Time-Frequency Analysis

M.Sc. Thesis

by

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Under the guidance of

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DEPARTMENT OF MATHEMATICS

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A THESIS

Submitted in partial fulfillment of the requirements for the award of the degree

of

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"Time-Frequency Analysis"** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF MATHEMATICS**, **INDIAN INSTITUTE OF TECHNOLOGY INDORE**, is an authentic record of my own work carried out during the time period from July 2024 to May 2025 under the supervision of **Dr. Vijay Kumar Sohani**, Assistant Professor, Department of Mathematics, IIT Indore. The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

Bhanu Pratap Sharma
29-05-2025

Signature of the student with date

(**Bhanu Pratap Sharma**)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

V.K. Sohani

Signature of Thesis Supervisor with date

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Bhanu Pratap Sharma has successfully given his M.Sc. Oral Examination on 29th May, 2025.

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Date: 29/05/2025

V.K. Sohani

Signature of Convener, DPGC

Date: 29/05/2025

Dedicated to my

Family

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Abstract

This thesis presents a detailed study of time-frequency analysis, a modern area within harmonic analysis concerned with the simultaneous treatment of time and frequency information. The work begins with foundational concepts from measure theory and functional analysis, including L^p spaces and bounded linear operators. We then explore essential tools from Fourier analysis, setting the stage for a deeper investigation of time-frequency methods.

The uncertainty principle is introduced to highlight the inherent limitations of simultaneous time and frequency localization. Building on this, the short-time Fourier transform is developed as a central tool, alongside other time-frequency representations such as the Wigner distribution, spectrogram, and the ambiguity function. We then delve into Gabor analysis and frame theory, emphasizing the construction and existence of Gabor frames in $L^2(\mathbb{R}^d)$, supported by results such as Walnut's representation.

Throughout, the thesis focuses on the mathematical structures underlying time-frequency analysis rather than its applications, aiming to provide a rigorous and self-contained exposition of the subject.

Contents

CHAPTER 1

Introduction

Time-frequency analysis presents a beautiful intersection of ideas in modern analysis, combining notions of translation, modulation, and localization in a way that feels both natural and powerful. As I explored this area more deeply, I was drawn to the way it extends the classical tools of Fourier analysis to settings where both time and frequency need to be considered together. The desire to understand how functions behave not just globally, but locally in both domains, is what motivates much of the theory. This perspective leads to rich mathematical structures and a range of elegant results that form the foundation of this thesis.

The topics covered range from the elementary theory of the short-time Fourier transform and classical results about the Wigner distribution to the modern theory of Gabor frames and quantitative methods in time-frequency analysis. While many of these ideas have important applications in fields such as quantum mechanics and signal analysis, the orientation here is firmly rooted in the mathematical structures themselves.

Chapters 2–8 present the core material for time-frequency analysis in the Hilbert space setting of $L^2(\mathbb{R}^d)$. We begin with foundational tools from measure theory and functional analysis, followed by key results from Fourier analysis in Chapter 3. Chapter 4 introduces the uncertainty principle, which provides a crucial motivation for the development of time-frequency methods. Chapter 5 discusses the short-time Fourier transform and its connection to complex analysis. In Chapter 6, we study quadratic time-frequency representations, particularly the Wigner distribution. Chapters 7 and 8 are devoted to frame theory and the existence of Gabor frames, with an emphasis on Walnut's representation and its implications.

A recurring theme throughout this thesis is the pursuit of joint time-frequency representations, and the prominent role played by the uncertainty principle and Gaussian functions in shaping this field.

CHAPTER 2

Preliminaries

Before going to start our discussion about time-frequency analysis we first see some basic concepts. In this chapter we gather information related to L^p spaces and will see some approximation theorems of measure theory. In the later half we discuss about the bounded operators in brief.

2.1 L^p Space

Definition 2.1. *[5] Let $(X, \mathcal{B}, \mu)$ be a measure space and $f : X \to \mathbb{C}$ be a function, then $f \in L^p(X), 1 \leq p < \infty$, if f is a μ-measurable function and $\int_X |f(x)|^p d\mu(x) < \infty$.*

- $L^p(X)$ is a Banach space over the field $\mathbb{C}$ with the norm $\| \cdot \|_p$ define as
$$\|f\|_p := \left(\int_X |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}.$$
- For $p = 2, L^2(X)$ is a Hilbert space with the inner product given by
$$\langle f, g \rangle = \int_X f(x)\overline{g(x)} d\mu(x).$$

Definition 2.2. *[5] Let $(X, \mathcal{B}, \mu)$ be a measure space and $f : X \to \mathbb{C}$ be a function, then $f \in L^\infty$ if f is a μ-measurable function and is essentially bounded i.e. $\exists\, M > 0$ s.t. $|f(x)| \leq$*

M a.e.

- L^∞ is a Banach space over the field $\mathbb{C}$ with the norm $\|\cdot\|_\infty$ defined as

$$\|f\|_\infty := \underset{x\in X}{\text{ess sup}}|f(x)| = \inf\{M \geq 0 : |f(x)| \leq M \text{ a.e.}\}.$$

Theorem 2.1. *(**Lusin**[2]) Let $E \subset \mathbb{R}^d$ be a measurable set of finite measure and $f : E \to \mathbb{R}$ be a measurable function. Let $\varepsilon > 0$ be given. Then, there exists $\varphi \in C_c(\mathbb{R}^d)$ such that*

$$\mu(\{x \in E \mid \varphi(x) \neq f(x)\}) < \varepsilon.$$

Further, if f is bounded then we can ensure that

$$\|\varphi\|_\infty \leq \|f\|_\infty.$$

Here $C_c(\mathbb{R}^d)$ is the collection of all continuous functions defined on $\mathbb{R}^d$ with compact support.

Definition 2.3. *(σ**-finite measure space**[2]) Let $(X, \mathcal{A}, \mu)$ be a measure space The measure μ is called a σ-finite measure, if there exists $A_1, A_2, \cdots \in \mathcal{A}$ with $\mu(A_n) < \infty$ $\forall n \in \mathbb{N}$ s.t. $\bigcup A_n = X$. And $(X, \mathcal{A}, \mu)$ is called as σ-finite measure space.*

Theorem 2.2. *(**Fubini**[2]) If X and Y are σ-finite measure spaces, and if f is a measurable function on $X \times Y$, then*

$$\int_X \left(\int_Y |f(x,y)|\, dy \right) dx = \int_Y \left(\int_X |f(x,y)|\, dx \right) dy = \int_{X\times Y} |f(x,y)|\, d(x,y).$$

Furthermore, if any one of these integrals is finite, then

$$\int_X \left(\int_Y f(x,y)\, dy \right) dx = \int_Y \left(\int_X f(x,y)\, dx \right) dy = \int_{X\times Y} f(x,y)\, d(x,y).$$

Definition 2.4. *(**Simple function**[2]) Let $(X, \mathcal{A})$ be a measurable space then a function $f : X \to \mathbb{C}$ is known as simple fucntion if it is of the form*

$$f(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x), \tag{2.1}$$

where $A_1, A_2, ...An \in \mathcal{A}$ and $a_1, a_2, ...a_n \in \mathbb{C}$.

Theorem 2.3. *[2] Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to [0, +\infty]$ be a measurable function, then there exists an increasing sequence of real-valued simple functions $\{s_n\}_{n=1}^\infty$ on X s.t. $s_n \to f$ (pointwise).*

Theorem 2.4. *[5] $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$.*

Proof. Let $f \in L$ s.t. $f \geq 0$. Then by theorem 2.3 there exists a sequence of simple functions $\{s_n\}_{n=1}^\infty$ s.t. $0 \leq s_1 \leq s_2, \leq ... \leq f$ and $s_n \to f$ as $n \to \infty$ and $s_n \in L^p(\mathbb{R}^d)$ as $s_n^p \leq f^p$. Now since $\|f - s_N\|_p \to 0$ as $n \to \infty$, then for any given $\epsilon > 0 \; \exists \quad N \in \mathbb{N}$ s.t. $\|f - s_N\|_p < \epsilon$.

Now since $s_N \in L^p(\mathbb{R}^d)$ then $\mu(\{x : s_N(x) \neq 0\}) < \infty$. By Lusin's theorem $\exists g \in C_c(\mathbb{R}^d)$ s.t.

$$\mu(\{x : s_N(x) \neq g(x)\}) < \epsilon$$

$$\|g\|_\infty \leq \|s_N\|_\infty.$$

Now

$$\begin{aligned}
\|s_N - p\|_p^p &= \int_{\mathbb{R}^d} |s_N(x) - g(x)|^p \mathrm{d}x. \\
&= \int_{\{x: s_N(x) \neq g(x)\}} |s_N(x) - g(x)|^p \mathrm{d}x. \\
&\leq \int_{\{x: s_N(x) \neq g(x)\}} (\|s_N\|_\infty^p + \|g\|_\infty^p) \mathrm{d}x. \\
&\leq 2^p \|s_N\|_\infty \cdot \epsilon.
\end{aligned}$$

Now $\|f - g\|_p \leq \|f - s_N\|_p + \|s_N - g\|_p \leq \epsilon(1 + 2^p \|s_N\|_\infty)$.

Now that is, for any given $\epsilon > 0 \; \exists g \in C_c(\mathbb{R}^d)$ s.t. $\|f - g\|_p < \epsilon$.So $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$. □

Proposition 2.5. *(**Holder's inequality**[5]) Let f, g be two measurable functions on X. Let p and p' are conjugate exponent i.e. $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 \leq p \leq \infty$. Then, we have*

$$\int_X |f(x)g(x)| \mathrm{d}\mu(x) \leq \left(\int_X |f(x)|^p \mathrm{d}\mu(x) \right)^{\frac{1}{p}} \left(\int_X |g(x)|^{p'} \mathrm{d}\mu(x) \right)^{\frac{1}{p'}}. \tag{2.2}$$

Proposition 2.6. *(**Minkowski's Integral Inequaltiy**[5]) Let $(X, \mu, \mathcal{A}_1)$ and $(Y, v, \mathcal{A}_2)$ be measure spaces and $F : X \times Y \to \mathbb{C}$ be a measurable funcion. Then, we have for $1 \leq p < \infty$,*

$$\left[\int_X \left| \int_Y F(x, y) \mathrm{d}y \right|^p \mathrm{d}x \right]^{\frac{1}{p}} \leq \int_Y \left[\int_X |F(x, y) \mathrm{d}x| \right]^{\frac{1}{p}}. \tag{2.3}$$

2.2 Operators on Banach space

Definition 2.5. *(**Bounded Operator**[3]) Let $(X, \|\cdot\|_1), (Y, \|\cdot\|_2)$ be two normed linear spaces over the field $\mathbb{F}$. Then the linear operator $A : X \to Y$ is said to be a bounded operator if $\exists\, c > 0$ s.t.*

$$\|Ax\|_2 \leq c\|x\|_1, \forall x \in X. \tag{2.4}$$

Note : A linear operator T is bounded iff it is continuous.

