathbf{u})\log(C_N(\mathbf{u})) + C(\mathbf{u})\log(C(\mathbf{u}))\right| = 0,$$

almost surely as $N \to \infty$. Since the CCE is always bounded, using the dominated convergence theorem, the result immediately follows. □

Now, we will illustrate the consistency property of the fractional empirical beta CCE and empirical beta CCIGF by considering various bivariate and trivariate copulas available in the literature. We consider the following multivariate copulas for the illustration purpose:

1. Product Copula: $\Pi(\mathbf{u}) = \Pi_{i=1}^{p} u_i$.
2. Clayton Copula: $C(\mathbf{u}) = \max\left\{\sum_{i=1}^{p} u_i^{\alpha} - p + 1, 0\right\}^{-1/\alpha}, \alpha \in [-1, \infty) \setminus \{0\}$.
3. Gumbel-Hougaard Copula: $C(\mathbf{u}) = \exp\left\{-\left(\sum_{i=1}^{p}(-\log(u_i))^{\phi}\right)^{1/\phi}\right\}, \phi \geq 1$.

4. Frank Copula: $C(\mathbf{u}) = -\theta^{-1}\log\left(1+\frac{\prod_{i=1}^{p} e^{-\theta u_i - 1}}{e^{-\theta}-1}\right), \theta \in \mathbb{R}\setminus\{0\}$.
5. Joe Copula: $C(\mathbf{u}) = 1-\left(1-\left[1-(1-u_1)^{\theta}\right]\cdots\left[1-(1-u_p)^{\theta}\right]\right)^{1/\theta}$, where $\theta \geq 1$.
6. Normal copula: $C(\mathbf{u}) = \mathbf{\Phi}_{\boldsymbol{\rho}}\left(\phi(u_1), \phi(u_2), \ldots, \phi(u_p)\right)$, where $\mathbf{\Phi}_{\boldsymbol{\rho}}(\cdot)$ is the CDF of multivariate normal distribution with zero mean and correlation matrix $\boldsymbol{\rho} = [\rho_{ij}]$ with each $|\rho_{ij}| < 1$ and $\rho_{ij} = \rho_{ji}$ for $i \neq j = 1, 2, 3, \ldots, p$.

We generated 1000 random samples from the copula and computed the fractional empirical beta CCE and empirical beta CCIGF, comparing them with the actual values. Since no closed-form expression can be obtained for the empirical CCE, we evaluate the integrals numerically using the `adaptIntegrate` function in the `cubature` package of R (version 4.2.2). Figures 4.1 and 4.2 illustrate the consistency of the nonparametric estimates of the fractional CCE for the Clayton, Gumbel–Hougaard, and Gaussian copulas in both bivariate and trivariate cases. These figures demonstrate that the shape of the fractional CCE varies with the copula dimension. Figures 4.3 and 4.4 show the consistency between the empirical beta CCIGF and the corresponding theoretical values for various values of s in both bivariate and trivariate settings. The following theorem provides a bound for the empirical information-generating function.

The following theorem will provide a bound for the empirical information-generating function.

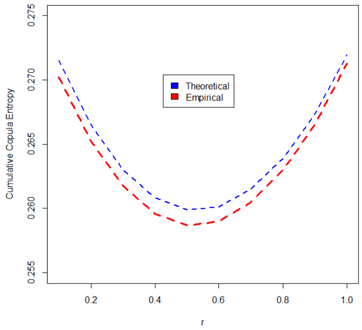
Theorem 4.5.5. *For any p-dimensional copula C,*

$$\mathcal{G}_{\hat{C}_N}(s) \begin{cases} \geq \left[N^{-1}\sum_{i=1}^{N}\prod_{j=1}^{p}\sum_{y=R_{i,j}}^{N}\beta(y+1, N-y+1)\right]^{s}, & \text{if } s > 1 \\ \leq \left[N^{-1}\sum_{i=1}^{N}\prod_{j=1}^{p}\sum_{y=R_{i,j}}^{N}\beta(y+1, N-y+1)\right]^{s}, & \text{if } 0 \leq s \leq 1. \end{cases}$$

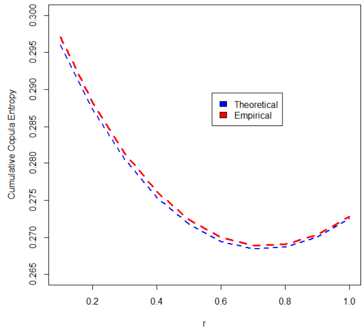
Proof. Consider the integral,

$$\begin{aligned}\int_{\mathbb{I}^p}\hat{C}_N(\mathbf{u})d\mathbf{u} &= \int_0^1\int_0^1\cdots\int_0^1 \frac{1}{N}\sum_{i=1}^{N}\prod_{j=1}^{p}\sum_{y=R_{i,j}}^{N}\binom{N}{y}u_j^{y}(1-u_j)^{N-y}du_1du_2\ldots u_p \\ &= \sum_{i=1}^{N}\prod_{j=1}^{p}\sum_{y=R_{i,j}}^{N}\beta(y+1, N-y+1).\end{aligned}$$

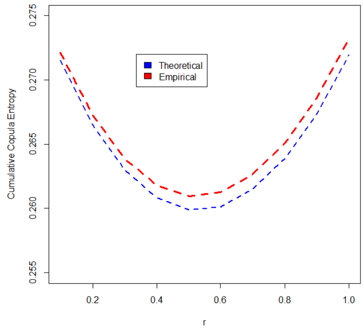
The proof now follows from Theorem 4.3.2. □



(A) Clayton copula with parameter $\alpha = 0.6$

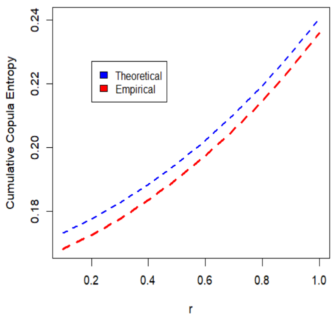


(B) Gumbel-Hougaard copula with parameter $\phi = 2$

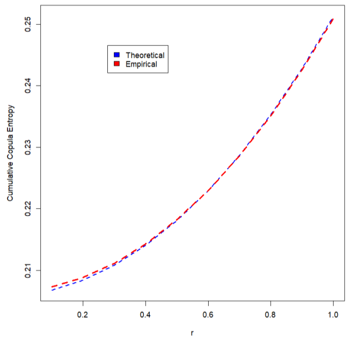


(C) Gaussian copula with parameters $\rho_1 = 0.2, \rho_2 = 0.6$

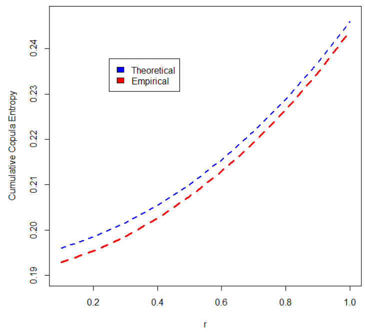
FIGURE 4.1. The fractional empirical beta CCE and theoretical fractional CCE of various bivariate copulas.



(A) Clayton copula with parameter $\alpha = 0.6$



(B) Gumbel-Hougaard copula with parameter $\phi = 2$

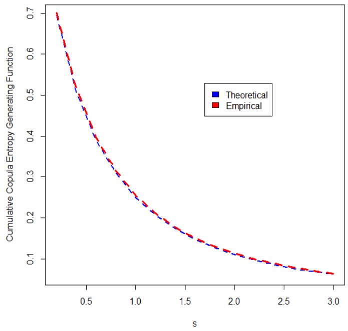


(C) Normal copula with parameters $\rho_1 = 0.2, \rho_2 = 0.6, \rho_3 = 0.9$

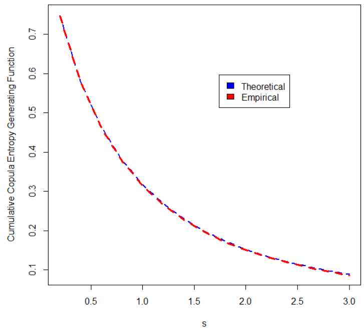
FIGURE 4.2. The fractional empirical beta CCE and theoretical fractional CCE of various trivariate copulas.

4.6 Cumulative Copula Kullback-Leibler Divergence and its Application

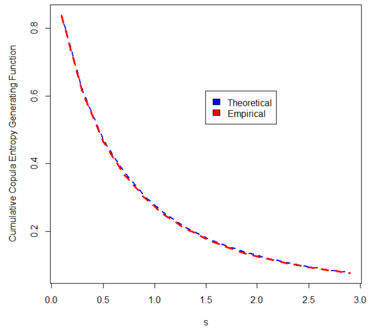
In this section, we propose a new divergence measure between copulas based on the Kullback-Leibler divergence proposed by Kullback and Leibler (1951). Kullback and Leibler (1951) proposed a discrimination measure between two random variables X and



(A) Product copula

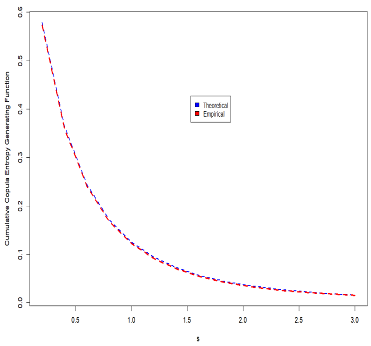


(B) Frank copula with parameter $\theta = 8$

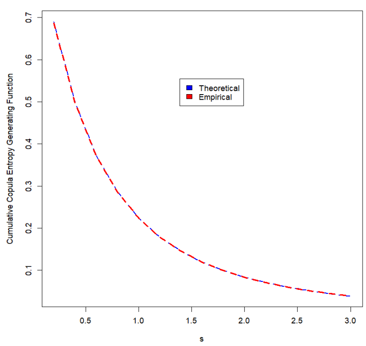


(C) Joe copula with parameter $\theta = 1.5$

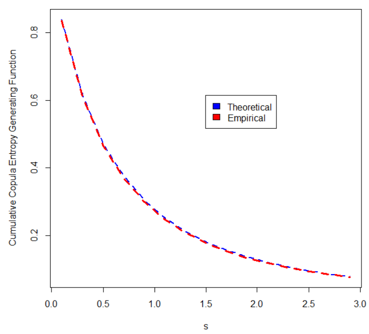
FIGURE 4.3. The empirical beta and theoretical CCIGF of various bivariate copulas.



(A) Product copula



(B) Frank copula with parameter $\theta = 8$



(C) Joe copula with parameter $\theta = 1.5$

FIGURE 4.4. The empirical beta and theoretical CCIGF of various trivariate copulas.

Y, having PDF $f(x)$ and $g(x)$ respectively, defined as

$$KL(f||g) = \int_{-\infty}^{\infty} f(x) \log\left(\frac{f(x)}{g(x)}\right) dx. \tag{4.14}$$

This measure is also known as relative entropy. Let $c_1(\cdot)$ and $c_2(\cdot)$ be the underlying copula density corresponds to random vectors $\mathbf{X}$ and $\mathbf{Y}$, respectively. Assume that the each component of $\mathbf{X}$ and $\mathbf{Y}$ are identically distributed, then the Kullback-Leibler divergence between two random vectors is

$$KL(f||g) = \int_{\mathbb{R}^p} f(\mathbf{x}) \log\left(\frac{f(\mathbf{x})}{g(\mathbf{x})}\right) d\mathbf{x} = \int_{\mathbb{I}^p} c_1(\mathbf{u}) \log\left(\frac{c_1(\mathbf{u})}{c_2(\mathbf{u})}\right) d\mathbf{u}. \tag{4.15}$$

Thus, the Kullback-Leibler divergence between two random vectors can be expressed in terms of copula density under certain conditions. The main limitation of this divergence measure is the existence of copula density. In many situations copula density need not exist, this motivated us to propose a new divergence measure in terms of cumulative copula which measure the divergence between two copulas. Baratpour and Rad (2012) proposed a new divergence measure based on the survival function of two non-negative random variables. Let $\bar{F}(x)$ and $\bar{G}(x)$ be the survival functions of X and Y, respectively. Then, the cumulative residual Kullback-Leibler (CRKL) divergence of X and Y is given by

$$CRKL(F||G) = \int_0^\infty \bar{F}(x) \log\left(\frac{\bar{F}(x)}{\bar{G}(x)}\right) dx - [\mathbb{E}(X) - \mathbb{E}(Y)].$$

Baratpour and Rad (2012) used this measure for the goodness of fit test for exponential distribution. Inspired by the work of Baratpour and Rad (2012), we propose a new divergence measure between two copulas of the same dimension.

Definition 4.6.1. *Let $C_1(\mathbf{u})$ and $C_2(\mathbf{u})$ be two copulas of the same dimension, then the cumulative copula Kullback-Leibler (CCKL) divergence of $C_1(\mathbf{u})$ and $C_2(\mathbf{u})$ is defined as*

$$CCKL(C_1||C_2) = \int_{\mathbb{I}^p} C_1(\mathbf{u}) \log\left(\frac{C_1(\mathbf{u})}{C_2(\mathbf{u})}\right) d\mathbf{u} - \left[\frac{\rho_p^-(C_1) - \rho_p^-(C_2)}{2^p n(p)}\right]. \tag{4.16}$$

The following theorem confirms the proposed CCKL divergence, a well-defined divergence measure, between two copulas.

Theorem 4.6.2. *$CCKL(C_1||C_2) \geq 0$ and equality holds if and only if $C_1(\mathbf{u}) = C_2(\mathbf{u})$, $\forall \mathbf{u} \in \mathbb{I}^p$.*

Proof. Using the inequality $x \log\left(\frac{x}{y}\right) \geq x - y$ for every non-negative x and y and by definition of the multivariate version of Spearman's Rho, we have

$$\int_{\mathbb{I}^p} C_1(\mathbf{u}) \log\left(\frac{C_1(\mathbf{u})}{C_2(\mathbf{u})}\right) d\mathbf{u} - \left[\frac{\rho_p^-(C_1) - \rho_p^-(C_2)}{2^p n(p)}\right] \geq \int_{\mathbb{I}^p} C_1(\mathbf{u}) - C_2(\mathbf{u}) d\mathbf{u} - \left[\frac{\rho_p^-(C_1) - \rho_p^-(C_2)}{2^p n(p)}\right]$$
$$\geq 0.$$

It is straight forward that if $C_1(\mathbf{u}) = C_2(\mathbf{u})$ then $CCKL(C_1||C_2) = 0$. Conversely, suppose that $CCKL(C_1||C_2) = 0$, it follows that

$$\begin{aligned} 0 &= \int_{\mathbb{I}^p} C_1(\mathbf{u}) \log\left(\frac{C_1(\mathbf{u})}{C_2(\mathbf{u})}\right) d\mathbf{u} - \left[\frac{\rho_p^-(C_1) - \rho_p^-(C_2)}{2^p n(p)}\right] \\ &= \int_{\mathbb{I}^p} C_1(\mathbf{u}) \log\left(\frac{C_1(\mathbf{u})}{C_2(\mathbf{u})}\right) - [C_1(\mathbf{u}) - C_2(\mathbf{u})]\, d\mathbf{u} \\ &= \int_{\mathbb{I}^p} \left[\frac{C_2(\mathbf{u})}{C_1(\mathbf{u})} - \log\left(\frac{C_2(\mathbf{u})}{C_1(\mathbf{u})}\right) - 1\right] C_1(\mathbf{u}) d\mathbf{u}. \end{aligned}$$

It is easy to verify that $g(z) = z - \log(z) - 1$ is non-negative for every $z \geq 0$ and $g(z) = 0$ if and only if $z = 1$. It follows that $C_1(\mathbf{u}) = C_2(\mathbf{u})$, for every $\mathbf{u} \in \mathbb{I}^p$. □

Now, we will consider the CCKL divergence of some well-known copulas.

Example 4.6.1. *Consider the Fréchet-Hoeffding lower bound copula* $W(u_1, u_2) = \max\{u_1 + u_2 - 1, 0\}$ *and the product copula* $\Pi(u_1, u_2) = u_1 u_2$. *It is obvious that* $\rho_2^-(W) = -1$ *and* $\rho_2^-(\Pi) = 0$. *Moreover,*

$$\int_0^1 \int_0^1 \max\{u_1 + u_2 - 1, 0\} \log\left(\frac{\max\{u_1 + u_2 - 1, 0\}}{u_1 u_2}\right) du_1 du_2 = \frac{1}{36},$$

and $n(2) = 3$ *(using Eq. 4.16). It follows that the CCKL divergence between the Fréchet-Hoeffding lower bound copula and product copula is* $\frac{1}{36} + \frac{1}{12} = \frac{1}{9}$.

Example 4.6.2. *Consider the bivariate product copula* $\Pi(u_1, u_2) = u_1 u_2$ *and the Gumbel-Barnett copula*

$$C(u_1, u_2) = u_1 u_2 \exp\{-\theta \log(u_1) \log(u_2)\}, \theta \in \mathbb{I}.$$

Since $\rho_2^-(C) = -12\left(\theta^{-1} e^{4/\theta} E_i(-4/\theta)\right) - 3$ *(see Yela and Cuevas (2018)), where* $E_i(\cdot)$ *is the usual exponential integral function. Consider the integral*

$$\int_0^1 \int_0^1 u_1 u_2 \log\left(\frac{u_1 u_2}{u_1 u_2 \exp\{-\theta \log(u_1) \log(u_2)\}}\right) du_1 du_2 = \frac{\theta}{16}.$$

Therefore, the CCKL divergence between the product copula and the Gumbel-Barnett copula is given by

$$CCKL(\Pi||C) = \frac{\theta}{16} - \frac{12\left(\theta^{-1} e^{4/\theta} E_i(-4/\theta)\right) - 3}{2^2 n(2)} = \frac{\theta}{16} - \left(\theta^{-1} e^{4/\theta} E_i(-4/\theta) + \frac{1}{4}\right), \theta \in \mathbb{I}.$$

Example 4.6.3. *Consider the $p-$dimensional product copula $\Pi(\mathbf{u}) = u_1u_2\ldots u_p$ and the minimum copula $M(\mathbf{u}) = \min\{u_1, u_2, \ldots, u_p\}$. We have $\rho_p^-(\Pi) = 0$ and $\rho_p^-(M)$ is*

$$\begin{aligned}
\rho_p^-(M) =& n(p)\left[2^p \int_{\mathbb{I}^p} M(\mathbf{u})d\mathbf{u} - 1\right] \\
=& n(p)\left[2^p \int_0^1 pu(1-u)^{p-1}du - 1\right] \qquad \textit{(using Eq. (4.3))} \\
=& n(p)\left[2^p\beta(2,p) - 1\right],
\end{aligned}$$

where $\beta(p, q)$ is the usual beta function. Moreover,

$$\begin{aligned}
\int_{\mathbb{I}^p} \Pi(\mathbf{u}) \log\left(\frac{\Pi(\mathbf{u})}{M(\mathbf{u})}\right) d\mathbf{u} =& \int_0^1\int_0^1 \cdots \int_0^1 u_1u_2\ldots u_p \log\left(\frac{u_1u_2\ldots u_p}{\min\{u_1,u_2,\ldots,u_p\}}\right) du_1du_2\ldots du_p \\
=& p!\int_0^1\int_0^{u_1}\int_0^{u_2}\cdots\int_0^{u_{p-1}} u_1u_2\ldots u_p \log\left(u_2u_3\ldots u_p\right) du_1du_2\ldots du_p \\
=& \sum_{i=2}^p J_i,
\end{aligned}$$

where for each $i = 2, 3, \cdots, p$, $J_i = p!\int_0^1\int_0^{u_p}\int_0^{u_{p-1}}\cdots\int_0^{u_2} u_1u_2\ldots u_p \log(u_i)\, du_1du_2\ldots du_p = -2^{-p-1}\sum_{n=i}^p n^{-1}$. Then the CCKL divergence between the product copula and minimum copula is given by

$$\begin{aligned}
CCKL(\Pi||M) =& \int_{\mathbb{I}^p} \Pi(\mathbf{u}) \log\left(\frac{\Pi(\mathbf{u})}{M(\mathbf{u})}\right) d\mathbf{u} - \left[\frac{\rho_p^-(\Pi) - \rho_p^-(M)}{2^pn(p)}\right]. \\
=& \sum_{i=2}^p J_i + \beta(2,p) - 2^{-p}.
\end{aligned}$$

Example 4.6.4. *Consider the p-dimensional Cuadras-Augé copula defined in Eq. (4.4) and the minimum copula. The multivariate Spearman's Rho of Cuadras-Augé copula is given by*

$$\begin{aligned}
\rho_p^-(C) =& n(p)\left[2^p \int_{\mathbb{I}^p} \prod_{i=1}^p u_{(i)}^{\theta_i} d\mathbf{u} - 1\right] \\
=& n(p)\left[\frac{2^p}{\prod_i^p k(i)} - 1\right],
\end{aligned}$$

where $u_{(1)} \leq u_{(2)} \cdots \leq u_{(p)}$, $\theta_1 = 1$, $\theta_i = \prod_{j=1}^{i-1}(1-\alpha_{ij})$, *and* $k(i) = k(i-1) + \theta_i + 1$ *with* $k(1) = 2$, *for* $i = 2, 3, \ldots, p$. *Furthermore,*

$$\begin{aligned}\int_{\mathbb{I}^p} C(\mathbf{u}) \log\left(\frac{C(\mathbf{u})}{M(\mathbf{u})}\right) d\mathbf{u} =& \int_0^1 \int_0^1 \cdots \int_0^1 u_{(1)} u_{(2)}^{\theta_2} \ldots u_{(p)}^{\theta_p} \log\left(\frac{u_{(1)} u_{(2)}^{\theta_2} \ldots u_{(p)}^{\theta_p}}{u_{(1)}}\right) du_1 du_2 \ldots du_p \\ =& p! \sum_{j=2}^{p} \int_0^1 \int_0^{u_p} \int_0^{u_{k-1}} \cdots \int_0^{u_2} u_1 u_2^{\theta_2} \ldots u_p^{\theta_p} \log\left(u_j^{\theta_j}\right) du_1 du_2 \ldots du_p \\ =& - p! \sum_{j=2}^{p} \theta_j I_j,\end{aligned}$$

where $I_j = \frac{1}{\prod_{i=1}^{p} k(i)} \left(\sum_{i=j}^{p} \frac{1}{k(j)}\right)$ *for every* $j = 2, 3, \ldots, p$. *The CCKL divergence between the Cuadras-Augé copula and the minimum copula is given by*

$$CCKL(C||M) = -p! \sum_{j=2}^{p} \theta_j I_j - \frac{1}{\prod_i^p k(i)} + \beta(2, p).$$

In the literature, several bootstrapping test procedures exist for the goodness-of-fit test for copulas (see Panchenko (2005), Genest et al. (2009), and Kojadinovic et al. (2011)). In the following subsection, we propose a goodness-of-fit test procedure for copulas based on the cumulative copula Kullback-Leibler divergence as an application.

A Goodness of fit test for copula

Let $\{C_\theta : \theta \in \Theta\}$ be a family of copula functions. We want to test the hypothesis

$$H_0 \ : \ C = C_\theta, \quad vs \quad H_A \ : \ C \neq C_\theta.$$

Now, using the definition of CCKL divergence between two copulas, the above hypothesis is equivalent to the hypothesis

$$H_0 \ : \ CCKL(C||C_\theta) = 0, \quad vs \quad H_A \ : \ CCKL(C||C_\theta) > 0.$$

The CCKL divergence between C and C_θ is given by

$$\begin{aligned}CCKL(C||C_\theta) =& \int_{\mathbb{I}^p} C(\mathbf{u}) \log\left(\frac{C(\mathbf{u})}{C_\theta(\mathbf{u})}\right) d\mathbf{u} - \left[\frac{\rho_p^-(C) - \rho_p^-(C_\theta)}{2^p n(p)}\right] \\ =& \int_{\mathbb{I}^p} C(\mathbf{u}) \log\left(\frac{C(\mathbf{u})}{C_\theta(\mathbf{u})}\right) - C(\mathbf{u}) + C_\theta(\mathbf{u}) \ d\mathbf{u} \\ =& - \xi(C) - \int_{\mathbb{I}^p} C(\mathbf{u}) \log\left(C_\theta(\mathbf{u})\right) d\mathbf{u} - \int_{\mathbb{I}^p} C(\mathbf{u}) d\mathbf{u} + \int_{\mathbb{I}^p} C_\theta(\mathbf{u}) d\mathbf{u}.\end{aligned}$$

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_N$ be a random sample of size N from a p-variate distribution with underlying copula C. We approximate the copula C by empirical beta copula $\hat{C}_N$ which yields the following test statistic

$$T_N = -\xi(\hat{C}_N) - \int_{\mathbb{I}^p} \hat{C}_N(\mathbf{u}) \log\left(C_\theta(\mathbf{u})\right) d\mathbf{u} - \int_{\mathbb{I}^p} \hat{C}_N(\mathbf{u}) d\mathbf{u} + \int_{\mathbb{I}^p} C_\theta(\mathbf{u}) d\mathbf{u}. \tag{4.17}$$

Theorem 4.6.3. *The test statistic based on CCKL given in Eq. (4.17) is consistent for the goodness of fit test for copula.*

Proof. Using theorem 4.5.4, we showed that $\xi(\hat{C}_N)$ is a consistent estimator for $\xi(C_\theta)$ under H_0. It implies that $T_N \xrightarrow{P} 0$ under H_0. Moreover, since $\hat{C}_N$ is also a valid copula and by theorem 4.6.2, we have $CCKL(\hat{C}_N||C) > 0$, under H_A. Consequently, $P(T_N > 0) = 1$ as $N \to \infty$ under H_A. Therefore, the test based on the test statistic T_N is a consistent test. □

We reject the null hypothesis H_0 at significance level α if $T_N \geq T_{N,1-\alpha}$, where $T_{N,1-\alpha}$ is the $100(1-\alpha)$th percentile of T_N under H_0. The distribution of T_N under H_0 can't be obtained analytically, so the Monte Carlo simulation method will be used to determine the value of $T_{N,1-\alpha}$. Since computing the test statistic T_N in Eq. (4.17) is often time-consuming, especially for large values of N, we need to approximate T_N by its sample counterpart. Note that the right-hand side (RHS) of Eq. (4.17) can be expressed as

$$\mathbb{E}\left[C_N(\mathbf{U}) \log\left(C_N(\mathbf{U})\right) - C_N(\mathbf{U}) \log\left(C_\theta(\mathbf{U})\right) - C_N(\mathbf{U}) + C_\theta(\mathbf{U})\right],$$

where $U_1, U_2, \ldots, U_p$ are k independently and identically distributed random variables from a uniform distribution over $\mathbb{I}$ and $\mathbf{U} = (U_1, U_2, \ldots, U_p)$. We approximate the expectation by the sample mean, which yields the approximate value of the test statistic T_N given by

$$T_N = \frac{1}{N}\sum_{i=1}^{N}\left[\hat{C}_N(\mathbf{e}_i) \log\left(\hat{C}_N(\mathbf{e}_i)\right) - \hat{C}_N(\mathbf{e}_i) \log\left(\hat{C}_\theta(\mathbf{e}_i)\right) - \hat{C}_N(\mathbf{e}_i) + C_\theta(\mathbf{e}_i)\right], \tag{4.18}$$

where $\mathbf{e}_i = (e_{i,1}, e_{i,2}, \ldots, e_{i,p})$, with $e_{i,j} = \frac{R_{i,j}}{N+1}$, and $R_{i,j}$ is the rank of the i-th observation of the j-th component $X_{i,j}$ for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, p$. The following algorithm will provide an estimated p-value and $100(1-\alpha)$th percentile of T_N for the proposed test based on the statistic T_N.

1. Estimate the copula parameter θ from the given data of size N. We can use any consistent estimator $\hat{\theta}_N$ of the copula parameter θ.
2. Compute the value of the test statistic T_N given in the Eq. (4.18).

3. Generate M random samples of size N from the copula with copula parameter $\hat{\theta}_N$, estimate θ by the same consistent estimator used in Step 1, and calculate the test statistic for each random sample.
4. Let $T_{N_{(1)}}, T_{N_{(2)}}, \ldots, T_{N_{(M)}}$ be the ordered values of the computated test statistic T_N in Step 3. Then the estimated $100(1-\alpha)$th percentile value of T_N is $T_{N_{[(1-\alpha)M)]}}$, where $[\cdot]$ denotes greatest integer function.
5. The estimated p-value associated with the observed test statistic T_N can be computed by

$$\frac{1}{M}\sum_{j=1}^{M}\mathbf{1}\left\{r : T_{N_{(r)}} \geq T_N\right\}.$$

4.7 Simulation Study and Data Analysis

In this section, an extensive simulation study is conducted to estimate the 95th percentile of the test statistic T_N for various sample sizes under different copula models. The simulation study is performed using R software (version 4.2.2). For the simulation study, we generated $10,000$ samples of size $N = 100, 150, 200$ and 250 from different copulas.

First, we estimate the 95th percentile of T_N based on sample sizes $N = 100, 150, 200$ and 250 under Clayton, Frank, Gumbel-Hougaarad, Joe, Normal, and product copulas. The estimated 95th percentile of the statistic T_N for the bivariate and trivariate versions of the considered copulas are reported in Table 4.1 and Table 4.2, respectively.

To calculate size and power, we generated $10,000$ samples of size $N = 100, 150, 200$ and 250 from the specific copula and estimated the size and power of the test based on whether or not the original data came from the assumed copula family under the null hypothesis. It is to be noted that in each bootstrapping sample, we assume that the copula parameters are known in advance, so we are not estimating the copula parameters. Tables [4.3, 4.4, 4.5] show the size and power of the test for some bivariate copula models and Tables [4.6, 4.7, 4.8] shows that size and power (in percentage) of the test for some trivariate copula models. Note that the size of the proposed test is given in bold format, and the copula model parameter values are mentioned in brackets next to each copula model.

Table 4.1. Estimated values of the 95th percentile of T_N for various bivariate copula models

Model	Parameter	Sample Size			
		100	150	200	250
Clayton	$\alpha = 0.5$	1.4062×10^{-3}	9.9775×10^{-4}	7.4645×10^{-4}	5.9598×10^{-4}
	$\alpha = 2$	7.5188×10^{-4}	4.8142×10^{-4}	3.5280×10^{-4}	2.8429×10^{-4}
	$\alpha = 6$	3.8466×10^{-4}	2.2941×10^{-4}	1.6636×10^{-4}	1.2723×10^{-4}
Frank	$\theta = 3$	1.3110×10^{-3}	8.9599×10^{-4}	7.1967×10^{-4}	5.7994×10^{-4}
	$\theta = 5$	1.05197×10^{-3}	7.0819×10^{-4}	5.2954×10^{-4}	4.3991×10^{-4}
	$\theta = 14$	4.9869×10^{-4}	3.3738×10^{-4}	2.5221×10^{-4}	2.0387×10^{-4}
Gumbel-Hougaarad	$\phi = 1.5$	1.3802×10^{-3}	9.6353×10^{-4}	7.3333×10^{-4}	5.8284×10^{-4}
	$\phi = 2$	1.0113×10^{-3}	7.0783×10^{-4}	5.3709×10^{-4}	4.2715×10^{-4}
	$\phi = 4$	5.0686×10^{-4}	3.2607×10^{-4}	2.3746×10^{-4}	1.1916×10^{-4}
Joe	$\theta = 1.5$	1.6839×10^{-3}	1.1841×10^{-3}	9.2153×10^{-4}	7.4284×10^{-4}
	$\theta = 3$	1.1915×10^{-3}	8.5074×10^{-4}	6.4118×10^{-4}	5.2301×10^{-4}
	$\theta = 7$	7.0373×10^{-4}	4.8555×10^{-4}	3.6759×10^{-4}	3.0002×10^{-4}
Normal	$\rho = 0.4$	1.4116×10^{-3}	9.3985×10^{-4}	7.2460×10^{-4}	5.7744×10^{-4}
	$\rho = 0.7$	8.7301×10^{-4}	5.9016×10^{-4}	4.4601×10^{-4}	3.5730×10^{-4}
	$\rho = 0.9$	4.6217×10^{-4}	3.0072×10^{-4}	2.2217×10^{-4}	1.7607×10^{-4}
Product		2.0239×10^{-3}	1.3582×10^{-3}	1.0457×10^{-3}	8.3575×10^{-4}

Table 4.2. Estimated values of the 95th percentile of T_N for various trivariate copula models

Model	Parameter	Sample Size			
		100	150	200	250
Clayton	$\alpha = 0.5$	2.7319×10^{-3}	1.8752×10^{-3}	1.3945×10^{-3}	1.11761×10^{-3}
	$\alpha = 2$	1.5505×10^{-3}	9.9921×10^{-4}	7.2811×10^{-4}	5.5819×10^{-4}
	$\alpha = 6$	8.3354×10^{-4}	4.8876×10^{-4}	3.4780×10^{-4}	2.6978×10^{-4}
Frank	$\theta = 3$	2.3452×10^{-3}	1.6175×10^{-3}	1.2467×10^{-3}	9.9976×10^{-4}
	$\theta = 5$	1.9130×10^{-3}	1.3126×10^{-3}	9.7085×10^{-4}	8.0744×10^{-4}
	$\theta = 14$	1.0311×10^{-3}	6.7041×10^{-4}	5.0303×10^{-4}	4.0423×10^{-4}
Gumbel-Hougaarad	$\phi = 1.5$	2.5235×10^{-3}	1.7071×10^{-3}	1.2952×10^{-3}	1.0597×10^{-3}
	$\phi = 2$	1.9748×10^{-3}	1.31551×10^{-3}	1.0025×10^{-3}	8.0576×10^{-4}
	$\phi = 4$	1.0649×10^{-3}	6.8730×10^{-4}	5.0277×10^{-4}	3.898×10^{-4}
Joe	$\theta = 1.5$	2.9452×10^{-3}	2.0107×10^{-3}	1.5395×10^{-3}	1.2463×10^{-3}
	$\theta = 3$	2.23767×10^{-3}	1.5386×10^{-3}	1.1609×10^{-3}	9.3941×10^{-4}
	$\theta = 7$	1.4641×10^{-3}	9.5634×10^{-4}	7.3481×10^{-4}	5.9205×10^{-4}
Normal	$\rho = (0.1, 0.2, 0.3)$	2.7463×10^{-3}	1.9346×10^{-3}	1.4515×10^{-3}	1.1740×10^{-3}
	$\rho = (0.4, 0.5, 0.6)$	2.1174×10^{-3}	1.4448×10^{-3}	1.0955×10^{-3}	9.0260×10^{-4}
	$\rho = (0.7, 0.8, 0.9)$	1.2762×10^{-3}	8.1735×10^{-4}	6.2295×10^{-4}	4.6812×10^{-4}
Product		3.066207×10^{-3}	2.1759×10^{-3}	1.6934×10^{-3}	1.3827×10^{-3}

TABLE 4.3. Percentage of rejection of H_0 for different bivariate copula models

Copula under H_0	True Copula	Sample Size			
		100	150	200	250
Clayton(0.5)	Clayton(0.5)	**4.88**	**5.14**	**5.17**	**4.81**
	Frank(3)	26.57	48.59	65.42	79.61
	Gumbel-Hougaarad(1.5)	33.28	57.34	80.06	88.67
	Joe(1.5)	57.9	80.54	91.82	97.15
	Normal(0.4)	11.2	16.73	23.33	32.4
	Product	93.64	99.12	99.92	99.98
Frank(3)	Clayton(0.5)	41.21	61.87	75.95	85.6
	Frank(3)	**4.79**	**5.13**	**4.92**	**5.27**
	Gumbel-Hougaarad(1.5)	7.29	8.62	8.83	10.43
	Joe(1.5)	59.96	76.14	85.77	92.41
	Normal(0.4)	13.26	17.33	23.11	31.43
	Product	99.4	99.97	100	100
Gumbel-Hougaarad(1.5)	Clayton(0.5)	41.33	64.49	80.4	88.88
	Frank(3)	4.92	5.31	6.4	7.35
	Gumbel-Hougaarad(1.5)	**5.97**	**5.16**	**4.94**	**4.85**
	Joe(1.5)	52.02	67.91	80.61	88.18
	Normal(0.4)	11.43	14.56	19.2	22.15
	Product	99.33	99.97	100	100
Joe(1.5)	Clayton(0.5)	56.79	75.55	87.98	94.56
	Frank(3)	44.82	65.32	80.01	97.79
	Gumbel-Hougaarad(1.5)	40.41	58.22	71.96	83.04
	Joe(1.5)	**5.23**	**4.96**	**4.83**	**5.28**
	Normal(0.4)	28.41	44.37	57.38	69.64
	Product	71.68	88.85	95.67	98.74
Normal(0.4)	Clayton(0.5)	14.3	21.6	30.74	36.78
	Frank(3)	6.06	9.91	13.54	17.09
	Gumbel-Hougaarad(1.5)	8.01	12.14	15.39	19.71
	Joe(1.5)	38.97	52.23	66.78	77.47
	Normal(0.4)	**4.98**	**5.24**	**5.03**	**4.88**
	Product	78.87	92.72	97.72	99.42
Product	Clayton(0.5)	91.28	98.32	99.76	99.97
	Frank(3)	62.98	99.03	99.93	100
	Gumbel-Hougaarad(1.5)	98.82	99.97	99.99	100
	Joe(1.5)	66.39	85.64	95.2	98.62
	Normal(0.4)	100	100	100	100
	Product	**5.2**	**4.99**	**5.15**	**5.97**

