Linear Fluctuations In Gravity

M.Sc. Thesis

By

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Linear Fluctuations In Gravity

A THESIS

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Master of Science

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"Linear Fluctuations In Gravity"** in partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF PHYSICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2024 to May 2025 under the supervision of **Dr. Manavendra N. Mahato**, Associate Professor, Department of Physics, IIT Indore.

The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

Faisal
20-05-2025

Signature of the student with date
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This is to certify that the above statement made by the candidate is correct to the best of my knowledge. .

20/5/2025

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Date: 20-05-25

Dedicated to my family

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Abstract

This study investigates gravitational waves as linear perturbations in curved spacetimes. Starting from the linearized Einstein equations, we derive gravitational wave equations in both spatially flat and closed FLRW universes, analyzing their evolution in expanding and contracting scenarios. The study further extends to a five-dimensional anisotropic cosmological model, where gravitational wave behavior is examined using separation of variables and gauge conditions. The results illustrate how spacetime geometry influences wave propagation, offering insights into gravitational dynamics beyond flat backgrounds.

Conventions

In this work, certain mathematical conventions are used and the analysis relies on standard results from Einstein's framework of gravitation.

- Metric signature is mostly positive, i.e., $(-,+,+,+)$.
- Throughout this work, natural units are used by setting the gravitational constant and the speed of light equal to one ($G = c = 1$).
- Indices corresponding to spatial directions are represented by Latin symbols, e.g., x., whereas spacetime 4-vectors (e.g., t, x) are denoted by Greek indices (e.g., μ, ν). Unless otherwise stated, the Einstein summation convention is assumed.

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Chapter 1

Introduction

Albert Einstein first predicted gravitational waves in 1916 as a natural outcome of his general theory of relativity, represent a profound aspect of modern astrophysics and theoretical physics. Gravitational waves are ripples in the fabric of spacetime generated by the movement or acceleration of massive bodies, such as binary systems of neutron stars or black holes. As these cosmic events unfold, they emit ripples that travel at the speed of light, transmitting crucial information about their sources and the underlying properties of gravity.

Gravitational waves were purely theoretical for almost 100 years until they were directly observed by the Laser Interferometer Gravitational Wave Observatory (LIGO) in September 2015. This landmark achievement confirmed Einstein's predictions and opened a new observational window into the universe, ushering in the field of gravitational wave astronomy. Since then, multiple events have been detected, providing insights into phenomena such as black hole mergers, neutron star collisions, and potentially offering a glimpse into the early universe.

The study of gravitational waves poses both theoretical and experimental challenges. Understanding their propagation requires a deep comprehension of general relativity, while the detection relies on highly sensitive instruments capable of measuring minuscule changes in distance caused by passing waves. The implications of gravitational wave observations extend beyond astrophysics; they touch upon fundamental questions regarding the nature of spacetime, the validity of general relativity in extreme conditions, and the possibility of new physics beyond our current understanding.

This report aims to explore the theoretical foundations of gravitational waves, detailing their generation, propagation, and detection.

1.1 What Makes Gravitational Waves Important?

The detection of gravitational waves is crucial for two primary reasons. First, it promises to revolutionize observational astronomy by providing a new perspective on the universe. Gravitational waves carry information distinct from that of electromagnetic waves, enhancing our understanding of cosmic phenomena. This new avenue will allow scientists to study the intricate structure of spacetime around black holes, directly observe the formation and mergers of black holes and neutron stars, search for rapidly spinning neutron stars, explore the early moments of the universe, and investigate the supermassive black holes residing at the centers of galaxies. These potential discoveries represent just a glimpse of the significant advancements expected in the early 21st century. Second, the confirmation of gravitational waves will validate a key prediction of general relativity made 85 years ago, deepening our understanding of fundamental physics. Additionally, by comparing the arrival times of light and gravitational waves from events such as supernovae, we can test Einstein's assertion that both travel at the same speed and confirm their predicted polarization as described by general relativity.

1.2 Methods for Detecting Gravitational Waves

Gravitational waves are observed through the use of extremely precise and sensitive instruments called interferometers, with the most prominent examples being Laser Interferometer Gravitational Wave Observatory (LIGO) along with the Virgo detector. The core technology behind gravitational wave detection is interferometry. Interferometers use the principle of interference of light beams to measure minute changes in distance.

LIGO and Virgo consist of two long arms arranged in an L-shape, each several kilometers long.A powerful laser beam is split into two beams that travel down the two arms of the interferometer. Each beam reflects off mirrors located at the ends of the arms and returns to a central point where they recombine.When a gravitational wave passes through the detector, it distorts spacetime, causing one arm of the interferometer to stretch while the other contracts. This change in length is incredibly small—on the order of one-thousandth the diameter of a proton. The interference pattern of the combined laser beams is sensitive to these changes, any variation indicates that a gravitational wave has passed through.The detected signal is processed using advanced algorithms to distinguish genuine gravitational wave events from background noise. Researchers study the wave's frequency and strength to gain insights into their origins, including events like black hole or neutron star mergers.

LIGO operates with two detectors located in different geographic locations (one in Louisiana and the other in Washington) to improve the accuracy of the signals and help triangulate the source of the gravitational waves. Virgo, located in Italy, collaborates with LIGO to enhance detection capabilities.Improvements in technology, such as increased laser power, better mirror quality, and active vibration isolation systems, have enhanced sensitivity. Future detectors, like the planned Einstein Telescope and LISA (Laser Interferometer Space Antenna), aim to detect a wider range of gravitational wave frequencies and sources.

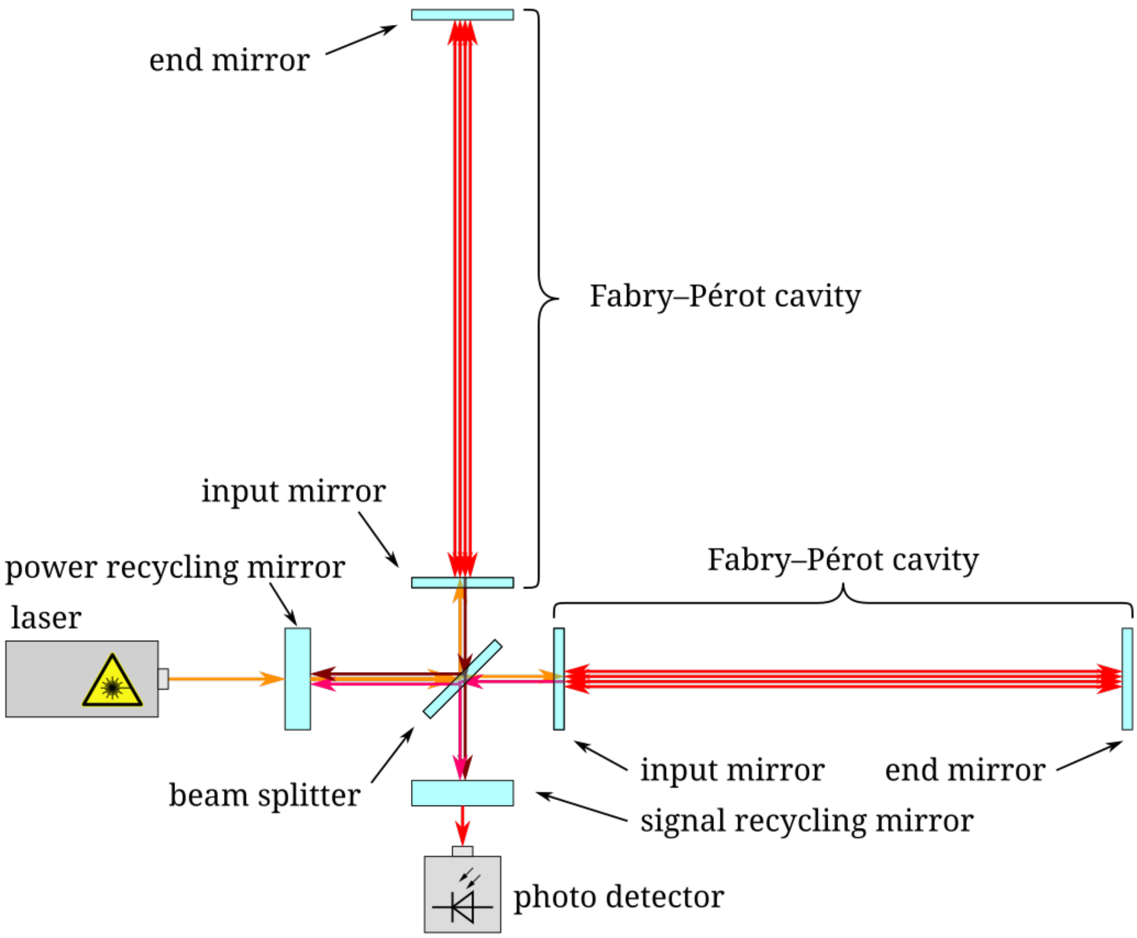


Figure 1.1: A simplified schematic of the LIGO detector.

1.3 Properties of gravitational waves

Gravitational waves are transverse in nature, with their oscillations occurring at right angles to the direction in which the wave travels. As they pass through space, they stretch and compress distances in the fabric of spacetime.These waves travel at the speed of light c. This property is in agreement with the expectations set forth by Einstein's general relativity. Gravitational waves exhibit two polarization states, often referred to as "plus" (+) and "cross" (×) polarizations. These polarization's describe how the waves affect distances in different orientations as they propagate. The amplitude of a Gravitational wave indicates the strength of the wave and is

related to the energy of the event that generated it. The frequency of the waves can vary widely, with different sources producing different frequency ranges. For example, merging black holes typically emit higher-frequency waves compared to the lower frequencies associated with supernovae.
Gravitational waves carry energy away from their sources, providing a unique means of understanding extreme astrophysical phenomena. The information encoded in these waves can reveal details about their origins, such as the masses, spins, and distances of the objects involved. Due to their minimal interaction with matter, gravitational waves can pass through celestial bodies like stars and planets almost unaffected. This property enables them to provide information about distant cosmic events that would otherwise be obscured by intervening matter. The characteristics of gravitational waves depend on their sources. For instance, binary systems of black holes and neutron stars produce distinct waveform during their inspiral, merger, and ringdown phases, allowing astrophysicists to identify the type of event and extract relevant physical parameters.
These properties make gravitational waves a powerful tool for exploring the universe, offering new insights into astrophysics, cosmology, and fundamental physics.

1.4 Why Gravitational waves are bit Complicated to describe Mathematically ?

Gravitational waves are bit complicated to describe mathematically and this is for two main reasons. The first reason is that the Einstein's equations are non-Linear, as a result we can't write the exact solution in most cases. We have to work with solutions that are only approximate. The another consequence of the non-linearity is that the waves can't be superimposed.

The second main reason why gravitational wave theory is complicated is because general relativity contains a large number of gauge degrees freedom, meaning that there are multiple equivalent mathematical descriptions of the same physical situation. This flexibility can lead to ambiguity in the mathematical formulation, making it challenging to identify the most useful or intuitive representation of gravitational waves.

Chapter 2

Theory of Gravitational Waves

Equations used :

- Einstein's Field Equations ;
  At the heart of Einstein's general relativity lie the Einstein field equations. The equations express how the distribution of matter and energy alters spacetime's geometry, determining the paths objects follow.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{2.1}$$

  Where, $G_{\mu\nu} = 8\pi T_{\mu\nu}$ and $T_{\mu\nu}$ is the stress energy-momentum tensor.
  $G_{\mu\nu}$ is The Einstein Tensor and we used Natural Units that is we take c=1, G=1 and $\hbar = 1$.

- Conservation Laws for the Stress-Energy Tensor ;
  The conservation of stress-energy-momentum is a fundamental principle that describes how energy, momentum, and stress are conserved inside a curved fabric of spacetime. This principle is encapsulated in the equation:

$$\nabla_\mu T^{\mu\nu} = 0$$

- Christoffel symbols ;
  The Christoffel symbols are mathematical objects used in general relativity to describe how coordinates change in curved spacetime. They are essential for defining the connection and curvature of a manifold.They are given as :

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta} \right) \tag{2.2}$$

- When the gravitational field is weak, the spacetime metric can be written as a Minkowski metric plus a slight disturbance ;

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.3}$$

Where the perturbation is small as $|h_{\mu\nu}| << 1$. Starting with the definition of the inverse metric, we determine that to first order in h;

$$g^{\mu\nu} = n^{\mu\nu} - h^{\mu\nu} \tag{2.4}$$

- Riemann tensor;
  General relativity relies heavily on the Riemann curvature tensor as a fundamental quantity that measures the intrinsic curvature of a manifold, providing insight into how spacetime is shaped by mass and energy. The Riemann tensor is given as :
  $$R^{\alpha}_{\mu\beta\nu} = \partial_\beta \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\alpha}_{\sigma\beta} - \Gamma^{\sigma}_{\mu\beta}\Gamma^{\alpha}_{\sigma\nu} \tag{2.5}$$

2.1 Linearized Einstein Equation ;

The linearized Einstein equations are a simplified form of Einstein equations used in general relativity, applicable when the gravitational field is weak. In the presence of a weak gravitational field, the metric tensor $g_{\mu\nu}$ is represented as a minor disturbance $h_{\mu\nu}$ superimposed on the flat metric $\eta_{\mu\nu}$ expressed as ;

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

In order to calculate the linearized Einstein equation let us start from Einstein tensor;

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

We first need to calculate the Riemann Tensor for calculating Einstein Tensor.
For Riemann Tensor let us first calculate Christoffel symbol given by equation (2.2):

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right)$$

From equation (2.4) we use the value of $g^{\mu\nu}$ we get ;

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}(n^{\mu\nu} - h^{\mu\nu}) \left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right)$$

or

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}(n^{\mu\nu} - h^{\mu\nu}) \left(\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}\right)$$

So in order to keep our equation linear in h we drop $h^{\mu\nu}$ term, our equation becomes;

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}(n^{\mu\nu}) \left(\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}\right) \tag{2.6}$$

Now Riemann Tensor from equation (2.5) is given as ;

$$R^{\alpha}_{\mu\beta\nu} = \partial_\beta \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\alpha}_{\sigma\beta} - \Gamma^{\sigma}_{\mu\beta}\Gamma^{\alpha}_{\sigma\nu}$$

Since Christoffel symbols are linear in the perturbation h, The $\Gamma\Gamma$ terms are quadratic in h,to linear order in small perturbation we ignore these terms. we write as ;

$$R^{\alpha}_{\mu\beta\nu} = \partial_\beta \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} \tag{2.7}$$

Now Ricci tensor $R_{\mu\nu}$ is calculated by contracting the Riemann Tensor as ;

$$R^{\alpha}_{\mu\alpha\nu} = R_{\mu\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha} \tag{2.8}$$

Now calculating the value of $\Gamma^{\alpha}_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\alpha}$ from equation 2.6 we get ;

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}(n^{\alpha\beta})\left(\partial_\mu h_{\nu\beta} + \partial_\nu h_{\mu\beta} - \partial_\beta h_{\mu\nu}\right)$$

and

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2}(n^{\alpha\beta})\left(\partial_\mu h_{\alpha\beta} + \partial_\alpha h_{\mu\beta} - \partial_\beta h_{\mu\alpha}\right)$$

Using these two expressions in equation (2.8) we get ;

$$\begin{aligned} R_{\mu\nu} =& \partial_\alpha[\frac{1}{2}(n^{\alpha\beta})\left(\partial_\mu h_{\nu\beta} + \partial_\nu h_{\mu\beta} - \partial_\beta h_{\mu\nu}\right)] \\ &-\partial_\nu[\frac{1}{2}(n^{\alpha\beta})\left(\partial_\mu h_{\alpha\beta} + \partial_\alpha h_{\mu\beta} - \partial_\beta h_{\mu\alpha}\right)] \end{aligned}$$

Calculating the above equation we get ;

$$R_{\mu\nu} = \frac{1}{2}[\partial_\alpha \partial_\mu h^{\alpha}_{\nu} - \partial_\alpha \partial^\alpha h_{\mu\nu} - \partial_\nu \partial_\mu h^{\alpha}_{\alpha} - \partial_\nu \partial^\alpha h_{\mu\alpha}] \tag{2.9}$$

The Ricci Tensor is symmetric tensor that is $R_{\mu\nu} = R_{\nu\mu}$.
Now let us calculate the Ricci scalar R by raising one index and setting it equal to the other index;

$$R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu}$$

Or

$$R = \frac{1}{2}[\partial_\alpha \partial_\mu g^{\mu\nu} h^{\alpha}_{\nu} - \partial_\alpha \partial^\alpha g^{\mu\nu} h_{\mu\nu} - \partial_\nu \partial_\mu g^{\mu\nu} h^{\alpha}_{\alpha} + \partial_\nu \partial^\alpha g^{\mu\nu} h_{\mu\alpha}]$$

Or

$$R = \partial_\alpha \partial_\mu h^{\alpha\mu} - \partial_\alpha \partial^\alpha h^{\mu}_{\mu} \tag{2.10}$$

Now from Equation (2.1) Einstein's tensor is given as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Or

$$G_{\mu\nu} = \frac{1}{2}[\partial_\alpha\partial_\mu h^\alpha_\nu - \partial_\alpha\partial^\alpha h_{\mu\nu} - \partial_\mu\partial_\nu h^\alpha_\alpha$$
$$-\partial_\alpha\partial_\nu h^\alpha_\nu - n_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} + n_{\mu\nu}\partial_\alpha\partial^\alpha h^\beta_\beta]$$

Or

$$G_{\mu\nu} = \frac{1}{2}[\partial_\alpha\partial_\nu h^\alpha_\nu + \partial_\alpha\partial_\nu h^\alpha_\mu - \partial_\alpha\partial^\alpha h_{\mu\nu}$$
$$-\partial_\mu\partial_\nu h^\alpha_\alpha - n_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} + n_{\mu\nu}\partial_\alpha\partial^\alpha h^\beta_\beta] \tag{2.11}$$

Since Ricci scalar is linear in h and our calculations are restricted to linear order of h, so we used $g_{\mu\nu} = n_{\mu\nu}$.
Now let us define **Trace-Reversed-Perturbation** as ;

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}n_{\mu\nu}h^\alpha_\alpha$$

Its called Trace reversed perturbation because $\bar{h}^\alpha_\alpha = -h^\alpha_\alpha$.
We can also write ;

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}n_{\mu\nu}\bar{h}^\alpha_\alpha \tag{2.12}$$

Using equation (2.12) in equation (2.11), We get;

$$G_{\mu\nu} = \frac{1}{2}[\partial_\mu\partial_\alpha\bar{h}^\alpha_\nu + \partial_\nu\partial_\alpha\bar{h}^\alpha_\mu - \partial_\alpha\partial^\alpha\bar{h}_{\mu\nu} - n_{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{\alpha\beta}]$$

