A Study on Covering Number of Finite Groups

M.Sc. Thesis

by

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A Study on Covering Number of Finite Groups

A THESIS

Submitted in partial fulfillment of the requirements for the award of the degree of

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by

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"A study on Covering Number of finite groups"** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF MATHEMATICS**, **INDIAN INSTITUTE OF TECHNOLOGY INDORE**, is an authentic record of my own work carried out during the time period from July 2024 to May 2025 under the supervision of **Dr. Sumit Chandra Mishra**, Assistant Professor, Department of Mathematics, IIT Indore. The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

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23/5/25

Signature of the student with date

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Date: 23-05-2025

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Signature of Convener, DPGC

Date: 23/05/2025

Dedicated to my

Family

“ Those times when you get up early and you work hard, those times when you stay up late and you work hard, those times when you don’t feel like working, you’re too tired, you don’t want to push yourself, but you do it anyway. That is actually the dream. That’s the dream. It’s not the destination, it’s the journey. And if you guys can understand that, then what you’ll see happen is you won’t accomplish your dreams, your dreams won’t come true; something greater will.”

– ***Kobe Bryant***

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Abstract

This thesis presents a detailed investigation into the covering number $\sigma(G)$ of finite groups. A covering of a group G is defined as a collection of proper subgroups whose union equals G, and the minimal such number is called the covering number. We begin by discussing foundational results, such as the non-existence of a covering for cyclic groups and the impossibility of a covering number of two for any group.

Special focus is given to the characterization of groups with covering number three, including a detailed structural proof that such groups G satisfy $G/K \cong C_2 \times C_2$ for some normal subgroup K [4]. Furthermore, we discuss computation of exact covering numbers for symmetric groups S_n with odd $n > 1$, we discuss the proof of the result $\sigma(S_n) = 2^{n-1}$, except for the case $n = 9$ as in [3].

Contents

CHAPTER 1

Introduction

The concept of a covering number arises from the study of how a finite group can be expressed as a union of its proper subgroups. A **covering** of a group G is defined as a collection of proper subgroups whose union equals the entire group, i.e., $G = \bigcup_{i=1}^{n} H_i$ where each H_i is a proper subgroup of G. A group that can be covered in this way is said to be *coverable*. The smallest number of proper subgroups required to cover a group G is known as its **covering number**, denoted by $\sigma(G)$.

It is a well-established result that a group can be covered by proper subgroups if and only if it is not cyclic [4]. Moreover, no group can be covered by just two proper subgroups, implying that the minimum possible covering number for any group is at least three [4]. These foundational results highlight how the algebraic structure of a group influences the behavior and complexity of its subgroup coverings.

The study of covering numbers not only provides insight into the subgroup structure of finite groups but also connects with broader themes in group theory such as normality, simplicity, and transitivity. In this thesis, we aim to study the groups with covering number three and study the covering numbers for symmetric and alternating groups."Throughout this thesis, all our groups are assumed to be finite." The readers can refer to [4] for more detailed introduction.

CHAPTER 2

Preliminaries

In this section, we recall some fundamental definitions from group theory that are essential for our study.

Definition 2.1. *A group covering is a set of proper subgroups of a given group whose union is the whole group. The group is said to be covered by these subgroups.*

Theorem 2.1. *[4, Theorem 1.3] A group has a covering if and only if it is not cyclic.*

Proof. : Assume to the contrary that there is a covering for some cyclic group, then one of these subgroups must contain a generating element. Therefore, that subgroup is the group itself and thus not a proper subgroup. This contradicts our definition of covering.

Suppose G is a non-cyclic group. Therefore, no single element generates it. collect the subgroups generated by each element.Thus we have created a set of proper subgroups such that every element of G is contained in at least one of them, hence getting a covering for G.

□

Definition 2.2. *The covering number of a group, denoted $\sigma(G)$, is the minimal number of proper subgroups required to cover the group. If a group cannot be covered, then $\sigma(G) = \infty$.*

Theorem 2.2. *[4, Theorem 1.5] No group has a covering number of two.*

Proof. Let G be any group such that its covering number is two. Therefore, there are two proper subgroups, which we will call H_1 and H_2 such that $G = H_1 \cup H_2$. Pick an elements $h_1 \in H_1$ such that $h_1 \notin H_2$ and likewise $h_2 \in H_2$ such that $h_2 \notin H_1$. If this cannot be done then either all elements of H_1 are contained in H_2 or vice versa. This means that one of the groups is not necessary for the covering, contradicting that two subgroups are needed. If $h_1 h_2 \in H_1$, then since $h_1^{-1} \in H_1$, we have $h_2 \in H_1$. This is a contradiction. Similarly, $h_1 h_2 \notin H_2$. Therefore, G cannot be covered by only two proper subgroups.

□

Lemma 2.3. *[4, Lemma 2.1] If $N \trianglelefteq G$, then $\sigma(G) \leq \sigma(G/N)$.*

Proof. Let us consider a minimal covering of G/N, labeled $\{H_1/N,\ H_2/N, \ldots,\ H_n/N\}$ for some $n = \sigma(G/N)$ where each H_i is a subgroup of G containing N. We have $G/N = \bigcup\limits_{i=1}^{n}$ (H$_i$/N) then $G/N = (\bigcup\limits_{i=1}^{n} H_i)/N$ and so $G = \bigcup\limits_{i=1}^{n} H_i$. Hence $\sigma(G)$ is at most n. □

Corollary 2.4. *[4, Corollary 2.2] Given a surjective homomorphism $\varphi : G \to H$, $\sigma(G) \leq \sigma(H)$.*

Proof. By First isomorphism theorem. we have $G/ker\varphi \cong \varphi(G) = H$.
Now the result follow from Lemma 2.3, $\sigma(G) \leq \sigma(H)$. □

Corollary 2.5. *[4, Corollary 2.3] $\sigma(H \times K) \leq$ min($\sigma(H)$, $\sigma(K)$).*

Proof. Consider a mapping $\varphi : H \times K \to H$, defined by $\varphi(h,k) = h$. This is a surjective homomorphism. So, $\sigma(H \times K) \leq \sigma(H)$. Similarly, considering a subjective homomorphism $\vartheta : H \times K \to K$, defined by $\vartheta(h,k) = k$, we get $\sigma(H \times K) \leq \sigma(K)$. Hence, $\sigma(H \times K) \leq \min(\sigma(H), \sigma(K)$.