Table 4.4. Percentage of rejection of H_0 for different bivariate copula models

Copula under H_0	True Copula	Sample Size			
		100	150	200	250
Clayton(2)	Clayton(2)	**5.11**	**4.93**	**5.26**	**4.99**
	Frank(5)	94.81	99.69	99.98	100
	Gumbel-Hougaarad(2)	95.81	99.71	100	100
	Joe(3)	99.95	100	100	100
	Normal(0.7)	70.81	90.19	97.62	99.19
	Product	100	100	100	100
Frank(5)	Clayton(2)	97.47	99.75	100	100
	Frank(5)	**81.9**	**93.07**	**97.92**	**99.37**
	Gumbel-Hougaarad(2)	7.36	10.96	13.2	16.56
	Joe(3)	31.61	50.13	65.91	81.25
	Normal(0.7)	10.56	21.08	32.08	41.89
	Product	100	100	100	100
Gumbel-Hougaarad(2)	Clayton(2)	88.47	98.88	99.99	100
	Frank(5)	7.41	10.77	14.74	17.88
	Gumbel-Hougaarad(2)	**5.13**	**5.22**	**4.83**	**5.31**
	Joe(3)	26.32	33.49	50.9	63.7
	Normal(0.7)	5.08	8.9	13.72	19.2
	Product	100	100	100	100
Joe(3)	Clayton(2)	99.85	100	100	100
	Frank(5)	19.76	33.25	49.75	60.54
	Gumbel(2)	18.55	30.27	41.27	52.6
	Joe(3)	**4.77**	**5.15**	**5.03**	**5.11**
	Normal(0.7)	52.83	77.45	90.72	96.14
	Product	100	100	100	100
Normal(0.7)	Clayton(2)	47.17	79.79	93.54	97.98
	Frank(5)	26.08	37.19	48.03	56.66
	Gumbel(2)	19.49	25.64	29.65	38.99
	Joe(3)	68.4	89.71	96.63	99.05
	Normal(0.7)	**5.17**	**5.19**	**4.97**	**5.08**
	Product	100	100	100	100
Product	Clayton(2)	100	100	100	100
	Frank(5)	100	100	100	100
	Gumbel-Hougaarad(2)	100	100	100	100
	Joe(3)	100	100	100	100
	Normal(0.7)	94.77	99.28	99.95	100
	Product	**4.97**	**5.06**	**4.84**	**5.21**

Table 4.5. Percentage of rejection of H_0 for different bivariate copula models

Copula under H_0	True Copula	Sample Size			
		100	150	200	250
Clayton(6)	Clayton(6)	**5.15**	**5.06**	**4.91**	**4.88**
	Frank(14)	97.61	99.32	100	100
	Gumbel-Hougaarad(4)	99.5	100	100	100
	Joe(7)	100	100	100	100
	Normal(0.9)	97.57	99.89	100	100
	Product	100	100	100	100
Frank(14)	Clayton(6)	100	100	100	100
	Frank(14)	**5.02**	**5.21**	**4.93**	**4.87**
	Gumbel-Hougaarad(4)	16.87	29.17	43.47	59.96
	Joe(7)	59.16	78.61	88.58	95.55
	Normal(0.9)	59.16	78.61	88.58	95.55
	Product	100	100	100	100
Gumbel-Hougaarad(4)	Clayton(6)	86.12	97.8	99.62	99.99
	Frank(14)	10.32	21.95	33.57	47.88
	Gumbel-Hougaarad(4)	**4.96**	**5.22**	**5.19**	**4.86**
	Joe(7)	57.88	78.57	89.43	95.86
	Normal(0.9)	6.77	8.67	11.28	17.07
	Product	100	100	100	100
Joe(7)	Clayton(6)	100	100	100	100
	Frank(14)	20.86	46.27	67.1	82.65
	Gumbel-Hougaarad(4)	32.6	65.06	83.02	92.35
	Joe(7)	**5.04**	**5.02**	**4.94**	**4.91**
	Normal(0.9)	69.29	93.75	99	99.82
	Product	100	100	100	100
Normal(0.9)	Clayton(6)	21.01	87.3	99.82	99.98
	Frank(14)	25.58	49.7	73.12	87.86
	Gumbel-Hougaarad(4)	10.73	15.08	21.73	29.03
	Joe(7)	82.06	97.5	99.7	99.99
	Normal(0.9)	**5.13**	**4.85**	**4.93**	**5.19**
	Product	100	100	100	100
Product	Clayton(6)	100	100	100	100
	Frank(14)	100	100	100	100
	Gumbel-Hougaarad(4)	100	100	100	100
	Joe(7)	100	100	100	100
	Normal(0.9)	100	100	100	100
	Product	**5.09**	**5.04**	**4.99**	**5.12**

TABLE 4.6. Percentage of rejection of H_0 for different trivariate copula models

Copula under H_0	True Copula	Sample Size			
		100	150	200	250
Clayton(0.5)	Clayton(0.5)	**4.78**	**4.83**	**5.1**	**4.89**
	Frank(3)	41	70.7	88.65	96.03
	Gumbel-Hougaarad(1.5)	50.46	78.26	92.66	97.55
	Joe(1.5)	83.41	96.77	99.45	99.94
	Normal(0.1, 0.2, 0.3)	62.8	80.15	91.54	96.94
	Product	99.79	99.99	100	100
Frank(3)	Clayton(0.5)	63.19	86.03	94.62	98.14
	Frank(3)	**4.94**	**4.91**	**5.09**	**5.05**
	Gumbel-Hougaarad(1.5)	8.73	9.85	10.46	13.83
	Joe(1.5)	81.9	93.07	97.92	99.37
	Normal(0.1, 0.2, 0.3)	89.47	97.95	99.6	99.93
	Product	100	100	100	100
Gumbel-Hougaarad(1.5)	Clayton(0.5)	57.05	82.82	94.97	98.33
	Frank(3)	4.85	4.96	5.48	6.83
	Gumbel-Hougaarad(1.5)	**5.14**	**5.21**	**4.89**	**4.96**
	Joe(1.5)	74.33	89.29	95.52	98.31
	Normal(0.1, 0.2, 0.3)	88.47	97.21	99.55	99.93
	Product	99.4	99.99	100	100
Joe(1.5)	Clayton(0.5)	79.73	93.73	98.63	99.77
	Frank(3)	68.91	88.06	95.55	98.69
	Gumbel-Hougaarad(1.5)	66.71	84.56	92.61	97.17
	Joe(1.5)	**4.99**	**4.83**	**5.18**	**4.91**
	Normal(0.1, 0.2, 0.3)	17.47	23.55	36.72	49.04
	Product	93.09	99.17	99.92	99.98
Normal(0.1, 0.2, 0.3)	Clayton(0.5)	59.44	80.12	89.17	95.21
	Frank(3)	83.75	96.26	99.3	99.85
	Gumbel-Hougaarad(1.5)	82.16	96.06	99.18	99.8
	Joe(1.5)	16.99	27.04	39.63	55.51
	Normal(0.1, 0.2, 0.3)	**5.19**	**4.95**	**5.12**	**5.08**
	Product	80.13	93.05	97.41	99.47
Product	Clayton(0.5)	99.45	99.97	100	100
	Frank(3)	80.61	100	100	100
	Gumbel-Hougaarad(1.5)	100	100	100	100
	Joe(1.5)	83.41	96.77	99.45	99.97
	Normal(0.1, 0.2, 0.3)	75.22	91.04	96.68	98.86
	Product	**4.98**	**5.09**	**4.99**	**4.93**

Table 4.7. Percentage of rejection of H_0 for different trivariate copula models

Copula under H_0	True Copula	Sample Size			
		100	150	200	250
Clayton(2)	Clayton(2)	**5.03**	**4.98**	**4.91**	**4.99**
	Frank(5)	94.81	99.69	99.98	100
	Gumbel-Hougaarad(2)	99.38	99.99	100	100
	Joe(3)	100	100	100	100
	Normal$(0.4, 0.5, 0.6)$	99.96	100	100	100
	Product	100	100	100	100
Frank(5)	Clayton(2)	99.88	100	100	100
	Frank(5)	**5.11**	**5.17**	**4.84**	**5.15**
	Gumbel-Hougaarad(2)	9.52	14.29	21.34	30.71
	Joe(3)	49.63	77.6	90.36	97.24
	Normal$(0.4, 0.5, 0.6)$	62.34	79.4	90.48	96.36
	Product	100	100	100	100
Gumbel-Hougaarad(2)	Clayton(2)	97.33	99.86	100	100
	Frank(5)	9.25	11.76	18.89	25.86
	Gumbel-Hougaarad(2)	**4.97**	**4.87**	**4.91**	**4.93**
	Joe(3)	39.86	58.63	72.5	85.18
	Normal$(0.4, 0.5, 0.6)$	63.75	83.55	94.43	98.73
	Product	100	100	100	100
Joe(3)	Clayton(2)	100	100	100	100
	Frank(5)	29.85	51.43	70.19	83.57
	Gumbel-Hougaarad(2)	28.93	49.53	64.89	77.01
	Joe(3)	**5.01**	**5.18**	**4.96**	**5.11**
	Normal$(0.4, 0.5, 0.6)$	81.11	97.16	99.7	99.98
	Product	100	100	100	100
Normal$(0.4, 0.5, 0.6)$	Clayton(2)	99.61	100	100	100
	Frank(5)	43.56	72.26	89.59	94.96
	Gumbel-Hougaarad(2)	53.71	81.1	93.52	97.99
	Joe(3)	87.08	98.83	99.93	99.99
	Normal$(0.4, 0.5, 0.6)$	**5.02**	**5.19**	**4.81**	**5.29**
	Product	100	100	100	100
Product	Clayton(2)	100	100	100	100
	Frank(5)	100	100	100	100
	Gumbel-Hougaarad(2)	100	100	100	100
	Joe(3)	100	100	100	100
	Normal$(0.4, 0.5, 0.6)$	100	100	100	100
	Product	**4.97**	**5.11**	**4.85**	**5.04**

TABLE 4.8. Percentage of rejection of H_0 for different trivariate copula models

Copula under H_0	True Copula	Sample Size			
		100	150	200	250
Clayton(6)	Clayton(6)	**4.99**	**4.93**	**4.89**	**5.01**
	Frank(14)	99.71	100	100	100
	Gumbel(4)	99.98	100	100	100
	Joe(7)	100	100	100	100
	Normal(0.7, 0.8, 0.9)	100	100	100	100
	Product	100	100	100	100
Frank(14)	Clayton(6)	100	100	100	100
	Frank(14)	**5.23**	**4.97**	**4.86**	**5.09**
	Gumbel-Hougaarad(4)	21.68	40.24	61.75	78.87
	Joe(7)	90.55	97.01	99.51	99.98
	Normal(0.7, 0.8, 0.9)	97.87	99.82	100	100
	Product	100	100	100	100
Gumbel-Hougaarad(4)	Clayton(6)	84.46	99.9	100	100
	Frank(14)	14.19	31.04	50.42	67.88
	Gumbel-Hougaarad(4)	**5.34**	**4.95**	**4.9**	**5.02**
	Joe(7)	73.51	92.36	97.86	99.6
	Normal(0.7, 0.8, 0.9)	86.57	98.19	99.85	100
	Product	100	100	100	100
Joe(7)	Clayton(6)	100	100	100	100
	Frank(14)	31.14	64.39	84.83	94.35
	Gumbel-Hougaarad(4)	54.3	86.37	95.85	98.94
	Joe(7)	**4.95**	**5.28**	**5.05**	**4.98**
	Normal(0.7, 0.8, 0.9)	88.43	99.16	99.93	99.99
	Product	100	100	100	100
Normal(0.7, 0.8, 0.9)	Clayton(6)	99.97	100	100	100
	Frank(14)	94.31	99.46	100	100
	Gumbel-Hougaarad(4)	60.5	95.81	99.77	100
	Joe(7)	98.1	99.98	100	100
	Normal(0.7, 0.8, 0.9)	**4.76**	**5.32**	**5.24**	**4.89**
	Product	100	100	100	100
Product	Clayton(6)	100	100	100	100
	Frank(14)	100	100	100	100
	Gumbel-Hougaarad(4)	100	100	100	100
	Joe(7)	100	100	100	100
	Normal(0.7, 0.8, 0.9)	100	100	100	100
	Product	**5.04**	**4.89**	**4.99**	**5.08**

It is observed that as the dimension of the copula increases, the power of the proposed test also increases in most cases. In order to continue our discussion, in the following subsection, we use the proposed test for the copula selection problem in a real data set.

4.7.1 Selection of an Appropriate Copula for "Pima Indians Diabetes" Data

In this subsection, we analyze a real dataset to demonstrate the practical utility of the copula selection problem. We consider the "Pima Indians Diabetes" data. The US National Institute of Diabetes and Digestive and Kidney Diseases collected diabetes data from women aged 21 and above, who were of Pima Indian descent and lived around Phoenix, Arizona. The data is freely available in the R software within the `pdp` package. We consider the variables "glucose", "pressure", and "mass" from the dataset, which represent plasma glucose concentration, diastolic blood pressure (mm Hg), and body mass index, respectively. All missing values were removed, resulting in a trivariate dataset with 724 entries. The copulas considered in this study include Clayton, Frank, Gumbel-Hougaarad, Joe, Normal and product copula. The marginal CDF are estimated by empirical distribution, and copula parameters are estimated using the maximum pseudo-likelihood (MPL) estimation method. We use our proposed method for the goodness of fit test, and the p-values of the proposed test are estimated for each copula model. We use the copula having the least CCKL divergence defined in Eq. (4.16) between the empirical beta copula is considered as the model selection criteria. We generated 1000 random samples of size $N = 724$ for estimating the p-values. The MPL estimates of the copula parameters, CCKL value and p-values are reported in Table 4.9. From Table 4.9, Frank copula has the least CCKL divergence between empirical beta copula with p-value$= 0.448$. It follows that Frank copula can be considered as an appropriate choice for modelling the given dataset.

4.8 Conclusion and Future Direction

In this chapter, we introduce multivariate cumulative copula entropy (CCE) and study its various mathematical properties. Furthermore, we propose the cumulative copula information generating function (CCIGF) and explore its properties. Using fractional calculus, we also introduce a fractional version of the multivariate cumulative copula entropy. We proved that the CCE of the weighted arithmetic mean of copulas always exceeds the weighted arithmetic mean of the CCE of copulas. The results are valid for the

TABLE 4.9. MPL estimates of the copula, CCKL divergence and p-values of the proposed test

Copula	Estimate	CCKL	p-value
Clayton	0.23907	5.9516×10^{-4}	0.029
Frank	**1.37762**	$\mathbf{2.4595 \times 10^{-4}}$	**0.488**
Gumbel-Hougaarad	1.15423	5.2801×10^{-4}	0.036
Joe	1.19773	1.4895×10^{-3}	0
Normal	0.22994, 0.22644, 0.28999	3.7083×10^{-4}	0.13
Product		1.4921×10^{-3}	0

CCIGF and fractional cumulative copula entropy (FCCE). We showed that positive lower orthant dependent (PLOD) ordering for two copulas never implies entropy ordering by a counter-example and provides conditions for the entropy ordering of two copulas. The results are valid for FCCE. However, in the case of CCIGF, PLOD ordering preserves the ordering of corresponding CCIGF. We also showed that the CCIGF of the weighted geometric mean of copulas never exceeds the weighted geometric mean of the CCIGF of copulas. We provide a nonparametric estimate of the FCCE and CCIGF using the empirical beta copula. We showed that the proposed nonparametric estimate converges almost surely to the true FCCE and CCIGF, theoretically and numerically. We define a new divergence measure between two copulas using the Kullback-Leibler divergence. Furthermore, using the proposed divergence measure, a goodness-of-fit test procedure is proposed for copulas. A copula selection procedure is discussed through the "Pima Indians Diabetes" dataset to illustrate the applications of the new divergence measure. Since this chapter discusses copula-based information measures using Shannon entropy, and various entropy variants are available in the literature, the work can be naturally extended in that direction as a potential avenue for future research.

CHAPTER 5

Copula-Based Information Measures Using Tsallis Entropy

This chapter introduces non-additive copula-based information measures using Tsallis entropy, offering enhanced flexibility for quantifying uncertainty. A cumulative copula Tsallis entropy is proposed, along with its properties and bounds. A nonparametric version is developed and validated using coupled periodic and chaotic maps. Kerridge's inaccuracy measure and Kullback–Leibler (KL) divergence are extended to the cumulative copula framework. Using the relationship between KL divergence and mutual information, a new cumulative mutual information (CMI) measure is proposed. A test procedure for mutual independence is formulated based on CMI. The effectiveness of the proposed CMI measure is demonstrated using real bivariate financial time series data.

5.1 Introduction

In the context of thermodynamics, when a system is out of equilibrium or its component states exhibit strong interdependence, non-additive entropy provides a more appropriate measure for quantifying the uncertainty involved in the system. Tsallis (1988) proposed a non-additive entropy and can be defined for an absolutely continuous random variable X with PDF $f(\cdot)$ as

$$\mathcal{T}_\alpha(X) = -\int_{-\infty}^{\infty} f(x)\log_{[\alpha]}(f(x))\,dx, \quad \alpha \in \mathcal{A},$$

where $\mathcal{A} = (0,1) \cup (1,\infty)$ and $\log_{[\alpha]}(r) = \frac{r^{\alpha-1}-1}{\alpha-1}, r \geq 0$, for every $\alpha \in \mathcal{A}$. It is noteworthy that $\lim_{\alpha\to1} \log_{[\alpha]}(r) = \log(r)$, implying that $\log_{[\alpha]}(\cdot)$ serves as a fractional generalization of the natural logarithm. Consequently, Tsallis entropy reduces to Shannon entropy as $\alpha \to 1$. Calì et al. (2017) introduced the cumulative Tsallis entropy (CTE), extending the cumulative entropy (CE) defined by Di Crescenzo and Longobardi (2009). Rajesh and Sunoj (2019) further generalized the cumulative residual entropy (CRE) to propose the cumulative residual Tsallis entropy (CRTE), defined as

$$\mathcal{TR}_\alpha(X) = -\int_0^\infty \bar{F}(x) \log_{[\alpha]}(\bar{F}(x))\, dx, \quad \alpha \in \mathcal{A}.$$

Raju et al. (2020) and Raju et al. (2023) subsequently extended both CTE and CRTE to the bivariate setting. Several applications of Tsallis entropy and its variants have been explored in the literature; for comprehensive discussions, see Cartwright (2014), De Albuquerque et al. (2004), Sparavigna (2015), Singh et al. (2017), Mohamed et al. (2022), Toomaj and Atabay (2022), among others. Additionally, Mao et al. (2020) extended the cumulative residual Kullback–Leibler divergence using Tsallis entropy, highlighting its relevance in finance. Recently, Raju et al. (2024) generalized the cumulative inaccuracy measure introduced by Kumar and Taneja (2015) within the Tsallis entropy framework.

In multivariate data analysis, quantifying the uncertainty in the dependence structure is essential, and copula-based information measures play a pivotal role in this context. However, the existing literature on such measures remains relatively limited. Chapter 4 explored copula-based information measures derived from Shannon entropy. Motivated by the importance of Tsallis entropy, this chapter focuses on measuring the uncertainty associated with the dependence structure of multivariate random variables using Tsallis entropy. The chapter further highlights the practical relevance of copula-based information measures through illustrative applications. The main contributions of this chapter are as follows:

- Copula-based information measures are proposed using Tsallis entropy, referred to as cumulative copula Tsallis entropy (CCTE).
- The validity of the proposed CCTE is illustrated in the context of Rulkov maps within chaos and bifurcation theory.
- A non-parametric estimator for CCTE is introduced, and its almost sure convergence is established.

- A new inaccuracy measure for copulas is introduced, with an exploration of its mathematical properties. This measure extends the work of Hosseini and Nooghabi (2021).
- Inspired by Mao et al. (2020), a cumulative copula Tsallis divergence is proposed, derived from cumulative Tsallis divergence.
- To address cases where existing mutual information may not be well-defined, a mutual information measure called Cumulative Mutual Information (CMI) is introduced as an alternative, relying on the relationship between KL divergence and mutual information.
- Two specific applications of the proposed mutual information measure are presented:
    1. Testing the independence of several random variables.
    2. Analyzing multivariate financial time series, where the proposed MI serves as an economic indicator.

The remaining structure of this chapter is organized into three main parts. In Section 5.2, we introduce the cumulative copula Tsallis entropy, examine its mathematical properties, and provide examples using well-known copulas. Section 5.3 presents a non-parametric estimator for the proposed dependence entropy, provides a theoretical proof of its almost sure convergence, and validates the results using Monte Carlo simulations. Section 5.4 discusses the validation of the proposed dependence entropy by applying it to Rulkov maps. The second part of the chapter, starting with Section 5.5, introduces a copula-based inaccuracy measure and explores related inequalities and ordering properties. Section 5.6 presents a newly developed cumulative copula divergence based on the Tsallis divergence, highlighting its properties and introducing a new mutual information measure. The final part of the chapter discusses the applications of the proposed mutual information measure. Section 5.7 is divided into two subsections. Subsection 5.7.1 proposes a new testing procedure for the mutual independence among the components of a multivariate random variable. The proposed test is compared with existing procedures based on Cramér-von Mises and Kolmogorov-Smirnov distance measures. The proposed test is applied to real data to demonstrate its practical utility. Subsection 5.7.2 illustrates the use of the proposed mutual information measure as an economic indicator in analyzing multivariate financial time series. The chapter concludes in Section 5.8 with a summary of the findings and a discussion of potential directions for future research.

5.2 Cumulative Copula Tsallis Entropy

In this section, we propose the cumulative copula Tsallis entropy (CCTE), defined as follows

$$\xi_\alpha(C) = -\int_{\mathbb{I}^p} C(\mathbf{u}) \log_{[\alpha]}(C(\mathbf{u}))\, d\mathbf{u}, \quad \alpha \in \mathcal{A}, \tag{5.1}$$

where $\log_{[\alpha]}(r) = \frac{r^{\alpha-1}-1}{\alpha-1}$ and $\mathcal{A} = (0,1)\cup(1,\infty)$. It is easy to show that for any $\alpha \in \mathcal{A}$, the function $h(r) = -r\log_{[\alpha]}(r)$ is bounded by $0 \le h(r) \le \alpha^{\alpha/1-\alpha}$ for every $r \in \mathbb{I}$. It follows that $0 \le \xi_\alpha(C) \le \alpha^{1/1-\alpha} \le 1$, for every $\alpha \in \mathcal{A}$. Moreover,

$$\lim_{\alpha\to 1} \xi_\alpha(C) = -\int_{\mathbb{I}^p} C(\mathbf{u}) \log\left(C(\mathbf{u})\right) d\mathbf{u} = \xi(C).$$

In the following subsection, we present typical examples of the CCTE for various well-known bivariate and multivariate copulas.

5.2.1 Examples

Example 5.2.1. *For any bivariate copula, the Fréchet-Hoeffding lower bound copula, defined as*

$$W(u_1,u_2) = \max\{u_1+u_2-1, 0\},$$

provides the lower bound for every bivariate copula. The CCTE corresponding to the Fréchet-Hoeffding lower bound copula is given by

$$\begin{aligned}\xi_\alpha(W) &= -\int_0^1\int_0^1 \max\{u_1+u_2-1,0\}\log_{[\alpha]}\Big(\max\{u_1+u_2-1,0\}\Big)\,du_1\,du_2\\ &= \frac{1}{\alpha-1}\int_0^1\int_0^{u_1}\Big(u_2-u_2^\alpha\Big)\,du_2\,du_1\\ &= \frac{\alpha+4}{6(\alpha+1)(\alpha+2)}.\end{aligned}$$

Example 5.2.2. *Consider the Marshall–Olkin copula defined by*

$$C(u_1,u_2) = u_1^{1-\beta_1}u_2^{1-\beta_2}\min\{u_1^{\beta_1}, u_2^{\beta_2}\}, \quad \beta_1,\beta_2 \in \mathbb{I}.$$

Then the CCTE for the Marshall-Olkin copula is given by

$$\begin{aligned}\xi_\alpha(C) &= \frac{1}{\alpha-1}\Bigg[\int_0^1\int_0^{u_2^{\beta_2/\beta_1}} \left(u_1u_2^{1-\beta_2} - \left(u_1u_2^{1-\beta_2}\right)^\alpha\right)\, du_1\, du_2 \\ &\qquad + \int_0^1\int_0^{u_1^{\beta_1/\beta_2}} \left(u_1^{1-\beta_1}u_2 - \left(u_1^{1-\beta_1}u_2\right)^\alpha\right)\, du_2\, du_1\Bigg] \\ &= \frac{(\beta_1+\beta_2)\left(\omega(1)-\omega(\alpha)\right)}{\alpha^2-1},\end{aligned}$$

where $\omega(x) = \frac{1}{(x+1)(\beta_1+\beta_2)-x\beta_1\beta_2}$.

Example 5.2.3. *The underlying copula corresponding to the mutual independence of random variables is the product copula, defined as*

$$\Pi(\mathbf{u}) = u_1u_2\ldots u_p.$$

The CCTE for the product copula is given by

$$\begin{aligned}\xi_\alpha(\Pi) &= \frac{1}{\alpha-1}\int_0^1\int_0^1\cdots\int_0^1 \left(u_1u_2\ldots u_p - (u_1u_2\cdots u_p)^\alpha\right) du_1\, du_2\ldots du_p \\ &= \frac{(\alpha+1)^p - 2^p}{2^p(\alpha^2-1)(\alpha+1)^{p-1}}.\end{aligned}$$

Example 5.2.4. *For any p-dimensional copula, the Fréchet-Hoeffding upper bound copula, defined as*

$$M(\mathbf{u}) = \min\{u_1, u_2, \ldots, u_p\},$$

provides the upper bound for every p-dimensional copula. The CCTE corresponding to the Fréchet-Hoeffding upper bound copula is given by

$$\begin{aligned}\xi_\alpha(M) &= \frac{1}{\alpha-1}\int_0^1\int_0^1\cdots\int_0^1 \min\{u_1,u_2,\ldots,u_p\} - \left(\min\{u_1,u_2,\ldots,u_p\}\right)^\alpha\, du_1\, du_2\ldots du_p \\ &= \frac{p}{\alpha-1}\int_0^1 \left(u-u^\alpha\right)(1-u)^{p-1}\, du \\ &= \frac{p}{\alpha-1}\left(\beta(2,p)-\beta(\alpha+1,p)\right),\end{aligned}$$

where $\beta(q_1, q_2)$ *is the well-known beta function. Note that the transformation of the above multiple integrals into a single integral uses the concept of order statistics. The multiple integral in the above equation can be expressed as* $\mathbb{E}\left(U_{[1]} - U_{[1]}^\alpha\right)$, *where* $U_1, U_2, \ldots, U_p$ *is a random sample of size p from the uniform distribution over* $\mathbb{I}$, *and* $U_{[1]} = \min\{U_1, U_2, \ldots, U_p\}$.

Next, we explore several inequalities associated with the CCTE, which establishes the bounds for the measure.

5.2.2 Inequalities

Theorem 5.2.1. *For every p-dimensional copula C with $\xi_\alpha(C)$, the following inequalities hold:*

$$\xi_\alpha(C) \begin{cases} \geq \xi(C), & \text{if } \alpha \in (0,1), \\ \leq \xi(C), & \text{if } \alpha \in (1,\infty). \end{cases}$$

Proof. For any $r \geq 0$, it holds that $1 - r \leq -\log(r)$. Consequently, for any $r \in \mathbb{I}$,

$$-r\log_{[\alpha]}(r) = \frac{r\left(1 - r^{\alpha-1}\right)}{\alpha - 1} \begin{cases} \geq -r\log(r), & \text{if } \alpha \in (0,1), \\ \leq -r\log(r), & \text{if } \alpha \in (1,\infty). \end{cases}$$

The result follows by substituting $r = C(\mathbf{u})$ and integrating over $\mathbb{I}^p$. □

Theorem 5.2.2. *Let $\xi_\alpha(C)$ be the CCTE of a copula C, then*

$$\xi_\alpha(C) \begin{cases} \geq \xi_2(C), & \text{if } \alpha \in (0,2] \setminus \{1\}, \\ \leq \xi_2(C), & \text{if } \alpha \in (2,\infty). \end{cases}$$

Proof. For $\alpha \in (0,2] \setminus \{1\}$, the function

$$g(r) = -\log_{[\alpha]}(r) + \log_{[2]}(r) = \frac{1 - r^{\alpha-1}}{\alpha - 1} - 1 + r$$

attains its minimum at $r = 1$. For $\alpha \in (2,\infty)$, $g(r)$ attains its maximum at $r = 1$. Thus, $g(r) \geq 0$ if $\alpha \in (0,2] \setminus \{1\}$ and $g(r) \leq 0$ if $\alpha \in (2,\infty)$. Substituting $r = C(\mathbf{u})$ and integrating over $\mathbb{I}^p$, the result follows. □

Let $C(\mathbf{u})$ be a p-dimensional copula. The multivariate version of Spearman's correlation can be defined as

$$\rho_p^-(C) = n(p)\left(2^p \int_{\mathbb{I}^p} C(\mathbf{u})d\mathbf{u} - 1\right), \tag{5.2}$$

where $n(p) = \dfrac{p+1}{2^p - p - 1}$. For more details, we refer to Schmid et al. (2010) and Bedő and Ong (2016)). The following theorem provides the relation between multivariate Spearman's correlation coefficient and CCTE.

Theorem 5.2.3. *Let C be a p-dimensional copula with multivariate Spearman's correlation coefficient $\rho_p^-(C)$. Then for any $\alpha \in \mathcal{A}$,*

$$\xi_\alpha(C) \leq g_p(C)\log_{[\alpha]}\left(g_d(C)\right),$$

where $g_p(C) = \left(\rho_p^-(C) + n(p)\right) n(p)^{-1} 2^{-p}$.

Proof. For any $\alpha \in \mathcal{A}$, $h(r) = -r \log_{[\alpha]}(r) = \dfrac{r - r^\alpha}{\alpha - 1}$ is concave for $r \in \mathbb{I}$. The result follows, using Jensen's inequality on the concave function. □

Now, we will focus on the CCTE of the weighted arithmetic mean (WAM) of copulas. It is important to note that the WAM of copulas with the same dimension is also a copula. The following theorem shows the uncertainty involved in the WAM of copulas.

Theorem 5.2.4. *Let $C_1, C_2, \ldots, C_m$ be m copulas of dimension p with corresponding CCTE values $\xi_\alpha(C_1), \xi_\alpha(C_2), \ldots, \xi_\alpha(C_m)$. Define $C^\Sigma(\mathbf{u}) = \sum_{j=1}^m l_j C_j(\mathbf{u})$ as the WAM of these copulas, where $l_j \in \mathbb{I}$ for $j = 1, 2, \ldots, m$ and $\sum_{j=1}^m l_j = 1$. Let $\xi_\alpha(C^\Sigma)$ denote the CCTE of C^Σ. Then the following inequality holds*

$$\sum_{j=1}^m l_j \, \xi_\alpha(C_j) \leq \xi_\alpha(C^\Sigma).$$

Proof. The function $h(r) = -r \log_{[\alpha]}(r)$ is concave, which implies that

$$\sum_{j=1}^m l_j h(r_j) \leq h\left(\sum_{j=1}^m l_j r_j\right),$$

for every $r_j \in \mathbb{I}$. The result follows by substituting $r_j = C_j(\mathbf{u})$ and integrating over $\mathbb{I}^p$.

□

Let $C_1(\mathbf{u})$ and $C_2(\mathbf{u})$ be two p-dimensional copulas. Then, C_1 is less positive lower orthant dependent (PLOD) than C_2, denoted by $C_1 \overset{\text{PLOD}}{\prec} C_2$, if $C_1(\mathbf{u}) \leq C_2(\mathbf{u})$ for every $\mathbf{u} \in \mathbb{I}^p$. Now, we will show that PLOD ordering does not necessarily imply the corresponding CCTE ordering through a counterexample by considering

$$C_1(u_1, u_2) = \left(1 + \left[(u_1^{-1} - 1)^2 + (u_2^{-1} - 1)^2\right]^{0.5}\right)^{-1},$$

and

$$C_2(u_1, u_2) = \min\{u_1, u_2\}.$$

The difference $\xi_\alpha(C_1) - \xi_\alpha(C_2)$ is shown in Figure 5.1, which illustrates that the inequality is not preserved for PLOD ordering.

In the following subsection, we establish the uniform convergence property of CCTE.

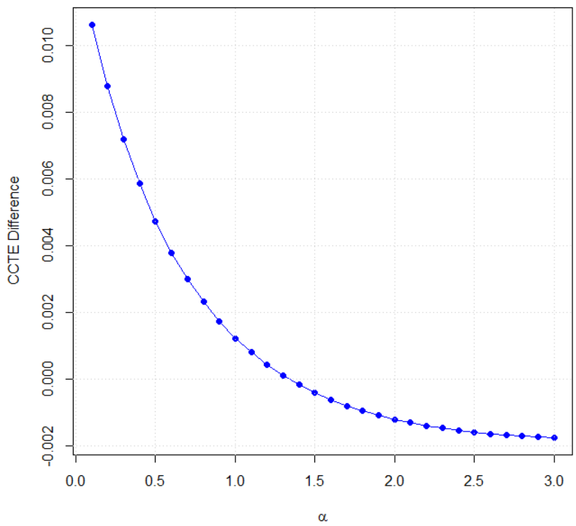


FIGURE 5.1. $\xi_\alpha(C_1) - \xi_\alpha(C_2)$ for different values of α

5.2.3 Uniform Convergence

Theorem 5.2.5. *Let $\{C_N\}$ be a sequence of p-dimensional copulas with CCTE $\xi_\alpha(C_N)$, and let C be a p-dimensional copula with CCTE $\xi_\alpha(C)$. If C_N converges uniformly to C, then $\xi_\alpha(C_N)$ converges uniformly to $\xi_\alpha(C)$ for all $\alpha \in \mathcal{A}$.*

Proof. The function $h(r) = -r\log_{[\alpha]}(r)$ is bounded and uniformly continuous on $\mathbb{I}$. Thus, for any $\delta > 0$, there exists $\eta > 0$ such that for any $r_1, r_2 \in \mathbb{I}$ satisfying $|r_1 - r_2| < \eta$, we have

$$|h(r_1) - h(r_2)| < \delta. \tag{5.3}$$

Substituting $r_1 = C_N(\mathbf{u})$ and $r_2 = C(\mathbf{u})$ in Eq. (5.3), it follows that

$$|h\left(C_N(\mathbf{u})\right) - h\left(C(\mathbf{u})\right)| < \delta, \tag{5.4}$$

whenever

$$|C_N(\mathbf{u}) - C(\mathbf{u})| < \eta.$$

If C_N converges uniformly to C, then for any $\eta > 0$, there exists a natural number $m \geq N$ such that

$$|C_N(\mathbf{u}) - C(\mathbf{u})| < \eta, \tag{5.5}$$

for every $\mathbf{u} \in \mathbb{I}^p$. Using Eq. (5.4) and Eq. (5.5), and applying the bounded convergence theorem, the result follows. □

5.3 Empirical Cumulative Copula Tsallis Entropy

In this section, we use the empirical copula to propose a non-parametric estimator for CCTE. A non-parametric estimate of CCE based on the empirical copula was introduced by Sunoj and Nair (2025). Let $\mathbf{X_j} = (X_{j,1}, X_{j,2}, \ldots, X_{j,p})\,; j = 1, 2, \ldots, n$ be a random sample of size n from a multivariate population. Based on these samples, the empirical copula $\hat{C}_n$ can be used to estimate the underlying copula, defined as

$$\hat{C}_n(\mathbf{u}) = \frac{1}{n} \sum_{j=1}^{n} \prod_{k=1}^{p} \mathbf{I}\left(\frac{R_{j,k}}{n+1} \leq u_k\right), \tag{5.6}$$

where $R_{j,k}$ is the rank of the k-th component of the j-th observation $X_{j,k}$, and $\mathbf{I}(\cdot)$ denotes the indicator function (see Deheuvels (1979), Nelsen (2006), Panchenko (2005), and Durante and Sempi (2016)). Now, using the definition of empirical copula, we define the empirical CCTE as

$$\xi_\alpha(\hat{C}_n) = -\int_{\mathbb{I}^p} \hat{C}_n(\mathbf{u}) \log_{[\alpha]}\left(\hat{C}_n(\mathbf{u})\right) d\mathbf{u}, \quad \alpha \in \mathcal{A}. \tag{5.7}$$

The following theorem provides the upper bound for the empirical CCTE.

Theorem 5.3.1. *Let $\hat{C}_n$ be the empirical copula based on the random sample $\mathbf{X_1}, \mathbf{X_2} \ldots, \mathbf{X_n}$ from a multivariate distribution of dimension p. Let $\xi_\alpha(\hat{C}_n)$ be the empirical CCTE defined in Eq. (5.7). Then, for any $\alpha \in \mathcal{A}$,*

$$\xi_\alpha(\hat{C}_n) \leq -\frac{1}{n}\mathcal{R} \log_{[\alpha]}(\mathcal{R}),$$

where $\mathcal{R} = \left\{\frac{1}{n} \sum_{j=1}^{n} \prod_{k=1}^{p} \left(1 - \frac{R_{j,k}}{n+1}\right)\right\}$.