This is the **linear approximation of Einstein's tensor**, expanded to the first order in the perturbation h.
It can also be denoted as ;

$$G^{(l)}_{\mu\nu} = \frac{1}{2}[\partial_\mu\partial_\alpha\bar{h}^\alpha_\nu + \partial_\nu\partial_\alpha\bar{h}^\alpha_\mu - \partial_\alpha\partial^\alpha\bar{h}_{\mu\nu} - n_{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{\alpha\beta}] \tag{2.13}$$

Where **(L)** denotes that the Einstein tensor is written in linear order of h.
What we actually did is we broke the Einstein tensor as;

$$G_{\mu\nu}(g) = G_{\mu\nu}(n+h) = G_{\mu\nu}(n) + G^{(l)}_{\mu\nu}(n,h) + G^{(Q)}_{\mu\nu}(n,h) + O(h)$$

$G_{\mu\nu} = 0$ for flat metric, $G^{(Q)}_{\mu\nu}(n,h)$ is Einstein tensor which is quadratic in h and O(h) denotes higher order terms of h, $G^{(l)}_{\mu\nu}(\eta,h)$ Is linearized Einstein gravity.
Likewise, we expand the stress-energy tensor $T_{\mu\nu}$ in terms of the perturbation h;

$$T_{\mu\nu}(g) = T_{\mu\nu}(n+h) = T_{\mu\nu}(n) + T^{(l)}_{\mu\nu}(n,h) + T^{(Q)}_{\mu\nu}(n,h) + O(h)$$

We assume that the energy, momentum and stresses produced by matter fields are small, that is $T_{\mu\nu}$ is small (same order small as h). Also we say that the linear

order term of the above expansion is same order as the h^2 and so on.
The Einstein equations written in order h are written as ;

$$G^{(l)}_{\mu\nu}(\eta, h) = 8\pi T_{\mu\nu}(\eta)$$

or

$$\frac{1}{2}[\partial_\mu \partial_\alpha \bar{h}^\alpha_\nu + \partial_\nu \partial_\alpha \bar{h}^\alpha_\mu - \partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} - n_{\mu\nu} \partial_\alpha \partial_\beta \bar{h}^{\alpha\beta}] = 8\pi T_{\mu\nu}(\eta, h)$$

These are called the **linearized Einstein equation.** These equations describe how weak gravitational fields propagate and lead to the prediction of gravitational waves.
We know conservation of stress-energy momentum is written as ;

$$\nabla_\mu T^{\mu\nu} = 0$$

Expanding this in order of h, we have $\partial_\mu T^{\mu\nu}(\eta) = 0$ to order one in small quantities. Therefore from linearized Einstein equation we can write ;

$$\partial_\mu G^{(l)\,\mu\nu}(\eta, h) = 0$$

2.2 Gauge freedom of Linearized Gravity

In linearized gravity, the concept of gauge freedom refers to the ability to make coordinate transformations that do not affect the physical predictions of the theory. Specifically,this freedom allows for the adjustment of the perturbation $h_{\mu\nu}$ without changing the underlying physics.
Consider two metric $g_{\mu\nu}$ and $g'_{\mu\nu}$ both are perturbation on flat metric $\eta_{\mu\nu}$ given as;

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

and

$$g'_{\mu\nu} = \eta_{\mu\nu} + h'_{\mu\nu}$$

One can write the general coordinate transformation as;

$$x^\mu = x^\mu(x')$$

let's write the coordinate transformation that is close to identity ;

$$x^\mu = x^{\mu'} + \xi^\mu(x')$$

ξ^μ is small, same order as h or h'. Now we can write as;

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}[x(x')]$$

where x^α and x^β are functions of x which inturn are functions of x'. Now $g'_{\mu\nu}$ in terms of flat metric and perturbation is given as;

$$\eta_{\mu\nu} + h'_{\mu\nu} = [\delta^\alpha_\mu + \partial_\mu \xi^\alpha(x')][\delta^\beta_\nu + \partial_\nu \xi^\beta(x')][\eta_{\alpha\beta} + h_{\alpha\beta}(x')]$$

neglecting the higher order terms we get;

$$\begin{aligned}\eta_{\mu\nu} + h'_{\mu\nu} =& \delta^\alpha_\mu \delta^\beta_\nu \eta_{\alpha\beta} + \delta^\alpha_\mu \delta^\beta_\nu h_{\alpha\beta}(x') \\ &+ \partial_\mu \xi^\alpha(x') \delta^\beta_\nu \eta_{\alpha\beta} + \delta^\alpha_\mu \partial_\nu \xi^\beta(x') \eta_{\alpha\beta}\end{aligned}$$

Or

$$\eta_{\mu\nu} + h'_{\mu\nu}(x') = \eta_{\mu\nu} + \partial_\mu \xi_\nu(x') + \partial_\nu \xi_\mu(x') + h_{\mu\nu}(x')$$

Or

$$h'_{\mu\nu}(x') = h_{\mu\nu}(x') + \partial_\mu \xi_\nu(x') + \partial_\nu \xi_\mu(x')$$

recall that x and x' are related as ;

$$x^\mu(x) = x^{\mu'} + \xi^\mu(x')$$

so to leading order;

$$x^\mu \approx x'^\mu$$

Therefore ;

$$h'_{\mu\nu}(x) = \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) + h_{\mu\nu}(x) \tag{2.14}$$

This expression represents the **gauge transformation for linearized gravity.** Hence we can say $h_{\mu\nu}$ and $h'_{\mu\nu}$ describe the same spacetime geometry as they are not physically distinct.

2.3 Gauge transformation in terms of Trace reverse perturbation

We write gauge transformations in terms of trace reversed perturbation because it helps in identifying physical degrees of freedom, making it easier to analyze gravitational waves and their propagation. From gauge transformation equation we write;

$$h'_{\mu\nu}(x) = \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) + h_{\mu\nu}(x)$$

Also from equation (2.12) the trace reversed perturbation is given as ;

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} n_{\mu\nu} \bar{h}^\alpha_\alpha$$

Using $h_{\mu\nu}$ in gauge transformation, we get trace reversed gauge transformation as ;

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\alpha \xi^\alpha$$

raising indices and operating ∂_μ on both sides of the equation, we get ;

$$\partial_\mu \bar{h}^{\mu\nu\,\prime} = \partial_\mu \bar{h}^{\mu\nu} + \partial_\mu \partial^\mu \xi^\nu + \partial_\mu \partial^\nu \xi^\mu - \partial^\nu \partial_\alpha \xi^\alpha$$

Or

$$\partial_\mu \bar{h}^{\mu\nu\,\prime} = \partial_\mu \bar{h}^{\mu\nu} + \partial_\mu \partial^\mu \xi^\nu$$

Or

$$\partial_\mu \partial^\mu \xi^\nu = \partial_\mu \bar{h}^{\mu\nu\,\prime} - \partial_\mu \bar{h}^{\mu\nu} \tag{2.15}$$

where $\partial_\mu \partial^\mu$ is called the flat-space wave operator.
Equation (2.15) can also be written as ;

$$\Box \xi^\nu = \partial_\mu \bar{h}^{\mu\nu\,\prime} - \partial_\mu \bar{h}^{\mu\nu}$$

Thus we can say each component of ξ^ν satisfies the wave equation, hence solutions exist.

2.4 Linearized Einstein equation in Lorentz Gauge

Gauge transformations in the linearized Einstein equations provide a mechanism to simplify the analysis of weak gravitational fields, ensuring that physical predictions are preserved while allowing for flexibility in mathematical treatment.
By choosing the gauge parameter ξ^μ satisfying

$$\partial_\mu \partial^\mu \xi^\mu = -\partial_\mu \bar{h}^{\mu\nu}$$

and from equation (2.15) we can write;

$$\partial_\mu \bar{h}^{\mu\nu} = 0 \tag{2.16}$$

This constraint is called the **Lorentz gauge.**
Now linearized Einstein equation is written as ;

$$G^{(l)}_{\mu\nu}(\eta, h) = 8\pi T_{\mu\nu}(\eta)$$

$$\frac{1}{2}[\partial_\mu \partial_\alpha \bar{h}^\alpha_\nu + \partial_\nu \partial_\alpha \bar{h}^\alpha_\mu - \partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} - n_{\mu\nu} \partial_\alpha \partial_\beta \bar{h}^{\alpha\beta}] = 8\pi T_{\mu\nu}(\eta, h)$$

Using gauge condition we get ;

$$\frac{1}{2}\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -8\pi T_{\mu\nu}$$

Or

$$\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{2.17}$$

Therefore;

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$$

This is **Einstein equation in lorentz gauge**.The Einstein equation in lorentz gauge is therefore simply a **wave equation** for each component of trace reverse metric perturbation. Under the lorentz gauge condition, the equations can be interpreted as wave equations, indicating that the gravitational field propagates as waves at the speed of light.

Chapter 3

Gravitational Waves in Vacuum regions of Spacetime

Gravitational waves in vacuum regions of spacetime are ripples in the curvature of spacetime caused by accelerating masses, propagating through a vacuum without the influence of matter. In vacuum regions of space time we take;

$$T_{\mu\nu} = 0$$

We are not assuming $T_{\mu\nu}$=0 everywhere in the universe, but there might be matter fields outside this region.We will consider radiative or Wave like solutions to the linearized Einstein equation. There can be non-radiative solutions as well for example if we have a black hole outside our region, but we don't consider them for now.

3.1 Radiative (Wave-like) solution

The radiative solution of gravitational waves in vacuum regions of spacetime describes how these waves propagate and behave in areas devoid of matter. This solution is derived from the linearized form of Einstein's equations. From equations (2.16) and (2.17) we get know;

$$\partial_\mu \bar{h}^{\mu\nu} = 0$$

$$\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$$

Using the value of $T_{\mu\nu}$=0, we have ;

$$\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = 0$$

Or

$$\Box \bar{h}^{\mu\nu} = 0$$

where $\Box$ is d'Alembert operator. Solution to this equation can be considered as monochromatic plane wave, as ;

$$\bar{h}^{\mu\nu} = Re(A^{\mu\nu} e^{i\mathbf{k}\cdot\mathbf{x}})$$

where $A^{\mu\nu}$ is the amplitude expressed as a complex quantity, k is wave vector and x represents spacetime coordinate.
$k \cdot x$ is a four vector dot product written as ;

$$k \cdot x = -\omega t + \vec{k} \cdot \vec{x}$$

Therefore ;

$$\bar{h}^{\mu\nu} = Re(A^{\mu\nu} e^{\iota(\vec{k}\cdot\vec{x} - \omega t)}) \tag{3.1}$$

Now impose lorentz gauge condition given by equation (2.16), we get;
For ν=0 ; i=1,2,3

$$Re[(-\iota\omega A^{00} + \iota k^i A^{i0})] e^{\iota(\vec{k}\cdot\vec{x} - \omega t)} = 0$$

For ν=j ; j=1,2,3

$$Re[(-\iota\omega A^{0j} + \iota k^i A^{ij})] e^{\iota(\vec{k}\cdot\vec{x} - \omega t)} = 0$$

These equations tell us that components of the amplitude $A^{\mu\nu}$ are not arbitary, they must follow ;

$$A^{00} = \frac{k^i}{\omega} A^{i0} \text{ and } A^{0j} = \frac{k^i}{\omega} A^{ij}$$

combining the two equation we get;

$$A^{00} = \frac{k^i}{\omega}\frac{k^j}{\omega} A^{ij}$$

but we know ω= $|\vec{k}|$ and $\hat{k} = \frac{k^i}{\omega}$, represents unit vector in the direction of the propagation. Thus the amplitudes can be written as ;

$$A^{00} = \hat{k^i}\hat{k^j} A^{ij} \text{ and } A^{0j} = \hat{k^j} A^{ij}$$

Now using above results of amplitude in equation (3.1) we get ;

$$\bar{h}^{00} = Re(\hat{k^i}\hat{k^j} A^{ij} e^{\iota(\vec{k}\cdot\vec{x} - \omega t)})$$

$$\bar{h}^{0i} = Re(\hat{k^j} A^{ij} e^{\iota(\vec{k}\cdot\vec{x} - \omega t)})$$

$$\bar{h}^{ij} = Re(A^{ij} e^{\iota(\vec{k}\cdot\vec{x} - \omega t)})$$

These three equations represent the general solution for a monochromatic plane gravitational wave under the lorentz gauge. Now we can superimpose these plane wave equations to obtain arbitrary radiative solution in the vacuum region,which is written as ;

$$\bar{h}^{00} = Re \int d^3k \;\; \hat{k^i}\hat{k^j} A^{ij} e^{\iota(\vec{k}\cdot\vec{x} - \omega t)} \tag{3.2}$$

$$\bar{h}^{0i} = Re \int d^3k \;\; \hat{k^j} A^{ij} e^{\iota(\vec{k}\cdot\vec{x} - \omega t)} \tag{3.3}$$

$$\bar{h}^{ij} = Re \int d^3k \;\; A^{ij} e^{\iota(\vec{k}\cdot\vec{x}-\omega t)} \tag{3.4}$$

The above three equations of $\bar{h}^{00}$, $\bar{h}^{0i}$ and $\bar{h^{ij}}$ represent our arbitrary radiative solution in vacuum regions.

3.2 Radiative solutions in the Transverse-Tracless TT Gauge.

Recall the expression for the trace reversed gauge transformation ;

$$\bar{h'}^{\mu\nu} = \partial^{\mu}\xi^{\nu} + \partial^{\nu}\xi^{\mu} - \eta^{\mu\nu}\partial_{\alpha}\xi^{\alpha} + \bar{h}^{\mu\nu}$$

We can transform any $h_{\mu\nu}$ into lorentz gauge, $\partial_{\mu}\partial^{\alpha} = 0$ by above given equation with ;

$$\partial_{\alpha}\partial^{\alpha}\xi^{\mu} = -\partial_{\alpha}\bar{h}^{\alpha\mu}$$

we know ξ^{μ} is not unique, we can add to ξ^{μ} any set of functions ζ^{μ} that satisfy source free wave equation as;

$$\partial_{\alpha}\partial^{\alpha}\zeta^{\mu} = 0$$

therefore the general radiative solution of this equation is written as;

$$\zeta^{\mu} = Re \int d^3k \;\; C^{\mu} e^{\iota(\vec{k}\cdot\vec{x}-\omega t)}$$

Now the gauge transformation for ζ^{μ} is written as ;

$$\bar{h'}^{\mu\nu} = \bar{h}^{\mu\nu} + \partial^{\mu}\zeta^{\nu} + \partial^{\nu}\zeta^{\mu} - \eta^{\mu\nu}\partial_{\alpha}\zeta^{\alpha}$$

let's consider the i, j components of this transformation and insert the wave solution for $\bar{h}^{\mu\nu\prime}$, $\bar{h}^{\mu\nu}$ and ζ^{μ} we get;

$$A^{ij\prime} = A^{ij} + \iota[k^i C^j + k^j C^i - \delta^{ij}(-\omega C^0 + \vec{k}\cdot\vec{C})]$$

Where all A's and C's are functions of $\vec{k}$. We can always choose C^0 and C^i such that

$$\hat{k}^j A^{ij} = 0 \quad \text{Transverse condition} \tag{3.5}$$

And

$$A^{ij} = 0 \quad \text{Traceless Condition} \tag{3.6}$$

This is called **Transverse Traceless Gauge**.
Equation (3.5) indicates that the amplitudes are perpendicular to the wave's propagation direction, represented by $\vec{k}$. equation (3.6) tells us that the amplitudes Aij are traceless.
Consequence of equation (3.2) is ;

$$\hat{k}^i\hat{k}^j A^{ij} = 0$$

Using transverse traceless gauge conditions on Equations (3.2), (3.3) and (3.4) respectively, we get;

$$\bar{h}^{00} = 0$$
$$\bar{h}^{0i} = 0$$
$$\bar{h}^{ij} = Re \int d^3k \;\; A^{ij} e^{\iota(\vec{k}\cdot\vec{x}-\omega t)}$$

These represent general solution of linearized vacuum Einstein equation in transverse traceless gauge.
Consider now a simple monochromatic plane gravitational wave in TT gauge;

$$h^{ij} = Re(A^{ij} \; e^{\iota(\vec{k}\cdot\vec{x}-\omega t)})$$

where $h^{i0} = h^{00} = 0$. in general symmetric 3x3 matrix A^{ij} have six independent components but using the transverse trace less condition leaves only two independent components in A^{ij}.
let's now define a set of spatial basis vectors e_1^i and e_2^i such that;
These basis vectors are orthogonal to wave vector: $e_1^i k^i = e_2^i k^i = 0$ and orthogonal to each other as well: $e_2^i e_2^i = e_2^i e_2^i = 1$
Then we can write the transverse traceless amplitude in terms of independent components as ;

$$A^{ij} = A_+(e_1^i e_1^j - e_2^i e_2^j) + A_\times(e_1^i e_2^j + e_2^i e_1^j) \tag{3.7}$$

Here A_+ and $A_\times$ are two independent components of A^{ij}, with the amplitude A^{ij} written this way h^{ij} is written as;

$$\begin{aligned} h^{ij} =& Re(A_+ \;\; e^{\iota(\vec{k}\cdot\vec{x}-\omega t)})(e_1^i e_1^j - e_2^i e_2^j) \quad \text{plus polarization} \\ &+ Re(A_\times \;\; e^{\iota(\vec{k}\cdot\vec{x}-\omega t)})(e_1^i e_2^j + e_2^i e_1^j) \quad \text{cross polarization} \end{aligned}$$

This finding clearly demonstrates that gravitational waves possess two distinct polarization states one is the $+$**Polarization** and other is the $\times$**polarization**.

3.3 Stress-Energy-Momentum Description of Gravitational Waves

Gravitational waves are oscillations in the metric of spacetime. Mathematically, spacetime is described by a metric tensor, which encodes the geometry of space and time. In the presence of gravitational waves, this metric is perturbed, or "wiggled," in a way that stretches and compresses distances along the direction of the wave's propagation. Gravitational waves carry energy because they transport distortions in spacetime. These distortions affect the curvature of spacetime itself, which corresponds to energy in the framework of general relativity.