□

Definition 2.3. *A group G is called primitive if there exists no normal subgroup N for which $\sigma(G) = \sigma(G/N)$.*

Lemma 2.6. *[4, Lemma 2.5] A minimal covering for any given group G can always be expressed as a union of maximal subgroups.*

Proof. Since we assume that G is finite, any proper subgroup is contained in a maximal subgroup. Therefore, any non-maximal proper subgroup used in a covering can be substituted with a maximal subgroup that contains it. The number of subgroups has not increased, and so this is still a minimal covering. □

Let G be a group acting on a set X

Definition 2.4. *A permutation group G is called transitive group if its group action is transitive.*

Note that group action is transitive iff it has exactly one orbit.

Definition 2.5. *An intransitive group G refers to a permutation group that does not act transitively on a given set. We say that G acts intransitively on X if there exist at least two distinct orbits under the action of G.*

Example 2.1. *Consider $S_3 = \{e, (12), (13), (23), (123), (132)\}$.Consider the natural action of S_3 on $X = \{1, 2, 3\}$.*

$\langle(12)\rangle = \{e, (12)\}$, $\langle(13)\rangle = \{e, (13)\}$, $\langle(23)\rangle = \{e, (23)\}$ $\langle(123)\rangle = \{e, (123), (132)\}$

These are transitive subgroups of S_3.

For G acting on X, we define for $g \in G$, $gX = \{g \cdot x | x \in X \ , g \in G\}$.

Example 2.2. *For $g = (1,2) \in S_3$ and $X = \{1,2,3\}$,$gX = X$.*

Let G be a group acting on X.

Definition 2.6. *A non-empty subset A of X is said to be block, if $\forall g \in G$, Either $gA = A$ or $gA \cap A = \phi$.*

Definition 2.7. *A group G is said to be primitive group if the only blocks are singletons and full set.*

Definition 2.8. *Given two groups N and H, and a homomorphism $\phi :\ H \rightarrow Aut(N)$, the semidirect product of N and H with respect to ϕ, denoted by $N \rtimes H$. The group operation is defined as $(n_1, h_1)(n_2, h_2) = (n_1\phi(h_1)(n_2), h_1h_2)$.*
where $n_1, n_2 \in N$,and $h_1, h_2 \in H$, $\phi(h_1)(n_2) = h_1n_2h_1^{-1}$. We have:-

1. *$N \trianglelefteq G$.*

2. *$H \cap N = e$.*

Definition 2.9. *Let A and B be two groups and let B act on a set Ω. The wreath product of A and B denoted by $A \wr B$ is defined as the semidirect product. where,*

1. *A^Ω is the direct product of copies of A, one copy for each element of Ω. A^Ω can be thought of as the set of all functions from Ω to A.*

2. *B acts on A^Ω by permuting coordinates as follows:- For $b \in B$. and For $f \in A^\Omega$, define $(b.f)(w) = f(b^{-1}.w)$.*

3. *The group operation in $A \wr B$ is $(f_1, b_1)(f_2, b_2) = (f_1.(b_1.f_2), b_1b_2)$. If B acts on itself by regular permutation, we get the regular wreath product.*

CHAPTER 3

Covering Number Three

In this chapter, we focus on ginite groups whose covering number is exactly three. That is, we investigate groups G for which $\sigma(G) = 3$. This chapter is based on [4] and we refer the reader to [4] for more detail.

We begin by considering simple examples and gradually build toward a general characterization of such groups. In particular, we examine the quaternion group Q_8 and use it as a motivating example to analyze the subgroup structure involved in such coverings. This analysis leads us to a key structural theorem: a group has covering number three if and only if it has a normal subgroup K such that $G/K \cong C_2 \times C_2$.

Before we proceed to formal results, let us recall some basic observations and notations that will be used throughout this chapter. We start with a group G such that $\sigma(G) = 3$. So, $G = A \cup B \cup C$, where A, B and C are proper subgroups of G. Now we take an example $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, $H_1 = \{\pm 1, \pm i\}$, $H_2 = \{\pm 1, \pm j\}$, $H_3 = \{\pm 1, \pm k\}$, where H_1 ,H_2 ,H_3 are proper subgroups of Q_8.

Let K denote the intersection of the three proper subgroups. Being the intersection of groups, K is also a subgroup of G. As well, let $H_1' = H_1 \setminus (H_2 \cup H_3)$, H_2' $= H_2 \setminus (H_1 \cup H_3)$ and $H_3' = H_3 \setminus (H_1 \cup H_2)$.
See Figure 1.