Remark 2.1. *$L(X, Y) = \{T : X \to Y | T$ is linear $\}$ forms a vector space over field the $\mathbb{F}$ and $BL(X, Y) = \{T : X \to Y | T$ is bounded linear operator $\}$ then $BL(X, Y)$ is a subspace for $L(X, Y)$.*

Note : When $Y = \mathbb{F}$, then $L(X)$ and $BL(X)$ are known as algebraic dual and dual of X respectively and we denote by X^* and X' respectively.

Remark 2.2. *$(BL(X,Y), \|\cdot\|_{op})$ is a norm linear space where the norm $\|\cdot\|_{op}$ is given by*

$$\|T\|_{op} = \sup_{x \in X_{x \neq 0}} \frac{\|Tx\|_2}{\|x\|_1}. \tag{2.5}$$

Definition 2.6. *(**Adjoint of a bounded operator**[3]) Let H_1 and H_2 be two Hilbert spaces, and $T : H_1 \to H_2$ be a bounded linear operator. Then a linear map $T^* : H_2 \to H_1$ is the adjoint of T if $\forall x \in H_1, \forall y \in H_2$*

$$\langle T(x), y\rangle_{H_2} = \langle x, T^*(y)\rangle_{H_1}. \tag{2.6}$$

- On a Hilbert space, the adjoint exists for any bounded linear operator and it is unique and bounded. Moreover,

$$\|T\|_{op} = \|T^*\|_{op}.$$

- A bounded linear operator T on a Hilbert-space H is self adjoint if

$$T = T^*.$$

Theorem 2.7. *[3] If H is a Hilbert-space and $T : H \to H$ is a linear operator that satisfies*

$$\langle T(x), y\rangle = \langle x, T(y)\rangle, \forall x, y \in H$$

then T is bounded.

Definition 2.7. *(**Adjoint of an unbounded operator**[3]) Let H be a Hilbert space and $\mathcal{D} \subseteq H$ be a dense subspace. Consider a linear operator $T : \mathcal{D}(T) \to H$. Then the Hilbert adjoint operator of T is $T^* : \mathcal{D}(T^*) \to H$, defined by*

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle, \forall x \in \mathcal{D}(T), \forall y \in \mathcal{D}(T^*), \text{ where}$$

$$\mathcal{D}(T^*) = \{y \in H : \exists y^* \in H \; s.t. \forall x \in \mathcal{D}(T), \langle T(x), y\rangle = \langle x, y^*\rangle\}.$$

Definition 2.8. *(**Multiplication operator**) Let $X : \mathcal{D}(X) \to L^2(\mathbb{R})$ be a linear operator defined as:*

$$(Xf)(x) := xf(x), \quad \forall x \in \mathbb{R},$$

where $\mathcal{D}(X) := \{f \in L^2(\mathbb{R}) : Xf \in L^2(\mathbb{R})\}$.

Note : The operator X is self-adjoint.

Proof.

$$\langle Xf, g\rangle = \int_{\mathbb{R}} xf(x)\overline{g(x)}\,dx = \int_{\mathbb{R}} f(x)\overline{xg(x)}\,dx = \langle f, Xg\rangle.$$

□

Remark 2.3. *X is an unbounded linear operator on $L^2(\mathbb{R})$.*

Proof. Consider a sequence of functions (f_n) given by:

$$f_n(t) = \begin{cases} 1 & \text{if } n \leq t < n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$Xf_n(t) = \begin{cases} t & \text{if } n \leq t < n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\|f_n\|_2 = 1$, and

$$\begin{aligned} \|Xf_n\|_2^2 &= \int_n^{n+1} t^2 \mathrm{d}t. \\ &= \left[\frac{t^3}{3}\right]_n^{n+1}. \\ &= \frac{1}{3}\left[(n+1)^3 - n^3\right] \\ &= \frac{n^2+n+1}{3} > n^2. \\ \implies \frac{\|Tf_n\|_2^2}{\|f_n\|_2^2} &> n^2. \end{aligned}$$

So X is an unbounded opeartor on $L^2(\mathbb{R})$. □

Definition 2.9. *(**Schwartz Space**[4]) The Schwartz space is defined as $S(\mathbb{R}^d) := \big\{ f \in C^\infty(\mathbb{R}^d) : \forall \alpha, \beta$ where $\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^d} |x^\alpha (D^\beta f)(x)|$, with*

$$D^\beta = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \frac{\partial^{\beta_2}}{\partial x_2^{\beta_2}} \cdots \frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}} \quad \textit{and} \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_d^{\alpha_d}.$$

Definition 2.10. *(**Differentiation Operator**) Let $P : \mathcal{D}(P) \to L^2(\mathbb{R})$ be a linear operator defined as:*

$$(Pf)(x) := \frac{1}{2\pi i} f'(x), \quad x \in \mathbb{R},$$

where $\mathcal{D}(P) := \{f \in L^2(\mathbb{R}) : Pf \in L^2(\mathbb{R})\}$.

Note : P is an unbounded linear operator.

Proof. Restrict P on $Y = \mathcal{D}(\mathbb{P}) \cap L^2(\mathbb{R}) \subseteq L^2(\mathbb{R})$ Consider the sequence of functions

$$f_n(t) = \begin{cases} 1 - nt & \text{if } 0 \le t \le \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

The Image sequence

$$P(f_n)(t) = \begin{cases} \frac{-n}{2\pi i} & \text{if } 0 \le t < \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \|f_n\|_2^2 &= \int_0^{\frac{1}{n}} (1 - nt)^2 \, \mathrm{d}t \\ &= \left[t + \frac{n^2 t^3}{3} - nt^2 \right]_0^{\frac{1}{n}} = \frac{1}{3n}. \end{aligned}$$

$$\text{Now } \|Pf_n\|_2^2 = \frac{n^2}{4\pi} \int_0^{\frac{1}{n}} 1 \cdot \mathrm{d}t = \frac{n}{4\pi}.$$

$$\implies \frac{\|Pf_n\|_2^2}{\|f_n\|_2^2} = \frac{3n^2}{4\pi} > \frac{n^2}{5}.$$

□

Note : P is self-adjoint.

Proof.

$$\begin{aligned} \langle Pf, g \rangle &= \int_{-\infty}^{\infty} \frac{1}{2\pi i} f'(x) \overline{g(x)} \mathrm{d}x. \\ &= \frac{1}{2\pi i} \left[f(x)\overline{g(x)} \Big|_{\infty}^{-\infty} - \int_{-\infty}^{\infty} f(x) \overline{g'(x)} \mathrm{d}x \right]. \\ &= \int_{-\infty}^{\infty} f(x) \overline{\left(\frac{g'(x)}{2\pi i} \right)} dx = \langle f, Pg \rangle. \end{aligned}$$

□

Definition 2.11. *(**Hilbert-Schmidt Operator** [6]) A bounded operator $A : \mathcal{H} \to \mathcal{H}$ where $\mathcal{H}$ is a seperable Hilbert space, is called a Hilbert–Schmidt operator if*

$$\sum_{n=1}^{\infty} \|Ae_n\|_{\mathcal{H}}^2 < \infty$$

for some orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of $\mathcal{H}$. The Hilbert–Schmidt norm of A is given by

$$\|A\|_{H.S.} = \left(\sum_{n=1}^{\infty} \|Ae_n\|_{\mathcal{H}}^2 \right)^{1/2},$$

and this quantity is independent of the choice of the orthonormal basis $\{e_n\}$.

Theorem 2.8. *[6] If A is an integral operator with kernel k, that is, $Af(x) = \int_{\mathbb{R}^d} k(x,y)f(y)\,dy$, then A is Hilbert–Schmidt operator if and only if $k \in L^2(\mathbb{R}^{2d})$, and in this case $\|A\|_{H.S.} = \|k\|_2$.*

CHAPTER 3

Some Basics of Fourier Analysis

3.1 Definition of the Fourier Transform

Definition 3.1. *[8] Let $f \in L^1(\mathbb{R}^d)$ then Fourier transform $\mathcal{F}$ maps f to a function $\mathcal{F}f \in L^\infty(\mathbb{R}^d)$ which is given by*

$$(\mathcal{F}f)(\omega) = \hat{f}(\omega) := \int_{\mathbb{R}^d} f(t)e^{-2\pi i\langle\omega,t\rangle}\mathrm{d}t. \tag{3.1}$$

Clearly, $\hat{f} \in L^\infty(\mathbb{R}^d)$ when $f \in L^1(\mathbb{R}^d)$ as

$$|\hat{f}(\omega)| \leq \int_{\mathbb{R}^d} |f(t)e^{-2\pi i\langle\omega,t\rangle}|\mathrm{d}t \leq \int_{\mathbb{R}^d} |f(t)|\mathrm{d}t = \|f\|_1.$$

Note : $\mathcal{F} : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ is a linear transformation.

Proof. Let $f, g \in L^1(\mathbb{R}^d)$ and $\alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned}\mathcal{F}(\alpha f + \beta g)(\omega) &= \int_{\mathbb{R}^d} (\alpha f + \beta g)(t)e^{-2\pi i\langle\omega,t\rangle}\mathrm{d}t \\ &= \int_{\mathbb{R}^d} (\alpha f(t) + \beta g(t))e^{-2\pi i\langle\omega,t\rangle}\mathrm{d}t \\ &= \alpha \int_{\mathbb{R}^d} f(t)e^{-2\pi i\langle\omega,t\rangle}\mathrm{d}t + \beta \int_{\mathbb{R}^d} f(t)e^{-2\pi i\langle\omega,t\rangle}\mathrm{d}t.\end{aligned}$$

(By the linearity of Lebesgue integral.)

$$= \alpha\mathcal{F}f(\omega) + \beta\mathcal{F}g(\omega).$$

Now, $|\mathcal{F}f(\omega)| = |\hat{f}(\omega)| \leq \int_{\mathbb{R}^d} |f(t)|\mathrm{d}t = \|f\|_1$. □

Note : $\mathcal{F} : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ is not onto, the following lemma will give us the explanation of it.

Lemma 3.1. *(**Reimann-Lebesgue**) If $f \in L^2(\mathbb{R}^d)$, then $\hat{f} \in C_0(\mathbb{R}^d)$. Where $C_0(\mathbb{R}^d) := \{f|f : \mathbb{R}^d \to \mathbb{C}, f$ is uniformly continuous and $\lim_{|x|\to\infty} |f(x)| = 0\}$*

Proof. Let $f \in L^1(\mathbb{R}^d)$ then $\hat{f} \in L^\infty(\mathbb{R}^d)$.

$$\begin{aligned}
|\hat{f}(\omega+h) - \hat{f}(\omega)| &= \left| \int_{\mathbb{R}^d} f(t) e^{-2\pi i \langle \omega+h, t\rangle} \mathrm{d}t - \int_{\mathbb{R}^d} f(t) e^{-2\pi i \langle \omega, t\rangle} \mathrm{d}t \right| \\
&= \left| \int_{\mathbb{R}^d} f(t) e^{-2\pi i \langle \omega, t\rangle} (e^{-2\pi i \langle h, t\rangle} - 1) \mathrm{d}t \right| \\
&\leq \int_{\mathbb{R}^d} |f(t)||e^{-2\pi i \langle h, t\rangle} - 1| \mathrm{d}t.
\end{aligned}$$

Take the limit $h \to 0$ both sides.

$$\lim_{h\to 0} |\hat{f}(\omega+h) - \hat{f}(\omega)| \leq \lim_{h\to 0} \int_{\mathbb{R}^d} |f(t)||e^{-2\pi i \langle h, t\rangle} - 1| \mathrm{d}t.$$

Since $f \in L^1(\mathbb{R}^d)$ and $|e^{-2\pi i \langle h, t\rangle} - 1| \leq 2,$ so the right side integral is finite.

