GRAVITY AND BLACK HOLES IN STRING THEORY

M.Sc. Project Report

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **Gravity and black holes in string theory** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DISCIPLINE OF PHYSICS**, Indian Institute of Technology Indore, is an authentic record of my own work carried out during the time period from **July 2023** to **May 2025** under the supervision of **Dr.Mritunjay Kumar Verma, Assistant Professor Department of Physics IIT Indore**.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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18-05-2025

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. . . .

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Deepam Sharma has successfully given his M.Sc. Oral Examination held on __**13 May 2025**______.

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ABSTRACT

The aim of this project is to study gravity and black holes in string theory. We investigate the low energy limit of string theory by considering the massless string states and analyze the beta functions which automatically gives us general relativity. Next we analyze tachyon and massless string states in the formalism and compute the beta functions and hence obtain the equation of motion of tachyon and massless field following renormalization group approach.

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Chapter 1

INTRODUCTION

Humans have always tried to understand the universe, we are always curious about the things happening around us and this curiosity has brought us a long way in our civilization. There are some truths of the universe that we try to explain via numbers and mathematics. One such truth is the concept of gravity. The fundamental question is why everything that goes up comes down. Some people tried to explain these observations but the most interesting answers came from Sir Issac Newton in his book *Principia Mathematica*, Newton said that objects having mass attract each other and the force of attraction is proportional to the masses and inversely proportional to the distance square between them. This picture continued for centuries and led the foundation for classical mechanics, however, it did not explain why there must be a force that brings masses together and the biggest problem with Newton's gravity was that it didn't go hand in hand with special relativity which was one of the most prominent theories of the 19^{th} century.

Einstein in the early 19^{th} century formulated a general picture of gravity called *General Relativity*, Einstein said that gravity is not just a force but is fundamentally the curvature of space-time due to mass and energy, this explained everything explained by Newtonian gravity and also en-cooperated special relativity in it. Einstein's picture is a more general picture of gravity that we follow to date. On the other hand, the early 19^{th} century laid the framework for a new and mysterious branch of physics known as quantum mechanics, which revealed that the universe doesn't operate by the straightforward laws of classical mechanics and there is more fundamental theory that describes reality, known as Quantum Mechanics. In the second half of the 19^{th} century, people tried to describe all known physics in a way that was consistent with quantum mechanics, but unfortunately, the theory of gravity didn't go hand in hand with quantum mechanics.

Here comes string theory which promises to unify gravity with string theory and to unify all the physics in the same framework. String theory is a framework that promotes fundamental particles from being point-like (0-Dimensional) to string-like (1-Dimensional), however, the length of fundamental strings is of the order of $l_s \approx 10^{-34}m$ which corresponds to the energy scale of $10^{19}Gev$ which is way beyond physics of fundamental particles are, therefore to recreate the physics of fundamental particles as we understand them today, we focus on massless string states and examine the low-energy effective theory.

This report attempts to regenerate the physics of gravity, given by Einstein, using the quantum version of strings. Initially, we start with a classical string action (Polyakov action) and look at the properties and invariances of the corresponding action, we see that string theory in its classical domain has a very interesting symmetry and that is conformal symmetry. This symmetry governs the physics of string worldsheet to a large extent. Now the next step is to quantize the string, there are various procedures used to derive a quantum theory of string from the classical theory but the most useful one from the point of view of string theory is path integral quantization, and we will employ the same. Unfortunately while looking at the properties of quantum effective action (before actually doing the actual quantization procedure) we find that the quantum theory of strings does not respect one of the most holy classical invariances which is the conformal invariance, this is called the conformal anomaly. The anomaly manifests itself as the expectation value of the stress-energy tensor. To solve this problem we shift our stress energy tensor in such a way that the anomaly vanishes.

Next, we want to study string theory in a nontrivial background, the background we consider is string states, since we are concerned low energy limit of string theory therefore we will only consider massless string states as background. Three massless string states arise from a closed string (Graviton, Kalb–Ramond field, dilaton). We will look at string physics choosing these three fields as a background. Eventually, we will analyze how solving the conformal anomalies that arose from the quantum version of string theory beautifully led us to general relativity equation.

Our next step step would be the study of black holes, black holes are one of the most mysterious objects in the universe, there are a lot of mysteries of black holes that are yet unsolved, one of such mysteries is the informal loss paradox, the information that goes inside the black hole seems to be lost. To solve this paradox, string theory has a lot to offer, we will analyze how string theory can help in solving the mysteries of the universe, again to do this systematically we would require a good understanding of the massive string states, therefore our attempt would be to study the beta function of the massive string states and hence the equation of motion of massive states. To do that we will develop a formalism in which the beta function of any general tensor field could be studied, be it massive or massless, and we will verify the formalism by calculating the beta function of tachyon field and massless field.

Chapter 2

String Actions and invariance

This chapter is focused on introducing classical string action and analyzing the invariances that it possesses, we will see how these invariances gives rise to beautiful properties of stress energy tensor.

2.1 Polyakov Action

The simplest action one can write for a string propagating in D dimensional spacetime is proportional to the area of the world-sheet swept by the propagating string.

$$S_N \propto \int dA$$

The corresponding Action will therefore be (A.1)

$$S = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{(\dot{X}.\dot{X})(X'.X') - (\dot{X}.X')^2} \tag{2.1}$$

The action is famously called Nambu-Goto Action

The factor of $\frac{1}{2\pi\alpha'}$ is a constant to make the action dimensionless, also in general if we consider D dimensions then our $\overrightarrow{X}$ becomes X^μ where $\mu = 0, 1, 2, ...D$ and if we consider D dimensional curved spacetime then $\delta^{\mu\nu}$ in 3D spacetime would be promoted into $g^{\mu\nu}$ so in general

$$\dot{X} = \frac{\partial X}{\partial \tau} \quad X' = \frac{\partial X}{\partial \sigma} \quad \text{and} \quad \dot{X}.X' = g_{\mu\nu}\dot{X}^\mu X'^\nu$$

The coordinates X^μ have the dimension of length, so α' must have the dimension of (length)^2.Usually the length scale depends upon the physics of the theory itself, but since we are dealing with quantum theory of gravity we usually work with plank scale which is of the order of $10^{-34}m$. The Nambu Goto action described is hard to deal with because the equation of motion would involve square root and it is difficult to quantize the action involving square root.Therefore we define an equivalent action called the Polyakov Action.

$$S_P = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} \tag{2.2}$$

Here the $\gamma_{\alpha\beta}$ are the auxiliary fields introduced in our theory. It may seem that the auxiliary fields are the new degree of freedom in our theory, but actually it turns out that they are not. Of course the string action that we have written must not depend upon the parameters that we have chosen, so for that to happen the auxiliary field $\gamma_{\alpha\beta}$ must transform like a second rank tensor. Consider the reparametrization of

$$\sigma^\alpha \longrightarrow \sigma'^\alpha(\sigma)$$

Our theory is reparametrization invariant if [8]

$$X^\mu \longrightarrow X'^\mu = X^\mu \qquad\qquad \gamma_{ab} \longrightarrow \gamma'_{ab} = \frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}\gamma_{cd} \tag{2.3}$$

2.2 Field equation of γ_{ab} and equivalence of actions

To see the field equation of γ_{ab} we need to vary action with respect to γ_{ab} itself, to do that we first note that the variation of determinant of any second rank tensor is given by

$$\delta\gamma = -\gamma\gamma_{ab}\delta\gamma^{ab} \tag{2.4}$$

Now if we vary our action we obtain (see(A.2))

$$\delta S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\gamma}\delta\gamma^{ab} g_{\mu\nu}\left(\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\gamma_{ab}\partial^c X^\mu \partial_c X^\nu\right) \tag{2.5}$$

Now we can extract T_{ab} from the definition

$$\delta S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\gamma} T_{ab}\delta\gamma^{ab} \tag{2.6}$$

$$T_{ab} = g_{\mu\nu}\left(\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\gamma_{ab}\partial^c X^\mu \partial_c X^\nu\right) \tag{2.7}$$

But considering least action principle the variation of the action must be zero for the particle who is following the equation of motion.This puts a constrain on T_{ab} which is $T_{ab} = 0$. Eq. (2.7) puts a constrains on the field X^μ meaning the field cannot take any form but is constrained by the above equation. Also this equation fixes the auxiliary field γ_{ab} and if we put the field in the action then **Polyakov Action** becomes **Nambu-Goto action**. There are problems with both the actions, Nambu-Goto Action has only the variable that we want X^μ's but it is difficult to quantize the action on the other hand Polyakov Action can be quantized easily but for that we need to deal with extra degrees of freedom which can be resolved if the particle follows the equation of motion or to put it other way the energy momentum tensor must vanish.

2.3 Theory Invariances

We know that the constraint equation is given by $T_{ab} = 0$ meaning there must be four independent equations related to it (as interpreted from Eq (2.7)). But since T_{ab} is symmetric so there must be only 3 independent constrain equation. Again in classical theory the trace of Energy Momentum tensor vanishes, but it is not necessarily true in quantum theory [1]. Consider a two dimensional theory with action $A(X, \gamma)$, there are certain invariances that an action must follow, one of such invariance is reparametrization invariance. Consider an infinitesimal reparametrization of worldsheet coordinates as,

$$\sigma'^{\alpha} = \sigma^{\alpha} + v^{\alpha}(\sigma)$$

We know that, for reparametrization invaiance Eq (2.3) must hold true, one can easily verify that under this transformation our theory is reparametrization invariant. Applying the infinitesimal transformation we get

$$\delta X^\mu = v^\alpha \nabla_\alpha X^\mu \tag{2.8}$$

Similarly for the variation of metric is given by (A.3)

$$\delta\gamma_{ab} = -(\nabla_a v_b + \nabla_b v_a) \tag{2.9}$$

Therefore if our theory is reparametrization invariant then X^μ and γ_{ab} must satisfy the above equation under infinitesimal transformation. The total change in the action under such infinitesimal transformation is given by

$$\delta S = \int d^2\sigma \left(\frac{\delta S}{\delta\gamma_{ab}} \delta\gamma_{ab} + \frac{\delta S}{\delta X^\mu} \delta X^\mu \right)$$

When the equation of motion of the fields X^μ is assumed the variation of action will be zero meaning both the terms will be individually become zero. Let us for now focus on the first term. Using Eq (2.9) and Eq (2.6) and integrating by parts we get

$$\delta S = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{\gamma} v^b \nabla^a T_{ab}$$

For the reparametrization invariance of Polyakov action δS vanishes and hence,

$$\nabla^a T_{ab} = 0$$

We can see how reparametrization invariance led us to the fact that $\nabla^a T_{ab} = 0$, therefore it is the reparametrization invariance that imposes divergence-free condition in our classical theory. Our theory is also invariant in another class of transformation and this is something that we do not demand it pops out automatically,the transformation is called *Weyl Transformation*. Weyl transformation is the transformation of the metric, basically it is the local rescaling of the metric conserving angles.

$$\delta\gamma_{ab} = \delta\phi(\sigma)\gamma_{ab}$$

To obtain this kind of transformation consider the transformation of metric as,

$$\gamma'_{ab} = e^{\phi(\sigma)}\gamma_{ab} \qquad\qquad \det(\gamma'_{ab}) = \gamma' = e^{2\phi}$$

Therefore,

$$\sqrt{\gamma'}\gamma'^{ab} = \sqrt{\gamma}\gamma^{ab}$$

which makes our polyakov action invariant with respect to weyl transformation. If our action has this invariance then,

$$\delta S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}T_{ab}\delta\gamma^{ab} = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}T_{ab}\gamma^{ab}\delta\phi(\sigma)$$

For theory invariance $\delta S = 0$ and for that

$$T_{ab}\gamma^{ab} = T^a_a = 0$$

Hence the immediate consequence of weyl invariance is that the trace of the Energy Momentum tensor is zero. But unfortunately we cannot maintain this invariance in quantum theory of strings.

2.4 Conformal Gauge

Let us now coordinate transform to a gauge which is suitable for us. From reparametrization invariance of polyakov action we can choose two parameters (σ, τ) by our convenience keeping the action invariant, therefore we can use that to transform ourselves into a coordinate system in which the metric γ_{ab} has only one independent parameter, a convenient choice would be conformal gauge metric.

$$\gamma_{ab} = e^{\phi(\sigma)}\delta_{ab}$$

If we choose to work with complex variables z in place of real variables ($z = \sigma + i\tau \quad \bar{z} = \sigma - i\tau$). The metric, connection coefficients and the curvature scalar is given by,

$$\gamma^{ab} = \begin{pmatrix} 0 & 2e^{-\phi} \\ 2e^{-\phi} & 0 \end{pmatrix} \qquad \Gamma^{z}_{zz} = \partial_z \phi \quad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \partial_{\bar{z}}\phi \qquad R = -4e^{-\phi}\partial_z\partial_{\bar{z}}\phi \tag{2.10}$$

We will be using conformal gauge for our calculations from now on, for more detailed calculations see (A.4)

Chapter 3

Quantum Theory of Strings

In this chapter we will look at the quantum version of string theory. We will quantize the classical action and we will look at the properties of the quantum effective action starting from a general action $A(X, \gamma)$ after analyzing some properties we will do it for specific actions.[1]

3.1 The problem with Quantum Theory

Now we move on to Quantum mechanical path integral and discuss the effect of the conformal transformation on it. The field integral over the field X's gives the partition function of the system represented as,

$$Z[J] = \int D[X] e^{-A(X,\gamma)+J.X} = e^{-W[J]} \tag{3.1}$$

Here we only integrate over the X's and take the worldsheet metric to be fixed(but arbitrary). The effective action W, depends upon γ only as we have integrated out X's. Since we want our physical quantities to be invariant under reparametrization therefore we demand that $Z[J]$ and hence $W[\gamma]$ to be invariant under reparametrization, for that we want $D[X]$ and $A(X, \gamma)$ to be invariant under reparametrization. We now need an inner product that we can use to measure distances and hence volumes in the function space of X's so that we can carry out our path integral, but we must do it in reparametrization invariant way, to do it in reparametrization invariant way from the point of view of the worldsheet, we would inevitably need the two dimensional metric γ to define the inner product metric. When we work with polyakov action a natural choice of inner product would be

$$|\delta X| = \int d^2\sigma \sqrt{\gamma'(\sigma)} \delta X^\mu \delta X^\nu g_{\mu\nu} \tag{3.2}$$

We are writing γ' to distinguish with the determinant of the conformal gauge metric which we will be denoting by γ for notational consistency. This is because $\int d^2\sigma\sqrt{\gamma'(\sigma)}$ is invariant with a change in parameter σ. For now let us discuss some general properties of Quantum effective action. Since the effective action W depends upon two dimensional metric therefore

$$\delta W = \int d^2\sigma \frac{\delta W}{\delta \gamma'^{ab}}\delta\gamma'^{ab}$$

Reparametrization invariance of W demands us that the variation of quantum effective action with respect to the metric to be zero $\frac{\delta W}{\delta\gamma'^{ab}} = 0$, therefore using Eq (2.9) for the variation of metric under reparametrization in the above equation we get,

$$\int d^2\sigma \frac{\delta W}{\delta\gamma'^{ab}}\left(\nabla^a v^b\right) = 0$$

Performing integration by parts in conformal gauge we get,

$$\nabla_z\left(\frac{1}{\sqrt{\gamma}}\frac{\delta W}{\delta\phi}\right) = \nabla^z\left(\frac{1}{\sqrt{\gamma}}\frac{\delta W}{\delta\gamma^{zz}}\right) \qquad \nabla_{\bar{z}}\left(\frac{1}{\sqrt{\gamma}}\frac{\delta W}{\delta\phi}\right) = \nabla^{\bar{z}}\left(\frac{1}{\sqrt{\gamma}}\frac{\delta W}{\delta\gamma^{\bar{z}\bar{z}}}\right) \quad (3.3)$$

This equation is the classical analogue of the conservation of the stress energy tensor the only difference is that now the action is the new effective action W. Also, the variation of the effective action W with the two-dimensional action plays the role of the stress energy / energy momentum tensor. We can argue that the right-hand side of the above equation is the z-th derivative of the quantum expectation value of the zz component of the energy momentum tensor. The partition function Eq (3.1) depends on γ in two ways through the path integral measure $X's$ and trough the classical action $A(X,\gamma)$. The variation of the classical action reduces the factor of the energy momentum tensor to the path integral,which, when divided by Z, gives us the expectation value.

$$\frac{4\pi}{\sqrt{\gamma}}\frac{\delta W}{\delta\gamma^{ab}} = \langle T_{ab}\rangle \quad (3.4)$$

But the question is that if any unwanted term arise from the variation of the path integral measure. From equation Eq (3.2) we can say that the path integral measure only depends upon the determinant of the two dimensional metric. Variation of the determinant of a matric is given by,

$$\delta\gamma = -\gamma\gamma_{ab}\delta\gamma^{ab}$$
$$= -(\gamma\gamma_{zz}\delta\gamma^{zz} + \gamma\gamma_{z\bar{z}}\delta\gamma^{z\bar{z}} + \gamma\gamma_{\bar{z}z}\delta\gamma^{\bar{z}z} + \gamma\gamma_{\bar{z}\bar{z}}\delta\gamma^{\bar{z}\bar{z}})$$

But since γ_{zz} and $\gamma_{\bar{z}\bar{z}}$ are zero in the conformal gauge, the first-order variation of the determinant with respect to γ_{zz} and $\gamma_{\bar{z}\bar{z}}$ is zero. Therefore, the path integral measure is invariant under variation and hence no extra term arises due to the variation of the path integral measure hence we can write,

$$\nabla^z \left(\frac{4\pi}{\sqrt{\gamma}} \frac{\delta W}{\delta \gamma^{zz}} \right) = \nabla^z \langle T_{zz} \rangle \tag{3.5}$$

Now let us analyze the significance of the right-hand side of equation (3.3), the right hand side can be explicitly written as,

$$-\frac{1}{2} \left(\frac{\gamma^{\bar{z}z}}{Z} \frac{\delta Z}{\delta \gamma^{\bar{z}z}} + \frac{\gamma^{\bar{z}z}}{Z} \frac{\delta Z}{\delta \gamma^{\bar{z}z}} \right) = \frac{\delta W}{\delta \phi}$$

Hence with the similar argument as before that the variation of the partition function brings down the energy momentum tensor and the division with the partition function gives us the expectation value, but now the metric γ^{ab} in front gives us the trace of the energy momentum tensor. As we have seen before that the trace of the Energy-Momentum tensor vanishes during conformal transformation, but Eq (3.3) suggests that the variation of the effective action with respect to ϕ is not in general zero telling us that the conformal transformations are in general anomalous in 2-D field theories.Also using Eq (3.3) we can get the form of the anomalous form of the effective action itself. Similarly the left hand side of the Eq (3.3) can be interpreted as ∇_z of something we assume that it is local in the worldsheet. Since it is local in the worldsheet therefore we need to construct something that is made up of γ^{ab}. The right hand side is a tensor of t_z even under conformal reparametrization and is of scaling dimension one. The only local function of γ_{ab} that has these conformal properties is the zth derivative of scalar curvature R. The assumption of locality and with dimensional analysis we can therefore tell that the left hand side of the Eq (3.3) must be of the form, [4]

$$\nabla_z \left(\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta \phi} \right) = \frac{\lambda}{48\pi} \nabla_z R \tag{3.6}$$

We cannot determine the constant of proportionality λ by general arguments as it is the characteristics of the theory. Integrating both sides with respect to z we get

$$\frac{\delta W}{\delta \phi} = \frac{\lambda}{48\pi} \sqrt{\gamma} \left(R + \mu^2 \right)$$

μ^2 is the constant of integration. During a conformal transformation $\gamma_{ab} = e^{\phi} \tilde{\gamma}_{ab}$ and in conformal gauge the $\tilde{\gamma}_{ab}$ is chosen to be flat metric. We can see that the equation is satisfied by the form of W given below (See (B.2))

$$W = \frac{\lambda}{48\pi}\int d^2\sigma\sqrt{\tilde{\gamma}}\left(\frac{1}{2}\tilde{\gamma}^{ab}\partial_a\phi\partial_b\phi + \mu^2 e^{\phi}\right) \tag{3.7}$$

The assumption that the anomaly is local has enabled us to characterize the conformally non invariant part of the quantum effective action by one dimensionless parameter λ and a dimensional parameter μ. The term in W that depends upon ϕ is called *Liouville action*.

3.2 Operator Product Expansion And Virasoro algebra

From (3.3) , (3.6) and (3.5) we can write,

$$\frac{\lambda}{48\pi}\nabla_z R = \nabla^z \langle T_{zz}\rangle$$

Since the partial derivative of $\langle T_{zz}\rangle$ with respect to $\bar{z}$ is non zero (due to weyl anomaly) so the vacuum expectation value of T_{zz} is not analytic. But it is possible to make the stress energy tensor analytic, by improving stress energy tensor. In conformal gauge the derivative of curvature tensor becomes (See (B.3))

$$\nabla_z R = \nabla^z(-2\partial_z\partial_z\phi + \partial_z\phi\partial_z\phi) \tag{3.8}$$

Therefore,

$$\frac{\lambda}{48\pi}\nabla^z(-2\partial_z\partial_z\phi + \partial_z\phi\partial_z\phi) = \nabla^z \langle T_{zz}\rangle$$

If we redefine the zz component of stress energy tensor as,

$$T^0_{zz} = T_{zz} + \frac{\lambda}{48\pi}\left(2\partial_z^2\phi - (\partial\phi)^2\right) \tag{3.9}$$

Here T^0_{zz} is called the improved stress energy tensor. We first note that the above equation is valid for a specific metric, the equation holds for any kind of metric, and we can do variation with respect to metric The second variation of the W with respect to γ gives us the expectation value of two stress energy tensors. Also we would have the variation of the covariant derivative and the Ricci scalar which would give us other terms. Say that the variation with respect to γ_{zz} is being done at a point w which is different from which the original $T_{zz}(z)$ was evaluated, then the two point function $\langle T^0_{zz}T^0_{ww}\rangle$ turns out to be [7]

$$\left\langle T^0_{zz}T^0_{ww}\right\rangle = \frac{\lambda}{2}\frac{1}{(z-w)^4} + \frac{\langle T^0_{ww}\rangle}{(z-w)^2} + \frac{\partial_w \langle T^0_{ww}\rangle}{(z-w)} + \text{regular terms..} \tag{3.10}$$

The above expression looks similar to the OPE of the Energy-Momentum tensor in conformal field theory,therefore it can be used to generate virasoro algebra, to do that we can Laurent expand $T^0(z)$.Appendix (B.4)

$$T(z) = \sum_{-\infty}^{+\infty} z^{-n-2}L_n \qquad\qquad L_n = \frac{1}{2\pi i}\oint dz z^{n+1}T_{zz}(z) \tag{3.11}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\lambda}{12}(m^3-m)\delta_{m,-n} \tag{3.12}$$

Therefore it turns out that the operators L_n satisfies the Virasoro Algebra.Similarly the another independent component of stress energy tensor $T_{\bar{z}\bar{z}}$ (which can be defined using another set of equations involving the variation of w with $\gamma_{\bar{z}\bar{z}}$) can be defined with another set of operators $\tilde{L}_n$ also satisfies virasoro algebra among themselves but commute with L_n. Due to conformal anomalies the trace in general of the stress energy tensor fails to become zero, in spite of that if the anomaly has a local form which can be characterized by a number λ the expectation value of the independent component of energy momentum tensor behaves like analytic objects. The trace in general which was non vanishing as given by Eq (3.3) and its interpretations but if we shift our stress energy tensor to the form given by Eq (3.9) then the $\nabla^z \langle T^0_{zz}\rangle$ vanishes which means vanishing of the trace of the stress energy tensor as given by Eq (3.3) and its interpretation of trace and expectation value of stress energy tensor. The remaining components of the energy momentum tensor T_{zz} and $T_{\bar{z}\bar{z}}$ vanishing (Coming from reparametrization invariance which is still holy in quantum theory) gives us other remaining constraint equation, but if it is so L_n and $\tilde{L}_n$ must be zero as interpreted from Eq (3.11) quantum mechanically it means that the expectation value of the operators must be zero, it therefore means that L_n and $\tilde{L}_n$ acting on any state must annihilates the state. But the algebra does not allows us to do so

$$\left\langle\psi\middle|[L_m, L_n]\middle|\psi\right\rangle = (m-n)\left\langle\psi\middle|L_{m+n}\middle|\psi\right\rangle + \frac{\lambda}{12}(n(n^2-1))\delta_{m,-n}\left\langle\psi\middle|\psi\right\rangle$$

$$0 = \frac{\lambda}{12}(n(n^2-1))\delta_{m,-n}$$

The left hand side of the equation is zero for any value of m and n but the right hand side cannot be satisfied for any value of $morn$ therefore we have to demand the condition that,

$$\hat{L}_m|\psi\rangle = 0 = \hat{\tilde{L}}_m|\psi\rangle \quad \text{only for} \quad m \geq 1$$

Therefore we say that the physical states are those states which are annihilated by L_m and $\tilde{L}_m$ with $m > 0$, therefore the expectation value of all Virasoro generators (except L_0 and $\tilde{L}_0$) for all the physical states are zero, therefore the expectation value of the analytic and the anti analytic component of the energy momentum tensor is zero. Hence even if there are anomalies they do not prove to be complete disaster and there are other things to compensate its effect and in all the cases we come up with will keep the physics of 2-D conformal field theory intact.

3.3 Renormalizable sigma model

In this section we will be looking how the string couples with other strings and will typically looking at the interaction they possesses, also we will be looking at how string couples with the background, since most of the string states are massive and the masses of these states are of the order of 10^{19} Gev corresponding to plank length $l_p \approx 10^{-34}$m therefore in the low energy limit only massless string states would be present, if we consider closed strings there are three massless string states Graviton (described by spacetime field $g_{\mu\nu}$), Kalb-Ramond field (described by an antisymmetric tensor $B_{\mu\nu}$), Dilaton (described by dilaton field Φ). Therefore studying low energy limit of string theory would be equivalent to studying the behavior of string in these background. The polyakov action is power counting renormalizable since the constant α' has the dimension of $[L]^2$ and therefore $\frac{1}{\alpha'}$ which acts as a coupling constant would be of dimension $[M]^2$, also the action is weyl invariant and reparametrization invariant, therefore the terms that we add in our action must have all these properties.