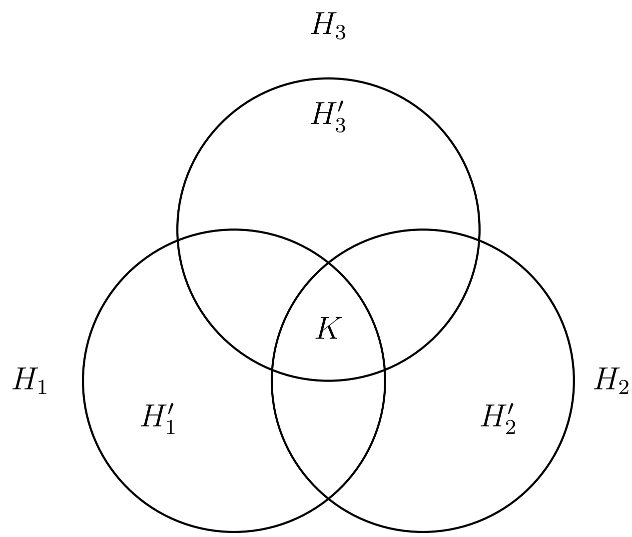

Figure 3.1: Venn diagram of covering groups H_1, H_2 and H_3 with labeled regions

The sets H_1', H_2', H_3', K are non-empty. We only need to observe that if one of H_i' is empty, say H_1', then $G = H_2 \cup H_3$, which contradicts our assumption of needing three. Therefore, H_1', H_2', and H_3' are all non-empty. We have e $\in K$, so K is also non-empty. With those four regions labeled, we are left with only three more: $(H_1 \cap H_2) \setminus H_3$,$(H_1 \cap H_3) \setminus H_2$, and $(H_2 \cap H_3) \setminus H_1$. These sets can be thought of as elements belonging to exactly two of the subgroups.

Theorem 3.1. *[4, Theorem 3.1] The set $(H_2 \cap H_3) \setminus H_1$ is empty.*

Proof. Let X be the set $(H_2 \cap H_3) \setminus H_1$. Assume that X is non empty and let $x \in X$. Then, pick an element h_1 from the set H_1'. We have $h_1 x \in G$, and so $h_1 x$ is contained in H_1, H_2 or H_3. If $h_1 x \in H_1$, then we know $h_1^{-1} \in H_1$, and so $x = h_1^{-1} h_1 x \in H_1$, a contradiction. If $h_1 x \in H_2$, then since $x^{-1} \in B$,

$h_1 = h_1 x x^{-1} \in H_2$, but $h_1 \in H_1$, once again a contradiction. Therefore, X is empty. □

Lemma 3.2. *[4, Lemma 3.2] Let $G = H_1 \cup H_2 \cup H_3$ be a finite group with $\sigma(G) = 3$, and define the sets:$H_1' = H_1 \setminus (H_2 \cup H_3)$, $H_2' = H_2 \setminus (H_1 \cup H_3)$, $H_3' = H_3 \setminus (H_1 \cup H_2)$. Suppose $h_1' \in H_1'$, $h_2' \in H_2'$, and $h_3' \in H_3'$. Then the following hold:*

$$h_1' h_2' \in H_3', \quad h_2' h_3' \in H_1', \quad h_3' h_1' \in H_2'.$$

Proof. Let us see the proof of the first statement: $h_1' h_2' \in H_3'$. The proofs for the other two are similar. Since $h_1' \in H_1$ and $h_2' \in H_2$, their product $h_1' h_2' \in G$. We now examine which subgroup it could belong to.

Suppose $h_1' h_2' \in H_1$. Then multiplying on the left by $(h_1')^{-1} \in H_1$, we get:$h_2' = (h_1')^{-1}(h_1' h_2') \in H_1$, which contradicts $h_2' \notin H_1$ (since $h_2' \in H_2'$). So, $h_1' h_2' \notin H_1$.

Similarly, if $h_1' h_2' \in H_2$, then multiplying on the right by $(h_2')^{-1} \in H_2$, we would get:$h_1' = (h_1' h_2')(h_2')^{-1} \in H_2$, which contradicts $h_1' \in H_1'$. Hence, $h_1' h_2' \notin H_2$ either.

Therefore, the product $h_1' h_2'$ must lie in H_3, but not in $H_1 \cup H_2$. So: $h_1' h_2' \in H_3 \setminus (H_1 \cup H_2) = H_3'$. The same argument applies for the products $h_2' h_3' \in H_1'$ and $h_3' h_1' \in H_2'$, completing the proof. □

Lemma 3.3. *[4, Lemma 3.3] Each of the sets H_1', H_2', H_3' is closed under taking inverses. That is, if an element belongs to one of the unique parts of the subgroups, then its inverse also belongs to that same part.*

Proof. Let us see the proof for H_1'; the argument is exactly the same for H_2' and H_3'.

Let $h \in H_1'$. Since $h \in H_1$, and H_1 is a subgroup, the inverse $h^{-1} \in H_1$ as well. We now need to check that $h^{-1} \notin H_2 \cup H_3$, because that would force $h^{-1} \notin H_1'$. Suppose, for contradiction, that $h^{-1} \in H_2$. Then since $h = (h^{-1})^{-1} \in H_2$, it would follow that $h \in H_1 \cap H_2$, contradicting the fact that $h \in H_1'$, which is defined to exclude all elements from H_2 and H_3. A similar contradiction arises if we suppose $h^{-1} \in H_3$. Hence, $h^{-1} \notin H_2 \cup H_3$, which means $h^{-1} \in H_1'$, as required. Therefore, H_1' is closed under inverses, and similarly for H_2' and H_3'. □

Lemma 3.4. *[4, Lemma 3.4] An element belongs to one of the unique parts H_1', H_2', or H_3' if and only if it can be written as the product of elements from the other two.*

Proof. We already know from Lemma 3.2 that if you take one element from each of two different sets among H_1', H_2', H_3', their product lies in the third. This gives us the "if" direction.