Then by the dominated convergence theorem

$$\begin{aligned}
\lim_{h\to 0} |\hat{f}(\omega+h) - \hat{f}(\omega)| &\leq \int_{\mathbb{R}^d} \lim_{h\to 0} |f(t)||e^{-2\pi i \langle h, t\rangle} - 1| \mathrm{d}t. \\
&\leq 0.
\end{aligned}$$

Since $|\hat{f}(\omega+h) - \hat{f}(\omega)| \geq 0, \lim_{h\to 0} |\hat{f}(\omega+h) - \hat{f}(\omega)| = 0.$

So $\hat{f}$ is uniformly continuous on $\mathbb{R}^d$.

Now, let $f \in L^1(\mathbb{R}^n)$ be continuous with compact support. Then

$$\begin{aligned}
\hat{f}(x) &= \int_{\mathbb{R}} f(t) e^{-2\pi i \langle x, t\rangle} \mathrm{d}t \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(t_1, \ldots, t_n) e^{-2\pi i (x_1 t_1 + \cdots + x_n t_n)} \mathrm{d}t_1 \cdots \mathrm{d}t_n.
\end{aligned}$$

$$\begin{aligned}
\text{Now consider } I &= \int_{-\infty}^{\infty} \left(f(t_1, \ldots, t_n)\right) e^{-2\pi i (x_1 t_1 + \cdots + x_n t_n)} \mathrm{d}t_1 \\
&= \int_{-\infty}^{\infty} \left(f\left(t_1 + \frac{1}{2x_1}, t_2, \ldots, t_n\right)\right) e^{-2\pi i \left(x_1\left(t_1 + \frac{1}{2x_1}\right) + x_2 t_2 + \cdots + x_n t_n\right)} \mathrm{d}t_1. \\
&= \int_{-\infty}^{\infty} f\left(t_1 + \frac{1}{2x_1}, t_2, \ldots, t_n\right) e^{-2\pi i \left(t_1 + \frac{1}{2x_1}\right) x_1} e^{-2\pi i (x_2 t_2 + \cdots + x_n t_n)} \mathrm{d}t_1 \\
&= \int_{-\infty}^{\infty} f\left(t_1 + \frac{1}{2x_1}, t_2, \ldots, t_n\right) e^{-2\pi i t_1 x_1} e^{-\pi i} e^{-2\pi i (x_2 t_2 + \cdots + x_n t_n)} \mathrm{d}t_1 \\
&= -\int_{-\infty}^{\infty} f\left(t_1 + \frac{1}{2x_1}, t_2, \ldots, t_n\right) e^{-2\pi i (x_1 t_1 + x_2 t_2 + \cdots + x_n t_n)} \mathrm{d}t_1.
\end{aligned}$$

Now we got two expressions for I. Taking averages of both the expressions, we get
$$I = \frac{I+I}{2} = \frac{1}{2}\int_{-\infty}^{\infty} \left(f(t_1,\ldots,t_n) - f\left(t_1 + \frac{1}{2x_1}, t_2, \ldots, t_n\right)\right) e^{-2\pi i(x_1t_1+\cdots+x_nt_n)}\, dt_1.$$
Since f is continuous with compact support, by the Dominated Convergence Theorem
$$\lim_{|x_1|\to\infty} I = \lim_{|x_1|\to\infty} \int_{-\infty}^{\infty} \left(f(t_1,\ldots,t_n) - f\left(t_1 + \frac{1}{2x_1}, t_2, \ldots, t_n\right)\right) e^{-2\pi i(x_1t_1+\cdots+x_nt_n)}\, dt_1 = 0.$$
Now if $|x| \to \infty$ implies at least one $|x_i| \to \infty$ and since f has compact support we get
$$\lim_{|x|\to\infty} \hat{f}(x) = 0.$$

Now we use the fact that the set of continuous functions with compact support is dense in $L^1(\mathbb{R}^d)$. Let $f \in L^1(\mathbb{R}^d)$ and $\epsilon > 0$. Then there exists a continuous function with compact support, g, such that
$$\|f - g\|_1 \leq \epsilon.$$

Now,
$$|\hat{f}(x)| \leq |\hat{f}(x) - \hat{g}(x)| + |\hat{g}(x)| \leq \|f-g\|_1 + |\hat{g}(x)|.$$
Taking limit on both sides we get
$$\limsup_{|x|\to\infty} |\hat{f}(x)| \leq \epsilon.$$

□

Note : Till now we have seen the Fourier transform on the $L^1(\mathbb{R}^d)$ space, now we will try to write the Fourier transform on $L^2(\mathbb{R}^d)$ space. But the integral definition of Fourier transform won't work for a function $f \in L^2(\mathbb{R}^d)$. In the later part of this chapter we will see the definition of Fourier transform for $L^2(\mathbb{R}^d)$ functions.

3.2 Some Fundamental Operators

Definition 3.2. *(**Translation Operator**[1]) Let $f : \mathbb{R}^d \to \mathbb{C}$ be a map. Then for $x \in \mathbb{R}^d$, the translation of f by x is the map $T_x f : \mathbb{R}^d \to \mathbb{C}$ defined as*
$$T_x f(t) := f(t-x) \quad t \in \mathbb{R}^d.$$

Definition 3.3. *(**Modulation Operator**[1]) Let $f : \mathbb{R}^d \to \mathbb{C}$ be a map. Then for $\omega \in \mathbb{R}^d$, the modulation of f by ω is the map $M_\omega f : \mathbb{R}^d \to \mathbb{C}$ defined as*
$$M_\omega f(t) := e^{2\pi i\langle\omega,t\rangle} f(t), t \in \mathbb{R}^d.$$

Note : T_x is also known as time shift and M_ω is also known as frequency shift. Operators $T_x M_\omega, M_\omega T_x$ are known as time-frequency shifts.

Lemma 3.2.

$$T_x M_\omega = e^{-2\pi i \langle \omega, t \rangle} M_\omega T_x. \tag{3.2}$$

Proof. Let f be a function defined on $\mathbb{R}^d$ then,

$$\begin{aligned} T_x M_\omega f(t) &= (M_\omega f)(t-x) \\ &= e^{2\pi i \langle t-x, \omega \rangle} f(t-x). \\ &= e^{-2\pi i \langle x, \omega \rangle} e^{2\pi i \langle t, \omega \rangle} (T_x f)(t). \\ &= e^{-2\pi i \langle x, \omega \rangle} M_\omega (T_x f)(t). \\ &= e^{-2\pi i \langle x, \omega \rangle} (M_\omega T_x f)(t). \end{aligned}$$

□

Note : Clearly T_x and M_ω doesn't commute in general. T_x, M_ω commutes with each other *iff* $\langle x, \omega \rangle \in \mathbb{Z}$.

- Time-frequency shifts are isometries on $L^p(\mathbb{R}^d)$ for $1 \le p \le \infty$.

Proof. Let $f \in L^p(\mathbb{R}^d)$ when $1 \le p < \infty$ then

$$\begin{aligned} \int_{\mathbb{R}^d} |T_x M_\omega f(t)|^p \mathrm{d}t &= \int_{\mathbb{R}^d} |M_\omega f(t-x)|^p \mathrm{d}t. \\ &= \int_{\mathbb{R}^d} |e^{2\pi i \langle \omega, t-x \rangle} f(t-x)|^p \mathrm{d}t. \\ &= \int_{\mathbb{R}^d} |f(t-x)|^p \mathrm{d}t. \end{aligned}$$

Substitute $t - x \to u$ then $\mathrm{d}t = \mathrm{d}u$.

$$\begin{aligned} &= \int_{\mathbb{R}^d} |f(u)|^p \mathrm{d}u. \\ &= \|f\|_p^p < \infty. \end{aligned}$$

$$\implies T_x M_\omega f \in L^p(\mathbb{R}^d) \text{ and } \|T_x M_\omega f\|_p = \|f\|_p.$$

Now let $f \in L^\infty(\mathbb{R}^d)$, then

$$|T_x M_\omega f(t)| = |e^{-2\pi i \langle t-x, \omega \rangle} f(t-x)| = |f(t-x)| \le \|f\|_\infty \quad \forall x \in \mathbb{R}^d.$$

□

- Let $f \in L^1(\mathbb{R}^d)$. Then

$$(T_x f)\hat{} = M_{-x}\hat{f} \tag{3.3}$$

Proof. Since $f \in L^1(\mathbb{R}^d)$ then $T_x f \in L^1(\mathbb{R}^d)$ and

$$\begin{aligned}(T_x f)\hat{}(\omega) &= \int_{\mathbb{R}^d} (T_x f)(t) e^{-2\pi i \langle t,\omega\rangle} \mathrm{d}t. \\ &= \int_{\mathbb{R}^d} f(t-x) e^{-2\pi i \langle t,\omega\rangle} \mathrm{d}t.\end{aligned}$$

Substitute $t - x \to u$ then $\mathrm{d}t = \mathrm{d}u$.

$$\begin{aligned} &= \int_{\mathbb{R}^d} f(u) e^{-2\pi i \langle x+u,\omega\rangle} \mathrm{d}u. \\ &= e^{-2\pi i \langle u,\omega\rangle} \int_{\mathbb{R}^d} f(u) e^{-2\pi i \langle u,\omega\rangle} \mathrm{d}u. \\ &= e^{-2\pi i \langle x,\omega\rangle} \hat{f}(\omega) \\ &= (M_{-x}\hat{f})(\omega).\end{aligned}$$

□

- If $f \in L^1(\mathbb{R}^d)$. Then

$$(M_\omega f)\hat{} = T_\omega \hat{f}. \tag{3.4}$$

Proof. Since $f \in L^1(\mathbb{R}^d)$. Then $M_\omega f \in L^1(\mathbb{R}^d)$ and

$$\begin{aligned}(M_\omega f)\hat{}(\alpha) &= \int_{\mathbb{R}^d} (M_\omega f)(t) e^{-2\pi i \langle t,\alpha\rangle} \mathrm{d}t. \\ &= \int_{\mathbb{R}^d} f(t) e^{2\pi i \langle t,\omega\rangle} e^{-2\pi i \langle t,\alpha\rangle} \mathrm{d}t. \\ &= \int_{\mathbb{R}^d} f(t) e^{-2\pi i \langle t,\alpha-\omega\rangle} \mathrm{d}t. \\ &= \hat{f}(\alpha - \omega). \\ &= (T_\omega \hat{f})(\alpha).\end{aligned}$$

□

Note : Equation **??** explains why modulation is known as frequency shifts.