Gravitational wave causes the background to become curved. so in order to calculate the energy for gravitational waves we must allow background metric to be curved we generalize the metric perturbation as:

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$$

Where $\tilde{g}_{\mu\nu}$ represents the curved background metric.
Now expanding the Einstein tensor as :

$$G_{\mu\nu}(g) = G_{\mu\nu}(\tilde{g} + h) = G_{\mu\nu}(\tilde{g}) + G^{(l)}_{\mu\nu}(\tilde{g}, h) + G^{(Q)}_{\mu\nu}(\tilde{g}, h) + O(h)$$

Now in this case where the background metric is curved the Eintein tensor $G_{\mu\nu}(\tilde{g})$ is not necessarily zero that is $G_{\mu\nu}(\tilde{g}) \neq 0$.
In the same way, the stress-energy tensor $T_{\mu\nu}$ is expressed as a series expansion in h;

$$T_{\mu\nu}(g) = T_{\mu\nu}(\tilde{g} + h) = T_{\mu\nu}(\tilde{g}) + T^{(l)}_{\mu\nu}(\tilde{g}, h) + T^{(Q)}_{\mu\nu}(\tilde{g}, h) + O(h)$$

Now we know from Einstein equation:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

Therefore, we can write :

$$G_{\mu\nu}(\tilde{g}) + G^{(l)}_{\mu\nu}(\tilde{g}, h) + G^{(Q)}_{\mu\nu}(\tilde{g}, h) + O(h) = 8\pi t_{\mu\nu}(g)$$

Ignoring terms of higher order, we obtain ;

$$G_{\mu\nu}(\tilde{g}) + G^{(l)}_{\mu\nu}(\tilde{g}, h) = 8\pi \left[t_{\mu\nu}(g)\right] - \left[G^{(Q)}_{\mu\nu}(\tilde{g}, h)\right]$$

The metric $\tilde{g}_{\mu\nu}$ can be separated into low frequency(long wavelength) background and high frequency(shorter wavelength) perturbation. This approach allows us to separate slowly varying spacetime geometry from rapidly oscillating gravitational wave components.

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}(lowfrequency) + h_{\mu\nu}(highfrequency)$$

Low-frequency component $\tilde{g}_{\mu\nu}$:Represents the background metric, such as a cosmological spacetime or the metric near massive objects like stars or black holes. This background changes slowly over large distances.
High-frequency component $h_{\mu\nu}$: Represents the gravitational wave exhibiting much quicker oscillations, with a wavelength far shorter than the characteristic length scale of the surrounding spacetime.
Now Einstein equation can be explained as :

$$G_{\mu\nu}(\tilde{g}) + G^{(l)}_{\mu\nu}(\tilde{g}, h) = 8\pi[t_{\mu\nu}(g) - \frac{1}{8\pi} G^{(Q)}_{\mu\nu}(\tilde{g}, h)] \tag{3.8}$$

where, $G_{\mu\nu}(\tilde{g})$ is low frequency term, $G^{(l)}_{\mu\nu}(\tilde{g},h)$ is high frequency term, $T_{\mu\nu}(g)$ and $G^{(Q)}_{\mu\nu}(\tilde{g},h)$ contains both low and high frequency terms. Thus Einstein equation (4.1) can be split into two equations one is low frequency part and other is high frequency part given below ;

Low frequency :

$$G_{\mu\nu}(\tilde{g}) = 8\pi[t_{\mu\nu}(g)]_{low} - [G^{(Q)}_{\mu\nu}(\tilde{g},h)]_{low}$$

Curvature in the background spacetime arises from the low-frequency parts of $G^{(Q)}_{\mu\nu}$ and $T_{\mu\nu}$.
High frequency ;

$$G^{(l)}_{\mu\nu}(\tilde{g},h) = 8\pi[t_{\mu\nu}(g)]_{low} - [G^{(Q)}_{\mu\nu}(\tilde{g},h)]_{high}$$

The propagation of gravitational waves is affected by the high-frequency contributions of $G^{(Q)}_{\mu\nu}$ and $T_{\mu\nu}$.
Thus the low frequency part of $-\frac{1}{8\pi}G^{(Q)}_{\mu\nu}(\tilde{g},h)$ in Einstein tensor acts as a source for background curvature just as the stress-energy momentum tensor in matter fields. Therefore stress-energy momentum tensor for gravitational waves $T_{\mu\nu}$ is given as ;

$$T_{\mu\nu} = -\frac{1}{8\pi}[G^{(Q)}_{\mu\nu}(\tilde{g},h)]_{low} \tag{3.9}$$

Thus the low frequency part of Einstein tensor, corresponding to the background $\tilde{g}_{\mu\nu}$, can be expressed as :

$$G_{\mu\nu}(\tilde{g}) = 8\pi[(T_{\mu\nu})_{low} + t_{\mu\nu}] \tag{3.10}$$

Recall the Einstein tensor for any metric always satisfies the contracted **Bianchi identity** that is given as :

$$\tilde{\nabla}_{\mu}G^{\mu\nu}(\tilde{g}) = 0$$

where $\tilde{\nabla}_{\mu}$ is covariant derivative for background $\tilde{g}$. Putting the value of $G^{\mu\nu}(\tilde{g})$ from equation (4.3), we get :

$$\tilde{\nabla}_{\mu}[(T_{\mu\nu})_{low} + t_{\mu\nu}] = 0 \tag{3.11}$$

This equation represents the principle of **local energy-momentum conservation**.
In this case the conserved quantity is not just the energy-momentum in matter fields but it's combination of the energy-momentum in the matter fields and gravitational waves.
We know stress-energy momentum tensor for gravitational waves $T_{\mu\nu}$ is given as ;

$$T_{\mu\nu} = -\frac{1}{8\pi}[G^{(Q)}_{\mu\nu}(\tilde{g},h)]_{low}$$

In order to extract the low frequency part of $G^{(Q)}_{\mu\nu}(\tilde{g}, h)$ We perform an averaging over a spacetime region that is large relative to the high frequency components but small compared to the low-frequency ones. Consequently, the stress-energy tensor for gravitational waves is expressed as:

$$T_{\mu\nu} = -\frac{1}{8\pi}\langle G^{(Q)}_{\mu\nu}(\tilde{g}, h)\rangle$$

Expanding the $G^{(Q)}_{\mu\nu}(\tilde{g}, h)$ and assuming background to be nearly flat $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$, we can write it's average as :

$$\langle G^{(Q)}_{\mu\nu}(\tilde{g}, h)\rangle = \langle -\frac{1}{4}\partial_\mu \bar{h}^{\alpha\beta}\partial_\nu \bar{h}_{\alpha\beta} + \frac{1}{8}\partial_\nu \bar{h}^{\alpha}_{\alpha}\partial_\nu \bar{h}^{\beta}_{\beta} + \frac{1}{4}\partial_\alpha \bar{h}^{\alpha}_{\beta}(\partial_\mu \bar{h}^{\beta}_{\nu} + \partial_\nu \bar{h}^{\beta}_{\mu})\rangle$$

Where $\bar{h}^{\alpha\beta}$ refers to the metric perturbation in its trace reversed form, which is written as:

$$\bar{h}^{\alpha\beta} = h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}h^{\mu}_{\mu}$$

Therefore, the stress-energy momentum tensor for gravitational waves is written as:

$$T_{\mu\nu} = -\frac{1}{8\pi}\langle -\frac{1}{4}\partial_\mu \bar{h}^{\alpha\beta}\partial_\nu \bar{h}_{\alpha\beta} + \frac{1}{8}\partial_\nu \bar{h}^{\alpha}_{\alpha}\partial_\nu \bar{h}^{\beta}_{\beta} + \frac{1}{4}\partial_\alpha \bar{h}^{\alpha}_{\beta}(\partial_\mu \bar{h}^{\beta}_{\nu} + \partial_\nu \bar{h}^{\beta}_{\mu})\rangle$$

In lorentz gauge where $\partial_\mu \bar{h}^{\mu\nu} = 0$, we can write the stress energy momentum tensor as :

$$T_{\mu\nu} = \frac{1}{32\pi}\langle \partial_\mu \bar{h}^{\alpha\beta}\partial_\nu \bar{h}_{\alpha\beta} - \frac{1}{2}\partial_\mu \bar{h}^{\alpha}_{\alpha}\partial_\nu \bar{h}^{\beta}_{\beta}\rangle \tag{3.12}$$

3.4 Stress-Energy momentum tensor for plane monochromatic gravitational wave:

For radiative solution in vacuum region, impose the ransverse-traceless gauge condition,$i = 1, 2, 3$:

$$h^{00} = 0 = h^{0i} \quad \text{and} \quad \partial_i h^{ij} = 0$$

The stress-energy momentum tensor takes the form :

$$T_{\mu\nu} = \frac{1}{32\pi}\langle \partial_\mu h^{ij}\partial_\nu h_{ij}\rangle \tag{3.13}$$

Now we know for monochromatic plane wave in transverse-traceless gauge:

$$h^{ij} = Re(A^{ij}e^{\iota(\vec{k}\cdot\vec{x}-\omega t)}) \quad \text{and} \quad K^{ij}A^{ij} = 0 = A^{ii}$$

applying ∂_μ on h^{ij}, we get:

$$\partial_\mu h^{ij} = k_\mu Re(A^{ij}e^{\iota(\vec{k}\cdot\vec{x}-\omega t)}) \quad \text{where} \quad k_\mu = (\vec{k}, \omega)$$

Or

$$\partial_\mu h^{ij} = k_\mu[-(ReA^{ij})\sin(\vec{k}\cdot\vec{x}-\omega t) - (ImgA^{ij})\cos(\vec{k}\cdot\vec{x}-\omega t)]$$

Similarly we can calculate $\partial_\nu h_{ij}$ as :

$$\partial_\nu h_{ij} = k_\nu[-(ReA_{ij})\sin(\vec{k}\cdot\vec{x}-\omega t) - (ImgA_{ij})\cos(\vec{k}\cdot\vec{x}-\omega t)]$$

Now

$$\begin{aligned}\partial_\mu h^{ij}\partial_\nu h_{ij} =& k_\mu k_\nu[Re(A^{ij})Re(A_{ij})\sin^2(\vec{k}\cdot\vec{x}-\omega t)\\ &+Img(A^{ij})Img(A_{ij})\cos^2(\vec{k}\cdot\vec{x}-\omega t)\\ &+2Re(A^{ij})Img(A_{ij})\sin(\vec{k}\cdot\vec{x}-\omega t)\cos(\vec{k}\cdot\vec{x}-\omega t)]\end{aligned}$$

Taking the average $\langle ....\rangle$ on both sides of the above equation:

$$\begin{aligned}\langle\partial_\mu h^{ij}\partial_\nu h_{ij}\rangle = &\langle k_\mu k_\nu[Re(A^{ij})Re(A_{ij})\sin^2(\vec{k}\cdot\vec{x}-\omega t)\\ &+Img(A^{ij})Img(A_{ij})\cos^2(\vec{k}\cdot\vec{x}-\omega t)\\ &+2Re(A^{ij})Img(A_{ij})\sin(\vec{k}\cdot\vec{x}-\omega t)\cos(\vec{k}\cdot\vec{x}-\omega t)]\rangle\end{aligned}$$

Using,

$$\langle\sin^2(\vec{k}\cdot\vec{x}-\omega t)\rangle = \langle\cos^2(\vec{k}\cdot\vec{x}-\omega t)\rangle = \frac{1}{2}$$

and

$$\langle\sin(\vec{k}\cdot\vec{x}-\omega t)\cos(\vec{k}\cdot\vec{x}-\omega t)]\rangle = 0$$

We get:

$$\langle\partial_\mu h^{ij}\partial_\nu h_{ij}\rangle = \frac{1}{2}k_\mu k_\nu[Re(A^{ij})Re(A_{ij}) + Img(A^{ij})Img(A_{ij})] \tag{3.14}$$

Now from equation (3.7) we have;

$$A^{ij} = A_+(e_1^i e_1^j - e_2^i e_2^j) + A_\times(e_1^i e_2^j + e_2^i e_1^j)$$

Let $(e_1^i e_1^j - e_2^i e_2^j)= e_+^{ij}$ and $(e_1^i e_2^j + e_2^i e_1^j)= e_\times^{ij}$, we can write A^{ij} as ;

$$A^{ij} = A_+e_+^{ij} + A_\times e_\times^{ij}$$

also we know $e_+^{ij}e_\times^{ij} = 0$ and $e_+^{ij}e_+^{ij} = 2 = e_\times^{ij}e_\times^{ij}$
Therefore,

$$Re(A^{ij})Re(A_{ij}) = [Re(A_+e_\times^{ij}) + Re(A_\times e_\times^{ij})][Re(A_+e_\times^{ij}) + Re(A_\times e_\times^{ij})]$$

or

$$Re(A^{ij})Re(A_{ij}) = 2Re(A_+)^2 + 2Re(A_\times)^2$$

Using these calculations in equation(3.14) we get;

$$\langle\partial_\mu h^{ij}\partial_\nu h_{ij}\rangle = k_\mu k_\nu(|A_+|^2 + |A_\times|^2)$$

Using in equation(3.13) we have :

$$T_{\mu\nu} = \frac{1}{32\pi} k_\mu k_\nu (|A_+|^2 + |A_\times|^2) \tag{3.15}$$

This gives the **stress-energy momentum for a plane monochromatic gravitational wave.** This effective stress-energy tensor represents the average energy density, momentum density, and fluxes of gravitational waves. It serves as an effective source term in the Einstein equations, allowing gravitational waves to interact with the background curvature by contributing to the overall energy and momentum content of spacetime, despite not being directly tied to a conventional matter source.

Chapter 4

Gravitational wave sources:

Gravitational waves are produced by asymmetrical acceleration of masses. Examples of gravitational wave sources are binary black holes, binary neutron stars, supernovae, neutron stars etc.
We know linearized Einstein equation is written as ;

$$G^{(l)}_{\mu\nu}(\eta, h) = 8\pi T_{\mu\nu}(\eta)$$

and under the lorentz gauge, the linearized Einstein equation is represented by ;

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$$

where $\bar{h}_{\mu\nu}$ is trace-reverse metric perturbation.
Also we know radiative solution in the vacuum region, which is written as ;

$$\bar{h}^{00} = Re \int d^3k \;\; \hat{k^i}\hat{k^j} A^{ij} e^{\iota(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\bar{h}^{0i} = Re \int d^3k \;\; \hat{k^j} A^{ij} e^{\iota(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\bar{h}^{ij} = Re \int d^3k \;\; A^{ij} e^{\iota(\vec{k}\cdot\vec{x} - \omega t)}$$

In transverse-traceless gauge (TT) we write these equations as;

$$\bar{h}^{00}_{TT} = 0$$

$$\bar{h}^{0i}_{TT} = 0$$

$$\bar{h}^{ij}_{TT} = Re \int d^3k \;\; A^{ij}_{TT} e^{\iota(\vec{k}\cdot\vec{x} - \omega t)}$$

The relation between amplitude in lorentz gauge and amplitude in TT gauge is given as:

$$A^{ij}_{TT} = P^{ik}P^{jl}A^{kl} - \frac{1}{2}P^{ij}P^{kl}A^{kl}$$

Where

$$P^{ij} = \delta^{ij} - \hat{k}^i\hat{k}^j, \qquad P^{ij} \quad \text{gives tranverse projection}$$

As a result, the trace-reversed metric perturbation under the TT gauge is written as ;

$$\bar{h}^{ij}_{TT}(\vec{x},t) = Re\int d^3k \ \ (P^{ik}P^{jl} - \frac{1}{2}P^{ij}P^{kl})A^{kl}e^{\iota(\vec{k}\cdot\vec{x}-\omega t)} \tag{4.1}$$

Assume propagation in z direction then $\hat{k}^i = \delta^i_3$, projection operator P^{ij} can be written as;

$$P^{ij} = \delta^{ij}\delta^i_3\delta^j_3 = \delta^i_1\delta^j_1 + \delta^i_2\delta^j_2$$

$$P^{ij} = \delta^i_a\delta^j_a \qquad (\text{a, b \ range \ } 1,2)$$

Therefore equation(4.1) can be written as;

$$\bar{h}^{ij}_{TT}(\vec{x},t) = Re\int d^3k \ \ (\delta^i_a\delta^k_a\delta^j_b\delta^l_b - \frac{1}{2}\delta^i_a\delta^j_a\delta^k_b\delta^l_b)A^{kl}e^{\iota(\vec{k}\cdot\vec{x}-\omega t)}$$

Each of the modes in sum over k propagate in z direction, hence all projection operators in bracket are independent of k. Therefore we can write ;

$$\bar{h}^{ij}_{TT} = (\delta^i_a\delta^k_a\delta^j_b\delta^l_b - \frac{1}{2}\delta^i_a\delta^j_a\delta^k_b\delta^l_b) \ \ \bar{h}^{ij}$$

or

$$\bar{h}^{ij}_{TT} = \delta^i_a\delta^j_b \ \bar{h}^{ab} - \frac{1}{2}\delta^i_a\delta^j_a \ \bar{h}^{bb}$$

If one of the indices is 3 then right hand side of the above equation becomes zero that is:

$$\bar{h}^{13}_{TT} = \bar{h}^{23}_{TT} = \bar{h}^{33}_{TT} = 0$$

Also when $i = 1$ and $j = 1$ we have;

$$\bar{h}^{11}_{TT} = \bar{h}^{11} - \frac{1}{2}(\bar{h}^{11} + \bar{h}^{22}) = \frac{1}{2}(\bar{h}^{11} - \bar{h}^{22}) \tag{4.2}$$

similarly when $i = 2$ and $j = 2$ we have;

$$\bar{h}^{22}_{TT} = \bar{h}^{22} - \frac{1}{2}(\bar{h}^{11} + \bar{h}^{22}) = \frac{1}{2}(\bar{h}^{22} - \bar{h}^{11}) \tag{4.3}$$

and,

$$\bar{h}^{12}_{TT} = \bar{h}^{12}$$

Also we know from trace-reverse metric perturbation and TT gauge ,

$$\bar{h}^{00}_{TT} = \bar{h}^{01}_{TT} = \bar{h}^{02}_{TT} = \bar{h}^{03}_{TT} = 0$$

From above results we can say trace-reversed metric perturbation in TT gauge is trace free,

$$\bar{h}^{ij}_{TT} = \quad \text{trace \ free}$$

Therefore,

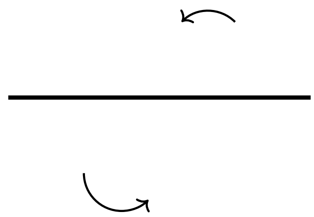
$$\bar{h}^{ij}_{TT} = h^{ij}_{TT}$$

Thus in our previous results we can drop the bar on $\bar{h}^{ij}_{TT}$.
Also from equation(4.2) and equation (4.3) we can say that in TT gauge, the trace-reversed metric perturbation depends solely on the transverse components of the trace-reversed perturbation from the lorentz gauge.
This is useful because;

- Linearized equations are easy to solve in lorentz gauge.
- Far from source solution is wave like and waves propagate in a single direction.
- Gravitational waves are expressed most simply in transverse-traceless gauge.