Now we discuss the "only if" direction: that is, every element in H_1' can be written as a product of one element from H_2' and one from H_3', and similarly for the other two sets. Let $h_1 \in H_1'$. Pick any element $h_3 \in H_3'$. Since H_3' is closed under inverses (as shown in Lemma 3.3), we know that $h_3^{-1} \in H_3'$. Now consider the product $h_1 h_3^{-1}$. This element lies in G, and more specifically: $h_1 h_3^{-1} \in H_1 \cdot H_3 \subseteq G$. Let us determine where this product lies. Since $h_1 \in H_1$ and $h_3^{-1} \in H_3$, the product $h_2 := h_1 h_3^{-1}$ lies in some H_i. But by the same kind of reasoning used in Lemma 3.2, it cannot lie in H_1 or H_3, because that would contradict the uniqueness of h_1 and h_3. Therefore, $h_2 \in H_2'$. Now solving for h_1, we get: $h_1 = h_2 \cdot h_3$, where $h_2 \in H_2'$ and $h_3 \in H_3'$. Thus, h_1 is a product of elements from the other two sets. By symmetry, the same holds for any element in H_2' or H_3'. So every element in one of the H_i' sets is a product of elements from the other two. This completes the proof. □

Lemma 3.5. *[4, Lemma 3.5] If $h_1, h_2 \in H_1'$, then their product $h_1 h_2 \in K$, where $K = H_1 \cap H_2 \cap H_3$.*

Proof. Let $h_1, h_2 \in H_1'$, meaning both elements lie in H_1, but not in $H_2 \cup H_3$. Since H_1 is a subgroup, their product $h_1 h_2 \in H_1$. We want to discuss that

$h_1h_2 \in H_2$ and $h_1h_2 \in H_3$, which would imply that $h_1h_2 \in K$, the intersection of all three subgroups.

From Lemma 3.4, we know that any element of H_1' can be written as a product of elements from H_2' and H_3'. So write: $h_2 = ab, \quad$ with $a \in H_2'$, $b \in H_3'$. Then: $h_1h_2 = h_1ab$. Group this as $(h_1a)b$. Since $h_1 \in H_1'$ and $a \in H_2'$, their product $h_1a \in H_3'$ by Lemma 3.2. So now we have: $h_1h_2 = (h_1a)b$, where both $h_1a \in H_3'$ and $b \in H_3'$. Since $H_3' \subseteq H_3$ and is closed under group operation (as shown in Lemma 3.3), their product lies in H_3. So we now know that $h_1h_2 \in H_1 \cap H_3$. A similar argument (e.g., writing $h_1 = a'b'$ with $a' \in H_2', b' \in H_3'$) shows that $h_1h_2 \in H_2$ as well.

Hence: $h_1h_2 \in H_1 \cap H_2 \cap H_3 = K$. This completes the proof. $\square$

Lemma 3.6. *[4, Lemma 3.6] The subgroup $K = H_1 \cap H_2 \cap H_3$ is a normal subgroup of G.*

Proof. To show that K is normal in G, we must prove that for every $g \in G$ and $k \in K$, the conjugate $gkg^{-1} \in K$. Since $G = H_1 \cup H_2 \cup H_3$, any element $g \in G$ must belong to at least one of the subgroups H_1, H_2, H_3. We can see that conjugation by elements from each of these subgroups preserves K; this will be enough to conclude that $K \trianglelefteq G$.

Let us first consider conjugation by an element $h_1 \in H_1$. Take any $k \in K$. Since H_1 is a subgroup, the product $h_1kh_1^{-1} \in H_1$. Also, because $k \in K \subseteq H_2$ and H_2 is a subgroup, $h_1kh_1^{-1} \in H_2$ if and only if conjugation preserves subgroup membership (which holds if H_2 is normal, but we do not assume that). However, we do know that if $h_1 \in K$, then $h_1kh_1^{-1} \in K$, since K is a subgroup and hence closed under conjugation by its own elements.

Now suppose $h_1 \in H_1'$, i.e., in H_1 but not in $H_2 \cup H_3$. Then $h_1k \in H_1$, and since $k \in K$, we know $h_1k \notin H_2 \cup H_3$ only if $h_1 \notin K$, which is allowed.

We use the fact from Lemma 3.3 that H_1' is closed under inverses, so $h_1^{-1} \in H_1'$. Now the product $h_1kh_1^{-1} \in H_1$ (since subgroups are closed under conjugation), and we also want to check if this lies in H_2 and H_3. Since $k \in K \subseteq H_2 \cap H_3$, and $h_1 \in H_1$, we argue similarly that $h_1 k h_1^{-1} \in H_2$ and H_3. Although we don't assume normality of the H_i, the intersection K is preserved because $h_1 \in G$ and the structure established by previous lemmas ensures closure of K under such conjugations.

Therefore, for any $g \in G$, whether in H_1, H_2, or H_3, we have $gkg^{-1} \in K$. So K is invariant under conjugation by all elements of G, which means $K \trianglelefteq G$. $\square$

Lemma 3.7. *[4, Lemma 3.7] Let $H_1' = H_1 \setminus (H_2 \cup H_3)$ and $K = H_1 \cap H_2 \cap H_3$. Then the left coset hK for any $h \in H_1'$ lies entirely within H_1. In fact,*

$$H_1' K = H_1.$$

Proof. In this lemma we will see that multiplying every element of H_1' by each element of K gives all of H_1, and nothing more.

First we show that $H_1'K \subseteq H_1$. Let $h \in H_1'$ and $k \in K$. Since both h and k are in H_1, and H_1 is a subgroup, their product $hk \in H_1$. Hence, every element in $H_1'K$ lies in H_1.