Definition 3.4. *(**Convolution**[8]) Let $f, g \in L^1(\mathbb{R}^d)$ then convolution of f and g is a function $f * g$ defined on $\mathbb{R}^d$ given by*

$$(f * g)(x) := \int_{\mathbb{R}^d} f(y) g(x-y) \mathrm{d}y. \tag{3.5}$$

Lemma 3.3. *If $f, g \in L^1(\mathbb{R}^d)$, then $f * g \in L^1(\mathbb{R}^d)$. And*

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1. \tag{3.6}$$

Proof. Since $f, g \in L^1(\mathbb{R}^d)$ then

$$\begin{aligned}\int_{\mathbb{R}^d} |(f * g)(x)|\mathrm{d}x &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(y)g(x-y)\mathrm{d}y \right| \mathrm{d}x.\\ &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(y)g(x-y)|\mathrm{d}y\mathrm{d}x.\\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(y)||(T_y g)(x)|\mathrm{d}y\mathrm{d}x.\end{aligned}$$

Since $g \in L^1(\mathbb{R}^d)$ then $T_y g \in L^1(\mathbb{R}^d)$,$|T_y g|$ is integrable. Now by using Fubini's theorem

$$= \int_{\mathbb{R}^d} |f(y)| \left(\int_{\mathbb{R}^d} |T_y g(x)|\mathrm{d}x \right) \mathrm{d}y.$$

Since time-frequency shifts are isometries on $L^1(\mathbb{R}^d)$.

$$\begin{aligned}&= \|g\|_1 \int_{\mathbb{R}^d} |f(y)|\mathrm{d}y.\\ &= \|g\|_1 \|f\|_1 < \infty.\end{aligned}$$

$\Longrightarrow$ $(f * g) \in L^1(\mathbb{R}^d)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. ◻

Lemma 3.4. *If $f, g \in L^1(\mathbb{R}^d)$. Then*

$$(f * g)\hat{} = \hat{f}\hat{g} \tag{3.7}$$

Proof. Since $f, g \in L^1(\mathbb{R}^d)$, $(f * g)\hat{} \in L^1(\mathbb{R}^d)$ and

$$\begin{aligned}(f * g)\hat{}(\omega) &= \int_{\mathbb{R}^d} (f * g)(t) e^{-2\pi i \langle t, \omega \rangle} \mathrm{d}t.\\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x)g(t-x)\mathrm{d}x \right) e^{-2\pi i \langle x, \omega \rangle} e^{-2\pi i \langle t-x, \omega \rangle} \mathrm{d}t.\end{aligned}$$

Since $f * g \in L^1(\mathbb{R}^d)$ then by Fubini's theorem we can interchange the order of integral.

$$\begin{aligned}&= \int_{\mathbb{R}^d} f(x) \left(\int_{\mathbb{R}^d} g(t-x) e^{-2\pi i \langle t-x, \omega \rangle} \mathrm{d}t \right) e^{-2\pi i \langle x, \omega \rangle} \mathrm{d}x.\\ &= \hat{g}(\omega) \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \omega \rangle} \mathrm{d}x.\\ &= \hat{f} \cdot \hat{g}(\omega).\end{aligned}$$

◻

Theorem 3.5. ***(Young)*** *If $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, then $f * g \in L^r(\mathbb{R}^d)$ and*

$$\|f * g\|_r \leq (A_p A_q A r')^d \|f\|_p \|g\|_q, \tag{3.8}$$

where $A_p = \left(\frac{p^{\frac{1}{p}}}{P'^{\frac{1}{p'}}}\right)^{\frac{1}{2}}$.

Proof. Consider $r = \infty$, then

$$|f * g(x)| = \left|\int_{\mathbb{R}^d} f(y)g(x-y)\mathrm{d}y.\right| \leq \int_{\mathbb{R}^d} |f(y)g(x-y)|\mathrm{d}y.$$

By Holder's inequality

$$\leq \|f\|_p \|g\|_q.$$

□

Definition 3.5. *(**Involution Operator**[1]) Let f be a function defined on $\mathbb{R}^d$, then involution of f is a function f^* defined on $\mathbb{R}^d$ and given by*

$$f^*(x) = \overline{f(-x)}. \tag{3.9}$$

Definition 3.6. *(**Reflection Operator**[1]) Let f be a function defined on $\mathbb{R}^d$, then reflection of f is a function $\mathcal{I}f$ defined on $\mathbb{R}^d$ and is given by*

$$(\mathcal{I}f)(x) = f(-x). \tag{3.10}$$

Lemma 3.6. *Let $f \in L^1(\mathbb{R}^d)$. Then*

$$\widehat{f^*} = \overline{\hat{f}} \text{ and } \widehat{\mathcal{I}f} = \mathcal{I}\hat{f}. \tag{3.11}$$

Proof. Clearly if $f \in L^1(\mathbb{R}^d)$, then $f^*, \mathcal{I}f \in L^1(\mathbb{R}^d)$.

$$\widehat{f^*}(\omega) = \int_{\mathbb{R}^d} f^*(t)e^{-2\pi i\langle t,\omega\rangle}\mathrm{d}t. = \int_{\mathbb{R}^d} \overline{f(-t)}e^{-2\pi i\langle t,\omega\rangle}\mathrm{d}t.$$

Substitute $-t \to u$ then $\mathrm{d}t = \mathrm{d}u$.

$$= \int_{\mathbb{R}^d} \overline{f(u)}e^{2\pi i\langle u,\omega\rangle}\mathrm{d}u. = \overline{\int_{\mathbb{R}^d} f(u)e^{-2\pi i\langle u,\omega\rangle}\mathrm{d}u.} = \overline{\hat{f}(\omega)}.$$

$$\begin{aligned}
\text{Now, } \widehat{\mathcal{I}f}(\omega) &= \int_{\mathbb{R}^d} \widehat{\mathcal{I}f}(t) e^{-2\pi i\langle t,\omega\rangle} \mathrm{d}t. \\
&= \int_{\mathbb{R}^d} f(-t) e^{-2\pi i\langle t,\omega\rangle} \mathrm{d}t. \\
\text{Substitute } -t \to u \text{ then } \mathrm{d}t = \mathrm{d}u. \\
&= \int_{\mathbb{R}^d} f(u) e^{-2\pi i\langle u,-\omega\rangle} \mathrm{d}u. \\
&= \hat{f}(-\omega) = \mathcal{I}\hat{f}(\omega)
\end{aligned}$$

□

Theorem 3.7. *Suppose $f \in L^1(\mathbb{R}^d)$ and $x_k f(\cdot) \in L^1(\mathbb{R}^d)$, where x_k is the k-th coordinate function. Then $\hat{f}$ is differentiable with respect to x_k and*

$$\frac{\partial \hat{f}}{\partial x_k}(x) = (-2\pi i t_k f(t))^\wedge(x). \tag{3.12}$$

Proof. Let $h = (0, 0, \ldots, h_k, \ldots, 0)$ be a nonzero vector along the k-th coordinate axis. Then using Lebesgue dominated convergence theorem

$$\begin{aligned}
\lim_{h\to 0} \frac{\hat{f}(x_h) - \hat{f}(x)}{h_k} &= \lim_{h\to 0} \int_{\mathbb{R}^d} f(t) e^{-2\pi i t\cdot x} \left(\frac{e^{-2\pi i h\cdot t} - 1}{h_k}\right) dt. \\
&= \int_{\mathbb{R}^d} f(t) e^{-2\pi i t\cdot x} \lim_{h\to 0} \left(\frac{e^{-2\pi i h\cdot t} - 1}{h_k}\right) dt. \\
&= \int_{\mathbb{R}^d} f(t) e^{-2\pi i t\cdot x} (-2\pi i t_k) dt. \\
&= (-2\pi i t_k f(t))^\wedge(x).
\end{aligned}$$

□

Definition 3.7. *$f \in L^p(\mathbb{R}^d)$ is differentiable in the L^p norm with respect to x_k if there exist a function $g \in L^p(\mathbb{R}^d)$ such that*

$$\left(\int_{\mathbb{R}^d} \left|\frac{f(x+h) - f(x)}{h_k} - g(x)\right|^p dx\right)^{1/p} \to 0 \quad \text{as } h_k \to 0.$$

Theorem 3.8. *If $f \in L^1(\mathbb{R}^d)$ and g is the partial derivative of f with respect to x_k in the L^1 norm then*

$$\hat{g}(x) = 2\pi i x_k \hat{f}(x).$$

Remark. The above theorem can be extended to higher derivatives. For an n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ of nonnegative integers let

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \quad \text{and} \quad D^\alpha = \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

P be a polynomial of n variables $x_1, x_2, \ldots, x_n$ and $P(D)$ be the associated differential operator. Then

$$P(D)\hat{f}(x) = (P(-2\pi it)f(t))^\wedge(x).$$

$$(P(D)f)^\wedge(x) = P(2\pi ix)\hat{f}(x).$$

3.3 Fourier Series

Definition 3.8. *(**Periodic Function**) Let f be a function defined on $\mathbb{R}^d$ and $S \subseteq \mathbb{R}^d$. Then f is said to be periodic on S if*

$$f(x+s) = f(x) \quad \forall x \in \mathbb{R}^d \quad \forall s \in S.$$

We say f is $\mathbb{Z}^d$-periodic if $f(x+k) = f(x), \forall x \in \mathbb{R}^d$ and $\forall k \in \mathbb{Z}^d$.

Note : A $\mathbb{Z}^d$-periodic functions can be uniquely determined by it's restriction to the cube $[0,1]^d$.

Lemma 3.9. *Consider $S = \{e_n(x) = e^{2\pi i\langle n,x\rangle} : n \in \mathbb{Z}^d\}$, where $e_n s$ are functions defined on $[0,1]^d$. Then S be an orthonormal basis for $L^2([0,1]^d)$.*

Proof.

$$\begin{aligned}
\text{Consider, } \|e_n\|_{L^2([0,1]^d)} &= \langle e_n, e_n\rangle_{L([0,1]^d)} \\
&= \int_{[0,1]^d} e_n(x)\overline{e_n(x)}\mathrm{d}x. \\
&= \int_{[0,1]^d} |e_n(x)|^2\mathrm{d}x \\
&= \int_{[0,1]^d} |e^{2\pi i\langle n,x\rangle}|^2\mathrm{d}x \\
&= \int_{[0,1]^d} 1 \cdot \mathrm{d}x = 1.
\end{aligned}$$

Now consider $m \neq n$, then

$$\begin{aligned}
\langle e_n, e_m\rangle_{L^2([0,1]^d)} &= \int_{[0,1]^d} e_n(x)\overline{e_m(x)}\mathrm{d}x \\
&= \int_{[0,1]^d} e^{2\pi i\langle n,x\rangle}\overline{e^{2\pi i\langle m,x\rangle}}\mathrm{d}x \\
&= \int_{[0,1]^d} e^{2\pi i\langle n,x\rangle}e^{-2\pi i\langle m,x\rangle}\mathrm{d}x \\
&= \int_{[0,1]^d} e^{2\pi i\langle n-m,x\rangle}\mathrm{d}x = 0.
\end{aligned}$$

So S is an orthonormal set of functions for $L^2([0,1]^d)$.