3.3.1 Antisymmetric Action

We can add a term in the polyakov action which has the properties as described above, one of such terms is given by Eq (3.13), $B_{\mu\nu}$ is antisymmetric in spacetime indices.

$$S_{AS} = \frac{1}{4\pi\alpha'} \int d^2\sigma \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu} \tag{3.13}$$

Where ϵ^{ab} is two dimensional Levi-Ci-vita tensor. Since ϵ^{ab} is a tensor density which will work in the same way as the factor $\sqrt{\gamma}$ for making the theory reparametrization invariant, the ϵ^{ab} is used in place of $\sqrt{\gamma}\gamma^{ab}$ because of the antisymmeteric nature of $B_{\mu\nu}$. We can think $B_{\mu\nu}$ as analogous to gauge potential A_μ in electromagnetism,and the action S_{AS} tells us how string is electrically charged under the influence of the background field $B_{\mu\nu}$, See (B.5). Similarly to the one form A_μ the two form $B_{\mu\nu}$ the action is invariant under the transformation,

$$B_{\mu\nu} \longrightarrow B_{\mu\nu} + (\partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu) \tag{3.14}$$

Similar to $F_{\mu\nu}$ in electromagnetism we can construct $H_{\mu\nu\rho}$ by anti symmetrizing it and using the antisymmetric property of $B_{\mu\nu}$

$$H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \tag{3.15}$$

3.3.2 Dilaton Action

There is another reparametrization invariant terms that we can add in our theory,

$$S_D = \frac{1}{4\pi} \int d^2\sigma \sqrt{\gamma} R \Phi(X) \tag{3.16}$$

This action vanishes in flat worldsheet metric (since $R = 0$ in flat worldsheet metric), second the action that we have written is not weyl invariant. Since we want our classical theory to be weyl invariant, therefore we want S_D to have non trivial contributions only in higher orders of the perturbation theory as compared to the other terms. We want the tree level weyl variation caused by this term to cancel out the one loop weyl anomalies which arises from other terms and so on. Since, $\Phi(X)$ is dimensionless in worldsheet point of view therefore we do not need α' to make action dimensionless. Since α' is loop counting parameter of our theory and higher order of α' means higher loop diagrams. Therefore this suggests that S_D first contributes on the one loop level as compared to classical level. There are no other reparametrization invariant terms of dimension two that we can add to our action. In bosonic string theory the coupling function $G_{\mu\nu}, B_{\mu\nu}, \Phi(X)$ corresponds to vacuum expectation value of modes of string of graviton, antisymmetric tensor and dilaton.

3.4 Background expansion And Riemann Normal Coordinates

We consider two dimensional field theory with classical actions which included three actions described before,

$$A[X, \gamma] = S_P + S_{AS} + S_D$$

This approach facilitates the study of string theory in non-trivial background fields. Since the quantities, $G_{\mu\nu}(X), B_{\mu\nu}(X)$ and $\Phi(X)$ transforms covariantly under general spacetime coordinate transformation, also as we have seen before S_{AS} is invariant to gauge transformation of the $B_{\mu\nu}$. Therefore it is important that the pertubative calculations are performed in a way that explicitly represents the symmetries of the spacetime. We can achieve this using a trick called *covariant background field expansion*. In this technique we separate the fields into Classical Background part and Quantum Part.

$$X^\mu(\sigma) = X_0^\mu(\sigma) + \pi^\mu(\sigma)$$

And we shift the path integral to be over the quantum fields π^μ only, the background field lives in the 2-Dimensional worldsheet and not in the D-dimensional spacetime, doing so we define the background field generating functional as,

$$\Omega[X_0, \gamma] = \int D[\pi] e^{-A[X_0+\pi]-A[X_0]-\int d^2\sigma \frac{\delta A}{\delta X_0^\mu(\sigma)} \pi^\mu(\sigma)}$$

The next step is to expand the classical action in terms of the quantum field π^μ, and derive the Feynman rules for the diagrams. $\Omega[X_0, \gamma]$ can be viewed as a generating functional for "loop diagrams" with all external legs amputated. To obtain the explicit form of the $\Omega[X_0, \gamma]$ we can expand $A[X_0+\pi, \gamma]$ around π the background field expansion will eventually led us to a well defined perturbation theory, and would give us correct results but since the quantum field π^μ is defined as a coordinate difference in spacetime coordinate and therefore does not transform as a vector in general (Transformation properties of π^μ is restricted by that of X^μ and X_0^μ). Therefore we need to do a little work before we proceed further. We need to replace π^μ with an integration variable that transforms like a vector under general spacetime coordinate transformation. The plan is to expand $\pi^\mu(\sigma)$ as a power series of η^μ which transforms covariantly under reparametrization of the worldsheet coordinates. The tangent vector to the geodesic $\lambda^\mu(t)$ that connects X_0^μ and $X_0^\mu + \pi^\mu$ would contain all the information of the coordinate difference and also transforms like a vector, so tangent vector would do the job perfectly.(See (B.6))

3.5 Covariant Expansion of Effective Action

Knowing Eq (B.14) we can expand various terms to do the pertubative calculations. Let us start by expanding the spacetime metric $g_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$, and the dilaton field Φ, then we will expand the corresponding actions.Expanding the polyakov action (See (B.8),(B.7))

$$S_P(X_0^\mu + \pi^\mu) = S_P(X_0) + \frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\gamma^{ab}g_{\mu\nu}(X_0)\partial_a X_0^\mu \nabla_b \eta^\nu$$
$$+ \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\gamma^{ab}\left(g_{\mu\nu}\nabla_a\eta^\mu\nabla_b\eta^\nu + R_{\mu\alpha\beta\nu}(X_0)\partial_a X_0^\mu \partial_b X_0^\nu \eta^\alpha\eta^\beta\right)$$
$$+ \frac{1}{3\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\gamma^{ab}R_{\mu\alpha\beta\nu}(X_0)\partial_a X_0^\mu \nabla_b\eta^\nu\eta^\alpha\eta^\beta$$
$$+ \frac{1}{12\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\gamma^{ab}R_{\mu\alpha\beta\nu}(X_0)\nabla_a\eta^\mu\nabla_b\eta^\nu\eta^\alpha\eta^\beta. \tag{3.17}$$

Similarly, we want to expand the antisymmetric action S_{AS} in terms of η^μ(B.9),

$$S_{AS}(X_0^\mu + \pi^\mu) = S_{AS}(X_0) + \int d^2\sigma\epsilon^{ab}\frac{1}{2\pi\alpha'}\left[B_{\mu\nu}\partial_a X_0^\mu\nabla_b\eta^\nu + \frac{1}{2}\nabla_\alpha B_{\mu\nu}\partial_a\partial_b\eta^\alpha\right]$$
$$+ \int d^2\sigma\epsilon^{ab}\frac{1}{4\pi\alpha'}\left[B_{\mu\nu}\nabla_a\eta^\mu\nabla_b\eta^\mu + 2\nabla_\alpha B_{\mu\nu}\partial_a X_0^\mu\nabla_b\eta^\nu\eta^\alpha\right]$$
$$+ \frac{1}{2}\left[\nabla_\alpha\nabla_\beta B_{\mu\nu} + B_{\mu\rho}R^\rho_{\alpha\beta\nu} + B_{\rho\nu}R^\rho_{\alpha\beta\mu}\right]\partial_a X_0^\mu\partial_b X_0^\nu\eta^\alpha\eta^\beta \tag{3.18}$$

Now let us also expand the dilaton action given by Eq (3.16), since the dilaton coupling function $\Phi(X)$ is scalar in spacetime, keeping this in mind we expand Eq (B.11) for a scalar,

$$\Phi(X_0+\eta) = \Phi(X_0) + \nabla_{\mu_1}\Phi(X_0)\eta^{\mu_1} + \frac{1}{2}\nabla_{\mu_1}\nabla_{\mu_2}\Phi(X_0)\eta^{\mu_1}\eta^{\mu_2} \tag{3.19}$$

The dilation action Eq (3.16) can now be expanded using Eq (3.19),

$$S_D(X_0^\mu + \pi^\mu) = \frac{1}{4\pi}\int d^2\sigma\sqrt{\gamma}R\Phi(X_0) + \frac{1}{4\pi}\int d^2\sigma\sqrt{\gamma}R\nabla_\alpha\Phi(X_0)\eta^\alpha$$
$$+\frac{1}{4\pi}\int d^2\sigma\sqrt{\gamma}R\nabla_\alpha\nabla_\beta\Phi(X_0)\eta^\alpha\eta^\beta \tag{3.20}$$

The term which involves two covariant derivative of quantum field η^μ in Eq (3.17) is the kinetic term of the theory, and the propagator which is derived from the kinetic term will be non standard as it involves the spacetime metric $g_{\mu\nu}$, which is one of the coupling function of our theory. To solve this issue we define an n-bein (also called vielbein) e^i_μ which is like a matrix which relates the vectors in curved space η^μ to the local flat Lorentz frames therefore dealing with the curved space becomes equivalent to dealing with the local flatness, which makes the kinetic term diagonal in η^i coordinate system. Thus the $\eta^i$and the covariant derivative of η^i is defined as,

$$\eta^i = e^i_\mu \eta^\mu \qquad \nabla_a \eta^i = \partial_a \eta^i + \omega^{ij}_\mu \partial_a X^\mu_0 \eta^j$$

Therefore,

$$g_{\mu\nu} \nabla_a \eta^\mu \nabla_b \eta^\nu = \delta_{ij} \nabla_a \eta^i \nabla_b \eta^j = \nabla_a \eta^i \nabla_b \eta^i \tag{3.21}$$

Our theory remains invariant under local Lorentz transformations which is $SO(D-1,1)$ symmetry, this symmetry is an internal symmetry as it doesn't directly act on spacetime X^μ_0 itself but acts on the internal degrees of freedom (η^i). The field $A^{ij}_\mu = \omega^{ij}_\mu(X_0)\partial_a X^\mu_0$ transforms like a Yang-Mill gauge potential under local Lorentz transformation. However we need to break the gauge symmetry in order to define a propagator because without breaking (or fixing) a gauge there are redundant degrees of freedom of the fields η^i which makes the propagators for the field ill defined. Although we break the gauge symmetry to ensure a well defined propagator we maintain enough gauge covariance to ensure that the theory behaves consistently under gauge transformations of background fields, it makes our calculations simpler because if we demand that the background fields are gauge covariant then the diagrams involving gauge potential must combine is such a way that the resulting expression is gauge covariant the combination that doesn't respect this will give vanishing result, this is because the background field is gauge covariant and the fluctuations combining with it must also be gauge covariant. Therefore it is a good idea to work with η^i local Lorentz frames as it simplifies the propagator and also makes our calculations simpler. To do so we need to change variables in the integral and integrate over η^i, since the path integral measure is defined in reparametrization invariant way therefore the change in variables would not effect the path integral measure therefore when we change $\pi^\mu \longrightarrow \eta^i$ doesn't effect it. Before proceeding further we modify the form of Eq (3.18) and write its quadratic term in terms of the antisymmetric tensor field strength $H_{\mu\nu\alpha}$ as defined in Eq (3.15), we want to do so because the gauge invariance Eq (3.14) the physics of antisymmetric field only depends on its field strength, therefore any term except $H_{\mu\nu\alpha}$'s would vanish.

$$\frac{1}{4\pi\alpha'} \int d^2\sigma \epsilon^{ab} \left[H_{\mu\alpha\beta}(X_0)\partial_a X^\mu_0 \nabla_b \eta^\alpha \eta^\beta + \frac{1}{2} \nabla_\alpha H_{\mu\nu\beta}(X_0) \partial_a X^\mu_0 \partial_b X^\nu_0 \eta^\alpha \eta^\beta \right] \tag{3.22}$$

3.6 One Loop Calculation of Weyl Anomaly

We will calculate one loop weyl anomaly of the sigma model using the formalism that we have developed earlier called the covariant background field expansion, after all to calculate weyl anomaly we already say that we need to calculate the variation of metric with the scaling factor ϕ of the theory. Using Eq (3.3) and Combining this with Eq (3.4) the conservation equation takes beautiful form

$$\nabla^{\bar{z}} \langle T_{\bar{z}z} \rangle + \nabla^{z} \langle T_{zz} \rangle = 0 \tag{3.23}$$

If we assume that $\langle T_{zz} \rangle$ is finite and well defined and somehow if we could find it then Eq (3.23) could be used to find $\langle T_{\bar{z}z} \rangle$ which precisely gives us the weyl anomaly of the sigma model. We begin to do our one loop calculations, for simplicity we consider flat worldsheet metric and the curvature of worldsheet will be taken care later. It is convenient to work with light cone coordinates and momentum space for our calculations. In momentum space Eq (3.24) takes the form,

$$q_+ \langle T_{-+} \rangle + q_- \langle T_{++} \rangle = 0 \tag{3.24}$$

3.7 Contribution from S_P

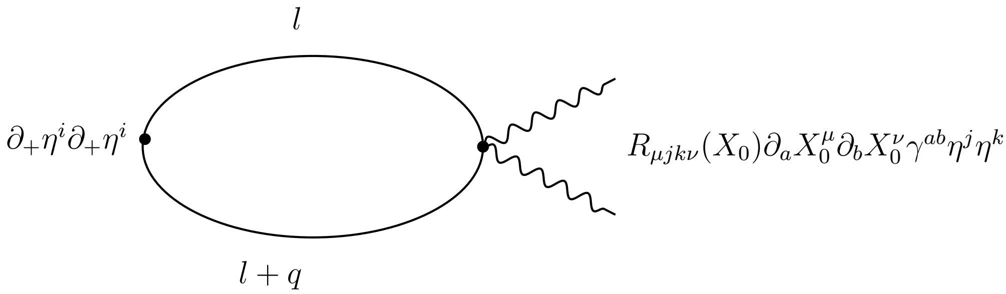


Figure 3.1: Feynman diagram representing the contribution to T_{++}

First we want to compute the contribution to $\langle T_{++} \rangle$ that comes from S_P. The $\partial_+\eta^i\partial_+\eta^i$ comes from an insertion of T_{++}, of course we would expect other terms also since stress energy tensor is the variation of the action with respect to the metric therefore it would have remaining terms coming from $g_{\mu\nu}\nabla_a\eta^\mu\nabla_b\eta^\nu = \nabla_a\eta^i\nabla_b\eta^i$ and the terms coming from the Riemann tensor see Eq (3.17), however the remaining terms from $\nabla_a\eta^i\nabla_b\eta^i$ the $SO(D-1,1)$ invariant gauge potential does not contribute to insertion because the because it does not give us gauge covariant terms and the second term in the insertion would be thrown away because two scalar curvature one from the insertion and the other from the T_{++} in the left and other from S_P on the right can be neglected for small quantum fluctuations. We insert T_{++} on the left with a momentum q and the momentum is carried away by the background fields represented by double lines in the right, propagated by the propagator of η^i. The only interaction between X_0 and η^i is given by the term $R_{\mu jk\nu}(X_0)\partial_a X_0^\mu \partial_b X_0^\nu \gamma^{ab}\eta^j\eta^k$. The contribution of the diagram would be given by

$$\frac{1}{2\pi}\int d^2l\frac{l_+(l_++q_+)}{l^2(l+q)^2}\left[R_{\mu\nu}\partial_aX_0^\mu\partial^aX_0^\nu\right](q) \tag{3.25}$$

Solving it and putting it in Eq (3.24)

$$\langle T_{-+}\rangle = -\frac{q_-}{q_+}\langle T_{++}\rangle = \frac{1}{4}R_{\mu\nu}(X_0)\partial_aX_0^\mu\partial^aX_0^\nu \tag{3.26}$$

This implies that the trace of the stress energy tensor is non zero in quantum theory even though we start with a conformally invariant action classically and the anomaly depends upon the curvature of spacetime, our goal now is to remove the conformal anomaly by adjusting the spacetime metric $g_{\mu\nu}(X_0)$ in such a way that the Ricci tensor $R_{\mu\nu}(X_0)$ vanishes therefore making the trace of stress energy tensor zero. But before doing that we need to find the full form of anomaly because this anomaly is only because of the polyakov action. The anomaly has the power of α^0 in the front of it, this is because the insertion of Energy-Momentum tensor and the interaction term gives a factor of $\frac{1}{\alpha'}$ in the front and the propagator gives a factor of α' there are two propagators therefore we have a factor of α^0 in the front. This suggests α' to be a loop counting parameter as in tree level T_{ab} has a factor of $\frac{1}{\alpha'}$ in the front.

3.8 Contributions from S_{AS}

We will have contributions from S_{AS} also in the conformal anomaly, S_{AS} will contribute two diagrams.

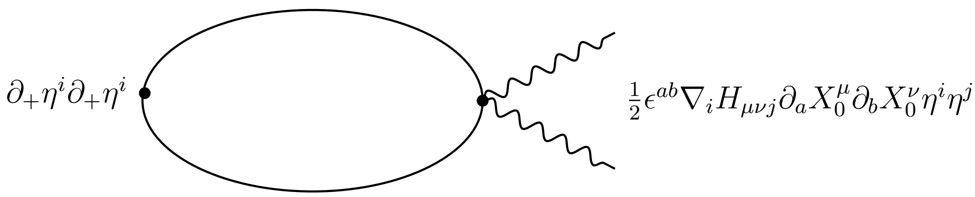


Figure 3.2: First contribution of S_{AS} in one loop anomaly.

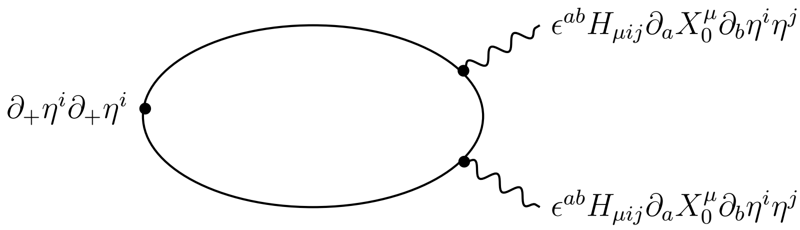


Figure 3.3: Second contribution of S_{AS} in one loop anomaly.

The contribution of the first diagram Fig:(3.2) is similar to Eq (3.25) and the calculations are same therefore we can directly write the result as [1],

$$\langle T_{-+} \rangle = \frac{1}{8} \nabla^{\lambda} H_{\lambda\mu\nu}(X_0) \partial_a X_0^{\mu} \partial_b X_0^{\nu} \epsilon^{ab} \tag{3.27}$$

Similarly the contribution from second diagram Fig:(3.3) is given by[1],

$$\langle T_{-+} \rangle = -\frac{1}{16} H_{\mu\lambda\sigma}(X_0) H_{\nu}^{\lambda\sigma}(X_0) \partial_a X_0^{\mu} \partial^a X_0^{\nu} \tag{3.28}$$

3.9 Contribution from Dilaton Coupling

We have included the contribution from S_P and S_{AS}, now its the turn of S_D. Initially we calculated the contributions due spacetime background fields on a flat worldsheet, but the dilaton action involves R which is Ricci Scalar on 2-D worldsheet , we can still calculate the contribution of S_D to weyl anomaly without going in the curved worldsheet, even though dilaton coupling itself vanishes in the flat worldsheet limit, but the variation of it with respect to metric will not, meaning that even though we have added dilaton coupling in curved worldsheet still the stress energy tensor on a flat metric would change due to dilaton coupling. We can find that the shift in the stress energy tensor is of the form (B.11),

$$T^d_{ab} = (\partial_a \partial_b - \delta_{ab} \Box) \, \Phi(X) \tag{3.29}$$

The off diagonal component of stress energy tensor,

$$T^d_{-+} = \Box \Phi(X) \tag{3.30}$$

We want the tree level non vanishing part of the trace of the stress energy tensor will cancel one loop level weyl anomalies arising from Polyakov action and antisymmetric action. We simply need to calculate the classical trace corresponding to Eq (3.30) which cancels the weyl anomaly arising from one loop contribution of Polyakov action and antisymmetric action.

$$\Box \Phi(X_0) = \Box X_0^{\mu} \, \partial_{\mu} \Phi(X_0) + \partial_a X_0^{\mu} \, \partial^a X_0^{\nu} \, \partial_{\mu} \partial_{\nu} \Phi(X_0). \tag{3.31}$$

The terms are not covariant from the point of view of the spacetime. Now we rewrite this expression using the classical equation of motion of X_0^{μ}, the classical equation of motion is therefore,

$$\Box' X_0^{\mu} = \Gamma^{\mu}_{\alpha\beta} \partial_a X_0^{\alpha} \partial^a X_0^{\beta} - \frac{1}{2} H^{\mu}_{\alpha\beta} \partial_a X_0^{\alpha} \partial^a X_0^{\beta} \epsilon^{ab} \tag{3.32}$$

Here $\Box'$ denotes d'Alembertian operator in spacetime indices. Plugging

Eq (3.32) into Eq (3.31) we get spacetime covariant results.

$$\Box\Phi = \nabla'_\mu \nabla'_\nu \Phi(X_0)\partial_a X_0^\mu \partial^a X_0^\nu - \frac{1}{2}\nabla'^\lambda \Phi(X_0) H_{\lambda\mu\nu}(X_0)\partial_a X_0^\mu \partial_b X_0^\nu \epsilon^{ab} \tag{3.33}$$

Therefore the trace of the full energy momentum tensor is the sum of all the three contributions,

$$\langle T_{-+}\rangle = \frac{1}{4}\beta^G_{\mu\nu}\partial_a X_0^\mu \partial_b X_0^\nu \sqrt{\gamma}\gamma^{ab} + \frac{1}{4}\beta^H_{\mu\nu}\partial_a X_0^\mu \partial_b X_0^\nu \epsilon^{ab} \tag{3.34}$$

where

$$\beta^G_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}H^2_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi \tag{3.35}$$

$$\beta^H_{\mu\nu} = \frac{1}{2}\nabla^\lambda H_{\lambda\mu\nu} - \nabla^\lambda \Phi H_{\lambda\mu\nu} \tag{3.36}$$

The only way $\langle T_{-+}\rangle$ can vanish is when $\beta^G_{\mu\nu}$ and $\beta^H_{\mu\nu}$ individually vanish. This will ensure that there is no trace anomaly in the quantum theory. Hence, we set $\beta^G_{\mu\nu} = 0 = \beta^H_{\mu\nu}$. The first condition can be written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{4}\left[H^2_{\mu\nu} - \frac{1}{6}g_{\mu\nu}H^2\right] - 2\nabla_\mu \nabla_\nu \Phi + 2g_{\mu\nu}\nabla^2\Phi \tag{3.37}$$

Which is the standard Einstein's equation of General Relativity. The right hand side of the above equation is sourced by the dilaton and Kalb Ramond fields. If we just consider the background of gravitons, the above equation would reduce to the Einstein's equation in the vacuum. The above derivation shows that string can only propagate in those spacetime which satisfy Einstein's equation.

3.10 The complete story

Now we need to generalize the above concept to calculate the trace of stress energy tensor in the curved metric, as we have evaluated it earlier in the flat worldsheet metric. It turns out that we can get a missing piece of the T_{-+} which comes by considering any general curved worldsheet by simply calculating the two point function of the stress energy tensor in the flat worldsheet. Since we have used the reparametrization invariance property of the theory to fix the metric to the form

$$\gamma^{ab} = e^\phi \delta_{ab}$$

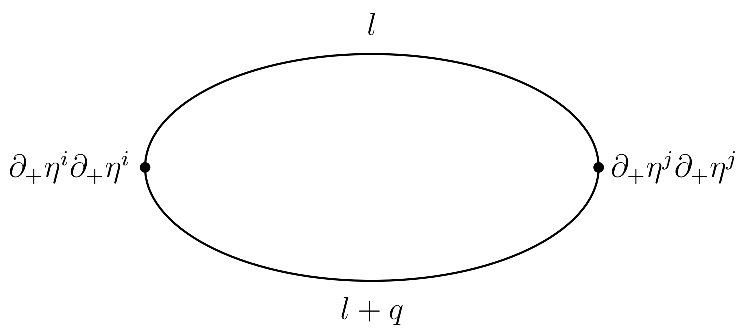


Figure 3.4: Feynman diagram for dilaton coupling

The trace of the stress energy tensor must vanish regardless of the scaling factor ϕ. Expanding $\langle T_{-+}(\phi)\rangle$ we get

$$\langle T_{-+}(\phi)\rangle = \langle T_{-+}(0)\rangle + \left.\frac{\delta\langle T_{-+}\rangle}{\delta\phi}\right|_{\phi=0}\phi + \frac{1}{2!}\left.\frac{\delta^2\langle T_{-+}\rangle}{\delta\phi^2}\right|_{\phi=0}\phi^2 + \cdots$$

The minimum requirement so that the overall trace of the stress energy tensor is zero in any arbitrary metric is that the first variation of the stress energy tensor must be zero therefore,

$$\left.\frac{\delta}{\delta\phi(\sigma)}\langle T_{-+}(0)\rangle_{e^{\phi}\delta_{ab}}\right|_{\phi=0} = 0 \tag{3.38}$$

Evaluating the above equation on the flat worldsheet we obtain,

$$\left.\frac{\delta}{\delta\phi(\sigma)}\langle T_{-+}(0)\rangle_{e^{\phi}\delta_{ab}}\right|_{\phi=0} = -\frac{1}{4\pi}\left.\langle T_{-+}(\sigma)T_{-+}(0)\rangle\right|_{\delta_{ab}} \tag{3.39}$$

The two point function in Eq (3.39) must vanish at the classical level if the theory never had weyl anomaly but since our theory has weyl anomaly therefore it will contribute at classical level to cancel the one loop anomaly generated from other action contribution. At the classical level the only non vanishing two point function of the stress energy tensor is $\langle T_{++}T_{++}\rangle$ and $\langle T_{--}T_{--}\rangle$. Therefore only evaluating one diagram is enough, which is given in Fig (3.10) The contributions of the diagram is given by,

$$\langle T_{++}(q)T_{++}(-q)\rangle = 2D\int d^2l\frac{l_+^2(l_++q_+)^2}{l^2(l+q)^2} \tag{3.40}$$

The integral could be solved and is given by

$$\langle T_{++}(q)T_{++}(-q)\rangle = -\frac{\pi D}{6}\frac{q_+^3}{q_-} \tag{3.41}$$

Now we can use the conservation equation to obtain,

$$\langle T_{-+}(q)T_{-+}(-q)\rangle = -\frac{\pi D}{6}q_+q_- \tag{3.42}$$

Since the two point function given by Eq (3.42) is non zero therefore anomaly is present in our theory. If we transform back to the coordinate space we will find that the product q_+q_- is the D'Alembertian operator. Therefore if we take the Fourier transform of the Eq (3.42) then we would obtain

$$\langle T_{-+}(\sigma)T_{-+}(0)\rangle = \frac{\pi D}{12}\Box\delta^2(\sigma) \tag{3.43}$$

$$\langle T_{-+}\rangle\bigg|_{e^{\phi}\delta_{ab}} = -\frac{D}{48}\Box\phi \tag{3.44}$$

Therefore,

$$\langle T_{-+}\rangle_{e^{\phi}\delta_{ab}} = \frac{D}{24}\sqrt{\gamma}^{(2)}R \tag{3.45}$$

This contribution to the trace anomaly is a little bit different from that of the Polyakov and the antisymmetric contribution. It is proportional to the 2-Dimensional scalar curvature of the worldsheet which looks similar to the dilaton coupling. We write this in terms of

$$\langle T_{-+}\rangle_{e^{\phi}\delta_{ab}} = -\beta_\Phi\left(\frac{1}{4}\sqrt{\gamma}^{(2)}R\right)$$

Where $\beta_\Phi = \frac{D}{6}$ only depends upon the spacetime dimension. So far we have obtained the one loop diagram considering the curvature of worldsheet which is of the order of α'^0. If we calculate the two loop diagram we would obtain β_Φ up to the order of α'. At the two loop level the coupling function like $H_{\mu\nu\rho}$ $g_{\mu\nu}$ and Φ comes into picture. If we go to the two loop the interaction vertices would now be given by S_P and S_{AS}. The calculations are much more involved therefore we avoid to do this at this stage, however the momentum structure of the diagrams would turn out to be the same i.e. q_+^3/q_-, but now the coefficients depends upon the spacetime coupling functions. We will find terms involving R and H^2 after doing the loop integral and making the use of the conservation of stress energy tensor given by Eq (3.23). Also at the order of α' the dilaton action would also have some contribution because while considering the two point function of the stress energy tensor we considered it to be classically weyl invariant, but there is an explicit contribution of the dilaton term given by Eq (3.30). Since the Dilaton action has an extra power of α' in the front therefore the stress energy tensor derived from the dilaton action would also have an extra power of α' in the front therefore the dilaton contribution in first order in α' can be written in two ways

1. Tree level two point function of T^d_{-+} with itself
2. One loop diagram with one insertion of T_{-+} and other T^d_{-+}

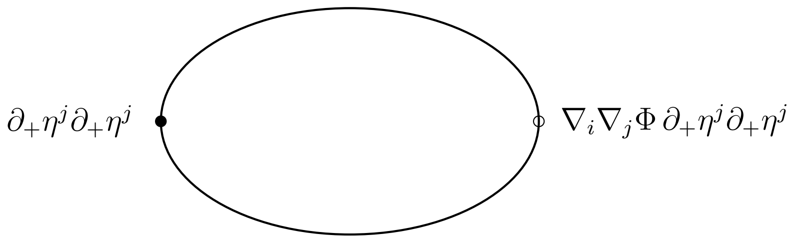


Figure 3.5: One-loop diagram giving dilaton contribution to β^Φ.