Proof. By Jensen's inequality, we have

$$\begin{aligned}
\xi_\alpha(\hat{C}_n) &\leq \frac{1}{\alpha-1}\left\{\int_{\mathbb{I}^p} \hat{C}_n(\mathbf{u})\, d\mathbf{u} - \left(\int_{\mathbb{I}^p} \hat{C}_n(\mathbf{u})\, d\mathbf{u}\right)^\alpha\right\} \\
&= \frac{1}{\alpha-1}\left\{\int_{\mathbb{I}^p} \frac{1}{n}\sum_{j=1}^{n}\prod_{k=1}^{p} \mathbf{I}\left(\frac{R_{j,k}}{n+1} \leq u_k\right) d\mathbf{u} - \left(\int_{\mathbb{I}^p} \frac{1}{n}\sum_{j=1}^{n}\prod_{k=1}^{p} \mathbf{I}\left(\frac{R_{j,k}}{n+1} \leq u_k\right) d\mathbf{u}\right)^\alpha\right\} \\
&= \frac{1}{\alpha-1}\left\{\frac{1}{n}\sum_{j=1}^{n}\prod_{k=1}^{p}\left(1 - \frac{R_{j,k}}{n+1}\right) - \left(\frac{1}{n}\sum_{j=1}^{n}\prod_{k=1}^{p}\left(1 - \frac{R_{j,k}}{n+1}\right)\right)^\alpha\right\} \\
&= -\frac{1}{n}\mathcal{R}\log_{[\alpha]}(\mathcal{R}).
\end{aligned}$$

□

We now focus on the consistency of the proposed non-parametric estimator. The following theorem asserts the convergence of the empirical CCTE.

Theorem 5.3.2. *The empirical CCTE converges almost surely to the true CCTE. Specifically, for any $\alpha \in \mathcal{A}$, as $n \to \infty$, we have*

$$\hat{\xi}_\alpha(C_n) \to \xi_\alpha(C) \quad a.s.$$

Proof. By the Glivenko-Cantelli theorem for empirical copulas, we have that as $n \to \infty$,

$$\sup_{\mathbf{u}\in\mathbb{I}^p} |C_n(\mathbf{u}) - C(\mathbf{u})| \to 0, \quad \text{a.s.} \tag{5.8}$$

For further details, see Deheuvels (1979), Kiefer (1961), Shorack and Wellner (2009), and Janssen et al. (2012). Using the continuous mapping theorem of almost sure convergence, along with the bounded convergence theorem, the result follows. □

Now, we illustrate this theorem through a simulation study for various copulas, specifically considering the following:

- Clayton copula:

$$C(\mathbf{u}) = \max\left\{\sum_{i=1}^{p} u_i^\theta - p + 1, 0\right\}^{-1/\theta}, \quad \theta \in [-1, \infty) \setminus \{0\}.$$

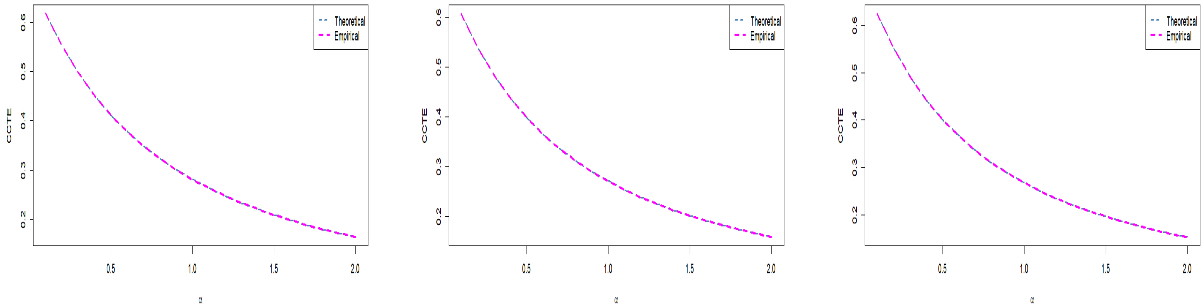
- Gumbel-Hougaard copula:

$$C(\mathbf{u}) = \exp\left\{-\left(\sum_{i=1}^{p}(-\log(u_i))^\theta\right)^{1/\theta}\right\}, \quad \theta \geq 1.$$

- Frank copula:

$$C(\mathbf{u}) = -\frac{1}{\theta}\log\left(1 + \frac{\prod_{i=1}^{p} e^{-\theta u_i} - 1}{e^{-\theta} - 1}\right), \quad \theta \in \mathbb{R} \setminus \{0\}.$$

We generated 1,000 random numbers from each of the copulas mentioned above and computed the empirical CCTE, comparing these estimates with the actual values. Due to the absence of a closed-form expression for the empirical CCTE, we evaluated the integrals numerically using the `adaptIntegrate` function from the `cubature` package in R (version 4.2.2). Figures 5.2 and 5.3 illustrate the convergence of the non-parametric estimate of CCTE for the Clayton copula, Gumbel-Hougaard copula, and Frank copula in both bivariate and trivariate cases. From these figures, it is evident that the shape of the CCTE varies with the dimension of the copula.

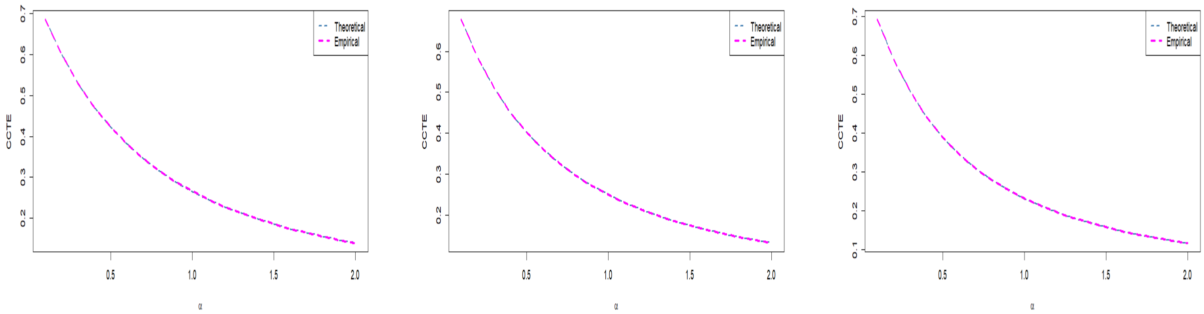


(A) Clayton copula with parameter $\theta = 1.5$

(B) Gumbel-Hougaard copula with parameter $\theta = 2$

(C) Frank copula with parameters $\theta = 2.5$

FIGURE 5.2. The empirical CCTE and theoretical CCE of various bivariate copulas.



(A) Clayton copula with parameter $\theta = 1.5$

(B) Gumbel-Hougaard copula with parameter $\theta = 2$

(C) Frank copula with parameters $\theta = 2.5$

FIGURE 5.3. The empirical CCTE and theoretical CCE of various trivariate copulas.

5.4 Validity of Cumulative Copula Tsallis Entropy with Chaotic Theory

Here, we validate our entropy measure using chaotic theory. We consider the identical Rulkov maps given by the system of equations:

$$x_{n+1} = \frac{\beta}{1 + x_n^2} + \delta + \gamma(y_n - x_n), \quad y_{n+1} = \frac{\beta}{1 + y_n^2} + \delta + \gamma(x_n - y_n).$$

where γ is the coupling parameter. For more details, refer to Rulkov (2001) and Bashkirtseva and Pisarchik (2019). It has been shown that for $\beta = 4.1$, $\gamma = 0.131$ specific values of $\delta = 2, 0.2, -0.8, -2, -2.5$, the coupled map exhibits periodicity of 1, 2, 4, and two chaotic sequences, respectively.

We perform numerical simulations on the Rulkov maps and consider the first 2000 observations with initial values $x_0 = 0.1$ and $y_0 = 0.5$. We plot the bifurcation diagram with respect to y_n, which is presented in Figure 5.4, for verification purposes. Using the empirical copula, we calculate the empirical CCTE. As per the theory, for periodic cases, the dependence entropy tends to be lower. Even as the period increases, the CCTE increases, while in chaotic cases, the CCTE remains higher than in the periodic cases. The results are shown in Figure 5.5, where we observe that the CCTE increases with the periodicity and is greater in the chaotic case than in the periodic case.

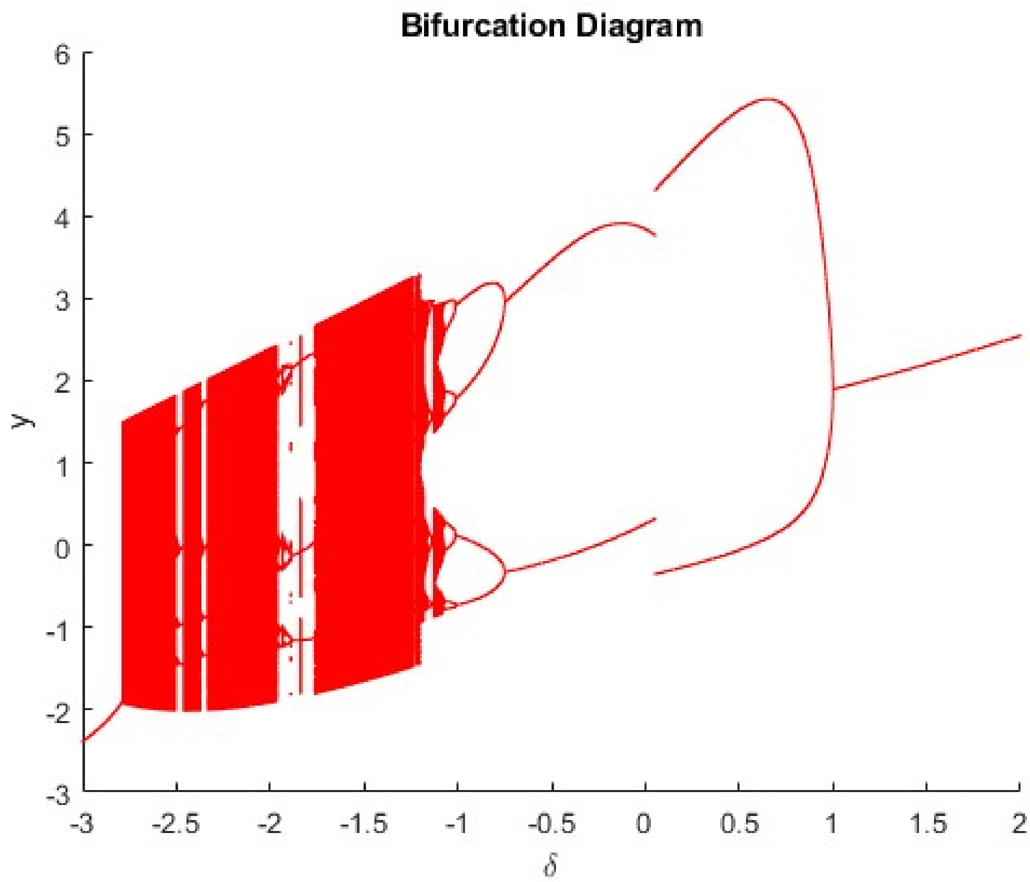


FIGURE 5.4. Bifurcation diagram of identical Rulkov maps

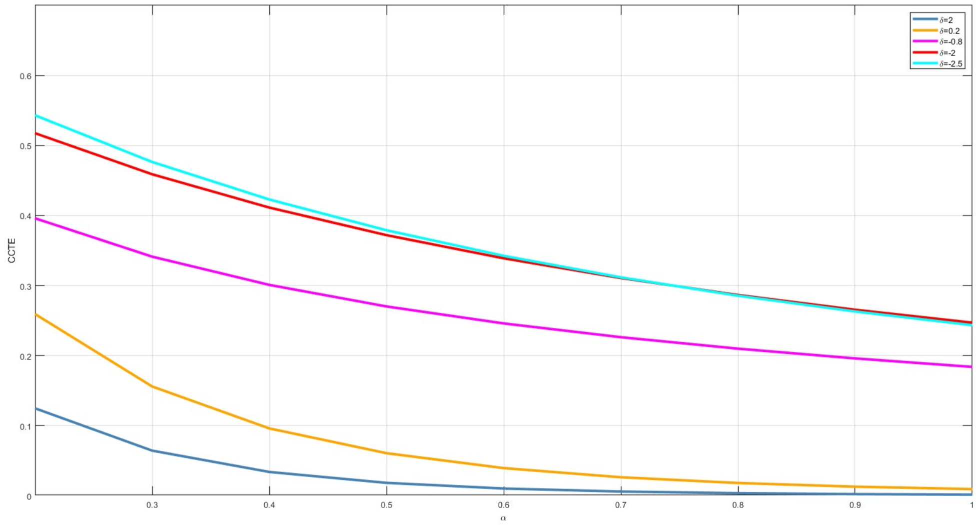


FIGURE 5.5. CCTE of identical Rulkov maps

5.5 Cumulative Copula Tsallis Inaccuracy Measure

Apart from entropy, several information measures for uncertainty are available in the literature. In this section, we introduce a new measure known as the cumulative copula Tsallis inaccuracy (CCTI) measure, which generalizes the inaccuracy measure proposed by Hosseini and Ahmadi (2019). Let $C_1(\mathbf{u})$ and $C_2(\mathbf{u})$ be two copulas of the same dimension p. If an experimenter uses C_2 to model the dependence structure among random variables instead of the true copula C_1, the copula-based inaccuracy measure quantifies the error introduced by this incorrect assumption is well-known as misspecification in literature. This incorrect assumption may be due to experimental error or wrong observations, or maybe both. Let C_1 be the true copula, and suppose the experimenter uses C_2 instead of C_1. Then, the CCTI measure corresponds to the copula C_1 and C_2 is defined as

$$\mathcal{I}_\alpha(C_1|C_2) = -\int_{\mathbb{I}^p} C_1(\mathbf{u}) \log_{[\alpha]}(C_2(\mathbf{u}))\, d\mathbf{u}. \tag{5.9}$$

Note that $\lim_{\alpha\to 1} \mathcal{I}_\alpha(C_1|C_2) = \mathcal{I}(C_1|C_2)$ (copula-based inaccuracy measure proposed by Hosseini and Ahmadi (2019)), and when $C_1 = C_2 = C$ then $\mathcal{I}_\alpha(C_1|C_2) = \xi_\alpha(C)$. Thus, the proposed inaccuracy measure can be viewed as a generalization of the inaccuracy measure proposed by Hosseini and Ahmadi (2019), and the parameter α will give flexibility in quantifying the uncertainty involved in the experimental error. We will now go through a few examples based on commonly used copula in the literature.

Example 5.5.1. *The CCTI measure corresponding to the Fréchet-Hoeffding lower bound copula $W(u_1, u_2)$ and the product copula $\Pi(u_1, u_2)$ is given by*

$$\mathcal{I}_\alpha(W|\Pi) = \frac{1}{6(\alpha-1)} + \frac{\beta(\alpha, \alpha+2) + (\alpha+1)\beta(\alpha, 2) - 1}{\alpha(\alpha^2-1)}.$$

The Fréchet-Hoeffding lower bound copula is used for modelling strongly negatively dependent bivariate data. Thus, the above inaccuracy measure quantifies the uncertainty involved in incorrectly assuming independence when the data exhibits strong negative dependence.

Example 5.5.2. *Consider the FGM copula given by:*

$$C(u_1, u_2) = u_1 u_2 \left(1 + \theta(1-u_1)(1-u_2)\right),$$

where $\theta \in [-1, 1]$. The CCTI measure corresponding to the FGM copula and the product copula is given by

$$\mathcal{I}_\alpha(C|\Pi) = \frac{\theta+9}{36(\alpha-1)} - \frac{1}{(\alpha^2-1)(\alpha+1)} - \frac{\theta\,\beta(\alpha+1, 2)}{\alpha-1}.$$

Example 5.5.3. *The CCTI measure corresponding to the Fréchet-Hoeffding upper bound copula $M(u_1, u_2)$ and the product copula is given by*

$$\mathcal{I}_\alpha(M|\Pi) = \frac{1}{(p+1)(\alpha-1)} + \frac{p!}{(\alpha^2-1)\prod_{j=2}^{p}(j\alpha+1)}.$$

The Fréchet-Hoeffding upper bound copula is used for modelling strongly positively dependent data. The above inaccuracy measure quantifies the uncertainty involved in incorrectly assuming independence when the data exhibits strong positive dependence.

Example 5.5.4. *The p-variate Cuadras-Augé copula, proposed by Cuadras (2009), is given by*

$$C(\mathbf{u}) = \prod_{i=1}^{p} u_{[i]}^{\gamma_i}, \tag{5.10}$$

where $\gamma_1, \gamma_2, \ldots, \gamma_p$ are copula parameters such that $C(\mathbf{u})$ in Eq. (5.10) is a valid copula, and $u_{[1]}, u_{[2]}, \ldots, u_{[p]}$ are the ordered values of $u_1, u_2, \ldots, u_p$ in ascending order. For more details, see Nadarajah et al. (2017) and Cuadras (2009). The CCTI measure corresponding to the Cuadras-Augé copula and the product copula is given by

$$\mathcal{I}_\alpha(\Pi|C) = \frac{1}{2^p(\alpha-1)} - \frac{1}{\alpha-1}\prod_{j=1}^{p}\frac{1}{\left(\sum_{i=1}^{j}\delta(i) + j\right)},$$

where $\delta(i)$ satisfies the recurrence relation $\delta(i) = \delta(i-1) + (\alpha - 1)\gamma_i + 2$ for $i = 2, \ldots, p$ with $\delta(1) = \theta_1(\alpha - 1) + 2$.

We now discuss some mathematical properties of the CCTI measure. Similar to Theorem 5.2.1 and 5.2.2, we have the following result. The proof is similar to the proof of Theorem 5.2.1 and 5.2.2, so we omitted.

Theorem 5.5.1. *Let C_1 and C_2 be two copulas of the same dimension. Let $\mathcal{I}_\alpha(C_1|C_2)$ be the inaccuracy measure by incorrect use of C_2, instead of C_1. Then*

$$\mathcal{I}_\alpha(C_1|C_2) \begin{cases} \geq \mathcal{I}(C_1|C_2), & \text{if } \alpha \in (0,1), \\ \leq \mathcal{I}(C_1|C_2), & \text{if } \alpha \in (1,\infty). \end{cases}$$

Theorem 5.5.2. *Let $\mathcal{I}_\alpha(C_1|C_2)$ be the CCTI measure with respect to the copulas C_1 and C_2 of the same dimension d, then the following inequalities hold.*

$$\mathcal{I}_\alpha(C_1|C_2) \begin{cases} \geq \mathcal{I}_2(C_1|C_2), & \text{if } \alpha \in (0,2] \setminus \{1\}, \\ \leq \mathcal{I}_2(C_1|C_2), & \text{if } \alpha \in (2,\infty). \end{cases}$$

Theorem 5.5.3. *Let $C_1, C_2, \ldots, C_m$ be m p-dimensional WGM copulas, and let $C^\Sigma(\mathbf{u}) = \sum_{j=1}^m l_j C_j(\mathbf{u})$ be the WAM of these copulas, where $l_j \in \mathbb{I}$ for $j = 1, 2, \ldots, m$ with $\sum_{j=1}^m l_j = 1$. Let C be any p-dimensional copula, then*

$$\mathcal{I}_\alpha\left(C|C^\Sigma\right) \begin{cases} \leq \sum_{j=1}^m l_j \mathcal{I}_\alpha(C|C_j), & \text{if } \alpha \in (0,2] \setminus \{1\}, \\ \geq \sum_{j=1}^m l_j \mathcal{I}_\alpha(C|C_j), & \text{if } \alpha \in (2,\infty). \end{cases}$$

Proof. Since the function $-\log_{[\alpha]}(y) = \frac{1-y^{\alpha-1}}{\alpha-1}$ is convex (concave) in $y \geq 0$ if $\alpha \in (0,2]\setminus\{1\}$ ($\alpha \in (2,\infty)$), it follows that for fixed $x \in \mathbb{I}$,

$$-x\log_{[\alpha]}(z) \begin{cases} \leq \sum_{j=1}^m l_j \left(-x\log_{[\alpha]}(y_j)\right), & \text{if } \alpha \in (0,2] \setminus \{1\}, \\ \geq \sum_{j=1}^m l_j \left(-x\log_{[\alpha]}(y_j)\right), & \text{if } \alpha \in (2,\infty), \end{cases}$$

where $z = \sum_{j=1}^m l_j y_j$ and $y_1, y_2, \ldots, y_m \in \mathbb{I}$. Substituting $x = C(\mathbf{u})$ and $y_j = C_j(\mathbf{u})$ for every $j = 1, 2, \ldots, m$, and integrating over $\mathbb{I}^p$, we obtain the required result. □

Now, we will discuss the inaccuracy measure related to the weighted geometric mean (WGM) of copulas. Let $C_1, C_2, \ldots, C_m$ represent m p-dimensional copulas. The WGM of these copulas is defined as:

$$C^{\Pi}(\mathbf{u}) = \prod_{j=1}^{m} C_j(\mathbf{u})^{q_j}, \tag{5.11}$$

where $q_j \in \mathbb{I}$ for $j = 1, 2, \ldots, m$, and $\sum_{j=1}^{m} q_j = 1$. It is important to note that $C^{\Pi}(\mathbf{u})$ defined in Eq. (5.11) is not always a valid copula. However, under specific conditions, it can satisfy the requirements of a copula. For more details, see Cuadras (2009), Zhang et al. (2013) and Diaz and Cuadras (2022).

The following theorem provides an upper bound for the inaccuracy measure associated with the WGM of copulas.

Theorem 5.5.4. *Let $C^{\Pi}(\mathbf{u}) = \prod_{j=1}^{m} C_j(\mathbf{u})^{q_j}$ be the WGM of m copulas of dimension p defined in Eq. (5.11), and let C be any p-dimensional copula. Then*

$$\mathcal{I}_\alpha\left(C^{\Pi}|C\right) \leq \prod_{j=1}^{m} \left[\mathcal{I}_\alpha\left(C_j|C\right)\right]^{q_j}.$$

Proof. Let $g : \mathbb{I}^p \to \mathbb{R}_+$ be a function. For any $t \neq 0$, we have

$$\int_{\mathbb{I}^p} g(\mathbf{u}) \left[C^{\Pi}(\mathbf{u})\right]^t d\mathbf{u} = \int_{\mathbb{I}^p} \prod_{j=1}^{m} g(\mathbf{u})^{q_j} \left[C_j(\mathbf{u})^{q_j t}\right] d\mathbf{u}.$$

By applying the generalized Hölder's inequality (see Kufner et al. (1977), Finner (1992)), we obtain

$$\int_{\mathbb{I}^p} g(\mathbf{u}) \left[C^{\Pi}(\mathbf{u})\right]^t d\mathbf{u} \leq \prod_{j=1}^{m} \left(\int_{\mathbb{I}^p} g(\mathbf{u}) \left[C_j(\mathbf{u})\right]^t d\mathbf{u}\right)^{q_j}. \tag{5.12}$$

The CCTI measure associated with C^{Π} and C is given by

$$\mathcal{I}_\alpha\left(C^{\Pi}|C\right) = \int_{\mathbb{I}^p} C^{\Pi}(\mathbf{u}) \log_{[\alpha]}\left(C(\mathbf{u})\right) d\mathbf{u}.$$

Substituting $g(\mathbf{u}) = \log_{[\alpha]}\left(C(\mathbf{u})\right)$ and $t = 1$ into inequality (5.12), we obtain

$$\begin{aligned}\mathcal{I}_\alpha\left(C^{\Pi}|C\right) &\leq \prod_{j=1}^{m} \left(\int_{\mathbb{I}^p} \frac{1 - C^{\alpha-1}(\mathbf{u})}{\alpha - 1} \cdot C_j(\mathbf{u}) d\mathbf{u}\right)^{q_j} \\ &= \prod_{j=1}^{m} \left(\mathcal{I}_\alpha\left(C_j|C\right)\right)^{q_j}.\end{aligned}$$

This completes the proof. □

Now, we will discuss some results for CCTI based on the PLOD property of copulas.

Theorem 5.5.5. *Let $C_1 \overset{PLOD}{\prec} C_2$. Then, for any $\alpha \in \mathcal{A}$, the inequality $\mathcal{I}_\alpha(C_1|C_2) \leq \mathcal{I}_\alpha(C_2|C_1)$ holds.*

Proof. By the assumption $C_1 \overset{\text{PLOD}}{\prec} C_2$, for any $\alpha \in \mathcal{A}$, we have

$$\begin{aligned}\mathcal{I}_\alpha(C_1|C_2) - \mathcal{I}_\alpha(C_2|C_1) &= \int_{\mathbb{I}^p} \frac{C_1(\mathbf{u})\left(1 - C_2^{\alpha-1}(\mathbf{u})\right)}{\alpha - 1} - \frac{C_2(\mathbf{u})\left(1 - C_1^{\alpha-1}(\mathbf{u})\right)}{\alpha - 1} d\mathbf{u} \\ &\le \int_{\mathbb{I}^p} C_1(\mathbf{u}) \left(\frac{C_1^{\alpha-1}(\mathbf{u}) - C_2^{\alpha-1}(\mathbf{u})}{\alpha - 1}\right) d\mathbf{u} \le 0.\end{aligned}$$

□

Theorem 5.5.6. *Let C_1 and C_2 and C_3 be three p-dimensional copulas. If $C_1 \overset{PLOD}{\prec} C_2$, then for any $\alpha \in \mathcal{A}$, the following triangle inequalities hold.*

1. $\mathcal{I}_\alpha(C_3|C_1) + \mathcal{I}_\alpha(C_1|C_2) \ge \mathcal{I}_\alpha(C_3|C_2)$
2. $\mathcal{I}_\alpha(C_1|C_2) + \mathcal{I}_\alpha(C_2|C_3) \ge \mathcal{I}_\alpha(C_1|C_3)$.

Proof. We will prove the part (a) of the theorem. The proof of the part (b) is similar to that of the first part and is therefore omitted. Under the assumption of $C_1 \overset{\text{PLOD}}{\prec} C_2$, we have

$$\begin{aligned}\mathcal{I}_\alpha(C_3|C_1) + \mathcal{I}_\alpha(C_1|C_2) &= \int_{\mathbb{I}^p} C_1(\mathbf{u}) \log_{[\alpha]}(C_2(\mathbf{u})) - C_3(\mathbf{u}) \log_{[\alpha]}(C_1(\mathbf{u}))\, d\mathbf{u} \\ &= \int_{\mathbb{I}^p} \frac{C_3(\mathbf{u})\left(1 - C_1^{\alpha-1}(\mathbf{u})\right)}{\alpha - 1} - \frac{C_1(\mathbf{u})\left(1 - C_2^{\alpha-1}(\mathbf{u})\right)}{\alpha - 1} d\mathbf{u} \\ &\ge \int_{\mathbb{I}^p} (C_3(\mathbf{u}) + C_1(\mathbf{u}) \left(\frac{1 - C_2^{\alpha-1}(\mathbf{u})}{\alpha - 1}\right) d\mathbf{u} \\ &\ge \int_{\mathbb{I}^p} C_3(\mathbf{u}) \left(\frac{1 - C_2^{\alpha-1}(\mathbf{u})}{\alpha - 1}\right) d\mathbf{u} = \mathcal{I}_\alpha(C_3|C_2).\end{aligned}$$

□

The proofs of the following theorem is similar to the proof of Theorem 5.5.6, so we left out here.

Theorem 5.5.7. *Let C_1, C_2, and C_3 be three copulas of same dimension. Then for any $\alpha \in \mathcal{A}$, we have*

1. *If $C_1 \overset{PLOD}{\prec} C_2$, then $\mathcal{I}_\alpha(C_1|C_3) \le \mathcal{I}_\alpha(C_2|C_3)$ and $\mathcal{I}_\alpha(C_3|C_2) \le \mathcal{I}_\alpha(C_3|C_1)$.*
2. *If $C_1 \overset{PLOD}{\prec} C_2$ and $C_1 \overset{PLOD}{\prec} C_3$, then $\mathcal{I}_\alpha(C_1|C_3) \le \mathcal{I}_\alpha(C_2|C_3) \le \mathcal{I}_\alpha(C_2|C_1)$.*
3. *If $C_1 \overset{PLOD}{\prec} C_3$ and $C_2 \overset{PLOD}{\prec} C_3$, then $\mathcal{I}_\alpha(C_1|C_3) \le \mathcal{I}_\alpha(C_1|C_2) \le \mathcal{I}_\alpha(C_3|C_1)$.*
4. *If $C_1 \overset{PLOD}{\prec} C_2 \overset{PLOD}{\prec} C_3$, then*

$$\max\{I_\alpha(C_2|C_1), I_\alpha(C_3|C_2)\} \le \mathcal{I}_\alpha(C_3|C_2) \quad \textit{and} \quad \min\{I_\alpha(C_1|C_2), I_\alpha(C_2|C_3)\} \ge \mathcal{I}_\alpha(C_1|C_3).$$

5.6 Cumulative Copula Tsallis Divergence and Mutual Information

In this section, we propose a new divergence measure between two copulas based on Tsallis divergence, along with a new mutual information (MI) measure derived from the cumulative copula. The concept of divergence plays a crucial role in the field of statistics, particularly in statistical inference.

Let $\mathbf{X_1}$ and $\mathbf{X_2}$ be two multivariate random variables with identical marginals but with underlying copulas that are not necessarily the same. Let $\mathbf{f_1}(\cdot)$ and $\mathbf{f_2}(\cdot)$ denote the joint PDF of $\mathbf{X_1}$ and $\mathbf{X_2}$ and, c_1 and c_2 denote the underlying copula densities corresponding to $\mathbf{X_1}$ and $\mathbf{X_2}$, respectively. Then, the KL divergence between $\mathbf{X_1}$ and $\mathbf{X_2}$ is equivalent to the KL divergence between the two copula densities, as discussed in Ghosh and Sunoj (2024). That is,

$$KL(\mathbf{f_1}||\mathbf{f_2}) = \int_{\mathbb{I}^p} c_1(\mathbf{u}) \log\left(\frac{c_1(\mathbf{u})}{c_2(\mathbf{u})}\right) d\mathbf{u}.$$

As outlined in the introduction, the above copula density divergence may not be suitable in certain cases. In Chapter 4, we discuss the cumulative copula Kullback-Leibler divergence. Motivated by the works of Mao et al. (2020), we now propose the following divergence measure between two copulas based on Tsallis divergence:

$$\Delta_\alpha(C_1||C_2) = \int_{\mathbb{I}^p} C_1(\mathbf{u}) \log_{[\alpha]}\left(\frac{C_1(\mathbf{u})}{C_2(\mathbf{u})}\right) d\mathbf{u} - \left[\frac{\rho_p^-(C_1) - \rho_p^-(C_2)}{2^p n(p)}\right], \tag{5.13}$$

where $\alpha \in \mathcal{A}$. We refer to $\Delta_\alpha(C_1||C_2)$ as the cumulative copula Tsallis divergence (CCTD). It is straightforward to show that

$$\lim_{\alpha\to 1} \Delta_\alpha(C_1||C_2) = CCKL(C_1||C_2),$$

where $CCKL(C_1||C_2)$ is the CCKL divergence between two copulas defined in Eq. (4.16). Moreover, when $\alpha = 2$, CCTD reduces to

$$\Delta_2(C_1||C_2) = \int_{\mathbb{I}^p} \frac{(C_1(\mathbf{u}) - C_2(\mathbf{u}))^2}{C_2(\mathbf{u})} d\mathbf{u},$$

which we call as the χ^2 divergence between two copulas, C_1 and C_2. We denote it as $\chi^2(C_1||C_2)$. The χ^2 divergence between two copula densities is discussed in Ghosh and Sunoj (2024). As the copula density may not exist in certain cases, the proposed measure can be considered as an alternative. The following theorem shows that CCTD is always non-negative and zero whenever $C_1 = C_2$ almost surely.

Theorem 5.6.1. *Let $\Delta_\alpha(C_1||C_2)$ be the CCTD between two copulas C_1 and C_2, then for any $\alpha \in \mathcal{A}$, $\Delta_\alpha(C_1||C_2)$ is always non-negative, and $\Delta_\alpha(C_1||C_2) = 0$ whenever $C_1 = C_2$ almost surely.*

Proof. By definition, we have

$$\begin{aligned}\Delta_\alpha(C_1||C_2) &= \int_{\mathbb{I}^p} C_1(\mathbf{u}) \log_{[\alpha]}\left(\frac{C_1(\mathbf{u})}{C_2(\mathbf{u})}\right) d\mathbf{u} - \left[\frac{\rho_p^-(C_1) - \rho_p^-(C_2)}{2^p n(p)}\right] \\ &= \int_{\mathbb{I}^p} C_1(\mathbf{u}) \log_{[\alpha]}\left(\frac{C_1(\mathbf{u})}{C_2(\mathbf{u})}\right) - C_1(\mathbf{u}) + C_2(\mathbf{u})\ d\mathbf{u} \\ &= \int_{\mathbb{I}^p} C_1(\mathbf{u}) f_\alpha\left(\frac{C_2(\mathbf{u})}{C_1(\mathbf{u})}\right)\ d\mathbf{u},\end{aligned}$$

where $f_\alpha(r) = \dfrac{r^{1-\alpha}-1}{\alpha-1} + r - 1$. Using elementary calculus, one can easily show that for any $\alpha \in \mathcal{A}$, the function $f_\alpha(r)$ is always non-negative for every $r \geq 0$ and $f(r)$ attains its minimum at $r = 1$. It follows that $C_1(\mathbf{u}) f_\alpha\left(\dfrac{C_2(\mathbf{u})}{C_1(\mathbf{u})}\right)$ is always non-negative and equal to zero if and only if $C_1(\mathbf{u}) = C_2(\mathbf{u})$ for every $u \in \mathbb{I}^p$, which concludes the proof. □

Now, we will discuss a few mathematical properties associated with the proposed divergence measure. The following theorem discusses how CCTD relates to CCKLD and χ^2 divergence.

Theorem 5.6.2. *Let C_1 and C_2 be two copulas of the same dimension, then the following inequalities hold.*

1. $\Delta_\alpha(C_1||C_2) \begin{cases} \geq \Delta(C_1||C_2), & \text{if } \alpha \in (0,1), \\ \leq \Delta(C_1||C_2), & \text{if } \alpha \in (1,\infty). \end{cases}$
2. $\Delta_\alpha(C_1||C_2) \begin{cases} \geq \chi^2(C_1||C_2), & \text{if } \alpha \in (0,2] \setminus \{1\}, \\ \leq \chi^2(C_1||C_2), & \text{if } \alpha \in (2,\infty). \end{cases}$

Theorem 5.6.3. *Let $C^\Sigma(\mathbf{u}) = \sum_{j=1}^m l_j C_j(\mathbf{u})$ represent the WAM of m copulas, $C_1, C_2, \ldots, C_m$, of the dimension p, where $l_j \in \mathbb{I}$ for $j = 1, 2, \ldots, m$, satisfying $\sum_{j=1}^m l_j = 1$. Let C be any p-dimensional copula, then for any $\alpha \in \mathcal{A}$, we have*

1. $\Delta_\alpha\left(C||C^\Sigma\right) \leq \sum_{j=1}^m l_j \Delta_\alpha\left(C||C_j\right)$
2. $\Delta_\alpha\left(C^\Sigma||C\right) \leq \sum_{j=1}^m l_j \Delta_\alpha\left(C_j||C\right).$

Proof of 1. For any fixed $\alpha \in \mathcal{A}$, the function $f_\alpha(r) = \dfrac{r^{1-\alpha} - 1}{\alpha - 1} + r - 1$ is a convex function for every $x \geq 0$. It follows that for every $x_j \geq 0$, $j = 1, 2, \ldots, m$, we have $f_\alpha\left(\sum_{j=1}^{m} l_j r_j\right) \leq \sum_{j=1}^{m} f_\alpha(l_j r_j)$. Substituting $r_j = \dfrac{C_2(\mathbf{u})}{C_1(\mathbf{u})}$ and using the definition of CCTD, we obtain

$$\begin{aligned}
\Delta_\alpha\left(C||C^{\Sigma}\right) &= \int_{\mathbb{I}^p} C(\mathbf{u}) f_\alpha\left(\frac{C^{\Sigma}(\mathbf{u})}{C(\mathbf{u})}\right) d\mathbf{u} \\
&\leq \sum_{j=1}^{p} \int_{\mathbb{I}^p} C(\mathbf{u}) f_\alpha\left(\frac{C_j(\mathbf{u})}{C(\mathbf{u})}\right) d\mathbf{u} \\
&= \sum_{j=1}^{m} l_j \Delta_\alpha\left(C||C_j\right).
\end{aligned}$$

□

Proof of 2. For any $\alpha \in \mathcal{A}$, we define the function $g_\alpha(r) = \dfrac{k^{1-\alpha} r^{\alpha} - r}{\alpha - 1} + r - k$, where $k \geq 0$ is fixed. It is easy to show that the $g_\alpha(r)$ is a convex function for every $x \geq 0$ for fixed $k \geq 0$. Now, substituting $k = C_2(\mathbf{u})$ and $r = C_1(\mathbf{u})$ and the similar argument of the proof of part (a), we obtain the required result. □

Now, we will discuss the ordering property of CCTD based on the PLOD ordering of copula.

Theorem 5.6.4. *If* $C_1 \overset{PLOD}{\prec} C_2$, *then* $\Delta_\alpha(C_1||C_2) \begin{cases} \geq \Delta_\alpha(C_2||C_1), & \text{if } \alpha \in \left(0, \frac{1}{2}, \right] \\ \leq \Delta_\alpha(C_2||C_1), & \text{if } \alpha \in \left(\frac{1}{2}, \infty\right) \setminus \{1\}. \end{cases}$

Proof. For every fixed $\alpha \in \mathcal{A}$, define the function $h_\alpha : \mathbb{I} \to \mathbb{R}$ as

$$h_\alpha(r) = r \log_{[\alpha]}(r) - \log_{[\alpha]}\left(\frac{1}{r}\right) - 2r + 2, \quad r \in \mathbb{I}.$$

It is easy to show that $h_\alpha(r)$ is an increasing (decreasing) function in $r \in \mathbb{I}$ if $\alpha \in \left(0, \frac{1}{2}\right]$ $\left(\alpha \in \left(\frac{1}{2}, \infty\right) \setminus \{1\}\right)$. It follows that for every $r \in \mathbb{I}$, we have

$$h_\alpha(r) \begin{cases} \geq 0, & \text{if } \alpha \in \left(0, \frac{1}{2}\right], \\ \leq 0, & \text{if } \alpha \in \left(\frac{1}{2}, \infty\right) \setminus \{1\}. \end{cases} \tag{5.14}$$

Note that if $C_1 \overset{\text{PLOD}}{\prec} C_2$, then $\Delta_\alpha(C_1||C_2) - \Delta_\alpha(C_2||C_1) = \int_{\mathbb{I}^p} C_2(\mathbf{u}) h_\alpha\left(\frac{C_1(\mathbf{u})}{C_2(\mathbf{u})}\right) d\mathbf{u}$. Now, the result follows from inequality (5.14). □

Analogous to Theorem 5.5.6, we also have triangle inequality for CCTD.