Now, assume $f \in L^2([0,1]^d)$, s.t. $f \in S$. Then

$$\begin{aligned}\langle f, e_n\rangle &= \int_{[0,1]^d} f(x)\overline{e_n(x)}\mathrm{d}x = 0, \forall n \in \mathbb{Z}^d \\ &= \int_{[0,1]^d} f(x)e^{-2\pi i\langle n,x\rangle}\mathrm{d}x = 0. \\ &= \hat{f}(n) = 0.\end{aligned}$$

We know that Fourier transform is an unitary map on $L^2(\mathbb{R}^d)$. So if $\hat{f} = 0$, then $f = 0$. So $f = 0$ is the only choice of a function in $L^2([0,1]^d)$, which is orthogonal to S. So S is the maximal set of orthonormal functions in $L^2([0,1]^d)$. Hence S is orthonormal basis for $L^2([0,1]^d)$. □

Theorem 3.10. ***(Plancherel)*** *Let $f \in L^2([0,1]^d)$ and let*

$$\hat{f}(n) = \int_{[0,1]^d} f(x)e^{-2\pi i\langle n,x\rangle}\mathrm{d}x \tag{3.13}$$

be the n-th Fourier coefficient. Then f can be expended into the Fourier series

$$f(x) = \sum_{n\in\mathbb{Z}^d} \hat{f}(n)e^{2\pi i\langle n,x\rangle} \tag{3.14}$$

with the convergence as an orthonormal expansion, and we have

$$\int_{[0,1]^d} |f(x)|^2\mathrm{d}x = \|f\|_{L^2([0,1]^d)} = \sum_{n\in\mathbb{Z}^d} |\hat{f}(n)|^2. \tag{3.15}$$

Proof. By lemma 3.9we know that $S = \{e_n(x) = e^{2\pi i\langle n,x\rangle} : n \in \mathbb{Z}^d\}$ is an orthonormal basis for $L^2([0,1]^d)$, and $L^2([0,1]^d)$ is an seperable Hilbert space, so each $f \in L^2([0,1]^d)$, can be uniquely written in linear combination of $e_{n's}$.

$$f(x) = \sum_{n\in\mathbb{Z}^d} \langle f, e_n\rangle e_n(x),$$

where $\langle f, e_n\rangle = \int_{[0,1]^d} f(x)e^{-2\pi i\langle n,x\rangle}\mathrm{d}x = \hat{f}(n)$ is n-th Fourier coefficient of f w.r.t orthonormal basis S. Then,

$$f(x) = \sum_{n\in\mathbb{Z}^d} \hat{f}(n)e^{2\pi i\langle n,x\rangle}.$$

Now by the Parseval relation

$$\|f\|_{L^2([0,1]^d)} = \sum_{n\in\mathbb{Z}^d} |\hat{f}(n)|^2.$$

□

Definition 3.9. ***(Lattice)*** *Let $\Lambda \subseteq \mathbb{R}^d$, then we say Λ is a lattice if it is a discrete subgroup of $\mathbb{R}^d$ of the form $\Lambda = A\mathbb{Z}^d$, where A is an invertible $d \times d$-matrix over $\mathbb{R}$. The volume of Λ is defined as $vol(\Lambda) = |det(A)|$. The lattice $\Lambda^\perp = (A^{-1})^T\mathbb{Z}^d$ is called the dual lattice of Λ.*

Lemma 3.11. *Let f be a $\Lambda = A\mathbb{Z}^d$-periodic function. Then Fourier series expansion of f can be given as*

$$f(x) = \frac{1}{|det(A)|} \sum_{n \in \mathbb{Z}^d} \hat{f}(u) e^{2\pi i \langle u, x \rangle}, \tag{3.16}$$

where $u = (A^{-1})^T n$.

Proof. Define, $g(x) = f(Ax)$ and f is $\Lambda = A\mathbb{Z}^d$-periodic then g is $\mathbb{Z}^d$-periodic. Let $n \in \mathbb{Z}^d$ then $An \in \Lambda$.

And $g(x+n) = f(A(x+n)) = f(Ax + An) = f(Ax) = g(x)$.

By theorem 3.10,

$$g(x) = \sum_{n \in \mathbb{Z}^d} \hat{g}(n) e^{2\pi i \langle n, x \rangle},$$

where

$$\hat{g}(n) = \int_{[0,1]^d} g(x) e^{-2\pi i \langle n, x \rangle} \mathrm{d}x.$$

$$= \int_{[0,1]^d} f(Ax) e^{-2\pi i \langle n, x \rangle} \mathrm{d}x.$$

Subsitute $x \to A^{-1}t$ then $\mathrm{d}x = |det(A^{-1})| \mathrm{d}t$.

$$= \frac{1}{|det(A)|} \int_{A[0,1]^d} f(t) e^{-2\pi i \langle n, A^{-1}t \rangle} \mathrm{d}t.$$

$$= \frac{1}{|det(A)|} \int_{A[0,1]^d} f(t) e^{-2\pi i \langle (A^T)^{-1} n, t \rangle} \mathrm{d}x.$$

$$= \frac{1}{|det(A)|} \hat{f}((A^T)^{-1} n).$$

Since $f(x) = g(A^{-1}x)$. Then,

$$f(x) = g(A^{-1}x) = \sum_{n \in \mathbb{Z}^d} \hat{g}(n) e^{2\pi i \langle n, A^{-1}x \rangle}.$$

$$= \frac{1}{|det(A)|} \sum_{u \in \Lambda^{\perp}} \hat{f}(u) e^{2\pi i \langle u, x \rangle}. \ (u = (A^T)^{-1} n).$$

Now, let us consider rectangular lattice $\Lambda = \alpha \mathbb{Z}^d$ and aassume that f is Λ-periodic then

$$f(x) = \alpha^{-d} \sum_{n \in \mathbb{Z}^d} f\left(\frac{n}{\alpha}\right) e^{2\pi i \left\langle n, \frac{x}{\alpha} \right\rangle}. \tag{3.17}$$

Furthermore,

$$\|f\|_{L^2([0,\alpha]^d)} = \int_{[0,\alpha]^d} |f(x)|^2 \mathrm{d}x = \alpha^d \sum_{n \in \mathbb{Z}^d} |\hat{f}(n)|. \tag{3.18}$$

□

Lemma 3.12. *Let* $f \in L^1(\mathbb{R}^d)$*, then* $\forall \alpha > 0$,

$$\int_{\mathbb{R}^d} f(x)\mathrm{d}x = \int_{[0,\alpha]} \left(\sum_{n\in\mathbb{Z}^d} f(x+\alpha n) \right) \mathrm{d}x. \tag{3.19}$$

Proof.

$$f(x) = \sum_{n\in\mathbb{Z}^d} \int_{\alpha n + [0,\alpha]^d} f(x)\mathrm{d}x.$$

Substitute $x \to t + \alpha n$ then $\mathrm{d}x = \mathrm{d}t$.

$$= \sum_{n\in\mathbb{Z}^d} \int_{[0,\alpha]^d} f(t+\alpha n)\mathrm{d}t.$$

Since $f \in L^1(\mathbb{R}^d)$ then by using Fubini's theorem.

$$= \int_{[0,\alpha]} \left(\sum_{n\in\mathbb{Z}^d} f(x+\alpha n) \right) \mathrm{d}x.$$

□

Theorem 3.13. *Let* f *be a function defined on* $\mathbb{R}^d$ *s.t. for some* $\epsilon > 0$ *and* $C > 0$

$$|f(x)| \le C(1+|x|)^{-d-\epsilon} \text{ and } |\hat{f}(\omega)| \le C(1+|\omega|)^{-d-\epsilon}.$$

Then

$$\sum_{n\in\mathbb{Z}^d} f(x+n) = \sum_{n\in\mathbb{Z}^d} \hat{f}(n) e^{2\pi i\langle n,x\rangle}. \tag{3.20}$$

The above identity holds pointwise $\forall x \in \mathbb{R}^d$*, and both sums converge absolutely* $\forall x \in \mathbb{R}^d$.

Proof. Define $\psi(x) = \sum_{n\in\mathbb{Z}^d} f(x+n)$, then for $\lambda \in \mathbb{Z}^d$

$$\psi(x+\lambda) = \sum_{n\in\mathbb{Z}^d} f(x+\lambda+n) = \sum_{n\in\mathbb{Z}^d} f(x+n) = \psi(x).$$

Now consider the integral

$$\int_{\mathbb{R}^d} |f(x)|\mathrm{d}x \le C \int_{\mathbb{R}^d} (1+|x|)^{-d-\epsilon}\mathrm{d}x.$$

By using polar cordinate system. Consider $x = r\cdot\omega$ where $r > 0$ and $\omega \in S^{d-1}$.

Then $|x| = r$ and $0 \le r \le \infty$.

$$= C\int_{S^{d-1}} \int_0^\infty (1+r)^{-d-\epsilon} r^{d-1}\mathrm{d}r\mathrm{d}\omega.$$

Now consider the integral

$$\int_0^\infty (1+r)^{-d-\epsilon} r^{d-1} \mathrm{d}r\mathrm{d}\omega = \int_0^1 (1+r)^{-d-\epsilon} r^{d-1} \mathrm{d}r\mathrm{d}\omega. + \int_1^\infty (1+r)^{-d-\epsilon} r^{d-1} \mathrm{d}r.$$
$$\leq \int_0^1 1\mathrm{d}r + \int_1^\infty (2r)^{-d-\epsilon} r^{d-1} \mathrm{d}r.$$
$$= 1 + 2^{-d-\epsilon} \int_1^\infty r^{-\epsilon-1} \mathrm{d}r.$$
$$\leq 1 - \left[(\epsilon+1) r^{-\epsilon-2}\right]_1^\infty .$$
$$= 2 + \epsilon.$$
$$\implies C \int_{S^{d-1}} \int_0^\infty (1+r)^{-d-\epsilon} r^{d-1} \mathrm{d}r\mathrm{d}\omega \leq C(2+\epsilon) \int_{S^{d-1}} 1 \cdot \mathrm{d}\omega.$$
$$= C(2+\epsilon).$$
$$\implies \int_{\mathbb{R}^d} |f(x)| \mathrm{d}x \leq C(2+\epsilon) < \infty,$$

So $f \in L^1(\mathbb{R}^d)$ and then $\psi \in L^1([0,1]^d)$. Now look at the Fourier coefficient of ψ.

$$\hat{\psi}(n) = \int_{[0,1]^d} \psi(x) e^{-2\pi i \langle n,x \rangle} \mathrm{d}x.$$
$$= \int_{[0,1]^d} \left(\sum_{n \in \mathbb{Z}^d} f(x+k) e^{-2\pi i \langle n,x+k \rangle} \right) \mathrm{d}x.$$