The only one loop diagram that is relevant for β_Φ is Fig: (3.5). The left hand side is obviously the insertion of the T_{++} coming from the S_P and S_{AS} and the right hand side is the dilation's contribution of the stress energy tensor arising from the expansion of the dilaton action given by Eq (3.20). Following the similar calculations to obtain Eq (3.42) the contribution of this diagram is given by,

$$\langle T_{++}(q)T_{++}(-q)\rangle = \alpha'\nabla^2\Phi\frac{\pi q_+^3}{2q_-}$$

Therefore,

$$\left\langle T_{-+}(\sigma)T^d_{-+}(0)\right\rangle = -\pi\alpha'\nabla^2\Phi\Box\delta^2(\sigma) \tag{3.46}$$

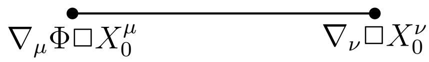


Figure 3.6: Tree diagram contribution for β_Φ

The other diagram contributing to β_Φ is the one loop diagram which is given by Fig: (3.6). The propagator cancels one of the D'Alembertian and contributes a factor of α'. We convert the D'Alembertian as the momenta in the momentum space, also the ∇_μ is the derivative with respect to spacetime fields therefore the tree level contribution is given by,

$$\nabla_\nu\Phi\nabla_\mu\Phi\ (q_+q_-)(q_+q_-)(-2\pi\alpha')\frac{g^{\mu\nu}}{q^2}$$

$$= -\pi\alpha'(\nabla\Phi)^2q_+q_-$$

Re converting into the position space we finally obtain

$$\left\langle T^d_{-+}(\sigma)T^d_{-+}(0)\right\rangle = \pi\alpha'(\nabla\Phi)^2\Box\delta^2(\sigma) \tag{3.47}$$

Combining all the pieces together we would finally obtain the β_Φ till the order of α'

$$\beta_\Phi = \frac{D}{6} + \frac{\alpha'}{2}\Big[-R + \frac{H^2}{12} + 4(\nabla\Phi)^2 - 4\nabla^2\Phi\Big] \tag{3.48}$$

we have obtained β_Φ to the order of α' but β^G and β^H are evaluated up to order α', we can however add them because the coefficients which are the coupling functions of the theory have been calculated till same order. In 2-D sigma models there are only three independent structures of dimension two are present and β_Φ , β^G and β^H are the objects that multiply these structure.

3.11 Consistency of Weyl Anomaly Conditions

Since we want weyl anomaly to vanish therefore we want $\beta^G_{\mu\nu}$, $\beta^H_{\mu\nu}$ and β_Φ to vanish. The first term in the Eq (3.48) is non zero even in flat spacetime because it only depends upon the spacetime dimensions D and since string theory is defined in 26 Dimensional flat spacetime therefore we want the fist term of β_Φ to vanish somehow. It is canceled if we take into account conformal ghost fields, which comes into picture when we fix the $2-D$ metric. They contribute a constant term to β_Φ which is given by $-\frac{26}{6}$ which exactly removes the constant term in the Eq (3.48) if we choose our spacetime dimensions to be 26. Now we removed the constant term from the weyl anomaly coefficients, and we want to remove the other terms also, we want a specific configuration of spacetime field $g_{\mu\nu}$, the antisymmetric field $H_{\mu\nu}$ and dilaton field Φ so that all the terms vanishes.

Now if we rearrange the weyl anomaly coefficients given by Eq (3.36) Eq (3.35) and Eq (3.48) after setting them to zero (for weyl anomaly to vanish in 2-D curved worldsheet) we get

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{4}[H^2_{\mu\nu} - \frac{1}{6}g_{\mu\nu}H^2] + 2g_{\mu\nu}(\nabla\Phi)^2 \tag{3.49}$$

$$\nabla^\lambda H_{\lambda\mu\nu} = 2\nabla^\lambda\Phi H_{\lambda\mu\nu} \tag{3.50}$$

$$\nabla^2\Phi - 2(\nabla\Phi)^2 = -\frac{1}{12}H^2 \tag{3.51}$$

Eq (3.49) is a familiar equation which is the Einstein's equation of gravity, implying that if we want the weyl anomaly to vanish then $g_{\mu\nu}$ must have certain restrictions meaning only those values of $g_{\mu\nu}$ are allowed which satisfies Einstein's General relativity equation. The other two equation are similarly restrictions of $B_{\mu\nu}$ and Φ field or putting other way these are the equations of motion of $B_{\mu\nu}$ and Φ fields. The spacetime stress-energy tensor is given by the right hand side of Eq (3.49)

$$\Theta_{\mu\nu} = \frac{1}{4}[H^2_{\mu\nu} - \frac{1}{6}g_{\mu\nu}H^2] + 2g_{\mu\nu}(\nabla\Phi)^2 \tag{3.52}$$

Of course it is a symmetric tensor and since the L.H.S of Eq (3.49) is conserved therefore the right hand side must also be conserved. For general $H_{\mu\nu\sigma}$ and Φ , $\Theta_{\mu\nu}$ given by Eq (3.52) has no reason to be conserved but since the L.H.S of the Eq (3.49) is conserved $\nabla^\mu(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = 0$ therefore $\nabla^\mu\Theta_{\mu\nu} = 0$. We can define a D-Dimensional covariant action whose variation gives us the equation of motion given by Eq (3.49) Eq (3.50) and Eq (3.51)

$$S_D = \int d^DX\sqrt{g}e^{-2\Phi}\left[R + 4(\nabla\Phi)^2 - \frac{1}{12}H^2\right] \tag{3.53}$$

We can do a weyl transformation on the spacetime metric and write Eq (3.53) in a more standard form to obtain (See Appendix (B.13))

$$S_D = \int d^DX\sqrt{\tilde{g}}\left[\tilde{R} - \frac{4}{D-2}(\nabla\tilde{\Phi})^2 - \frac{1}{12}e^{-\frac{8\Phi}{D-2}}\tilde{H}^2\right] \tag{3.54}$$

Looking at the Eq (3.54), the first term reminds of the Einstein-Hilbert action followed by the second term which looks like the kinetic term for the Dilaton field, and finally the third term is similar to the maxwell like kinetic term for antisymmetric B field with the coupling constant that depends to the dilaton field.

We finally derived an action whose variation with respect to the fields $g_{\mu\nu}, B_{\mu\nu}$ and Φ gives rise to the weyl anomaly condition. Of course all the work done here is up to the order of α', but the question is are the weyl anomaly condition true for all order in α', we exactly don't know the complete answer but it has been consistent till three loops, and based on that it is generally believed that there exists a master action which can give rise to the weyl anomaly coefficients in the power series of α'. Of course the action depends upon these spacetime function and their higher derivatives and the higher order terms gives rise to the short distance corrections to the Einstein's equation and the equations of motion of the dilaton and the antisymmetric fields.

Chapter 4

Black Holes and String Theory

4.1 Black Hole information Loss Paradox

Analyzing quantum mechanics of gravitational fields can be excellently done using black holes. While quantizing the standard G.R action one runs into a problem, the problem is that the theory is non-renormalizable, beyond the one loop level, to consider the matter fields in general we need to include the one-loop corrections. Another complication with the theory is that non rotating and uncharged black holes with mass M emit radiation beyond the temperature

$$T = \frac{1}{8\pi GM}$$

This effect is called Hawking effect. Any quantum mechanical object or a quantum state that goes inside the black hole cannot escape it. The information about the object is inaccessible inside the event horizon, therefore we can say that for an observer point of view who is outside the event horizon of the black hole a pure quantum mechanical state evolves into a mixed state. This itself is not a paradox because the observer is outside the black hole and he/she do not choose to obtain all the information about the original quantum state but in principle observer can go inside the black hole to obtain all the information that is required to reconstruct the original quantum state. However if an observer chooses to do so it eventually reaches a spacetime singularity which is guaranteed inside the event horizon of the black hole by the virtue of the singularity theorems, which is satisfied if

$$R_{ab} l^a l^b \geq 0$$

Where l^a are null or time-like vectors and

$$R_{ab} = 8\pi \left(T_{ab} - \frac{1}{2} T g_{ab} \right)$$

These conditions are satisfied in general relativity provided String energy condition is satisfied meaning that condition that the matter must gravitate towards matter and the cosmological constant is either zero or negative.

As the black hole radiates the black hole will loose its mass and as a result the temperature of black hole also increases, which leads to black hole loosing more mass because the energy of radiation $E \propto T^4$ this process continues till the black hole completely evaporates or due to some phenomenon the radiation turns off. If the black hole evaporates the information inside the black hole is lost, therefore the incoming pure state has actually been converted into a mixed state, meaning that the remaining information that went inside the black hole in order to reconstruct the initial state cannot be obtained (not even in principle). But according to quantum mechanics a pure state cannot evolve into a mixed state, this is because evolution of a state is described by a unitary operator (Time evolution operator must be unitary for probability to be conserved). Unitary evolution is deterministic and reversible $Tr(\rho^2) = 1$, meaning that we must be able to reconstruct the original quantum state from the evolved one. Therefore the evolution of a pure state into mixed state is not possible due to loss of information as mixed represents the lack of the complete information $Tr(\rho^2) < 1$. This leads to something called Black hole information loss paradox, which arises because of information loss as purely thermal radiation does not contain any information.

A possible scenario could be that the black hole stops radiating the radiation after certain time, for that the relation between the mass of the black hole and the Hawking temperature must be modified. One of such situations could arise when the temperatures reaches its maximum and eventually decreases to zero as the mass decreases stopping the evaporation. The remanent object still may have an event horizon and the necessary information in order to reconstruct the original state could be hidden inside the event horizon.

Another possibility could be that the radiation coming out of black hole is not thermal radiation, but such a radiation that could potentially carry the initial state of the system and it appears to be thermal in the certain limits in which we study these radiations. The key difference between black hole and any other ordinary physical system emitting thermal radiation for example sun is that the quantum state of the system (sun) is accessible and is not destroyed meaning even though the radiation coming out of the system (sun) is purely thermal but the information about the sun can be further retraced by observation of the system (sun) therefore leading to no paradoxes, but in contrast black holes gets eventually evaporated and this leads to the loss of information contained inside black holes. Small black holes makes the paradox even worse, the entropy of a black hole is related to the surface area of black hole's event horizon by Bekenstein-Hawking formula

$$S = \frac{k_b A}{4 l_p^2}$$

Where A is the area of event horizon of black holes and l_p is plank length. This formula implies that the bigger black holes have more entropy and since the entropy is related to the no of micro states therefore the size of black holes is proportion to the no of its internal states. The problem with the smaller black holes is that due to its small size at the later part of evaporation it may have very less or negligible internal states to take into account, therefore it is very hard to believe that such black holes have sufficient internal states to match with the emitted radiation.

4.1.1 How String Theory can Help

String theory has a remarkable feature of having a maximum temperature called the Hagedorn temperature, which can potentially solve the issue of information loss paradox. The mass of an open string in bosonic string theory is given by,

$$\alpha' M^2 = \left(\sum_{n=1}^{\infty} n a_n^{i\dagger} a_n^i - 1 \right)$$

The degeneracy of the n^{th} level can be calculated as,

$$\rho(n) \approx \frac{1}{\sqrt{2}} n^{-27/4} e^{4\pi\sqrt{n}}$$

Therefore the partition function is defined in the units of $\hbar = c = k_B = 1$

$$Z = \sum_{n=1}^{\infty} \rho(n) e^{-M/T}$$
$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{2}} n^{-27/4} e^{4\pi\sqrt{n} - \frac{M}{T}}$$

As the temperature increases $\frac{M}{T}$ decreases and eventually we will reach a temperature $T_{\max}$ at which the factor $4\pi\sqrt{n} - \frac{E}{T}$ becomes positive, meaning that the factor $\frac{E}{T}$ fails to bound $\rho(n)$

$$T_{\max} = \frac{M}{4\pi\sqrt{n}}$$

Therefore,

$$T_{\max} = \frac{1}{4\pi\sqrt{\alpha'}}$$

This suggests that in string theory we expect black holes (which can be characterized by massive string states) to have a maximum temperature.

4.2 String Theory and Black Holes

While studying the low energy limit of string theory using sigma model approach we have neglected the massive string modes and worked only with the massless modes of strings. The leading order equations of motion of dilaton and graviton are

$$R_{ab} + 2\nabla_a \nabla_b \Phi + \lambda R_{acde} R_b^{cde} = 0 \tag{4.1}$$

$$\Box\Phi - (\nabla\Phi)^2 + \frac{1}{4}R + \frac{1}{8}\lambda R_{abcd} R^{abcd} = 0 \tag{4.2}$$

Where $\lambda = \frac{1}{2}\alpha', 0$ for bosonic and supersymmeteric string theory. The action that can generate these equations of motion is given by,

$$\frac{1}{16\pi G}\int d^D X \sqrt{-g} e^{-2\phi}\left(R + 4(\nabla\Phi)^2 + \frac{1}{2}\lambda R_{abcd} R^{abcd}\right) \tag{4.3}$$

Here G is the newton's constant in D dimensions. Our main goal would be to examine the black hole solutions of Eq (4.3). When the curvature of the spacetime is small as compared to the string scale $\frac{1}{\alpha'}$ solutions of string theory will approximate the Einstein's equations, but when the curvature of spacetime is larger then the we cannot only consider the leading order approximation in sigma models and therefore the higher order terms in α' will also contribute making string solutions deviating much from Einstein's equation. Since Eq (4.3) is written based upon the beta functions which is derived up to the order of α' therefore it is not the full action but just the leading order approximation of the full action which can be fully written as an infinite series in α', therefore it cannot fully promise to give a detailed information that how string theory could resolve the singularities that arises in string theory, however since Eq (4.3) yields to a field equation, which does not necessarily satisfy time like convergence condition (potentially due to higher order term like $R_{abcd}R^{abcd}$), therefore singularities are not guaranteed in these theories. In case of Black holes the regions of finite curvature must be considered to study the event horizon and singularities (with infinite curvature) are avoided for the analysis if we consider sufficiently large black holes for which the gravitational length scale Gm is much larger than the string scale α'(α' is of the order of l_p^2) $(Gm)^2 >> \alpha'$ then the curvature of the event horizon would be small this is because the curvature near the event horizon is related to the length scale by $R \propto \frac{1}{(Gm)^2}$, therefore the terms like $R_{abcd}R^{abcd}$ could be considered as small perturbations of the Einstein's equation.

We will begin with a constant dilaton field and Black hole solution of Einstein's equation as a background fields and then solve Eq (4.1), by the pertubative expansion of metric in λ, but Eq (4.2) tells us that the curvature of the black hole acts like the source of the dilaton field, therefore we must expand the dilaton field also perturbatively. Also we would set $B_{\mu\nu}$ to consider spherically symmetric nature of black hole and torsion free condition. We consider a 4-D static Black hole, since the dimension of spacetime in string theory is 26, but we observe universe as a $4-D$ spacetime therefore the remaining dimensions are compactified so that from our length scale only 4 dimensions are visible (Consider for an example of a hair it looks like $1-D$ when observed via human eyes but if we observe it using a microscope it becomes a $3-D$ object). Therefore we consider the metric in string theory to be a direct product of $4-D$ black hole metric and the metric corresponding to compact internal spacetime.

$$g_{AB}(x,y) = \begin{pmatrix} g_{ab}(x) & 0 \\ 0 & \tilde{g}_{\tilde{a}\tilde{b}(y)} \end{pmatrix}$$

Similarly we can do it for dilaton

$$\Phi(x,y) = \phi(x) + \tilde{\phi}(y)$$

Where x and y represents internal and external coordinates. Now if we use these equations in Eq (4.1) we obtain

$$\begin{pmatrix} R_{ab}(x) & 0 \\ 0 & \tilde{R}_{ab}(y) \end{pmatrix} + 2\begin{pmatrix} g_{ab}(x)\nabla_i\nabla^i\big(\phi(x)+\tilde{\phi}(y)\big) & 0 \\ 0 & \tilde{g}_{ab}(y)\nabla_j\nabla^j\big(\phi(x)+\tilde{\phi}(y)\big) \end{pmatrix}$$
$$+\begin{pmatrix} \lambda R_{acde}R_a^{cde}(x) & 0 \\ 0 & \lambda\tilde{R}_{acde}\tilde{R}_a^{cde} \end{pmatrix} = 0$$

Where i runs to external coordinates and j to internal coordinates, therefore

$$\begin{pmatrix} R_{ab}(x) & 0 \\ 0 & \tilde{R}_{ab}(y) \end{pmatrix} + 2\begin{pmatrix} \nabla_a\nabla_b\phi & 0 \\ 0 & \tilde{\nabla}_a\tilde{\nabla}_b\tilde{\phi} \end{pmatrix} + \begin{pmatrix} \lambda R_{acde}R_a^{cde}(x) & 0 \\ 0 & \lambda\tilde{R}_{acde}\tilde{R}_a^{cde} \end{pmatrix} = 0$$

Therefore,

$$R_{ab} + 2\nabla_a\nabla_b\phi + \lambda R_{acde}R_b^{cde} = 0 = \beta^g_{ab}(g,\phi) \tag{4.4}$$

$$\tilde{R}_{ab} + 2\tilde{\nabla}_a\tilde{\nabla}_b\tilde{\phi} + \lambda\tilde{R}_{acde}\tilde{R}_b^{cde} = 0 = \tilde{g}_{\tilde{a}\tilde{b}}(\tilde{g},\tilde{\phi}) \tag{4.5}$$

Similarly the Eq (4.2) could be written as,

$$\Box\phi - (\nabla\phi)^2 + \frac{1}{4}R + \frac{1}{8}\lambda R_{abcd}R^{abcd} + \Box\tilde{\phi} - (\nabla\tilde{\phi})^2 + \frac{1}{4}\tilde{R} + \frac{1}{8}\lambda\tilde{R}_{abcd}\tilde{R}^{abcd} = 0$$

$$\beta_(g,\phi) = -\tilde{\beta}^{\phi}(\tilde{g},\tilde{\phi}) = 0 \tag{4.6}$$

We will assume that the internal metric and the dilaton field satisfy their own equations of motion independently, therefore we are free to study the black hole space. This decoupling can be done in any order in α' and the higher order contribution will not interfere the 4-D solutions.

We begin with a static spherically symmetric metric in four dimensions.

$$ds^2 = -f^2dt^2 + g^2dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{4.7}$$

Therefore the metric is given by,

$$\begin{pmatrix} -f^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2sin^2\theta \end{pmatrix} \tag{4.8}$$

Since it is static which implies two conditions one that the metric components are independent of time and other there are no time space cross terms which is guaranteed by time reversal invariance of dt^2 but not of the cross terms, therefore static black hole implies that the coefficients f and g are not dependent on t also it is spherically symmetric they also wouldn't have angular and azimuthal dependence. Therefore first order expansion of f, g and ϕ must look like

$$\begin{aligned} f &= f_0(1 + \lambda\mu(r)) \\ g &= g_0(1 + \lambda\epsilon(r)) \\ \phi &= \phi_0 + \lambda\varphi(r) \end{aligned} \tag{4.9}$$

Where ϕ is not azimuthal angle but is the component of dilaton field depending on the 4-D coordinates. Now we can use Einstein's Equation to evaluate the zeroth order components of the metric, the procedure is to evaluate the Christopher's symbol and therefore calculate the Riemann and Ricci tensor and therefore set the Ricci tensor to zero. The calculations are exactly similar to the Schwarzwald metric and the coefficients f_0 and g_0 if given by

$$f_0^2 = g_0^{-2} = 1 - \frac{2Gm}{r} \tag{4.10}$$

Since we want the spacetime to be flat at larger distances from the black hole therefore we choose the boundary conditions

$$\mu, \epsilon, \varphi(r) \longrightarrow 0 \quad \text{when} \quad r \longrightarrow \infty$$

The perturbed solutions must be such that the event horizon ($r = 2Gm$) remain non-singular in the unperturbed space-time, of course it seems that the metric described by Eq (4.7) with Eq (4.9) is singular for the unperturbed spacetime but it is not the case as the metric is coordinate dependent and therefore we can transform in a coordinate in which $r = 2Gm$ does not led to singularities(in our coordinate system we choose the event horizon to be at $r = 2Gm$), one way to test it is by using scalars (which are coordinate independent) like $R = g^{\mu\nu}R_{\mu\nu}$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, $R_{\mu\nu}R^{\mu\nu}$, if these scalars blows up at certain points it is definitely a singular point. Explicit calculations shows that none of these scalars blows up ar $r = 2Gm$ but there is definitely a singularity at $r = 0$ as

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \propto \frac{1}{r^6}$$

Using the metric of the form Eq (4.8) we can calculate all the necessary things like the Riemann tensor, Ricci tensor and finally when we plug it in Eq (4.1) we get few non trivial components of Eq (4.1) as,

$$\begin{aligned}
(tt) \quad &: \quad \frac{r-2Gm}{r}\mu'' + \frac{2r-Gm}{r^2}\mu' - \frac{Gm}{r^2}\epsilon' - \frac{2Gm}{r^2}\varphi' - \frac{12(Gm)^2}{r^6} = 0 \\
(rr) \quad &: \quad \frac{r-2Gm}{r}(\mu'' - 2\varphi'') + \frac{3Gm}{r^2}\mu' - \frac{2r-3Gm}{r^2}\epsilon' - \frac{2Gm}{r^2}\varphi' - \frac{12(Gm)^2}{r^6} = 0 \\
(\theta\theta) = (\phi\phi) \quad &: \quad \frac{r-2Gm}{r^2}(2\varphi' + \epsilon' - \mu') + \frac{2}{r^2}\epsilon + \frac{12(Gm)^2}{r^6} = 0
\end{aligned} \tag{4.11}$$

Now if we take the trace of Eq (4.1) and plug it in Eq (4.2) we would obtain,

$$\begin{aligned}
g^{ab}R_{ab} + 2g^{ab}\nabla_a\nabla_b\Phi + \lambda g^{ab}R_{acde}R_b^{cde} = 0 \\
R = -2\Box\Phi - \lambda R_{abcd}R^{abcd}
\end{aligned} \tag{4.12}$$

Plugging R in Eq (4.2) we obtain

$$\Box\Phi - (\nabla\Phi)^2 - \frac{1}{4}(2\Box\Phi + \lambda R_{abcd}R^{abcd}) + \frac{1}{8}\lambda R_{abcd}R^{abcd} = 0$$

$$\Box\Phi - 2(\nabla\Phi)^2 - \frac{1}{4}\lambda R_{abcd}R^{abcd} = 0 \tag{4.13}$$

Solving this to the order of λ we would obtain

$$\frac{r-2Gm}{r}\varphi'' + \frac{2r-2Gm}{r^2}\varphi - \frac{12(Gm)^2}{r^6} \tag{4.14}$$

Solving this equation we obtain

$$\varphi = -\frac{2\,Gm}{3\,r^3} - \frac{1}{2\,r^2} - \frac{1}{2\,Gm\,r} + C_1 + \frac{(1-2GmC_2)\log(r)}{4\,Gm^2} + \frac{(-1+2GmC_2)\log(-2Gm+r)}{4\,Gm^2}$$

But since $\varphi \longrightarrow 0$ as $r \longrightarrow \infty$ and at $r = 2Gm$ φ is finite therefore $C_1 = 0$ and C_2 must be such that $1 - 2GmC_2 = 0$, therefore

$$\varphi(r) = -\frac{2\,Gm}{3\,r^3} - \frac{1}{2\,r^2} - \frac{1}{2\,Gm\,r} \tag{4.15}$$

Therefore the first and the second derivative of φ with respect to r is given by

$$\varphi' = \frac{2Gm}{r^4} + \frac{1}{r^3} + \frac{1}{2Gmr^2} \tag{4.16}$$

$$\varphi'' = -\frac{-8Gm}{r^5} - \frac{3}{r^4} - \frac{1}{Gmr^3} \tag{4.17}$$

Subtracting first two equations of Eq (4.11) we obtain

$$2\frac{r-2Gm}{r}\varphi'' + \left(\frac{2r-Gm}{r^2} - \frac{3Gm}{r^2}\right)\mu' - \left(\frac{Gm}{r^2} - \frac{2r-3Gm}{r^2}\right)\epsilon' = 0$$

$$2\frac{r-2Gm}{r}\varphi'' + 2\frac{r-2Gm}{r^2}\mu' - 2\frac{r-2Gm}{r^2}\epsilon' = 0$$

One of the solutions of the above equation is,

$$r\varphi'' + \mu' - \epsilon' = 0$$

$$r\varphi'' = \epsilon' - \mu'$$

Integrating the above equation we obtain

$$\mu = -\epsilon - r\varphi + \varphi \tag{4.18}$$

Now if we plug it in Third equation of Eq (4.11) we obtain,

$$\epsilon' + \frac{1}{r-2Gm}\epsilon = -\varphi' - \frac{1}{2}r\varphi'' - \frac{6(Gm)^2}{r^4(r-2Gm)} \tag{4.19}$$

Plugging in the expression for φ from Eq (4.15) we obtain

$$\epsilon' + \frac{1}{r - 2Gm}\epsilon = \frac{8\,(Gm)^2 + 12\,G^2\,m^2 - 2\,Gm\,r - r^2}{4\,Gm\,r^4 - 2\,r^5}$$

Solving this differential equation with respect to the boundary condition as mentioned above would give us

$$\epsilon = -\frac{5Gm}{3r^3} - \frac{7}{12r^2} - \frac{1}{24Gmr} \tag{4.20}$$

Plugging the value of Eq (4.20) and Eq (4.15) in Eq (4.18) we obtain

$$\frac{-24Gm + \frac{r^2}{Gm} + 2r\left(-11 - \frac{12r}{Gm}\right)}{24r^3}$$

$$\mu(r) = -\frac{Gm}{r^3} - \frac{23}{24Gm\,r} - \frac{11}{24r^2} \tag{4.21}$$

Consider the tt component of the metric given by Eq (4.8)

$$g_{tt} = -f^2 = \left(\frac{2Gm}{r} - 1\right)\left[(1 + \lambda\mu(r))\right]^2$$

For $r \longrightarrow \infty$ $\mu \longrightarrow 0$, therefore for the regions far away from the black hole we can write,

$$g_{tt} = \left(\frac{2Gm}{r} - 1\right)(1 + 2\lambda\mu(r)) = \frac{2Gm}{r} - 2\lambda\mu(r) - 1$$

Plugging the value of $\mu(r)$ from Eq (4.21) we obtain

$$g_{tt} = -1 + \frac{2GM_G}{r} \tag{4.22}$$

Where

$$M_G = m\left(1 + \frac{23}{6}\frac{\lambda}{(2Gm)^2}\right) \tag{4.23}$$

The mass of a black holes in our pertubative expansion has a λ correction and the mass M_G is called the gravitational mass that would be felt by a point like test particle which is following the geodesic in space. Similarly the rr component of the metric is given by

$$g_{rr} = \left(1 - \frac{2Gm}{r}\right)^{-1}(1 + \lambda\epsilon(r))^2$$

Similarly considering the regions far away from the black hole

$$= \left(1 + \frac{2Gm}{r}\right)(1 + 2\lambda\epsilon(r))$$

$$g_{rr} \approx 1 + \frac{2Gm}{r} + 2\lambda\epsilon = 1 + \frac{2GM_I}{r}$$

Where

$$M_I = m + \frac{r\lambda\epsilon}{G}$$

Plugging in the value of $\epsilon(r)$ and retaining only till $\frac{1}{r}$ we obtain,

$$M_I = m\left(1 - \frac{1}{6}\frac{\lambda}{(2Gm)^2}\right) \tag{4.24}$$

The quantity M_I is called the Inertial mass. One point to be noted before proceeding further is that the gravitational mass is greater than the mass of black hole $M_G \geq m$ and the inertial mass is lesser than the mass of the black hole $M_I \leq m$

4.3 Temperature of Black Holes

We now rotate out time component to imaginary time to define something called euclidean time $t \longrightarrow i\tau$.