Theorem 5.6.5. *Let C_1 and C_2 and C_3 be three p-dimensional copulas.*

1. *If $C_1(\mathbf{u}) \le \min\{C_2(\mathbf{u}), C_3(\mathbf{u})\}$ for every $\mathbf{u} \in \mathbb{I}^p$, then $\Delta_\alpha(C_3||C_1) + \Delta_\alpha(C_1||C_2) \ge \Delta_\alpha(C_3||C_2)$*
2. *If $C_1 \overset{PLOD}{\prec} C_2 \overset{PLOD}{\prec} C_3$, then $\Delta_\alpha(C_1||C_2) + \Delta_\alpha(C_2||C_3) \le \Delta_\alpha(C_1||C_3)$.*

Proof. We will prove the first part of the theorem. Since the second part of the proof is similar to the first part, so we left out here. Assume that $C_1(\mathbf{u}) \le \min\{C_2(\mathbf{u}), C_3(\mathbf{u})\}$ for every $\mathbf{u} \in \mathbb{I}^p$. Now consider

$$\begin{aligned}
\Delta_\alpha(C_3||C_1) + \Delta_\alpha(C_1||C_2) - \Delta_\alpha(C_3||C_2) &= \int_{\mathbb{I}^p} C_3(\mathbf{u}) \log_{[\alpha]} \left(\frac{C_3(\mathbf{u})}{C_1(\mathbf{u})}\right) + C_1(\mathbf{u}) \log_{[\alpha]} \left(\frac{C_1(\mathbf{u})}{C_2(\mathbf{u})}\right) \\
&\quad - C_3(\mathbf{u}) \log_{[\alpha]} \left(\frac{C_3(\mathbf{u})}{C_2(\mathbf{u})}\right) d\mathbf{u} \\
&= \int_{\mathbb{I}^p} \left(\frac{C_3^\alpha(\mathbf{u}) - C_1^\alpha(\mathbf{u})}{\alpha - 1}\right) \left[\frac{1}{C_1^{\alpha-1}(\mathbf{u})} - \frac{1}{C_2^{\alpha-1}(\mathbf{u})}\right] d\mathbf{u} \\
&\ge 0.
\end{aligned}$$

□

Now, we proceed to discuss the mutual information (MI) of a multivariate random vector. Let X_1 and X_2 be two continuous random variables with joint PDF $f(x_1, x_2)$ and marginal PDFs $f_1(x_1)$ and $f_2(x_2)$, respectively. The MI between X_1 and X_2 is defined as

$$MI(X_1, X_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2) \log \left(\frac{f(x_1, x_2)}{f_1(x_1) f_2(x_2)}\right) dx_1 dx_2.$$

Note that $MI(X_1, X_2)$ is equivalent to the KL divergence between the joint PDF of (X_1, X_2) and the product of the marginal PDFs of X_1 and X_2. For further details, we refer readers to Cover (1999), Ash (2012), and Murphy (2022).

Joe (1987) extended the notion of mutual information to higher dimensions. Let $\mathbf{X}$ be a d-variate continuous random variable with joint PDF $f(\cdot)$ and marginal CDFs (PDFs) $F_i(\cdot)$ $(f_i(\cdot))$, $i = 1, 2, \ldots, p$, where the marginals need not be identical. Let $c(\cdot)$ denote the copula density corresponding to $\mathbf{X}$. By Sklar's theorem, the joint PDF $f(\mathbf{x})$ can be expressed as

$$f(\mathbf{x}) = c\left(F_1(x_1), F_2(x_2), \ldots, F_p(x_p)\right) \prod_{j=1}^{p} f_j(x_j),$$

where $\mathbf{x} \in \mathbb{R}^p$. It follows that the MI corresponding to $\mathbf{X}$ is given by

$$MI(\mathbf{X}) = \int_{\mathbb{I}^p} c(\mathbf{u}) \log\left(c(\mathbf{u})\right) d\mathbf{u}.$$

The relationship between MI and copula entropy has been discussed independently by Blumentritt and Schmid (2012) and Ma and Sun (2011). However, the term "copula entropy" was first introduced in Ma and Sun (2011).

If the underlying copula is not absolutely continuous (e.g., the minimum copula), the copula density does not exist, and estimating MI non-parametrically in such cases becomes challenging. Using the relationship between KL divergence and MI, and based on the proposed cumulative copula Tsallis divergence (CCTD), we introduce an alternative MI measure called cumulative mutual information (CMI) of order α. Let C_1 denote the underlying copula of a multivariate random vector $\mathbf{X}$, and let $\Pi(\mathbf{u}) = \prod_{j=1}^{p} u_j$ represent the product copula. For any $\alpha \in \mathcal{A}$, the CMI of order α is defined as

$$\mu_\alpha(C) = \Delta_\alpha(C\|\Pi) = \int_{\mathbb{I}^p} C(\mathbf{u}) \log_{[\alpha]} \left(\frac{C(\mathbf{u})}{\Pi(\mathbf{u})}\right) d\mathbf{u} - \frac{\rho_p^-(C)}{2^p n(p)},$$

where $\rho_p^-(\cdot)$ is the multivariate Spearman's correlation. In the limiting case as $\alpha \to 1$, we have

$$\mu(C) = \int_{\mathbb{I}^p} C(\mathbf{u}) \log \left(\frac{C(\mathbf{u})}{\Pi(\mathbf{u})}\right) d\mathbf{u} - \frac{\rho_p^-(C)}{2^p n(p)}.$$

This limiting case is referred to as cumulative mutual information. The proposed CMI provides an alternative to existing correlation measures. The existing correlation measures, such as Pearson's correlation, are limited to linear relationships, while Spearman's and Kendall's correlations capture monotonic relationships but are primarily suited for bivariate cases. The proposed measure, on the other hand, quantifies deviations from independence to stronger dependence in any dimension, making it a robust candidate for dependency analysis in multivariate contexts. The application of CMI of order α is illustrated in the subsequent section. We conclude this section by presenting a few examples of the proposed CCTD and CMI for well-known copulas.

Example 5.6.1. *The CCTD measure between the FGM copula*

$$C(u_1, u_2) = u_1 u_2 \left(1 + \theta(1-u_1)(1-u_2)\right)$$

and Fréchet-Hoeffding upper bound copula $M(u_1, u_2) = \min\{u_1, u_2\}$ *is*

$$\begin{aligned}\Delta_\alpha(C||M) =& \frac{2}{\alpha-1} \sum_{t=0}^{\infty} \binom{\alpha+t-1}{t} \theta^t \left[\frac{\beta(\alpha+1, t+1)}{(t+1)(t+2)} - \frac{\beta(\alpha+1, 2t+1)}{(t+1)}\right. \\ &\left. + \frac{\beta(\alpha+1, 2t+2)}{(t+2)(t+2)}\right] - \frac{\alpha(\theta+9)}{36(\alpha-1)} + \frac{1}{3}.\end{aligned}$$

Example 5.6.2. *The CCTD of the Gumbel-Barnett copula*

$$C(u_1, u_2) = u_1 u_2 \exp\{-\phi \log(u_1) \log(u_2)\}, \theta \in \mathbb{I},$$

then

$$\Delta_\alpha(C||\Pi) = \mu_\alpha(C) = -\frac{e^{\frac{4}{(\alpha-1)\phi}} E_i\left(\frac{-4}{\phi(\alpha-1)}\right)}{\phi} + \frac{\alpha e^{\frac{4}{(\alpha-1)\phi}} E_i\left(\frac{-4}{\phi}\right)}{(\alpha-1)\phi} + \frac{1}{4}.$$

We use the results of Yela and Cuevas (2018) for computing the above intergals and $E_i(\cdot)$ is the well-known exponential integral function.

Example 5.6.3. *The CCTD between the $p-$variate product copula and the p-variate Fréchet-Hoeffding upper bound copula is*

$$\Delta_\alpha(\Pi||M) = \frac{p!}{2(\alpha-1)\prod_{j=1}^{p-1}(j(\alpha+1)+2)} - \frac{\alpha}{2^p(\alpha-1)} + \frac{1}{p+1}.$$

Example 5.6.4. *The CCTD of the Cuadras-Augé copula $C(\mathbf{u}) = \prod_{i=1}^{p} u_{[i]}^{\gamma_i}$ is given by*

$$\Delta_\alpha(C||\Pi) = \mu_\alpha(C) = \frac{d!}{(\alpha-1)}\left[\frac{1}{\prod_{i=1}^{p}\omega_1(i)} - \frac{\alpha}{\prod_{i=1}^{p}\omega_2(i)}\right] + \frac{1}{2^p},$$

where $\omega_1(i)$ and $\omega_2(i)$ satisfies the recurrence relation given by

$$\begin{aligned}\omega_1(i) =& \omega_1(i-1) + (\theta_i - 1)\alpha + 1,\\ \omega_2(i) =& \omega_2(i-1) + \theta_i + 1,\end{aligned}$$

for $i = 2, 3, \ldots, p$ with $\omega_1(1) = (\theta_1 - 1)\alpha + 1$ and $\omega_2(1) = \theta_1 + 1$

5.7 Application

Here, we explore the applications of the proposed mutual information measure in two different areas: testing for the mutual independence of continuous random variables and its relevance in the finance sector as an economic indicator.

5.7.1 Test for the Mutual Independence of Continuous Random Variables

In multivariate data analysis, the assumption of mutual independence is frequently encountered. For such cases, Pearson's correlation test is commonly used under the assumption of bivariate normality. Non-parametric tests, such as Spearman's and Kendall's correlation tests, are often used when the relationship between variables is monotonic.

However, these tests are primarily designed to test the pairwise correlation for specific types of relationships and are often misused as tests for independence.

Current research focused on empirical copula process-based tests for independence. The foundational idea was introduced by Deheuvels (1979). The Cramér-von Mises and Kolmogorov-Smirnov functionals are widely used for testing mutual independence among random variables. For further details, we recommend Deheuvels (1979), Genest and Rémillard (2004), Genest et al. (2006), Kojadinovic and Holmes (2009), Belalia et al. (2017), Herwartz and Maxand (2020), and Nasri and Remillard (2024).

Further, we propose using the CMI measure as a test statistic for testing the mutual independence among continuous random variables. We also compare the power of the proposed test with existing independence tests based on the Cramér-von Mises and Kolmogorov-Smirnov statistics. To illustrate its practicality, we apply our test to a real dataset.

Let $\mathbf{X} = (X_1, X_2, \ldots, X_p)$ be a p-variate continuous random vector with an underlying copula C. The copula C can be approximated by the empirical copula $\hat{C}_n$, based on n random samples $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$, as defined in Eq. (5.6). To measure dependence, we consider the non-parametric cumulative mutual information (CMI). For mathematical simplicity, we take $\alpha = 2$, yielding

$$\begin{aligned}\mu_2(\hat{C}_n) &= \int_{\mathbb{I}^p} \hat{C}_n(\mathbf{u}) \log_{[2]}\left(\frac{\hat{C}_n(\mathbf{u})}{\Pi(\mathbf{u})}\right) d\mathbf{u} - \frac{\rho_p^-(\hat{C}_n)}{2^p n(p)} \\ &= \int_{\mathbb{I}^p} \frac{\left(\hat{C}_n(\mathbf{u}) - \Pi(\mathbf{u})\right)^2}{\Pi(\mathbf{u})} d\mathbf{u} \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{k=1}^{p}\left[-\log\left(\max\left\{\frac{R_{i,k}}{n+1}, \frac{R_{j,k}}{n+1}\right\}\right)\right] - \frac{2}{n}\sum_{i=1}^{n}\prod_{k=1}^{p}\left[1 - \frac{R_{i,k}}{n+1}\right] + \frac{1}{2^p},\end{aligned} \tag{5.15}$$

where $R_{i,k}$ represents the rank of the k-th component of the i-th observation.

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be n random samples from a common multivariate population. We aim to test the null hypothesis H_0 that the components of the multivariate population are mutually independent, i.e., the underlying copula is the product copula $\Pi(\mathbf{u}) = \prod_{k=1}^{p} u_k$. Using the definition of non-parametric CMI from Eq. (5.15), we propose the following test statistic

$$\chi^2_{\text{div}}(n) = n\mu_2(\hat{C}_n) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{k=1}^{p}\left[-\log\left(\max\left\{\frac{R_{i,k}}{n+1}, \frac{R_{j,k}}{n+1}\right\}\right)\right] - 2\sum_{i=1}^{n}\prod_{k=1}^{p}\left[1-\frac{R_{i,k}}{n+1}\right] + \frac{n}{2^p}. \tag{5.16}$$

Since we set $\alpha = 2$, we call this the χ^2 *divergence test for mutual independence*, and denote the test statistic by $\chi^2_{\text{div}}(n)$, where n is the sample size.

To study the asymptotic behavior of the proposed test under H_0, the following lemma, discussed in Fermanian et al. (2004), Tsukahara (2005), and Kojadinovic and Holmes (2009) is useful.

Lemma 5.7.1. *Let C be a p-dimensional copula. Let $L_\infty(\mathbb{I}^p)$ denote the Banach space of real-valued bounded functions defined on $\mathbb{I}^p$, equipped with the supremum norm. If C has continuous partial derivatives for every $\mathbf{u} \in \mathbb{I}^p$, then the empirical process*

$$\mathbb{Z}_n(\mathbf{u}) = \sqrt{n}\left(\hat{C}_n(\mathbf{u}) - C(\mathbf{u})\right)$$

converges weakly in $L_\infty(\mathbb{I}^p)$ to the tight centered Gaussian process

$$\mathbb{Z}(\mathbf{u}) = \Gamma(\mathbf{u}) - \sum_{i=1}^{p}\partial_i C(\mathbf{u})\Gamma(\mathbf{u}_i),$$

where $\partial_i C(\mathbf{u})$ is the i-th partial derivative of C, $\mathbf{u}_i = (1,\ldots,1,u_i,1,\ldots,1)$ with u_i in the i-th position, and $\Gamma(\mathbf{u})$ is a tight centered Gaussian process on $\mathbb{I}^p$ with covariance function

$$\Sigma(\mathbf{u},\mathbf{v}) = C(\mathbf{u}\wedge\mathbf{v}) - C(\mathbf{u})C(\mathbf{v}),$$

where $\mathbf{u}\wedge\mathbf{v} = \Big(\min(u_1,v_1),\ldots,\min(u_p,v_p)\Big)$.

Using the Lemma 5.7.1 and the application of the continuous mapping theorem, we have the following theorem.

Theorem 5.7.2. *Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be n random samples from a multivariate population. Then, under the null hypothesis of mutual independence, the test statistic $\chi^2_{div}(n)$ (as given in Eq. (5.16)) converges in distribution to*

$$\int_{\mathbb{I}^p}\frac{\mathbb{Z}^2(\mathbf{u})}{\Pi(\mathbf{u})}\,d\mathbf{u},$$

where

$$\mathbb{Z}(\mathbf{u}) = \Gamma\left(\mathbf{u}\right) - \sum_{i=1}^{p}\Gamma(\mathbf{u}_i)\prod_{\substack{j=1\\ j\neq i}}^{p}\Pi(\mathbf{u}_j),$$

is a tight centered Gaussian process with $\mathbf{u}_i = (1, \ldots, 1, u_i, 1, \ldots, 1)$ *represents the vector with the* i*-th component equal to* u_i *and all other components equal to 1 for* $i = 1, 2, \ldots, d$*. The process* $\Gamma(\mathbf{u})$ *is a tight centered Gaussian process on* $\mathbb{I}^p$ *with covariance function*

$$\Sigma(\mathbf{u}, \mathbf{v}) = \mathbb{E}\left[\Gamma(\mathbf{u})\Gamma(\mathbf{v})\right] = \Pi(\mathbf{u} \wedge \mathbf{v}) - \Pi(\mathbf{u})\Pi\mathbf{v}),$$

where $\mathbf{u} \wedge \mathbf{v} = (\min(u_1, v_1), \ldots, \min(u_p, v_d))$*.*

Now, we will discuss the computation of p-values of the proposed test. Since the the distribution of the proposed test statistic $\chi^2_{\text{div}}(n)$ based on n random samples is complex in nature, even in the asymptotic case, we employ the bootstrapping procedure to compute the approximate p-values. The validity of the proposed approach is discussed in Genest and Rémillard (2008). Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be n random samples from a multivariate population. Let $\mathbf{D} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)'$ be the data matrix. Then the procedure for computing the p-values is discussed as follows.

1. Convert the data matrix $\mathbf{D} = [\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n]$ to the rank matrix $\mathbf{R} = [R_{ik}]$, where $R_{i,k}$ is the rank of the k-th component of the i-th observation (i.e., $\mathbf{X}_i$). If ties occur, break them randomly.
2. Calculate the test statistic $\chi^2_{\text{div}}(n)$ using the formula

$$\chi^2_{\text{div}}(n) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{k=1}^{p}\left[-\log\left(\max\left\{\frac{R_{i,k}}{n+1}, \frac{R_{j,k}}{n+1}\right\}\right)\right] - 2\sum_{i=1}^{n}\prod_{k=1}^{p}\left[1 - \frac{R_{i,k}}{n+1}\right] + \frac{n}{2^p}.$$

3. Generate B random samples of size n from the product copula. For each random sample, compute the test statistic $\chi^2_{\text{div}}(n_b)$, $b = 1, 2, \ldots, B$.
4. Arrange the computed bootstrap test statistics in ascending order

$$\chi^2_{\text{div}}(n_{(1)}) \le \chi^2_{\text{div}}(n_{(2)}) \le \cdots \le \chi^2_{\text{div}}(n_{(B)}).$$

5. Estimate the p-value associated with the observed test statistic $\chi^2_{\text{div}}(n)$ as:

$$\text{p-value} = \frac{1}{B}\sum_{b=1}^{B}\mathbf{I}\left\{\chi^2_{\text{div}}(n_{(b)}) \ge \chi^2_{\text{div}}(n)\right\},$$

   where $\mathbf{I}\{\cdot\}$ is the indicator function.

Now, we conduct the simulation study to evaluate the performance of the proposed model. We generate $10,000$ samples of various sizes and compute the power of the proposed test with the alternative copula such as Clayton, FGM, Frank, Normal and Student t of various

Kendall's Tau. We compare our results with Cramér-von Mises (CVM) statistics, which are given by

$$S_n = \frac{n}{3^p} - \frac{1}{2^{p-1}} \sum_{i=1}^{n} \prod_{k=1}^{p} \left(1 - \left(\frac{R_{i,k}}{n+1^2}\right)^2\right) + \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{k=1}^{p} \left(1 - \max\left\{\frac{R_{i,k}}{n+1}, \frac{R_{j,k}}{n+1}\right\}\right)$$

and Kolmogorov-Smirnov (KS) statistics given by

$$K_n = \sqrt{n} \sup_{\mathbf{u} \in \mathbf{I}^p} |\hat{C}_n(\mathbf{u}) - C(\mathbf{u})|.$$

Note that the explicit statistic of K_n is often challenging, and we approximate it by its sample counterparts. The results are presented in Table 5.1.

From Table 5.1, it is clear that the proposed test rejection power is superior when the alternative copula is Clayton compared to other tests. The results show a significant improvement over CVM and KS tests. The proposed test also performs better for the Student's t copula across various Kendall's τ. For the remaining copulas, the power of the proposed test is comparable, making it a strong candidate for testing the mutual independence of random variables.

We now apply our test to a real dataset. This dataset comprises 249 observations of the volatility-adjusted log returns (VALR) of two banks, Citigroup and Bank of America, for the year 2012. The "Banks" dataset is freely accessible in the R software within the `gofCopula` package. The scatterplot of the data is presented in Figure 5.6. From Figure 5.6, it is evident that there is a strong positive dependence between the VALR of the two banks. The proposed test supports this observation, yielding a test statistic of 12.089 and a p-value approximately equal to zero.

5.7.2 Application in Financial Time Series

In this subsection, we present our proposed CMI as an economic indicator. We consider the daily price returns of Crude Oil and the S&P 500 index during the period from January 2, 2005, to December 31, 2022. The plots of daily data and daily returns are shown in Figures 5.7 and 5.8, respectively.

The data for Crude Oil was obtained from the Federal Reserve Economic Data (FRED), and the data for the S&P 500 index was sourced from Yahoo Finance. We compute the proposed CMI using an overlapping sliding time window of 200 data points, with a shift size of 100 points. This approach reveals the evolution of the series over time and identifies any mutual information between the daily price returns of WTI Crude Oil and the S&P

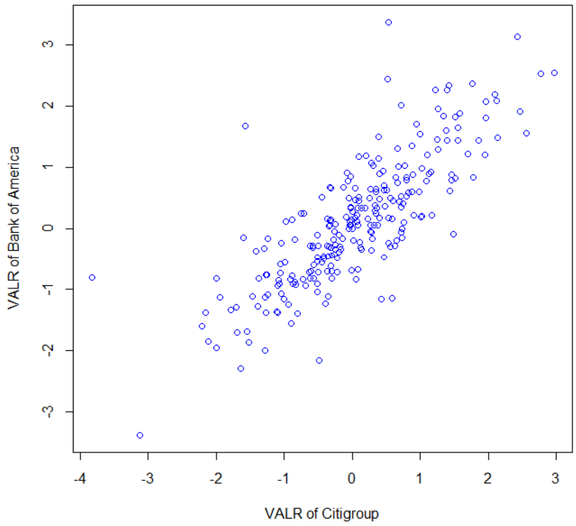


FIGURE 5.6. Scatterplot of volatility-adjusted log returns of Citigroup and Bank of America.

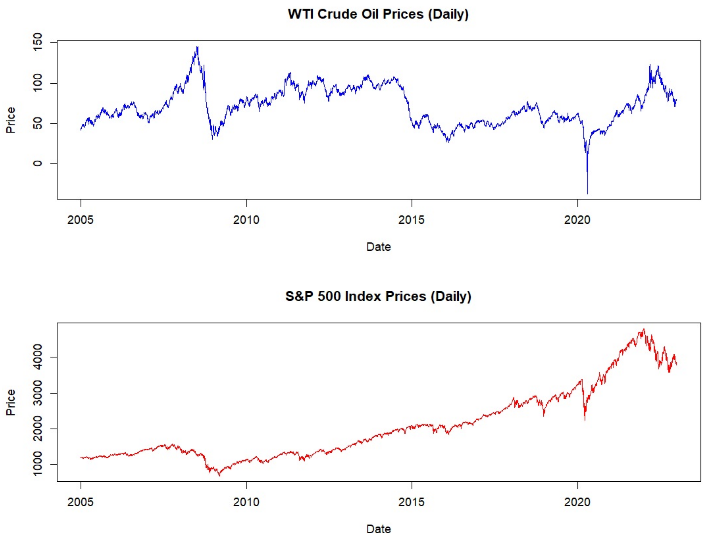


FIGURE 5.7. Daily data of Crude Oil and S&P 500 index.

500 index. The contour plot of the proposed CMI for different values of α is shown in Figure 5.9.

During the period from 2005 to 2022, several significant financial events occurred:

- The global economic recession from 2008 to 2009 and slow recovery impacted crude oil prices from 2010 to 2012.

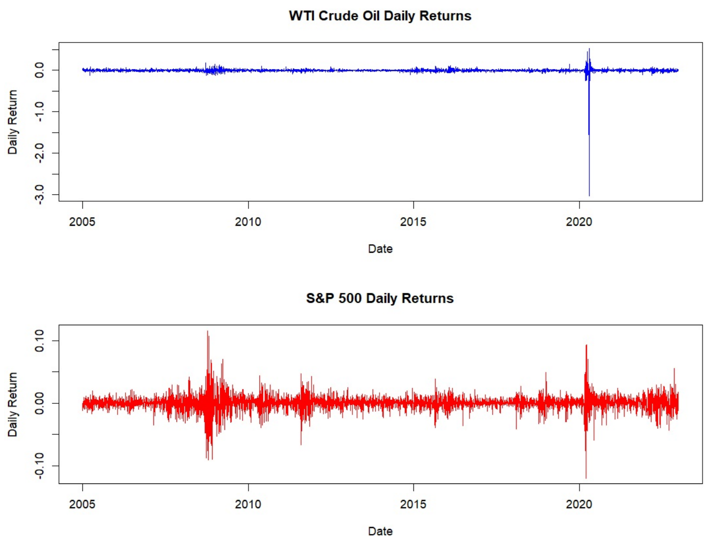


FIGURE 5.8. Daily returns of Crude Oil and S&P 500 index.

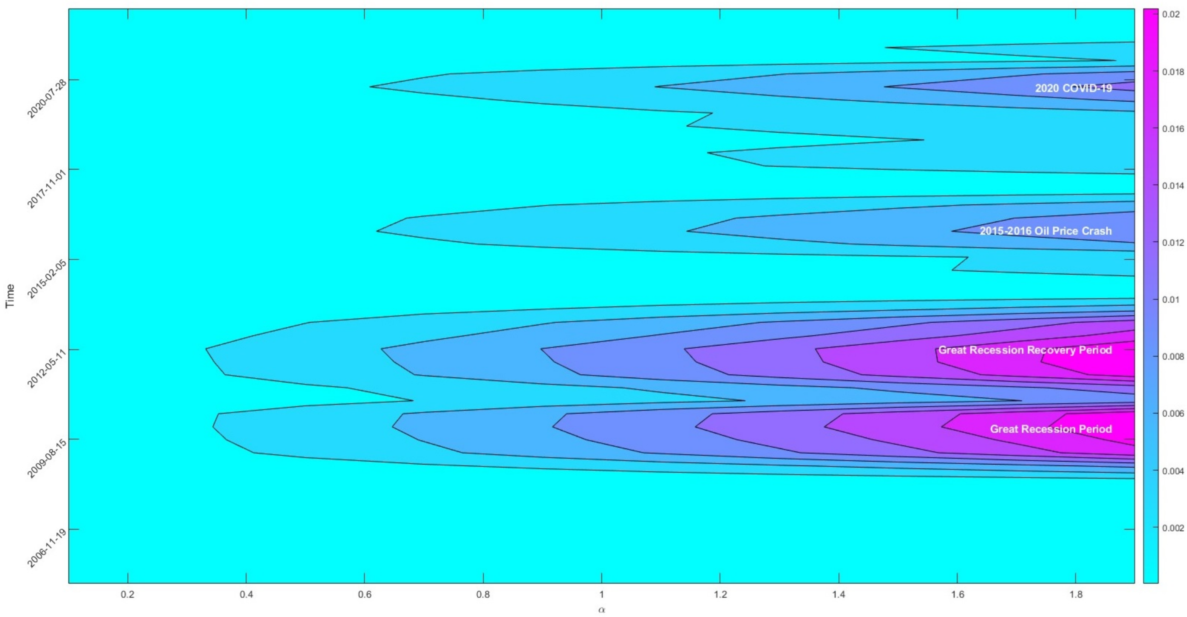


FIGURE 5.9. Contour plot of the proposed CMI for different values of α.

- The oil price crash occurred between 2015 and 2016.
- The COVID-19 pandemic caused significant financial disruptions in 2020.

For more details, we refer to Lyu et al. (2021), Stocker et al. (2018), and news articles during these financial crisis periods. The proposed CMI measure effectively captures the financial crises, as evident from the results. For higher values of α, the CMI increases, further supporting its potential as an economic indicator for financial crises.

Table 5.1. Power comparison of tests for different true copulas, Kendall's τ, and sample sizes n.

True Copula	Test	$\tau=-0.2$			$\tau=-0.1$			$\tau=0$			$\tau=0.1$			$\tau=0.2$		
		$n=50$	$n=100$	$n=150$	$n=50$	$n=100$	$n=150$	$n=50$	$n=100$	$n=150$	$n=50$	$n=100$	$n=150$	$n=50$	$n=100$	$n=150$
Clayton	CVM Test	0.5394	0.8427	0.8603	0.1694	0.2984	0.4341	0.0370	0.0490	0.0518	0.1476	0.2776	0.4098	0.4984	0.8104	0.9386
	KS Test	0.1143	0.4531	0.4779	0.0267	0.0860	0.1663	0.0494	0.0487	0.0477	0.1743	0.2813	0.3779	0.4892	0.4926	0.8991
	Proposed Test	0.7065	0.9624	0.9692	0.2184	0.4250	0.5923	0.0580	0.0513	0.0492	0.2142	0.3952	0.5572	0.6491	0.8148	0.9838
FGM	CVM Test	0.5443	0.8344	0.8483	0.1646	0.2901	0.4383	0.0537	0.0471	0.0502	0.1646	0.3033	0.4387	0.5248	0.8501	0.9542
	KS Test	0.1387	0.7562	0.5187	0.0298	0.0947	0.1969	0.0511	0.0516	0.0489	0.0298	0.3155	0.4282	0.5263	0.8080	0.9196
	Proposed Test	0.5336	0.9187	0.8290	0.1669	0.2820	0.3958	0.0585	0.0486	0.0513	0.1292	0.2478	0.3523	0.4313	0.7740	0.9151
Frank	CVM Test	0.5220	0.8267	0.8386	0.1675	0.2978	0.4336	0.0499	0.0512	0.0512	0.1646	0.3038	0.4375	0.5170	0.8350	0.9489
	KS Test	0.1359	0.8080	0.5107	0.0270	0.0984	0.1955	0.0514	0.0590	0.0483	0.2015	0.3217	0.4281	0.5231	0.7977	0.9223
	Proposed Test	0.5078	0.7740	0.8092	0.1673	0.2973	0.3893	0.0545	0.0612	0.0499	0.1347	0.2571	0.3596	0.4375	0.7667	0.9178
Normal	CVM Test	0.5086	0.8052	0.8187	0.1570	0.2731	0.4037	0.0621	0.0492	0.0511	0.1586	0.2824	0.4066	0.4897	0.8012	0.9312
	KS Test	0.1083	0.4287	0.4258	0.0234	0.0790	0.1586	0.0624	0.0481	0.0486	0.1858	0.2884	0.3790	0.4727	0.7321	0.8739
	Proposed Test	0.5105	0.7985	0.8111	0.1649	0.2799	0.3814	0.0550	0.0465	0.0541	0.1311	0.2556	0.3604	0.4434	0.7638	0.9106
Student's t	CVM Test	0.4987	0.7885	0.8036	0.1805	0.2937	0.4305	0.0580	0.0521	0.0478	0.1772	0.3050	0.4260	0.4958	0.7927	0.9261
	KS Test	0.1126	0.4318	0.4378	0.0292	0.0863	0.1743	0.0610	0.0528	0.0498	0.2093	0.3063	0.3926	0.4926	0.7600	0.8825
	Proposed Test	0.4960	0.7638	0.7878	0.1930	0.3026	0.4087	0.0590	0.0492	0.0501	0.1909	0.3527	0.4785	0.4976	0.7947	0.9279

5.8 Conclusion and Future Direction

In this chapter, we introduced a new non-additive dependence entropy called the cumulative copula Tsallis entropy. We discussed its mathematical properties, including bounds, copula ordering, and uniform convergence results. Using the empirical copula, we proposed a non-parametric estimator for the entropy and established its theoretical convergence as well as its convergence through Monte Carlo simulations.

To validate the utility of the proposed entropy in quantifying uncertainty in dependence structures, we examined Rulkov maps. Our findings indicate that the proposed entropy increases with periodicity and reaches its maximum in chaotic cases. To address the uncertainty arising from incorrect copula assumptions, we proposed a copula-based Kerridge inaccuracy measure, studied its properties (including triangular inequalities), and demonstrated its generalization of the results presented in Hosseini and Nooghabi (2021). These concepts were illustrated using well-known copulas.

Furthermore, we introduced cumulative copula divergence using Tsallis divergence. Based on this, a new mutual information measure, termed cumulative mutual information, was proposed by leveraging its relationship with Kullback-Leibler divergence. This approach overcomes limitations in the existing copula density-based mutual information. The utility of this mutual information measure was demonstrated in two important statistical applications:

- **Hypothesis testing**, specifically for mutual independence among random variables.
- **Finance**, as an economic indicator for multivariate time series, providing a robust alternative to traditional correlation measures.

While our study focused on Tsallis entropy, the proposed methodology can be extended to other entropies, such as Rényi entropy. Moreover, recent advancements using the Möbius decomposition of the empirical copula process have been shown to improve the power of tests based on the Cramér-von Mises statistic. For further details, see Deheuvels (1979), Genest and Rémillard (2004), and Kojadinovic and Holmes (2009). Incorporating similar techniques into our proposed test could significantly enhance its power, making it an interesting direction for future research.

Chapter 6

Smooth Estimation of Bivariate Mean Residual Life Function

This chapter presents a smooth estimation procedure for the bivariate residual life function. An extensive simulation study is conducted to evaluate the performance of the proposed estimator and illustrate its applicability using bivariate warranty data.

6.1 Introduction

In the automobile industry, warranty policy design primarily considers two key factors: the age and the usage of the vehicle. Excessive usage beyond the expected limit can significantly contribute to vehicle failure. Additionally, as the vehicle ages, its reliability tends to decline. Therefore, in formulating effective warranty policies, manufacturers must determine both the age limit and the usage limit of the product, which together define the expiration criteria for warranty claims. In many practical situations, the age and usage of the product exhibit a high degree of dependence.

Several approaches have been proposed in the literature to model and analyze bivariate warranty data. One notable method involves treating usage as a random function of product age, enabling the estimation of product reliability as a basis for developing warranty policies. This methodology is commonly referred to as the conditional approach.

For further details on this line of research, we refer the reader to Lawless et al. (1995), Ahn et al. (1998), and Duchesne and Lawless (2000).

An alternative approach assumes a bivariate lifetime distribution for the product's age and usage, allowing for the estimation of bivariate reliability. For instance, Jung and Bai (2007) modelled the joint distribution using a bivariate Weibull distribution with a Gumbel-Hougaard copula to capture the dependence structure. Similarly, Wu (2014) and Anderson et al. (2017) employed copula functions to estimate bivariate reliability. Yuan (2018) proposed a generalized moment estimator for the bivariate Weibull distribution and demonstrated its applicability in the analysis of bivariate warranty data. More recently, Gupta and Bhattacharya (2022) introduced a nonparametric estimation procedure for bivariate reliability based on kernel estimation, with a focus on censored bivariate warranty data.

In order to measure the reliability of a product, in addition to the joint survival function, three reliability measures are commonly used: the joint density function, the bivariate hazard rate function, and the bivariate mean residual life (BMRL) function. The bivariate hazard rate function evaluates the instantaneous failure rate, whereas the BMRL function provides information about the average remaining lifetime of a product, given that both its age and usage have survived beyond some time t.

The BMRL measure plays a crucial role in formulating warranty policies, as it offers insight into the remaining life of a product and the duration for which it is expected to operate without failure. Let X_1 and X_2 represent the age and usage of a product, respectively. Then the BMRL function $(m_1(x_1, x_2), m_2(x_1, x_2))$ is defined by the vector

$$(m_1(x_1, x_2), m_2(x_1, x_2)) = \left(\frac{\int_{x_1}^{\infty} \bar{F}(t, x_2)\, dt}{\bar{F}(x_1, x_2)}, \frac{\int_{x_2}^{\infty} \bar{F}(x_1, t)\, dt}{\bar{F}(x_1, x_2)} \right), \tag{6.1}$$

where $\bar{F}(x_1, x_2)$ denotes the joint survival function of age and usage. It is to be noted that the concept of the BMRL function was introduced by Arnold and Zahedi (1988).

One approach to estimating the BMRL function is the parametric method, which involves assuming a specific bivariate lifetime distribution. However, if the assumed distribution does not reflect the true underlying distribution, it may lead to significant bias in the estimates. This, in turn, can result in inaccurate warranty claims and potential financial losses for the manufacturers. Therefore, it is essential to consider nonparametric approaches for estimating the BMRL function, which offer greater flexibility and robustness without relying on strict distributional assumptions.

Kulkarni and Rattihalli (2002) proposed a nonparametric estimator for the BMRL function using the bivariate empirical survival function. Let (X_{1i}, X_{2i}), $i = 1, 2, \ldots, n$ be a bivariate random sample of size n with joint survival function $\bar{F}(x_1, x_2)$. The empirical survival function for $\bar{F}(x_1, x_2)$ based on the sample is given by

$$R_n(x_1, x_2) = \frac{\sum_{i=1}^{n} \mathbf{I}(X_{1i} > x_1,\ X_{2i} > x_2)}{n}, \tag{6.2}$$

where $\mathbf{I}(E)$ denotes the indicator function, which takes value 1 if E holds; takes value 0, otherwise.