Next we show that $H_1 \subseteq H_1'K$. Let $x \in H_1$. If $x \in H_1'$, then clearly $x = x \cdot e \in H_1'K$, where $e \in K$ is the identity. Now suppose $x \in H_1 \setminus H_1'$. That means $x \in H_2 \cup H_3$, and hence $x \in K$ (because if $x \in H_1 \cap H_2$ or $H_1 \cap H_3$, and not in the unique parts, then it's in all three subgroups). In that case, pick any $h \in H_1'$. Then $h^{-1} \in H_1'$ as well (by Lemma 3.3), and so: $x = (hh^{-1})x = h \cdot (h^{-1}x)$. Since $h^{-1}x \in H_1$, and both h^{-1} and $x \in H_1$, their product is in H_1. But because $x \in K$, and $h^{-1} \in H_1'$, then $h^{-1}x \in H_1$ implies that the product lies in $H_1' \cdot K$. So, in short, every $x \in H_1$ can be written as $h \cdot k$ for some $h \in H_1'$, $k \in K$.Therefore,$H_1 = H_1'K$. This completes the proof. $\square$

Theorem 3.8. *[4, Lemma 3.8] Let G be a finite group. Then $\sigma(G) = 3$ if and only if there exists a normal subgroup $K \trianglelefteq G$ such that*

$$G/K \cong C_2 \times C_2.$$

Proof. ($\Rightarrow$) Suppose $\sigma(G) = 3$, so there exist three proper subgroups $H_1, H_2, H_3 < G$ such that $G = H_1 \cup H_2 \cup H_3$. Define the intersection $K := H_1 \cap H_2 \cap H_3$. Let us consider the sets $H_1' := H_1 \setminus (H_2 \cup H_3)$, $\quad H_2' := H_2 \setminus (H_1 \cup H_3)$, $\quad H_3' := H_3 \setminus (H_1 \cup H_2)$. Each H_i' is non-empty, otherwise the union of the other two subgroups would cover G, contradicting minimality.

Next we show that the set $(H_2 \cap H_3) \setminus H_1$ is empty. Assume there exists $x \in (H_2 \cap H_3) \setminus H_1$, and choose $h_1 \in H_1'$. Since $G = H_1 \cup H_2 \cup H_3$, the element $h_1 x$ must lie in one of the H_i. If $h_1 x \in H_1$, then $x = h_1^{-1}(h_1 x) \in H_1$, contradiction. If $h_1 x \in H_2$, then $h_1 = (h_1 x)x^{-1} \in H_2$, again contradiction. Similarly for H_3. Thus, such x cannot exist, so the set is empty. Now observe that: $H_1' \cdot H_2' \subseteq H_3'$, $\quad H_2' \cdot H_3' \subseteq H_1'$, $\quad H_3' \cdot H_1' \subseteq H_2'$, and each H_i' is closed under inverses. The product of two elements in the same H_i' lies in K. These relations show that the cosets K, $h_1 K$, $h_2 K$, $h_3 K$ (for representatives $h_i \in H_i'$) form a subgroup of G/K of order 4 with the same multiplication structure as the Klein four-group $C_2 \times C_2$. Therefore: $G/K \cong C_2 \times C_2$. Also, one can show $K \trianglelefteq G$ by verifying that conjugation by elements of H_1, H_2, H_3 leaves K invariant, using closure and symmetry properties already established.

($\Leftarrow$) Now suppose $K \trianglelefteq G$ and $G/K \cong C_2 \times C_2$. Since $C_2 \times C_2$ has three proper subgroups of index 2, whose union is the entire group, let us take their preimages under the canonical projection $\pi : G \to G/K$. Denote them as: $H_1 := \pi^{-1}(\langle x \rangle)$, $\quad H_2 := \pi^{-1}(\langle y \rangle)$, $\quad H_3 := \pi^{-1}(\langle xy \rangle)$, where $G/K = \{1, x, y, xy\}$. Then H_1, H_2, H_3 are proper subgroups of G whose union covers G. Hence, $\sigma(G) \leq 3$. Since no group can be covered by fewer than three proper subgroups, it follows that $\sigma(G) = 3$. $\square$

CHAPTER 4

Covering Number of S_n

In this chapter, we investigate the covering number $\sigma(S_n)$ of the symmetric group S_n, which consists of all permutations on n elements. While previous chapters focused on specific small values of n, such as $n = 5$, our goal here is to understand the behavior of the covering number as n increases. This chapter is based on [3] and we refer the reader to [3] for more details.

We begin by considering the case where $n > 1$ is odd and present a known result that $\sigma(S_n) = 2^{n-1}$ for such n, with the exception of $n = 9$. This result relies on the structure of maximal intransitive subgroups and their ability to cover elements formed by products of at most two disjoint cycles.

The analysis also highlights a deep connection between the group's action on sets, the structure of its subgroups, and the types of permutations it contains. Moreover, we use results from the literature to support the claim that the union of A_n and all maximal intransitive subgroups forms an optimal and, in many cases, unbeatable covering on certain subsets of S_n.

This chapter sets the stage for further research into the generalization of covering numbers for even values of n, which remains an open and compelling direction. We now present the key results and supporting arguments for the case when n is odd.

Theorem 4.1. *[3, Theorem 3.1] If $n > 1$ is odd ,then $\sigma(S_n) = 2^{n-1}$ unless $n = 9$.*

Proof. The set-theoretic union of A_n and all maximal intransitive subgroups of S_n is S_n. Structure of maximal intransitive subgroup is of the type $S_k \times S_{n-k}$. Number of such type of subgroups $= (1/2)(\sum_{k=1}^{n-1}\binom{n}{k})$. $\sigma(S_n) \leq 1 + (1/2)(\sum_{k=1}^{n-1}\binom{n}{k}) = 2^{n-1}$. The upper bound is known to be exact for $n = 3$ and 5, (I have already proved this in the previous presentation) so assume that $n \geq 7$. Now let Π be the set of all permutations of S_n,which are the product of at most two disjoint cycles.,we need atleast 2^{n-1} subgroups of S_n to cover all the elements of Π. For $n \geq 11$, this is the direct consequence of the fact that the set consisting of A_n and of all maximal intransitive subgroups of S_n is definitely unbeatable on Π .This is proved in two steps.