After rotation to euclidean time the manifold we would obtain would be smooth near the horizon, if the imaginary time is periodic $\tau = \tau + \beta$. The periodicity of the imaginary time will then help us find the temperature of the black hole. To study the system in thermal equilibrium with a bath of temperature T we generally draw correspondence between statistical physics and quantum field theory, one of such correspondence is given by

$$it = \tau \longrightarrow \frac{1}{T}$$

Therefore the periodicity in imaginary time β. The periodicity in our case would be

$$\beta = \frac{2\pi}{\kappa}$$

Where κ is the surface gravity of the horizon and for our case β is given by,

$$\beta = 8\pi Gm\left(1 + \lambda(\epsilon - \mu)\big|_{r=2Gm}\right)$$

Therefore the temperature is given by

$$T = \frac{1}{\beta} = \frac{1}{8\pi Gm}\left(\frac{1}{(1 + \lambda(\epsilon - \mu)}\right) = \frac{1}{8\pi Gm}\left(1 - \lambda(\epsilon - \mu)\right)$$

Plugging in the value of ϵ and μ as given by Eq (4.20) and Eq (4.21) we would obtain

$$T = \frac{1}{8\pi Gm}\left[1 - \frac{11}{6}\frac{\lambda}{(2Gm)^2}\right] \tag{4.25}$$

We can express T in terms of gravitational and inertial masses, by rearranging Eq (4.23) and Eq (4.24)

$$m = \frac{M_G}{\left(1 + \frac{23}{6}\frac{\lambda}{(2Gm)^2}\right)}$$

Plugging it in Eq (4.25) we obtain,

$$T = \frac{1}{8\pi GM_G}\left[1 + \frac{23}{6}\frac{\lambda}{(2Gm)^2}\right]\left[1 - \frac{11}{6}\frac{\lambda}{(2Gm)^2}\right]$$

$$T \approx \frac{1}{8\pi GM_G}\left[1 + 2\frac{\lambda}{(2GM_G)^2}\right] \tag{4.26}$$

By similar procedure we can calculate T in terms of inertial mass

$$T \approx \frac{1}{8\pi M_I}\left[1 - 2\frac{\lambda}{(2GM_I)^2}\right] \tag{4.27}$$

This means that the temperature of black hole is lower in string theory as compared temperature arising from Einstein's equation with the same inertial mass, but with the gravitational mass the temperature arising from string theory is higher.

More systematic study of black holes requires the study of massive string states, therefore we first analyze the massive string states and calculate their beta functions and hence their equation of motions, which will help us get a deep understanding about the black hole dynamics. In the next chapter we will study how we can calculate the beta functions of any string state be ti massive or massless, we will verify it for tachyons and massless fields and then eventually continue studying black holes after analyzing the massive state beta functions.

Chapter 5

Including Tachyon and Massive Fields

5.1 Why The Picture Is Not Complete

The propagation of strings in the background can be described in various ways, one of the standard approach is the non-linear sigma model in which we describe the string propagation in the presence of some background fields which are nothing but modes of string vibrations, for the non linear sigma model approach to be consistent the theory must have been weyl invariant. Classically the theory was weyl invariant, however as soon as we tried to quantize the theory weyl anomaly started appearing on the theory, therefore we computed the beta functions of the string background namely dilaton, graviton and anti-symmetric field (Kalb-Ramond field) and demanded that these beta functions must be set to zero for the weyl anomaly to vanish which gave us the equation of motion of the background fields. Till now we did not bother much about the renormalization of the theory because the theory was renormalizable up-to any finite order in loop expansion but new ultraviolet divergences comes into the picture if we try to sum up the contributions of all the loop order. To remove these divergences we would require the counter-terms involving infinite number of tensor fields, which can be thought of the vacuum expectation value of all the modes of string vibration, however this means that our analysis of low energy effective field theory is incomplete without in-cooperating the infinite tensor fields.

In this section we try to get a systematic way in order to get the beta function when we include arbitrary background fields (which may include the infinite tensor fields), the beta functions we would obtain would be calculated non perturbatively using the weak field expansion around a flat background.

5.2 β functions

Consider a closed string propagating in the flat spacetime background, the worldsheet is also taken to be flat with the choice of coordinate system to be the light cone coordinate system, a typical string action would look like in the above setup

$$S_0 = \frac{1}{2\pi\alpha'} \int d^2\sigma \; \partial_+ X^\mu \partial_- X_\mu \tag{5.1}$$

Now we add an interaction action which governs the interaction of the string and involves the tensor fields.

$$S = S_0 + S_{int}$$

$$S_{int} = \frac{1}{2\pi\alpha'} \int d^2\sigma \; \left[\frac{\alpha'}{\epsilon^2} \Phi(X) + \partial_+ X^\mu \partial_- X^\nu A_{\mu\nu}(X) + \dots \right] \tag{5.2}$$

The $\Phi(X)$ here denotes the tachyon field $A_{\mu\nu}$ denotes massless two index tensor field and $\dots$ denotes other tensor fields. The factor of $\frac{1}{\alpha'}$ in the first term is present because we want to regularize the divergences using point splitting regularization procedure in which we consider fields to be not on the same point in the spacetime rather ϵ distance apart, for example a typical $\phi^2(X)$ term would look like $\phi(X)\phi(X+\epsilon)$ where $\epsilon \longrightarrow 0$. The dimension of α' from the worldsheet point of view is $[L]^2$ similarly the dimension of $d^2\sigma$ is also $[L]^2$ also the partial derivatives ∂_+ is $[L]^{-1}$ therefore the dimension of X^μ is $[L]^1$ to make the action dimensionless, from this analysis in the interaction action the dimension of $\Phi(X)$ turns out to be $[L]^0$ because the ϵ would have a dimension of $[L]^1$, similarly the dimension of $A_{\mu\nu}$ turns out to be $[L]^0$. We expand the background into two parts, classical part X_0 and the quantum fluctuations ξ,

$$X^\mu = X_0^\mu + \xi^\mu$$

We define the generating functional for the background field which gives us the beta function hence the equation of motion of these fields is given by,

$$\Omega(X_0) = \int D[\xi] e^{-S_0(\xi) - S_{int}(X_0+\xi)} = \left\langle e^{-S_{int}(X_0+\xi)} \right\rangle \tag{5.3}$$

The definition of the beta functions are given by the

$$\mu \frac{d}{d\mu} f(g,\mu) = \left(\mu \frac{d}{d\mu} + \sum_i g_i \frac{\partial}{\partial \mu} \right) f$$

Therefore the beta functions can be given by the expression given below

$$-\epsilon\frac{d}{d\epsilon}\log\Omega[X_0] = \frac{1}{2\pi\alpha'}\int d^2\sigma\left[\frac{\alpha'}{\epsilon^2}\beta_\phi(X_0) + \partial_+X_0^\mu\partial_-X_0^\nu\beta_{\mu\nu}(X_0) + \dots\right] \tag{5.4}$$

5.3 β function for Tachyon field

We will now perform a weak field expansion around the background X_0 considering $\Phi(X)$ as a weak field and in order to calculate the beta functions of tachyon field we will set the massless tensor field $A_{\mu\nu}$ to zero. In order to calculate the beta functions from Eq (5.4) we take the Fourier transform of the tachyon field.

$$\Phi(X) = \int \frac{d^Dk}{(2\pi)^D}\ \tilde{\Phi}(k)\ e^{ikx} \tag{5.5}$$

Therefore $\Omega[X_0]$ becomes

$$\Omega[X_0] = \left\langle e^{-S_{int}[X_0]}\right\rangle = \left\langle e^{-\frac{1}{2\pi}\int d^2\sigma\frac{1}{\epsilon^2}\Phi(X)}\right\rangle$$

Using the standard taylor expansion of the exponential functions given by,

$$\left\langle e^{-\frac{1}{2\pi}\int d^2\sigma\frac{1}{\epsilon^2}\Phi(X)}\right\rangle = \sum_{N=0}^{\infty}\frac{1}{N!}\left(\frac{-1}{2\pi}\right)^N\int d^2\sigma_1.....d^2\sigma_N\frac{1}{\epsilon^{2N}}\left\langle\Phi(X(\sigma_1)).....\Phi(X(\sigma_N)\right\rangle$$

Using Eq (5.5) in the above equation we would obtain

$$\left\langle e^{-\frac{1}{2\pi}\int d^2\sigma\frac{1}{\epsilon^2}\Phi(X)}\right\rangle = \sum_{N=0}^{\infty}\frac{1}{N!}\left(\frac{-1}{2\pi}\right)^N\int\frac{d^Dk_1}{(2\pi)^D}\cdots\frac{d^Dk_n}{(2\pi)^D}\int d^2\sigma_1\dots d^2\sigma_N\tilde{\Phi}(k_1)\dots\tilde{\Phi}(k_N)$$
$$\times\frac{1}{\epsilon^{2N}}\left\langle e^{ik_1X(\sigma_1)}\dots e^{ik_nX(\sigma_1)}\right\rangle \tag{5.6}$$

Therefore in order to obtain the beta function of Tachyon field up to any order we just need to obtain the factor

$$\frac{1}{\epsilon^{2N}}\left\langle e^{ik_1X(\sigma_1)}\dots e^{ik_nX(\sigma_1)}\right\rangle \tag{5.7}$$

This factor is called the **Koba-Nielsen factor**

5.3.1 Koba-Nielsen factor for N=1

For $N = 1$ the factor given in the above equation becomes

$$\frac{1}{\epsilon^2}\left\langle e^{ikX(\sigma)}\right\rangle$$

To evaluate this we use the standard result from the conformal field theory [12]

$$\left\langle \xi(\sigma)\xi(\sigma+\epsilon)\right\rangle = -\frac{\alpha'}{2}\delta^{\mu\nu}\ \log\epsilon^2 \tag{5.8}$$

$\left\langle e^{ik.X(\sigma)}\right\rangle$ could be evaluated as done in Appendix (C.1)

$$\frac{1}{\epsilon^2}\left\langle e^{ik.X}\right\rangle = \epsilon^{\frac{\alpha'}{2}k^2-2}\ e^{ikX_0} \tag{5.9}$$

5.4 Koba-Nielsen factor for N=2

Moving on to $N = 2$ in Eq (5.7), we need to evaluate

$$\frac{1}{\epsilon^4}\left\langle e^{ik_1.X(\sigma_1)}\ e^{ik_2.X(\sigma_2)}\right\rangle$$

$$\frac{1}{\epsilon^4}e^{ik_1.X_0(\sigma_1)}\ e^{ik_2.X_0(\sigma_2)}\left\langle e^{ik_1.\xi(\sigma_1)}\ e^{ik_2.\xi(\sigma_2)}\right\rangle$$

The above calculation can be calculated as done in Appendix (C.2)

$$\frac{1}{\epsilon^4}\left\langle e^{ik_1.X(\sigma_1)}\ e^{ik_2.X(\sigma_2)}\right\rangle = e^{ik_1X_0(\sigma_1)}e^{ik_2X_0(\sigma_2)}\epsilon^{\frac{\alpha'}{2}(k_1^2+k_2^2)-4}\ |\sigma_1-\sigma_2|^{\alpha' k_1.k_2} \tag{5.10}$$

5.4.1 Singularities for N=2

Now in order to extract the U.V divergences from $|\sigma_1 - \sigma_2|$ we use the following identity

$$\int_{\epsilon<|\sigma_1-\sigma_2|} d^2\sigma e^{ip.\sigma}\left(|\sigma_1-\sigma_2|^2\right)^{-a} \sim -\sum_{n=0}\frac{\pi}{4^n(n!)^2}\frac{\epsilon^{2(-a+n+1)}}{(-a+n+1)}(-p^2)^n \tag{5.11}$$

Where σ is the distance between σ_1 and σ_2. Taking the inverse Fourier transform of the above equation

$$(|\sigma_1-\sigma_2|^2)^{-a} \sim -\sum_{n=0}\frac{\pi}{4^n(n!)^2}\frac{\epsilon^{2(-a+n+1)}}{(-a+n+1)}(\partial^2)^n\delta(\sigma_1-\sigma_2) \tag{5.12}$$

Now if we plug in $a = -\frac{1}{2}\alpha' k_1.k_2$ we would obtain

$$(|\sigma_1-\sigma_2|^{\alpha' k_1.k_2} \sim -\sum_{n=0}\frac{2\pi}{4^n(n!)^2}\frac{\epsilon^{(\alpha' k_1.k_2+2n+2)}}{(\alpha' k_1.k_2+2n+2)}(\partial^2)^n\delta(\sigma_1-\sigma_2) \tag{5.13}$$

also the ∂^2 is with respect to $\sigma = |\sigma_1 - \sigma_2|$ therefore, Now using Eq (5.13) in Eq (5.10) we get,

$$\frac{1}{\epsilon^4}\left\langle e^{ik_1.X(\sigma_1)}\, e^{ik_2.X(\sigma_2)}\right\rangle = -e^{ik_1X_0(\sigma_1)}e^{ik_2X_0(\sigma_2)}\epsilon^{\frac{\alpha'}{2}(k_1^2+k_2^2)-4}$$

$$\times\sum_{n=0}\frac{2\pi}{4^n(n!)^2}\frac{\epsilon^{(\alpha' k_1.k_2+2n+2)}}{(\alpha' k_1.k_2+2n+2)}(\partial^2)^n\delta(\sigma_1-\sigma_2) \tag{5.14}$$

For $n = 0$ the equation take the form

$$= -e^{ik_1.X_0(\sigma_1)}e^{ik_2.X_0(\sigma_2)}\ \epsilon^{\frac{\alpha'}{2}\left(k_1^2+k_2^2+2k_1.k_2\right)-2}\frac{2\pi}{\alpha' k_1.k_2+2}\delta(\sigma_1-\sigma_2)$$

Therefore,

$$\int d^2\sigma_1 d^2\sigma_2\frac{1}{\epsilon^4}\left\langle e^{ik_1.X(\sigma_1)}e^{ik_2.X(\sigma_2)}\right\rangle = -\frac{\pi}{\epsilon^4}\frac{\epsilon^{2\left(\frac{\alpha'}{4}(k_1+k_2)^2+1\right)}}{\frac{\alpha'}{2}k_1k_2+1}\int d^2\sigma\ e^{i(k_1+k_2)X_0(\sigma)} \tag{5.15}$$

5.5 Tachyon Beta Functions

After getting the Koba-nelson factors we can now easily obtain the beta functions for tachyons with the general definition of β-function, we will do it order by order, but before doing that we make our calculations easier by obtaining the general expression for $\log\Omega(X_0)$. To calculate the beta functions we need to calculate the factor $\log\Omega$, this is because the quantum effective action and the partition functions are related as $\Omega = e^{-W}$, therefore for regularizing the quantum effective action we need to regularize the quantity $\log\Omega$, which can be calculated as,

$$\Omega(X_0) = \left\langle e^{-S_{int}[X_0+\xi]}\right\rangle = \int d^2\xi\ e^{-S_0[\xi]}e^{-S_{int}[X_0+\xi]}$$

$$\Omega(X_0) = \Big\langle\sum_{N=0}^{\infty}\frac{1}{N!}(-S_{int})^N\Big\rangle = \left(1+\big\langle -S_{int}\big\rangle+\frac{1}{2}\big\langle(-S_{int})^2\big\rangle+\dots\right)$$

Therefore,

$$\log\Omega(X_0) = \log\big(1+\big\langle -S_{int}\big\rangle+\frac{1}{2}\big\langle(-S_{int})^2\big\rangle+\dots\big)$$

Expanding the logarithmic series we obtain,

$$\log\Omega(X_0) = \left(\big\langle -S_{int}\big\rangle+\frac{1}{2}\big\langle(-S_{int})^2\big\rangle+\dots\right)-\frac{1}{2}\left(\big\langle -S_{int}\big\rangle+\frac{1}{2}\big\langle(-S_{int})^2\big\rangle+\dots\right)^2$$

$$+\frac{1}{3}\Big(\big\langle -S_{int}\big\rangle+\frac{1}{2}\big\langle(-S_{int})^2\big\rangle+\dots\Big)^3+\dots$$

Now if we recollect the terms, first we do it for the order of $\Phi(X)$

$$\mathcal{O}(\Phi)=\big\langle -S_{int}\big\rangle$$

$$\mathcal{O}(\Phi^2)=\big\langle -S_{int}\big\rangle+\frac{1}{2}\big\langle(-S_{int})^2\big\rangle-\frac{1}{2}\big\langle(-S_{int})\big\rangle^2$$

Therefore regularizing, $\log\Omega$ which is the quantum effective action is equivalent to regularizing $\big\langle S_{int}\big\rangle$ at first order in fields and $\big\langle S_{int}^2\big\rangle$ at second order in fields.

5.5.1 β-function for N=1

For $N=1$ we can write,

$$\log\Omega=\big\langle -S_{int}\big\rangle$$

Using the expression for S_{int} for tachyon part from Eq (5.2) we get

$$-\big\langle\frac{1}{2\pi\alpha'}\int d^2\sigma\ \ \frac{\alpha'}{\epsilon^2}\Phi(X)\big\rangle$$

Taking the Fourier transform of tachyon field as in Eq (5.5) we obtain

$$-\big\langle\frac{1}{2\pi\alpha'}\int d^2\sigma\int\frac{d^Dk}{(2\pi)^D}\ \ \frac{\alpha'}{\epsilon^2}\tilde{\Phi}(X)e^{ik.X(\sigma)}\big\rangle$$

Now using Eq (5.9)

$$=-\frac{1}{2\pi}\int d^2\sigma\int\frac{d^Dk}{(2\pi)^D}\ \Big(\tilde{\Phi}(k)\epsilon^{\frac{\alpha'}{2}k^2-2}\ e^{ikX_0}\Big)$$

Redefining field $\tilde{\Phi}(k)$ by absorbing infinities such that

$$\tilde{\Phi}_R(k)=\epsilon^{k^2\frac{\alpha'}{2}-2}\tilde{\Phi}(k)$$

Differentiating the above equation with respect to ϵ and multiplying ϵ and using the definition of beta function as $\beta_{g_i}=\epsilon\frac{d}{d\epsilon}g_i$

$$\epsilon\frac{d\tilde{\Phi}_R}{d\epsilon}=\epsilon\frac{d}{d\epsilon}\Big(\epsilon^{\frac{\alpha'}{2}k^2-2}\tilde{\Phi}\Big)$$

The renormalized field does not vary with the energy scale therefore the L.H.S of the above equation becomes zero. Therefore,

$$\beta_{\tilde{\Phi}}=\frac{1}{2}\Big(-\alpha'k^2+4\Big)\tilde{\Phi}$$

In position space it reads

$$\beta_\Phi = \int \frac{d^D k}{(2\pi)^D} e^{ik.X} \beta_{\tilde{\Phi}}$$

$$\beta^T_\Phi = \frac{1}{2}\big(\alpha'\partial^\mu\partial_\mu + 4\big)\Phi \tag{5.16}$$

5.5.2 β-function for N=2

Now we will do the similar calculations for $N = 2$ and to do we need to look for $\mathcal{O}(\Phi^2)$ term in the effective action which is $\log\Omega$, as calculated earlier the $\mathcal{O}(\Phi^2)$ includes terms like $\langle S_{int}\rangle$, $\langle S^2_{int}\rangle$ and $\langle S_{int}\rangle\langle S_{int}\rangle$, but since $\langle S_{int}\rangle$ is already regularized therefore we only need to regularize $\langle S^2_{int}\rangle$ in order to regularize the effective action till $\mathcal{O}(\Phi^2)$

$$\langle S^2_{int}\rangle = \frac{1}{(2\pi\alpha')^2}\ \epsilon\frac{d}{d\epsilon}\Big\langle \int d^2\sigma_1 d^2\sigma_2 \frac{\alpha'^2}{\epsilon^4}\Phi(X(\sigma_1))\Phi(X(\sigma_2))\Big\rangle$$

Again taking the Fourier transform of the $\Phi(X)$ field as in Eq (5.5) we get,

$$\langle S^2_{int}\rangle = \frac{1}{(2\pi\alpha')^2}\ \epsilon\frac{d}{d\epsilon}\int d^2\sigma_1 d^2\sigma_2 \frac{d^D k_1}{(2\pi)^D}\ \frac{d^D k_2}{(2\pi)^D}\ \frac{\alpha'^2}{\epsilon^4}\tilde{\Phi}(k_1)\tilde{\Phi}(k_2)\ \Big\langle e^{ik_1.X(\sigma_1)}\ e^{ik_2.X(\sigma_2)}\Big\rangle$$

Now Using Eq (5.15) we get

$$= -\frac{1}{4\pi}\Bigg(\int d^2\sigma \frac{d^D k_1}{(2\pi)^D}\ \frac{d^D k_2}{(2\pi)^D}\ \frac{\epsilon^{2\left(\frac{\alpha'}{4}(k_1+k_2)^2-1\right)}}{\frac{\alpha'}{2}k_1k_2+1} e^{i(k_1+k_2)X_0(\sigma)}\Bigg)\tilde{\Phi}(k_1)\tilde{\Phi}(k_2)$$

Writing $k = k_1 + k_2$ and $k_2 = k - k_1$ and introducing a delta function on the k_1 integral we obtain,

$$\langle S^2_{int}\rangle = -\frac{1}{4\pi}\Bigg(\int d^2\sigma \frac{d^D k}{(2\pi)^D}\epsilon^{\frac{\alpha'}{2}k^2-2} e^{ik.X_0(\sigma)} \int \frac{d^D k_1}{(2\pi)^D}\frac{d^D k_2}{(2\pi)^D}\frac{\delta(k-k_1-k_2)}{\frac{\alpha'}{2}k_1k_2+1}\Bigg)\tilde{\Phi}(k_1)\tilde{\Phi}(k_2)$$

Now we redefine our fields such that the term $\langle S^2_{int}\rangle$ gets regularized

$$\tilde{\Phi}^{(2)}_R = \epsilon^{\frac{\alpha'}{2}k^2-2}\tilde{\Phi}(k) - \frac{1}{4\pi}\int \frac{d^D k_1}{(2\pi)^D}\frac{d^D k_2}{(2\pi)^D}\frac{\delta(k-k_1-k_2)}{\frac{\alpha'}{2}k_1k_2+1}\tilde{\Phi}(k_1)\tilde{\Phi}(k_2)\epsilon^{\frac{\alpha'}{2}k^2-2}$$

The beta function could be obtained as (C.3)

$$\beta_\Phi = \frac{1}{2}\big(\alpha'\partial^\mu\partial_\mu + 4\big)\Phi + \Phi^2 \tag{5.17}$$

5.6 Tachyon and Massless fields

Now we consider the Tachyon field along with the massless background field $A_{\mu\nu}$ which was initially set to zero in order to derive the beta functions for Tachyons only, doing so we would obtain the beta functions of both the fields $A_{\mu\nu}$ and Φ. The massless tensor field $A_{\mu\nu}$ includes all the massless field excluding the dilaton field therefore $A_{\mu\nu}$ is traceless. In order to do so we need to evaluate,

$$\left\langle e^{-S_{int}(X_0+\xi)}\right\rangle = \int D\xi \;\; e^{-S_0(\xi)-S_{int}(X_0+\xi)}$$