Using the definition in Eq. (6.2), Kulkarni and Rattihalli (2002) proposed the following nonparametric estimator for the BMRL function

$$\hat{m}_r(x_1, x_2) = \begin{cases} \dfrac{\sum_{i=1}^{n} (X_{ji} - x_j)\, \mathbf{I}(X_{1i} > x_1,\ X_{2i} > x_2)}{\sum_{i=1}^{n} \mathbf{I}(X_{1i} > x_1,\ X_{2i} > x_2)}, & \text{if } X_{1[n]} > x_1 \text{ and } X_{2[n]} > x_2, \\ 0, & \text{otherwise,} \end{cases} \tag{6.3}$$

where $X_{r[n]} = \max\{X_{r1}, X_{r2}, \ldots, X_{rn}\}$ for $r = 1, 2$. We call this estimator as natural nonparametric estimator. It is important to note that a key limitation of the above estimator is that $\hat{m}_r(x_1, x_2)$ becomes undefined if $\sum_{i=1}^{n} \mathbf{I}(X_{1i} > x_1,\ X_{2i} > x_2) = 0$, i.e., when there are no observations such that both $X_{1i} > x_1$ and $X_{2i} > x_2$. This may lead to significant bias in the estimation. Furthermore, for an absolutely continuous joint survival function, the BMRL function is always continuous; however, the aforementioned nonparametric estimator is not continuous. These limitations motivate us to propose a smooth estimator for the BMRL function.

In recent years, considerable efforts have been made to propose smooth estimators. Using Bernstein polynomials, Leblanc (2012) proposed a smooth nonparametric estimator of the distribution function by smoothing the empirical distribution. Along similar lines, Babu and Chaubey (2006) extended the idea to the multivariate case. However, a key limitation of these methods is that the support of the random vector is restricted to the unit hypercube. To overcome this limitation, Chaubey and Sen (1996) introduced a smooth estimator for univariate survival and density functions of non-negative random variables using Poisson weights.

Let $S_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{I}(X_i > x)$ denote the empirical survival function based on a random sample $\{X_1, X_2, \ldots, X_n\}$ of size n. Let $p(k;\mu) = \frac{e^{-\mu}\mu^k}{k!}$ denote the probability mass function of a Poisson random variable with rate parameter μ. Then, the smooth estimator is given by

$$S_n^p(x) = \sum_{k=0}^{\infty} p(k; x\mu_n)\, S_n\left(\frac{k}{\mu_n}\right), \quad x > 0,$$

where μ_n is chosen such that $\mu_n \to \infty$ and $\mu_n/n \to 0$ as $n \to \infty$. In practice, Chaubey and Sen (1996) suggested taking $\mu_n = \frac{n}{\max\{X_1,X_2,\ldots,X_n\}}$. Then, the smooth estimator reduces to

$$S_n^p(x) = \sum_{k=0}^{n} p(k; x\mu_n)\, S_n\left(\frac{k}{\mu_n}\right), \quad x > 0. \tag{6.4}$$

The authors also studied the asymptotic properties of the smooth estimator and showed that, like the empirical estimator, the smooth estimator is a consistent estimator of the true survival function. Using the smooth survival estimator in Eq. (6.4), Chaubey and Sen (1999) further proposed a smooth estimator for the univariate mean residual life (MRL) function $m(x) = \mathbb{E}(X - x \mid X > x)$ for a non-negative random variable, given by

$$\hat{m}^p(x) = \frac{(1/\mu_n)\sum_{k=0}^{n} Q(k; x\mu_n)\, S_n\left(\frac{k}{\mu_n}\right)}{\sum_{k=0}^{n} p(k; x\mu_n)\, S_n\left(\frac{k}{\mu_n}\right)}, \quad x > 0, \tag{6.5}$$

where $Q(k; x\mu_n) = \sum_{r=0}^{k} p_{x\mu_n}(r)$ denotes the cumulative distribution function (CDF) of a Poisson random variable with rate parameter $x\mu_n$. Unlike the natural estimator of the MRL based on the empirical survival function, the smooth estimator allows estimation of $m(x)$ beyond the largest observed value in the sample. The method was further extended to accommodate censored samples in Chaubey and Sen (2008).

Motivated by the work of Chaubey and Sen (1999), we extend the idea to higher dimensions. The main contributions of this chapter are summarized below:

- A smooth estimator for the bivariate mean residual life (BMRL) function is proposed, and it is shown that the estimator is consistent.
- An extensive simulation study is conducted for various bivariate distributions to compare the performance of the proposed smooth estimator with the natural estimator of Kulkarni and Rattihalli (2002).
- A real bivariate warranty dataset is analyzed, and the mean residual life of product age under various warranty policies is discussed.

The remainder of the chapter is organized as follows. In Section 6.2, a new smooth estimator for BRML is proposed, and its asymptotic properties are discussed. Section 6.3 presents a simulation study comparing the performance of the proposed estimator with the estimator of Kulkarni and Rattihalli (2002) under different sample sizes and bivariate distributions. Section 6.4 analyzes a real bivariate warranty dataset on traction motors and estimates the mean residual life under various warranty limits. Finally, the chapter is concluded in Section 6.5.

6.2 Smooth Estimator of Bivariate Mean Residual Life Function

Chaubey and Sen (2002) proposed a smooth estimator for the multivariate survival function by smoothing the empirical joint survival function. For the sake of convenience, we consider the bivariate case only. Let $(X_{1i}, X_{2i}),\ i = 1, 2, \ldots, n$ be a bivariate random sample of size n with joint survival function $\bar{F}(x_1, x_2)$. Let $R_n(x_1, x_2)$ be the bivariate empirical survival function as defined in Eq. (6.2). Then, the smooth estimator for the joint survival function, as proposed by Chaubey and Sen (2002), is given by

$$R_n^p(x_1, x_2) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p(j; x_1\mu_{1n})\, p(k; x_2\mu_{2n})\, R_n\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right), \quad x_1 > 0,\ x_2 > 0, \tag{6.6}$$

where $\mu_{rn} \to \infty$ and $\mu_{rn}/n \to 0$ as $n \to \infty$ for $r = 1, 2$. The authors proved that $R_n^p(x_1, x_2)$ is a consistent estimator of the true survival function, provided that the survival function $\bar{F}(x_1, x_2)$ is absolutely continuous. As suggested by Chaubey and Sen (1996) in the univariate case, Chaubey and Sen (2002) also recommended, for practical purposes, choosing $\mu_{rn} = \frac{n}{X_{r[n]}}$, where $X_{r[n]} = \max\{X_{r1}, X_{r2}, \ldots, X_{rn}\}$ for $r = 1, 2$. Now, by plugging $R_n^p(x_1, x_2)$ into Eq. (6.1), we obtain the smooth estimators for the Bivariate Mean Residual Life (BMRL) functions, given by

$$\hat{m}_1^p(x_1, x_2) = \frac{(1/\mu_{1n}) \sum_{j=0}^{n}\sum_{k=0}^{n} Q(j; x_1\mu_{1n})\, p(k; x_2\mu_{2n})\, R_n\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right)}{\sum_{j=0}^{n}\sum_{k=0}^{n} p(j; x_1\mu_{1n})\, p(k; x_2\mu_{2n})\, R_n\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right)}, \quad x_1 > 0,\ x_2 > 0, \tag{6.7}$$

$$\hat{m}_2^p(x_1, x_2) = \frac{(1/\mu_{2n}) \sum_{j=0}^{n}\sum_{k=0}^{n} p(j; x_1\mu_{1n})\, Q(k; x_2\mu_{2n})\, R_n\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right)}{\sum_{j=0}^{n}\sum_{k=0}^{n} p(j; x_1\mu_{1n})\, p(k; x_2\mu_{2n})\, R_n\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right)}, \quad x_1 > 0,\ x_2 > 0, \tag{6.8}$$

where $Q(k; x_r\mu_{rn}) = \sum_{t=0}^{k} p(t; x_r\mu_{rn})$ denotes the CDF of a Poisson random variable with rate parameter $x_r\mu_{rn}$, for $r = 1, 2$. It is important to note that the proposed smooth estimator is capable of estimating $m_r(x_1, x_2)$, for $r = 1, 2$, beyond the largest order statistic of either component. Moreover, it is a continuous function. Thus, the proposed estimator overcomes the limitations of the natural estimator proposed by Kulkarni and Rattihalli (2002). In the following proposition, we will show that the proposed estimator is a consistent estimator. Before that, we first present a lemma that will be useful in the proof.

Lemma 6.2.1 (Hille's Theorem)**.** *Let $\{H_n(\mathbf{z}; \boldsymbol{\theta})\}$ be a sequence of distribution functions defined on $\mathbb{R}^p$, where $\boldsymbol{\theta} \in \mathbb{R}^p$ is the parameter vector. Let $\mathbf{Z}_n = (Z_{1n}, Z_{2n}, \ldots, Z_{pn})$ be a random vector with multivariate CDF $H_n(\mathbf{z}; \boldsymbol{\theta})$ such that:*

(i) $\mathbb{E}(\mathbf{Z}_n) = \boldsymbol{\theta}$,

(ii) For every fixed $\boldsymbol{\theta} \in \mathbb{R}^p$,

$$\max\{\mathrm{Var}(Z_{1n}), \mathrm{Var}(Z_{2n}), \ldots, \mathrm{Var}(Z_{pn})\} \to 0 \quad \text{as } n \to \infty,$$

(iii) $H_n(\mathbf{z}; \boldsymbol{\theta})$ is a continuous function of $\boldsymbol{\theta}$.

Then, for any bounded function $g : \mathbb{R}^p \to \mathbb{R}$, define

$$g^{H_n}(\boldsymbol{\theta}) = \int_{\mathbb{R}^p} g(\mathbf{z})\, dH_n(\mathbf{z}; \boldsymbol{\theta}).$$

The following hold:

(i) $g^{H_n}(\boldsymbol{\theta}) \to g(\boldsymbol{\theta})$ for every $\boldsymbol{\theta}$ in any compact subset of $\mathbb{R}^p$.

(ii) The convergence is uniform over any subset on which $g(\cdot)$ is uniformly continuous.

(iii) If $g(\cdot)$ is monotone, then the convergence holds uniformly over the entire space $\mathbb{R}^p$.

For more details and proof of Lemma 6.2.1, we refer to Feller (1991), Chaubey and Sen (1996) and Chaubey and Sen (2002). In the following proposition, we assume that μ_{rn}, $r = 1, 2$, are non-stochastic parameters such that $\mu_{rn} \to \infty$ and $\frac{\mu_{rn}}{n} \to 0$ as $n \to \infty$.

Proposition 6.2.2. *Let (X_1, X_2) be an absolutely continuous bivariate random vector with joint survival function $\bar{F}(x_1, x_2)$. Assume that $m_r(x_1, x_2) < \infty$ for every $(x_1, x_2) \in \mathbb{R}_+^2 = [0, \infty)^2$ and for every $r = 1, 2$. Let $\mathcal{J} \subset \mathbb{R}_+^2$ be a compact set such that $\bar{F}(x_1, x_2) > 0$*

for all $(x_1, x_2) \in \mathcal{J}$. *Then, for every* $r = 1, 2$,

$$\sup_{(x_1,x_2)\in\mathcal{J}} |\hat{m}_r^p(x_1, x_2) - m_r(x_1, x_2)| \xrightarrow{a.s.} 0 \quad as\ n \to \infty.$$

Proof. Under the assumption that μ_{rn} for $r = 1, 2$ is non-stochastic, the smooth estimator for the BMRL function can be written as

$$\hat{m}_1^p(x_1, x_2) = \frac{(1/\mu_{1n}) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} Q(j; x_1\mu_{1n})\, p(k; x_2\mu_{2n})\, R_n\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right)}{\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} p(j; x_1\mu_{1n})\, p(k; x_2\mu_{2n})\, R_n\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right)},$$

$$\hat{m}_2^p(x_1, x_2) = \frac{(1/\mu_{2n}) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} p(j; x_1\mu_{1n})\, Q(k; x_2\mu_{2n})\, R_n\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right)}{\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} p(j; x_1\mu_{1n})\, p(k; x_2\mu_{2n})\, R_n\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right)}.$$

Let us now consider the numerator of $\hat{m}_1^p(x_1, x_2)$, denoted by $\hat{N}_1^p(x_1, x_2)$, which can be expressed as

$$\begin{aligned}\hat{N}_1^p(x_1, x_2) &= \frac{1}{\mu_{1n}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} p(k; x_2\mu_{2n})\, R_n\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right) \sum_{u=0}^{j} p(u; x_1\mu_{1n}) \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} p(j; x_1\mu_{1n})\, p(k; x_2\mu_{2n})\, Z_{1n}^p\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right),\end{aligned}$$

where

$$Z_{1n}^p\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right) = \frac{1}{\mu_{1n}} \sum_{s=j}^{\infty} R_n\left(\frac{s}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right).$$

Let $Z_{1n}(x_1, x_2) = \int_{x_1}^{\infty} R_n(u, x_2)\, du$. Then, we can establish the following inequality:

$$\frac{1}{\mu_{1n}} \sum_{s=j+1}^{\infty} R_n\left(\frac{s}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right) \le Z_{1n}\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right) \le \frac{1}{\mu_{1n}} \sum_{s=j}^{\infty} R_n\left(\frac{s}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right),$$

or equivalently,

$$Z_{1n}^p\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right) - \frac{1}{\mu_{1n}} R_n\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right) \le Z_{1n}\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right) \le Z_{1n}^p\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right). \tag{6.9}$$

It follows from (6.9) that

$$\sup_{(x_1,x_2)\in\mathcal{J}} \left| Z_{1n}^p\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right) - Z_{1n}\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right) \right| \xrightarrow{\text{a.s.}} 0 \quad \text{as } n \to \infty.$$

This implies that

$$\sup_{(x_1,x_2)\in\mathcal{J}} \left| \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} p(j; x_1\mu_{1n})\, p(k; x_2\mu_{2n}) \left(Z_{1n}^p\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right) - Z_{1n}\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right) \right) \right| \xrightarrow{\text{a.s.}} 0.$$

Under the assumption $m_1(x_1, x_2) < \infty$, it follows that $Z_{1n}(x_1, x_2)$ is bounded almost surely over $\mathcal{J}$. Also, $Z_{1n}(x_1, x_2)$ is non-decreasing and continuous in x_1. Let H_n be the joint CDF defined in Lemma 6.2.1 which places mass $p(j; x_1\mu_{1n})\, p(k; x_2\mu_{2n})$ at the point $\left(\frac{j}{\mu_{1n}}, \frac{k}{\mu_{2n}}\right)$. Then, applying Lemma 6.2.1, we obtain

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p(j; x_1\mu_{1n})\, p(k; x_2\mu_{2n})\, Z_{1n}\left(\tfrac{j}{\mu_{1n}}, \tfrac{k}{\mu_{2n}}\right) \to Z_{1n}(x_1, x_2).$$

From Lemma A.1 of Kulkarni and Rattihalli (2002),

$$\sup_{(x_1,x_2)\in\mathcal{J}} \left| Z_{1n}(x_1, x_2) - \int_{x_1}^{\infty} \bar{F}(t, x_2)\, dt \right| \xrightarrow{\text{a.s.}} 0.$$

Also, by Chaubey and Sen (2002),

$$\sup_{(x_1,x_2)\in\mathbb{R}_+^2} |R_n^p(x_1, x_2) - R_n(x_1, x_2)| = O(n^{-3/4}(\log n)^{1+\alpha}) \xrightarrow{\text{a.s.}} 0,$$

for a suitable choice of $\alpha > 0$. Further, Lemma A.2 of Kulkarni and Rattihalli (2002) gives

$$\sup_{(x_1,x_2)\in\mathbb{R}_+^2} \left| R_n(x_1, x_2) - \bar{F}(x_1, x_2) \right| \xrightarrow{\text{a.s.}} 0.$$

Combining the above results, we conclude that

$$\sup_{(x_1,x_2)\in\mathbb{R}_+^2} \left| R_n^p(x_1, x_2) - \bar{F}(x_1, x_2) \right| \xrightarrow{\text{a.s.}} 0.$$

Since both the numerator and denominator of $\hat{m}_1^p(x_1, x_2)$ are consistent estimators of the corresponding components of $m_1(x_1, x_2)$, we conclude that $\hat{m}_1^p(x_1, x_2)$ is a consistent estimator of $m_1(x_1, x_2)$. A similar argument establishes the consistency of $\hat{m}_2^p(x_1, x_2)$ for $m_2(x_1, x_2)$.

□

6.3 Simulation Study

In this section, we compare the proposed smooth estimator with the natural nonparametric estimator. We consider two bivariate distributions with exponential marginals having means 2 and 3, respectively. The dependence structures are modelled using the Clayton copula and the Gumbel–Hougaard copula. The corresponding survival function of the bivariate Clayton–exponential distribution is given by

$$\bar{F}(x_1, x_2; \delta) = \left[e^{x_1\delta/2} + e^{x_2\delta/3} - 1\right]^{-1/\delta}, \quad x_1 > 0, x_2 > 0, \delta > 0,$$

and similarly, the survival function of the bivariate Gumbel–Hougaard exponential distribution is given by

$$\bar{F}(x_1, x_2; \phi) = \exp\left\{-\left[\left(\frac{x_1}{2}\right)^{1/\phi} + \left(\frac{x_2}{3}\right)^{1/\phi}\right]^{\phi}\right\}, \quad x_1 > 0, x_2 > 0, \phi \geq 1.$$

We evaluate the performance of the proposed smooth estimator by computing the bias and mean square error (MSE) for various sample sizes. Furthermore, we assess the improvement over the natural nonparametric estimator using the relative percentage absolute bias improvement (RPABI) and the relative percentage mean square improvement (RPMSI).

Let B_0 and B_U denote the bias of the natural nonparametric estimator and the proposed smooth estimator, respectively. Then RPABI is defined as

$$\text{RPABI} = 100\left(\frac{|B_0| - |B_U|}{|B_0|}\right)\%.$$

Similarly, let M_0 and M_E denote the MSE of the natural nonparametric and the proposed estimator, respectively. Then RPMSI is defined as

$$\text{RPMSI} = 100\left(\frac{M_0 - M_E}{M_0}\right)\%.$$

Positive values of RPABI and RPMSI indicate an improvement by the proposed estimator, while negative values imply better performance by the natural nonparametric estimator. Each experiment is repeated 1,000 times, and the results are summarized in Table 6.1, Table 6.2, Table 6.3, and Table 6.4.

It has been observed that, in most cases, the proposed estimator outperforms the natural nonparametric estimator. Although, in a few instances, the natural nonparametric estimator performs better, the improvements are relatively small.

In the case of the *bivariate Clayton exponential distribution*, the maximum percentage improvements in RPMSI (RPABI) of the proposed estimator $\hat{m}_1^p(x_1, x_2)$ over the natural estimator are observed to be up to 56% (31%), 53% (28%), and 51% (26%) for sample sizes of 50, 75, and 100, respectively. Similarly, the improvements for $\hat{m}_2^p(x_1, x_2)$ are up to 43% (24%), 37% (19%), and 33% (15%) for the same sample sizes.

In the case of the *bivariate Gumbel-Hougaard exponential distribution*, the maximum improvements in RPMSI (RPABI) of $\hat{m}_1^p(x_1, x_2)$ over the natural estimator are up to 31% (14%), 28% (13%), and 19% (11%) for sample sizes of 50, 75, and 100, respectively. For

$\hat{m}_2^p(x_1, x_2)$, the corresponding improvements are up to 29% (13%), 22% (10%), and 17% (7%).

Moreover, the proposed estimator exhibits significant improvement near the largest order statistic of each component, highlighting its practical relevance and effectiveness in real-life applications.

6.4 Real Data Application

In this section, we apply the proposed smooth estimator to predict the BMRL function for a bivariate warranty dataset. The data, obtained from Eliashberg et al. (1997), consist of 40 observations recording the maintenance history of locomotive traction motors. Each observation provides the time since the inception of service (age, in days) and the miles accumulated (usage, in miles) by a traction motor before it failed and was returned to the maintenance depot. To facilitate efficient analysis, we rescale the data by dividing the age by 100, denoted by X_1, and the usage by $10,000$, denoted by X_2. Consequently, age is expressed in units of 100 days and usage in units of $10,000$ miles. Before proceeding to estimate the BMRL function, we first examine the boxplots of the dataset to identify any potential outliers, as shown in Figure 6.4. From the figure, it is evident that outliers are present, which could introduce significant bias into the estimator. Therefore, we omitted three extreme data points. The revised dataset, used for subsequent analysis, is provided in Table 7.6 in the appendix for reference.

We then conduct an exploratory data analysis. Table 6.5 presents the basic descriptive statistics along with three dependence measures for the bivariate warranty data. The high correlation observed between age and usage underscores the importance of conducting a bivariate analysis.

We now proceed to estimate the BMRL function using the proposed smooth estimator. The surface and contour plots of $\hat{m}_1^p(x_1, x_2)$ and $\hat{m}_2^p(x_1, x_2)$ are displayed in Figures 6.2 and 6.3, respectively. Next, we estimate $m_1(x_1, x_2)$, which represents the average remaining life of the traction motor given that it has already survived up to time x_1 and the usage has exceeded x_2 (per 10,000 miles).

This estimator not only provides insight into the overall lifetime of the traction motor but also offers valuable information about the expected remaining life, making it

TABLE 6.1. Bias, MSE, RPABI, and RPMSI for $\hat{m}_1^p(x_1, x_2)$ of bivariate Clayton exponential distribution

δ	x_1	x_2	Sample Size											
			$n=50$				$n=75$				$n=100$			
			Bias	**MSE**	**RPABI** (%)	**RPMSI** (%)	**Bias**	**MSE**	**RPABI** (%)	**RPMSI** (%)	**Bias**	**MSE**	**RPABI** (%)	**RPMSI** (%)
	0.3	2.2	-0.211716	0.241153	6.02	12.42	-0.227235	0.160125	6.81	11.97	-0.237471	0.143919	5.73	11.03
	0.6	2.4	-0.267230	0.286770	10.13	17.75	-0.292748	0.203881	10.51	18.01	-0.308431	0.191825	8.87	16.72
	0.9	2.6	-0.287151	0.319688	13.27	22.89	-0.315974	0.229786	11.78	20.85	-0.335047	0.218136	9.94	18.10
	1.2	2.8	-0.286325	0.343746	14.01	24.57	-0.312844	0.243031	12.29	21.71	-0.335977	0.229294	9.89	18.57
	1.5	3	-0.273459	0.364694	13.27	23.60	-0.294964	0.252299	12.69	23.86	-0.323500	0.234774	10.37	19.25
	1.8	3.2	-0.253793	0.387111	14.51	26.74	-0.269671	0.263021	13.79	24.15	-0.302922	0.239877	11.86	21.72
	2.1	3.4	-0.230732	0.413952	15.82	30.22	-0.241559	0.278585	13.71	24.38	-0.276948	0.247780	12.43	23.13
2	2.4	3.6	-0.207017	0.447353	18.09	33.27	-0.213252	0.302113	13.73	25.44	-0.248086	0.260405	13.66	25.80
	2.7	3.8	-0.185035	0.489684	19.68	36.64	-0.185784	0.336423	14.81	26.84	-0.218782	0.279340	13.17	26.14
	3	4	-0.166900	0.543982	21.17	39.51	-0.159474	0.383968	18.67	35.97	-0.191385	0.306928	14.64	28.77
	3.3	4.2	-0.154432	0.613076	21.08	42.77	-0.134859	0.447337	20.21	38.36	-0.167958	0.345584	14.66	30.56
	3.6	4.4	-0.149049	0.698193	23.59	45.31	-0.113006	0.529263	22.28	43.93	-0.149785	0.396735	15.82	30.60
	3.9	4.6	-0.151810	0.798588	25.87	48.26	-0.095337	0.631667	21.57	42.66	-0.136950	0.462021	19.47	37.20
	4.2	4.8	-0.163524	0.912213	29.14	54.06	-0.083401	0.753996	25.66	49.94	-0.128721	0.544565	23.31	50.70
	4.5	5.0	-0.184551	1.036389	31.03	55.69	-0.078791	0.893248	27.41	52.72	-0.124637	0.648669	25.87	55.04
	0.3	2.2	-0.155963	0.217263	4.84	8.37	-0.169146	0.135389	3.64	6.08	-0.171846	0.117632	2.53	5.03
	0.6	2.4	-0.194457	0.249395	8.15	13.00	-0.218514	0.164174	7.40	12.02	-0.228278	0.148534	7.76	13.41
	0.9	2.6	-0.199641	0.271907	11.10	18.68	-0.229199	0.180011	10.07	17.80	-0.244640	0.164097	8.57	15.43
	1.2	2.8	-0.183820	0.288310	12.54	20.73	-0.213597	0.186814	11.48	19.46	-0.234471	0.168280	10.22	18.73
	1.5	3	-0.156340	0.304498	11.52	19.44	-0.183211	0.192689	12.15	22.47	-0.209888	0.169139	11.10	20.63
	1.8	3.2	-0.123563	0.324880	13.86	24.30	-0.146187	0.202536	13.55	23.44	-0.177545	0.171882	11.87	21.98
	2.1	3.4	-0.089817	0.351819	13.39	24.28	-0.108048	0.218937	13.73	22.63	-0.141430	0.179817	11.87	22.48
3	2.4	3.6	-0.058313	0.386286	14.36	26.32	-0.072122	0.243770	13.95	25.23	-0.104821	0.194595	12.99	22.80
	2.7	3.8	-0.031467	0.429101	16.63	30.17	-0.039884	0.278661	13.87	25.35	-0.070500	0.216765	13.27	23.85
	3	4	-0.011122	0.481777	18.58	35.69	-0.011842	0.325005	16.85	32.43	-0.040824	0.246844	14.05	24.92
	3.3	4.2	0.001321	0.546149	19.46	37.56	0.011622	0.383970	16.98	31.74	-0.017610	0.285529	13.65	25.98
	3.6	4.4	0.004961	0.623399	19.53	37.42	0.030019	0.456758	19.02	37.83	-0.001743	0.333324	14.36	26.55
	3.9	4.6	-0.000679	0.713480	22.61	42.49	0.042835	0.544919	20.43	40.87	0.007162	0.391451	15.61	29.02
	4.2	4.8	-0.015895	0.815372	26.12	48.71	0.049493	0.649470	22.80	47.17	0.010406	0.462493	19.69	37.98
	4.5	5.0	-0.040743	0.927482	29.29	53.44	0.049169	0.769500	24.70	48.95	0.009108	0.549686	21.16	41.45
	0.3	2.2	-0.115406	0.202327	2.21	4.20	-0.124391	0.121021	0.60	2.06	-0.122753	0.102442	1.57	3.49
	0.6	2.4	-0.140173	0.227411	4.11	6.98	-0.160233	0.141090	3.84	7.99	-0.166052	0.122778	4.46	8.22
	0.9	2.6	-0.134954	0.246295	6.73	11.15	-0.163014	0.152874	8.50	15.37	-0.175504	0.133584	7.33	12.99
	1.2	2.8	-0.109508	0.261929	9.59	15.28	-0.140739	0.159146	10.09	16.68	-0.159144	0.136809	9.02	16.78
	1.5	3	-0.072979	0.279620	9.49	14.09	-0.103728	0.166568	10.73	19.30	-0.127907	0.138723	9.73	17.16
	1.8	3.2	-0.032384	0.303119	10.90	17.41	-0.060597	0.179231	10.95	17.45	-0.089310	0.144025	10.10	17.39
	2.1	3.4	0.007085	0.333875	11.44	18.70	-0.017809	0.199094	10.80	17.52	-0.048536	0.155591	11.71	19.39
4	2.4	3.6	0.041625	0.371691	12.48	22.98	0.020546	0.227164	11.31	19.22	-0.009627	0.174420	11.29	18.27
	2.7	3.8	0.068664	0.416164	13.88	26.71	0.052621	0.264139	11.38	20.01	0.024391	0.200149	11.10	19.21
	3	4	0.086645	0.467858	15.04	30.43	0.077949	0.310652	13.89	25.97	0.051340	0.232015	12.27	21.86
	3.3	4.2	0.094777	0.528220	15.58	30.84	0.096570	0.367181	14.64	28.46	0.069927	0.269523	12.17	20.96
	3.6	4.4	0.092927	0.598763	16.93	32.20	0.108705	0.434306	15.72	31.80	0.079960	0.312856	11.57	19.91
	3.9	4.6	0.081411	0.680361	19.57	38.53	0.114695	0.513326	18.34	37.08	0.082483	0.363821	12.70	23.05
	4.2	4.8	0.060624	0.773139	22.92	44.65	0.114780	0.605881	21.24	45.33	-0.949050	1.847955	13.19	26.63
	4.5	5	0.030934	0.876545	27.52	51.42	0.108787	0.712477	22.82	46.90	0.072212	0.501744	20.25	38.64

TABLE 6.2. Bias, MSE, RPABI, and RPMSI for $\hat{m}_2^p(x_1, x_2)$ of bivariate Clayton exponential distribution

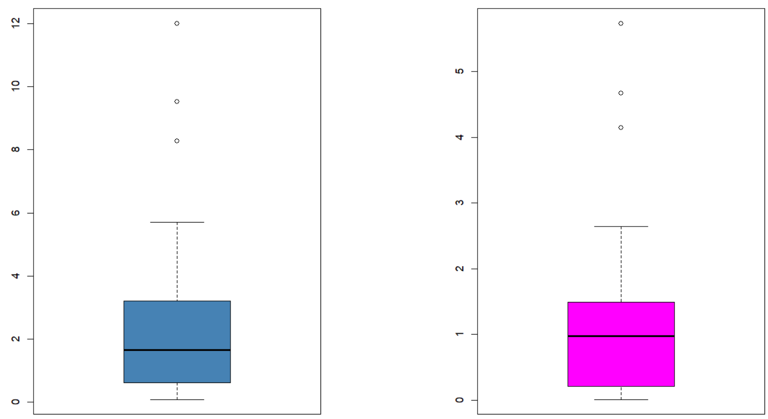
			Sample Size											
δ	x_1	x_2	$n = 50$				$n = 75$				$n = 100$			
			Bias	**MSE**	**RPABI** (%)	**RPMSI** (%)	**Bias**	**MSE**	**RPABI** (%)	**RPMSI** (%)	**Bias**	**MSE**	**RPABI** (%)	**RPMSI** (%)
	0.3	2.2	0.027690	0.339503	7.05	10.98	0.043336	0.266124	5.79	9.24	0.002716	0.174212	5.31	9.30
	0.6	2.4	-0.006259	0.379764	9.85	15.75	0.011546	0.295384	5.95	7.85	-0.033401	0.192249	7.26	13.78
	0.9	2.6	-0.041698	0.432573	10.34	17.18	-0.014123	0.337033	6.06	10.13	-0.065077	0.216704	6.85	14.64
	1.2	2.8	-0.087969	0.498858	10.45	19.54	-0.044905	0.387607	7.39	12.57	-0.102630	0.251592	9.81	17.83
	1.5	3	-0.147917	0.583772	12.15	23.34	-0.086594	0.448039	9.25	17.12	-0.151202	0.300095	11.86	20.98
	1.8	3.2	-0.221280	0.692266	14.93	27.74	-0.141883	0.524117	11.06	20.58	-0.213731	0.365520	12.34	22.53
	2.1	3.4	-0.307019	0.830577	14.83	29.16	-0.211927	0.624416	11.62	22.02	-0.291462	0.453536	13.19	24.35
2	2.4	3.6	-0.404421	1.008187	16.77	31.59	-0.297235	0.757281	12.05	22.59	-0.384196	0.572138	13.56	24.95
	2.7	3.8	-0.513446	1.237994	16.75	33.00	-0.397684	0.932053	14.98	28.13	-0.491008	0.732186	14.63	27.30
	3	4	-0.634831	1.536362	17.79	37.70	-0.512469	1.161856	14.84	27.78	-0.610922	0.947110	13.89	26.64
	3.3	4.2	-0.769710	1.922018	21.20	41.04	-0.640587	1.462697	16.46	31.95	-0.743145	1.230650	12.50	24.22
	3.6	4.4	-0.918799	2.414150	22.04	43.70	-0.781503	1.850599	17.62	33.88	-0.887237	1.596912	12.03	23.64
	3.9	4.6	-1.082286	3.028680	22.78	43.50	-0.935344	2.340861	17.95	35.19	-1.043353	2.063195	11.34	23.83
	4.2	4.8	-1.260613	3.774622	23.84	43.80	-1.102793	2.950738	19.27	37.40	-1.211881	2.650739	14.84	29.97
	4.5	5.0	-1.454553	4.657337	23.71	42.90	-1.284981	3.699594	18.82	37.05	-1.392695	3.381540	15.44	32.58
	0.3	2.2	0.053557	0.340903	6.43	10.94	0.068877	0.266865	5.25	7.40	0.023973	0.174071	4.84	9.36
	0.6	2.4	0.044791	0.377761	7.32	12.92	0.060679	0.294966	4.76	7.22	0.009181	0.188452	5.58	10.96
	0.9	2.6	0.040747	0.425048	8.30	14.39	0.064648	0.332836	5.84	8.50	0.006322	0.207861	5.98	11.00
	1.2	2.8	0.029286	0.481045	8.12	14.68	0.067396	0.376915	5.01	7.43	0.002433	0.233778	6.17	10.73
	1.5	3	0.003612	0.546623	9.60	17.77	0.059225	0.425617	6.41	10.36	-0.012272	0.266346	7.32	13.03
	1.8	3.2	-0.039273	0.624267	10.79	20.10	0.034237	0.480918	6.18	11.10	-0.044125	0.306096	9.24	16.86
	2.1	3.4	-0.100583	0.719716	12.51	24.57	-0.011033	0.548087	8.55	15.15	-0.096617	0.357123	10.83	19.10
3	2.4	3.6	-0.180876	0.843053	14.12	26.55	-0.078576	0.634487	9.52	16.87	-0.171125	0.427715	13.00	22.85
	2.7	3.8	-0.280475	1.008845	15.63	29.59	-0.169059	0.751272	12.66	23.92	-0.267630	0.530313	13.21	25.73
	3	4	-0.399797	1.235729	17.50	33.80	-0.281839	0.915016	13.56	25.19	-0.385389	0.680794	13.79	26.21
	3.3	4.2	-0.539249	1.545136	20.02	38.01	-0.415386	1.145951	17.07	30.73	-0.523432	0.896460	13.55	25.59
	3.6	4.4	-0.698663	1.960177	19.74	39.46	-0.568093	1.464289	16.99	32.87	-0.680927	1.195491	14.55	26.76
	3.9	4.6	-0.877300	2.504130	20.08	41.19	-0.738874	1.888377	15.88	32.22	-0.857237	1.598977	13.38	26.65
	4.2	4.8	-1.074718	3.197846	22.71	43.35	-0.927247	2.437711	17.36	33.32	-1.051383	2.131532	13.17	27.18
	4.5	5.0	-1.290982	4.057778	22.17	42.30	-1.133025	3.135493	16.66	32.22	-1.261527	2.817930	13.98	27.91
	0.3	2.2	0.069639	0.349318	4.55	7.83	0.087714	0.269285	3.24	4.74	0.038318	0.175201	4.85	8.99
	0.6	2.4	0.074993	0.386395	5.78	8.89	0.091349	0.296763	4.49	5.94	0.034315	0.188998	5.51	11.19
	0.9	2.6	0.089886	0.435133	4.07	6.28	0.110411	0.334471	3.67	4.29	0.046044	0.208518	4.27	7.40
	1.2	2.8	0.101534	0.493025	4.32	8.23	0.132939	0.380592	3.98	4.28	0.062089	0.234546	2.88	4.69
	1.5	3	0.100545	0.557846	4.86	9.03	0.147339	0.432012	3.29	2.79	0.070250	0.265719	4.33	7.63
	1.8	3.2	0.080977	0.628563	6.90	13.25	0.144831	0.486741	3.48	3.69	0.061098	0.299589	4.64	7.41
	2.1	3.4	0.038985	0.707749	8.98	17.37	0.119113	0.545416	6.79	10.73	0.028447	0.336968	6.33	12.59
4	2.4	3.6	-0.027837	0.803742	11.39	20.35	0.066001	0.612236	7.24	11.37	-0.031177	0.384116	8.79	16.24
	2.7	3.8	-0.120653	0.931611	14.12	27.39	-0.016445	0.697786	10.40	17.17	-0.119093	0.453406	11.27	21.41
	3	4	-0.239734	1.112167	16.64	31.45	-0.127922	0.820543	10.95	20.32	-0.235027	0.562657	12.60	24.65
	3.3	4.2	-0.384630	1.369572	19.36	38.23	-0.266383	1.004522	14.44	25.91	-0.377693	0.732969	13.53	25.53
	3.6	4.4	-0.553943	1.729817	18.65	37.67	-0.429025	1.274869	16.02	30.49	-0.545471	0.987011	14.14	25.57
	3.9	4.6	-0.745585	2.219940	20.35	42.34	-0.613258	1.654609	15.25	30.38	-0.736681	1.349626	13.41	26.90
	4.2	4.8	-0.957745	2.866569	21.43	42.44	-0.817140	2.165592	16.45	32.13	-0.949050	1.847955	13.19	26.63
	4.5	5	-1.189347	3.692897	21.05	41.40	-1.039135	2.830693	15.44	29.30	-1.179236	2.508275	14.88	29.46

TABLE 6.3. Bias, MSE, RPABI, and RPMSI for $\hat{m}_1^p(x_1, x_2)$ of bivariate Gumbel- Hougaard exponential distribution