4.1 Types of sources:

1. Weak, slow moving sources with negligible internal gravity.
   Examples ; Rotating rod of length L rotating with angular velocity ω

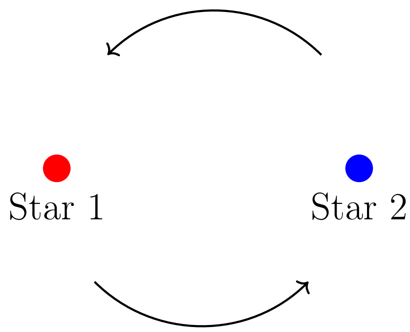


$$\frac{GM}{c^2} \ll L \qquad (\omega L \ll c)$$

   we can also write;

$$\frac{GM}{L^2} \ll M\omega^2 L$$

2. Weak and slow moving sources with non-negligible internal gravity, we can't ignore internal gravity.
   Example; binary star system with separation L and angular velocity ω



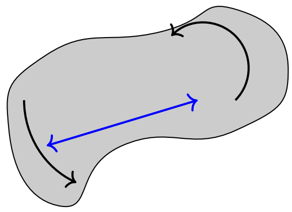
$$\frac{GM}{c^2} \ll L \qquad (\omega L \ll c)$$

3. Strong, slow moving sources.
   Example; binary black hole, neutron star system

4. Sources with strong internal gravity and fast moving.
   Example; final stages of binary black hole or binary neutron star mergers.

4.2 Weak and slow moving sources with negligible internal gravity:

The result of this analysis will be a formula for gravitational waves that is an expression for spatial components of the metric perturbation in TT gauge (h^{ij}_{TT}) in terms of the source motion.
For sources traveling at speeds much less than the speed of light $v \ll c$



If we have a compact source of *size* $\sim L$, the only thing matter can do in this compact region is oscillate back and forth or circulate around. Therefore we can write;

$$v \sim \frac{2\pi L}{period} = 2\pi L f$$

The variable f represents the frequency of the source's motion, which sets the frequency of the gravitational waves. Since these waves propagate at the speed of light c, their frequency is given by $f = \frac{c}{\lambda}$

$$v = \frac{2\pi Lc}{\lambda} = \frac{Lc}{\bar{\lambda}} \qquad \Rightarrow \frac{v}{c} \sim \frac{L}{\bar{\lambda}}$$

For $v \ll c$ we can say $L \ll \bar{\lambda}$ (reduced wavelength)

$$\therefore \text{Size of source} \ll \text{gravitational wave wavelength}$$

Because gravitational effects are weak, it is appropriate to use the linearized Einstein equation under the lorentz gauge condition ;

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$$

from electromagnetism we write;

$$\Box A_\mu = -4\pi J_\mu$$

by analogous to the solution in electromagnetism we can write $x^{'}$ is source point ;

$$\bar{h}^{ij}(\vec{x},t) = 4\int \frac{d^3x^{'}\ \ T^{ij}(x^{'},t^{'}=t-(x-x^{'})}{|\vec{x}-\vec{x^{'}}|}$$

For field point far away from the source we can write ;

$$\bar{h}^{ij}(\vec{x},t) = \frac{4}{r}\int d^3x^{'}\ \ T^{ij}(x^{'},t^{'})|_{t^{'}=t-r}$$

The above equation holds only for case first that is rotating rods.
If we instead use conservation of energy momentum $\partial_{\mu}T^{\mu\nu}=0$ we get;

$$\bar{h}^{ij}(\vec{x},t) = \frac{4}{r}\int d^3x^{'}\ \frac{1}{2}\ \partial_0^{'}\ \partial_0^{'}T^{00}\ X^{i^{'}}\ X^{j^{'}}|_{t^{'}=t-r}$$

Now putting the time derivative outside the integral and define second moment mass distribution as:

$$I^{ij}(t^{'}) = \int d^3x^{'}\ T^{00}(\vec{x^{'}},t^{'})\ X^{i^{'}}\ X^{j^{'}}$$

Therefore;

$$\bar{h}^{ij}(\vec{x},t) = \frac{2}{r}\frac{\partial^2}{\partial t^2}I^{ij}(t-r)$$

we know the metric perturbation in TT gauge in terms of lorentz gauge is given as;

$$h^{ij}_{TT}(\vec{x},t) = (P^{ik}P^{jl}-\frac{1}{2}P^{ij}P^{kl})\bar{h}^{kl}$$

also using ;

$$\bar{I}^{ij} = (P^{ik}P^{jl}-\frac{1}{2}P^{ij}P^{kl})I^{kl}$$

Where $\bar{I}^{ij}$ is called reduced quadrupole moment
Therefore;

$$h^{ij}_{TT}(\vec{x},t) = \frac{2}{r}\frac{\partial^2}{\partial t^2}\bar{I}^{ij}_{TT}(t-r) \tag{4.4}$$

This is the general result connecting the source motion to the gravitational waves that they produce and this result is useful for weak and slow moving sources with non-negligible internal gravity.
Now we know stress-energy momentum tensor is expressed as ;

$$T_{\mu\nu} = \frac{1}{32\pi}\langle\partial_{\mu}h^{ij}_{TT}\partial_{\nu}h^{ij}_{TT}\rangle$$

Energy density is given as;

$$T_{00} = \frac{1}{32\pi}\langle\dot{h}^{ij}_{TT}\dot{h}^{ij}_{TT}\rangle$$

Using equation(4.4) we get;

$$T_{00} = \frac{1}{8\pi r^2}\langle \dddot{\bar{I}}^{ij}_{TT}\,\dddot{\bar{I}}^{ij}_{TT}\rangle|_{t-r} \qquad \text{evaluated at retarded time}$$

Energy flux (momentum density) in radial direction;

$$T_{0k}n^k = \frac{1}{32\pi}\langle \dot{h}^{ij}_{TT}\partial_k h^{ij}_{TT}\rangle n^k$$

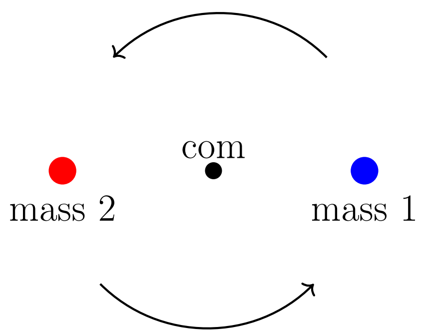
Therefore **Energy flux in radial direction** is written as;

$$T_{0k}n^k = -\frac{1}{8\pi r^2}\langle \dddot{\bar{I}}^{ij}_{TT}\,\dddot{\bar{I}}^{ij}_{TT}\rangle|_{t-r} \qquad \text{evaluated at retarded time} \tag{4.5}$$

For gravitational waves $T_{0k}n^k = -T_{00}$.
From energy flux we can compute the rate at which the source losses energy through gravitational radiation, gravitational wave luminosity ;

$$L_{GW} = \int_{\text{Sphere of large r}} t_{0k}n^k r^2 d\omega = -\frac{1}{8\pi}\int_{\text{Sphere of large r}} d\omega\langle \dddot{\bar{I}}^{ij}_{TT}\,\dddot{\bar{I}}^{ij}_{TT}\rangle|_{t-r}$$

4.3 Binary Star System :



Binary star system consist of stars M_1 and M_2 in circular orbits about one another. $L = r_1 + r_2$ is the separation between the two stars , r_1 and r_2 is the radius of orbit M_1 and M_2.
Reduced mass can be written as;

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

For star 1 and star 2 we can write ;

$$r_1\omega^2 = \frac{m_2}{L^2} \qquad\qquad r_2\omega^2 = \frac{m_1}{L^2}$$

adding the two we get;

$$L^3\omega^2 = m_1 + m_2 \quad \text{Therfore} \;\; r_1 = \frac{\mu}{m_1}L \;\; \text{and} \;\; r_2 = \frac{\mu}{m_2}L$$

We want to compute $h^{ij}_{TT}(\vec{x},t) = \frac{2}{r}\ddot{\bar{I}}^{ij}_{TT}(t-r)$ Thus we need ;

$$I^{ij}(t) = \int d^3x\, T^{00}(\vec{x},t)\; X^i\; X^j$$

since our motion is slow and the stresses are very small, the energy density is dominated by the rest mass density $T^{00} \approx \rho_0$(rest mass density).
At time t and location $\vec{x}$,the rest mass density ρ_0 is defined as ;

$$\begin{aligned}\rho_0 =& m_1\delta(x - r_1\cos\omega t)\;\delta(y - r_1\sin\omega t)\;\delta(z)\\ &+ m_2\delta(x + r_2\cos\omega t)\;\delta(y + r_2\sin\omega t)\;\delta(z)\end{aligned}$$

Calculating the second moment of mass distribution ;

$$I^{11} = m_1(r_1\cos\omega t)^2 + m_2(r_2\cos\omega t)^2$$

$\therefore$

$$I^{11} = \mu L^2\cos^2\omega t$$

similarly ;

$$I^{22} = \mu L^2\sin^2\omega t \qquad\qquad I^{12} = \mu L^2\cos\omega t\sin\omega t$$

$$I^{13} = I^{23} = I^{33} = 0$$

Therefore Trace is written as $I^{kk} = I^{11} + I^{22} + I^{33} = \mu L^2$
Now reduced quadrupole moment $\bar{I}^{ij} = I^{ij} - \frac{1}{3}\,\delta^{ij}\; I^{kk}$
For binary star system ;

$$\bar{I}^{11} = \mu L^2(\cos^2\omega t - \frac{1}{3})$$

$$\bar{I}^{22} = \mu L^2(\sin^2\omega t - \frac{1}{3})$$

$$\bar{I}^{12} = \mu L^2\sin\omega t\cos\omega t$$

$$\bar{I}^{33} = -\frac{1}{3}\mu L^2 \qquad\qquad \bar{I}^{13} = \bar{I}^{23} = 0$$

Let's now calculate the transverse traceless part $\bar{I}^{ij}$;

$$\bar{I}^{ij}_{TT} = (P^{ik}P^{jl} - \frac{1}{2}P^{ij}P^{kl})\bar{I}^{kl} \qquad\qquad \text{but}\;\; \bar{I}^{kl} = I^{kl}$$

Calculating I^{11} components ;

$$\begin{aligned}\bar{I}^{11}_{TT} =& \frac{\mu L^2}{64}[\cos(2\omega t - 2\phi)[-5 + 4\cos 2\theta + cos 4\theta + (19 + 12\cos 2\theta + \cos 4\theta)\cos 2\phi\,]\\ &-16\sin(2\omega t - 2\phi)(1 + \cos 2\theta)\sin 2\phi\,] + \text{t independent terms}\end{aligned}$$

t-independent terms won't contribute because the metric perturbation depends on time derivative.
Consider introducing a set of basis vectors, e^i_θ in direction of increasing θ, e^i_ϕ in the direction of increasing ϕ and n^i is the normal vector at the point of observation.

$$n^i = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} \qquad e^i_\theta = \begin{bmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{bmatrix} \qquad e^i_\phi = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix}$$

computing the $\bar{I}_{TT}$ in these basis;

$$\bar{I}^{\theta\theta}_{TT} = \bar{I}^{ij}_{TT}\ e^i_\theta\ e^j_\theta$$

or,

$$\bar{I}^{\theta\theta}_{TT} = e^i_\theta\ e^j_\theta(P^{ik}P^{jl} - \frac{1}{2}P^{ij}P^{kl})\ \bar{I}^{kl}$$

but $e^i_\theta n^i = 0$ that tells us $e^i_\theta P^{ij} = e^j_\theta$, we write ;

$$\bar{I}^{\theta\theta}_{TT} = e^k_\theta e^l_\theta \bar{I}^{kl} - \frac{1}{2}P^{kl}\bar{I}^{kl} = (e^k_\theta e^l_\theta + \frac{1}{2}n^k n^l)\ \bar{I}^{kl}$$

$$\bar{I}^{\theta\theta}_{TT} = (e^k_\theta e^l_\theta + \frac{1}{2}n^k n^l)\ \bar{I}^{kl}$$

similarly ;

$$\bar{I}^{\phi\phi}_{TT} = (e^k_\phi e^l_\phi + \frac{1}{2}n^k n^l)\ \bar{I}^{kl}$$

$$\bar{I}^{\theta\phi}_{TT} = e^k_\theta\ e^l_\phi\ \bar{I}^{kl} \qquad \bar{I}^{nn}_{TT} = \bar{I}^{ij}_{TT}\ n^i\ n^j = 0 \qquad \bar{I}^{n\theta}_{TT} = \bar{I}^{n\phi}_{TT} = 0$$

Therefore;

$$\bar{I}^{\theta\theta}_{TT} = -\bar{I}^{\phi\phi}_{TT} = \frac{\mu L^2}{4}[(1+\cos^2\theta)\cos(2\omega t - 2\phi) - \sin^2\theta\ ]$$

$$\bar{I}^{\theta\phi} = \frac{\mu L^2}{2}\cos\theta\ \sin(2\omega t - 2\phi)$$

The metric perturbation is constructed from the second time derivative of TT projection of the reduced quadrupole moment hence ;

$$\ddot{\bar{I}}^{\theta\theta}_{TT} = -\ddot{\bar{I}}^{\phi\phi}_{TT} = -\mu L^2\omega^2(1+\cos^2\theta)\cos(2\omega t - 2\phi)$$

Therefore ;

$$\ddot{\bar{I}}^{\theta\theta}_{TT} = -\ddot{\bar{I}}^{\phi\phi}_{TT} = -\frac{m_1 m_2}{L}(1+\cos^2\theta)\cos(2\omega t - 2\phi)$$

Similarly ;

$$\ddot{\bar{I}}^{\theta\phi}_{TT} = -\frac{2m_1 m_2}{L}\cos\theta\sin(2\omega t - 2\phi)$$

Finally at this stage, the metric perturbation can be written in the TT gauge form as ;

$$h^{ij}_{TT}(\vec{x},t) = \frac{2}{r}\ddot{I}^{ij}_{TT}(t-r)$$

or

$$h^{\theta\theta}_{TT}(t,\vec{x}) = -h^{\phi\phi}_{TT}(t,\vec{x}) = -\frac{2m_1m_2}{rL}\cos[2\omega(t-r)-2\phi](1+\cos^2\theta) \tag{4.6}$$

and ;

$$h^{\theta\phi}_{TT}(\vec{x},t) = -\frac{4m_1m_2}{rL}\cos\theta\sin(2\omega(t-r)-2\phi) \tag{4.7}$$

Hence equation (4.6) and equation (4.7) define **Plus** and **Cross** polarizations of gravitational wave.

$$h_+ = h^{\theta\theta}_{TT} \qquad \text{and} \quad h_\times = h^{\theta\phi}_{TT}$$

Thus **Plus** and **Cross** polarization of gravitational waves is given as;

$$h_+ = -\frac{2m_1m_2}{rL}\cos[2\omega(t-r)-2\phi](1+\cos^2\theta)$$

$$h_\times = -\frac{4m_1m_2}{rL}\sin[2\omega(t-r)-2\phi]\cos\theta$$

So a binary system consisting of masses m_1 and m_2 in circular orbit about one another meets gravitational waves with both plus and cross polarization. The observed strength of the two polarization modes varies with the angle of observation θ. The waves are strongest along z-axis where $\theta = 0$, these These directions lie orthogonal to the orbital plane and exhibit the least strength within the plane itself, particularly when $\theta = \frac{\pi}{2}$, in fact the cross polarization vanishes in orbital plane. Also the frequency of gravitational waves is two times the orbital frequency of two stars.

4.3.1 Energy emitted by Binary star system:

We know energy flux is given by;

$$T_{0k}n^k = -\frac{1}{8\pi r^2}\langle \dddot{I}^{ij}_{TT}\dddot{I}^{ij}_{TT}\rangle|_{t-r}$$

In the radial direction, the energy flux is described by the stress-energy tensor formulated with upper indices, thus ;

$$T^{0k}n^k = \frac{1}{8\pi r^2}\langle \dddot{I}^{ij}_{TT}\dddot{I}^{ij}_{TT}\rangle|_{t-r}$$

Now carry out the contraction over these indices i and j using basis e_θ,e_ϕ and n we get ;

$$T^{0k}n^k = \frac{1}{8\pi r^2}\langle\, (\dddot{I}^{\theta\theta}_{TT})^2 + (\dddot{I}^{\phi\phi}_{TT})^2 + 2(\dddot{I}^{\theta\phi}_{TT})^2\,\rangle$$

but $\bar{I}^{\theta\theta}_{TT} = -\bar{I}^{\phi\phi}_{TT}$, we can write ;

$$T^{0k}n^k = \frac{1}{4\pi r^2}\langle\ (\dddot{\bar{I}}^{\theta\theta}_{TT})^2 + (\dddot{\bar{I}}^{\theta\phi}_{TT})^2\ \rangle|_{t-r}$$

Applying equations (4.6) and (4.7) we obtain ;

$$T^{0k}n^k = \frac{1}{2\pi r^2}\ \frac{Mm_1^2m_2^2}{L^5}[1 + 6\cos^2\theta + \cos^4\theta]$$

Thus we can say energy flux depends on θ, maximum when $\theta = 0, \pi$.

4.3.2 Luminosity:

Luminosity refers to the rate at which a binary system emits energy through gravitational waves.

$$L_{GW} = \int_{\text{Sphere of large r}} t_{0k}n^kr^2d\omega$$

or,

$$L_{GW} = \int d\theta d\phi \sin\theta \frac{1}{2\pi}\frac{Mm_1^2m_2^2}{L^5}(1 + 6\cos^2\theta + \cos^4\theta)$$

Therefore energy emitted per unit time by the gravitational waves is given by;

$$L_{GW} = \frac{32}{5}\ \frac{Mm_1^2m_2^2}{L^5} \tag{4.8}$$

Although we assumed the binary star system consists of stars in circular orbits, the loss of energy through gravitational radiation will cause the orbits to decay and the stars will spiral together. The decay is very slow for weak/slow moving sources , hence our approximation of circular orbits is reasonable.

4.3.3 Rate of orbital decay:

The system's total energy is represented by;

$$E = \frac{1}{2}m_1(r_1\omega)^2 + \frac{1}{2}m_2(r_2\omega)^2 - \frac{m_1m_2}{L}$$

using Kepler's third law $\omega^2 = \frac{m_1+m_2}{L^3}$,$r_1 + r_2 = L$ we have ,

$$E = -\frac{1}{2}\frac{m_1m_2}{L}$$

The rate at which the system's energy changes is equal to the rate of gravitational wave emission ;

$$\frac{dE}{dt} = -L_{GW}$$

Or,

$$\frac{d}{dt}(-\frac{1}{2}\frac{m_1 m_2}{L}) = -\frac{32}{5}M\frac{(m_1^2 m_2^2)}{L^5}$$

or,

$$\dot{L} = -\frac{64}{5}M\frac{(m_1 m_2)}{L^3}$$

In terms of orbital period $\tau = \frac{2\pi}{\omega} = 2\pi(\frac{M}{L^3})^{-\frac{1}{2}}$, rate of change of orbital period is given as ;

$$\frac{\dot{\tau}}{\tau} = -\frac{96}{5}\frac{Mm_1 m_2}{L^4}$$

Therefore, the shrinking of the orbit in a binary star system directly results from energy carried away by gravitational waves. As the system radiates gravitational waves, it loses orbital energy, causing the stars to move closer together over time. The rate of orbital contraction is proportional to the mass product of the two stars, the orbital frequency, and an inverse power of the orbital separation. Observations of systems like the Hulse-Taylor binary pulsar confirm this decay, aligning precisely with predictions from general relativity.