Claim 4.0.1. *[3, Claim 3.1] Let H_1 and H_2 be A_n or a maximal intransitive subgroup of S_n.If $H_1 \neq H_2$,then $\Pi \cap H_1 \cap H_2 = \phi$.*

Proof. $A_n \cap \Pi$ is the set of all n-cycles, while $S_k \times S_{n-k} \cap \Pi$ is the set of all permutations of the form $\pi = \delta.\bar{\delta}$ with δ a k-cycle from S_k and $\bar{\delta}$ a $(n-k)$-cycle from S_{n-k}. □

Claim 4.0.2. *[3, Claim 3.2] Suppose that $n \geq 11$ is odd. Let H be A_n or a maximal intransitive subgroup of S_n, and let S be any subgroup of S_n different from A_n and different from any maximal intransitive subgroup. Then $|S \cap \Pi| \leq |H \cap \Pi|$.*

Proof. It can be assumed that S is maximal in S_n. First let $n \geq 17$. If S is primitive, then $|S \cap \Pi| \leq |S| \leq e^n$ follows from [2] $e^n \leq ((n-1)/2)! \cdot ((n-3)/2)! \leq |H \cap \Pi|$. If S is imprimitive ,then $|S \cap \Pi| \leq |S| \leq (n/p)!^p p! \leq ((n-1)/2)! \cdot ((n-3)/2)! \leq |H \cap \Pi|$ holds where p is the smallest prime divisor of n. if $n = 11$ or 13, then S is primitive and $S \leq ((n-1)/2)! \cdot ((n-3)/2)!$ follows from [2] If n=15, then by [2], S is conjugate to a maximal imprimitive group with five blocks of imprimitivity, to a maximal imprimitive group with three blocks of imprimitivity, or to S_6 acting on the sets of distinct pairs of points. In the first and the third case we have $|S| \leq (3)!^5 5! \leq 6!7! \leq |H \cap \Pi|$.Let S be a maximal imprimitive subgroup of S_15 with three blocks of imprimitivity.So, we get $|S \cap \Pi| \leq 2338560 \leq 6!7!$. The remaining cases, $n = 7, 9$, are handled separately.

Case $n = 7$**.** We have $\sigma(\Pi) \geq 64$. We will show $\sigma(\Pi) \leq 64$. Let $\mathcal{L}$ be a set of $\sigma(S_7)$ maximal subgroups of S_7 covering S_7. Since there is exactly one maximal subgroup (an intransitive one) containing a given (3,4)- or (2,5)-cycle, all $\binom{7}{3} + \binom{7}{2} = 56$ maximal intransitive groups which do not stabilize any point are contained in $\mathcal{L}$. The group A_7 is also contained in $\mathcal{L}$, else the 7-cycles (which number 6!) could only be covered by at least 5! maximal primitive groups conjugate to $\mathrm{AGL}(1,7)$, giving $\sigma(\Pi) \geq 56 + 5! > 64$. We discuss that $\mathcal{L}$ contains all 7 one-point stabilizers as well. To see this, consider the (1,6)-cycles in Π. A maximal subgroup of S_7 containing such a permutation is either a stabilizer of a point or conjugate to $\mathrm{AGL}(1,7)$. Suppose $\mathcal{L}$ does not contain the stabilizer of point x. Then the 6-cycles of $S_7 \setminus \{x\}$ require at least 60 affine groups to cover, yielding $\sigma(\Pi) \geq 56 + 60 = 116 > 64$.

Case $n = 9$**.** We have $\sigma(\Pi) \geq 256$. Partition Π into three sets: Π_1 is the set of (4,5)-cycles, Π_2 the set of (3,6)-cycles, and $\Pi_3 = \Pi \setminus (\Pi_1 \cup \Pi_2)$. We show $\sigma(\Pi) \leq \sigma(\Pi_1 \cup \Pi_3) = 172$. There is no subgroup intersecting both Π_1 and Π_3, so $\sigma(\Pi_1 \cup \Pi_3) = \sigma(\Pi_1) + \sigma(\Pi_3)$. Since each (4,5)-cycle lies in a unique subgroup $S_4 \times S_5$, we have $\sigma(\Pi_1) = 126$. Now the set $\mathcal{H}$ of subgroups A_9 together with all maximal intransitive subgroups isomorphic to $S_1 \times S_8$ or $S_2 \times S_7$ is definitely unbeatable on Π_3, since these subgroups cover Π_3 in a disjoint way and for all $S \notin \mathcal{H}$, $|S \cap \Pi_3| \leq 6! \leq |H \cap \Pi_3|$ for $H \in \mathcal{H}$. Therefore, $\sigma(\Pi) \leq 126+46 = 172$. $\square$

$\square$

Theorem 4.2. *[3, Theorem 3.2] If $n > 2$ is even, then $\sigma(S_n) \sim \frac{1}{2}\binom{n}{n/2}$. More precisely, for any $\varepsilon > 0$ there exists N such that if $n > N$, then*

$$\frac{1}{2}\binom{n}{n/2} + \left(\frac{1}{2} - \varepsilon\right) \sum_{i=0}^{\lfloor n/3 \rfloor} \binom{n}{i} < \sigma(S_n) \leq \frac{1}{2}\binom{n}{n/2} + \sum_{i=0}^{\lfloor n/3 \rfloor} \binom{n}{i}.$$

Proof. The set-theoretic union of all maximal imprimitive subgroups conjugate to $S_{n/2} \wr S_2$, all maximal intransitive subgroups conjugate to some $S_i \times S_{n-i}$ with $i \leq \lfloor n/3 \rfloor$, and A_n is S_n. This gives

$$\sigma(S_n) \leq \frac{1}{2}\binom{n}{n/2} + \sum_{i=0}^{\lfloor n/3 \rfloor} \binom{n}{i}.$$

Let Π_0 be the set of all n-cycles of S_n. For each $(n-2)/4 < i < \lfloor n/3 \rfloor$ with i odd, let Π_i be the set of all $(i, i+1, n-2i-1)$-cycles. Moreover, let $\mathcal{H}_0$ be the set of all maximal imprimitive subgroups of S_n conjugate to $S_{n/2} \wr S_2$. For each $i > 0$, let $\mathcal{H}_i$ be the set of all maximal intransitive subgroups of S_n conjugate to $S_i \times S_{n-i}$. The following two claims are to show that if n is sufficiently large, then $\mathcal{H}_0$ is definitely unbeatable on Π_0, and for each $i > 0$ the set $\mathcal{H}_i$ is definitely unbeatable on Π_i.