δ	x_1	x_2	Sample Size											
			$n = 50$				$n = 75$				$n = 100$			
			Bias	**MSE**	**RPABI** (%)	**RPMSI** (%)	**Bias**	**MSE**	**RPABI** (%)	**RPMSI** (%)	**Bias**	**MSE**	**RPABI** (%)	**RPMSI** (%)
	0.3	2.2	0.158151	0.222036	3.75	9.24	0.146411	0.160135	1.67	3.25	0.126949	0.120051	0.79	1.99
	0.6	2.4	0.219598	0.271859	2.04	4.19	0.204316	0.194696	-0.59	-1.48	0.179755	0.150465	0.91	1.71
	0.9	2.6	0.254899	0.317721	0.13	0.27	0.236536	0.226524	-1.20	-1.58	0.205855	0.174165	0.16	0.70
	1.2	2.8	0.271865	0.357328	1.04	4.60	0.251201	0.255164	-2.72	-2.95	0.215312	0.191290	-2.66	-5.02
	1.5	3	0.275946	0.392093	-0.59	3.36	0.254799	0.281240	-1.30	-0.49	0.214746	0.205966	-0.46	-1.39
	1.8	3.2	0.271053	0.424878	1.26	5.28	0.251246	0.305949	0.34	1.64	0.207947	0.220509	-1.48	-1.31
	2.1	3.4	0.260131	0.457728	1.53	5.87	0.242679	0.331005	-0.08	0.42	0.197458	0.236361	-0.48	-1.55
2	2.4	3.6	0.245162	0.491886	1.40	3.75	0.230525	0.357848	1.54	2.52	0.184330	0.254371	1.44	1.14
	2.7	3.8	0.227163	0.529176	2.31	6.77	0.215861	0.387357	3.55	7.27	0.168520	0.274363	3.59	4.40
	3	4	0.206625	0.572718	2.43	5.87	0.199536	0.420741	4.18	9.27	0.150081	0.296138	3.77	5.25
	3.3	4.2	0.183983	0.626213	5.80	12.25	0.181942	0.459966	4.46	10.77	0.129536	0.320466	4.69	7.08
	3.6	4.4	0.159660	0.692530	9.03	21.14	0.162811	0.506724	6.63	13.05	0.107467	0.348700	7.11	13.42
	3.9	4.6	0.133700	0.772726	10.16	25.85	0.141463	0.561600	8.54	17.57	0.084456	0.382597	7.99	14.81
	4.2	4.8	0.105565	0.866027	12.63	29.96	0.117205	0.624171	12.36	23.37	0.061281	0.424238	7.35	15.94
	4.5	5	0.074324	0.970567	14.40	31.00	0.089640	0.693604	13.46	27.67	0.038793	0.475668	10.62	19.55
	0.3	2.2	0.062260	0.181340	4.42	10.21	0.059805	0.126952	5.24	9.04	0.044923	0.094001	3.65	6.04
	0.6	2.4	0.119698	0.213938	2.90	6.00	0.109881	0.147665	1.92	5.20	0.089897	0.110756	2.55	3.89
	0.9	2.6	0.162654	0.249633	1.74	3.66	0.146534	0.171512	-0.73	-1.00	0.120188	0.128448	0.30	0.95
	1.2	2.8	0.188830	0.283540	-0.28	-1.67	0.167127	0.194354	-2.43	-4.54	0.134257	0.144006	-0.85	-2.18
	1.5	3	0.200671	0.315159	-2.38	-6.48	0.174609	0.215540	-4.05	-7.08	0.136312	0.156802	-2.33	-4.73
	1.8	3.2	0.201490	0.345751	-1.80	-3.16	0.172678	0.236065	-2.63	-7.49	0.130006	0.168229	-2.67	-6.20
	2.1	3.4	0.194351	0.376440	-1.28	-1.11	0.164572	0.256772	0.32	-2.18	0.118213	0.180177	-1.31	-3.98
3	2.4	3.6	0.181684	0.408297	-1.33	-2.30	0.152469	0.278560	-0.53	-4.84	0.103240	0.193930	0.42	-2.51
	2.7	3.8	0.165147	0.442866	-0.76	-2.10	0.137271	0.302954	1.61	-2.66	0.086221	0.210015	2.70	4.02
	3	4	0.145609	0.482201	3.21	4.52	0.119646	0.331858	4.40	6.09	0.067485	0.228366	4.33	4.53
	3.3	4.2	0.123364	0.528648	6.05	6.85	0.100640	0.366737	4.09	5.85	0.047178	0.249236	6.16	7.81
	3.6	4.4	0.098588	0.584860	6.14	10.31	0.081123	0.407904	4.48	6.80	0.025504	0.273658	7.53	11.05
	3.9	4.6	0.071679	0.653671	9.24	21.21	0.061255	0.454933	8.41	12.12	0.002804	0.303018	7.28	11.03
	4.2	4.8	0.043173	0.737159	10.43	21.72	0.040562	0.507654	8.17	12.86	-0.020301	0.338933	7.13	11.60
	4.5	5	0.013381	0.835593	12.51	23.46	0.018283	0.566551	10.82	18.79	-0.042893	0.383526	10.36	18.99
	0.3	2.2	0.019533	0.169077	6.43	12.39	0.020676	0.116938	3.72	8.48	0.009128	0.086687	4.36	8.71
	0.6	2.4	0.066482	0.193109	4.95	9.31	0.060079	0.131688	2.35	4.06	0.042382	0.097342	3.92	5.43
	0.9	2.6	0.113345	0.223844	1.80	3.86	0.099745	0.151691	1.01	1.61	0.075211	0.111234	1.86	2.39
	1.2	2.8	0.150303	0.258073	-2.35	-5.48	0.129746	0.173709	-2.24	-5.53	0.098116	0.126268	-0.80	-3.07
	1.5	3	0.174258	0.293198	-5.12	-12.41	0.147409	0.195296	-3.93	-11.23	0.109398	0.140378	-1.44	-5.70
	1.8	3.2	0.185750	0.327734	-4.67	-11.14	0.153953	0.216184	-4.12	-11.45	0.110927	0.153170	-1.04	-5.42
	2.1	3.4	0.186891	0.360884	-2.46	-7.59	0.152053	0.237067	-2.05	-9.46	0.105313	0.166086	-3.47	-10.26
4	2.4	3.6	0.180021	0.392986	-1.36	-5.86	0.144110	0.258855	-0.69	-7.30	0.095030	0.180321	0.80	-1.69
	2.7	3.8	0.167240	0.425954	0.90	-1.63	0.116448	0.310023	3.50	0.18	0.081497	0.195926	2.47	2.42
	3	4	0.150295	0.462901	2.77	0.44	0.099277	0.341668	3.97	2.91	0.065316	0.212985	4.69	6.79
	3.3	4.2	0.130408	0.506915	3.38	1.76	0.099277	0.341668	3.97	2.91	0.047105	0.232206	5.31	6.49
	3.6	4.4	0.108291	0.560293	6.58	12.26	0.081078	0.378196	6.25	7.48	0.027687	0.254645	7.20	9.90
	3.9	4.6	0.084421	0.625069	6.84	10.42	0.062257	0.419587	6.47	9.04	0.007831	0.281390	7.46	10.70
	4.2	4.8	0.059150	0.703449	9.95	22.80	0.042913	0.466044	8.78	12.26	-0.011880	0.313633	9.64	14.18
	4.5	5	0.032518	0.796945	12.31	27.04	0.022918	0.518497	10.21	14.87	-0.030992	0.353268	11.14	20.55

TABLE 6.4. Bias, MSE, RPABI, and RPMSI for $\hat{m}_2^p(x_1, x_2)$ of bivariate Gumbel- Hougaard exponential distribution

			Sample Size											
δ	x_1	x_2	$n = 50$				$n = 75$				$n = 100$			
			Bias	**MSE**	**RPABI** (%)	**RPMSI** (%)	**Bias**	**MSE**	**RPABI** (%)	**RPMSI** (%)	**Bias**	**MSE**	**RPABI** (%)	**RPMSI** (%)
	0.3	2.2	0.190931	0.401987	1.41	2.09	0.148623	0.272468	0.01	-1.99	0.101261	0.197627	1.75	0.75
	0.6	2.4	0.227707	0.473442	1.16	0.14	0.183314	0.317498	-0.97	-4.58	0.131021	0.231962	2.19	0.99
	0.9	2.6	0.253769	0.551616	-0.23	-1.19	0.207810	0.370142	-0.92	-2.44	0.150330	0.269560	1.10	1.09
	1.2	2.8	0.274275	0.638233	0.79	1.11	0.227381	0.432076	-1.87	-2.29	0.165928	0.311691	0.01	-0.68
	1.5	3	0.292269	0.734467	1.74	2.48	0.246036	0.503696	-0.39	1.19	0.182315	0.360542	0.78	0.62
	1.8	3.2	0.309738	0.840950	2.90	5.25	0.266314	0.584941	-0.08	0.31	0.201635	0.417825	1.69	2.50
	2.1	3.4	0.328177	0.959332	2.06	5.57	0.290075	0.677393	0.80	0.74	0.224999	0.485570	1.05	2.41
2	2.4	3.6	0.348574	1.092282	2.25	5.73	0.318628	0.783833	1.47	1.88	0.252848	0.566737	0.17	1.82
	2.7	3.8	0.371259	1.243376	1.59	5.51	0.352490	0.907417	2.80	5.96	0.285188	0.663436	2.54	4.02
	3	4	0.396036	1.418305	3.70	7.85	0.391618	1.052031	2.00	4.74	0.321980	0.776229	3.04	4.65
	3.3	4.2	0.422495	1.625569	5.53	11.93	0.435515	1.223184	3.86	8.42	0.363208	0.905419	3.03	6.40
	3.6	4.4	0.450157	1.875405	7.41	15.03	0.483239	1.427256	5.61	12.26	0.408665	1.052083	3.53	8.70
	3.9	4.6	0.478299	2.177356	8.63	21.02	0.533553	1.669262	7.32	19.43	0.458127	1.219127	4.43	9.77
	4.2	4.8	0.505846	2.537525	11.02	26.42	0.585023	1.950815	9.07	21.04	0.511761	1.413283	5.22	11.66
	4.5	5.0	0.531571	2.957788	13.01	28.94	0.636141	2.271095	9.82	22.29	0.570139	1.645708	6.66	17.28
	0.3	2.2	0.169721	0.390291	0.04	-1.69	0.134073	0.263593	1.84	1.00	0.085015	0.185977	0.54	-0.27
	0.6	2.4	0.207683	0.450245	-1.35	-5.27	0.165046	0.299397	-0.05	-2.55	0.109769	0.210744	0.46	-1.31
	0.9	2.6	0.235715	0.516840	0.07	-4.37	0.187494	0.340172	-1.04	-5.61	0.125139	0.238016	1.23	-0.59
	1.2	2.8	0.252969	0.587645	-1.01	-5.53	0.199914	0.384264	-1.45	-6.71	0.131210	0.267149	0.58	-1.09
	1.5	3	0.261850	0.663373	-1.44	-8.14	0.205074	0.432605	-1.27	-7.16	0.131869	0.298999	0.54	-1.08
	1.8	3.2	0.265000	0.745626	1.29	-1.79	0.206691	0.486527	-0.36	-4.90	0.130546	0.334472	-0.00	-0.81
	2.1	3.4	0.264790	0.835728	1.40	1.13	0.208202	0.547289	0.97	-1.81	0.130292	0.374857	1.26	0.05
3	2.4	3.6	0.263402	0.934842	1.91	2.12	0.211982	0.616609	-0.36	-4.14	0.133679	0.421779	1.73	0.22
	2.7	3.8	0.262738	1.045047	0.71	-0.64	0.218850	0.696640	0.97	-1.40	0.141885	0.476854	1.93	3.28
	3	4	0.263893	1.170509	1.32	1.49	0.229074	0.790024	2.09	3.90	0.154906	0.541222	1.76	3.02
	3.3	4.2	0.266849	1.317443	2.21	3.27	0.243439	0.900013	0.42	-0.18	0.172341	0.616085	2.93	3.33
	3.6	4.4	0.270962	1.493697	3.12	7.60	0.262968	1.030167	0.49	-0.22	0.193978	0.703329	3.09	6.41
	3.9	4.6	0.275754	1.708942	5.06	17.26	0.287975	1.184390	4.60	8.39	0.219993	0.805027	3.65	7.42
	4.2	4.8	0.281140	1.973223	6.22	18.33	0.317572	1.366843	5.64	10.61	0.251083	0.924350	3.34	7.78
	4.5	5.0	0.287061	2.293876	9.07	21.90	0.350055	1.581773	8.15	18.18	0.288372	1.068279	7.12	14.41
	0.3	0.157155	0.385500	2.48	1.99	1.99	0.122621	0.259609	1.52	0.03	0.076010	0.183517	2.53	4.07
	0.6	2.4	0.193197	0.439139	0.68	-2.27	0.151313	0.291393	-0.48	-5.43	0.097189	0.204110	0.39	-0.68
	0.9	2.6	0.228434	0.503071	-2.24	-6.85	0.180000	0.330434	-0.88	-5.38	0.118156	0.228960	0.11	-2.35
	1.2	2.8	0.254954	0.573396	-4.24	-14.23	0.200612	0.372867	-3.11	-10.45	0.131780	0.256262	-0.90	-3.89
	1.5	3	0.270293	0.648476	-5.11	-16.39	0.210998	0.417070	-2.12	-12.37	0.136919	0.285044	-0.34	-5.06
	1.8	3.2	0.275190	0.727598	-2.95	-11.60	0.212770	0.463656	-1.65	-10.20	0.135520	0.315596	0.65	-3.40
	2.1	3.4	0.271845	0.810343	-0.49	-7.11	0.209186	0.514596	-0.55	-8.39	0.130592	0.349774	-0.46	-4.63
4	2.4	3.6	0.262805	0.897570	0.46	-4.41	0.203284	0.572578	-0.90	-7.37	0.125348	0.389400	2.09	0.68
	2.7	3.8	0.250568	0.992353	1.85	-0.90	0.197112	0.640162	-0.03	-4.37	0.121809	0.435021	3.43	4.13
	3	4	0.237316	1.100191	1.89	-0.47	0.192202	0.718807	1.27	-1.94	0.120788	0.486860	3.15	5.34
	3.3	4.2	0.224475	1.228052	1.22	0.56	0.189949	0.809342	2.59	1.84	0.122962	0.545673	2.98	3.73
	3.6	4.4	0.212595	1.382993	3.86	11.32	0.191447	0.913335	2.70	4.19	0.129150	0.612933	2.85	5.62
	3.9	4.6	0.201742	1.572417	3.71	9.72	0.197222	1.033216	3.47	5.44	0.140052	0.690910	3.25	5.77
	4.2	4.8	0.191815	1.805157	7.01	21.70	0.207157	1.172116	6.21	11.70	0.156265	0.783299	5.59	10.36
	4.5	5	0.182337	2.089925	9.18	26.94	0.220630	1.334259	7.83	16.59	0.178338	0.896651	7.52	18.96



(A) Boxplot of X_1 (B) Boxplot of X_2

FIGURE 6.1. Surface plots of the functions $\hat{m}_1^p(x_1, x_2)$ and $\hat{m}_2^p(x_1, x_2)$.

TABLE 6.5. Descriptive statistics and measures of dependence of the bivariate warranty data.

Statistics	X_1		X_2
Minimum	0.090		0.0095
Maximum	5.710		2.6433
1st Quantile	0.590		0.2041
Mean	1.894		0.9261
Median	1.640		0.8607
3rd Quantile	2.610		1.3827
Skewness	0.7183		0.6635
Kurtosis	2.81		2.5948
Variance	2.0349		0.5572
Pearson's correlation		0.9298	
Spearman's rho		0.9568	
Kendall's tau		0.8422	

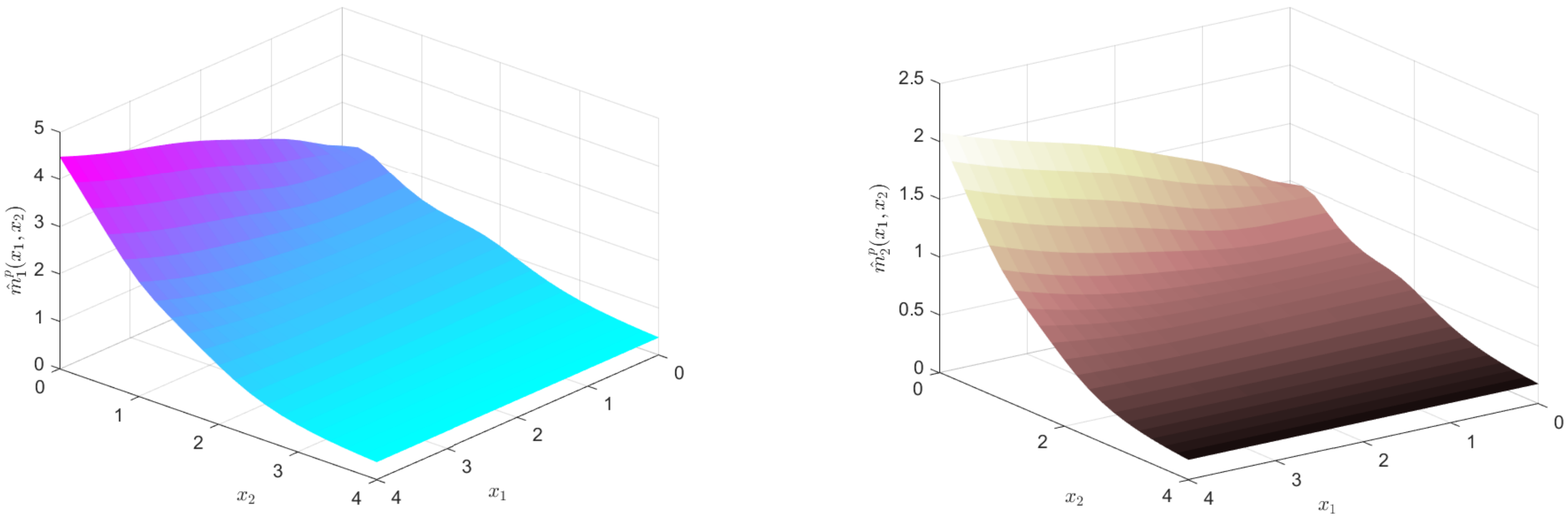
more informative than simply computing the survival probability. Such information is particularly useful for manufacturers, as it enables them to design warranty policies such as offering a first free maintenance service—that are cost-effective and tailored to the motor's actual usage without incurring significant losses.

Let A and U denote the age and usage limits, respectively, for the bivariate warranty. We consider four warranty policies: the first based on the first quartile, the second based on the median, the third based on the mean, and the fourth representing a one-year or 12,000 miles warranty.

Additionally, we estimate the reliability using the smooth estimator defined in Eq. (6.6). The corresponding results are presented in Table 6.6. Note that the warranty limits shown in square brackets are expressed in terms of days and miles, as the original data have been rescaled. Based on the constraints associated with the manufacturer's maintenance policy, an optimal warranty strategy can be designed by considering both the expected remaining life of the motor and the cost constraints involved in offering the first free maintenance service.

6.5 Conclusion and Future Direction

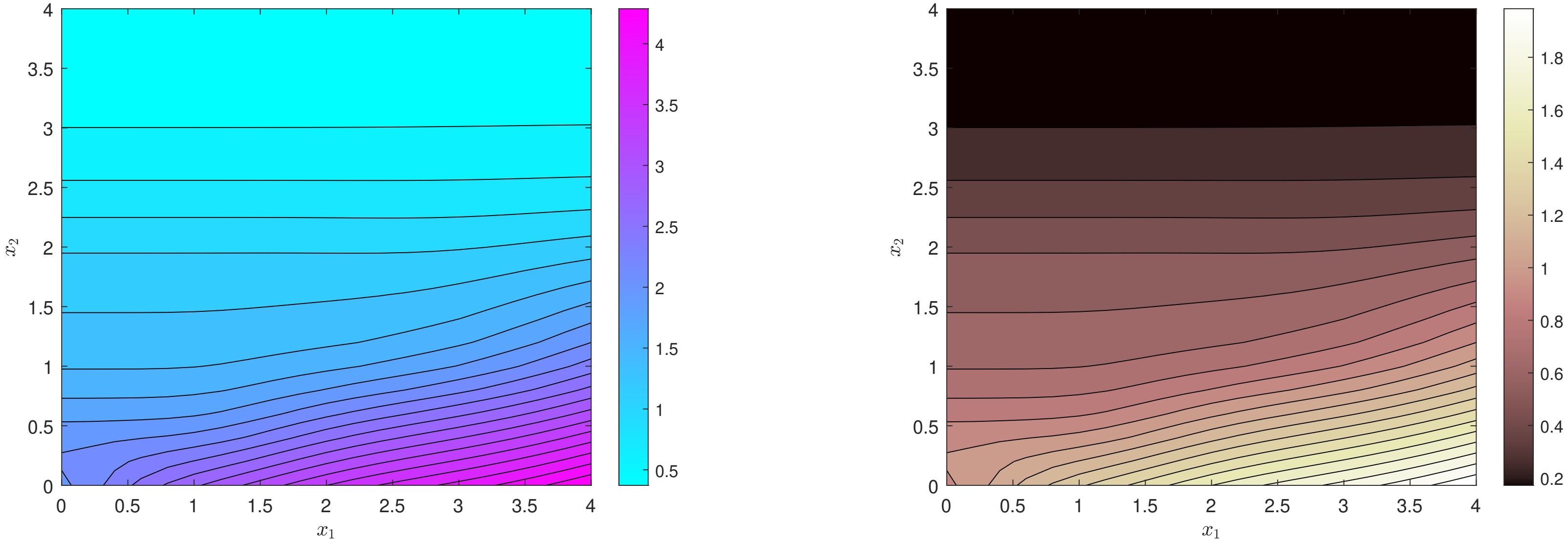
We propose a smooth estimator for the bivariate mean residual life (BMRL) function by smoothing the natural nonparametric estimator introduced by Kulkarni and Rattihalli (2002). The proposed smooth estimator extends the work of Chaubey and Sen (1999) from the univariate to the bivariate setup. We establish the uniform consistency property of the proposed estimator and compare its efficiency with that of Kulkarni and Rattihalli (2002) through Monte Carlo simulation experiments. The results indicate that the proposed estimator outperforms the natural nonparametric estimator in most cases, while in some cases, the performances are slightly better than the proposed estimator. Moreover, the estimator shows significant improvement over the natural estimator when analyzing the remaining life beyond the maximum observed values in the bivariate data. Finally, we conduct an extensive data analysis using bivariate warranty data, formulating four warranty policies based on (i) the first quartile, (ii) the median, (iii) the mean, and (iv) a one-year or 12,000-mile warranty. We also compute the BMRL function based on these limits. Given the remaining life and cost constraints, future work could extend this study into an optimization framework, which remains an open problem in this direction.



(A) Surface plot of $\hat{m}_1^p(x_1, x_2)$

(B) Surface plot of $\hat{m}_2^p(x_1, x_2)$

FIGURE 6.2. Surface plots of the functions $\hat{m}_1^p(x_1, x_2)$ and $\hat{m}_2^p(x_1, x_2)$.



(A) Contour plot of $\hat{m}_1^p(x_1, x_2)$

(B) Contour plot of $\hat{m}_2^p(x_1, x_2)$

FIGURE 6.3. Contour plots of the functions $\hat{m}_1^p(x_1, x_2)$ and $\hat{m}_2^p(x_1, x_2)$.

TABLE 6.6. Estimates of reliability and bivariate mean residual life functions for various bivariate warranty limits based on the warranty data

Bivariate Warranty Limits (A, W)	**Reliability** $R_n^p(A, W)$	$\hat{m}_1^p(A, W)$	$\hat{m}_2^p(A, W)$
(0.6, 0.2) [60 days, 2,000 miles]	0.7066	2.0162 [201.62 days]	1.0973 [10,973 miles]
(1.6, 0.86) [160 days, 8,600 miles]	0.4388	1.7272 [172.72 days]	0.8136 [8,136 miles]
(1.9, 0.9) [190 days, 9,000 miles]	0.3934	1.5963 [159.63 days]	0.8260 [8,260 miles]
(3.65, 1.2) [365 days, 12,000 miles]	0.1343	1.0923 [109.23 days]	0.9073 [9,073 miles]

CHAPTER 7

Nonparametric Estimation of a Bivariate Mean Inactivity Time Function

This chapter presents a nonparametric estimation procedure for estimating the bivariate mean inactivity time function. The asymptotic properties such as bias, consistency and asymptotic normality are established. A Monte Carlo simulation study is conducted to evaluate the performance of the estimator and discuss its application in medical sciences.

7.1 Introduction

In lifetime data analysis, researchers are mainly using two classes of measures. The first includes the survival probability function and the mean residual life function. The second comprises the failure probability function and the mean inactivity time function (MITF), also known as the mean past time function. The MITF plays a vital role when the exact failure time of a unit is of interest. It tells us the average amount of time a system has been non-functional, given that it was found to have failed at a specific time. For example, in medical science, consider a scenario where an individual is infected with a contagious disease such as the HIV virus. The exact time of infection may not be known; however, the individual becomes aware of their condition only upon testing positive at a later date. In such cases, it becomes crucial for healthcare professionals to estimate the incubation period to assess the severity of the patient's health condition. Direct testing methods to determine the incubation period are often costly and time-consuming. In such situations, the MITF serves as a useful alternative by offering an estimate of the incubation period based on retrospective data collected from patients with similar medical conditions, as recorded in hospital databases. Hence, MITF provides critical information for clinical decision-making, especially when direct measurements of infection time are unavailable or impractical. The concept is also valuable in forensic science for estimating the time of death, among other applications. Importantly, the utility of MITF extends beyond these disciplines into various other fields. For a detailed discussion on the applications of MITF, we refer to Jayasinghe and Zeephongsekul (2013) and the references therein.

Let X represent the lifetime of a system. Given that the system has already failed at some time x, the MITF is defined as $\mathbb{E}(x - X \mid X \leq x)$, which quantifies the average duration of inactivity prior to time x. Nanda et al. (2003) explored various properties of the MITF and established several characterization theorems based on stochastic ordering. The concept of MITF has been extended to higher dimensions by Nair and Asha (2008). For a bivariate non-negative random vector (X_1, X_2) with joint cumulative distribution function (CDF) $F(x_1, x_2)$, the bivariate mean inactivity time function (BMITF) is defined by the vector

$$(r_1(x_1, x_2), r_2(x_1, x_2)) = \mathbb{E}\left(x_1 - X_1,\, x_2 - X_2 \,|\, X_1 \leq x_1,\, X_2 \leq x_2\right). \qquad (7.1)$$

This concept is particularly useful in medical sciences, where the incubation period of a disease may depend on multiple covariates. For instance, individuals with diabetes often exhibit elevated cholesterol levels. Incorporating such covariate information can lead to more efficient and accurate prediction of the incubation period for diabetes. In many real-world scenarios, the primary study variable and covariates are highly dependent, which underscores the relevance of MITF in higher dimensions.

Now, the problem of estimation arises. If the joint CDF $F(x_1, x_2)$ is known in advance, one can estimate the BMITF using a parametric approach. However, incorrect assumptions about F may lead to significant bias and misleading conclusions. This highlights the importance of nonparametric estimation procedures, which do not rely on strict distributional assumptions. Jayasinghe and Zeephongsekul (2013) proposed a nonparametric smooth estimator for the univariate MITF and discussed its applications in reliability engineering and medical sciences. Kulkarni and Rattihalli (2002) proposed a nonparametric estimator for the bivariate mean residual life function, which was later extended to right-censored observations by Efromovich (2025). However, to the best of our knowledge, no work has addressed the nonparametric estimation of the BMITF. Considering the practical relevance of BMITF, this chapter proposes a novel nonparametric estimator of BMITF. The main contributions of this chapter are summarized as follows:

- A nonparametric estimator of the bivariate mean inactivity time function is proposed.
- The expression for the bias of the proposed estimator is derived, and its consistency and asymptotic normality properties are established.
- The performance of the proposed estimator is evaluated through Monte Carlo simulations under various bivariate copula models and different values of Kendall's tau.
- The proposed estimator is applied to pink eye disease data to estimate the BMITF of the two infected eyes.

The rest of the chapter is organized as follows. In Section 7.2, the smooth estimator is proposed, and various asymptotic properties such as bias, consistency, and normality are studied. Section 7.3 presents a detailed simulation study under different copula models to evaluate the performance of the proposed estimator. In Section 7.4, an application to pink eye disease data is discussed. Finally, the conclusion and future research directions are provided in Section 7.5.

7.2 Nonparametric estimator for bivariate mean inactivity time function

Let (X_1, X_2) be a non-negative bivariate random vector with joint CDF $F(x_1, x_2)$. Throughout this chapter, we assume that (X_1, X_2) possesses finite first-order moments. Note that the bivariate mean inactivity time function (BMITF) given in Eq. (1.44) can also be written as

$$(r_1(x_1, x_2), r_2(x_1, x_2)) = \left(\frac{\int_0^{x_1} F(t, x_2)\, dt}{F(x_1, x_2)}, \frac{\int_0^{x_2} F(x_1, t)\, dt}{F(x_1, x_2)} \right).$$

Let (X_{1i}, X_{2i}), for $i = 1, 2, \ldots, n$, be n independent copies of the random vector (X_1, X_2). The natural non-parametric estimator of the joint CDF based on this random sample is the bivariate empirical CDF, defined as

$$\hat{F}_n(x_1, x_2) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{I}(X_{1i} \le x_1,\ X_{2i} \le x_2), \tag{7.2}$$

where $\mathbf{I}(\cdot)$ denotes the indicator function.

Substituting the empirical CDF $\hat{F}_n(x_1, x_2)$ into the expression for the BMITF yields the following non-parametric estimators:

$$\begin{cases} \hat{r}_1(x_1, x_2) = \dfrac{\sum_{i=1}^{n} (x_1 - X_{1i}) \mathbf{I}(X_{1i} \le x_1,\ X_{2i} \le x_2)}{\sum_{i=1}^{n} \mathbf{I}(X_{1i} \le x_1,\ X_{2i} \le x_2)}, \\ \hat{r}_2(x_1, x_2) = \dfrac{\sum_{i=1}^{n} (x_2 - X_{2i}) \mathbf{I}(X_{1i} \le x_1,\ X_{2i} \le x_2)}{\sum_{i=1}^{n} \mathbf{I}(X_{1i} \le x_1,\ X_{2i} \le x_2)}. \end{cases} \tag{7.3}$$

The estimators in Eq. (7.3) are defined when $\sum_{i=1}^{n} \mathbf{I}(X_{1i} \le x_1,\ X_{2i} \le x_2) > 0$, and are set to zero otherwise. In the following theorem, we derive the expressions for the bias of the proposed estimators.

Theorem 7.2.1. *Let (X_1, X_2) be a bivariate non-negative random vector having joint CDF $F(x_1, x_2)$ with BMITF $(r_1(x_1, x_2), r_2(x_1, x_2))$. Let $(\hat{r}_1(x_1, x_2), \hat{r}_1(x_1, x_2))$ be the non-parametric estimator of the BMITF defined in Eq. (7.3) based on the random sample $(X_{1i}, X_{2i}); i = 1, 2, \ldots, n$. Then, for $j = 1, 2$, the bias of $\hat{r}_j(x_1, x_2)$ is given by*

$$\mathbb{E}[\hat{r}_j(x_1, x_2)] - r_j(x_1, x_2) = -r_j(x_1, x_2) \left(1 - F(x_1, x_2)\right)^n. \tag{7.4}$$

Proof. Let Λ_p denote the collection of all subsets of $\{1, 2, \ldots, n\}$ with cardinality p. For each element $\lambda_p \in \Lambda_p$, define the event

$$\mathcal{F}_{\lambda_p} = \{(X_{1i} \le x_1,\ X_{2i} \le x_2) \text{ for all } i \in \lambda_p \ \text{ such that either } X_{1i} > x_1, \ \text{ or } \ X_{2i} > x_2 \text{ for all } i \notin \lambda_p\}.$$

Note that $\{\mathcal{F}_{\lambda_p}; \lambda_p \in \Lambda_p\}$ forms a class of disjoint events for all p. It is clear that the $P\left(\bigcup_{p=1}^{n} \bigcup_{\lambda_p \in \Lambda_p} \mathcal{F}_{\lambda_p}\right)$ is the probability that at least one observation in the sample satisfies $(X_{1i} \le x_1, X_{2i} \le x_2)$. This can be expressed as

$$P\left(\bigcup_{p=1}^{n} \bigcup_{\lambda_p \in \Lambda_p} \mathcal{F}_{\lambda_p}\right) = 1 - P(\text{No observation failed at } (x_1, x_2)) = 1 - (1 - F(x_1, x_2))^n.$$

Now consider the expectation

$$\begin{aligned}
\mathbb{E}[\hat{r}_1(x_1,x_2)] &= \mathbb{E}\left[\frac{\sum_{i=1}^{n}(x_1-X_{1i})\mathbf{I}(X_{1i}\le x_1,\ X_{2i}\le x_2)}{\sum_{i=1}^{n}\mathbf{I}(X_{1i}\le x_1,\ X_{2i}\le x_2)}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left\{\frac{\sum_{i=1}^{n}(x_1-X_{1i})\mathbf{I}(X_{1i}\le x_1,\ X_{2i}\le x_2)}{\sum_{i=1}^{n}\mathbf{I}(X_{1i}\le x_1,\ X_{2i}\le x_2)}\,\middle|\,\mathcal{F}_{\lambda_p}\right\}\right] \\
&= \sum_{p=1}^{n}\sum_{\lambda_p\in\Lambda_p}\sum_{k\in\lambda_p}\frac{1}{p}\mathbb{E}\left[x_1-X_{1k}\,\middle|\,\mathcal{F}_{\lambda_p}\right]P(\mathcal{F}_{\lambda_p}) \\
&= r_1(x_1,x_2)\sum_{p=1}^{n}\sum_{\lambda_p\in\Lambda_p}P(\mathcal{F}_{\lambda_p}) \\
&= r_1(x_1,x_2)\cdot P\left(\bigcup_{p=1}^{n}\bigcup_{\lambda_p\in\Lambda_p}\mathcal{F}_{\lambda_p}\right) \\
&= r_1(x_1,x_2)\left(1-(1-F(x_1,x_2))^n\right).
\end{aligned}$$

It follows that, for $j=1,2$, the bias of $\hat{r}_j(x_1,x_2)$ is given by

$$\mathbb{E}[\hat{r}_j(x_1,x_2)] - r_j(x_1,x_2) = -r_j(x_1,x_2)\left(1-F(x_1,x_2)\right)^n.$$

□

From, Eq. (7.4), we can conclude that as $n\to\infty$, the proposed estimator is asymptotically unbiased. Now, we will establish the consistency property of the proposed estimator. Before that, we need a few definitions which are useful for proving the theorem. For more details on these definitions, we refer to Pollard (1984) and Van der Vaart and Wellner (1996).

Definition 7.2.2 (Vapnik–Chervonenkis Class). *Let $\mathcal{F}$ be a collection of subsets of a set Ω. A finite subset $W=\{\omega_1,\omega_2,\ldots,\omega_p\}$ of Ω is said to be **shattered** by $\mathcal{F}$ if for every subset W' of W, there exists a set $A\in\mathcal{F}$ such that $A\cap W=W'$. Then, the **Vapnik–Chervonenkis dimension** (VC-dimension) of $\mathcal{F}$ is defined as*

$$D(\mathcal{F}) = \max\{p\in\mathbb{N} : \textit{there exists } W\subset\Omega \textit{ with } |W|=p \;\textit{ such that } W \textit{ is shattered by } \mathcal{F}\}.$$

*If $D(\mathcal{F})<\infty$, we say that $\mathcal{F}$ is a **VC-class**.*

Definition 7.2.3 (Graph of a function). *Let $\Omega\subseteq\mathbb{R}^p$ and let $g:\Omega\to\mathbb{R}$ be a real-valued function defined on Ω. Then, the **graph** of the function g, denoted by $\mathcal{G}$, is defined as*

$$\mathcal{G} = \{(\mathbf{u},x)\in\Omega\times\mathbb{R} : 0\le x\le g(\mathbf{u}) \;\textit{or}\; g(\mathbf{u})\le x\le 0\}.$$

The concept of VC class and the graph of a function play a pivotal role in empirical process theory. Let g be any measurable function with respect to a probability measure P, and let P_n denote the empirical probability measure based on a random sample. Let $\mathcal{F}$ be a class of measurable functions g with the assumption that there exists a measurable function G satisfying $\int_{\mathbb{R}^p} G dP < \infty$ such that $|g| \leq G$. Often, G is chosen as the pointwise supremum of all $g \in \mathcal{F}$. If the graphs of the functions in $\mathcal{F}$ form a VC class, then the convergence

$$\sup_{g\in\mathcal{F}} \left| \int_{\mathbb{R}^p} g\, dP_n - \int_{\mathbb{R}^p} g\, dP \right| \xrightarrow{\text{a.s.}} 0 \quad \text{as } n \to \infty.$$

holds with probability one. For more details, refer to Corollary 17, Theorem 24, and the Approximation Lemma in Pollard (1984).