Chapter 5

Friedmann Space-time

Evidence indicates that the universe, on large scale, is the same everywhere (homogeneous) and also same in all directions (isotropic). We aim to determine the structure of the geometric framework in a reference system (t, x^α) where these properties are apparent. Breaking down the differential element of separation dS^2, we obtain the following:

$$dS^2 = g_{00}dt^2 + 2g_{0\alpha}dt\,dx^\alpha - \sigma_{\alpha\beta}dx^\alpha dx^\beta$$

Where, the term $g_{00}dt^2$ controls the flow of time, $2g_{0\alpha}dt\,dx^\alpha$ represents how space and time interact and $-\sigma_{\alpha\beta}dx^\alpha dx^\beta$ captures the spatial geometry.
For the universe to be isotropic, the metric must look the same in all directions from any point, therefore $g_{0\alpha}$ (which mixes time and space) must vanish.
In a coordinate system defined by fundamental observers, we can utilize the proper time measured by their clocks to define space-like surfaces. Choosing this proper time as the time coordinate t ensures that $g_{00} = -1$, simplifying spacetime interval to the form :

$$dS^2 = -dt^2 + \sigma_{\alpha\beta}dx^\alpha dx^\beta$$

$$\implies dS^2 = -dt^2 + dl^2$$

The line element dl^2 contains the three space components and since the universe is assumed to be isotropic around every point, it must also be spherically symmetric. This follows because if isotropy holds at every point, this eliminates any inherent directional preference, requiring the spatial component of the metric to be expressed in spherical coordinates as ;

$$dl^2 = g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

The angular part of the metric must take the form:

$$g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2 \sin^2\theta$$

because this is the only form that preserves rotational invariance. However, the radial part can have an arbitrary function $\lambda^2(r)$:

$$g_{rr} = \lambda^2(r).$$

Therefore, the line interval is written as ;

$$dl^2 = \lambda^2(r)a^2(t)\,dr^2 + r^2a^2(t)d\theta^2 + r^2a^2(t)\sin^2\theta\,d\phi^2$$

Here $a(t)$, depending only on time, is the scale factor that encodes cosmic expansion and $\lambda(r)$ is a function that determines the spatial curvature.
The Ricci scalar R for this 3D space is given by:

$$R = \frac{3}{2a^2r^3}\frac{d}{dr}\left[r^2\left(1 - \frac{1}{\lambda^2}\right)\right]$$

Homogeneity implies that all geometric properties are independent of r, meaning R must remain uniform. Assigning it a fixed value and performing integration, we arrive at:

$$\frac{\lambda^2r^2 - r^2}{\lambda^2} = kr^4 + B \qquad k, B = \text{constants.}$$

To ensure regularity at $r = 0$, the term B must vanish. As a result, we obtain ;

$$1 - kr^2 = \frac{1}{\lambda^2}$$

Therefore, space-time interval can be expressed as ;

$$dS^2 = -dt^2 + a^2(t)\left[\frac{1}{1 - kr^2}dr^2 + r^2d\theta^2 + (\sin\theta)^2\,r^2d\phi^2\right]$$

In the above equation of metric, $a(t)$ is called scale factor, which varies with cosmic time t that specifies the size of the universe at any moment and permits the universe to expand or contract with time. This metric, known as **FLRW (Friedmann-Lemaitre-Robertson-Walker)** metric, represents a universe that maintains both homogeneity and isotropy. Here k expresses the curvature of the space, where $k = 0$ means the universe is flat, $k = +1$ describes a closed, spherical geometry and $k = -1$ signifies an open, hyperbolic geometry.

5.1 Friedmann Equations :

The equations that describe the dynamics expansion of the universe are derived from Einstein's field equations in general relativity. These relations hold true in a universe that exhibits both homogeneity and isotropy, governing the evolution of scale factor $a(t)$ based on the composition of the universe, including matter, radiation, and dark energy. When the cosmos is modeled as being composed of a homogeneous fluid, the resulting equations take the form :

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p)$$

Here, energy density is represented by ρ and p denotes pressure of the ideal fluid. This equation provides double derrivative of time ($\frac{\partial^2}{\partial t^2}$) of scale factor or the expansion acceleration. Also, we have the Hubble parameter equation (first derivative of $a(t)$) given by ;

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{3}{8\pi}\rho - \frac{k}{a^2}$$

This is the first time derivative of $a(t)$, describing the cosmic expansion rate $H(t) = \frac{\dot{a}}{a}$, which quantifies how the universe evolves over time. The equation illustrates how this expansion rate is connected to the total energy content ρ of the cosmos and the curvature parameter k, which characterizes the spatial geometry of the universe.

5.2 Gravitational wave equation in curved spacetime background :

The equation describing metric perturbations in a general curved or non-flat spacetime is;

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

The inverse or the reciprocal form of the metric is expressed as ;

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu}$$

Using the above equations of metric perturbation in the formulation of the Christoffel symbol, we get ;

$$\begin{aligned}\Gamma^{\rho}_{\mu\nu} &= \bar{\Gamma}^{\rho}_{\mu\nu} + \frac{1}{2}\bar{g}^{\rho\sigma}\left(\nabla_\nu h_{\mu\sigma} + \nabla_\mu h_{\sigma\nu} - \nabla_\sigma h_{\mu\nu}\right) \\ &= \Gamma^{\rho(l)}_{\mu\nu} + \bar{\Gamma}^{\rho}_{\mu\nu}\end{aligned}$$

Thus in a curved background space-time, connection coefficient separate into two different parts, the first part corresponds to the linear perturbations and is expressed as $\Gamma^{(l)}$, while the second represents the background space-time and is denoted by $\bar{\Gamma}$. In a similar manner, the full curvature tensor is decomposed into its background form and a linear perturbation form as :

$$R^{\rho}_{\mu\nu\lambda} = R^{(l)\rho}_{\mu\nu\lambda} + \bar{R}^{\rho}_{\mu\nu\lambda}$$

Now the linearized form of Ricci tensor $R^{(l)}_{\alpha\beta}$ is expressed by the below equation :

$$R^{(l)}_{\alpha\beta} = \frac{\nabla_\rho \nabla_\alpha h^{\rho}_{\beta}}{2} - \frac{1}{2}\left(\Box h_{\alpha\beta} + \nabla_\beta \nabla_\alpha h\right) + \frac{\nabla_\rho \nabla_\beta h^{\rho}_{\alpha}}{2} \tag{5.1}$$

Where $h = h^{\mu}_{\mu}$ $(\mu = 0, 1, 2, 3)$ represents the trace components of metric perturbation.
The Einstein tensor and by extension, the field equations can be decomposed into an unperturbed part $\bar{G}_{\alpha\beta}$ and a linear perturbation $G^{(l)}_{\alpha\beta}$ as ;

$$G_{\alpha\beta} = \bar{G}_{\alpha\beta} + G^{(l)}_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

$$= 8\pi T^{(l)}_{\alpha\beta} + 8\pi \bar{T}_{\alpha\beta}$$

Here, $\bar{T}_{\alpha\beta}$ corresponds to the energy-momentum distribution of the background spacetime, while $T^{(l)}_{\alpha\beta}$ accounts for the linear perturbative correction.
In the short wavelength approximation, the first order correction $T^{(l)}_{\alpha\beta}$ to the energy momentum content is considered negligible, i.e., $T^{(l)}_{\alpha\beta} = 0$, which implies $G^{(l)}_{\alpha\beta} = 0$, because when the wavelength becomes negligible compared to the curvature scale of the background $(\lambda << L)$ gravitational waves oscillate rapidly relative to the characteristic scale of spacetime curvature. This rapid oscillation averages out the effect of stress-energy perturbations over time, meaning $T^{(l)}_{\alpha\beta}$ does not significantly contribute to the dynamics of these waves. Essentially, the energy-momentum fluctuations do not have time to respond to the rapidly oscillating metric perturbations.
In the long-wavelength limit $T^{(l)}_{\alpha\beta}$ becomes relevant because when the wavelength is comparable to the curvature scale $(\lambda \sim L)$ matter can respond to gravitational waves.
Taking the trace of Einstein equation, linearized Ricci scalar is written as ;

$$R^{(l)} = -8\pi T^{(l)} = 0 \qquad \Longrightarrow \qquad R^{(l)}_{\alpha\beta} = 0.$$

Therefore from equation(5.1) we can write ;

$$\nabla_{\rho}\nabla_{\alpha}h^{\rho}_{\beta} + \nabla_{\rho}\nabla_{\beta}h^{\rho}_{\alpha} - \nabla_{\beta}\nabla_{\alpha}h - \Box h_{\alpha\beta} = 0 \tag{5.2}$$

We define the trace-reversed metric perturbation $\tilde{h}_{\mu\nu}$ so that the metric perturbation $h_{\mu\nu}$ takes the form :

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\tilde{h}$$

where $\tilde{h} = \tilde{h}^{\mu}_{\mu}$ and $h = h^{\mu}_{\mu}$, replacing the metric perturbation with the trace-reversed metric perturbation, equation (5.2) simplifies to ;

$$\nabla_{\rho}\nabla_{\beta}\tilde{h}^{\rho}_{\alpha} + \frac{1}{2}\bar{g}_{\alpha\beta}\Box\tilde{h} + \nabla_{\rho}\nabla_{\alpha}\tilde{h}^{\rho}_{\beta} - \Box\tilde{h}_{\alpha\beta} = 0. \tag{5.3}$$

Using lorentz gauge, $\nabla_{\nu}\tilde{h}^{\mu\nu} = 0$, equation (5.3) modifies to ;

$$\bar{R}_{\delta\alpha}\tilde{h}^{\delta}_{\beta} + \bar{R}_{\delta\beta}\tilde{h}^{\delta}_{\alpha} - 2\bar{R}_{\beta\delta\alpha\rho}\tilde{h}^{\delta\rho} - \Box\tilde{h}_{\alpha\beta} + \frac{1}{2}\bar{g}_{\alpha\beta}\Box\tilde{h} = 0.$$

Imposing the traceless condition $\tilde{h} = \tilde{h}^{\alpha}_{\alpha} = 0$,($h_{\mu\nu} = \tilde{h}_{\mu\nu}$), we get ;

$$\Box\tilde{h}_{\beta\alpha} - \bar{R}_{\delta\alpha}\tilde{h}^{\delta}_{\beta} - 2\bar{R}_{\delta\beta\alpha\rho}\tilde{h}^{\delta\rho} - \bar{R}_{\delta\beta}\tilde{h}^{\delta}_{\alpha} = 0 \tag{5.4}$$

This describes the differential equation that controls ripples in spacetime geometry within a non-flat cosmological setting.
In the above equation(5.4) the term $\Box\tilde{h}_{\beta\alpha}$ represents the wave propagation, $-2\bar{R}_{\delta\beta\alpha\rho}\tilde{h}^{\delta\rho}$ indicates the interaction with the background curvature through the Riemann tensor. $-\bar{R}_{\delta\alpha}\tilde{h}^{\delta}_{\beta} - \bar{R}_{\delta\beta}\tilde{h}^{\delta}_{\alpha}$ indicates the interaction with the background curvature through the Ricci tensor.

5.3 Gravitational waves in FLRW universe :

The FLRW (Friedmann-Lemaitre-Robertson-Walker) spacetime is described by the following metric: ;

$$dS^2 = -dt^2 + a^2(t)\left[\frac{1}{1-kr^2}dr^2 + r^2 d\theta^2 + (\sin\theta)^2 r^2\, d\phi^2\right] \tag{5.5}$$

We are considering the signiture of the metric to be (-1,+1,+1,+1).

$$\implies \bar{g}_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)r^2\sin^2\theta \end{bmatrix}$$

We therefore can write $g^{\mu\nu}$ as ;

$$\bar{g}^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1-kr^2}{a^2(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{a^2(t)r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{a^2(t)r^2\sin^2\theta} \end{bmatrix}$$

Calculating non-zero Christoffel symbols for this metric, we get ;

$$\Gamma^0_{11} = a\frac{\dot{a}}{1-kr^2}, \quad \Gamma^0_{22} = ar^2\dot{a}, \quad \Gamma^0_{33} = ar^2\dot{a}\sin^2\theta, \quad \Gamma^1_{11} = k\frac{r}{1-kr^2}$$

$$\Gamma^1_{01} = \frac{\dot{a}}{a}, \quad \Gamma^1_{10} = \frac{\dot{a}}{a}, \quad \Gamma^2_{02} = \frac{\dot{a}}{a}, \quad \Gamma^2_{20} = \frac{\dot{a}}{a}, \quad \Gamma^3_{03} = \frac{\dot{a}}{a}, \quad \Gamma^3_{30} = \frac{\dot{a}}{a}$$

$$\Gamma^1_{22} = -(1-kr^2)r, \quad \Gamma^1_{33} = -r(\sin\theta)^2(1-kr^2)$$

$$\Gamma^2_{12} = \frac{1}{r}, \quad \Gamma^3_{13} = \frac{1}{r}, \quad \Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^3_{31} = \frac{1}{r}$$

$$\Gamma^3_{32} = \cot\theta = \Gamma^3_{23}, \quad \Gamma^2_{33} = -\frac{\sin 2\theta}{2} = -\cos\theta\sin\theta, \quad (0, 1, 2, 3 \quad \text{means} \quad t, r, \theta, \phi)$$

Using the above equations let's now calculate the Ricci tensor for background metric, Ricci tensor components are just the Riemann tensor components where we sum the upper index with lower middle index ;

$$R^\mu_{\sigma\mu\nu} = \partial_\mu \Gamma^\mu_{\nu\sigma} - \partial_\nu \Gamma^\mu_{\mu\sigma} + \Gamma^\alpha_{\nu\sigma}\Gamma^\mu_{\mu\alpha} - \Gamma^\beta_{\mu\sigma}\Gamma^\mu_{\nu\beta}$$

$$\bar{R}_{00} = \bar{R}^\mu_{0\mu0} = \partial_\mu \Gamma^\mu_{00} - \partial_0 \Gamma^\mu_{00} + \Gamma^\alpha_{00}\Gamma^\mu_{\mu\alpha} - \Gamma^\beta_{\mu0}\Gamma^\mu_{0\beta} = -\partial_0 \Gamma^\mu_{00} - \Gamma^\beta_{\mu0}\Gamma^\mu_{0\beta}$$

$$\bar{R}_{00} = -3\frac{\ddot{a}}{a}$$

$$\bar{R}_{11} = \partial_0 \Gamma^0_{11} + \Gamma^0_{11}\Gamma^1_{10} + 2\Gamma^1_{11}\Gamma^2_{21} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2}$$

$$\bar{R}_{22} = \partial_0 \Gamma^0_{22} + \partial_1 \Gamma^1_{22} + \Gamma^0_{22}\Gamma^1_{10} + \Gamma^1_{22}\Gamma^1_{11} + 1 = (a\ddot{a} + 2\dot{a}^2 + 2k)r^2$$

Similarly ;

$$\bar{R}_{33} = \partial_0 \Gamma^0_{33} + \partial_1 \Gamma^1_{33} + \Gamma^0_{33}\Gamma^1_{10} + \Gamma^1_{33}\Gamma^1_{11} + \sin^2\theta = (a\ddot{a} + 2\dot{a}^2 + 2k)r^2\sin^2\theta$$

Therefore Ricci scalar for background metric can be written as ;

$$\bar{R} = R_{00}\bar{g}^{00} + R_{11}\bar{g}^{11} + R_{22}\bar{g}^{22} + R_{33}\bar{g}^{33} = \frac{6(a\ddot{a} + \dot{a}^2 + k)}{a^2}$$

The non-zero elements of the Riemann curvature associated with the background geometry are as follows :

$$\bar{R}_{i0j0} = a\ddot{a}\delta^i_j, \qquad (i, j \text{ takes the value } 1, 2, 3)$$

$$\bar{R}_{1212} = \bar{R}_{1313} = \frac{ka^2 - \dot{a}^2}{(1 - kr^2)^2}, \qquad \bar{R}_{2323} = a^2 r^4 (k - \frac{\dot{a}^2}{a^2})$$

From equation(5.4), in curved space-time background the wave equation for gravitational wave is ;

$$\Box\tilde{h}_{\beta\alpha} - \bar{R}_{\delta\alpha}\tilde{h}^\delta_\beta - \bar{R}_{\delta\beta}\tilde{h}^\delta_\alpha - 2\bar{R}_{\delta\beta\alpha\rho}\tilde{h}^{\delta\rho} = 0$$

Where,

$$\Box h_{\beta\alpha} = \frac{1}{\sqrt{-g}}\,\partial_\mu\big(\sqrt{-g}\;g^{\mu\nu}\partial_\nu h_{\beta\alpha}\big)$$

For $\alpha = 0, \beta = 0$ and writing $h_{\alpha\beta}$ instead of $\tilde{h}_{\alpha\beta}$ and putting all the above values of Riemann tensor and Ricci tensor in gravitational wave equation(5.4) we get ;

$$\Box h_{00} - 6\frac{\ddot{a}}{a}h_{00} - 2\frac{\ddot{a}(1 - kr^2)}{a^3}h_{11} - 2\frac{\ddot{a}}{a^3 r^2}h_{22} - 2\frac{\ddot{a}}{a^3 r^2\sin^2\theta}h_{33} = 0 \tag{5.6}$$

Similarly for $\alpha = 1, \beta = 1$, we have ;

$$\Box h_{11} - 4\frac{\dot{a}^2}{a^2}h_{11} - 2\frac{\ddot{a}}{a}h_{11} + \frac{2\dot{a}^2 + 2k}{r^2a^2(1-kr^2)}h_{22} - 4\frac{k}{a^2}h_{11}$$
$$-2\frac{k + \dot{a}^2(\sin\theta)^{-2}}{(kr^2-1)a^2r^2}h_{33} + 2\frac{a\ddot{a}}{kr^2-1}h_{00} = 0 \tag{5.7}$$

Similarly for $\alpha = 2, \beta = 2$, we get ;

$$\Box h_{22} - \frac{4}{a^2}\left[\dot{a}^2 + k + \frac{a\ddot{a}}{2}\right]h_{22} + \frac{r^2(2\dot{a}^2+2k)(1-kr^2)}{a^2}h_{11}$$
$$+2\frac{(\dot{a}^2+k)}{a^2\sin^2\theta}h_{33} - 2r^2a\ddot{a}h_{00} = 0 \tag{5.8}$$

For $\alpha = 3, \beta = 3$ we have ;

$$\Box h_{33} - 2\left[2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2} + \frac{\ddot{a}}{a}\right]h_{33} + 2\sin^2\theta r^2\frac{(1-kr^2)r^2(\dot{a}^2+k)}{a^2}h_{11}$$
$$+2\sin^2\theta\frac{(\dot{a}^2+k)}{a^2}h_{22} - 2\sin^2\theta r^2a\ddot{a}h_{00} = 0 \tag{5.9}$$

Now using $\beta = 0, \alpha = i$, where i=1,2,3, we have ;

$$\Box h_{0i} - \frac{1}{a^2}\left[6\ddot{a}a + 2\dot{a}^2 + 2k\right]h_{0i} = 0 \tag{5.10}$$

For $\beta = m, \alpha = n$, where m and n take values from the spatial coordinates and are distinct $m \neq n$, we have ;

$$\Box h_{mn} - \left[2\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} + 6\frac{k}{a^2}\right]h_{mn} = 0 \tag{5.11}$$

In all the above equations $i, j = 1, 2, 3$ corresponds to r, θ, ϕ components.