Claim 4.0.3. *[3, Claim 3.3] With the notations above we have the following:*

(i) $\Pi_0 \subseteq \bigcup_{H \in \mathcal{H}_0} H$;

(ii) $\Pi_i \subseteq \bigcup_{H \in \mathcal{H}_i} H$ *for all* $i > 0$;

(iii) *If* $H_1, H_2 \in \mathcal{H}_0$ *and* $H_1 \neq H_2$ *then* $\Pi_0 \cap H_1 \cap H_2 = \emptyset$;

(iv) *For all* $i \neq j$, $H_1 \in \mathcal{H}_i$, $H_2 \in \mathcal{H}_j$, *and* $H_1 \neq H_2$, *then* $\Pi_i \cap H_1 \cap H_2 = \emptyset$.

Claim 4.0.4. *[3, Theorem 3.4] Let $n \geq 14$ and let S be a maximal subgroup of S_n. Then*

(i) $|S \cap \Pi_0| < |H \cap \Pi_0|$ for all $S \notin \mathcal{H}_0$, $H \in \mathcal{H}_0$;

(ii) $|S \cap \Pi_i| < |H \cap \Pi_i|$ for all i and all $S \notin \mathcal{H}_i$, $H \in \mathcal{H}_i$.

Proof. (i) If S is primitive, then $|S \cap \Pi_0| \leq |S| < e^n < \frac{(n/2)!^2 \cdot 2}{n} = |H \cap \Pi_0|$. If S is imprimitive, then $|S \cap \Pi_0| \leq |S| \leq (n/d)!^d \cdot d! < \frac{(n/2)!^2 \cdot 2}{n} = |H \cap \Pi_0|$, where d is the smallest divisor of n greater than 2. If S is intransitive, then $S \cap \Pi_0 = \emptyset$. (ii) Fix an index i. If S is primitive, then $|S \cap \Pi_i| \leq |S| < e^n < \frac{([n/3]-2)! \cdot (n-[n/3]+1)!}{n(n-2[n/3]+1)} = |H \cap \Pi_i|$. If S is imprimitive, then $|S \cap \Pi_i| \leq |S| < (n/d)!^d \cdot d! < \frac{([n/3]-2)! \cdot (n-[n/3]+1)!}{[n/3](n-2[n/3]+1)} = |H \cap \Pi_i|$, where d is the smallest divisor of n greater than 2. Let S be intransitive. If S is contained in a group conjugate to $S_i \times S_{n-i-1}$, then $\frac{|S \cap \Pi_i|}{|H \cap \Pi_i|} = \frac{(i+1)! \cdot (n-i-1)!}{i!(n-i)!} = \frac{i+1}{n-i} < 1$. If S is contained in a group conjugate to $S_{n-2i-1} \times S_{2i+1}$, then $\frac{|S \cap \Pi_i|}{|H \cap \Pi_i|} = \frac{(n-2i-1)! \cdot (2i+1)!}{i!(n-i)!} \cdot \binom{n}{2i+1}^{-1} < 1$. Finally, if S is contained neither in a group conjugate to $S_i \times S_{n-i-1}$ nor in a group conjugate to $S_{n-2i-1} \times S_{2i+1}$, then $S \cap \Pi_i = \emptyset$. $\square$

Claim 4.0.5. *[3, Claim 3.5] With the notations above, we have $\Pi = \mathcal{H}_0 \cup \bigcup_i \mathcal{H}_i$ whenever $n \geq 14$.*

Proof. Let $\mathcal{H}'$ be the set of all intransitive groups in $\mathcal{H}$ together with all maximal imprimitive subgroups of $\mathcal{H}$ conjugate to $S_{n/2} \wr S_2$. For each $S \in \mathcal{H}'$, there exists a unique j such that $S \subseteq H_j$. Moreover, for all i and all $S \in \mathcal{H}'$, $H_i \in \mathcal{H}_i$, we have $|S \cap \Pi_i| \leq |H_i \cap \Pi_i|$. This means that the union of all subgroups in $\mathcal{H}'$ does not contain at least $\min\left\{\frac{(n/2)!^2}{n}, \quad \frac{(\lfloor n/3 \rfloor-2)! \cdot (n-\lfloor n/3 \rfloor+1)!}{\lfloor n/3 \rfloor \cdot (n-2\lfloor n/3 \rfloor+1)}\right\}$ elements of Π. If this expression is 0, then by Claims 3.3 and 3.4 we are finished. Otherwise, these elements cannot be in $\mathcal{H}_0 \cup \bigcup \mathcal{H}_i \setminus \mathcal{H}'$, i.e., in $\mathcal{H}$'s transitive groups neither of which is conjugate to $S_{n/2} \wr S_2$. But this is impossible since $\max\{e^n, (n/d)^d \cdot d!\} < \min\left\{\frac{(n/2)!^2}{n}, \quad \frac{(\lfloor n/3 \rfloor-2)! \cdot (n-\lfloor n/3 \rfloor+1)!}{\lfloor n/3 \rfloor \cdot (n-2\lfloor n/3 \rfloor+1)}\right\}$, where d is the smallest divisor of n with $d > 2$. $\square$