Now by lemma 3.12

$$= \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle n,x \rangle} \mathrm{d}x.$$
$$= \hat{f}(n).$$

$$\implies \textstyle\sum_{n \in \mathbb{Z}^d} f(x+n) = \psi(x) = \sum_{n \in \mathbb{Z}^d} \hat{\psi}(n) e^{-2\pi i \langle n,x \rangle} = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{2\pi i \langle n,x \rangle}.$$ □

Note : Let $\Lambda = A\mathbb{Z}^d$ and $\Lambda^\perp = (A^T)^{-1}\mathbb{Z}^d$ be the dual lattice. Then

$$\sum_{\lambda \in \Lambda} f(x+\lambda) = \frac{1}{|detA|} \sum_{n \in \mathbb{Z}^d} \hat{f}((A^{-1})^T n) e^{2\pi i (A^{-1})^T \langle n,x \rangle}. \tag{3.21}$$

Proof.

$$\sum_{\lambda \in \Lambda} f(x+\lambda) = \sum_{n \in \mathbb{Z}^d} (f \circ A)(A^1 x + n).$$

By 3.20

$$= \sum_{n \in \mathbb{Z}^d} (f \circ A)\hat{\ }(n) e^{2\pi i \langle n, A^{-1}x \rangle}.$$

$$= \frac{1}{|detA|} \sum_{n \in \mathbb{Z}^d} \hat{f}((A^{-1})^T n) e^{2\pi i \langle (A^{-1})^T n, x \rangle}.$$

Now for the rectangular lattice $\Lambda = \alpha \mathbb{Z}^d$, the Poisson summation formula is

$$\sum_{n \in \mathbb{Z}^d} f(x + \alpha n) = \alpha^{-d} \sum_{n \in \mathbb{Z}^d} \hat{f}\left(\frac{n}{\alpha}\right) e^{2\pi i \left\langle n, \frac{x}{\alpha} \right\rangle}. \tag{3.22}$$

□

3.4 Gaussians and Inverse Fourier Transform

Definition 3.10. *The non-normalised Guassian function ψ_a is a function defined on $\mathbb{R}^d$ of the form*

$$\psi_a(x) = e^{-\frac{\pi \|x\|^2}{a}}. \tag{3.23}$$

Here $a > 0$ is known as width of ψ_a.

Lemma 3.14. *Let ψ_a be a non-normalised Gaussian function with width $a > 0$ Then*

$$\widehat{\psi_a}(\omega) = a^{\frac{d}{2}} \psi_{\frac{1}{a}}(\omega). \tag{3.24}$$

In particular, when $a = 1$, then $\widehat{\psi_1} = \psi_1$.

Proof. Let us prove it for $d = 1$. Consider $f(t) = e^{-at^2}$.

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-at^2} e^{-2\pi i t\omega} \mathrm{d}t.$$

$$= \int_{-\infty}^{\infty} e^{-(at^2 + 2\pi i t\omega)} \mathrm{d}t$$

$$= \int_{-\infty}^{\infty} e^{-\left(\sqrt{a}t + \frac{\pi i \omega}{\sqrt{a}}\right)^2 + \frac{\pi^2 \omega^2}{a}} \mathrm{d}t.$$

$$= e^{\frac{\pi^2 \omega^2}{a}} \int_{-\infty}^{\infty} e^{-\left(t + \frac{\pi i \omega}{\sqrt{a}}\right)^2} \frac{dt}{\sqrt{a}}$$

□

$$\hat{f}(\omega) = e^{\frac{\pi^2\omega^2}{a}}\sqrt{\frac{\pi}{a}}.$$

From the above equation we get

$$\int_{\mathbb{R}} e^{-\pi\alpha|y|^2} e^{-2\pi i t\cdot y}\mathrm{d}y = \alpha^{-1/2} e^{-\pi|t|^2/\alpha}.$$

Now our result follows easily.

$$\begin{aligned}\int_{\mathbb{R}^d} e^{-\pi\frac{\|y\|^2}{a}} e^{-2\pi i\langle ty\rangle}\mathrm{d}y &= \prod_{j=1}^{d}\int_{-\infty}^{\infty} e^{-2\pi\frac{y_j^2}{a}} e^{-2\pi i t_j y_j}\,\mathrm{d}y_j \\ &= a^{\frac{d}{2}} e^{-\pi a\|t\|^2}.\end{aligned}$$

Theorem 3.15. *For all $\epsilon > 0$ we have*

$$\int_{\mathbb{R}^d} e^{-2\pi i\|t\|\cdot\epsilon} e^{-2\pi i\langle t,x\rangle}\mathrm{d}t = c_d\frac{\epsilon}{(\epsilon^2+\|x\|^2)^{\frac{(d+1)}{2}}},$$

where

$$c_d = \Gamma\left(\frac{(d+1)}{2}\right)\pi^{\frac{(d+1)}{2}}.$$

Proof. We use two identities :

1. $\int_{\mathbb{R}^d} e^{-\pi\delta\|t\|^2} e^{-2\pi i\langle t,x\rangle}\mathrm{d}t = \delta^{-\frac{d}{2}} e^{-\pi\frac{\|x\|^2}{\delta}}, \delta > 0$
2. $e^{-\gamma} = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{\gamma^2}{4u}}\mathrm{d}u, \gamma > 0.$

Identity (1) can be seen from lemma 3.14. First we will prove identity (2).

Let $f(z) = \frac{e^{i\gamma z}}{1+z^2}$. C be the contour (positive direction) given below.

Then,

$$\int_C f(z)\mathrm{d}z = \int_{-R}^{R} f(z)\mathrm{d}z + \int_{C_R} f(z)\mathrm{d}z = 2\pi i\sum R^+.$$

Now by Jordan's lemma

$$\lim_{R\to\infty}\int_{C_R} f(z)\mathrm{d}z = 0.$$

Therefore

$$\lim_{R\to\infty}\int_C f(z)\mathrm{d}z = \int_{-\infty}^{\infty} f(z)\mathrm{d}z = 2\pi i Res_{z=i} f(z) = \pi e^{-\gamma}.$$

That is

$$e^{-\gamma} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{i\gamma x}}{1+x^2}\mathrm{d}x.$$

Now,

$$e^{-\gamma} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\gamma x}}{1+x^2}\mathrm{d}x = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{i\gamma x}\int_{0}^{\infty} e^{-(1+x^2)u}\mathrm{d}u\mathrm{d}x.$$
$$= \frac{1}{\pi}\int_{0}^{\infty} e^{-u}\int_{-\infty}^{\infty} e^{i\gamma x}e^{-x^2u}\mathrm{d}x\mathrm{d}u.$$
$$= \frac{1}{\pi}\int_{0}^{\infty} e^{-u}\int_{-\infty}^{\infty} e^{-2\pi i\gamma x}e^{-4\pi^2ux^2}2\pi\mathrm{d}x\mathrm{d}u.$$
$$= 2\int_{0}^{\infty} e^{-u}\sqrt{\frac{\pi}{4\pi^2u}}e^{-\frac{\pi^2\gamma^2}{4\pi^2u}}\mathrm{d}u.$$
$$= \frac{1}{\sqrt{\pi}}\int_{0}^{\infty} \frac{e^{-u}}{\sqrt{u}}e^{-\frac{\gamma^2}{4u}}\mathrm{d}u.$$

Now,

$$\int_{\mathbb{R}^d} e^{-2\pi i\|t\|\cdot\epsilon}e^{-2\pi i\langle t,x\rangle}\mathrm{d}t = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}^d}\left(\int_{0}^{\infty}\frac{e^{-u}}{\sqrt{u}}e^{-\pi^2\|t\|^2\frac{\epsilon^2}{u}}\mathrm{d}u\right)e^{-2\pi i\langle t,x\rangle}\mathrm{d}t.$$
$$= \frac{1}{\pi^{\frac{d+1}{2}}}\int_{0}^{\infty} e^{-u}e^{-\frac{\|x\|^2u}{\epsilon^2}}\epsilon^{-n}u^{\frac{d-1}{2}}\mathrm{d}u.$$
$$= \frac{\epsilon}{(\pi(\|x\|^2+\epsilon^2))^{\frac{d+1}{2}}}\int_{0}^{\infty} e^{-u}u^{\frac{d-1}{2}}\mathrm{d}u.$$
$$= \frac{c_d\epsilon}{(\|x\|^2+\epsilon^2)^{\frac{d+1}{2}}}.$$

□

Remark 3.1. *We shall denote the Fourier transform of* $e^{-4\pi\epsilon\|y\|^2}, \epsilon > 0$ *by* W *and* P *respectively. That is*

$$W(t,\epsilon) = (4\pi\epsilon)^{-\frac{d}{2}}e^{-\frac{\|t\|^2}{4\epsilon}}. \tag{3.25}$$

And

$$P(t,\epsilon) = C_d\left(\frac{\epsilon}{(\epsilon^2+\|t\|^2)^{\frac{(d+1)}{2}}}\right). \tag{3.26}$$

Theorem 3.16. *If* $f, g \in L^1(\mathbb{R}^d)$, *then*

$$\int_{\mathbb{R}^d}\hat{f}(x)g(x)\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\hat{g}(x)\mathrm{d}x. \tag{3.27}$$

Proof.

$$\int_{\mathbb{R}^d} \hat{f}(x)g(x)\mathrm{d}x = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(t)e^{-2\pi\langle t,x\rangle}\mathrm{d}t g(x)\mathrm{d}x.$$

By Fubini's theorem

$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(x)e^{-2\pi\langle t,x\rangle}\mathrm{d}x f(t)\mathrm{d}t.$$
$$= \int_{\mathbb{R}^d} \hat{g}(t)f(t)\mathrm{d}t.$$

□

Theorem 3.17. *Integrals of the Poission kernel and the Weistrass kernel is* 1*. That is*

$$\int_{\mathbb{R}^d} W(x,\epsilon)\mathrm{d}x = 1. \tag{3.28}$$

and

$$\int_{\mathbb{R}^d} P(x,\epsilon)\mathrm{d}x = 1. \tag{3.29}$$

Proof.

$$\int_{\mathbb{R}^d} (4\pi\epsilon)^{-\frac{d}{2}} e^{-\frac{\|x\|^2}{4\epsilon}}\mathrm{d}x. = \int_{\mathbb{R}^d} (4\pi)^{-\frac{d}{2}} e^{-\frac{\|x\|^2}{4}}\mathrm{d}x.$$
$$= (4\pi)^{-\frac{d}{2}} \prod_{j=1}^{d} \int_{-\infty}^{\infty} e^{-\frac{\|x_j\|^2}{4}}\mathrm{d}x_j.$$
$$= (4\pi)^{-\frac{d}{2}} \prod_{j=1}^{d} \int_{-\infty}^{\infty} e^{-\|x_j\|^2} 2\mathrm{d}x_j.$$
$$= (4\pi)^{-\frac{d}{2}} 2^n \pi^{\frac{d}{2}} = 1.$$