Keeping terms up-to $A_{\mu\nu}$ in the expression Eq (5.2) and expanding as before we get

$$\left\langle \exp\left[-S_{int}\right]\right\rangle = \left\langle \sum_{n=0}^{\infty}\frac{(-1)^N}{N!}\left[\frac{1}{2\pi\alpha'}\int d^2\sigma\frac{\alpha'}{\epsilon^2}\Phi(X)+\int d^2\sigma \;\partial_+X^\mu\partial_-X^\nu A_{\mu\nu}\right]^N\right\rangle$$

Taking the Fourier Transform of the $\Phi(X)$ as in Eq (5.5) and also Fourier transforming the massless tensor field as,

$$A_{\mu\nu}(X) = \int \frac{d^Dk}{(2\pi)^D}\tilde{A}_{\mu\nu}\; e^{ik.X(\sigma)} \tag{5.18}$$

$$\left\langle \exp\left[-S_{int}\right]\right\rangle = \sum_{N=0}^{\infty}\int d^2\sigma_1\ldots d^2\sigma_N\frac{d^Dk_1}{(2\pi)^D}\cdots\frac{d^Dk_N}{(2\pi)^D}\left(\frac{-1}{2\pi\alpha'}\right)^N \times\frac{1}{N!}\left\langle\left[\frac{\alpha'}{\epsilon^2}e^{ik.X(\sigma)}\tilde{\Phi}(k)+\partial_+X^\mu\partial_-X^\nu\tilde{A}_{\mu\nu}\; e^{ik.X(\sigma)}\right]^N\right\rangle$$

Binomial Expanding the above expression,

$$= \sum_{N=0}^{\infty}\int d^2\sigma_1\ldots d^2\sigma_N\frac{d^Dk_1}{(2\pi)^D}\cdots\frac{d^Dk_N}{(2\pi)^D}\left(\frac{-1}{2\pi\alpha'}\right)^N\frac{1}{N!} \times\sum_{r=0}^{N}{}^NC_r\left\langle\left(\frac{\alpha'}{\epsilon^2}\tilde{\Phi}(k)e^{ik.X(\sigma)}\right)^r\left(\partial_+X^\mu\partial_-X^\nu A_{\mu\nu}e^{ik.X(\sigma)}\right)^{N-r}\right\rangle$$

For the first order in fields $N=1$ we just need to evaluate the terms like

$$\frac{1}{\epsilon^2}\left\langle e^{ik.X(\sigma)}\right\rangle, \left\langle\partial_+X^\mu(\sigma)\partial_-X^\nu(\sigma+\epsilon)e^{ik.X(\sigma)}\right\rangle$$

Let us now focus on the second term, to do that we use the following

$$\left\langle\partial_+^\mu\xi(\sigma)\partial_+^\nu\xi(\sigma+\epsilon)\right\rangle = \left\langle\partial_-^\mu\xi(\sigma)\partial_-^\nu\xi(\sigma+\epsilon)\right\rangle = \left\langle\partial_+^\mu\xi(\sigma)\partial_-^\nu\xi(\sigma+\epsilon)\right\rangle = 0 \tag{5.19}$$

Using Eq (5.19) we can write

$$\left\langle \partial_+(X_0^\mu + \xi^\mu)\partial_-(X_0^\nu + \xi^\nu)e^{ik.X(\sigma)}\right\rangle = \partial_+ X_0^\mu \partial_- X_0^\nu \left\langle e^{ik.X}\right\rangle$$

Therefore,

$$\left\langle \partial_+ X^\mu(\sigma)\partial_- X^\nu(\sigma+\epsilon)e^{ik.X(\sigma)}\right\rangle = \partial_+ X_0^\mu \partial_- X_0^\nu e^{ik.X_0}\epsilon^{\frac{\alpha'}{2}k^2} \tag{5.20}$$

Plugging it in the above expression for $\left\langle e^{S_{int}}\right\rangle$ we obtain

$$\left\langle e^{-S_{int}}\right\rangle = \Omega = -\int d^2\sigma \frac{d^D k}{(2\pi)^D}\frac{1}{2\pi\alpha'}\left(\frac{\alpha'}{\epsilon^2}\left\langle e^{ik.X}\right\rangle\tilde{\Phi}(X) + \left\langle \partial_+ X^\mu \partial_- X^\nu e^{ik.X}\right\rangle \tilde{A}_{\mu\nu}\right)$$

Using the expression Eq (5.9) and Eq (5.20) we obtain,

$$\Omega = -\int d^2\sigma \frac{d^D k}{(2\pi)^D}\left(\frac{1}{2\pi}\epsilon^{\frac{\alpha'}{2}k^2-2}e^{ik.X_0}\tilde{\Phi}(X) + \frac{1}{2\pi\alpha'}\partial_+ X_0^\mu \partial_- X_0^\nu e^{ik.X_0}\epsilon^{\frac{\alpha'}{2}k^2}\tilde{A}_{\mu\nu}\right)$$

the effective action is given by $\log\Omega$ which is equal to $\left\langle -S_{int}\right\rangle$ as we have calculated before, which is precisely the first term in the expansion of $\left\langle e^{-S_{int}}\right\rangle$ which we have written above, therefore in order to regularize the effective action we need to regularize the $\left\langle -S_{int}\right\rangle$ which can de done by hiding the singularities in the fields $\tilde{\Phi}(X)$ and $\tilde{A}_{\mu\nu}$

$$\tilde{\Phi}_R = \epsilon^{\alpha'/2\,k^2-2}\tilde{\Phi} \qquad \tilde{A}'_{\mu\nu} = \epsilon^{\frac{\alpha'}{2}k^2}\tilde{A}_{\mu\nu} \tag{5.21}$$

Differentiating the above fields with respect to ϵ and multiplying ϵ will give us the standard definition of beta functions, which is given by $\epsilon\frac{d}{d\epsilon}g_i$

$$\epsilon\frac{d\tilde{\Phi}_R}{d\epsilon} = \epsilon\frac{d}{d\epsilon}\left(\epsilon^{\frac{\alpha'}{2}k^2-2}\tilde{\Phi}\right)$$

The renormalized field is the actual field therefore, the L.H.S of the above equation becomes zero, therefore

$$\beta_\Phi = \frac{1}{2}\left(\partial^\mu\partial_\mu + 4\right)\Phi(X_0)$$

Similarly doing it for the field $A_{\mu\nu}$

$$\beta_{\mu\nu} = \frac{1}{2}\partial^\rho\partial_\rho A_{\mu\nu}$$

Chapter 6

Results and Future Outlook

6.0.1 Einstein's Equation from String Theory

In the first part of thesis we have successfully developed and applied the covariant background field expansion method in order to derive the general relativity equation from string theory. First we looked in the classical version fo string theory, there we found an interesting property of classical strings i.e. the worldsheet stress energy tensor was traceless, but when we tried to quantize the classical version of string action we found out that the stress-energy tensor is not traceless, which was a matter of concern, the non vanishing trace was called *trace anomaly* or the *weyl anomaly*. We then used methods to solve the trace anomaly, at tree level the trace of stress-energy tensor was vanishing therefore we solve it first at one loop level and solving the issue of trace anomaly we found out that if we want the worldsheet theory to be conformally invariant then the spacetime fields must be restricted by some equation of motion, which is obtained when we set the beta functions to zero, the equation of motion of the metric $g_{\mu\nu}$, precisely turns out to be the Einstein's equation of general relativity.

6.0.2 String theory and black hole information loss

We also saw how string theory helps in solving the mystery of information loss paradox in black holes, and we also saw how string theory could help explain the Hawking effect, because of remarkable feature of temperature cutoff that string theory has, we then see how the equation of motion of background fields that we derived in the earlier section could help us analyze black hole dynamics, finally we calculated the temperature of black holes in terms of the mass of the black hole which came out to be slightly lower in string theory as compared to the one from Einstein's equation.

6.0.3 Including Tachyon and massive string states

After looking how string theory could help us analyzing the dynamics of black and potentially solve the issue of information loss paradox we now need a more systematic way to analyze the massive string states which helps us analyze the black holes in more systematic way, for that we need the equation of motion of massive string states, which is given by the beta function of these states. To calculate the beta functions and hence the equation of motion of massive states we develop a general formalism that helps us get the beta functions of any general tensor field, be it massive or massless, we then verified the beta function and hence the equation of motion of the tachyon field and the massless field from the new formalism developed, also using this procedure we can get beta functions of any kind of fields when all the tensor fields (which could be thought of as condensates of string states) are present, which helps us in re-normalization of the theory.

Appendix A

Details of Chapter 1

A.1 Worldsheet Area and Action

We want to calculate the area of world-sheet and the corresponding action of string, to do so consider a function $f(x, y)$, a small change in f would be given by the relation

$$df_x(x, y) = \frac{f(x + dx, y) - f(x, y)}{dx} dx \quad = \frac{\partial f}{\partial x} dx$$

$$df_y(x, y) = \frac{f(x, y) - f(x, y + dy)}{dy} dy \quad = \frac{\partial f}{\partial y} dy$$

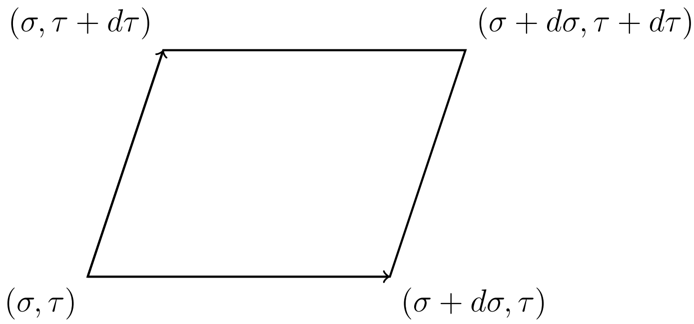
The total change is therefore

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

This is called the total derivative of the function f

Now consider $\overrightarrow{X}(\tau, \sigma)$ describe any point in the string worldsheet.(Here (τ, σ) are used to parameterize the string worldsheet).

Consider an infinitesimal parallelogram formed on the string worldsheet with the vertices $(\sigma, \tau), (\sigma + d\sigma), (\sigma + d\sigma, \tau + d\tau), (\sigma, d\tau)$



The two vertices of the parallelogram formed would be given by

$$\overrightarrow{A} = \frac{\partial \overrightarrow{X}}{\partial \sigma} d\sigma$$

$$\overrightarrow{B} = \frac{\partial \overrightarrow{X}}{\partial \tau} d\tau$$

Area of a parallelogram is given by $Ar = |\overrightarrow{A}||\overrightarrow{B}|sin\theta$

$$Ar = \sqrt{A^2B^2(1 - cos^2\theta)}$$

$$Ar = \sqrt{A^2B^2 - (A.B)^2}$$

Hence we can write our infinitesimal area as

$$dA = \sqrt{\left(\frac{\partial \overrightarrow{X}}{\partial \sigma} d\sigma\right)^2 \left(\frac{\partial \overrightarrow{X}}{\partial \tau} d\tau\right)^2 - \left(\left(\frac{\partial \overrightarrow{X}}{\partial \sigma} d\sigma\right) . \left(\frac{\partial \overrightarrow{X}}{\partial \tau} d\tau\right)\right)^2}$$

$$= d\sigma d\tau \sqrt{(\dot{X}.\dot{X})(X'.X') - (\dot{X}.X')^2}$$

Where,

$$\dot{X} = \frac{\partial X}{\partial \tau} \quad X' = \frac{\partial X}{\partial \sigma} \quad \text{and} \quad \dot{X}.X' = g_{\mu\nu}\dot{X}^{\mu}X'^{\nu}$$

A.2 Variation of Action

The variation of polyakov action with respect to the metric could be calculated as below,

$$\begin{aligned}
\delta S &= \delta\left(\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\gamma^{ab}\partial_a X^\mu\partial_b X^\nu g_{\mu\nu}\right)\\
&= \frac{1}{4\pi\alpha'}\int d^2\sigma\left(\delta\sqrt{\gamma}\gamma^{ab}+\sqrt{\gamma}\delta\gamma^{ab}\right)\partial_a X^\mu\partial_b X^\nu g_{\mu\nu}\\
&= \frac{1}{4\pi\alpha'}\int d^2\sigma\left(\frac{1}{2\sqrt{\gamma}}\delta\gamma\gamma^{ab}+\sqrt{\gamma}\delta\gamma^{ab}\right)\partial_a X^\mu\partial_b X^\nu g_{\mu\nu}\\
&= \frac{1}{4\pi\alpha'}\int d^2\sigma\left(\frac{1}{2\sqrt{\gamma}}(-\gamma\gamma_{ab}\delta\gamma^{ab})\gamma^{ab}+\sqrt{\gamma}\delta\gamma^{ab}\right)\partial_a X^\mu\partial_b X^\nu g_{\mu\nu}\\
&= \frac{1}{4\pi\alpha'}\int d^2\sigma\left(\frac{1}{2\sqrt{\gamma}}(-\gamma\gamma_{cd}\delta\gamma^{cd})\gamma^{ab}+\sqrt{\gamma}\delta\gamma^{ab}\right)\partial_a X^\mu\partial_b X^\nu g_{\mu\nu}\\
&= \frac{1}{4\pi\alpha'}\int d^2\sigma\left(-\frac{1}{2}\sqrt{\gamma}\gamma_{cd}\delta\gamma^{cd}\gamma^{ab}+\sqrt{\gamma}\delta\gamma^{ab}\right)\partial_a X^\mu\partial_b X^\nu g_{\mu\nu}\\
&= \frac{1}{4\pi\alpha'}\int d^2\sigma\left(-\frac{1}{2}\sqrt{\gamma}\gamma_{cd}\delta\gamma^{cd}\gamma^{ab}\partial_a X^\mu\partial_b X^\nu g_{\mu\nu}+\sqrt{\gamma}\delta\gamma^{ab}\partial_a X^\mu\partial_b X^\nu g_{\mu\nu}\right)\\
\delta S &= \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\delta\gamma^{ab}g_{\mu\nu}\left(\partial_a X^\mu\partial_b X^\nu-\frac{1}{2}\gamma_{ab}\partial^c X^\mu\partial_c X^\nu\right) \qquad \text{(A.1)}
\end{aligned}$$

A.3 Variation of metric

In order to vary the metric we remember,

$$\gamma'_{ab}(\sigma+v)=\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}\gamma_{cd}(\sigma)$$

$$\begin{aligned}
\gamma'_{ab}(\sigma)+v^c\frac{\partial\gamma'_{ab}}{\partial\sigma'^c} &= \frac{\partial(\sigma'^c-v^c)}{\partial\sigma'^a}\frac{\partial(\sigma'^d-v^d)}{\partial\sigma'^b}\gamma_{cd}(\sigma)\\
&= \left(\delta^c_a-\frac{\partial v^c}{\partial\sigma'^a}\right)\left(\delta^d_b-\frac{\partial v^d}{\partial\sigma'^b}\right)\gamma_{cd}(\sigma)
\end{aligned}$$

$$\gamma'_{ab}(\sigma)+v^c\partial'_c\gamma_{ab}=(\delta^c_a-\partial'_a v^c)(\delta^d_b-\partial'_b v^d)\gamma_{cd}(\sigma)$$
$$\gamma'_{ab}(\sigma)+v^c\partial'_c\gamma_{ab}=(\delta^c_a\delta^d_b-\partial'_a v^c\delta^d_b-\partial'_b v^d\delta^c_a+\mathcal{O}(v^2))\gamma_{cd}(\sigma)$$
$$\gamma'_{ab}(\sigma)+v^c\partial'_c\gamma_{ab}=\gamma_{ab}-\partial'_a v^c\gamma_{cb}-\partial'_b v^d\gamma_{ad}$$
$$\delta\gamma_{ab}=-(v^c\partial'_c\gamma_{ab}+\partial'_a v^c\gamma_{cb}+\partial'_b v^d\gamma_{ad})$$

Since this is a tensor equation so it will hold in any coordinate system so we can remove primes from the R.H.S

$$\delta\gamma_{ab} = -(v^c\partial_c\gamma_{ab} + \partial_a v^c\gamma_{cb} + \partial_b v^d\gamma_{ad})$$

It is convenient to write this equation in the form of covariant derivatives as,

$$\nabla_a v_b = \partial_a v_b - \Gamma^{\lambda}_{ab} v_\lambda$$

We know that

$$\partial_a v^c\gamma_{cb} + v^c\partial_a\gamma_{cb} = \partial_a(v^c\gamma_{cb}) = \partial_a v_b$$

$$\partial_a v^c\gamma_{cb} = \partial_a v_b - v^c\partial_a\gamma_{cb}$$

Similarly we can write,

$$\partial_b v^d\gamma_{ad} = \partial_b v_a - v^d\partial_b\gamma_{ad}$$

Therefore,

$$\delta\gamma_{ab} = -\partial_a v_b - \partial_b v_a - v^c\partial_c\gamma_{ab} + v^c\partial_a\gamma_{cb} + v^d\partial_b\gamma_{ad}$$

$$\begin{aligned}
\delta\gamma_{ab} &= -\partial_a v_b - \partial_b v_a - 2v_\lambda\big[\frac{1}{2}g^{\lambda c}(\partial_a\gamma_{cb} + \partial_b\gamma_{ac} - \partial_c\gamma_{ab})\big] \\
&= -\partial_a v_b - \partial_b v_a + 2v_\lambda(\Gamma^{\lambda}_{ab}) \\
&= -(\partial_a v_b - \Gamma^{\lambda}_{ab}v_\lambda) - (\partial_b v_a - \Gamma^{\lambda}_{ba}v_\lambda) \\
\delta\gamma_{ab} &= -(\nabla_a v_b + \nabla_b v_a) \qquad \text{(A.2)}
\end{aligned}$$

A.4 Conformal Gauge

We calculate some important quantities in conformal gauge. First we will calculate the metric in conformal gauge and then move on with the rest of the things, we know the fields γ_{ab} satisfies the following transformation equation

$$\gamma'^{ab} = \frac{\partial\sigma'^a}{\partial\sigma^c}\frac{\partial\sigma'^b}{\partial\sigma^d}\gamma^{cd}$$

$$\frac{\partial z}{\partial\sigma} = 1 \quad \frac{\partial z}{\partial\tau} = i$$

$$\frac{\partial\bar{z}}{\partial\sigma} = 1 \quad \frac{\partial\bar{z}}{\partial\tau} = -i$$

$$\gamma'^{zz} = \frac{\partial z}{\partial\sigma}\frac{\partial z}{\partial\sigma}e^{-\phi}\delta^{11} + \frac{\partial z}{\partial\tau}\frac{\partial z}{\partial\tau}e^{-\phi}\delta^{22}$$

$$= 0$$

Similarly,

$$\gamma'^{z\bar{z}} = \frac{\partial z}{\partial \sigma}\frac{\partial \bar{z}}{\partial \sigma} e^{-\phi}\delta^{11} + \frac{\partial z}{\partial \tau}\frac{\partial \bar{z}}{\partial \tau} e^{-\phi}\delta^{22}$$

$$= 2e^{-\phi}$$

$$\gamma'^{\bar{z}z} = \frac{\partial \bar{z}}{\partial \sigma}\frac{\partial z}{\partial \sigma} e^{-\phi}\delta^{11} + \frac{\partial \bar{z}}{\partial \tau}\frac{\partial z}{\partial \tau} e^{-\phi}\delta^{22}$$

$$= 2e^{-\phi}$$

With similar calculations one can show that

$$\gamma^{z\bar{z}} = 2e^{-\phi} = \gamma^{\bar{z}z} \quad \gamma^{zz} = 0 = \gamma^{\bar{z}\bar{z}}$$

Hence the metric would look like,

$$\gamma^{ab} = \begin{pmatrix} 0 & 2e^{-\phi} \\ 2e^{-\phi} & 0 \end{pmatrix}$$

Similarly,

$$\gamma_{ab} = \begin{pmatrix} 0 & \frac{1}{2}e^{\phi} \\ \frac{1}{2}e^{\phi} & 0 \end{pmatrix}$$

Now we can calculate the Christopher's connection coefficients using the formula,

$$\Gamma^{c}_{ab} = \frac{1}{2}\gamma^{cd}\left(\frac{\partial \gamma_{ad}}{\partial z^{b}} + \frac{\partial \gamma_{bd}}{\partial z^{a}} - \frac{\partial \gamma_{ab}}{\partial z^{d}}\right)$$

Here $z^1 = z \quad z^2 = \bar{z}$, the non zero Christopher's coefficient are therefore,

$$\Gamma^{1}_{11} = \frac{1}{2}\gamma^{12}\left(\frac{\gamma_{12}}{\partial z} + \frac{\gamma_{12}}{\partial z} - \frac{\gamma_{11}}{\partial \bar{z}}\right)$$

$$= \partial_z \phi$$

$$\Gamma^{2}_{22} = \frac{1}{2}\gamma^{21}\left(\frac{\gamma_{21}}{\partial \bar{z}} + \frac{\gamma_{21}}{\partial \bar{z}} - \frac{\gamma_{22}}{\partial z}\right)$$

$$= \partial_{\bar{z}} \phi$$

$$\Gamma^{1}_{11} = \partial_z \phi \quad \Gamma^{2}_{22} = \partial_{\bar{z}} \phi$$

We can also calculate Riemann tensor and Ricci scalar once we know the connection coefficients

$$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc}$$

In 2D Riemann tensor will have only one independent component.

$$R^1_{212} = \partial_1 \Gamma^1_{22} - \partial_2 \Gamma^1_{21} + \Gamma^1_{11}\Gamma^1_{22} - \Gamma^1_{e2}\Gamma^e_{21}$$

$$R^1_{212} = 0$$

Similarly,

$$R^2_{121} = \partial_2 \Gamma^2_{11} - \partial_1 \Gamma^2_{12} + \Gamma^2_{22}\Gamma^2_{11} - \Gamma^2_{e1}\Gamma^e_{12} = 0$$

Similarly the other components are given by

$$R^2_{212} = \partial_1 \Gamma^2_{22} - \partial_2 \Gamma^2_{21} + \Gamma^2_{e1}\Gamma^e_{22} - \Gamma^2_{22}\Gamma^2_{21}$$

$$= \partial_z \left(\partial_{\bar{z}} \phi\right)$$

Now the lower indices are given by,

$$R_{1212} = \gamma_{1\lambda} R^\lambda_{212}$$

$$R_{1212} = \gamma_{11} R^1_{212} + \gamma_{12} R^2_{212}$$
$$= \frac{1}{2} e^{\phi} \partial_z \left(\partial_{\bar{z}} \phi\right)$$

Also we know that

$$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}$$

So,

$$R_{11} = \gamma^{ab} R_{a1b1}$$

$$= \gamma^{12} R_{1121} + \gamma^{21} R_{2111}$$

$$= 0$$

$$R_{22} = \gamma^{ab} R_{a2b2}$$

$$= \gamma^{12} R_{1222} + \gamma^{21} R_{2212}$$

$$= 0$$

$$R_{12} = \gamma^{ab} R_{a1b2}$$

$$= \gamma^{12} R_{1122} + \gamma^{21} R_{2112}$$

Also we know from the Riemann tensor is antisymmetric under the exchange of the first index with the second $R_{2112} = -R_{1212}$ so,

$$R_{12} = -\gamma^{21} R_{1212}$$

$$= -2e^{-\phi}\left(\frac{1}{2}e^{\phi}\right)\partial_z\left(\partial_{\bar{z}}\phi\right)$$

$$R_{12} = -\partial_z(\partial_{\bar{z}}\phi)$$

Also because Ricci tensor is symmetric with the exchange of the two indices so we can write

$$R_{21} = R_{12}$$

$$R_{21} = -\partial_z(\partial_{\bar{z}}\phi)$$

Now we know the Scalar curvature is given by

$$\begin{aligned} R &= g^{\mu\nu} R_{\mu\nu} \\ &= \gamma^{ab} R_{ab} \\ &= \gamma^{12} R_{12} + \gamma^{21} R_{21} \\ &= -2(2e^{-\phi})\partial_z\left(\partial_{\bar{z}}\phi\right) \end{aligned}$$

$$R = -4e^{-\phi}\partial_z\partial_{\bar{z}}\phi \tag{A.3}$$

Once we are at conformal gauge, a general coordinate transformation could take us out of the conformal gauge, but there are certain type of coordinate transformations called conformal reparametrization which does not break us out of conformal gauge. A conformal reparametrization is a coordinate transformation characterized by $v^z(z)$ which is a function of z alone, and similarly $v^{\bar{z}}(\bar{z})$ (In general v^z may be a function of both z and $\bar{z}$). When v^z only becomes a function of z then $\nabla_{\bar{z}}v^z = 0$ and $\nabla_z v^{\bar{z}} = 0$,

$$\delta\gamma_{zz} = -(\nabla_z(\gamma_{zc}v^c) + \nabla_z(\gamma_{zd}v^d)) = 0 \qquad \delta\gamma_{\bar{z}\bar{z}} = -(\nabla_{\bar{z}}(\gamma_{\bar{z}c}v^c) + \nabla_{\bar{z}}(\gamma_{\bar{z}d}v^d)) = 0$$

Also we can clearly see that $\gamma_{\bar{z}z}$ is not zero, which means that this transformation does not change the form of the metric but leds to a change in scale factor ϕ

$$\delta\gamma_{\bar{z}z} = -\frac{1}{2}e^{-\phi}\left(\nabla_{\bar{z}}v^{\bar{z}} + \nabla_z v^z)\right) \tag{A.4}$$

On comparing Eq (A.4) with Eq (2.4) we can clearly see that

$$\delta\phi = \nabla_{\bar{z}} v^{\bar{z}} + \nabla_z v^z$$

In the conformal reparametrization from Eq (2.8)

$$\delta X^\mu = v^z \nabla_z X^\mu + v^{\bar{z}} \nabla_{\bar{z}} X^\mu$$

The conformal reparametrization is a coordinate change which changes both the metric and the fields, in fact it changes the scaling factor of the metric not the actual form of the metric. But in the other hand if we define a conformal transformation given by,

$$\delta X^\mu = v^z \partial_z X^\mu + v^{\bar{z}} \partial_{\bar{z}} X^\mu \qquad \delta\gamma_{ab} = 0$$

A conformal transformation is a transformation that only applies to the fields and does not change the metric whereas conformal reparametrization changes both. If one has a conformal reparametrization which acts on $X's$ and γ_{ab} accompanied by appropriate Weyl transformation which acts only on γ_{ab} the two taken together makes conformal transformation. It involves change of coordinates by an analytic function and local rescaling of the 2-D metric. Also since our theory is invariant under reparametrization, so it must be invariant under conformal reparametrization also our theory is invariant under weyl transformation, and conformal transformation is made up of Weyl transformation and conformal reparametrization so it means that our theory is invariant under conformal transformation.