Claim 4.0.6. *[3, Claim 3.6] If $n > 14$, then*

$$\frac{1}{2}\binom{n}{n/2} + \sum_{\substack{(n-2)/4<i<n/3 \\ i \text{ odd}}} \left(\binom{n}{i} - \binom{n}{2}\right) = \sigma(\Pi) < \sigma(S_n) \leq \frac{1}{2}\binom{n}{n/2} + \sum_{i=0}^{\lfloor n/3 \rfloor} \binom{n}{i}.$$

Proof. The first equality is a consequence of Claim 3.5. $\sigma(\Pi) < \sigma(S_n)$ follows from the fact that $\sigma(\Pi) \neq \sigma(S_n)$, since the union of all subgroups of $\mathcal{H}_0 \cup \bigcup \mathcal{H}_i$ does not contain all even permutations. The upper bound was already established. Finally, we need to show that for any fixed $0 < \varepsilon < 1/2$, there exists an integer N, so that $\left(\frac{1}{2} - \varepsilon\right) \sum_{i=0}^{\lfloor n/3 \rfloor} \binom{n}{i} < \sum_{\substack{(n-2)/4<i<n/3 \\ i \text{ odd}}} \binom{n}{i}$ holds whenever $n > N$. Indeed, for a fixed real number $0 < \varepsilon < 1/2$, a suitable N is an integer with the property that whenever $n > N$, then both $\sum_{(n-2)/4<i<n/3} \binom{n}{i} < (2+2\varepsilon) \sum_{\substack{(n-2)/4<i<n/3 \\ i \text{ odd}}} \binom{n}{i}$ and

$\sum_{0 \leq i \leq (n-2)/4} \binom{n}{i} \leq 2\epsilon \sum_{\substack{(n-2)/4<i<\lfloor n/3 \rfloor \\ i \text{ odd}}} \binom{n}{i}$ hold. $\square$

By Theorems 3.1 and 3.2, to complete the proof of part (1) of Theorem 1.1, we only need to show $\sigma(S_n) < 2^{n-2}$ for $4 \leq n \leq 12$ and n even, since if $n \geq 14$ we have $\frac{1}{2}\binom{n}{n/2} + \sum_{i=0}^{\lfloor n/3 \rfloor} \binom{n}{i} < 2^{n-2}$. If $n = 4$, then $\sigma(S_4) \leq 4$, since S_4 is the union of A_4 and the three Sylow 2-subgroups of S_4. For $n = 6$, we have $\sigma(S_6) < 16$, since S_6 is the union of all imprimitive subgroups conjugate to $S_3 \wr S_2$ and all intransitive subgroups conjugate to $S_1 \times S_5$. If $n = 8$, then S_8 is the union of all imprimitive subgroups conjugate to $S_4 \wr S_2$, all intransitive subgroups conjugate to $S_2 \times S_6$ and A_8, hence $\sigma(S_8) < 64$. For $n = 10$ we have $\sigma(S_{10}) < 256$, since S_{10} is the union of all imprimitive subgroups conjugate to $S_5 \wr S_2$, all intransitive subgroups conjugate to $S_1 \times S_9$ and all intransitive subgroups conjugate to $S_3 \times S_7$. Finally, $\sigma(S_{12}) < 761$, since S_{12} may be written as the union of all imprimitive subgroups conjugate to $S_6 \wr S_2$, all intransitive subgroups conjugate to $S_1 \times S_{11}, S_2 \times S_{10},$ or $S_3 \times S_9$, and A_{12}.

$\square$

CHAPTER 5

Conclusion and future work

5.1 Conclusion

In this thesis, we have explored the concept of group coverings and the minimal number of proper subgroups required to cover a finite group, denoted by $\sigma(G)$. We began with fundamental definitions and results that clarify when such coverings are possible, establishing that only non-cyclic groups admit coverings and that no group has a covering number equal to two.

A significant result is the characterization of groups with covering number three, where it is proved that such a group G must satisfy $G/K \cong C_2 \times C_2$ for some normal subgroup K. This result provides both a necessary and sufficient condition and is supported through careful group-theoretic and set-theoretic reasoning.

We have been computed covering numbers of symmetric groups S_n for odd n has been studied that $\sigma(S_n) = 2^{n-1}$ unless $n = 9$, and an outline of the proof using the notion of intransitive subgroups and permutation types has been given.

The study of covering numbers not only reveals intricate properties of group structure and subgroup interactions but also provides a foundation for further exploration in finite group theory. Future directions may include extending this analysis to other group families or exploring connections with probabilistic group theory and group generation problems.

5.2 Future Work

While this thesis has provided a strong foundation for understanding covering numbers of finite groups, several exciting directions for future research have emerged during this work. These directions reflect both natural extensions of the current results and areas that have personally motivated me for further exploration.

1. **Generalizing covering numbers for S_n with even n:** Most of the results discussed in this thesis focus on the symmetric group S_n for odd values of n. However, the behavior of covering numbers for even values of n remains less understood and potentially more complex due to different subgroup structures. I am particularly motivated to explore the cases of S_6, S_8, and beyond, with the goal of identifying patterns or even formulating a general result for $\sigma(S_n)$ when n is even.

2. **Finding and classifying groups with specific covering numbers (e.g., 8, 9, etc.):** Another intriguing area is to identify all groups that have a specific covering number, especially for values such as 8, 9, or higher. Understanding which groups achieve these values may reveal structural similarities or subgroup configurations that explain why these particular covering numbers arise. This line of research could eventually lead to a classification of finite groups by their covering numbers.

3. **Characterizing impossible covering numbers:** During my study, I was particularly fascinated by the fact that some covering numbers are not possible at all—like 2 and 7. This observation naturally leads to the question: which natural numbers cannot be realized as the covering number of any finite group? Investigating this further could lead to the discovery of new constraints or even a complete description of the set of "forbidden" covering numbers.

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