5.3.1 Spatially flat FLRW Universe ($k = 0$) :

A flat Friedmann-Lemaitre-Robertson-Walker universe corresponds to the FLRW metric with spatial curvature $k = 0$. This means that, at any fixed time slice, the universe is described by Euclidean geometry rather than spherical ($k > 0$) or hyperbolic ($k < 0$) geometry, hence we use cartesian coordinate system. Applying tranverse traceless synchronous gauge $h_{0\mu} = h_{1\mu} = 0$ and $h_{22} = h_{33}$ on the above equations (1.6,1.7,1.8,1.9,1.10,1.11), we get the non zero and independent wave equations as ;

$$\Box h_{22} - \left[2\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2}\right]h_{22} = 0 \tag{5.12}$$

$$\Box h_{23} - \left[2\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2}\right] h_{23} = 0 \tag{5.13}$$

Where the term $-\left[2\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2}\right] h_{22}$ and $-\left[2\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2}\right] h_{23}$ originate from the universe's expansion, $\frac{\ddot{a}}{a}$ gives acceleration of universe and $(\frac{\dot{a}}{a})$ is the Hubble parameter. These terms introduce damping to the gravitational waves due to the stretching of space. When the universe is flat and dominated by pressureless ($P = 0$) matter, the expansion rate is described by the scale factor growing in time as shown below ;

$$a = a_i t^{2/3}$$

a_i is a constant. Next, we assume that the perturbation components depend on both time t and and spatial coordinate x only, that is $h_{22} = \psi(x,t)$, we can write the equation(5.12) and equation(5.13) as ;

$$\frac{\partial^2 \Psi}{\partial t^2} + 3\frac{\dot{a}}{a}\frac{\partial \Psi}{\partial t} - \frac{1}{a^2}\frac{\partial^2 \Psi}{\partial x^2} - \left[2\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2}\right]\psi = 0$$

Using the value of $a = a_i t^{2/3}$ in above equation and solving we get ;

$$\frac{\partial^2 \Psi}{\partial t^2} + \frac{2}{t}\Psi - \frac{1}{a_i^2 t^{4/3}}\frac{\partial^2 \Psi}{\partial x^2} - \frac{20}{9t^2}\psi = 0$$

By applying the method of separating variables, we write $\Psi(x,t) = X_1(x)T_1(t)$, we can write the above equation in the form ;

$$a_i^2 t^{4/3}\left(\frac{1}{T_1}\frac{\partial^2 T_1}{\partial t^2} + \frac{2}{t}\frac{1}{T_1}\frac{dT_1}{dt} - \frac{20}{9t^2}T_1\right) = \frac{1}{X_1}\frac{d^2 T_1}{dx^2} = -\Omega^2 \tag{5.14}$$

Where Ω^2 is the separation constant.
The equation governing the time variable can be written as ;

$$\frac{d^2 T_1}{dt^2} + \left(\frac{\Omega^2}{a_i^2 t^{4/3}}T_1 - \frac{20}{9t^2}T_1\right) + \frac{2}{t}\frac{dT_1}{dt} = 0$$

The equation yields the following solution ;

$$T_1(t) = \frac{2\sqrt{\frac{2}{3}}\, a_i^{3/2}\left[C_2\Gamma\left(\frac{\sqrt{89}}{2}+1\right)J_{\frac{\sqrt{89}}{2}}\left(\frac{3\sqrt[3]{t}\Omega}{a_i}\right) + C_1\Gamma\left(1-\frac{\sqrt{89}}{2}\right)J_{-\frac{\sqrt{89}}{2}}\left(\frac{3\sqrt[3]{t}\Omega}{a_i}\right)\right]}{3\sqrt{t}\Omega^{3/2}}$$

In this expression, C_1 and C_2 represent integration constants, while $J_n(z)$ and $\Gamma(z)$ denote the Bessel function of the first kind with order n and the Gamma function, respectively. Figure(5.1) below illustrates the time evolution of the gravitational wave.

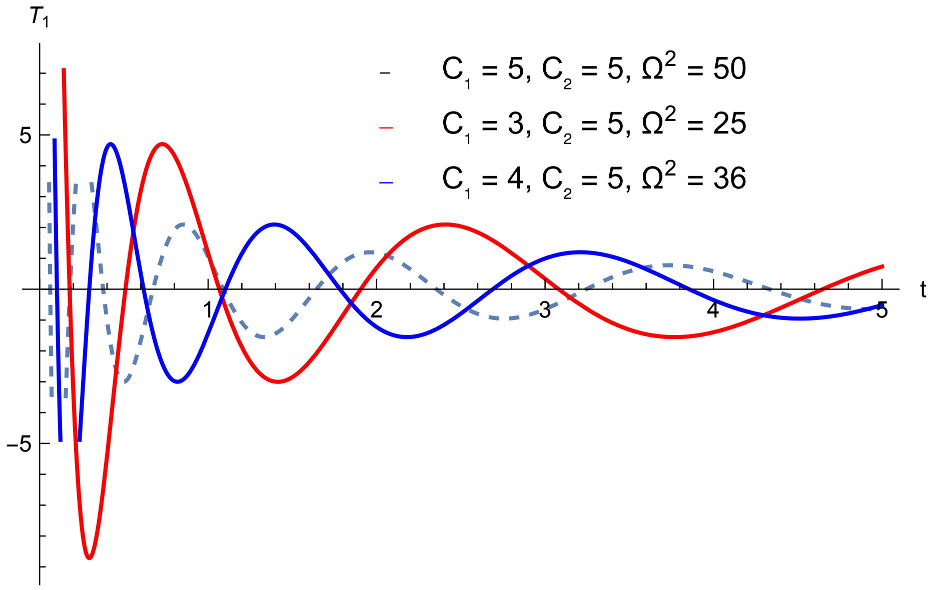


Figure 5.1: Gravitational wave amplitude versus time for a spatially flat, dust-dominated cosmos.

With a_i set to unity, the graph reveals that both the amplitude and the oscillation frequency of the wave diminish as time progresses. This attenuation can be attributed to cosmic expansion; in an expanding background, a comoving observer perceives gravitational wave signals that gradually redshift and weaken.

In a radiation dominated universe, characterized by the equation of state $P = \frac{1}{3}$,the scale factor evolves over time according to :

$$a = a_i t^{\frac{1}{2}}$$

Using the same varaiable separable method as above and putting $\Psi(t,x) = X_2(x)\,T_2(t)$, The time dependent behavior of the gravitational wave is described by the following equation :

$$\frac{d^2T_2}{dt^2} + \left(\frac{\Omega^2}{a_i^2 t}T_2 - \frac{1}{t^2}T_2\right) + \frac{3}{2t}\frac{dT_2}{dt} = 0$$

and whose solution is of the form ;

$$T_2(t) = \frac{\sqrt{a_i}\left[C_4\Gamma\left(\frac{\sqrt{17}}{2}+1\right)J_{\frac{\sqrt{17}}{2}}\left(\frac{2\sqrt{t}\Omega}{a_i}\right) + C_3\Gamma\left(1-\frac{\sqrt{17}}{2}\right)J_{-\frac{\sqrt{17}}{2}}\left(\frac{2\sqrt{t}\Omega}{a_i}\right)\right]}{\sqrt[4]{t}\sqrt{\Omega}}$$

Here C_3 and C_4 are integration constants.

The graph depicting the time evolution of the gravitational wave is shown in Figure(5.2) below ;

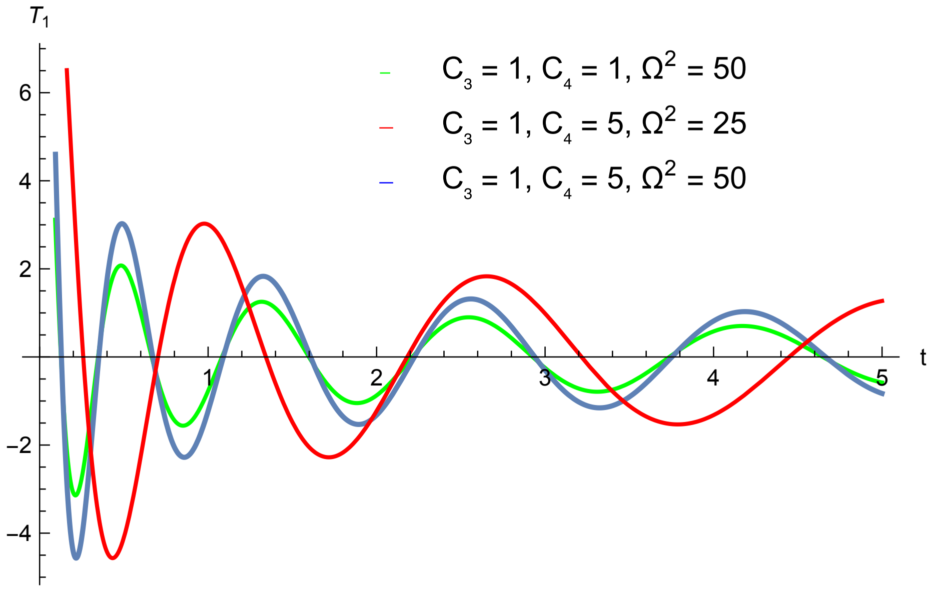


Figure 5.2: Time evolution of gravitational wave amplitude in a flat universe dominated by radiation.

As observed in the dust-dominated scenario, the gravitational wave's amplitude and frequency also diminish over time in the radiation-dominated case. However, as evident from Figure(5.2), the wave experiences less damping here than in the dust-dominated model. This outcome aligns with expectations, since the rate of cosmic expansion is slower in a radiation-dominated universe than in one dominated by dust.

5.3.2 Spatially Closed FLWR Universe $(k = +1)$:

A spatially closed FLRW universe refers to a Friedmann-Lemaitre-Robertson-Walker metric (FLRW) metric with positive spatial curvature $(k = +1)$.This means that at any fixed time slice, the spatial geometry is spherical rather than flat or hyperbolic. We take into consideration a wave moving outward along the radial coordinate. Using spherical polar coordinates and the transverse traceless synchronous gauge condition, we obtain the expressions governing the distinct non-zero perturbation components, given as ;

$$\Box h_{23} - \frac{2}{a^2}\left[a\ddot{a} + 3\dot{a}^2 + 3\right] h_{23} = 0 \tag{5.15}$$

$$\Box h_{22} - \frac{2}{a^2}\left[a\ddot{a} + 3\dot{a}^2 + 3\right] h_{22} = 0 \tag{5.16}$$

Where the extra term $\frac{6}{a^2}$ arises due to closed geometry.
In a spatially closed universe dominated by radiation (where the equation of state is $P/\rho = 1/3$), the scale factor changes over time according to the following behavior:

$$a(t) = \zeta\sqrt{1 - (1 - \frac{t}{\zeta})^2}$$

Where $\zeta = \sqrt{\frac{2q_o}{2q_o - 1}}$ is a constant and q_o is the deceleration parameter of universe and a suffix o indicates its present value.
Solving the equation(5.15) and equation(5.16) by using varaible separable method the equations turn into this ;

$$\frac{d^2T_3}{dt^2} + 3\frac{\dot{a}}{a}\frac{dT_3}{dt} + \left(2\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} + \frac{6 - \Omega^2}{a^2}\right)T_3 = 0$$

Putting the value of $a(t)$ and solving further we get ;

$$\frac{d^2T_3(t)}{dt^2} + 3\frac{\zeta - t}{2\zeta t - t^2}\frac{dT_3(t)}{dt} - 2\frac{\zeta^2}{2\zeta t - t^2}T_3(t) + 6\frac{(\zeta - t)^2}{(2\zeta t - t^2)^2}T_3(t) + \frac{6 - \Omega^2}{2\zeta t - t^2}T_3(t) = 0$$

The solution of the equation is ;

$$T3(t) = \frac{C_1 P^{\frac{i\sqrt{15}}{2}}_{\frac{1}{2}\left(2\sqrt{1-\Omega^2}-1\right)}\left(\frac{t}{\zeta} - 1\right)}{\sqrt[4]{t}\sqrt[4]{t - 2\zeta}} + \frac{C_2 Q^{\frac{i\sqrt{15}}{2}}_{\frac{1}{2}\left(2\sqrt{1-\Omega^2}-1\right)}\left(\frac{t}{\zeta} - 1\right)}{\sqrt[4]{t}\sqrt[4]{t - 2\zeta}} \tag{5.17}$$

C_1 and C_2 denote constants of integration, while P and Q represent the associated Legendre functions of the first and second kind, respectively.
To visualize how the gravitational wave evolves over time, we set the present day deceleration parameter q_0 equal to one and generate the corresponding plot shown below :

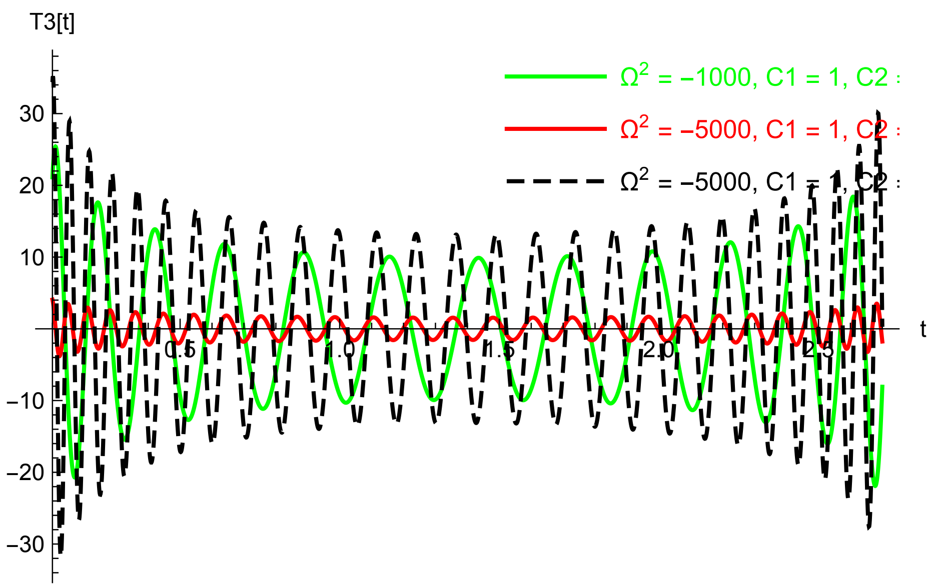


Figure 5.3: Time progression of gravitational waves in a spatially closed universe dominated by radiation.

It is observed that as the universe expands, the gravitational wave's amplitude and frequency gradually diminish, while they rise again during the universe's contraction phase. Furthermore, a significant point is that the amplitude increases dramatically near the end of this cycle, coinciding with the collapse of the universe into a singular state.

Chapter 6

Anisotropic 5D universe

To explore gravitational wave behavior in more general cosmological settings, we go beyond the standard 4D framework and consider a five-dimensional anisotropic and warped spacetime. We study a specific 5D vacuum solution presented in [arXiv:2402.13665v3, Eq.(20)], which describes a time-dependent anisotropic universe with exponential warping along an extra spatial dimension. The metric includes direction-dependent expansion rates and a warp factor $e^{-2\omega}$, which modifies the geometry across the fifth dimension. The 5D anisotropic warped metric is given by:

$$ds^2 = \frac{\mu}{\sinh^{2/3}(t)}\left[-\frac{dt^2}{9\sinh^2 t} + \sum_{i=1}^{3} e^{k_i t - 2\omega}(dX^i)^2 + d\omega^2\right] \tag{6.1}$$

Where k_i is given as ;

$$\kappa_i = \frac{2\sqrt{\frac{2}{3}}}{\sqrt{p^2+pq+q^2}}(p,\ q,\ -(p+q))$$

In particular, we focus on the case $k_1 = k, k_2 = -k, k_3 = 0$, where $k = 2\sqrt{\frac{2}{3}}$ leading to one expanding, one contracting and one static spatial direction. This setup allows us to examine how anisotropy and warping affect the evolution of gravitational waves.