Now

$$\int_{\mathbb{R}^d} c_d \frac{\epsilon}{(\epsilon^2 + \|x\|^2)^{\frac{(d+1)}{2}}}\mathrm{d}x. = c_d \int_{\mathbb{R}^d} \frac{\epsilon}{\epsilon^{n+1}(1+\frac{\|x\|^2}{\epsilon^2})^{\frac{(d+1)}{2}}}\mathrm{d}x.$$
$$= c_d \int_{\mathbb{R}^d} \frac{\epsilon}{\epsilon^{n+1}(1+\|x\|^2)^{\frac{(d+1)}{2}}}\mathrm{d}x.$$

By using polar cordinates

$$= c_d \int_{S^{d-1}} \int_0^\infty \frac{r^{d-1}}{(1+r^2)^{\frac{d+1}{2}}}\mathrm{d}r\mathrm{d}\omega.$$
$$= c_d |S^{n-1}| \int_0^\infty \frac{r^{d-1}}{(1+r^2)^{\frac{d+1}{2}}}\mathrm{d}r$$
$$= 1.$$

□

Theorem 3.18. ***(Fourier Inversion)*** *Let f and $\hat{f}$ belongs to $L^1(\mathbb{R}^d)$. Then for all x in the set of Lebesgue points of f, we have*

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(t)e^{2\pi i\langle t,x\rangle}\mathrm{d}t. \tag{3.30}$$

Proof. Let $\phi(x) = W(x, 1)$. Then

$$\phi_\epsilon(x) = \epsilon^{-n}\phi\left(\frac{x}{\epsilon}\right) = W(x, \epsilon^2).$$

Also W is a decreasing function in $|x|$ implies $\psi = \phi$ and hence is integrable. Therefore using approximation to the identity we get ,

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\mathbb{R}^d} \hat{f}(t)e^{2\pi i\langle t,x\rangle}e^{-4\pi^2\epsilon\|x\|^2}\mathrm{d}t &= \lim_{\epsilon\to 0}\int_{\mathbb{R}^d} f(t)W(t-x,\epsilon)\mathrm{d}t \\ &= \lim_{\epsilon\to 0}\int_{\mathbb{R}^d} f(t)W(x-t,\epsilon)\mathrm{d}t = f(x).\end{aligned}$$

Also since $\hat{f}$ is integrable using the dominated convergence theorem, we get

$$\begin{aligned}f(x) = \lim_{\epsilon\to 0}\int_{\mathbb{R}^d} \hat{f}(t)e^{2\pi i\langle t,x\rangle}e^{-4\pi^2\epsilon\|x\|^2}\mathrm{d}t &= \int_{\mathbb{R}^d} \lim_{\epsilon\to 0}\hat{f}(t)e^{2\pi i\langle t,x\rangle}e^{-4\pi^2\epsilon\|x\|^2}\mathrm{d}t \\ &= \int_{\mathbb{R}^d} \hat{f}(t)e^{2\pi i\langle t,x\rangle}\mathrm{d}t.\end{aligned}$$

□

Theorem 3.19. *Suppose $f \in L^1(\mathbb{R}^d)$ and $\hat{f} \geq 0$. If f is continuous at 0 then $\hat{f} \in L^1(\mathbb{R}^d)$ and*

$$f(t) = \int_{\mathbb{R}^d} \hat{f}e^{2\pi i\langle t,x\rangle}\mathrm{d}x$$

for almost every x. In particular

$$f(0) = \int_{\mathbb{R}^d} \hat{f}(x)\mathrm{d}x.$$

Proof. We only need to show that $\hat{f} \in L^1(\mathbb{R}^d)$. Consider the function $g_\epsilon(x) = e^{-4\pi\epsilon x}\hat{f}(x)$. Then by Fatous Lemma

$$\int_{\mathbb{R}^d} \hat{f}(x)\mathrm{d}x \leq \lim_{\epsilon} inf \int_{\mathbb{R}^d} e^{4\pi^2\epsilon\|x\|^2}\hat{f}(x)\mathrm{d}x = f(0),$$

since 0 is a Lebesgue point of f. □

Theorem 3.20. ***(Plancherel)*** *If $f \in C_c(\mathbb{R}^d)$ then $\|\hat{f}\|_2 = \|f\|_2$.*

Proof. Define $g(x) = \overline{f(-x)}$. Then $g \in L^1(\mathbb{R}^d)$. Now say $h = f * g$ then $h \in L^1(\mathbb{R}^d)$. Now by lemma 3.4 and 3.6

$$\hat{h}(\omega) = \hat{f}(\omega)\cdot\hat{g}(\omega) = \hat{f}(\omega)\cdot\overline{\widehat{f(-\omega)}} = |\hat{f}(\omega)|^2. \tag{3.31}$$

Now by theorem 3.19, we can say that $\hat{h} \in L^1(\mathbb{R}^d)$ and $h(0) = \int_{\mathbb{R}^d} \hat{h}(x)\mathrm{d}x$. We thus have

$$\begin{aligned}\int_{\mathbb{R}^d} |\hat{f}(x)|^2 \mathrm{d}x = \int_{\mathbb{R}^d} \hat{h}(x)\mathrm{d}x = h(0) = \int_{\mathbb{R}^d} f(x)g(-x)\mathrm{d}x.\\ = \int_{\mathbb{R}^d} f(x)\overline{f(x)}\mathrm{d}x = \int_{\mathbb{R}^d} |f(x)|^2 \mathrm{d}x.\\ \implies \|f\|_2 = \|\hat{f}\|.\end{aligned}$$

□

Remark 3.2. *Since we know that $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$, and $\mathcal{F}$ is continuous on $C_c(\mathbb{R}^d)$ so we can extend this to $L^2(\mathbb{R}^d)$.*

Definition 3.11. *(**Fourier transform on** $L^2(\mathbb{R}^d)$ **Space**) Let $f \in L^2(\mathbb{R}^d)$ and $f_n \in C_c(\mathbb{R}^d)$ be such that $f_n \to f$ in $L^2(\mathbb{R}^d)$.Then we define*

$$\mathcal{F}(f) := \lim_{n\to\infty} \hat{f}_n. \tag{3.32}$$

Note : The above definition is independent of the choice of sequence.

Lemma 3.21. *(**Time-frequency shift of Gaussians**) For all $\alpha > 0$ and for all $x, u, \omega, n \in \mathbb{R}^d$ we have*

$$\langle T_x M_\omega \psi_a, T_u M_n \psi_a \rangle = \left(\frac{a}{2}\right)^{\frac{1}{2}} e^{\pi i \langle u-x, n+\omega\rangle} \psi_{2a}(u-x)\psi_{\frac{2}{a}}(n-\omega). \tag{3.33}$$

Proof.

$$\begin{aligned}\langle \psi_a, M_\omega T_x \psi_a \rangle &= \int_{\mathbb{R}^d} \psi_a(t)\overline{M_\omega T_x \psi_a(t)}\mathrm{d}t\\ &= \int_{\mathbb{R}^d} e^{-\pi\frac{\|t\|^2}{a}} e^{-2\pi i\langle \omega, t\rangle} e^{-\pi\frac{\|t-x\|^2}{a}} \mathrm{d}t.\\ &= \int_{\mathbb{R}^d} e^{-\pi\frac{\|t\|^2}{a}} e^{-2\pi i\langle \omega, t\rangle} e^{-\pi\frac{\langle t-x, t-x\rangle}{a}} \mathrm{d}t.\\ &= \int_{\mathbb{R}^d} e^{-\pi\frac{\|t\|^2}{a}} e^{-2\pi i\langle \omega, t\rangle} e^{-\pi\frac{\|t\|^2 - 2\langle t,x\rangle + \|x\|^2}{a}} \mathrm{d}t.\\ &= e^{\frac{-\pi\|x\|^2}{a}} \int_{\mathbb{R}^d} e^{-2\pi\frac{\|t\|^2}{a}} e^{-2\pi i\langle \omega, t\rangle} e^{2\pi\frac{\langle x,t\rangle}{a}} \mathrm{d}t.\\ &= e^{\frac{-\pi\|x\|^2}{a}} \int_{\mathbb{R}^d} e^{-2\pi\frac{\langle t-\frac{x}{2}, t-\frac{x}{2}\rangle}{a}} e^{-2\pi\frac{\|t\|^2}{a}} e^{2\pi\frac{\langle t-\frac{x}{2}, t-\frac{x}{2}\rangle}{a}} e^{2\pi\frac{\langle x,t\rangle}{a}} e^{-2\pi i\langle \omega, t\rangle} \mathrm{d}t.\\ &= e^{\frac{-\pi\|x\|^2}{2a}} \int_{\mathbb{R}^d} e^{-\pi\frac{\|t-\frac{x}{2}\|^2}{\frac{a}{2}}} e^{-2\pi i\langle \omega, t\rangle} \mathrm{d}t.\\ &= \psi_{2a}(x)(T_{\frac{x}{2}}\psi_{\frac{a}{2}})\hat{\ }(\omega).\end{aligned}$$

□

By equation 3.3

$$\langle \psi_a, M_\omega T_x \psi_a \rangle = \psi_{2a}(x) M_{\frac{-x}{2}} \hat{\psi}_{\frac{a}{2}}(\omega).$$

By lemma 3.14

$$= \psi_{2a}(x) e^{2\pi i \langle \omega, \frac{-x}{2} \rangle} \left(\frac{a}{2}\right)^{\frac{d}{2}} \psi_{\frac{2}{a}}(\omega).$$
$$= \psi_{2a}(x) e^{-\pi i \langle \omega, x \rangle} \left(\frac{a}{2}\right)^{\frac{d}{2}} \psi_{\frac{2}{a}}(\omega).$$

Now by lemma 3.2

$$M_{-\omega} T_{u-x} M_n = M_\omega e^{-2\pi i \langle n, u-x \rangle} M_n T_{u-x}. = e^{-2\pi i \langle n, u-x \rangle} M_{n-\omega} T_{u-x}.$$

Now again consider,

$$\begin{aligned} \langle T_x M_\omega \psi_a, T_u M_n \psi_a \rangle &= \langle \psi_a, M_{-\omega} T_{u-x} M_n \psi_a \rangle. \\ &= e^{2\pi i \langle n, u-x \rangle} \langle \psi_a, M_{n-\omega} T_{u-x} \psi_a \rangle. \\ &= \left(\frac{a}{2}\right)^{d/2} e^{2\pi i \eta \cdot (u-x)} e^{-\pi i (u-x) \cdot (\eta - \omega)} \varphi_{2a}(u-x) \varphi_{\frac{2}{a}}(\eta - \omega) \\ &= \left(\frac{a}{2}\right)^{d/2} e^{\pi i (u-x) \cdot (\eta + \omega)} \varphi_{2a}(u-x) \varphi_{\frac{2}{a}}(\eta - \omega). \end{aligned}$$

CHAPTER 4

Uncertainty Principle

4.1 Introduction

In mathematics, uncertainty principles in the strict sense are inequalities that involve both a function f and its Fourier transform $\hat{f}$. We begin with the classical uncertainty principle in dimension $d = 1$, which is now commonly known as the Heisenberg-Pauli-Weyl inequality.