Theorem 7.2.4. *Let $\mathcal{J}$ be any compact set of $\mathbb{R}_+^2 = [0,\infty) \times [0,\infty)$. Then, for $j = 1, 2$,*

$$\sup_{(x_1,x_2)\in\mathcal{J}} |\hat{r}_j(x_1,x_2) - r_j(x_1,x_2)| \xrightarrow{a.s.} 0 \quad as\ n \to \infty.$$

Proof. Let us denote

$$N_1(x_1,x_2) = \int_0^{x_1} F(t,x_2)\, dt = E\left((x_1 - X_1)\mathbf{I}(X_1 \leq x_1, X_2 \leq x_2)\right) = r_1(x_1,x_2)F(x_1,x_2),$$

and its empirical counterpart,

$$\hat{N}_1(x_1,x_2) = \frac{1}{n}\sum_{i=1}^{n}(x_1 - X_{1i})\mathbf{I}(X_{1i} \leq x_1,\ X_{2i} \leq x_2) = \hat{r}_1(x_1,x_2)\hat{F}_n(x_1,x_2).$$

For each fixed $(x_1,x_2) \in \mathbb{R}_+^2$, define a measurable function on $\mathbb{R}_+^2$ as

$$g_{(x_1,x_2)}(t_1,t_2) = (x_1 - t_1)\mathbf{I}(t_1 \leq x_1,\ t_2 \leq x_2).$$

Then,

$$N_1(x_1,x_2) = \int_{\mathbb{R}_+^2} g_{(x_1,x_2)}(t_1,t_2)\, dF(t_1,t_2), \quad \hat{N}_1(x_1,x_2) = \int_{\mathbb{R}_+^2} g_{(x_1,x_2)}(t_1,t_2)\, d\hat{F}_n(t_1,t_2). \tag{7.5}$$

Let $\mathcal{F} = \{g_{(x_1,x_2)} : (x_1,x_2) \in \mathbb{R}_+^2\}$ denote the class of such measurable functions. For each $g_{(x_1,x_2)} \in \mathcal{F}$, its graph is defined as

$$\begin{aligned}\mathcal{H}_{g_{(x_1,x_2)}} &= \{(t_1,t_2,r) : 0 \leq r \leq g_{(x_1,x_2)}(t_1,t_2),\ (t_1,t_2) \in \mathbb{R}_+^2\} \\ &= \{(t_1,t_2,r) : 0 \leq r \leq x_1 - t_1,\ t_1 \leq x_1,\ t_2 \leq x_2,\ x_1,x_2 \geq 0\}.\end{aligned}$$

Thus, the class of graphs can be written as

$$\{\mathcal{H}_{g_{(x_1,x_2)}} : (x_1, x_2) \in \mathbb{R}_+^2\} = \{\mathcal{E}_{x_1} : x_1 \geq 0\} \times \{[0, x_2] : x_2 \geq 0\},$$

where

$$\mathcal{E}_{x_1} = \{(t_1, r) : 0 \leq r \leq x_1 - t_1,\ t_1 \leq x_1\}.$$

According to Corollary 17, Theorem 24, and the Approximation Lemma from Pollard (1984), a sufficient condition for the uniform consistency of $\hat{N}_1(x_1, x_2)$ is that the class $\{\mathcal{H}_{g_{(x_1,x_2)}}\}$ forms a VC-class. It is known that $\{[0, x_2] : x_2 \geq 0\}$ is a VC-class. Hence, it remains to show that $\{\mathcal{E}_{x_1} : x_1 \geq 0\}$ is also a VC-class. Suppose $x_{1a} \neq x_{1b}$ are two distinct non-negative numbers. Then there exist two points (t_{1a}, t_{2a}) and (t_{1b}, t_{2b}) such that:

1. $(t_{1a}, t_{2a}) \in \mathcal{E}_{x_{1a}}$ and $(t_{1a}, t_{2a}) \notin \mathcal{E}_{x_{1b}}$,
2. $(t_{1b}, t_{2b}) \in \mathcal{E}_{x_{1b}}$ and $(t_{1b}, t_{2b}) \notin \mathcal{E}_{x_{1a}}$.

Without loss of generality, assume $t_{1a} \leq t_{1b}$, and define the set

$$W' = \{(t_{1a}, t_{2a}),\ (t_{1b}, t_{2b})\}.$$

From (a), we have

$$x_{1b} - t_{1a} \leq t_{2a} \leq x_{1a} - t_{1a},$$

and from (b),

$$x_{1a} - t_{1b} \leq t_{2b} \leq x_{1b} - t_{1b}.$$

This leads to a contradiction as we assume $x_{1a} \neq x_{1b}$, and thus no two-point subset of $\mathbb{R}_+^2$ can be shattered by $\{\mathcal{E}_{x_1}\}$. Hence, $\{\mathcal{E}_{x_1}\}$ is a VC-class. It follows that $\{\mathcal{H}_{g_{(x_1,x_2)}}\}$ is also a VC-class. Therefore, we can conclude that

$$\sup_{(x_1,x_2)\in\mathcal{J}} \left|\hat{N}_1(x_1, x_2) - N_1(x_1, x_2)\right| \xrightarrow{\text{a.s.}} 0 \quad \text{as } n \to \infty. \tag{7.6}$$

By the Glivenko–Cantelli theorem (see Sen and Singer (1993), p. 187),

$$\sup_{(x_1,x_2)\in\mathcal{J}} \left|\hat{F}_n(x_1, x_2) - F(x_1, x_2)\right| \xrightarrow{\text{a.s.}} 0 \quad \text{as } n \to \infty. \tag{7.7}$$

Combining (7.6) and (7.7), and applying the continuous mapping theorem, we obtain

$$\sup_{(x_1,x_2)\in\mathcal{J}} |\hat{r}_1(x_1, x_2) - r_1(x_1, x_2)| \xrightarrow{\text{a.s.}} 0 \quad \text{as } n \to \infty.$$

The result follows similarly for the other component $r_2(x_1, x_2)$. □

Now, we discuss the asymptotic normality of the proposed estimator. To establish this property, we impose an additional assumption that the second-order moments of (X_1, X_2) exist and are finite. We use the following theorem, which is popularly known as multivariate delta method which is stated below. For more details, we refer to Lehmann and Casella (2006).

Lemma 7.2.5 (Multivariate Delta Method). *Let $\boldsymbol{\mu}_n$ be an estimator of the parameter vector $\boldsymbol{\mu} \in \mathbb{R}^p$ such that*

$$\sqrt{n}(\boldsymbol{\mu}_n - \boldsymbol{\mu})$$

converges to p-variate Gaussian distribution with zero mean vector and covariance matrix Σ, as $n \to \infty$. Suppose $\mathbf{h} : \mathbb{R}^p \to \mathbb{R}^q$ is a continuously differentiable function at $\boldsymbol{\mu}$, and let M denote the Jacobian matrix of $\mathbf{h}$ evaluated at $\boldsymbol{\mu}$ defined by

$$M = \left.\frac{\partial \mathbf{h}}{\partial \boldsymbol{\mu}}\right|_{\boldsymbol{\mu}}.$$

Then,

$$\sqrt{n}(\mathbf{h}(\boldsymbol{\mu}_n) - \mathbf{h}(\boldsymbol{\mu}))$$

converges to q-variate Gaussian distribution with zero mean vector and covariance matrix $M\Sigma M'$ as $n \to \infty$.

Theorem 7.2.6. *For every $(x_1, x_2) \in \mathbb{R}_+^2$ such that $F(x_1, x_2) > 0$,*

$$\sqrt{n}\left(\hat{r}_1(x_1, x_2) - r_1(x_1, x_2),\ \hat{r}_2(x_1, x_2) - r_2(x_1, x_2)\right)$$

converges in distribution to a bivariate Gaussian distribution with zero mean vector and covariance matrix $M\Sigma M'$, where $\Sigma = [\sigma_{ij}]$ is defined as

$$\sigma_{ij} = \mathrm{Cov}\left((x_i - X_i)\mathbf{I}(X_1 \le x_1,\ X_2 \le x_2), (x_j - X_j)\mathbf{I}(X_1 \le x_1, X_2 \le x_2)\right), \quad i, j = 1, 2,$$

$$\sigma_{i3} = r_i(x_1, x_2)F(x_1, x_2)\left[1 - F(x_1, x_2)\right], \quad i = 1, 2,$$

$$\sigma_{33} = F(x_1, x_2)\left[1 - F(x_1, x_2)\right],$$

and the matrix M is given by

$$M = \begin{bmatrix} \frac{1}{F(x_1,x_2)} & 0 & -\frac{r_1(x_1,x_2)}{F(x_1,x_2)} \\ 0 & \frac{1}{F(x_1,x_2)} & -\frac{r_2(x_1,x_2)}{F(x_1,x_2)} \end{bmatrix}.$$

Proof. Let us define the following functions for fixed $(x_1, x_2) \in \mathbb{R}_+^2$ such that $F(x_1, x_2) > 0$. Denote

$$\mathbf{g}_{x_1,x_2}(t_1, t_2) = \left(g_{1,(x_1,x_2)}(t_1, t_2),\ g_{2,(x_1,x_2)}(t_1, t_2),\ g_{3,(x_1,x_2)}(t_1, t_2)\right)$$

for every $t_1, t_2 \in \mathbb{R}_+^2$, defined by

$$g_{1,(x_1,x_2)}(t_1, t_2) = (x_1 - t_1)\mathbf{I}(t_1 \leq x_1, t_2 \leq x_2),$$

$$g_{2,(x_1,x_2)}(t_1, t_2) = (x_2 - t_2)\mathbf{I}(t_1 \leq x_1, t_2 \leq x_2),$$

$$g_{3,(x_1,x_2)}(t_1, t_2) = \mathbf{I}(t_1 \leq x_1, t_2 \leq x_2).$$

The expected value of $\mathbf{g}_{(x_1,x_2)}(X_1, X_2)$ is given by the vector $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$, where

$$\mu_1 = E[g_{1,(x_1,x_2)}(X_1, X_2)] = r_1(x_1, x_2)F(x_1, x_2),$$

$$\mu_2 = E[g_{2,(x_1,x_2)}(X_1, X_2)] = r_2(x_1, x_2)F(x_1, x_2),$$

$$\mu_3 = E[g_{3,(x_1,x_2)}(X_1, X_2)] = F(x_1, x_2).$$

The covariance matrix of $\mathbf{g}_{(x_1,x_2)}(X_1, X_2)$ is Σ.

Let (X_{1i}, X_{2i}), for $i = 1, 2, \ldots, n$, be a random sample of size n. Define the statistics

$$\bar{\mathbf{S}}_{(x_1,x_2)} = \left(\bar{S}_{1,(x_1,x_2)},\ \bar{S}_{2,(x_1,x_2)},\ \bar{S}_{3,(x_1,x_2)}\right),$$

where

$$\bar{S}_{j,(x_1,x_2)} = \frac{1}{n}\sum_{i=1}^{n} S_{j,(x_1,x_2)}(X_{1i}, X_{2i}), \quad j = 1, 2, 3.$$

By the multivariate central limit theorem,

$$\sqrt{n}\left(\bar{\mathbf{S}}_{(x_1,x_2)} - \boldsymbol{\mu}\right)$$

follows a trivariate Gaussian distribution with zero mean vector and covariance matrix Σ as $n \to \infty$.

Consider the transformation defined by

$$\mathbf{h}(y_1, y_2, y_3) = (h_1(y_1, y_2, y_3),\ h_2(y_1, y_2, y_3)) = \left(\frac{y_1}{y_3}, \frac{y_2}{y_3}\right).$$

It is straightforward to show that this transformation is continuously differentiable, and the corresponding Jacobian matrix of $\mathbf{h}$ evaluated at $\boldsymbol{\mu}$ is M. By applying the multivariate delta method, the result follows. □

7.3 Simulation Study

In this section, we evaluate the performance of the proposed estimator through a Monte Carlo simulation study. Different copula models with various levels of Kendall's τ are considered to assess the estimator's performance across a range of dependence structures, from low to high. The marginal distributions are assumed to follow exponential distributions with mean 1 for each component. Each experiment is repeated over 2,000 simulations, and the average bias and mean squared error (MSE) for both components of the BMITF estimator are computed at various values of (x_1, x_2). The results are presented in Table 7.1, Table 7.2, and Table 7.3.

It is observed from the tables that as the sample size increases, both the bias and the MSE decrease. Furthermore, the bias tends to zero with increasing sample size. The proposed estimator is also computationally efficient, producing estimates in most cases in a few minutes, even for large samples, making it both time-efficient and practically feasible. In the next section, we demonstrate the practical utility of the proposed estimator using a real dataset.

7.4 Application to Pink Eye Disease Data

In this section, we apply our proposed estimator to estimate the BMITF for pink eye disease data. The dataset, originally presented in Sankaran et al. (2012), is provided in the Appendix for reference. It consists of 40 observations, each representing the time (in weeks) at which an individual developed an infection in both the left and right eyes during a one-year follow-up study.

Pink eye disease, also known as conjunctivitis, is an infection that causes inflammation and redness in the transparent membrane lining the eyelids and covering the white part of the eyeball. The condition may result from bacterial or viral infections or allergic reactions to foreign substances. It can affect individuals across all age groups and typically resolves within one to two weeks.

Since pink eye is a contagious disease, transmitted through direct or indirect contact with an infected individual, understanding the incubation period is crucial. This knowledge aids in determining appropriate isolation durations for individuals who have been in contact

Table 7.1. Bias and mean squared error of the proposed nonparametric estimator of BMITF for different copula models with Kendall's $\tau = 0.25$.

Copula	(x_1, x_2)	**Sample Size**											
		$n = 25$				$n = 50$				$n = 100$			
		$\hat{r}_1(x_1x_2)$		$\hat{r}_2(x_1x_2)$		$\hat{r}_1(x_1x_2)$		$\hat{r}_2(x_1x_2)$		$\hat{r}_1(x_1x_2)$		$\hat{r}_2(x_1x_2)$	
		Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
	$(0.3, 0.6)$	-0.002844	0.002661	-0.004592	0.010969	-0.000845	0.001096	-0.000884	0.004161	0.000197	0.000489	0.001085	0.001786
	$(0.7, 0.4)$	-0.001440	0.009158	0.000499	0.003040	0.000868	0.003728	-0.000564	0.001313	-0.001039	0.001690	0.000586	0.000621
Gumbel	$(1.1, 0.8)$	0.003756	0.009135	-0.000625	0.004938	0.000363	0.004442	-0.000724	0.002338	-0.000395	0.002180	-0.000228	0.001171
	$(0.9, 1.2)$	-0.002644	0.006016	0.002823	0.009404	-0.002961	0.002677	-0.000867	0.004568	0.000557	0.001472	0.002071	0.002247
	$(1.3, 1.5)$	0.002663	0.008550	0.000200	0.010241	-0.000858	0.004136	0.000934	0.005358	-0.000780	0.002037	-0.000815	0.002680
Frank	$(0.3, 0.6)$	-0.001070	0.002509	-0.002840	0.009613	0.000628	0.001001	-0.002682	0.003555	0.000617	0.000467	-0.000927	0.001720
	$(0.7, 0.4)$	-0.000743	0.008012	-0.001341	0.002695	-0.000122	0.003709	-0.000116	0.001196	0.001235	0.001758	-0.000326	0.000597
	$(1.1, 0.8)$	0.002417	0.008310	0.001325	0.004874	-0.001806	0.004233	0.000216	0.002339	-0.001096	0.002107	-0.000197	0.001137
	$(0.9, 1.2)$	0.000373	0.005618	0.001463	0.009432	0.000556	0.002690	-0.000067	0.004352	-0.000123	0.001253	-0.000728	0.002145
	$(1.3, 1.5)$	-0.000565	0.008375	0.000161	0.010726	0.001651	0.004175	-0.000507	0.004977	-0.001943	0.002101	0.000530	0.002652
Joe	$(0.3, 0.6)$	-0.001967	0.002910	-0.008198	0.011111	-0.000270	0.001089	0.002909	0.004342	-0.000113	0.000511	0.000207	0.002102
	$(0.7, 0.4)$	-0.001712	0.009173	0.000241	0.003073	0.001071	0.003923	0.001275	0.001291	0.000494	0.001964	-0.000692	0.000618
	$(1.1, 0.8)$	-0.000343	0.009009	0.002271	0.004824	-0.002835	0.004385	-0.001198	0.002303	-0.000876	0.002035	0.000651	0.001142
	$(0.9, 1.2)$	0.002455	0.005660	-0.002295	0.009932	0.001591	0.002619	-0.000764	0.004777	0.000309	0.001414	-0.002331	0.002273
	$(1.3, 1.5)$	0.001457	0.008322	-0.000540	0.010306	0.000961	0.004075	-0.003249	0.005195	-0.000471	0.002016	0.000447	0.002520
Normal	$(0.3, 0.6)$	0.000153	0.002548	-0.000504	0.009987	0.000934	0.001034	0.000671	0.004028	0.000177	0.000451	-0.000272	0.001839
	$(0.7, 0.4)$	-0.002971	0.008841	-0.000890	0.002823	-0.000810	0.003937	0.000150	0.001280	0.000581	0.001849	0.000586	0.000658
	$(1.1, 0.8)$	-0.001447	0.009280	-0.001009	0.005047	0.001127	0.004409	-0.000298	0.002377	-0.000241	0.002191	0.000396	0.001131
	$(0.9, 1.2)$	0.002578	0.005591	-0.003101	0.010104	-0.000358	0.002707	0.000985	0.004453	0.001098	0.001381	-0.000695	0.002339
	$(1.3, 1.5)$	0.000049	0.008422	0.002022	0.010574	0.000591	0.003895	-0.001468	0.005399	-0.000893	0.002061	0.001236	0.002845

TABLE 7.2. Bias and mean squared error of the proposed nonparametric estimator of BMITF for different copula models with Kendall's $\tau = 0.5$.

Copula	(x_1, x_2)	**Sample Size**											
		$n = 25$				$n = 50$				$n = 100$			
		$\hat{r}_1(x_1x_2)$		$\hat{r}_2(x_1x_2)$		$\hat{r}_1(x_1x_2)$		$\hat{r}_2(x_1x_2)$		$\hat{r}_1(x_1x_2)$		$\hat{r}_2(x_1x_2)$	
		Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
	$(0.3, 0.6)$	-0.000155	0.001794	-0.001219	0.007433	0.000660	0.000751	0.000208	0.002766	-0.000124	0.000360	0.000718	0.001297
	$(0.7, 0.4)$	0.001702	0.005792	0.000712	0.002115	0.000165	0.002808	0.000197	0.001054	-0.000640	0.001312	0.000280	0.000478
Gumbel	$(1.1, 0.8)$	0.001188	0.007453	0.001643	0.004193	0.001501	0.003608	0.000471	0.002169	-0.000090	0.001708	-0.000936	0.000970
	$(0.9, 1.2)$	0.000062	0.005343	0.002878	0.008116	-0.001661	0.002373	0.000144	0.003721	0.000710	0.001214	0.001127	0.001776
	$(1.3, 1.5)$	0.001694	0.007618	-0.000355	0.009093	-0.000722	0.003737	0.000791	0.004489	0.000042	0.001859	-0.000237	0.002344
Frank	$(0.3, 0.6)$	-0.000165	0.001651	-0.000664	0.005674	0.000863	0.000733	-0.000058	0.002323	0.000620	0.000350	0.000255	0.001163
	$(0.7, 0.4)$	-0.000695	0.005255	-0.000450	0.002095	0.000984	0.002535	0.000525	0.000968	0.001038	0.001213	-0.000129	0.000473
	$(1.1, 0.8)$	0.001450	0.006346	0.001220	0.004161	-0.001234	0.003203	-0.000373	0.001921	-0.000743	0.001704	0.000261	0.001007
	$(0.9, 1.2)$	0.000828	0.004874	0.000423	0.007438	0.000361	0.002305	-0.000539	0.003369	-0.000273	0.001105	-0.000905	0.001758
	$(1.3, 1.5)$	0.000063	0.007376	-0.000474	0.008971	0.001791	0.003696	0.000841	0.004201	-0.001934	0.001856	-0.000527	0.002257
Joe	$(0.3, 0.6)$	-0.000639	0.001935	0.000166	0.006902	-0.000445	0.000866	0.000132	0.003223	-0.000040	0.000375	-0.000561	0.001439
	$(0.7, 0.4)$	-0.002298	0.006236	0.000408	0.002329	0.000458	0.002875	0.000514	0.001005	0.000072	0.001371	-0.000379	0.000497
	$(1.1, 0.8)$	-0.002306	0.007084	0.001303	0.003852	-0.001240	0.003596	-0.000524	0.002071	-0.000202	0.001642	-0.000077	0.000977
	$(0.9, 1.2)$	0.000272	0.004766	0.000075	0.007493	0.000998	0.002264	-0.000305	0.003586	0.001427	0.001161	-0.001603	0.001838
	$(1.3, 1.5)$	0.001799	0.007303	-0.000006	0.008836	-0.001166	0.003653	-0.004444	0.004554	-0.000038	0.001839	-0.000304	0.002185
Normal	$(0.3, 0.6)$	0.001569	0.001717	0.002300	0.005964	0.000769	0.000745	-0.000102	0.002719	0.000403	0.000353	-0.000485	0.001287
	$(0.7, 0.4)$	0.000478	0.005544	-0.000896	0.002105	0.000343	0.002777	0.000347	0.000974	0.000721	0.001274	0.000193	0.000493
	$(1.1, 0.8)$	-0.000618	0.007381	0.000759	0.004307	-0.000095	0.003485	-0.000490	0.001993	-0.000253	0.001681	-0.000209	0.000983
	$(0.9, 1.2)$	0.000521	0.004986	-0.002061	0.008169	0.000362	0.002425	0.002463	0.003647	0.001292	0.001191	-0.001003	0.001921
	$(1.3, 1.5)$	0.000087	0.007260	0.000668	0.009180	0.000348	0.003655	-0.001009	0.004535	-0.000631	0.001904	0.001037	0.002427

Table 7.3. Bias and mean squared error of the proposed nonparametric estimator of BMITF for different copula models with Kendall's $\tau = 0.75$.

Copula	(x_1, x_2)	Sample Size											
		$n = 25$				$n = 50$				$n = 100$			
		$\hat{r}_1(x_1x_2)$		$\hat{r}_2(x_1x_2)$		$\hat{r}_1(x_1x_2)$		$\hat{r}_2(x_1x_2)$		$\hat{r}_1(x_1x_2)$		$\hat{r}_2(x_1x_2)$	
		Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
	$(0.3, 0.6)$	0.000196	0.001332	0.000032	0.003925	0.000693	0.000658	0.000672	0.001675	0.000354	0.000300	0.000246	0.000793
	$(0.7, 0.4)$	0.000503	0.003743	-0.000905	0.001766	0.001966	0.001843	0.000826	0.000912	0.000142	0.000882	0.000335	0.000427
Gumbel	$(1.1, 0.8)$	-0.000167	0.005513	0.000538	0.003844	0.001579	0.002747	0.000427	0.001902	-0.000450	0.001284	-0.000429	0.000910
	$(0.9, 1.2)$	0.001845	0.004669	0.002309	0.006058	-0.001449	0.002163	-0.001092	0.002932	0.000624	0.001086	0.001084	0.001417
	$(1.3, 1.5)$	0.001439	0.007006	0.001380	0.007673	-0.000640	0.003554	-0.000165	0.003924	-0.000183	0.001745	-0.000331	0.002009
Frank	$(0.3, 0.6)$	-0.000064	0.001353	-0.000574	0.003028	0.000666	0.000636	0.000089	0.001428	0.000706	0.000301	0.000864	0.000666
	$(0.7, 0.4)$	-0.000193	0.003304	-0.000527	0.001788	0.000679	0.001554	0.000696	0.000861	0.001262	0.000759	0.000279	0.000409
	$(1.1, 0.8)$	0.001861	0.004863	0.001610	0.003865	-0.000503	0.002462	-0.000256	0.001761	-0.000216	0.001286	-0.000481	0.000920
	$(0.9, 1.2)$	0.000632	0.004615	0.000429	0.005962	-0.000039	0.002122	-0.000686	0.002769	-0.000624	0.001049	-0.001620	0.001366
	$(1.3, 1.5)$	0.000720	0.006877	0.001340	0.007841	0.002153	0.003564	0.001225	0.003865	-0.002074	0.001761	-0.001228	0.002005
Joe	$(0.3, 0.6)$	-0.001375	0.001408	-0.001771	0.004187	-0.000586	0.000639	0.000424	0.001818	0.000870	0.000295	-0.000161	0.000884
	$(0.7, 0.4)$	-0.001720	0.003942	-0.000722	0.001768	0.000974	0.001909	0.001228	0.000867	0.000502	0.000894	0.000013	0.000418
	$(1.1, 0.8)$	-0.000067	0.005325	0.001702	0.003729	0.000100	0.002473	-0.000070	0.001810	0.000069	0.001182	-0.000168	0.000925
	$(0.9, 1.2)$	0.000926	0.004281	0.002332	0.005478	0.000020	0.002156	-0.001038	0.002744	-0.000097	0.001041	-0.000823	0.001398
	$(1.3, 1.5)$	0.000722	0.007291	0.000734	0.007800	-0.002702	0.003659	-0.004127	0.004153	-0.000604	0.001769	-0.000890	0.001955
Normal	$(0.3, 0.6)$	0.000908	0.001342	0.002871	0.003333	-0.000004	0.000618	0.000303	0.001567	0.000074	0.000311	-0.000269	0.000785
	$(0.7, 0.4)$	0.000132	0.003461	-0.000772	0.001721	-0.000452	0.001809	-0.000090	0.000844	0.000347	0.000847	0.000499	0.000420
	$(1.1, 0.8)$	0.000403	0.005634	0.000914	0.003857	-0.000595	0.002537	-0.000483	0.001870	-0.000347	0.001278	-0.000912	0.000884
	$(0.9, 1.2)$	-0.000830	0.004576	-0.001322	0.006087	0.001231	0.002181	0.001647	0.002954	0.000698	0.001110	-0.000771	0.001579
	$(1.3, 1.5)$	0.000576	0.006680	0.001018	0.007676	0.000109	0.003375	-0.000219	0.003947	0.000629	0.001779	0.001746	0.002063

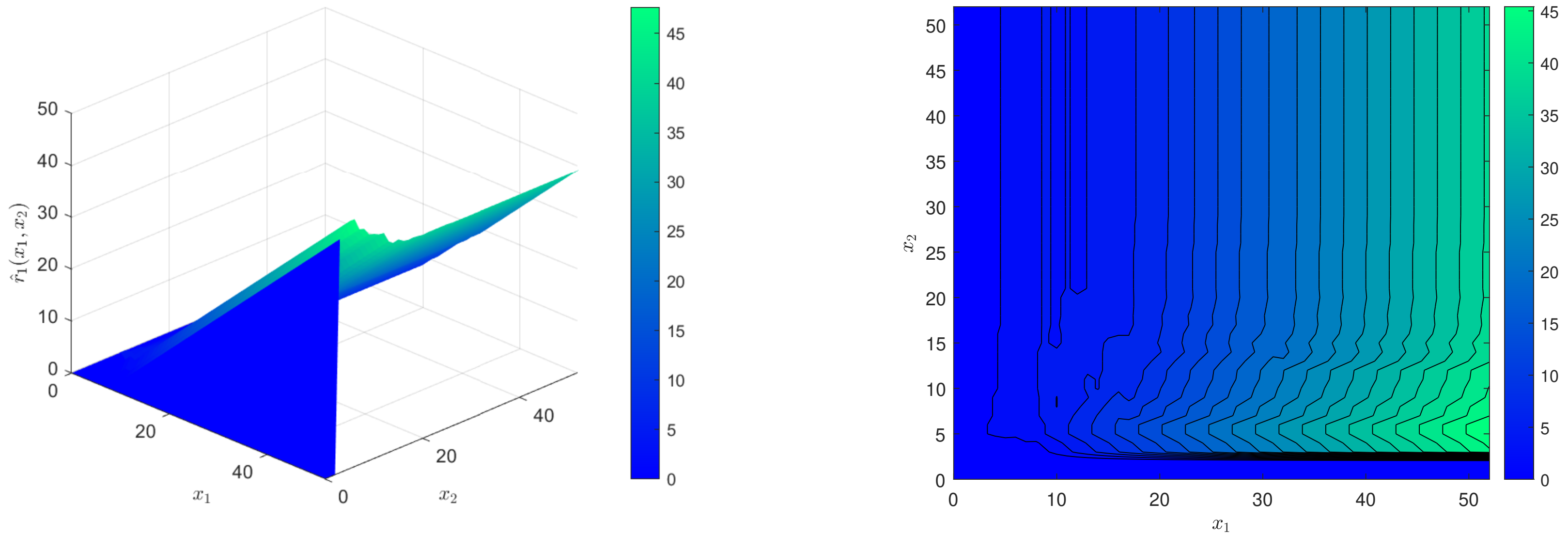
with infected persons, thereby helping to prevent further spread. Moreover, the incubation period varies depending on the underlying cause (e.g., viral, bacterial, or allergic), which in turn helps in implementing suitable treatment strategies.

As the BMITF provides provides insights into the time since infection for both the left and right eyes, our objective here is to estimate the BMITF for a dataset associated with pink eye disease using the proposed nonparametric estimator. Note that the *time since infection* refers to the duration from the exact time of exposure to the present time.

Let X_1 and X_2 represent the exact times (measured in weeks) at which an individual contracted pink eye in the left and right eyes, respectively, within a one-year period. Before estimating the BMITF, we first perform an exploratory data analysis. The summary statistics and various correlation measures of the dataset are presented in Table 7.4. Next, we estimate the time since infection for both the left and right eyes for different values of (x_1, x_2). The corresponding surface plots and contour plots of the estimators are shown in Figure 7.1 and Figure 7.2.

Table 7.4. Descriptive statistics of infection times for left eye (X_1) and right eye (X_2).

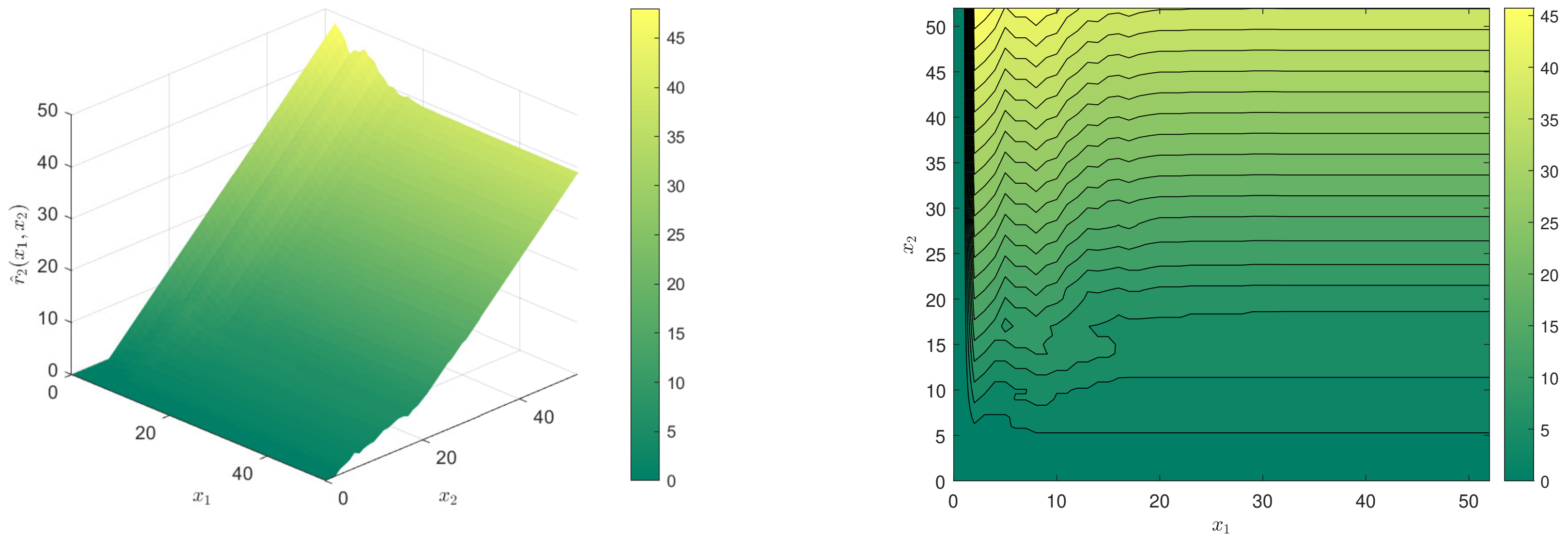
Statistics	**Left Eye (X_1)**	**Right Eye (X_2)**
Minimum	1.00	2.00
Maximum	31.00	28.00
1st Quartile	9.75	9.00
Mean	12.90	13.07
Median	12.50	14.00
3rd Quartile	16.00	16.00
Skewness	0.5966	0.1864
Kurtosis	4.2146	3.1790
Variance	36.74	30.62
Correlation Measures	**Value**	
Pearson's Correlation	0.9298	
Spearman's Rho	0.9568	
Kendall's Tau	0.8422	



(A) Surface plot of $\hat{r}_1(x_1, x_2)$

(B) Contour plot of $\hat{r}_1(x_1, x_2)$

FIGURE 7.1. Surface plot and contour plot of $\hat{r}_1(x_1, x_2)$.



(A) Surface plot of $\hat{r}_2(x_1, x_2)$

(B) Contour plot of $\hat{r}_2(x_1, x_2)$

FIGURE 7.2. Surace plot and contour plot of $\hat{r}_2(x_1, x_2)$.

7.5 Conclusion and Future Direction

In this chapter, we proposed a nonparametric estimator for estimating the bivariate mean inactivity time function (BMITF) as introduced by Nair and Asha (2008). We established that the proposed estimator is asymptotically biased, consistent, and asymptotically normally distributed. The performance of the estimator was evaluated through a simulation study under various copula models exhibiting low to high dependence structures.

The results indicate that the proposed estimator performs well across both weak and strong dependence scenarios. Furthermore, the bias and mean squared error (MSE) of the estimator decrease as the sample size increases. To demonstrate its practical applicability, we applied the estimator to a real dataset involving pink eye disease to estimate the time since infection period in the left and right eyes. Recent developments in the literature suggest various smoothing techniques, such as kernel-based estimators, to refine empirical estimators. A promising direction for future research would be to extend the proposed estimator using such smoothing techniques to enhance its performance and interpretability.

Appendix

Table 7.5. First goal times for Team-A (X) and Team-B (Y) of the UEFA Champions League football data reported in Meintanis (2007)

Sl. No.	X	Y	Sl. No.	X	Y
1	26	20	20	34	34
2	63	18	21	53	39
3	19	19	22	54	7
4	66	85	23	51	28
5	20	27	24	44	31
6	49	49	25	64	15
7	8	8	26	26	48
8	26	0	27	16	46
9	60	39	28	11	3
10	82	48	29	25	14
11	72	72	30	45	55
12	66	22	31	36	49
13	16	41	32	24	30
14	41	3	33	44	36
15	11	40	34	2	1
16	26	33	35	27	47
17	49	42	36	28	6
18	42	52	37	2	2
19	36	52			

Table 7.6. Scaled bivariate warranty data without outliers reported in Eliashberg et al. (1997)

No.	Age	Usage	No.	Age	Useage	No.	Age	Usage	No.	Age	Usage
1	1.66	0.9766	11	1.64	0.5992	21	0.31	0.1974	31	0.27	0.0095
2	0.35	0.2041	12	1.45	0.5932	22	0.65	0.203	32	4.02	1.26
3	2.49	1.2392	13	3.1	1.3827	23	2.61	1.2532	33	1.4	0.8607
4	1.97	0.9889	14	1.4	0.7553	24	0.13	0.0796	34	0.09	0.0105
5	0.27	0.0974	15	2.49	2.5014	25	3.16	1.4796	35	2.09	1.2302
6	0.41	0.1994	16	5.71	2.538	26	2.61	1.5062	36	0.48	0.0447
7	0.59	0.2128	17	4.9	2.6433	27	3.92	2.0688	37	1.66	0.9766
8	0.75	0.2158	18	3.4	1.6494	28	3.97	1.8688			
9	2.53	1.1817	19	1.6	0.7162	29	0.48	0.3099			
10	3.25	1.421	20	1.28	0.5922	30	0.01	0.1983			

TABLE 7.7. Infection duration (in Weeks) of left and right eyes for 40 Patients reported in Sankaran et al. (2012)

Patient No.	**Left Eye**	**Right Eye**	**Patient No.**	**Left Eye**	**Right Eye**
1	19	13	21	15	16
2	12	16	22	13	15
3	16	8	23	3	9
4	18	19	24	1	4
5	14	16	25	11	18
6	4	16	26	9	7
7	11	15	27	16	12
8	2	6	28	28	15
9	12	16	29	12	18
10	13	9	30	10	20
11	8	14	31	12	15
12	7	2	32	31	12
13	10	17	33	15	10
14	15	16	34	13	9
15	5	3	35	14	28
16	9	6	36	15	12
17	18	13	37	16	6
18	22	16	38	17	25
19	11	8	39	11	11
20	9	14	40	19	18

BIBLIOGRAPHY

Abd Elaal, M. K. and Jarwan, R. S. (2017). Inference of bivariate generalized exponential distribution based on copula functions. *Applied Mathematical Sciences*, 11(24):1155–1186.

Abramowitz, M. and Stegun, I. A. (1972). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. National Bureau of Standards Applied Mathematics Series 55. Tenth Printing.* ERIC, Washington.

Achcar, J. A., Moala, F. A., Tarumoto, M. H., and Coladello, L. F. (2015). A bivariate generalized exponential distribution derived from copula functions in the presence of censored data and covariates. *Pesquisa Operacional*, 35(1):165–186.

Ahn, C.-W., Chae, K.-C., and Clark, G. M. (1998). Estimating parameters of the power law process with two measures of failure time. *Journal of Quality Technology*, 30(2):127–132.

Alhadlaq, W. and Alzaid, A. (2020). Distribution function, probability generating function and Archimedean generator. *Symmetry*, 12(12):2108.

Almetwally, E. M. and Muhammed, H. Z. (2020). On a bivariate Fréchet distribution. *Journal of Statistics Applications and Probability*, 9(1):71–91.

Alvoni, E., Durante, F., Papini, P. L., and Sempi, C. (2007). Different types of convexity and concavity for copulas. In Štěpnička, M., Novák, V., and Bodenhofer, U., editors, *New Dimensions in Fuzzy Logic and Related Technologies: Proceedings of the 5th EUSFLAT Conference*, pages 185–189. University of Ostrava, Ostrava.

Alvoni, E. and Papini, P. L. (2007). Quasi-concave copulas, asymmetry and transformations. *Commentationes Mathematicae Universitatis Carolinae*, 48(2):311–319.