The equation that describes the behavior of gravitational waves within a curved spacetime framework is expressed as follows:

$$\Box\tilde{h}_{\beta\alpha} - \bar{R}_{\delta\alpha}\tilde{h}^{\delta}_{\beta} - 2\bar{R}_{\delta\beta\alpha\rho}\tilde{h}^{\delta\rho} - \bar{R}_{\delta\beta}\tilde{h}^{\delta}_{\alpha} = 0$$

For $\alpha = 0, \beta = 0$ and writing $h_{\alpha\beta}$ instead of $\tilde{h}_{\alpha\beta}$ and putting all the values of

Riemann tensor and Ricci tensor in gravitational wave equation we get ;

$$\Box h_{00}-\frac{16-3k^2+4k\coth(t)}{6}h^0_0+\frac{2\mu\ e^{kt-2\omega}}{3\sinh^{\frac{2}{3}}t}h^{11}+\frac{2\mu}{3\sinh^{\frac{2}{3}}t}h^{44}+\frac{2\mu\ e^{-2\omega}}{3\sinh^{\frac{2}{3}}t}h^{33}$$
$$-\frac{2\mu\ e^{kt-2\omega}(-4+3k^2-4k\coth(t))}{12\sinh^{\frac{2}{3}}t}h^{22}+k\ h^4_0=0$$

For $\alpha=1,\beta=1$ we get ;

$$\Box h_{11}+\frac{2\mu\ e^{kt-2\omega}}{3\sinh^{\frac{2}{3}}t}h^{00}-\frac{3\ e^{kt-2\omega}k\sinh(2t)}{2}h^1_1+2\ e^{kt-4\omega}\mu\sinh^{\frac{4}{3}}(t)\ h^{33}$$
$$+2\ e^{kt-2\omega}\mu\sinh^{\frac{4}{3}}(t)\ h^{44}+\ e^{-4\omega}\mu\sinh^{\frac{1}{3}}(t)(3k\cosh(t)+2\sinh t)\ h^{22}=0$$

For $\alpha=2,\beta=2$ we get ;

$$\Box h_{22}+\mu\sinh^{\frac{1}{3}}t\ (3k\cosh(t)+2\sinh t)\left[e^{-kt-4\omega}h^{33}+e^{-4\omega}h^{11}+e^{-kt-2\omega}h^{44}\right]$$
$$-\frac{3k\ e^{-kt-2\omega}\sinh t\ (2\cosh t+3k\sinh t)}{2}h^2_2-\frac{\mu\ k\ e^{-kt-2\omega}}{\sinh^{\frac{2}{3}}t}[h^{04}+h^{40}]$$
$$-\frac{e^{-kt-2\omega}\mu[-4k\cosh t+(-4+3k^2)\sinh t]}{6\sinh^{\frac{5}{3}}t}h^{00}=0$$

For $\alpha=3,\beta=3$ we get ;

$$\Box h_{33}+\frac{2\ \mu\ e^{-2\omega}}{3\sinh^{\frac{2}{3}}t}h^{00}-2\ \mu\ e^{-kt-4\omega}\sinh^{\frac{4}{3}}(t)\ h^{11}+2\ \mu\ e^{-2\omega}\sinh^{\frac{4}{3}}(t)\ h^{44}$$
$$+\mu\ e^{-kt-4\omega}\sinh^{\frac{1}{3}}(t)[3k\cosh t+2\sinh t]\ h^{22}-\frac{3\ k\ e^{-2\omega}\sinh 2t}{2}h^3_3=0$$

For $\alpha=4,\beta=4$ we get ;

$$\Box h_{44}+\frac{2\ \mu}{3\sinh^{\frac{2}{3}}t}h^{00}+2\ \mu\ e^{kt-2\omega}\sinh^{\frac{4}{3}}(t)\ h^{11}+2\ \mu\ e^{-2\omega}\sinh^{\frac{4}{3}}(t)\ h^{33}$$
$$+\mu\ e^{-kt-2\omega}\sinh^{\frac{1}{3}}(t)[3k\cosh t+2\sinh t]\ h^{22}+k\ h^0_4-\frac{3k\ \sinh 2t}{2}h^4_4=0$$

For $\alpha=1,\beta=0$ we get ;

$$\Box h_{01}-\frac{2\ \mu\ e^{kt-2\omega}}{3\sinh^{\frac{2}{3}}t}h^{10}+\frac{k}{2}h^4_1-\frac{16-3k^2+4k\cosh t}{12}h^0_1-\frac{3k\ e^{kt-2\omega}\sinh 2t}{4}h^1_0=0$$

For $\alpha=2,\beta=0$ we get ;

$$\Box h_{02}+\frac{k\ \mu\ e^{-kt-2\omega}}{\sinh^{\frac{2}{3}}t}h^{24}-\frac{e^{-kt-2\omega}\mu[4k\cosh t+(4-3k^2)\sinh t]}{6\sinh^{\frac{5}{3}}t}h^{20}+k\ h^4_2$$
$$-\frac{16-3k^2+4k\cosh t}{12}h^0_2-\frac{3\ k\ e^{-kt-2\omega}\sinh(t)\ [2\cosh t+3k\sinh t]}{4}\ h^2_0=0$$

For $\alpha=3,\beta=0$ we get ;

$$\Box h_{03}-\frac{2\ \mu\ e^{-2\omega}}{3\sinh^{\frac{2}{3}}t}h^{30}+\frac{k}{2}h^4_3-\frac{16-3k^2+4k\cosh t}{12}h^0_3-\frac{3\ k\ e^{-2\omega}\sinh 2t}{4}h^3_0=0$$

For $\alpha = 4, \beta = 0$ we get ;

$$\Box h_{04} - \frac{2\ \mu}{3\sinh^{\frac{2}{3}} t} h^{40} - \frac{k\ \mu\ e^{-kt-2\omega}}{\sinh^{\frac{2}{3}} t} h^{22} + \frac{k}{2}[h_4^4 + h_0^0] - \frac{16 - 3k^2 + 4k\cosh t}{12} h_4^0 \\ - \frac{3k\sinh 2t}{4} h_0^4 = 0$$

For $\alpha = 2, \beta = 1$ we get ;

$$\Box h_{12} - \mu\ e^{-4\omega} \sinh^{\frac{1}{3}}(t)[3k\cosh t + 2\sinh t]\ h^{21} - \frac{3\ k\ e^{kt-2\omega}\sinh 2t}{4} h_2^1 \\ - \frac{3\ k\ e^{-kt-2\omega}\sinh t[2\cosh t + 3k\sinh t]}{4} h_1^2 = 0$$

For $\alpha = 3, \beta = 1$ we get ;

$$\Box h_{13} - 2\ \mu\ e^{kt-4\omega} \sinh^{\frac{4}{3}}(t)\ h^{31} - \frac{3\ k\ e^{kt-2\omega}\sinh 2t}{4} h_3^1 - \frac{3\ k\ e^{-2\omega}\sinh 2t}{4} h_1^3 = 0$$

For $\alpha = 4, \beta = 1$ we get ;

$$\Box h_{14} - 2\ \mu\ e^{kt-2\omega} \sinh^{\frac{4}{3}}(t)\ h^{41} - \frac{3k\ e^{kt-2\omega}\sinh 2t}{4} h_4^1 + \frac{k}{2} h_1^0 - \frac{3k\ \sinh 2t}{4} h_1^4 = 0$$

For $\alpha = 3, \beta = 2$ we get ;

$$\Box h_{23} - \mu\ e^{-kt-4\omega} \sinh^{\frac{1}{3}}(t)[3k\cosh t + 2\sinh t]\ h^{32} - \frac{3\ k\ e^{-2\omega}\sinh 2t}{4} h_2^3 \\ - \frac{3\ k\ e^{-kt-2\omega}\sinh t\ [2\cosh t + 3k\sinh t]}{4} h_3^2 = 0$$

For $\alpha = 2, \beta = 3$ we get ;

$$\Box h_{34} - 2\ \mu\ e^{-2\omega} \sinh^{\frac{4}{3}}(t)\ h_3^4 - \frac{3k\ e^{-2\omega}\sinh 2t}{4} h_4^3 + \frac{k}{2}\ h_4^0 - \frac{3k\ \sinh 2t}{4} h_3^4 = 0$$

We will now focus on solving the two axial components h_{13} and h_{23} of the perturbation equations, we write ;

$$\Box h_{13} - 2\ \mu\ e^{kt-4\omega} \sinh^{\frac{4}{3}}(t)\ h^{31} - \frac{3\ k\ e^{kt-2\omega}\sinh 2t}{4} h_3^1 - \frac{3\ k\ e^{-2\omega}\sinh 2t}{4} h_1^3 = 0$$

Or

$$\Box h_{13} - \left(2\ e^{kt-2\omega}\sinh^2 t + \frac{3k\ e^{kt-2\omega}\sinh 2t}{2}\right) h_3^1 = 0 \qquad (6.2)$$

Where,

$$\Box h_{\beta\alpha} = \frac{1}{\sqrt{-g}}\,\partial_\mu\big(\sqrt{-g}\ g^{\mu\nu}\partial_\nu h_{\beta\alpha}\big)$$

In all the above equation $0, 1, 2, 3, 4$ corresponds to t, x_1, x_2, x_3, ω respectively.

Now solving $\Box h_{13}$ as follows ;

$$\Box h_{13} = \frac{1}{\sqrt{-g}}\,\partial_t\big(\sqrt{-g}\ g^{tt}\partial_t h_{13}\big) + \frac{1}{\sqrt{-g}}\,\partial_\omega\big(\sqrt{-g}\ g^{\omega\omega}\partial_\omega h_{13}\big)$$

or

$$\Box h_{13} = \frac{1}{\sqrt{-g}}\,\partial_t\big[\sqrt{-g}\ g^{tt}\partial_t(g_{11}h^1_3)\big] + \frac{1}{\sqrt{-g}}\,\partial_\omega\big[\sqrt{-g}\ g^{\omega\omega}\partial_\omega(g_{11}h^1_3)\big]$$

or

$$\Box h_{13} = -\,9\sinh^2 t\; e^{kt-2\omega}\left[(\frac{10}{9}\coth^2 t - 4k\coth t + \frac{2}{3} + k^2)\right]\ h^1_3$$
$$-9\sinh^2 t\; e^{kt-2\omega}\left[(2k - \frac{4}{3}\coth t)\partial_t h^1_3 + \partial^2_t h^1_3\right] + e^{kt-2\omega}\left[\partial^2_\omega h^1_3 - 7\partial_\omega h^1_3 + 10h^1_3\right]$$

Using the value of $\Box h_{13}$ in equation(6.2), we get ;

$$-\,9\sinh^2 t\; e^{kt-2\omega}\left[(\frac{10}{9}\coth^2 t - 4k\coth t + \frac{2}{3} + k^2)\right]\ h^1_3$$
$$+e^{kt-2\omega}\left[\partial^2_\omega h^1_3 - 7\partial_\omega h^1_3 + 10h^1_3\right] - 9\sinh^2 t\; e^{kt-2\omega}\left[(2k - \frac{4}{3}\coth t)\partial_t h^1_3 + \partial^2_t h^1_3\right]$$
$$-\left(2\ e^{kt-2\omega}\sinh^2 t + \frac{3k\ e^{kt-2\omega}\sinh 2t}{2}\right) h^1_3 = 0$$

$$\therefore\quad -\,9\sinh^2 t\left[(\frac{10}{9}\coth^2 t - 4k\coth t + \frac{2}{3} + k^2)\right]\ h^1_3 - \left(2\sinh^2 t + \frac{3k\sinh 2t}{2}\right) h^1_3$$
$$+\left[\partial^2_\omega h^1_3 - 7\partial_\omega h^1_3 + 10h^1_3\right] - 9\sinh^2 t\left[(2k - \frac{4}{3}\coth t)\partial_t h^1_3 + \partial^2_t h^1_3\right] = 0 \qquad (6.3)$$

6.1 Solution for $h^1_3(t) = T(t)$

Let's first consider $h^1_3 = T(t)$ only and using $k = 2\sqrt{\frac{2}{3}}$, equation (6.3) reduces to ;

$$\partial^2_t T(t) + \frac{12}{9}(\sqrt{6} - \coth t\ )\partial_t T(t) + \frac{10}{9}\coth^2(t)\ T(t) + \frac{32}{9}T(t)$$
$$-\frac{22\sqrt{6}}{9}\coth(t)T(t) - \frac{10}{9\sinh^2 t}T(t) = 0 \qquad (6.4)$$

For finding the stable solution let's further substitute $T(t) = P(t)e^{\alpha t}$, where α is a negative constant $\alpha = \frac{-12(\sqrt{6}-1)-27}{18}$, we get ;

$$\ddot{P} + \dot{P}\left[-2\alpha + \frac{12}{9}(\sqrt{6} - \coth t)\right] + P\left[\alpha^2 - \frac{12\alpha}{9}(\sqrt{6} - 1) + \frac{24\alpha}{9(e^{2t} - 1)}\right]$$
$$+P\left[\frac{42}{9} - \frac{22\sqrt{6}}{9} - \frac{44\sqrt{6}}{9(e^{2t} - 1)}\right] = 0 \qquad (6.5)$$

Below is the graph illustrating the solution to equation (6.5).

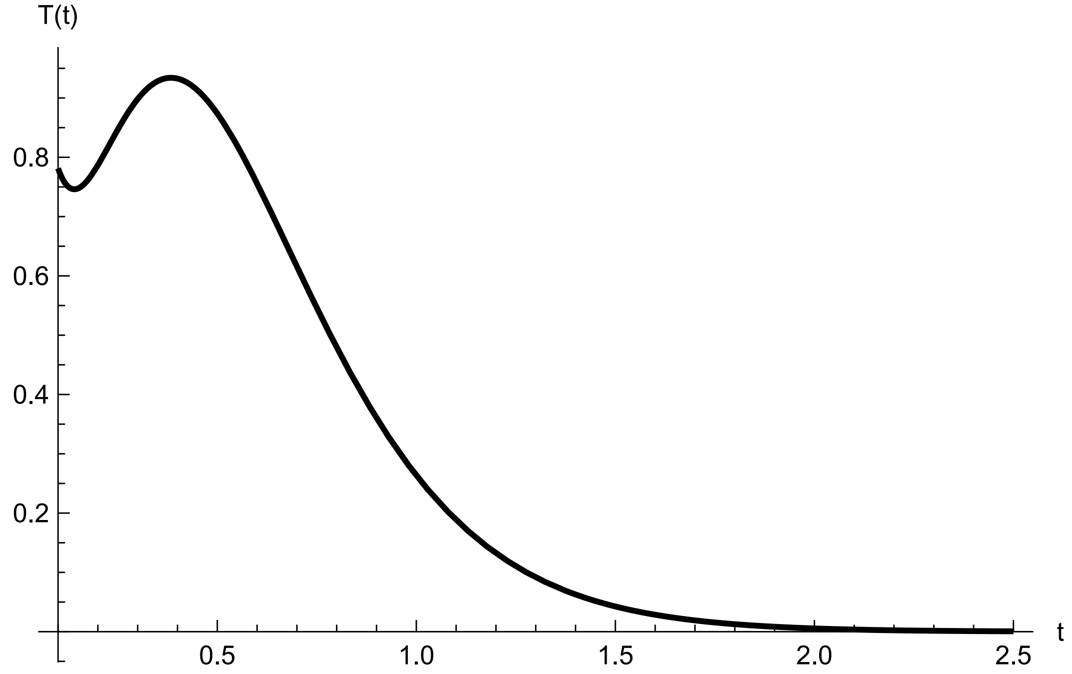


Figure 6.1: Time evolution of the gravitational wave component $h^1_3 = T(t)$

The decaying behavior of $T(t)$, representing the gravitational wave component h^1_3, reflects the damping of the wave amplitude over time in a five-dimensional anisotropic cosmological background.

6.2 Solution for $h^1_3 = h^1_3(t, \omega)$

Now let's consider the most general case $h^1_3(t, \omega)$, we get ;

$$-9\sinh^2 t\left[(\frac{10}{9}\coth^2 t - 4k\coth t + \frac{2}{3} + k^2)\right] h^1_3 - \left(2\sinh^2 t + \frac{3k\sinh 2t}{2}\right) h^1_3 \\ + \left[\partial^2_\omega h^1_3 - 7\partial_\omega h^1_3 + 10h^1_3\right] - 9\sinh^2 t\left[(2k - \frac{4}{3}\coth t)\partial_t h^1_3 + \partial^2_t h^1_3\right] = 0$$

By applying the method of separation of variables, we express $h^2_3(t, \omega)$ as the product of a time-dependent function $T(t)$ and a spatial function $W(\omega)$. The resulting spatial equation is:

$$\partial^2_\omega W(\omega) - 7\partial_\omega W(\omega) + \Omega^2 W(\omega) = 0$$

The equation governing the time dependent component is:

$$\partial_t^2 T(t) + \frac{4}{3}(\sqrt{6} - \coth t\,)\partial_t T(t) + \frac{2}{3}\coth(t)\, T(t) + \frac{14}{3}T(t) + \frac{8\sqrt{6}}{3}\coth(t) T(t) + \frac{\Omega^2}{9\sinh^2 t}T(t) = 0 \tag{6.6}$$

Here, Ω^2 represents the separation constant. The graph illustrating the solution to equation (6.6) is shown in the figure below :

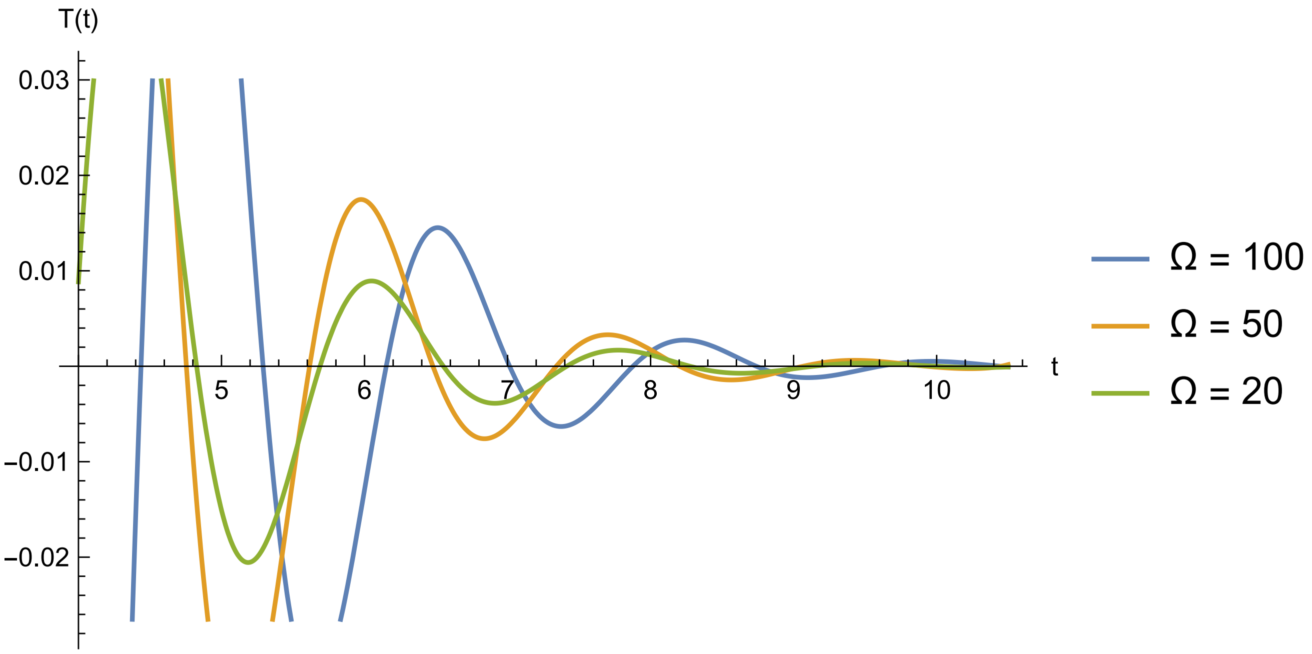


Figure 6.2: Time evolution of the gravitational wave component $h_3^1(t,\omega)$ for different values of the Ω.

As time t increases, the amplitude of $T(t)$ decreases for all values of Ω, indicating a damping of the gravitational wave signal. This decay in amplitude reflects the effect of an expanding anisotropic universe, where the stretching of spacetime causes gravitational waves to redshift and lose energy over time. Higher values of Ω show more rapid oscillations, but all modes exhibit a similar decaying trend, consistent with the dissipation of wave energy in both the observable and extra dimensions.