Appendix B

Details of Chapter 2

B.1 Quantum Conservation Equations

In order to derive the quantum conservation equation we consider the below equation and perform integration by parts in conformal gauge.

$$0 = \int d^2\sigma \sqrt{\gamma'} \frac{1}{\sqrt{\gamma'}} \frac{\delta W}{\delta \gamma'^{ab}} \left(\nabla^a v^b\right) = \int d^2 z \sqrt{\gamma} \frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta \gamma^{ab}} \left(\nabla^a v^b\right)$$

The term $\sqrt{\gamma}$ represents the determinant of the 2-dimensional metric in the conformal gauge. Now integrating by parts we get

$$\int d^2 z \sqrt{\gamma} \nabla^a \left(\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta \gamma^{ab}}\right) v^b = 0$$

$$\int d^2 z - \sqrt{\gamma} \left[\nabla^z \left(\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta \gamma^{zz}}\right) v^z + \nabla^z \left(\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta \gamma^{z\bar{z}}}\right) v^{\bar{z}} + \nabla^{\bar{z}} \left(\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta \gamma^{\bar{z}z}}\right) v^z + \nabla^{\bar{z}} \left(\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta \gamma^{\bar{z}\bar{z}}}\right) v^{\bar{z}}\right] = 0$$

Using,

$$\delta \gamma^{z\bar{z}} = \delta(2e^{-\phi}) = -2e^{-\phi} \delta\phi$$

Therefore,

$$\frac{\delta W}{\delta \gamma^{z\bar{z}}} = -\frac{\delta W}{\delta \phi} \gamma_{z\bar{z}}$$

Plugging it we get,

$$0 = \int d^2 z \sqrt{\gamma} \left[\left(\nabla_z \left(\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta \phi}\right) - \nabla^z \left(\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta \gamma^{zz}}\right)\right) v^z + \left(\nabla_{\bar{z}} \left(\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta \phi}\right) - \nabla^{\bar{z}} \left(\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta \gamma^{\bar{z}\bar{z}}}\right)\right)\right.$$

Since the functions v^z and $v^{\bar{z}}$ are arbitrary, their respective coefficients must be equal to zero, Hence

$$\nabla_z \left(\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta \phi} \right) = \nabla^z \left(\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta \gamma^{zz}} \right) \tag{B.1}$$

$$\nabla_{\bar{z}} \left(\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta \phi} \right) = \nabla^{\bar{z}} \left(\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta \gamma^{\bar{z}\bar{z}}} \right) \tag{B.2}$$

B.2 Luiovile's action

We want to check that the variation of the effective action W gets us to Eq (3.6) to do so consider a small variation in ϕ as

$$\phi \Longrightarrow \phi' = \phi + \delta\phi$$

Under this variation W changes as,

$$W' = \frac{\lambda}{48\pi} \int d^2\sigma \sqrt{\tilde{\gamma}} \left(\frac{1}{2} \tilde{\gamma}^{ab} \partial_a(\phi + \delta\phi) \partial_b(\phi + \delta\phi) + \mu^2 e^{(\phi+\delta\phi)} \right)$$

Considering $\delta\phi$ is small we can neglect the higher order terms.

$$W' = \frac{\lambda}{48\pi} \int d^2\sigma \sqrt{\tilde{\gamma}} \left(\frac{1}{2} \tilde{\gamma}^{ab} \left[\partial_a(\phi)\partial_b(\phi) + \partial_a\phi\partial_b\delta\phi + \partial_a\delta\phi\partial_b\phi + O(\delta\phi^2) \right] + \mu^2 e^{\phi}(1 + \delta\phi) \right)$$

$$W' = W + \frac{\lambda}{48\pi} \int d^2\sigma \sqrt{\tilde{\gamma}} \left(\frac{1}{2} \tilde{\gamma}^{ab} \left[\partial_a\phi\partial_b\delta\phi + \partial_a\delta\phi\partial_b\phi \right] + \mu^2 e^{\phi}\delta\phi \right)$$

$$\delta W = \frac{\lambda}{48\pi} \int d^2\sigma \sqrt{\tilde{\gamma}} \left(\tilde{\gamma}^{ab} (\partial_a\phi\partial_b\delta\phi) + \mu^2 e^{\phi}\delta\phi \right)$$

Now returning back in the conformal gauge and evaluating our integral there we would get,

$$\delta W = \frac{\lambda}{48\pi} \int d^2z \sqrt{\gamma} \left(\gamma^{z\bar{z}} (\partial_z\phi\partial_{\bar{z}}\delta\phi) + \gamma^{\bar{z}z} (\partial_{\bar{z}}\phi\partial_z\delta\phi) + \mu^2 e^{\phi}\delta\phi \right)$$

Now integrating by parts

$$\delta W = \frac{\lambda}{48\pi} \Bigg[\int dz \sqrt{\gamma} \gamma^{z\bar{z}} (\partial_z\phi)\delta\phi \Big|_{(boundary)} - \int d^2z \sqrt{\gamma} (\gamma^{z\bar{z}} \partial_{\bar{z}}\partial_z\phi\delta\phi) + \int d\bar{z} \gamma^{\bar{z}z} \partial_{\bar{z}}\phi\delta\phi \Big|_{boundary} - \int d^2z \sqrt{\gamma} (\gamma^{\bar{z}z} \partial_z\partial_{\bar{z}}\phi\delta\phi) + \int d^2z \sqrt{\gamma} \mu^2 e^{\phi}\delta\phi \Bigg]$$

Considering that $\delta\phi$ vanishes at the boundary,

$$\delta W = \frac{\lambda}{48\pi}\left[-\int d^2z\sqrt{\gamma}(\gamma^{z\bar{z}}\partial_{\bar{z}}\partial_z\phi\delta\phi) - \int d^2z\sqrt{\gamma}(\gamma^{\bar{z}z}\partial_z\partial_{\bar{z}}\phi\delta\phi) + \int d^2z\sqrt{\gamma}\mu^2 e^{\phi}\delta\phi\right]$$

Also we know that $\gamma^{z\bar{z}} = \gamma^{\bar{z}z}$ using this property we can write the above expression as,

$$\delta W = \frac{\lambda}{48\pi}\int d^2z\sqrt{\gamma}\left[-2(\gamma^{z\bar{z}}\partial_{\bar{z}}\partial_z\phi) + \mu^2 e^{\phi}\right]\delta\phi$$

Also we know that $\sqrt{\gamma} = \frac{1}{2}e^{\phi}$ and $\gamma^{z\bar{z}} = 2e^{-\phi}$

$$\delta W = \frac{\lambda}{48\pi}\int d^2z\sqrt{\gamma}\left[R + \mu^2 e^{\phi}\right]\delta\phi$$

B.3 Derivative of curvature scalar

We want to write curvature scalar in the upper z derivative so that we can improve the stress energy tensor, to do so we translate to conformal gauge and in conformal gauge,

$$\nabla_z R = \nabla_z(-4e^{-\phi}\partial_z\partial_{\bar{z}}\phi)$$

For scalars the Christopher's connection vanishes therefore,

$$\nabla_z R = \partial_z(-4e^{-\phi}\partial_z\partial_{\bar{z}}\phi)$$

$$= -4\left(\partial_z(e^{-\phi})\partial_z\partial_{\bar{z}}\phi + \partial_z(\partial_z\partial_{\bar{z}}\phi)e^{-\phi}\right)$$

$$= -4\left(-e^{-\phi}\partial_z\phi\partial_z\partial_{\bar{z}}\phi + (\partial_z^2\partial_{\bar{z}}\phi)e^{-\phi}\right)$$
$$= -4e^{-\phi}\left(-\partial_z\phi\partial_z\partial_{\bar{z}}\phi + \partial_z^2\partial_{\bar{z}}\phi\right)$$
$$= -4e^{-\phi}\left(-\frac{1}{2}\partial_{\bar{z}}(\partial_z\phi\partial_z\phi) + \partial_{\bar{z}}\partial_z^2\phi\right)$$
$$= -2\gamma^{z\bar{z}}\partial_{\bar{z}}\left(-\frac{1}{2}\partial_z\phi\partial_z\phi + \partial_z^2\phi\right)$$
$$= \partial^z\left(\partial_z\phi\partial_z\phi - 2\partial_z^2\phi\right)$$

$$\nabla_z R = \nabla^z(-2\partial_z\partial_z\phi + \partial_z\phi\partial_z\phi) \tag{B.3}$$

B.4 Virasoro Algebra

The Laurent series of $f(z)$ around z_0 is given by:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where the coefficients a_n are calculated as:

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

Therefore the Laurent expansion of $T_{zz}(z)$ is given by

$$T(z) = \sum_{-\infty}^{+\infty} z^{-n-2} L_n \qquad\qquad L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T_{zz}(z)$$

Here we have expanded about $z = 0$.The L_n are called Virasoro operators.

$$[L_m, L_n] = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} z^{m+1} w^{n+1} [T(z), T(w)]$$

We can choose specific contours one which includes $z = 0$ and one which which involves $z = w$ in those contours as we saw before the commutator becomes the radial ordered product.Therefore,

$$= \oint_{c(0)} \frac{dw}{2\pi i} w^{n+1} \oint_{c(w)} \frac{dz}{2\pi i} z^{m+1} R(T(z)T(w))$$

Recalling equation (3.6) we can say that,

$$= \oint_{c(0)} \frac{dw}{2\pi i} w^{n+1} \oint_{c(w)} \frac{dz}{2\pi i} z^{m+1} \left(\frac{\lambda}{2} \frac{1}{(z-w)^4} + \frac{T^0_{ww}}{(z-w)^2} + \frac{\partial_w T^0_{ww}}{(z-w)} \right)$$

Also we know that,

$$f^n(z_0) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z - z_0)^{n+1}}$$

For the first function $f(z) = \frac{\lambda}{2} z^{m+1}$ and $\left.\frac{f^3(z)}{3!}\right|_{z=w} = (m+1)m(m-1)w^{m-2}/2.3!$ ans similarly evaluating rest of the terms we get,

$$[L_m, L_n] = \oint_{c(0)} \frac{dw}{2\pi i} w^{n+1} \left(\frac{\lambda (m+1)m(m-1)w^{m-2}}{2.3!} + 2(m+1)w^m T(w) + w^{m+1} \partial_w T(w) \right)$$

$$= \oint \frac{dw}{2\pi i} \left(\frac{\lambda}{12} (m^3 - m) w^{m+n-1} + 2(m+1) w^{m+n+1} T(w) + w^{m+n+2} \partial_w T(w) \right)$$

The first term in the integral would exist if and only if $m = -n$ else there would be no singularities if w has positive powers, and for negative powers of w (say x) other than 1 would give us the $f^x(w) = 0$ therefore this integral returns value $\frac{\lambda}{12}(m^3 - m)$ for $m = -n$ and 0 elsewhere.

$$= \frac{\lambda}{12}(m^3 - m)\delta_{m,-n} + 2(m+1)L_{m+n} + \oint \frac{dw}{2\pi i} w^{m+n+2} \partial_w T(w)$$

Integrating by parts the last term we would get,

$$\oint \frac{dw}{2\pi i} w^{m+n+2} \partial_w T(w) = w^{m+n+2} T_w \big|_{w=0} - \oint \frac{dw}{2\pi i}(m+n+2) w^{m+n+1} T(w)$$

$$= (m+n+2)L_{m+n}$$

Therefore,

$$[L_m, L_n] = (2m + 2 - m - n - 2)L_{m+n} + \frac{\lambda}{12}(m^3 - m)\delta_{m,-n}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\lambda}{12}(m^3 - m)\delta_{m,-n} \tag{B.4}$$

B.5 Correspondence of $B_{\mu\nu}$ and A_μ

We know that for a point particle the action is given by,

$$S = \int d\tau \sqrt{\dot{X}^\mu \dot{X}_\mu}$$

Interaction with electromagnetic field is given by,

$$S' = -q \int d\tau \dot{X}^\mu A_\mu$$

Which is the Lagrangian that gives rise to Coulomb and Lorentz force laws for a charged particle.

This works for a point particle because the particle is 0-Dimensional and the world line is 1-D,and the action is such that the one form $A_\mu dX^\mu$ is evaluated along a path, since the worldsheet of the string is 2-D the analogous coupling must be of two indices in spacetime therefore it must be a 2-Form which is an antisymmetric tensor $B_{\mu\nu}$

(A differential p-form is a (0,p) tensor that is completely antisymmetric in its indices. Locally, a 1-form can be expressed as a linear combination of the differentials of the coordinates. For instance, if $(x^1, x^2, \ldots, x^n)$ are coordinates on a manifold, a 1-form ω can be written as:

$$\omega = f_1\, dx^1 + f_2\, dx^2 + \cdots + f_n\, dx^n,$$

where f_i are smooth functions and dx^i are the differentials of the coordinates.)

Evaluating $B_{\mu\nu} dX^\mu dX^\nu$ along a path onto the worldsheet gives us the action.

$$S'_{AS} = \int d^2\sigma B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \epsilon^{ab}$$

which is precisely the action S_{AS}, therefore we can say that the antisymmetric field $B_{\mu\nu}$ is analogous to gauge potential A_μ in electromagnetism

For a charged point particle we know that action is invariant under gauge transformation given by,

B.6 Riemann Normal Coordinates

The affine parameter t takes values $t = 0$ to $t = 1$ such that $\lambda^\mu(0) = X_0^\mu$ and $\lambda^\mu(1) = X_0^\mu + \pi^\mu$. Let η^μ be tangent to $\lambda^\mu(t)$ at X_0^μ, therefore we can write $\eta^\mu = \frac{d\lambda}{dt}\big|_{t=0}$. We define the magnitude of the tangent vector is equal to the square of the arc length s between two points X_0^μ to $X_0^\mu + \pi^\mu$ which is given as,

$$g_{\mu\nu}\eta^\mu\eta^\nu = s^2$$

Where s is given by, [?]

$$ds = \sqrt{g_{\mu\nu} d\lambda^\mu d\lambda^\nu}$$

$$ds = dt\sqrt{g_{\mu\nu}\frac{d\lambda^\mu}{dt}\frac{d\lambda^\nu}{dt}}$$

$$s = \int_0^1 dt\sqrt{g_{\mu\nu}\dot{\lambda}^\mu\dot{\lambda}^\nu}$$

. Also the geodesic equation for $\lambda^\mu(t)$ is given by,

$$\ddot{\lambda}^\mu(t) + \Gamma^\mu_{\nu\delta}\dot{\lambda}^\nu(t)\dot{\lambda}^\delta(t) = 0$$

Now we can expand $\lambda^\mu(t)$ around $t = 0$ assuming, X_0^μ and $X_0^\mu + \pi^\mu$ are close enough, we can use Taylor expansion to expand λ^μ around $t = 0$

$$\lambda^\mu(t) = \lambda^\mu(0) + \dot{\lambda}^\mu(0)t + \frac{1}{2}\ddot{\lambda}^\mu(0)t^2 + \frac{1}{6}\dddot{\lambda}^\mu(0)t^3 + \cdots$$

We know that $\lambda^\mu(0) = X_0^\mu$ similarly, $\dot{\lambda}^\mu(0) = \eta^\mu$ also, from geodesic equation we can write,

$$\ddot{\lambda}(t) = -\Gamma^{\mu}_{\nu\delta}\dot{\lambda}^{\nu}\dot{\lambda}^{\delta}$$

Similarly for the third order derivatives,

$$\dddot{\lambda} = \frac{d}{dt}\left(-\Gamma^{\mu}_{\nu\delta}\dot{\lambda}^{\nu}\dot{\lambda}^{\delta}\right)$$

$$= -\Gamma^{\mu}_{\nu\delta}\left(\ddot{\lambda}^{\nu}\dot{\lambda}^{\delta} + \dot{\lambda}^{\nu}\ddot{\lambda}^{\delta}\right) - \dot{\Gamma}^{\mu}_{\nu\delta}\dot{\lambda}^{\nu}\dot{\lambda}^{\delta}$$

$$= \Gamma^{\mu}_{\nu\delta}\left(\Gamma^{\nu}_{ab}\dot{\lambda}^{a}\dot{\lambda}^{b}\dot{\lambda}^{\delta} + \dot{\lambda}^{\nu}\Gamma^{\delta}_{ab}\dot{\lambda}^{a}\dot{\lambda}^{b}\right) - \dot{\Gamma}^{\mu}_{\nu\delta}\dot{\lambda}^{\nu}\dot{\lambda}^{\delta}$$

$$= \Gamma^{\mu}_{\nu\delta}\Gamma^{\nu}_{ab}\dot{\lambda}^{a}\dot{\lambda}^{b}\dot{\lambda}^{\delta} + \Gamma^{\mu}_{\nu\delta}\Gamma^{\delta}_{ab}\dot{\lambda}^{\nu}\dot{\lambda}^{a}\dot{\lambda}^{b} - \frac{d\Gamma^{\mu}_{\nu\delta}}{d\lambda^{c}}\dot{\lambda}^{c}\dot{\lambda}^{\nu}\dot{\lambda}^{\delta}$$

Exchanging ν and δ in the second term and manipulating last term such that all the indies of $\dot{\lambda}$ are the same we get

$$= \Gamma^{\mu}_{\nu\delta}\dot{\lambda}^{a}\dot{\lambda}^{b}\dot{\lambda}^{\delta} + \Gamma^{\mu}_{\delta\nu}\Gamma^{\nu}_{ab}\dot{\lambda}^{\delta}\dot{\lambda}^{a}\dot{\lambda}^{b} - \partial_{a}\Gamma^{\mu}_{b\delta}\dot{\lambda}^{a}\dot{\lambda}^{b}\dot{\lambda}^{\delta}$$

$$= \left(\Gamma^{\nu}_{ab}\Gamma^{\mu}_{\nu\delta} + \Gamma^{\nu}_{ab}\Gamma^{\mu}_{\nu\delta} - \partial_{a}\Gamma^{\mu}_{b\delta}\right)\dot{\lambda}^{a}\dot{\lambda}^{b}\dot{\lambda}^{\delta}$$

This is the definition of covariant derivative with respect to lower indices only,

$$\dddot{\lambda}^{\mu} = \Gamma^{\mu}_{\sigma_1\sigma_2\sigma_3}\eta^{\sigma_1}\eta^{\sigma_2}\eta^{\sigma_3}$$

where, we have used the notation

$$\nabla_{\sigma_1\sigma_2\ldots\sigma_n}\Gamma^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\sigma_1\sigma_2..\sigma_n\nu\rho}$$

Now we can write the full expression to be,

$$\lambda^{\mu}(t) = X_0^{\mu} + \eta^{\mu}t - \frac{1}{2}\Gamma^{\mu}_{\sigma_1\sigma_2}\eta^{\sigma_1}\eta^{\sigma_2}t^2 - \frac{1}{3!}\Gamma^{\mu}_{\sigma_1\sigma_2\sigma_3}\eta^{\sigma_1}\eta^{\sigma_2}\eta^{\sigma_3}t^3 + \ldots \tag{B.5}$$

At t=1

$$\lambda^{\mu}(1) = X_0^{\mu} + \eta^{\mu} - \frac{1}{2}\Gamma^{\mu}_{\sigma_1\sigma_2}\eta^{\sigma_1}\eta^{\sigma_2} - \frac{1}{3!}\Gamma^{\mu}_{\sigma_1\sigma_2\sigma_3}\eta^{\sigma_1}\eta^{\sigma_2}\eta^{\sigma_3} + \ldots$$

$$X_0^{\mu} + \pi^{\mu} = X_0^{\mu} + \eta^{\mu} - \frac{1}{2}\Gamma^{\mu}_{\sigma_1\sigma_2}\eta^{\sigma_1}\eta^{\sigma_2} - \frac{1}{3!}\Gamma^{\mu}_{\sigma_1\sigma_2\sigma_3}\eta^{\sigma_1}\eta^{\sigma_2}\eta^{\sigma_3} + \ldots$$

Therefore at t=1 we can regard Eq (B.5) as a coordinate transformation from $X_0^{\mu} + \pi^{\mu}$ near X_0 and to new coordinates η.

$$\pi^{\mu} = \eta^{\mu} - \frac{1}{2}\Gamma^{\mu}_{\sigma_1\sigma_2}\eta^{\sigma_1}\eta^{\sigma_2} - \frac{1}{3!}\Gamma^{\mu}_{(\sigma_1\sigma_2\sigma_3)}\eta^{\sigma_1}\eta^{\sigma_2}\eta^{\sigma_3} + \ldots \tag{B.6}$$

Here we have symmetrized the expression for Christopher's symbol of three indices and above as $\eta_1^\sigma \eta_2^\sigma \eta_3^\sigma$ is symmetric and any antisymmetric combination would give us null result.

For any two points $X_0^\mu + \pi^\mu$ and $X_0^\mu + \pi'^\mu$ on a common geodesic through ϕ^i will have normal coordinates η^μ and η'^μ and they will be related as $\frac{\eta}{\eta'} = \frac{s'}{s}$ (Since magnitude of the tangent vector is equal to s). This means that the geodesic in the normal coordinate is expressed as straight lines(because the curve joining tangent vectors is straight line it means that geodesic must also be a straight line), which means that in the expansion of π^μ only η^μ would survive, implying that the Christopher's symbol must vanish. The higher order Christopher's symbol must also vanish therefore this set of coordinates are called *Riemann Normal Coordinates* if we symmetrize w.r.t the lower indices.

Therefore in normal coordinates we can write,

$$\begin{aligned} \bar{\Gamma}^\mu_{\sigma_1\sigma_2} &= 0 \\ \bar{\Gamma}^\mu_{(\sigma_1\sigma_2\sigma_3,\ldots)} &= 0 \end{aligned} \tag{B.7}$$

Also we can show by induction that

$$\partial_{(\sigma_1}\partial_{\sigma_2}\ldots\partial_{\sigma_{n-2}}\bar{\Gamma}^\mu_{\sigma_{n-1}\sigma_n)} = 0$$

For n=3 it becomes

$$\partial_{(\sigma_1}\bar{\Gamma}^i_{\sigma_2\sigma_3)} = 0$$

$$\frac{1}{3!}\left(\partial_{\sigma_1}\bar{\Gamma}^i_{\sigma_2\sigma_3} + \partial_{\sigma_1}\bar{\Gamma}^i_{\sigma_3\sigma_2} + \partial_{\sigma_2}\bar{\Gamma}^i_{\sigma_3\sigma_1} + \partial_{\sigma_2}\bar{\Gamma}^i_{\sigma_1\sigma_3} + \partial_{\sigma_3}\bar{\Gamma}^i_{\sigma_1\sigma_2} + \partial_{\sigma_3}\bar{\Gamma}^i_{\sigma_2\sigma_1}\right) = 0$$

Since Christopher's symbol are symmetric w.r.t its lower indices therefore we can write

$$\partial_{\sigma_1}\bar{\Gamma}^i_{\sigma_2\sigma_3} + \partial_{\sigma_2}\bar{\Gamma}^i_{\sigma_3\sigma_1} + \partial_{\sigma_3}\bar{\Gamma}^i_{\sigma_1\sigma_2} = 0 \tag{B.8}$$

Eq (B.5) holds in any coordinate system but (B.7) may not be true in any general coordinate system, therefore the bar have been used to indicate that we will work with Riemann normal coordinates. The curvature tensor which is given by,

$$R^i_{jkl} = \partial_k\Gamma^i_{jl} - \partial_l\Gamma^i_{jk} + \Gamma^m_{jl}\Gamma^i_{km} - \Gamma^m_{jk}\Gamma^i_{lm}$$

simplifies to,

$$\bar{R}^i_{jkl} = \partial_k\bar{\Gamma}^i_{jl} - \partial_l\bar{\Gamma}^i_{jk} \tag{B.9}$$

Similarly we can write

$$\bar{R}^i_{lkj} = \partial_k\bar{\Gamma}^i_{jl} - \partial_j\bar{\Gamma}^i_{lk}$$

Adding $\bar{R}^i_{jkl}$ and $\bar{R}^i_{lkj}$ we get,

$$\bar{R}^i_{jkl} + \bar{R}^i_{lkj} = 2\partial_k \bar{\Gamma}^i_{jl} - \left(\partial_l \bar{\Gamma}^i_{jk} + \partial_j \bar{\Gamma}^i_{lk}\right)$$

$$\bar{R}^i_{jkl} + \bar{R}^i_{lkj} = 3\partial_k \bar{\Gamma}^i_{jl} - \left(\partial_l \bar{\Gamma}^i_{jk} + \partial_j \bar{\Gamma}^i_{lk} + \partial_k \bar{\Gamma}^i_{jl}\right)$$

Therefore,

$$\partial_k \bar{\Gamma}^i_{jl} = \frac{1}{3}\left(\bar{R}^i_{jkl} + \bar{R}^i_{lkj}\right) \tag{B.10}$$

The Taylor expansion of a general $(0, n)$ tensor is given by,

$$\bar{T}_{\sigma_1\sigma_2\sigma_3\ldots\sigma_n}(X_0 + \eta) = \sum_{m=0}^{\infty} \frac{1}{m!}\left(\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_n}\bar{T}_{\sigma_1\sigma_2\sigma_3\ldots\sigma_n}(X_0)\right)\eta^{\mu_1}\eta^{\mu_2}\ldots\eta^{\mu_3} \tag{B.11}$$

We want to write the Taylor expansion in terms of covariant derivative in normal coordinates, for that we express all the normal derivative term with the covariant derivatives, before that we note that

$$\partial_{\mu_1}\bar{T}_{\sigma_1\sigma_2\sigma_3\ldots\sigma_n} = \nabla_{\mu_1}\bar{T}_{\sigma_1\sigma_2\sigma_3\ldots\sigma_n}$$

This is because the Christopher's symbol vanishes in normal coordinates. Similarly for a two index tensor we can write

$$\nabla_{\mu_1}\nabla_{\mu_2}\bar{T}_{\sigma_1\sigma_2} = \partial_{\mu_1}(\nabla_{\mu_2}\bar{T}_{\sigma_1\sigma_2}) - \bar{\Gamma}^\rho_{\mu_1\mu_2}(\nabla_\rho\bar{T}_{\sigma_1\sigma_2}) - \bar{\Gamma}^\rho_{\mu_1\sigma_1}(\nabla_{\mu_2}\bar{T}_{\rho\sigma_2}) - \bar{\Gamma}^\rho_{\mu_1\sigma_2}(\nabla_{\mu_2}\bar{T}_{\sigma_1\rho})$$