Anderson, B., Chukova, S., and Hirose, Y. (2017). Using copulas to model and simulate two-dimensional warranty data. *Quality and Reliability Engineering International*, 33(3):595–605.

Arnold, B. C. and Zahedi, H. (1988). On multivariate mean remaining life functions. *Journal of Multivariate Analysis*, 25(1):1–9.

Arshad, M., Pathak, A. K., Azhad, Q. J., and Khetan, M. (2023). Modeling bivariate data using linear exponential and Weibull distributions as marginals. *Mathematica Slovaca*, 73(4):1075–1096.

Asadi, M. and Zohrevand, Y. (2007). On the dynamic cumulative residual entropy. *Journal of Statistical Planning and Inference*, 137(6):1931–1941.

Ash, R. B. (2012). *Information Theory*. Dover Publications, New York.

Babu, G. J. and Chaubey, Y. P. (2006). Smooth estimation of a distribution and density function on a hypercube using Bernstein polynomials for dependent random vectors. *Statistics and Probability Letters*, 76(9):959–969.

Bairamov, I. and Kotz, S. (2002). Dependence structure and symmetry of Huang-Kotz FGM distributions and their extensions. *Metrika*, 56(1):55–72.

Baratpour, S. and Rad, A. H. (2012). Testing goodness-of-fit for exponential distribution based on cumulative residual entropy. *Communications in Statistics-Theory and Methods*, 41(8):1387–1396.

Barbe, P., Genest, C., Ghoudi, K., and Rémillard, B. (1996). On Kendall's process. *Journal of Multivariate Analysis*, 58(2):197–229.

Barlow, R. B. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing: Probability Models*. Holt, Rinehart & Winston, New York.

Bashkirtseva, I. and Pisarchik, A. (2019). Variability and effect of noise on the corporate dynamics of coupled oscillators. In Pasheva, V., Popivanov, N., and Venkov, G., editors, *AIP Conference Proceedings*, volume 2172. AIP Publishing, Bulgaria.

Basu, A. P. (1971). Bivariate failure rate. *Journal of the American Statistical Association*, 66(333):103–104.

Bayramoglu, K. and Bayramoglu, I. (2014). Baker-Lin-Huang type bivariate distributions based on order statistics. *Communications in Statistics-Theory and Methods*, 43(10-12):1992–2006.

Bedő, J. and Ong, C. S. (2016). Multivariate Spearman's ρ for aggregating ranks using copulas. *Journal of Machine Learning Research*, 17(201):1–30.

Behboodian, J., Dolati, A., and Úbeda-Flores, M. (2007). A multivariate version of Gini's rank association coefficient. *Statistical Papers*, 48:295–304.

Bekrizadeh, H., Parham, G. A., and Zadkarami, M. R. (2015). An asymmetric generalized FGM copula and its properties. *Pakistan Journal of Statistics*, 31(1):95–106.

Belalia, M., Bouezmarni, T., Lemyre, F., and Taamouti, A. (2017). Testing independence based on Bernstein empirical copula and copula density. *Journal of Nonparametric Statistics*, 29(2):346–380.

Ben-Naim, A. (2008). *A Farewell to Entropy: Statistical Thermodynamics Based on Information: S*. World Scientific, London.

Blier-Wong, C., Cossette, H., and Marceau, E. (2023). Risk aggregation with FGM copulas. *Insurance: Mathematics and Economics*, 111:102–120.

Block, H. W., Savits, T. H., and Singh, H. (1998). The reversed hazard rate function. *Probability in the Engineering and Informational Sciences*, 12(1):69–90.

Blomqvist, N. (1950). On a measure of dependence between two random variables. *The Annals of Mathematical Statistics*, 21(4):593–600.

Blumentritt, T. and Schmid, F. (2012). Mutual information as a measure of multivariate association: analytical properties and statistical estimation. *Journal of Statistical Computation and Simulation*, 82(9):1257–1274.

Boltzmann, L. (1872). Weitere studien über das wärmegleichgewicht unter gasmolekülen. *Wiener Berichte*, 66:275–370.

Calì, C., Longobardi, M., and Ahmadi, J. (2017). Some properties of cumulative Tsallis entropy. *Physica A: Statistical Mechanics and its Applications*, 486:1012–1021.

Cartwright, J. (2014). Roll over, Boltzmann. *Physics World*, 27(05):31.

Chamizo, F., Fernández-Sánchez, J., and Úbeda-Flores, M. (2021). Construction of copulas with hairpin support. *Mediterranean Journal of Mathematics*, 18(4):155.

Chandra, N. K. and Roy, D. (2001). Some results on reversed hazard rate. *Probability in the Engineering and Informational Sciences*, 15(1):95–102.

Chaubey, Y. P. and Sen, A. (2008). Smooth estimation of mean residual life under random censoring. In Balakrishnan, N., Peña, E. A., and Silvapulle, M. J., editors, *Beyond Parametrics in Interdisciplinary Research: Festschrift in Honor of Professor Pranab K. Sen*, volume 1, pages 35–49. Institute of Mathematical Statistics.

Chaubey, Y. P. and Sen, P. K. (1996). On smooth estimation of survival and density functions. *Statistics and Risk Modeling*, 14(1):1–22.

Chaubey, Y. P. and Sen, P. K. (1999). On smooth estimation of mean residual life. *Journal of Statistical Planning and Inference*, 75(2):223–236.

Chaubey, Y. P. and Sen, P. K. (2002). Smooth estimation of multivariate survival and density functions. *Journal of Statistical Planning and Inference*, 103(1-2):361–376.

Chen, X. and Fan, Y. (2006). Estimation of copula-based semiparametric time series models. *Journal of Econometrics*, 130(2):307–335.

Cherubini, U., Luciano, E., and Vecchiato, W. (2004). *Copula Methods in Finance*. John Wiley & Sons, Chichester.

Chesneau, C. (2022). A note on a simple polynomial-sine copula. *Asian Journal of Mathematics and Applications*, 2:1–14.

Clausius, R. (1850). Ueber die bewegende Kraft der Wärme und die Gesetze, welche sich daraus für die Wärmelehre selbst ableiten lassen. *Annalen der Physik*, 155(3):368–397.

Clayton, D. G. (1978). A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. *Biometrika*, 65(1):141–151.

Cossette, H., Côté, M.-P., Marceau, E., and Moutanabbir, K. (2013). Multivariate distribution defined with Farlie–Gumbel–Morgenstern copula and mixed Erlang marginals: Aggregation and capital allocation. *Insurance: Mathematics and Economics*, 52(3):560–572.

Cover, T. M. (1999). *Elements of Information Theory*. John Wiley & Sons, New York.

Cuadras, C. M. (2009). Constructing copula functions with weighted geometric means. *Journal of Statistical Planning and Inference*, 139(11):3766–3772.

Cuadras, C. M. and Augé, J. (1981). A continuous general multivariate distribution and its properties. *Communications in Statistics-Theory and Methods*, 10(4):339–353.

Das, J., Jha, S., and Goyal, M. K. (2020). Non-stationary and copula-based approach to assess the drought characteristics encompassing climate indices over the Himalayan states in India. *Journal of Hydrology*, 580:124356.

De Albuquerque, M. P., Esquef, I. A., and Mello, A. G. (2004). Image thresholding using Tsallis entropy. *Pattern Recognition Letters*, 25(9):1059–1065.

Deheuvels, P. (1979). La fonction de dépendance empirique et ses propriétés. un test non paramétrique d'indépendance. *Bulletins de l'Académie Royale de Belgique*, 65(1):274–292.

DeVore, R. A. and Lorentz, G. G. (1993). *Constructive Approximation*. Springer, New York.

Di Crescenzo, A., Kayal, S., and Meoli, A. (2021). Fractional generalized cumulative entropy and its dynamic version. *Communications in Nonlinear Science and Numerical Simulation*, 102:105899.

Di Crescenzo, A. and Longobardi, M. (2006). On weighted residual and past entropies. *Scientiae Mathematicae Japonicae*, 64(2):255–266.

Di Crescenzo, A. and Longobardi, M. (2009). On cumulative entropies. *Journal of Statistical Planning and Inference*, 139(12):4072–4087.

Diaz, W. and Cuadras, C. M. (2022). An extension of the Gumbel–Barnett family of copulas. *Metrika*, 85(7):913–926.

Dolati, A., Amini, M., and Mirhosseini, S. M. (2014). Dependence properties of bivariate distributions with proportional (reversed) hazards marginals. *Metrika*, 77:333–347.

Drouet Mari, D. and Kotz, S. (2001). *Correlation and Dependence*. Imperial College Press, London.

Duchesne, T. and Lawless, J. (2000). Alternative time scales and failure time models. *Lifetime Data Analysis*, 6:157–179.

Durante, F., Quesada-Molina, J., and Sempi, C. (2006). Semicopulas: characterizations and applicability. *Kybernetika*, 42(3):287–302.

Durante, F. and Sempi, C. (2003). Copulæ and Schur-concavity. *International Mathematical Journal*, 3(9):893–905.

Durante, F. and Sempi, C. (2016). *Principles of Copula Theory*, volume 474. CRC Press, Boca Raton.

Ebaid, R., Elbadawy, W., Ahmed, E., and Abdelghaly, A. (2020). A new extension of the FGM copula with an application in reliability. *Communications in Statistics-Theory and Methods*, 51(9):2953–2961.

Ebrahimi, N., Kirmani, S. N. U. A., and Soofi, E. S. (2007). Multivariate dynamic information. *Journal of Multivariate Analysis*, 98(2):328–349.

Efromovich, S. (2025). Nonparametric estimation of bivariate mean residual life for right-censored observations. *Stat*, 14(1):e70053.

El-Sherpieny, E., Muhammed, H. Z., and Almetwally, E. M. (2018). FGM bivariate Weibull distribution. In *Proceedings of the Annual Conference in Statistics (53rd), Computer*

Science, and Operations Research, Institute of Statistical Studies and Research, pages 55–77. Cairo University.

Eliashberg, J., Singpurwalla, N. D., and Wilson, S. P. (1997). Calculating the reserve for a time and usage indexed warranty. *Management Science*, 43(7):966–975.

Emura, T., Nakatochi, M., Murotani, K., and Rondeau, V. (2017). A joint frailty-copula model between tumour progression and death for meta-analysis. *Statistical Methods in Medical Research*, 26(6):2649–2666.

Eyraud, H. (1936). Les principes de la mesure des correlations. *Annales de I'Université de Lyon, Séie III, Section A*, 1:30–47.

Farlie, D. J. G. (1960). The performance of some correlation coefficients for a general bivariate distribution. *Biometrika*, 47(3-4):307–323.

Feller, W. (1991). *An Introduction to Probability Theory and its Applications*, volume 2. John Wiley & Sons, New York.

Fermanian, J.-D., Radulovic, D., and Wegkamp, M. (2004). Weak convergence of empirical copula processes. *Bernoulli*, 10(5):847–860.

Finkelstein, M. S. (2002). On the reversed hazard rate. *Reliability Engineering and System Safety*, 78(1):71–75.

Finner, H. (1992). A generalization of Holder's inequality and some probability inequalities. *The Annals of Probability*, 20(4):1893–1901.

Foroghi, F., Tahmasebi, S., Afshari, M., and Lak, F. (2023). Extensions of fractional cumulative residual entropy with applications. *Communications in Statistics-Theory and Methods*, 52(20):7350–7369.

Frank, M. J. (1979). On the simultaneous associativity of $f(x, y)$ and $x + y - f(x, y)$. *Aequationes Mathematicae*, 19:194–226.

Galambos, J. (1975). Order statistics of samples from multivariate distributions. *Journal of the American Statistical Association*, 70(351a):674–680.

Genest, C. (1987). Frank's family of bivariate distributions. *Biometrika*, 74(3):549–555.

Genest, C. and Favre, A.-C. (2007). Everything you always wanted to know about copula modeling but were afraid to ask. *Journal of Hydrologic Engineering*, 12(4):347–368.

Genest, C. and Mackay, R. J. (1986). Copules archimédiennes et families de lois bidimensionnelles dont les marges sont données. *Canadian Journal of Statistics*, 14(2):145–159.

Genest, C. and Plante, J.-F. (2003). On Blest's measure of rank correlation. *Canadian Journal of Statistics*, 31(1):35–52.

Genest, C., Quessy, J.-F., and Rémillard, B. (2006). Local efficiency of a Cramér–von Mises test of independence. *Journal of Multivariate Analysis*, 97(1):274–294.

Genest, C. and Rémillard, B. (2004). Test of independence and randomness based on the empirical copula process. *Test*, 13:335–369.

Genest, C. and Rémillard, B. (2008). Validity of the parametric bootstrap for goodness-of-fit testing in semiparametric models. *Annales de I'IHP Probabilités et Statistiques*, 44(6):1096–1127.

Genest, C., Rémillard, B., and Beaudoin, D. (2009). Goodness-of-fit tests for copulas: A review and a power study. *Insurance: Mathematics and Economics*, 44(2):199–213.

Genest, C. and Rivest, L.-P. (1993). Statistical inference procedures for bivariate Archimedean copulas. *Journal of the American Statistical Association*, 88(423):1034–1043.

Georges, P., Lamy, A.-G., Nicolas, E., Quibel, G., and Roncalli, T. (2001). Multivariate survival modelling: a unified approach with copulas. *SSRN Electronic Journal*, pages 1–72. DOI: 10.2139/ssrn.1032559.

Ghalibaf, M. B. (2022). Stress-strength reliability with dependent variables based on copula function. *Journal of Dynamics and Games*, 9(3):283–297.

Ghosh, I. and Sunoj, S. (2024). Copula-based mutual information measures and mutual entropy: A brief survey. *Mathematical Methods of Statistics*, 33(3):297–309.

Golomb, S. (1966). The information generating function of a probability distribution (corresp.). *IEEE Transactions on Information Theory*, 12(1):75–77.

González-Barrios, J. M. and Hoyos-Argüelles, R. (2021). Estimating checkerboard approximations with sample d-copulas. *Communications in Statistics-Simulation and Computation*, 50(12):3992–4027.

Guiasu, S. and Reischer, C. (1985). The relative information generating function. *Information Sciences*, 35(3):235–241.

Gumbel, E. J. (1960). Bivariate exponential distributions. *Journal of the American Statistical Association*, 55(292):698–707.

Gupta, R. C. (2016). Reliability characteristics of Farlie–Gumbel–Morgenstern family of bivariate distributions. *Communications in Statistics-Theory and Methods*, 45(8):2342–2353.

Gupta, R. C. and Akman, O. H. (1995). Mean residual life function for certain types of non-monotonic ageing. *Communications in Statistics. Stochastic Models*, 11(1):219–225.

Gupta, S. K. and Bhattacharya, D. (2022). Non-parametric estimation of bivariate reliability from incomplete two-dimensional warranty data. *Reliability Engineering and System Safety*, 222:108385.

Hao, Z. and Singh, V. P. (2015). Integrating entropy and copula theories for hydrologic modeling and analysis. *Entropy*, 17(4):2253–2280.

Harris, T. E. (1948). Branching processes. *The Annals of Mathematical Statistics*, 19(4):474–494.

Henderson, S. B., Shahirinia, A. H., and Tavakoli Bina, M. (2021). Bayesian estimation of copula parameters for wind speed models of dependence. *IET Renewable Power Generation*, 15(16):3823–3831.

Herwartz, H. and Maxand, S. (2020). Nonparametric tests for independence: a review and comparative simulation study with an application to malnutrition data in India. *Statistical Papers*, 61:2175–2201.

Hofert, M., Kojadinovic, I., Mächler, M., and Yan, J. (2018). *Elements of Copula Modeling with R*. Springer, Cham.

Holland, P. W. and Wang, Y. J. (1987). Dependence function for continuous bivariate densities. *Communications in Statistics-Theory and Methods*, 16(3):863–876.

Hosseini, T. and Ahmadi, J. (2019). Results on inaccuracy measure in information theory based on copula function. In *Proceedings of the Fifth Seminar of Copula Theory and its Applications*, pages 22–32.

Hosseini, T. and Nooghabi, M. J. (2021). Discussion about inaccuracy measure in information theory using co-copula and copula dual functions. *Journal of Multivariate Analysis*, 183:104725.

Hougaard, P. (1984). Life table methods for heterogeneous populations: Distributions describing the heterogeneity. *Biometrika*, 71(1):75–83.

Huang, J. S. and Kotz, S. (1999). Modifications of the Farlie-Gumbel-Morgenstern distributions. A tough hill to climb. *Metrika*, 49(2):135–145.

Janssen, P., Swanepoel, J., and Veraverbeke, N. (2012). Large sample behavior of the Bernstein copula estimator. *Journal of Statistical Planning and Inference*, 142(5):1189–1197.

Jayasinghe, C. L. and Zeephongsekul, P. (2013). Nonparametric smooth estimation of the expected inactivity time function. *Journal of Statistical Planning and Inference*, 143(5):911–928.

Jia, X., Xing, L., and Li, G. (2018). Copula-based reliability and safety analysis of safety-critical systems with dependent failures. *Quality and Reliability Engineering International*, 34(5):928–938.

Joe, H. (1987). Majorization, randomness and dependence for multivariate distributions. *The Annals of Probability*, 15(3):1217–1225.

Joe, H. (1997). *Multivariate Models and Multivariate Dependence Concepts*. CRC Press, United Kingdom.

Johnson, N. L., Kemp, A. W., and Kotz, S. (2005). *Univariate Discrete Distributions*. John Wiley & Sons, New Jersey.

Johnson, N. L. and Kotz, S. (1975). A vector multivariate hazard rate. *Journal of Multivariate Analysis*, 5(1):53–66.

Jumarie, G. (2012). Derivation of an amplitude of information in the setting of a new family of fractional entropies. *Information Sciences*, 216:113–137.

Jung, M. and Bai, D. S. (2007). Analysis of field data under two-dimensional warranty. *Reliability Engineering and System Safety*, 92(2):135–143.

Kaishev, V. K., Dimitrova, D. S., and Haberman, S. (2007). Modelling the joint distribution of competing risks survival times using copula functions. *Insurance: Mathematics and Economics*, 41(3):339–361.

Karci, A. (2016). Fractional order entropy: New perspectives. *Optik*, 127(20):9172–9177.

Karlin, C. (1968). *Total Positivity*, volume 1. Stanford University Press, California.

Kayid, M. and Shrahili, M. (2022). Some further results on the fractional cumulative entropy. *Entropy*, 24(8):1037.

Keilson, J. and Sumita, U. (1982). Uniform stochastic ordering and related inequalities. *Canadian Journal of Statistics*, 10(3):181–198.

Kerridge, D. F. (1961). Inaccuracy and inference. *Journal of the Royal Statistical Society: Series B (Methodological)*, 23(1):184–194.

Kiefer, J. (1961). On large deviations of the empirical distribution function of vector chance variables and a law of the iterated logarithm. *Pacific Journal of Mathematics*, 11(2):649–660.

Kim, G., Silvapulle, M. J., and Silvapulle, P. (2007). Comparison of semiparametric and parametric methods for estimating copulas. *Computational Statistics and Data Analysis*, 51(6):2836–2850.

Kim, J. M., Sungur, E. A., Choi, T., and Heo, T. Y. (2011). Generalized bivariate copulas and their properties. *Model Assisted Statistics and Applications*, 6(2):127–136.

Kojadinovic, I. and Holmes, M. (2009). Tests of independence among continuous random vectors based on Cramér–von Mises functionals of the empirical copula process. *Journal of Multivariate Analysis*, 100(6):1137–1154.

Kojadinovic, I., Yan, J., and Holmes, M. (2011). Fast large-sample goodness-of-fit tests for copulas. *Statistica Sinica*, 21(2):841–871.

Kojadinovic, I. and Yi, B. (2024). A class of smooth, possibly data-adaptive nonparametric copula estimators containing the empirical beta copula. *Journal of Multivariate Analysis*, 201:105269.

Kufner, A., John, O., and Fucik, S. (1977). *Function Spaces*. Noordhoff, Leiden.

Kulkarni, H. V. and Rattihalli, R. (2002). Nonparametric estimation of a bivariate mean residual life function. *Journal of the American Statistical Association*, 97(459):907–917.

Kullback, S. and Leibler, R. A. (1951). On information and sufficiency. *The Annals of Mathematical Statistics*, 22(1):79–86.

Kumar, V. and Taneja, H. (2015). Dynamic cumulative residual and past inaccuracy measures. *Journal of Statistical Theory and Applications*, 14(4):399–412.

Kundu, A. and Kundu, C. (2017). Bivariate extension of (dynamic) cumulative past entropy. *Communications in Statistics-Theory and Methods*, 46(9):4163–4180.

Kundu, D. and Gupta, R. C. (2017). On Bivariate Birnbaum–Saunders distribution. *American Journal of Mathematical and Management Sciences*, 36(1):21–33.

Lai, C. D. and Balakrishnan, N. (2009). *Continuous Bivariate Distributions*. Springer, New York.

Latif, S. and Mustafa, F. (2020). Parametric vine copula construction for flood analysis for Kelantan river basin in Malaysia. *Civil Engineering Journal*, 6(8):1470–1491.

Lawless, J., Hu, J., and Cao, J. (1995). Methods for the estimation of failure distributions and rates from automobile warranty data. *Lifetime Data Analysis*, 1:227–240.

Leblanc, A. (2012). On estimating distribution functions using Bernstein polynomials. *Annals of the Institute of Statistical Mathematics*, 64:919–943.

Lehmann, E. L. (1966). Some concepts of dependence. *The Annals of Mathematical Statistics*, 37(5):1137–1153.

Lehmann, E. L. and Casella, G. (2006). *Theory of Point Estimation*. Springer Texts in Statistics. Springer, New York.

Lopes, A. M. and Machado, J. A. T. (2020). A review of fractional order entropies. *Entropy*, 22(12):1374.

Louzada, F., Suzuki, A. K., Cancho, V. G., Prince, F. L., and Pereira, G. A. (2012). The long-term bivariate survival FGM copula model: an application to a Brazilian HIV data. *Journal of Data Science*, 10(3):511–535.

Lyu, Y., Tuo, S., Wei, Y., and Yang, M. (2021). Time-varying effects of global economic policy uncertainty shocks on crude oil price volatility: New evidence. *Resources Policy*, 70:101943.

Ma, J. and Sun, Z. (2011). Mutual information is copula entropy. *Tsinghua Science and Technology*, 16(1):51–54.

MacKenzie, D. and Spears, T. (2012). The formula that killed wall street? The gaussian copula and the material cultures of modelling. *Social Studies of Science*, 44(3):393–417.

Mao, X., Shang, P., Wang, J., and Yin, Y. (2020). Fractional cumulative residual Kullback–Leibler information based on Tsallis entropy. *Chaos, Solitons and Fractals*, 139:110292.

Marcinowski, J. and Sadowski, M. (2020). Using the ERFI function in the problem of the shape optimization of the compressed rod. *International Journal of Applied Mechanics and Engineering*, 25(2):75–87.

Marri, F. and Moutanabbir, K. (2022). Risk aggregation and capital allocation using a new generalized Archimedean copula. *Insurance: Mathematics and Economics*, 102:75–90.

Marshall, A. W. and Olkin, I. (1967). A generalized bivariate exponential distribution. *Journal of Applied Probability*, 4(2):291–302.

Marshall, A. W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84(3):641–652.

Mathai, A. M. and Haubold, H. J. (2007). Pathway model, superstatistics, Tsallis statistics, and a generalized measure of entropy. *Physica A: Statistical Mechanics and its Applications*, 375(1):110–122.

Mehrali, Y. and Asadi, M. (2021). Parameter estimation based on cumulative Kullback–Leibler divergence. *REVSTAT-Statistical Journal*, 19(1):111–130.

Meintanis, S. G. (2007). Test of fit for Marshall–Olkin distributions with applications. *Journal of Statistical Planning and inference*, 137(12):3954–3963.

Mirhosseini, S. M., Amini, M., Kundu, D., and Dolati, A. (2015). On a new absolutely continuous bivariate generalized exponential distribution. *Statistical Methods and Applications*, 24:61–83.

Mohamed, M. S., Barakat, H. M., Alyami, S. A., and Abd Elgawad, M. A. (2022). Cumulative residual Tsallis entropy-based test of uniformity and some new findings. *Mathematics*, 10(5):771.

Mondal, S. and Kundu, D. (2020). A bivariate inverse Weibull distribution and its application in complementary risks model. *Journal of Applied Statistics*, 47(6):1084–1108.

Morgenstern, D. (1956). Einfache beispiele zweidimensionaler verteilungen. *Mitteilingsblatt fur Mathematische Statistik*, 8:234–235.

Morillas, P. M. (2005). A method to obtain new copulas from a given one. *Metrika*, 61:169–184.

Murphy, K. P. (2022). *Probabilistic Machine Learning: An Introduction*. MIT Press, United States.

Nadarajah, S., Afuecheta, E., and Chan, S. (2017). A compendium of copulas. *Statistica*, 77(4):279–328.

Nadarajah, S. and Zografos, K. (2005). Expressions for Rényi and Shannon entropies for bivariate distributions. *Information Sciences*, 170(2-4):173–189.

Nair, N. U. and Asha, G. (2008). Some characterizations based on bivariate reversed mean residual life. *ProbStat Forum*, 1:1–14.

Nair, N. U., Sankaran, P., and John, P. (2018). Modelling bivariate lifetime data using copula. *Metron*, 76:133–153.

Nair, N. U., Sunoj, S. M., and Rajesh, G. (2022). *Reliability Modelling with Information Measures*. CRC Press, Boca Raton.

Nanda, A. K., Singh, H., Misra, N., and Paul, P. (2003). Reliability properties of reversed residual lifetime. *Communications in Statistics-Theory and Methods*, 32(10):2031–2042.

Nasri, B. R. and Remillard, B. N. (2024). Tests of independence and randomness for arbitrary data using copula-based covariances. *Journal of Multivariate Analysis*, 201:105273.

Nelsen, R. B. (2002). Concordance and copulas: A survey. In Cuadras, C. M., Fortiana, J., and Rodriguez-Lallena, J. A., editors, *Distributions With Given Marginals and Statistical Modelling*, pages 169–177. Springer, Netherlands.

Nelsen, R. B. (2006). *An Introduction to Copulas*. Springer, New York.

Ning, S. and Shephard, N. (2018). A nonparametric Bayesian approach to copula estimation. *Journal of Statistical Computation and Simulation*, 88(6):1081–1105.

Olkin, I. and Liu, R. (2003). A bivariate beta distribution. *Statistics and Probability Letters*, 62(4):407–412.

Panchenko, V. (2005). Goodness-of-fit test for copulas. *Physica A: Statistical Mechanics and its Applications*, 355(1):176–182.

Papaioannou, T., Ferentinos, K., and Tsairidis, C. (2007). Some information theoretic ideas useful in statistical inference. *Methodology and Computing in Applied Probability*, 9:307–323.

Park, S. and Kim, I. (2014). On cumulative residual entropy of order statistics. *Statistics and Probability Letters*, 94:170–175.

Pathak, A. K., Arshad, M., J. Azhad, Q., Khetan, M., and Pandey, A. (2023). A novel bivariate generalized Weibull distribution with properties and applications. *American Journal of Mathematical and Management Sciences*, 42(4):279–306.

Pathak, A. K. and Vellaisamy, P. (2016a). A note on generalized Farlie-Gumbel-Morgenstern copulas. *Journal of Statistical Theory and Practice*, 10:40–58.

Pathak, A. K. and Vellaisamy, P. (2016b). Various measures of dependence of a new asymmetric generalized Farlie–Gumbel–Morgenstern copulas. *Communications in Statistics-Theory and Methods*, 45(18):5299–5317.

Pathak, A. K. and Vellaisamy, P. (2022). A bivariate generalized linear exponential distribution: properties and estimation. *Communications in Statistics-Simulation and Computation*, 51(9):5426–5446.

Pettere, G., Voronova, I., and Zariņa, I. (2018). Behaviour of multivariate tail dependence coefficients. *Acta et Commentationes Universitatis Tartuensis de Mathematica*, 22(2):299–310.

Pillai, R. N. and Jayakumar, K. (1995). Discrete Mittag-Leffler distributions. *Statistics and Probability Letters*, 23(3):271–274.

Pollard, D. (1984). *Convergence of Stochastic Processes*. Springer, New York.

Psarrakos, G. and Toomaj, A. (2017). On the generalized cumulative residual entropy with applications in actuarial science. *Journal of Computational and Applied Mathematics*, 309:186–199.

Rajesh, G., Abdul-Sathar, E., Nair, K. M., and Reshmi, K. (2014). Bivariate extension of dynamic cumulative residual entropy. *Statistical Methodology*, 16:72–82.

Rajesh, G., Sathar, A., and Nair, K. M. (2009). Bivariate extension of residual entropy and some characterization results. *Journal of Indian Statistical Association*, 47(1):91–107.

Rajesh, G. and Sunoj, S. M. (2019). Some properties of cumulative Tsallis entropy of order α. *Statistical Papers*, 60(3):933–943.

Raju, D. C., Sunoj, S., and Rajesh, G. (2024). Results and applications of a new inaccuracy measure based on cumulative Tsallis entropy. *Communications in Statistics-Theory and Methods*, 53(12):4334–4353.

Raju, D. C., Sunoj, S. M., and Rajesh, G. (2020). Some reliability properties of bivariate cumulative residual Tsallis entropy. *Journal of Statistical Theory and Practice*, 14:1–18.

Raju, D. C., Sunoj, S. M., and Rajesh, G. (2023). Bivariate cumulative Tsallis past entropy: Properties and applications. *American Journal of Mathematical and Management Sciences*, 42(1):30–50.

Rao, M., Chen, Y., Vemuri, B. C., and Wang, F. (2004). Cumulative residual entropy: a new measure of information. *IEEE transactions on Information Theory*, 50(6):1220–1228.

Rényi, A. (1961). On measures of entropy and information. In Neyman, J., editor, *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, pages 547–562. University of California Press, Berkeley.

Roy, D. (2002). A characterization of model approach for generating bivariate life distributions using reversed hazard rates. *Journal of the Japan Statistical Society*, 32(2):239–245.

Rudin, W. (1976). *Principles of Mathematical Analysis*. McGraw-Hill, New York.

Rulkov, N. F. (2001). Regularization of synchronized chaotic bursts. *Physical Review Letters*, 86(1):183.

Saha, S. and Kayal, S. (2023). Extended fractional cumulative past and paired ϕ-entropy measures. *Physica A: Statistical Mechanics and its Applications*, 614:128552.

Saha, S. and Kayal, S. (2024). General weighted information and relative information generating functions with properties. *IEEE Transactions on Information Theory*, 70(8):5498–5515.

Salmon, F. (2009). Recipe for disaster: The formula that killed wall street. *Wired Magazine*, 17(3):1–7.

Sancetta, A. and Satchell, S. (2004). The Bernstein copula and its applications to modeling and approximations of multivariate distributions. *Econometric Theory*, 20(3):535–562.

Sankaran, P. G., Manoharan, M., and Anisha, P. (2012). Nonparametric estimators for a survivor function of paired recurrent events. *Statistical Methodology*, 9(4):501–512.

Sarabia, J. M., Gómez-Déniz, E., Prieto, F., and Jordá, V. (2018). Aggregation of dependent risks in mixtures of exponential distributions and extensions. *ASTIN Bulletin: The Journal of the IAA*, 48(3):1079–1107.

Schmid, F., Schmidt, R., Blumentritt, T., Gaißer, S., and Ruppert, M. (2010). Copula-based measures of multivariate association. In Jaworski, P., Durante, F., Härdle, W., and Rychlik, T., editors, *Copula Theory and Its Applications: Proceedings of the Workshop Held in Warsaw, 25-26 September 2009*, pages 209–236. Springer, Berlin.

Schweizer, B. (1991). Thirty years of copulas. In Dall'Aglio, G., Kotz, S., and Salinetti, G., editors, *Advances in Probability Distributions with Given Marginals: Beyond the Copulas*, pages 13–50. Springer, Dordrecht.

Schweizer, B. and Sklar, A. (1958). Espaces métriques aléatories. *Comptes Rendus Hebdomadaires Des Seances De L Academie Des Sciences*, 247(23):2092–2094.

Segers, J., Sibuya, M., and Tsukahara, H. (2017). The empirical beta copula. *Journal of Multivariate Analysis*, 155:35–51.

Sen, P. K. and Singer, J. M. (1993). *Large Sample Methods in Statistics: An Introduction with Applications*. CRC Press, Boca Raton.

Shaked, M. (1975). A nots on the exchangeable generalized Farlie-Gumbel-Morgenstern distributions. *Communications in Statistics-Theory and Methods*, 4(8):711–721.

Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic Orders*. Springer, New York.

Shannon, C. E. (1948). A mathematical theory of communication. *The Bell System Technical Journal*, 27(3):379–423.

Shemyakin, A. and Kniazev, A. (2017). *Introduction to Bayesian Estimation and Copula Models of Dependence*. John Wiley & Sons, Hoboken.

Shih, J.-H. and Emura, T. (2019). Bivariate dependence measures and bivariate competing risks models under the generalized FGM copula. *Statistical Papers*, 60:1101–1118.

Shorack, G. R. and Wellner, J. A. (2009). *Empirical Processes with Applications to Statistics*. SIAM, Philadelphia.

Simard, C. and Rémillard, B. (2015). Forecasting time series with multivariate copulas. *Dependence Modeling*, 3(1):59–82.

Singh, V. P., Sivakumar, B., and Cui, H. (2017). Tsallis entropy theory for modeling in water engineering: A review. *Entropy*, 19(12):641.

Singh, V. P. and Zhang, L. (2018). Copula–entropy theory for multivariate stochastic modeling in water engineering. *Geoscience Letters*, 5(1):1–17.

Sklar, M. (1959). Fonctions de répartition à N dimensions et leurs marges. *Annales de l'ISUP*, 8(3):229–231.

Smitha, S., Kattumannil, S. K., and Sreedevi, E. P. (2023). Dynamic cumulative residual entropy generating function and its properties. *Communications in Statistics-Theory and Methods*, 53(16):5890–5909.

Sparavigna, A. C. (2015). On the role of Tsallis entropy in image processing. *International Scientific Research Journal*, 1(6):16–24.

Sreelakshmi, N. (2018). An introduction to copula-based bivariate reliability concepts. *Communications in Statistics-Theory and Methods*, 47(4):996–1012.

Sriboonchitta, S. and Kreinovich, V. (2018). Why are FGM copulas successful? A simple explanation. *Advances in Fuzzy Systems*, 2018(1):5872195.

Stocker, M., Baffes, J., and Vorisek, D. (2018). With the benefit of hindsight: The impact of the 2014–16 oil price collapse. *Global Economic Prospects–Special Focus*, 1:51–71.

Sunoj, S. M. and Nair, N. U. (2025). Survival copula entropy and dependence in bivariate distributions. *REVSTAT-Statistical Journal*, 23(1):101–115.

Toomaj, A. and Atabay, H. A. (2022). Some new findings on the cumulative residual Tsallis entropy. *Journal of Computational and Applied Mathematics*, 400:113669.

Trivedi, P. K., Zimmer, D. M., et al. (2007). Copula modeling: An introduction for practitioners. *Foundations and Trends® in Econometrics*, 1(1):1–111.

Tsallis, C. (1988). Possible generalization of Boltzmann-Gibbs statistics. *Journal of Statistical Physics*, 52:479–487.

Tsukahara, H. (2005). Semiparametric estimation in copula models. *Canadian Journal of Statistics*, 33(3):357–375.

Úbeda-Flores, M. (2005). Multivariate versions of Blomqvist's beta and Spearman's footrule. *Annals of the Institute of Statistical Mathematics*, 57:781–788.

Ubriaco, M. R. (2009). Entropies based on fractional calculus. *Physics Letters A*, 373(30):2516–2519.

Van der Vaart, A. W. and Wellner, J. A. (1996). *Weak Convergence and Empirical Processes: With Applications to Statistics*. Springer, New York.

Varma, R. S. (1966). Generalizations of Renyi's entropy of order α. *Journal of Mathematical Sciences*, 1(7):34–48.

Wang, W. and Wells, M. T. (2000). Model selection and semiparametric inference for bivariate failure-time data. *Journal of the American Statistical Association*, 95(449):62–72.

Wu, S. (2014). Construction of asymmetric copulas and its application in two-dimensional reliability modelling. *European Journal of Operational Research*, 238(2):476–485.

Xiong, H., Shang, P., and Zhang, Y. (2019). Fractional cumulative residual entropy. *Communications in Nonlinear Science and Numerical Simulation*, 78:104879.

Yang, S.-C. and Nachlas, J. A. (2001). Bivariate reliability and availability modeling. *IEEE Transactions on Reliability*, 50(1):26–35.

Yela, J. P. and Cuevas, J. R. T. (2018). Estimating the Gumbel-Barnett copula parameter of dependence. *Revista Colombiana de Estadística*, 41(1):53–73.

Yongjin, Z., Youchao, S., Longbiao, L., and Ming, Z. (2018). Copula-based reliability analysis for a parallel system with a cold standby. *Communications in Statistics-Theory and Methods*, 47(3):562–582.

Yuan, F. (2018). Parameter estimation for bivariate Weibull distribution using generalized moment method for reliability evaluation. *Quality and Reliability Engineering International*, 34(4):631–640.

Zhang, K., Lin, J., and Huang, C. (2013). Some new results on weighted geometric mean for copulas. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 21(02):277–288.

Zhang, Y. and Lam, J. S. L. (2016). A copula approach in the point estimate method for reliability engineering. *Quality and Reliability Engineering International*, 32(4):1501–1508.

Zhao, N. and Lin, W. T. (2011). A copula entropy approach to correlation measurement at the country level. *Applied Mathematics and Computation*, 218(2):628–642.