6.3 Solution for $h_3^2 = h_3^2(t,\omega)$

Now let's solve another axial component that is h_{23}, given as ;

$$\Box h_{23} - \mu\, e^{-kt-4\omega}\sinh^{\frac{1}{3}}(t)[3k\cosh t + 2\sinh t]\; h^{32} - \frac{3\; k\; e^{-2\omega}\sinh 2t}{4}h_2^3 - \frac{3\; k\; e^{-kt-2\omega}\sinh t\;[2\cosh t + 3k\sinh t]}{4}h_3^2 = 0$$

or

$$\Box h_{23} - e^{-kt-2\omega}(3k\ \sinh 2t + 8\ \sinh^2 t)\ h^2_3 = 0 \tag{6.7}$$

Now solving $\Box h_{23}$ as follows ;

$$\Box h_{23} = \frac{1}{\sqrt{-g}}\,\partial_t\big(\sqrt{-g}\ g^{tt}\partial_t h_{23}\big) + \frac{1}{\sqrt{-g}}\,\partial_\omega\big(\sqrt{-g}\ g^{\omega\omega}\partial_\omega h_{23}\big)$$

or

$$\Box h_{23} = \frac{1}{\sqrt{-g}}\,\partial_t\big[\sqrt{-g}\ g^{tt}\partial_t(g_{22}h^2_3)\big] + \frac{1}{\sqrt{-g}}\,\partial_\omega\big[\sqrt{-g}\ g^{\omega\omega}\partial_\omega(g_{22}h^2_3)\big]$$

or

$$\begin{aligned}\Box h_{23} =& -9\sinh^2 t\ e^{-kt-2\omega}\left[\left(\frac{10}{9}\coth^2 t + 4k\coth t - \frac{2}{3} + k^2\right)\right]\ h^2_3 \\ &-9\sinh^2 t\ e^{-kt-2\omega}\left[(-2k - \frac{4}{3}\coth t)\partial_t h^2_3 + \partial^2_t h^2_3\right] + e^{-kt-2\omega}\left[\partial^2_\omega h^2_3 - 7\partial_\omega h^2_3 + 10h^2_3\right]\end{aligned}$$

Using the value of $\Box h^2_3$ in equation(6.7), we get ;

$$\begin{aligned}&-9\sinh^2 t\ e^{-kt-2\omega}\left[\left(\frac{10}{9}\coth^2 t + 4k\coth t - \frac{2}{3} + k^2\right)\right]\ h^2_3 \\ &+e^{-kt-2\omega}\left[\partial^2_\omega h^2_3 - 7\partial_\omega h^2_3 + 10h^2_3\right] - 9\sinh^2 t\ e^{-kt-2\omega}\left[(-2k - \frac{4}{3}\coth t)\partial_t h^2_3 + \partial^2_t h^2_3\right] \\ &-e^{-kt-2\omega}(3k\ \sinh 2t + 8\ \sinh^2 t)\ h^2_3 = 0\end{aligned}$$

or

$$\begin{aligned}\therefore\quad & 9\sinh^2 t\left[\left(\frac{10}{9}\coth^2 t + 4k\coth t - \frac{2}{3} + k^2\right)\right]\ h^2_3 + (3k\ \sinh 2t + 8\ \sinh^2 t)\ h^2_3 \\ &-\left[\partial^2_\omega h^2_3 - 7\partial_\omega h^2_3 + 10h^2_3\right] + 9\sinh^2 t\left[(-2k - \frac{4}{3}\coth t)\partial_t h^2_3 + \partial^2_t h^2_3\right] = 0\end{aligned} \tag{6.8}$$

Using the separation of variables technique, we write $h^2_3(t,\omega)$ as the product of a time-dependent function $T(t)$ and a spatial function $W(\omega)$. The time-dependent portion then satisfies the following equation :

$$\begin{aligned}\partial^2_t T(t) - &\frac{4}{3}(\sqrt{6} - \coth t\ )\partial_t T(t) + \frac{12}{3}T(t) \\ &+\frac{28\sqrt{6}}{9}\coth(t)T(t) + \frac{\Omega^2}{9\sinh^2 t}T(t) = 0\end{aligned} \tag{6.9}$$

Ω^2 denotes the constant arising from the separation of variables.

The figure below illustrates the graphical solution of equation (6.9).

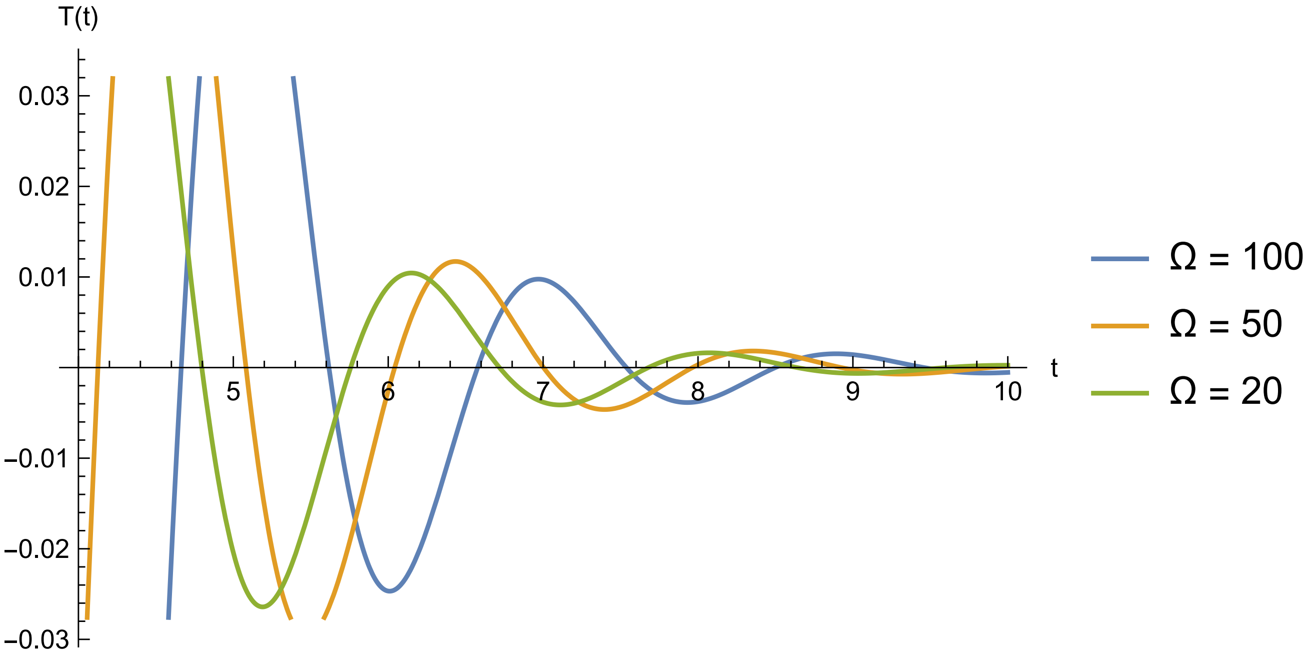


Figure 6.3: Time evolution of the gravitational wave component $h_3^2(t, \omega)$ for different values of the Ω.

Similar to the h_3^1 case, the amplitude of $T(t)$decreases as time progresses, signaling the damping of gravitational waves in an expanding five-dimensional anisotropic universe. The oscillatory nature remains evident across different Ωvalues, but the envelope of the wave decays with time. This behavior reflects the combined effects of cosmological expansion and energy dissipation into the extra spatial dimension, demonstrating how higher-dimensional geometry can influence the evolution and observability of gravitational waves.

Conclusion

We here presented a self contained description of gravitational waves and their properties. We also presented the explicit calculation of gravitational waves in the background of the FLRW universe. In the spatially open case, they behave as Bessel functions, and for the closed case, they are described by associated Legendre functions. We also analyzed gravitational waves in the case of an anisotropic 5D universe, where one of the spatial dimensions shrinks with time, effectively resulting in a 4D universe at late times. We obtained explicit, though numerical, solutions for the axial modes and plotted their time evolution. We find that their properties are significantly influenced by the anisotropy and warping of the background. In particular, the modes exhibit oscillatory behavior with frequencies that increase with the KK mass parameter Ω^2 and the amplitude shows direction dependent modulation due to the anisotropic expansion.

Scope for futute work

We can further extend this work by analyzing the longitudinal modes in the 5D anisotropic universe and studying their evolution in the same background. Additionally, calculating the quadrupole moment and luminosity of these gravitational waves will provide insights into their energy content and potential observational signatures. This would help in understanding how such modes might appear in gravitational wave detectors if the underlying geometry of our universe includes higher-dimensional anisotropic effects.

APPENDIX A

Bianchi Type-I universe

In this appendix, we study gravitational waves in the Bianchi type-I universe, which is the simplest anisotropic but homogeneous cosmological model. Unlike the FLRW metric, it allows for different scale factors along each spatial direction, making it suitable for analyzing anisotropic effects on gravitational wave propagation. We begin by writing down the Bianchi Type-I metric and then derive the corresponding gravitational wave equations using linear perturbation theory. The Bianchi type-I cosmological model represents the most straightforward homogeneous yet anisotropic model of cosmos, characterized by the following metric :

$$dS^2 = -dt^2 + a^2dx^2 + b^2dy^2 + c^2dz^2 \tag{6.10}$$

The functions $a(t), b(t), c(t)$ depend on cosmic time t and serve as scale factors along the spatial coordinates x, y, z respectively.

6.4 Equations Governing Gravitational Waves :

We know the general gravitational wave equation is given is given by equation (1.4) written as ;

$$\Box\tilde{h}_{\beta\alpha} - \bar{R}_{\delta\alpha}\tilde{h}^{\delta}_{\beta} - \bar{R}_{\delta\beta}\tilde{h}^{\delta}_{\alpha} - 2\bar{R}_{\delta\beta\alpha\rho}\tilde{h}^{\delta\rho} = 0 \tag{6.11}$$

The non-zero components of Christoffel symbol are given as ;

$$\Gamma^1_{10} = \frac{\dot{a}}{a}, \qquad \Gamma^2_{20} = \frac{\dot{b}}{b}, \qquad \Gamma^3_{30} = \frac{\dot{c}}{c}$$

$$\Gamma^0_{11} = a\dot{a}, \qquad \Gamma^0_{22} = b\dot{b}, \qquad \Gamma^0_{33} = c\dot{c}$$

Now, by evaluating the Ricci tensor, we obtain its non-zero components as follows :

$$\bar{R}^0_0 = -\frac{1}{abc}\left(bc\ddot{a} + \ddot{b}ac + \ddot{c}ab\right), \qquad \bar{R}^1_1 = -\frac{1}{abc}\left[bc\ddot{a} + c\dot{a}\dot{b} + b\dot{a}\dot{c}\right]$$

$$\bar{R}_2^2 = -\left[\frac{\ddot{b}}{b} + \frac{\dot{b}}{b}\left(\frac{\dot{c}}{c} + \frac{\dot{a}}{a}\right)\right], \qquad \bar{R}_3^3 = -\left[\frac{\ddot{c}}{c} + \frac{\dot{c}}{c}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right)\right]$$

From the above equations of Ricci tensor, Ricci scalar is written as ;

$$\bar{R} = \frac{2}{abc}\left(\dot{a}\dot{b}c + b\dot{c}\dot{a} + a\dot{b}\dot{c} + \ddot{a}bc + \ddot{b}ca + \ddot{c}ab\right)$$

Now, by evaluating the Riemann tensor, we obtain its non-zero components as follows :

$$\bar{R}^{01}_{01} = \frac{\ddot{a}}{a}, \qquad \bar{R}^{02}_{02} = \frac{\ddot{b}}{b}, \qquad \bar{R}^{03}_{03} = \frac{\ddot{c}}{c}$$

$$\bar{R}^{12}_{12} = -\frac{\dot{a}}{a}\frac{\dot{b}}{b}, \qquad \bar{R}^{23}_{23} = -\frac{\dot{b}}{b}\frac{\dot{c}}{c}, \qquad \bar{R}^{31}_{31} = -\frac{\dot{c}}{c}\frac{\dot{a}}{a}$$

By employing the Bianchi type-I metric (6.10), we have determined all the relevant Christoffel symbols, Ricci tensor, and Riemann tensor elements that are not equal to zero, as required in equation (6.20). Therefore the wave equation describing the evolution of each component of the metric perturbation can be expressed as :

$$\Box h_{11} - \frac{2}{abc}\left(\ddot{a}bc + \dot{a}\dot{b}c + \dot{a}\dot{c}b\right)h_{11} + 2a\dot{a}\left(\frac{\dot{b}}{b^3}h_{22} + \frac{\dot{c}}{c^3}h_{33}\right) - 2\left(a\ddot{a}\right)h_{00} = 0$$

$$\Box h_{00} - \frac{1}{abc}\left(2\ddot{a}bc + 2\ddot{b}ac + 2\ddot{c}ba\right)h_{00} - \frac{2}{a^3b^3c^3}\left(\ddot{a}b^3c^3h_{11} + \ddot{b}a^3c^3h_{22} + \ddot{c}a^3b^3h_{33}\right) = 0$$

$$\Box h_{22} - \frac{2}{abc}\left(\ddot{b}ca + \dot{a}\dot{b}c + \dot{b}\dot{c}a\right)h_{22} + \frac{2\dot{b}}{b^3c^3}\left(b\dot{a}c^3h_{11} + b\dot{c}b^3h_{33}\right) - 2\ddot{b}bh_{00} = 0$$

$$\Box h_{33} - \frac{2}{abc}\left(ab\ddot{c} + \dot{a}\dot{c}b + \dot{b}\dot{c}a\right)h_{33} + \frac{2c\dot{c}}{b^3}\left(\dot{a}h_{11} + \dot{b}h_{22}\right) - 2c\ddot{c}h_{00} = 0$$

$$\Box h_{01} - \frac{1}{abc}\left(\ddot{a}bc + \ddot{b}ca + \ddot{c}ab\right)h_{01} - \frac{2\ddot{a}}{a}h_{10} - \frac{1}{abc}\left(\ddot{a}bc + \dot{a}\dot{b}c + \dot{a}\dot{c}b\right)h_{10} = 0$$

$$\Box h_{02} - \frac{1}{abc}\left(\ddot{a}bc + \ddot{b}ca + \ddot{c}ab\right)h_{02} - \frac{2\ddot{b}}{b}h_{20} - \frac{1}{abc}\left(a\ddot{b}c + \dot{a}\dot{b}c + \dot{b}\dot{c}a\right)h_{20} = 0$$

$$\Box h_{03} - \frac{1}{abc}\left(\ddot{a}bc + \ddot{b}ca + \ddot{c}ab\right)h_{03} - \frac{2\ddot{c}}{c}h_{30} - \frac{1}{abc}\left(ab\ddot{c} + \dot{c}\dot{a}b + \dot{c}\dot{b}a\right)h_{30} = 0$$

$$\Box h_{12} - \frac{1}{cba}\left(ac\ddot{b} + \dot{a}\dot{b}c + \dot{b}\dot{c}a + 2\dot{a}\dot{b}c\right)h_{21} - \frac{1}{abc}\left[\dot{a}\left(c\dot{b} + b\dot{c}\right) + \ddot{a}bc\right]h_{12} = 0$$

$$\Box h_{13} - \frac{1}{abc}\left(b\dot{a}\dot{c} + \dot{b}\dot{c}a + 2\dot{a}\dot{c}b + \ddot{c}ba\right)h_{31} - \frac{1}{abc}\left(\ddot{a}bc + \dot{a}\dot{b}c + \dot{a}\dot{c}b\right)h_{13} = 0$$

$$\Box h_{23} - \frac{1}{abc}\left(ab\ddot{c} + \dot{a}\dot{c}b + \dot{b}\dot{c}a + 2a\dot{b}\dot{c}\right)h_{32} - \frac{1}{abc}\left(ac\ddot{b} + \dot{a}\dot{b}c + \dot{b}\dot{c}a\right)h_{23} = 0.$$

In all the above equations index $1, 2, 3$ correspond to x, y, z respectively.
Assuming the wave is moving in the x direction, further applying the transverse traceless gauge conditions ($h_{0\alpha} = 0,\ h_{1\alpha} = 0,\ \ h_{22} = -h_{33},\ \ h_{23} = h_{32}$), We are left with only two independent components of the metric perturbations, and their corresponding wave equations are given by :

$$\Box h_{22} - \frac{2}{abc}\left(\ddot{b}ca + \dot{a}\dot{b}c + \dot{b}\dot{c}a\right) h_{22} + \frac{2\dot{b}}{b^3c^3}\left(b\dot{a}c^3 h_{11} + b\dot{c}b^3 h_{33}\right) - 2\ddot{b}bh_{00} = 0 \quad (6.12)$$

$$\Box h_{23} - \frac{1}{abc}\left(ab\ddot{c} + \dot{a}\dot{c}b + \dot{b}\dot{c}a + 2a\dot{b}\dot{c}\right) h_{32} - \frac{1}{abc}\left(ac\ddot{b} + \dot{a}\dot{b}c + \dot{b}\dot{c}a\right) h_{23} = 0 \quad (6.13)$$

On further simplifying the above two equations on the basis of given condition, we get ;

$$\left[\Box - 2\frac{\dot{b}\dot{b}c}{c^3} - \frac{2}{b}\left(\frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{a} + \frac{\dot{b}\dot{c}}{c}\right)\right] h_{22} = 0 \quad (6.14)$$

$$\left[\Box - 2\frac{\dot{b}\dot{c}}{bc} - \frac{1}{abc}\left(ac\ddot{b} + \dot{a}\dot{b}c + \dot{b}\dot{c}a\right) - \frac{1}{abc}\left(ab\ddot{c} + \dot{a}\dot{c}b + \dot{b}\dot{c}a\right)\right] h_{32} = 0 \quad (6.15)$$

Where the d'Alembertian of h_{22} and h_{33} is given by ;

$$\Box h_{22} = -\ddot{h}_{22} - \frac{1}{abc}\left(\dot{a}bc + \dot{b}ca + \dot{c}ab\right)\dot{h}_{22} + \frac{1}{a^2}h''_{22}$$

$$\Box h_{33} = -\ddot{h}_{33} - \frac{1}{abc}\left(\dot{a}bc + \dot{b}ca + \dot{c}ab\right)\dot{h}_{33} + \frac{1}{a^2}h''_{33}$$

Where the double dot represents $\frac{\partial^2}{\partial t^2}$ and the double prime $''$ represents $\frac{\partial^2}{\partial x^2}$.
In conclusion, the Bianchi Type-I model allows us to explore the behavior of gravitational waves in an anisotropic but homogeneous universe. By applying the linearized Einstein equations to this background, we derived the complete set of wave equations for metric perturbations. Under the transverse-traceless gauge and assuming wave propagation along one direction, the analysis simplifies to two independent components, capturing the physical degrees of freedom. This study highlights how anisotropic expansion influences the dynamics of gravitational waves and serves as a useful generalization beyond the standard isotropic cosmologies.

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