All the terms involving free Christoffel symbols (by free we mean without derivative term) would be zero. The remaining terms are,

$$\nabla_{\mu_1}\nabla_{\mu_2}\bar{T}_{\sigma_1\sigma_2} = \partial_{\mu_1}\partial_{\mu_2}\bar{T}_{\sigma_1\sigma_2} - \partial_{\mu_1}(\bar{\Gamma}^\rho_{\mu_2\sigma_1})\bar{T}_{\rho\sigma_2} - \partial_{\mu_1}(\bar{\Gamma}^\rho_{\mu_2\sigma_2})\bar{T}_{\sigma_1\rho} \tag{B.12}$$

We use the fact that,

$$\partial_{\mu_1}\bar{\Gamma}^\rho_{\mu_2\sigma_1} = \frac{1}{3}\left(\bar{R}^\rho_{\mu_2\mu_1\sigma_1} + \bar{R}^\rho_{\sigma_1\mu_1\mu_2}\right) \qquad \partial_{\mu_1}\bar{\Gamma}^\rho_{\mu_2\sigma_2} = \frac{1}{3}\left(\bar{R}^\rho_{\mu_2\mu_1\sigma_2} + \bar{R}^\rho_{\sigma_2\mu_1\mu_2}\right)$$

Putting these in Eq (B.12) we get,

$$\nabla_{\mu_1}\nabla_{\mu_2}\bar{T}_{\sigma_1\sigma_2} = \partial_{\mu_1}\partial_{\mu_2}\bar{T}_{\sigma_1\sigma_2} - \frac{1}{3}\left(\bar{R}^\rho_{\mu_2\mu_1\sigma_1} + \bar{R}^\rho_{\sigma_1\mu_1\mu_2}\right)\bar{T}_{\rho\sigma_2} - \frac{1}{3}\left(\bar{R}^\rho_{\mu_2\mu_1\sigma_2} + \bar{R}^\rho_{\sigma_2\mu_1\mu_2}\right)\bar{T}_{\sigma_1\rho}$$

Therefore we can write

$$\begin{aligned}\nabla_{\mu_1}\nabla_{\mu_2}\bar{T}_{\sigma_1\sigma_2}\eta^{\mu_1}\eta^{\mu_2} = \partial_{\mu_1}\partial_{\mu_2}\bar{T}_{\sigma_1\sigma_2}\eta^{\mu_1}\eta^{\mu_2} &- \frac{1}{3}\left(\bar{R}^\rho_{\mu_2\mu_1\sigma_1}\eta^{\mu_1}\eta^{\mu_2} + \bar{R}^\rho_{\sigma_1\mu_1\mu_2}\eta^{\mu_1}\eta^{\mu_2}\right)\bar{T}_{\rho\sigma_2} \\ &- \frac{1}{3}\left(\bar{R}^\rho_{\mu_2\mu_1\sigma_2}\eta^{\mu_1}\eta^{\mu_2} + \bar{R}^\rho_{\sigma_2\mu_1\mu_2}\eta^{\mu_1}\eta^{\mu_2}\right)\bar{T}_{\sigma_1\rho}\end{aligned}$$

The term $\bar{R}^{\rho}_{\sigma_1\mu_1\mu_2}\eta^{\mu_1}\eta^{\mu_2}$ becomes zero because of the antisymmetric nature of the Riemann Tensor with the exchange of last two indices,

$$\bar{R}^{\rho}_{\sigma_1\mu_1\mu_2}\eta^{\mu_1}\eta^{\mu_2} = \gamma^{\delta\rho}\bar{R}_{\delta\sigma_1\mu_1\mu_2}\eta^{\mu_1}\eta^{\mu_2}$$

Now if we exchange the indices $\mu_1 \longleftrightarrow \mu_2$ we get,

$$\gamma^{\delta\rho}\bar{R}_{\delta\sigma_1\mu_1\mu_2}\eta^{\mu_1}\eta^{\mu_2} = \gamma^{\delta\rho}\bar{R}_{\delta\sigma_1\mu_2\mu_1}\eta^{\mu_2}\eta^{\mu_1} = -\gamma^{\delta\rho}\bar{R}_{\delta\sigma_1\mu_1\mu_2}\eta^{\mu_1}\eta^{\mu_2}$$

$$2\gamma^{\delta\rho}\bar{R}_{\delta\sigma_1\mu_1\mu_2}\eta^{\mu_1}\eta^{\mu_2} = 0$$

Therefore

$$\bar{R}^{\rho}_{\sigma_1\mu_1\mu_2}\eta^{\mu_1}\eta^{\mu_2} = 0$$

With the same logic $\bar{R}^{\rho}_{\sigma_1\mu_1\mu_2}\eta^{\mu_1}\eta^{\mu_2} = 0$, therefore,

$$\partial_{\mu_1}\partial_{\mu_2}\bar{T}_{\sigma_1\sigma_2}\eta^{\mu_1}\eta^{\mu_2} = \nabla_{\mu_1}\nabla_{\mu_2}\bar{T}_{\sigma_1\sigma_2}\eta^{\mu_1}\eta^{\mu_2} + \frac{1}{3}\bar{R}^{\rho}_{\mu_2\mu_1\sigma_1}\eta^{\mu_1}\eta^{\mu_2}\bar{T}_{\rho\sigma_2} + \frac{1}{3}\bar{R}^{\rho}_{\mu_2\mu_1\sigma_2}\eta^{\mu_1}\eta^{\mu_2}\bar{T}_{\sigma_1\rho} \tag{B.13}$$

Again using the antisymmetric nature of Riemann tensor with exchange of last two indices and substituting Eq (B.13) in Eq (B.11) we get,

$$\begin{aligned}\bar{T}_{\sigma_1\sigma_2}(X_0+\eta) = \bar{T}_{\sigma_1\sigma_2}(X_0) + \nabla_{\mu_1}\bar{T}_{\sigma_1\sigma_2}(X_0)\eta^{\mu_1} + \frac{1}{2}\nabla_{\mu_1}\nabla_{\mu_2}\bar{T}_{\sigma_1\sigma_2}(X_0)\eta^{\mu_1}\eta^{\mu_2} \\ - \frac{1}{6}\bar{R}^{\rho}_{\mu_1\sigma_1\mu_2}\bar{T}_{\rho\sigma_2}(X_0)\eta^{\mu_1}\eta^{\mu_2} - \frac{1}{6}\bar{R}^{\rho}_{\mu_1\sigma_2\mu_2}\bar{T}_{\rho\sigma_1}(X_0)\eta^{\mu_1}\eta^{\mu_2} + \dots\end{aligned} \tag{B.14}$$

Since this expression involves covariant derivative and this is a tensor equation, therefore it will hold in any coordinate system even though we derived it from normal coordinates. Therefore we can remove bars from the notation and write it for any general $T_{\sigma_1\sigma_2}(X_0)$.

B.7 Expansion of Partial derivative in Normal coordinates

To express the expansion of partial derivatives we differentiate with respect to a Eq (B.5) at t=1

$$\partial_a(X_0^{\mu} + \pi^{\mu}) = \partial_a X_0^{\mu} + \partial_a\eta^{\mu} - \frac{1}{2}\partial_a(\Gamma^{\mu}_{\sigma_1\sigma_2}\eta^{\sigma_1}\eta^{\sigma_2}) + \dots$$

$$\partial_a(X_0^{\mu}+\pi^{\mu}) = \partial_a X_0^{\mu} + \partial_a\eta^{\mu} - \frac{1}{2}\left(\partial_a\Gamma^{\mu}_{\sigma_1\sigma_2}\eta^{\sigma_1}\eta^{\sigma_2} + \Gamma^{\mu}_{\sigma_1\sigma_2}\partial_a\eta^{\sigma_1}\eta^{\sigma_2} + \Gamma^{\mu}_{\sigma_1\sigma_2}\eta^{\sigma_1}\partial_a\eta^{\sigma_2}\right)\dots$$

Since Christopher's connection is symmetric w.r.t lower two indices therefore,

$$\partial_a(X_0^\mu+\pi^\mu)=\partial_a X_0^\mu+\partial_a\eta^\mu-\frac{1}{2}\Big(\partial_a\Gamma^\mu_{\sigma_1\sigma_2}\eta^{\sigma_1}\eta^{\sigma_2}+2\Gamma^\mu_{\sigma_1\sigma_2}\partial_a\eta^{\sigma_1}\eta^{\sigma_2}\Big)+\dots$$

$$\partial_a(X_0^\mu+\pi^\mu)=\partial_a X_0^\mu+\partial_a\eta^\mu-\frac{1}{2}\frac{\partial}{\partial X_0^j}\Gamma^\mu_{\sigma_1\sigma_2}\eta^{\sigma_1}\eta^{\sigma_2}\partial_a X_0^j-\Gamma^\mu_{\sigma_1\sigma_2}\frac{\partial\eta^{\sigma_1}}{\partial X_0^j}\eta^{\sigma_2}\partial_a X_0^j+\dots$$

$$\partial_j\Gamma^\mu_{\sigma_1\sigma_2}=\frac{1}{3}\left(R^\mu_{\sigma_1 j\sigma_2}+R^\mu_{\sigma_2 j\sigma_1}\right)$$

$$\partial_a(X_0^\mu+\pi^\mu)=\partial_a X_0^\mu+\partial_a\eta^\mu-\frac{1}{6}\left(R^\mu_{\sigma_1 j\sigma_2}\eta^{\sigma_1}\eta^{\sigma_2}+R^\mu_{\sigma_2 j\sigma_1}\eta^{\sigma_1}\eta^{\sigma_2}\right)\partial_a X_0^j-\Gamma^\mu_{\sigma_1\sigma_2}\frac{\partial\eta^{\sigma_1}}{\partial X_0^j}\eta^{\sigma_2}\partial_a X_0^j+\dots$$

Exchanging σ_1 and σ_2 in the fourth term nd using the fact that $\eta^{\sigma_1}\eta^{\sigma_2}=\eta^{\sigma_2}\eta^{\sigma_1}$, also we use the asymmetric property of Riemann tensor while exchanging last two indices doing so we get,

$$\partial_a(X_0^\mu+\pi^\mu)=\partial_a X_0^\mu+\partial_a\eta^\mu+\frac{1}{3}R^\mu_{\sigma_1\sigma_2 j}\eta^{\sigma_1}\eta^{\sigma_2}\partial_a X_0^j-\Gamma^\mu_{\sigma_1\sigma_2}\frac{\partial\eta^{\sigma_1}}{\partial X_0^j}\eta^{\sigma_2}\partial_a X_0^j+\dots$$

Returning back to Riemann normal coordinates we know that the connection coefficients vanishes therefore we can write,

$$\partial_a(X_0^\mu+\pi^\mu)=\partial_a X_0^\mu+\nabla_a\eta^\mu+\frac{1}{3}R^\mu_{\sigma_1\sigma_2 j}\eta^{\sigma_1}\eta^{\sigma_2}\partial_a X_0^j+\dots \tag{B.15}$$

B.8 Polyakov Action Expansion

Before expanding polyakov action we need to expand the expression for the metric using Eq (B.14).

$$g_{\mu\nu}(X_0+\pi)=g_{\mu\nu}+\frac{1}{3}R_{\mu\alpha\beta\nu}\,\eta^\alpha\eta^\beta+\dots \tag{B.16}$$

To expand Polyakov action we use Eq (B.16) and Eq (B.15)

$$S_P(X_0^\mu+\pi^\mu)=\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\gamma^{ab}\partial_a(X_0^\mu+\pi^\mu)\partial_b(X_0^\nu+\pi^\nu)g_{\mu\nu}(X_0+\pi)$$

$$=\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\gamma^{ab}\left(\partial_a X_0^\mu+\nabla_a\eta^\mu+\frac{1}{3}R^\mu_{\alpha\beta j}\eta^\alpha\eta^\beta\partial_a X_0^j+\dots\right)$$
$$\times\left(\partial_b X_0^\nu+\nabla_b\eta^\nu+\frac{1}{3}R^\nu_{\alpha\beta j}\eta^\alpha\eta^\beta\partial_b X_0^j+\dots\right)\left(g_{\mu\nu}+\frac{1}{3}R_{\mu\alpha\nu\beta}\eta^\alpha\eta^\beta+\dots\right)$$

$$
\begin{aligned}
= \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\gamma}\gamma^{ab} \Big[& \partial_a X_0^\mu \partial_b X_0^\nu g_{\mu\nu}(X_0) + \partial_a X_0^\mu \nabla_b \eta^\nu g_{\mu\nu}(X_0) \\
& + \nabla_a \eta^\mu \partial_b X_0^\nu g_{\mu\nu}(X_0) + \nabla_a \eta^\mu \nabla_b \eta^\nu g_{\mu\nu}(X_0) \\
& + \frac{1}{3} R^\nu_{\alpha\beta j} \eta^\alpha \eta^\beta \partial_b X_0^j \partial_a X_0^\mu g_{\mu\nu} + \frac{1}{3} R^\mu_{\alpha\beta j} \eta^\alpha \eta^\beta \partial_a X_0^j \partial_b X_0^\mu g_{\mu\nu} \\
& + \frac{1}{3} \left(R^\mu_{\alpha\beta j} \eta^\alpha \eta^\beta \partial_a X_0^j \nabla_b \eta^\nu + R^\nu_{\alpha\beta j} \eta^\alpha \eta^\beta \partial_b X_0^j \nabla_a \eta^\mu \right) g_{\mu\nu} \\
& + \frac{1}{3} \left(\partial_a X_0^\mu \partial_b X_0^\nu R_{\mu\alpha\beta\nu} \eta^\alpha \eta^\beta + \partial_a X_0^\mu \nabla_b \eta^\nu R_{\mu\alpha\beta\nu} \eta^\alpha \eta^\beta \right) \\
& + \frac{1}{3} \nabla_a \eta^\mu \partial_b X_0^\nu R_{\mu\alpha\beta\nu} \eta^\alpha \eta^\beta + \frac{1}{3} \nabla_a \eta^\mu \nabla_b \eta^\nu R_{\mu\alpha\beta\nu} \eta^\alpha \eta^\beta + \dots \Big]
\end{aligned}
$$

Due to the symmetric nature of $g_{\mu\nu}$ and γ^{ab} we can write,

$$
\gamma^{ab} \partial_a X_0^\mu \nabla_b \eta^\nu g_{\mu\nu}(X_0) = \gamma^{ab} \nabla_a \eta^\mu \partial_b X_0^\nu g_{\mu\nu}(X_0)
$$

Similarly

$$
\begin{aligned}
\gamma^{ab} \frac{1}{3} R^\mu_{\alpha\beta j} \eta^\alpha \eta^\beta \partial_a X_0^j \nabla_b \eta^\nu g_{\mu\nu} = \gamma^{ab} \frac{1}{3} R^\nu_{\alpha\beta j} \eta^\alpha \eta^\beta \partial_b X_0^j \nabla_a \eta^\mu g_{\mu\nu} = \gamma^{ab} \frac{1}{3} \partial_a X_0^\mu \nabla_b \eta^\nu R_{\mu\alpha\beta\nu} \eta^\alpha \eta^\beta \\
= \gamma^{ab} \frac{1}{3} \nabla_a \eta^\mu \partial_b X_0^\nu R_{\mu\alpha\beta\nu} \eta^\alpha \eta^\beta
\end{aligned}
$$

Therefore we can write,

$$
\begin{aligned}
S_P(X_0^\mu + \pi^\mu) = S_P(X_0) & + \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{\gamma}\gamma^{ab} g_{\mu\nu}(X_0) \partial_a X_0^\mu \nabla_b \eta^\nu \\
& + \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\gamma}\gamma^{ab} \left(g_{\mu\nu} \nabla_a \eta^\mu \nabla_b \eta^\nu + R_{\mu\alpha\beta\nu}(X_0) \partial_a X_0^\mu \partial_b X_0^\nu \eta^\alpha \eta^\beta \right) \\
& + \frac{1}{3\pi\alpha'} \int d^2\sigma \sqrt{\gamma}\gamma^{ab} R_{\mu\alpha\beta\nu}(X_0) \partial_a X_0^\mu \nabla_b \eta^\nu \eta^\alpha \eta^\beta \\
& + \frac{1}{12\pi\alpha'} \int d^2\sigma \sqrt{\gamma}\gamma^{ab} R_{\mu\alpha\beta\nu}(X_0) \nabla_a \eta^\mu \nabla_b \eta^\nu \eta^\alpha \eta^\beta. \qquad \text{(B.17)}
\end{aligned}
$$

B.9 Antisymmetric Action Expansion

The expansion of antisymmetric action is similar to S_P, first we expand $B_{\mu\nu}(X_0^\mu + \pi^\mu)$ using Eq (B.14)

$$
B_{\mu\nu}(X_0^\mu + \pi^\mu) = B_{\mu\nu}(X_0) + \nabla_\alpha B_{\mu\nu}(X_0) \eta^\alpha + \frac{1}{2} \left[\nabla_\alpha \nabla_\beta B_{\mu\nu}(X_0) - \frac{1}{3} R^\rho_{\alpha\mu\beta} B_{\rho\nu} - \frac{1}{3} R^\rho_{\alpha\nu\beta} B_{\mu\rho} \right] \eta^\alpha \eta^\beta \qquad \text{(B.18)}
$$

Now we move on to derive the expansion for S_{AS}

$$S_{AS} = \frac{1}{4\pi\alpha'}\int d^2\sigma\epsilon^{ab}\partial_a(X_0^\mu + \pi^\mu)\partial_b(X_0^\nu + \pi^\nu)B_{\mu\nu}(X_0 + \pi)$$

$$\begin{aligned}
&= \frac{1}{4\pi\alpha'}\int d^2\sigma\epsilon^{ab}\left(\partial_a X_0^\mu + \nabla_a\eta^\mu + \frac{1}{3}R^\mu_{\alpha\beta j}\eta^\alpha\eta^\beta\partial_a X_0^j + \ldots\right)\\
&\times\left(\partial_b X_0^\nu + \nabla_b\eta^\nu + \frac{1}{3}R^\nu_{\alpha\beta j}\eta^\alpha\eta^\beta\partial_b X_0^j + \ldots\right)\\
&\times\left(B_{\mu\nu}(X_0) + \nabla_\alpha B_{\mu\nu}(X_0)\eta^\alpha + \frac{1}{2}\left[\nabla_\alpha\nabla_\beta B_{\mu\nu}(X_0) - \frac{1}{3}R^\rho_{\alpha\mu\beta}B_{\rho\nu} - \frac{1}{3}R^\rho_{\alpha\nu\beta}B_{\mu\rho}\right]\eta^\alpha\eta^\beta + \ldots\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4\pi\alpha'}\int d^2\sigma\epsilon^{ab}\Big[\partial_a X_0^\mu\partial_b X_0^\mu B_{\mu\nu} + \partial_a X_0^\mu\nabla_b\eta^\nu B_{\mu\nu} + \nabla_a\eta^\mu\partial_b X_0^\nu B_{\mu\nu} + \nabla_a\eta^\mu\nabla_b\eta^\nu B_{\mu\nu}\\
&+ \frac{1}{3}\left(\partial_a X_0^\mu\partial_b X_0^j R^\nu_{\alpha\beta j}\eta^\alpha\eta^\beta + \partial_b X_0^\nu\partial_a X_0^j R^\mu_{\alpha\beta j}\eta^\alpha\eta^\beta\right)B_{\mu\nu} + \partial_a X_0^\mu\partial_b X_0^\nu\nabla_\alpha B_{\mu\nu}\eta^\alpha\\
&+ \nabla_a\eta^\mu\partial_b X_0^\mu\nabla_\alpha B_{\mu\nu}\eta^\alpha + \nabla_b\eta^\nu\partial_a X_0^\mu\nabla_\alpha B_{\mu\nu}\eta^\alpha + \frac{1}{2}\partial_a X_0^\mu\partial_b X_0^\nu\nabla_\alpha\nabla_\beta B_{\mu\nu}\eta^\alpha\eta^\beta -\\
&\frac{1}{6}\partial_a X_0^\mu\partial_b X_0^\nu R^\rho_{\alpha\mu\beta}B_{\rho\nu}\eta^\alpha\eta^\beta - \frac{1}{6}\partial_a X_0^\mu\partial_b X_0^\nu R^\rho_{\alpha\nu\beta}B_{\mu\rho}\eta^\alpha\eta^\beta\Big]
\end{aligned}$$

Here we omitted all the higher order terms in η^μ as we only need second order terms in one loop calculations.

Since $\epsilon^{ab} = -\epsilon^{ba}$ and $B_{\mu\nu} = -B_{\nu\mu}$ therefore we can exchange $a \longrightarrow b$ and $\mu \longrightarrow \nu$ in the left hand side of the equation below to get

$$\epsilon^{ab}\partial_a X_0^\mu\nabla_b\eta^\nu B_{\mu\nu} = \epsilon^{ab}\partial_b X_0^\nu\nabla_b\eta^\nu B_{\mu\nu}$$

Similarly we can show that,

$$\epsilon^{ab}\nabla_\alpha B_{\mu\nu}\nabla_a\eta^\nu\partial_b X_0^\mu\eta^\alpha = \epsilon^{ab}\nabla_\alpha B_{\mu\nu}\nabla_b\eta^\nu\partial_a X_0^\mu\eta^\alpha$$

Now similarly exchanging the indices $j \longleftrightarrow \nu$ and suing antisymmetric property of the Riemann tensor we obtain

$$\epsilon^{ab}\partial_a X_0^\mu\partial_b X_0^j R^\nu_{\alpha\beta j}B_{\mu\nu}\eta^\alpha\eta^\beta = \epsilon^{ab}\partial_a X_0^\mu\partial_b X_0^\nu R^\rho_{\alpha\mu\beta}B_{\rho\nu}\eta^\alpha\eta^\beta$$

Using all these properties we finally obtain and arrange the equation in increasing order of η^α

$$
\begin{aligned}
S_{AS}(X_0^\mu+\pi^\mu)=S_{AS}(X_0) \quad &+ \quad \int d^2\sigma\epsilon^{ab}\frac{1}{2\pi\alpha'}\left[B_{\mu\nu}\partial_a X_0^\mu\nabla_b\eta^\nu+\frac{1}{2}\nabla_\alpha B_{\mu\nu}\partial_a\partial_b\eta^\alpha\right] \\
&+ \quad \int d^2\sigma\epsilon^{ab}\frac{1}{4\pi\alpha'}\left[B_{\mu\nu}\nabla_a\eta^\mu\nabla_b\eta^\mu+2\nabla_\alpha B_{\mu\nu}\partial_a X_0^\mu\nabla_b\eta^\nu\eta^\alpha\right] \\
&+ \quad \frac{1}{2}\left[\nabla_\alpha\nabla_\beta B_{\mu\nu}+B_{\mu\rho}R^\rho_{\alpha\beta\nu}+B_{\rho\nu}R^\rho_{\alpha\beta\mu}\right]\partial_a X_0^\mu\partial_b X_0^\nu\eta^\alpha\eta^\beta
\end{aligned}
\tag{B.19}
$$

B.10 Calculation of anomaly from S_P

We split the integral of Eq (3.25)

$$
I=\int d^2l\frac{l_+(l_++q_+)}{l^2(l+q)^2}=\int d^2l\frac{l_+l_+}{l^4+2l^3+l^2q^2}+\int d^2l\frac{l_+q_+}{l^4+2l^3+l^2q^2}
$$

The denominator can be written as $D=(l+q.l)^2$ which gives us the form of I as

$$
I=\int d^2l\frac{l_+l_+}{(l+q.l)^2}+\int d^2l\frac{l_+q_+}{(l+q.l)^2}
$$

To solve this integral we use the following formula

$$
\int d^Nl\,\frac{l_\mu}{(l^2+2p\cdot l)^A}=-\frac{\pi^{N/2}}{\Gamma(A)}\frac{\Gamma(A-N/2)}{p^{2A-N}}\,p_\mu \tag{B.20}
$$

$$
\int d^Nl\,\frac{l_\mu l_\nu}{(l^2+2p\cdot l)^A}=\frac{\pi^{N/2}}{\Gamma(A)p^{2A-N}}\left[\Gamma(A-N/2)p_\mu p_\nu-\frac{1}{2}p^2\delta_{\mu\nu}\Gamma(A-1-\frac{N}{2})\right] \tag{B.21}
$$

Putting $N=2,\ A=2,\ p=q/2$ we get

$$
q_+\int d^2l\frac{l_+}{(l+q.l)^2}=-\frac{\pi}{(q/2)^2}\frac{q_+q_+}{2}
$$

Similarly,

$$
\int d^2l\frac{l_+l_-}{(l+q.l)^2}=\frac{\pi}{(q/2)^2}\left[\frac{q_+}{2}\frac{q_+}{2}-\frac{1}{8}q^2\delta_{++}\right]
$$

Of course the Gamma function $\Gamma(A-1-\frac{N}{2})$ diverges for $A=2, N=2$ as $\Gamma(0)$ is not defined, therefore the theory needs to be properly renormalized, which can be done using dimensional regularization. We will not be doing the proper renormalization and will ignore the term and move on with our calculation.

Adding the above two equations we get,

$$I = -\pi \frac{q_+ q_+}{q^2} = -2\pi \frac{q_+ q_+}{q_+ q_-}$$

Hence,

$$\frac{1}{2\pi} \int d^2 l \frac{l_+(l_+ + q_+)}{l^2 (l+q)^2} = -\frac{q_+}{q_-} \tag{B.22}$$

Now we can use the conservation equation Eq (3.24) to calculate $\langle T_{-+} \rangle$ which gives us the weyl anomaly.

$$\langle T_{-+} \rangle = -\frac{q_-}{q_+} \langle T_{++} \rangle = \frac{1}{4} R_{\mu\nu}(X_0) \partial_a X_0^\mu \partial^a X_0^\nu \tag{B.23}$$

B.11 Stress Energy tensor contribution from dilaton action

The stress energy tensor is the response to the metric so varying the dilaton action we get the definition of the stress energy tensor

$$S_D = \frac{1}{8\pi} \int d^2\sigma \sqrt{\gamma} R \Phi(X)$$

$$\delta S_D = \frac{1}{8\pi} \int d^2\sigma \Big(\delta(\sqrt{\gamma}) R \Phi + \sqrt{\gamma} \delta(R) \Phi \Big)$$

We know that

$$R = \gamma^{ab} R_{ab} \qquad \delta R = \delta\gamma^{ab} R_{ab} + \gamma^{ab} \delta R_{ab}$$

$$\delta\sqrt{\gamma} = \frac{\delta\gamma}{2\sqrt{\gamma}} = -\frac{\gamma\gamma_{ab}\delta\gamma^{ab}}{2\sqrt{\gamma}}$$

We can explicitly show that, [6]

$$\gamma^{ab} \delta R_{ab} = \nabla_\sigma \left[\gamma_{ab} \nabla^\sigma \delta\gamma^{ab} - \nabla_b g^{\sigma b} \right]$$

Therefore combining all these we obtain the variation of the Dilaton action to be,

$$\delta S = -\frac{1}{8\pi} \int d^2\sigma \left[\frac{1}{2\sqrt{\gamma}} \gamma\gamma_{ab}\delta\gamma_{ab} R\Phi + \sqrt{\gamma} R_{ab} \delta\gamma^{ab} \Phi + (\gamma_{ab} \nabla_\sigma \nabla^\sigma \delta\gamma^{ab} - \nabla_a \nabla_b \delta\gamma^{ab}) \right]$$

In the flat worldsheet metric $R = 0$ and $R_{\mu\nu} = 0$ and $\gamma_{ab} = \delta_{ab}$ similarly the covariant derivatives could also be written as the ordinary derivatives. Making all these changes in the flat metric the stress energy tensor corresponding to the flat worldsheet metric coming from the dilaton action is,

$$T^d_{ab} = (\partial_a \partial_b - \delta_{ab}\Box)\,\Phi(X) \tag{B.24}$$

B.12 Vielbein and Spin connections

The spin connection in differential geometry describes how spinor field such as fermionic field behave under local Lorentz transformation in curved spacetime. Spin connection can be considered as a gauge field associated with local Lorentz, spin connection tells us how vielbein changes as we move along the spacetime.

The vielbein's spacetime indices could be raised and lowered via metric tensor

$$e^{\mu i} = g^{\mu\nu} e^i_\nu \qquad e_{\nu i} = \delta_{ij} e^j_\nu$$

η^i transforms as,

$$\eta^i \longrightarrow \eta'^i = \Lambda^i{}_j \eta^j$$

A general tensor that involves both coordinate indices and non coordinate indices transforms as,

$$T'^{\,i\mu}_{j\nu} = \Lambda^i{}_a \frac{\partial x^\mu}{\partial x^\alpha} \Lambda_j{}^b \frac{\partial x^\beta}{\partial x^\nu}$$

Earlier we used to define covariant derivatives as ordinary derivatives plus the Christopher's connections that canceled the non tonsorial part that was coming from the ordinary derivative, similarly in non coordinate basis we would do the same and the connection coefficients would be called *spin connections* ω^a_{bc}, therefore the derivatives are defined in non coordinate basis as,

$$\partial_\mu X^a_b = \partial_\mu X^a_b + \omega^a_{\mu c} X^c_b - \omega^c_{\mu b} X^a_c$$

For a mixed indices involving coordinate and non coordinate indices the derivatives would involve both Christopher's connection and spin connections,

If we demand that the the parallel transport and projection between i and μ indices to commute, (meaning that if we parallel transport a vector in curved spacetime and then project it to local flat Lorentz frame using vielbein is equivalent to projecting the vector in the local Lorentz frame and then parallel transporting it) then the covariant derivative of the vielbein must be zero $\nabla_\mu e^i_\mu = 0$, which gives us the definition of the covariant derivatives as,

$$\nabla_\mu e^i_\nu = \partial_\mu e^i_\nu - \Gamma^\rho_{\mu\nu} e^i_\rho - \omega^i_{\mu j} e^j_\nu = 0$$

B.13 Weyl Transformation in Effective Action

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} e^{\frac{4}{D-2}\Phi}$$

The determinant under the scaling transforms as,

$$\sqrt{g} = \sqrt{\tilde{g}} e^{\frac{2D\Phi}{D-2}}$$

Similarly the scalar curvature under the transformation $g_{\mu\nu} = \omega^2 \tilde{g}_{\mu\nu}$ is given as

$$R = \omega^{-2}\tilde{R} - 2(D-1)g^{\alpha\beta}\omega^{-3}(\nabla_\alpha \nabla_\beta \omega) - (D-1)(D-4)g^{\alpha\beta}\omega^{-4}(\nabla_\alpha \omega)(\nabla_\beta \omega). \tag{B.25}$$

Integrating by parts the second term of the above equation and plugging $\omega = e^{\frac{2}{D-2}\Phi}$

$$\begin{aligned} T_2 &= -(D-1)g^{\alpha\beta}\nabla_\alpha (e^{\frac{-6\Phi}{D-2}})\nabla_\beta (e^{\frac{2\Phi}{D-2}}) \\ &= \frac{24(D-1)}{D-2)(D-2)} e^{-\frac{4\Phi}{(D-2)}}(\nabla\Phi)^2 \end{aligned}$$

Similarly the third term would be given by

$$\begin{aligned} T_3 &= -(D-1)(D-4)g^{\alpha\beta}e^{-\frac{8\Phi}{D-2}}\nabla_\alpha (e^{\frac{2\Phi}{D-2}})\nabla_\beta (e^{\frac{2\Phi}{D-2}}) \\ &= -4\frac{(D-1)(D-4)}{(D-2)(D-2)} e^{\frac{-4\Phi}{D-2}}(\nabla\Phi)^2 \end{aligned}$$

Therefore the overall transformation would be given by

$$R = e^{\frac{-4\Phi}{D-2}}\left(\tilde{R} - \frac{4(D-1)}{(D-2)}(\nabla\Phi)^2\right)$$

Similarly, the $(\nabla\Phi)^2$ transforms as

$$(\nabla\Phi)^2 = g^{\mu\nu}\nabla_\mu \Phi \nabla_\nu \Phi \longrightarrow e^{-\frac{4\Phi}{D-2}}(\nabla\tilde{\Phi})^2$$

And similarly the H^2 transforms as

$$H^2 = H_{\mu\nu\rho}H^{\mu\nu\rho} = g^{a\mu}g^{b\nu}g^{c\rho}H_{abc}H_{\mu\nu\rho} \longrightarrow e^{-\frac{12\Phi}{D-2}}\tilde{H}^2$$

Therefore if we combine everything together,

$$S_D = \int d^D X \sqrt{\tilde{g}} e^{\frac{2D\Phi}{D-2}} e^{-2\Phi}\left[e^{\frac{-4\Phi}{D-2}}\left(\tilde{R} - \frac{4(D-2)}{(D-2)}(\nabla\Phi)^2\right) - e^{\frac{-4\Phi}{D-2}}(\nabla\Phi)^2 - e^{-\frac{12\phi}{D-2}}\tilde{H}^2\right]$$

$$S_D = \int d^D X \sqrt{\tilde{g}}\left[\tilde{R} - \frac{4}{D-2}(\nabla\tilde{\Phi})^2 - \frac{1}{12}e^{-\frac{8\Phi}{D-2}}\tilde{H}^2\right] \tag{B.26}$$

Appendix C

Details of Chapter 3

C.1 Koba-Nielson factor for N=1

We can expand the term $e^{ikX(\sigma)}$ in the following manner,

$$\begin{aligned}\left\langle e^{ikX(\sigma)}\right\rangle &= \left\langle e^{ikX_0}\left(1 + ik.\xi + \frac{1}{2!}(ik.\xi)^2 + \dots\right)\right\rangle \\ &= e^{ik.X_0}\left(1 + i\langle k.\xi\rangle - \frac{1}{2!}\langle (k.\xi)^2\rangle + \dots\right)\end{aligned} \tag{C.1}$$

All the odd point functions would vanish as a result of Wick theorem as the odd number of quantum fields would have no pair to contract therefore only the even terms like $(k.\xi)^2$ would survive. Therefore evaluating such terms we would get.

$$\langle (k.\xi)^2\rangle = -k_a k_b \langle \xi^a \xi^b\rangle$$

Using Eq (5.8) in the above equation we obtain,

$$\frac{1}{2!}\langle (k.\xi)^2\rangle = -\frac{1}{2!}\frac{\alpha'}{2}k_a k_b \delta^{ab}\ \log\epsilon^2 = \frac{\alpha'}{4}k^2\ \log\epsilon^2$$

Similarly the next even term in the expansion of Eq (C.1) could be evaluated as,

$$\langle (k.\xi)^4\rangle = -k_a k_b k_c k_d \langle \xi^a \xi^b \xi^c \xi^d\rangle$$

The number of possible ways to wick contract n (even n) operators is given by

$$W = \frac{n!}{(n/2)!\ 2^{n/2}}$$

This is because the first pair could be chosen in $n(n-1)$ ways the second one $(n-1)(n-2)$ ways and so on, but the order of pair does not matter that is why we divide with $(n/2)!$, also each pair can be swapped within themselves without changing anything so a factor of $2^{n/2}$ is needed.

$$\frac{1}{4!}\langle (k.\xi)^4 \rangle = \frac{1}{4!}\frac{4!}{2^2 \times 2!}\left(\frac{\alpha'}{2}k^2 \ \log \epsilon^2\right)^2 = \frac{1}{2!}\left(\frac{\alpha'}{4}k^2 \ \log \epsilon^2\right)^2$$

Any general term in the expression would therefore be

$$T = \frac{1}{(2n)!}\frac{(2n)!}{2^n \times n!}\left(\frac{\alpha'}{2}k^2 \ \log \epsilon^2\right)^n$$

$$T = \frac{1}{n!}\left(\frac{\alpha'}{4}k^2 \ \log \epsilon^2\right)^n$$

Therefore all the terms combined could be written as,

$$\langle e^{ikX(\sigma)} \rangle = e^{ikX_0}\left(1 + \frac{\alpha'}{4}k^2 \ \log \epsilon + \frac{1}{2!}\big(\frac{\alpha'}{4}k^2 \ \log \epsilon^2\big)^2 + \dots\right)$$

The above infinite series could be written as

$$= e^{ikX_0}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\alpha'}{4}k^2 \ \log \epsilon^2\right)^n = e^{ikX_0(\sigma)} \exp\left(\frac{\alpha'}{4}k^2 \ \log \epsilon^2\right)$$

$$= e^{ikX_0(\sigma)} \exp\big(\ \log \epsilon^{\frac{\alpha'}{2}k^2}\big)$$

$$= e^{ik.X_0}\epsilon^{\frac{\alpha'}{2}k^2}$$

Therefore,

$$\frac{1}{\epsilon^2}\langle e^{ik.X} \rangle = \epsilon^{\frac{\alpha'}{2}k^2 - 2} \ e^{ikX_0} \tag{C.2}$$

C.2 Koba Nelson factor for N=2

To calculate $\langle e^{ik_1.\xi(\sigma_1)} \, e^{ik_2.\xi(\sigma_2)} \rangle$ we could be Taylor expanded it as,

$$\langle e^{ik_1.\xi(\sigma_1)} \, e^{ik_2.\xi(\sigma_2}\rangle = \left\langle (1+ik_1.\xi(\sigma_1)+\frac{1}{2!}(ik_1.\xi(\sigma_1))^2+\dots)(1+ik_2.\xi(\sigma_2)+\frac{1}{2!}(ik_2.\xi(\sigma_2))^2+\dots) \right\rangle$$

Considering

$$(1 + ik_2.\xi(\sigma_2) + \frac{1}{2!}(ik_2.\xi(\sigma_2))^2 + \dots) = \mathcal{T}$$

$$\langle e^{ik_1.\xi(\sigma_1)} \, e^{ik_2.\xi(\sigma_2}\rangle = \langle \mathcal{T} \rangle + \langle ik_1.\xi(\sigma_1)\mathcal{T} \rangle + \langle \frac{1}{2!}(ik_1.\xi(\sigma_1))^2\mathcal{T} \rangle + \dots \tag{C.3}$$

The $\langle \mathcal{T} \rangle$ can be interpreted from Eq (5.9) as,

$$T_0 = \langle \mathcal{T} \rangle = \epsilon^{\frac{\alpha'}{2}k_2^2}$$

Similarly the term $\langle ik_1.\xi(\sigma_1)\mathcal{T}\rangle$ could be written as

$$\Big\langle \big(ik_1.\xi(\sigma_1)\big)\big(1 + ik_2.\xi_2(\sigma_2) + \frac{1}{2!}(ik_2.\xi(\sigma_2))^2 + \dots\big)\Big\rangle \tag{C.4}$$

Similar to the above discussion the odd point functions will eventually become zero, also we keep in mind the identity

$$\big\langle \xi(\sigma_1)\xi(\sigma_2)\big\rangle = \frac{\alpha'}{2}\ \log(|\sigma_1 - \sigma_2|^2)$$

Therefore the second term in the Eq (C.4) would be

$$T_1^1 = \frac{\alpha'}{2}k_1.k_2\ \log(|\sigma_1 - \sigma_2|^2)$$

Similarly the fourth term would be

$$\frac{1}{3!}\big\langle k_1.\xi(\sigma_1)(k_2.\xi(\sigma_2))^3\big\rangle = \frac{1}{3!}\Big(\frac{\alpha'}{2}k_2^2\ \log(\epsilon^2)\Big)\Big(\frac{\alpha'}{2}k_1.k_2\ \log(|\sigma_1 - \sigma_2|^2)\Big)$$

There are three ways of wick contracting the fourth term in the expression therefore it comes with the factor of 3 therefore the above expression becomes,

$$= \frac{3}{3!}\Big(\frac{\alpha'}{2}k_2^2\ \log(\epsilon^2)\Big)\Big(\frac{\alpha'}{2}k_1.k_2\ \log(|\sigma_1 - \sigma_2|^2)\Big)$$

$$T_1^{(2)} = \Big(\frac{\alpha'}{4}k_2^2\ \log(\epsilon^2)\Big)\Big(\frac{\alpha'}{2}k_1.k_2\ \log(|\sigma_1 - \sigma_2|^2)\Big)$$

Similarly any general term of Eq (C.4)would be given by

$$T_1^{(n)} = \Big(\frac{\alpha'}{2}k_1.k_2\ \log(|\sigma_1{-}\sigma_2|^2)\Big)\frac{(2n+2)!}{(n+1)!\ 2^{(n+1)}}\ \frac{1}{(2n+1)!}\Big(\frac{\alpha'}{2}k_2^2\ \log\epsilon^2\Big)^{(2n+1-1)/2}$$

This is because the no of ways fo contracting $2n{+}1$ odd operators and 1 operator is given by $[AAAAA...B]$

$$\frac{(2n+2)!}{(n+1)!\ 2^{(n+1)}}$$

Also the power of $\Big(\frac{\alpha'}{2}k_2^2\ \log\epsilon^2\Big)$ is $(2n{+}1{-}1)/2$ because $2n{+}1{-}1$ ξ_2 operators contract among themselves and 2 (ξ_1 and ξ_2) contract with each other.

$$= \Big(\frac{\alpha'}{2}k_1.k_2\ \log(|\sigma_1 - \sigma_2|^2)\Big)\frac{1}{(n)!\ 2^n}\ \Big(\frac{\alpha'}{2}k_2^2\ \log\epsilon^2\Big)^n$$

Therefore summing up all the terms would give us the result

$$T_1 = \left(\frac{\alpha'}{2} k_1.k_2 \;\log(|\sigma_1 - \sigma_2|^2)\right) \sum_{n=0}^{\infty} \;\frac{1}{n!}\left(\frac{\alpha'}{4} k_2^2 \;\log \epsilon^2\right)^n$$

$$T_1 = \left(\frac{\alpha'}{2} k_1.k_2 \;\log(|\sigma_1 - \sigma_2|^2)\right) \exp\left(\frac{\alpha'}{4} k_2^2 \;\log \epsilon^2\right)$$

$$T_1 = \left(\frac{\alpha'}{2} k_1.k_2 \;\log(|\sigma_1 - \sigma_2|^2)\right) \epsilon^{\alpha'/2\; k_2^2}$$

Now looking at the third term in the expression of Eq (C.3)

$$T_2 = -\frac{1}{2!}\big\langle (k_1.\xi(\sigma_1))^2 \mathcal{T} \big\rangle$$

$$= -\frac{1}{2!}\big\langle (k_1.\xi(\sigma_1))^2 \big(1 + ik_2.\xi(\sigma_2) + \frac{1}{2!}(ik_2.\xi(\sigma_2)^2 + \ldots\big)\big\rangle$$

Of course the odd point functions vanishes, also the above contractions could be done in two ways, one by self contracting $\xi(\sigma_1)\xi(\sigma_1)$ and then contracting $\xi(\sigma_2)\xi(\sigma_2)\ldots$ the other way is to mix contract the operators $\xi(\sigma_1)$ and $\xi(\sigma_2)$, doing the self contraction we would obtain

$$\tilde{T}_2 = \frac{\alpha'}{4} k_1^2 \;\log \epsilon^2 \,\epsilon^{\frac{\alpha'}{2} k_2^2}$$

For the mix terms we can use the same logic as before,

$$T_2' = \frac{1}{2! \times 2!}\big\langle (k_1.\xi(\sigma_1))^2 (k_2.\xi(\sigma_2))^2 + \ldots \big\rangle$$

$$= \left(\frac{\alpha'}{2} k_1.k_2 \;\log(|\sigma_1 - \sigma_2|^2)\right)^2 \frac{1}{2!}\left(1 + \frac{\alpha'}{4} k_2^2 \;\log \epsilon^2 + \ldots\right)$$

$$T_2' = \frac{1}{2!}\left(\frac{\alpha'}{2} k_1.k_2 \;\log(|\sigma_1 - \sigma_2|^2)\right)^2 \epsilon^{\frac{\alpha'}{2} k_2^2}$$

The second term in the expression of Eq (C.3) is the combination of T_2' and $\tilde{T}_2$

$$T_2 = \frac{1}{2!}\left(\frac{\alpha'}{2} k_1.k_2 \;\log(|\sigma_1 - \sigma_2|^2)\right)^2 \epsilon^{\frac{\alpha'}{2} k_2^2} + \frac{\alpha'}{4} k_1^2 \;\log \epsilon^2 \,\epsilon^{\frac{\alpha'}{2} k_2^2}$$

Similarly the fourth term in Eq (C.3) must be

$$\frac{1}{3!}\big\langle (ik_1.\xi(\sigma_1))^3 \big(1 + ik_2.\xi(\sigma_2) + \frac{1}{2!}(ik_2.\xi(\sigma_2))^2 + \ldots\big)\big\rangle$$

Similar to the above expression we can do the contractions in 2 different ways one by self contracting the $\xi(\sigma_1)$ and $\xi(\sigma_2)$ and the remaining term is contracted by $\xi(\sigma_2)$ and the other way in which we contract $\xi(\sigma_1)$ and $\xi(\sigma_2)$ together, the former would give us

$$\frac{1}{3!}\left\langle\left((k_1.\xi(\sigma_1))^3 k_2.\xi(\sigma_2)\right)\left(1+\frac{1}{3!}(k_1.\xi(\sigma_2))^2+\dots\right)\right\rangle$$

$$\tilde{T}_3=\frac{3}{3!}\left(\frac{\alpha'}{2}k_1^2\ \log(\epsilon^2)\right)\frac{\alpha'}{2}k_1.k_2\ \log(|\sigma_1-\sigma_2|^2)\epsilon^{\frac{\alpha'}{2}k_2^2}$$

The factor of 3 is used because the above contraction could be done in 3 possible ways

$$\tilde{T}_3=\left(\frac{\alpha'}{4}k_1^2\ \log\epsilon^2\right)\left(\frac{\alpha'}{2}k_2.k_2\ \log(|\sigma_1-\sigma_2|^2)\right)\epsilon^{\frac{\alpha'}{2}k_2^2}$$

Similarly complete mixed contraction would give us

$$T_3'=\frac{1}{3!}\left(\frac{\alpha'}{2}k_1.k_2\ \log(|\sigma_1-\sigma_2|^2)\right)^3\epsilon^{\frac{\alpha'}{2}k_2^2}$$

The overall term is the combination of these two terms

$$T_3=\left(\frac{\alpha'}{4}k_1^2\ \log\epsilon^2\right)\left(\frac{\alpha'}{2}k_2.k_2\ \log(|\sigma_1-\sigma_2|^2)\right)\epsilon^{\frac{\alpha'}{2}k_2^2}+\frac{1}{3!}\left(\frac{\alpha'}{2}k_1.k_2\ \log(|\sigma_1-\sigma_2|^2)\right)^3\epsilon^{\frac{\alpha'}{2}k_2^2}$$

Combining all the terms,

$$\left\langle e^{ik_1.\xi(\sigma_1)}\, e^{ik_2.\xi(\sigma_2}\right\rangle=T_0+T_1+T_2+T_3+\dots$$

$$=\epsilon^{\frac{\alpha'}{2}k_2^2}+\left(\frac{\alpha'}{2}k_1.k_2\log(|\sigma_1-\sigma_2|^2)\right)\epsilon^{\frac{\alpha'}{2}k_2^2}+\frac{1}{2!}\left(\frac{\alpha'}{2}k_1.k_2\log(|\sigma_1-\sigma_2|^2)\right)^2\epsilon^{\frac{\alpha'}{2}k_2^2}$$

$$+\frac{\alpha'}{4}k_1^2\log\epsilon^2\,\epsilon^{\frac{\alpha'}{2}k_2^2}+\left(\frac{\alpha'}{4}k_1^2\log\epsilon^2\right)\left(\frac{\alpha'}{2}k_2.k_2\log(|\sigma_1-\sigma_2|^2)\right)\epsilon^{\frac{\alpha'}{2}k_2^2}+\frac{1}{3!}\left(\frac{\alpha'}{2}k_1.k_2\log(|\sigma_1-\sigma_2|^2)\right)^3\epsilon$$

$$=\epsilon^{\frac{\alpha'}{2}k_2^2}\left[1+\left(\frac{\alpha'}{2}k_1.k_2\ \log(|\sigma_1-\sigma_2|^2)\right)+\frac{1}{2!}\left(\frac{\alpha'}{2}k_1.k_2\ \log(|\sigma_1-\sigma_2|^2)\right)^2+\dots\right]$$

$$+\epsilon^{\frac{\alpha'}{2}k_2^2}\left(\frac{\alpha'}{4}k_1^2\ \log\epsilon^2\right)\left[1+\frac{\alpha'}{2}k_1.k_2\ \log(|\sigma_1-\sigma_2|^2)+\dots\right]+\frac{1}{2!}\epsilon^{\frac{\alpha'}{2}k_2^2}\left(\frac{\alpha'}{4}k_1^2\ \log\epsilon^2\right)^2\left[1+\frac{\alpha'}{2}k_1.k_2..\right]+$$

$$\epsilon^{\frac{\alpha'}{2}k_2^2}\left[1+\left(\frac{\alpha'}{4}k_1^2\ \log\epsilon^2\right)+\frac{1}{2!}\left(\frac{\alpha'}{4}k_1^2\ \log\epsilon^2\right)^2\right]\left[1+\left(\frac{\alpha'}{2}k_1.k_2\ \log(|\sigma_1-\sigma_2|^2)\right)\right.$$

$$\frac{1}{2!}\left(\frac{\alpha'}{2}k_1.k_2\ \log(|\sigma_1-\sigma_2|^2)\right)^2+\dots\Bigg]$$

$$\epsilon^{\frac{\alpha'}{2}k_2^2}\exp\left[\frac{\alpha'}{4}k_1^2\ \log\epsilon^2\right]\exp\left[\frac{\alpha'}{2}k_1.k_2\ \log(|\sigma_1-\sigma_2|^2)\right]$$

$$\epsilon^{\frac{\alpha'}{2}k_2^2}\ \epsilon^{\frac{\alpha'}{2}k_1^2}\ |\sigma_1-\sigma_2|^{\alpha' k_1.k_2}$$

Therefore for N=2

$$\frac{1}{\epsilon^4}\langle e^{ik_1.X(\sigma_1)}\ e^{ik_2.X(\sigma_2)}\rangle = e^{ik_1X_0(\sigma_1)}e^{ik_2X_0(\sigma_2)}\epsilon^{\frac{\alpha'}{2}(k_1^2+k_2^2)-4}\ |\sigma_1-\sigma_2|^{\alpha' k_1.k_2} \tag{C.5}$$

C.3 Tachyon beta function for N=2

Now in order to obtain the β function we use the standard definition of beta function,

$$\epsilon\frac{d\tilde{\Phi}_R^{(2)}}{d\epsilon} = \big(\frac{\alpha'}{2}k^2-2\big)\epsilon^{\frac{\alpha'}{2}k^2-2}\tilde{\Phi}(k)+\tilde{\beta}(k)\epsilon^{\frac{\alpha'}{2}k^2-2}-\frac{1}{4\pi}\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{\delta(k-k_1-k_2)}{\frac{\alpha'}{2}k_1k_2+1}\epsilon^{\frac{\alpha'}{2}k^2-2}$$
$$\times\left[\big(\frac{\alpha'}{2}k^2-2\big)\tilde{\Phi}(k_1)\tilde{\Phi}(k_2)+\tilde{\beta}(k_1)\tilde{\Phi}(k_2)+\tilde{\Phi}(k_1)\tilde{\beta}(k_2)\right]=0 \tag{C.6}$$

The general renormalization group equations for a coupling g_i could be written as,

$$\beta_i=\frac{dg_i}{dt}=\lambda^i g_i+\alpha^i_{jk}g^jg^k+\gamma^i_{jkl}g^ig^jg^k+\dots \tag{C.7}$$

Where the first term is not summed and $t=\log\epsilon$, therefore the integral version of beta function could be written as,

$$\tilde{\beta}_\Phi(k)=a(k)\tilde{\Phi}(k)+\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}b(k,k_1,k_2)\tilde{\Phi}(k_1)\tilde{\Phi}(k_2)+\mathcal{O}(\tilde{\Phi}^3) \tag{C.8}$$

Substituting Eq (C.8) in Eq (C.6) and using $a(k)=(2-\frac{\alpha'}{2}k^2)$ we obtain

$$\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}b(k,k_1,k_2)=\frac{1}{2}\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{\delta(k-k_1-k_2)}{\frac{\alpha'}{2}k_1k_2+1}\left[\frac{\alpha'}{2}\big(k^2-k_1^2-k_2^2\big)+2\right]$$

Since $k=k_1+k_2$ and $k^2-k_1^2+k_2^2=2k_1k_2$ therefore,

$$b(k,k_1,k_2)=\delta(k-k_1-k_2)$$

Using $b(k,k_1,k_2)=\delta(k,k_1,k_2)$ in Eq (C.8) we get the tachyon beta function till order $\mathcal{O}(\Phi^2)$ as

$$\tilde{\beta}_\Phi(k) = -\big(\frac{\alpha'}{2}k^2 - 2\big) + \int \frac{d^D k_1}{(2\pi)^D}\frac{d^D k_2}{(2\pi)^D}\delta(k - k_1 - k_2)\tilde{\Phi}(k_1)\tilde{\Phi}(k_2) + \mathcal{O}(\Phi^3)$$

Which when Fourier transformed gives us

$$\beta_\Phi = \frac{1}{2}\big(\alpha'\partial^\mu\partial_\mu + 4)\Phi + \Phi^2 \tag{C.9}$$

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