4.2 Uncertainty Principles

Lemma 4.1. *Let A and B be (possibly unbounded) self adjoint operators on a Hilbert space $\mathcal{H}$. Then*

$$\|A - aI\|\|B - bI\| \geq \frac{1}{2} |\langle [A, B]f, f\rangle| \tag{4.1}$$

for all $a, b \in \mathbb{R}$ and for all f in the domain of AB and BA
In the above inequality equality holds iff $(A - aI) = ic(B - bI)$ for some $c \in \mathbb{R}$.here $[A, B] = AB - BA$, is the commutator of A and B.

Proof. Given that A, B be two self adjoint operators on Hilbert space $\mathcal{H}$, which are possibly unbounded and $[A, B]$ be the commutator of A and B s.t. $[A, B] = AB - BA$.

Claim $[A - aI, B - bI] = [A, B], \forall a, b \in \mathbb{R}$.

$$\begin{aligned}[A - aI, B - bI] &= (A - aI)(B - bI) - (B - bI)(A - aI).\\ &= AB - bA - aB + abI - BA + aB + bA - abI.\\ &= AB - BA.\end{aligned}$$

Now consider inner product

$$\begin{aligned}\langle [A, B]f, f\rangle &= \langle [A - aI, B - bI]f, f\rangle.\\ &= \langle \{(A - aI)(B - bI) - (B - bI)(A - aI)\}f, f\rangle\\ &= \langle (A - aI)(B - bI)f, f\rangle - \langle (B - bI)(A - aI)f, f\rangle.\end{aligned}$$

Since A, B are self adjoint operator

$$\begin{aligned}&= \langle (B - bI)f, (A - aI)f\rangle - \langle (A - aI)f, (B - bI)f\rangle.\\ &= \langle (B - bI)f, (A - aI)f\rangle - \overline{\langle (B - bI)f, (A - aI)f\rangle}.\\ &= 2iIm\langle (B - bI)f, (A - aI)f\rangle.\end{aligned}$$

$$|\langle [A, B]f, f\rangle| \leq 2|\langle (B - bI)f, (A - aI)f\rangle|.$$

The equality holds in above inequality if and only if $\langle (B-bI)f, (A-aI)f\rangle$ is purely imaginary By using Cauchy-Schwartz inequality.

$$\leq 2\|(B - bI)f\|_{\mathcal{H}}\|(A - aI)f\|_{\mathcal{H}}.$$

$$\frac{1}{2}|\langle [A, B]f, f\rangle| \leq \|(B - bI)f\|_{\mathcal{H}}\|(A - aI)f\|_{\mathcal{H}}.$$

The equality holds in above inequality if and only if $(A - aI)f = c(B - bI)f, c \in \mathbb{C}$.

So equality holds if and only if $(A - aI)f = ic(B - bI)f, c \in \mathbb{R}$.

□

Theorem 4.2. *Let $f \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$, then*

$$\left(\int_{\mathbb{R}} (x - a)^2|f(x)|^2\, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} (\omega - b)^2|\hat{f}(\omega)|^2\, d\omega\right)^{\frac{1}{2}} \geq \frac{1}{4\pi}\|f\|_2^2. \tag{4.2}$$

Proof. Consider the multiplication and differentiation operators, X and P, respectively.

$$Xf(x) = xf(x) \text{ and } Pf(x) = \frac{1}{2\pi i}f'(x).$$

The largest common domain for PX, XP, X, and P is

$$\{f \in L^2(\mathbb{R}) : Xf, Pf, XPf, PXf \in L^2(\mathbb{R})\}.$$

Clearly, the Schwartz class $\mathcal{S}(\mathbb{R})$ is the common domain.

$$[X, P]f(x) = \frac{1}{2\pi i}(xf'(x) - (xf)'(x)) = -\frac{1}{2\pi i}.$$

By lemma 4.1

$$\frac{1}{4\pi}\|f\|_2^2 = \frac{1}{2}|\langle [X,P]f, f\rangle| \leq \|(X-aI)f\|_2\|(P-bI)f\|_2. \tag{4.3}$$

By Plancherel's theorem

$$\|(P-bI)f\|_2 = \|\mathcal{F}(P-bI)f\|_2 = \left(\int_R (\omega-b)^2|\hat{f}(\omega)|^2 d\omega\right)^{\frac{1}{2}}. \tag{4.4}$$

$$\|(X-aI)f\|_2 = \left(\int_R (x-a)^2|f(x)|^2 dx\right)^{\frac{1}{2}}. \tag{4.5}$$

Now by equation 4.3, 4.4 and 4.5

$$\left(\int_{\mathbb{R}} (x-a)^2|f(x)|^2\, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} (\omega-b)^2|\hat{f}(\omega)|^2\, d\omega\right)^{\frac{1}{2}} \geq \frac{1}{4\pi}\|f\|_2^2.$$

□

Corollary 4.3. *For $f \in L^2(\mathbb{R})$,*

$$\|Xf\|_2^2 + \|Pf\|_2^2 \geq \frac{1}{2\pi}\|f\|_2^2.$$

Proof. By applying $2\alpha\beta \leq \alpha^2 + \beta^2$ in the theorem 4.2 with $a = b = 0$ and $\alpha = \|Xf\|_2, \beta = \|Pf\|_2$. □

4.3 The Uncertainty Principle of Donoho and Stark

Definition 4.1. *(ϵ-**concentrated function**[9]) Let $f \in L^2(\mathbb{R}^d)$ and $T \subseteq \mathbb{R}^d$ be a measurble set, then f is known as ϵ-concentrated on T if*

$$\left(\int_{T^c} |f(x)|^2 dx\right)^{\frac{1}{2}} \leq \epsilon\|f\|_2.$$

Theorem 4.4. *[9] Let $f \in L^2(\mathbb{R}^d)$ with $f \neq 0$ and T, Ω be two measurable subsets of $\mathbb{R}^d$. Suppose f is ϵ_T-concentrated and $\hat{f}$ is ϵ_Ω-concentrated. Then*

$$|T||\Omega| \geq (1-\epsilon_T-\epsilon_\Omega)^{\frac{1}{2}}.$$

Proof. Let $|T| = \infty$ or $|\Omega| = \infty$, then it is trivially true. Assume that T and Ω have finite measure.

Let $f \in L^2(\mathbb{R}^d)$, then define

$$P_T f := \chi_T f$$

and

$$Q_\Omega f(x) := \mathcal{F}^{-1}(\chi_\Omega \hat{f})(x) = \int_\Omega \hat{f}(\omega) e^{2\pi i \langle x,\omega\rangle} \mathrm{d}\omega.$$

We can observe that range of P_T is $L^2(T)$ and range of Q_Ω is collection of those $f \in L^2(\mathbb{R}^d)$ s.t. $supp \hat{f} \subseteq \Omega$, and with this set up we can write that f is ϵ_T-concentrated on T if and only if

$$\|f - P_T f\|_2 \leq \epsilon_T \|f\|_2, \tag{4.6}$$

and $\hat{f}$ is ϵ_Ω-concentrated on Ω if and only if

$$\|f - Q_\Omega f\|_2 = \|\mathcal{F}(f - Q_\Omega f)\|_2 = \|\chi_{\Omega^c} \hat{f}\|_2 \leq \epsilon_\Omega \|f\|_2. \tag{4.7}$$

(By Plancherel theorem.)

$$Q_\Omega^2 f = Q_\Omega(Q_\Omega f) = \mathcal{F}^{-1}(\chi_\Omega \mathcal{F}(Q_\Omega f)) = \mathcal{F}^{-1}(\chi_\Omega \mathcal{F}\mathcal{F}^{-1}(\chi_\Omega \hat{f})) = Q_\Omega f.$$

$\implies Q_\Omega$ is idempotent. Now again by Plancherel theorem

$$\begin{aligned}
\langle Q_\Omega f, g\rangle = \langle \mathcal{F}^{-1}(\chi_\Omega \hat{f}), g\rangle &= \langle \chi_\Omega \hat{f}, \hat{g}\rangle \\
&= \int_{\mathbb{R}^d} \chi_\Omega(\omega) \hat{f}(\omega) \overline{\hat{g}}(\omega) \mathrm{d}\omega \\
&= \int_{\mathbb{R}^d} \hat{f}(\omega) \overline{\chi_\Omega(\omega) \hat{g}}(\omega) \mathrm{d}\omega \\
&= \langle \hat{f}, \chi_\Omega \hat{g}\rangle. \\
&= \langle f, \mathcal{F}^{-1}(\chi_\Omega \hat{g})\rangle. \\
&= \langle f, Q_\Omega g\rangle.
\end{aligned}$$

$\implies Q_\Omega$ is self adjoint. So Q_Ω is an orthogonal projection on $L^2(\mathbb{R}^d)$. So $\|Q_\Omega\|_{op} \leq 1$, and by equation 4.7 and 4.3 we obtain that

$$\|f - Q_\Omega P_T f\|_2 \leq \|f - Q_\Omega f\|_2 + \|Q_\Omega(f - P_T f)\|_2 \leq (\epsilon_T + \epsilon_\Omega)\|f\|_2, \tag{4.8}$$

and consequently by 4.8

$$\|Q_\Omega P_T f\|_2 \geq \|f\|_2 - \|f - Q_\Omega P_T f\|_2 \geq (1 - \epsilon_\Omega - \epsilon_T)\|f\|_2. \tag{4.9}$$

Now we compute the integral kernel and then the Hilbert-Schmidt norm of $Q_\Omega P_T$. First,

$$\begin{aligned}
Q_\Omega P_T f(x) &= \mathcal{F}(\chi_\Omega (P_T f)\hat{\ })(x) \\
&= \int_\Omega \left(\int_T f(t) e^{-2\pi i \langle t,\omega\rangle} \right) e^{2\pi i \langle x,\omega\rangle} \mathrm{d}\omega.
\end{aligned}$$

Since we know that T and Ω both have finite measure and since $f \in L^2(T) \subseteq Ł^1(T)$, this double integral converge absolutely.So by Fubini's theorem we can interchange the order of integral and we obtain

$$Q_\Omega P_T f(x) = \int_{\mathbb{R}^d} \chi_T(t) \left(\int_\Omega e^{2\pi i \langle x-t,\omega\rangle} \mathrm{d}\omega \right) f(t)\mathrm{d}x.$$

Define the kernel $K(x.t) = \chi_T(t) \int_\Omega e^{2\pi i\langle x-t,\omega\rangle}\mathrm{d}\omega = \chi_T(t) T_t(\mathcal{F}^{-1}\chi_\Omega)(x)$.

Then, $Q_\Omega P_T f(x) = \int_{\mathbb{R}^d} K(x,t) f(t)\mathrm{d}t$.

Now calculate

$$\begin{aligned}
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |K(x,t)|^2 \mathrm{d}x\mathrm{d}t &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left| \chi_T(t) \int_\Omega e^{2\pi i\langle x-t,\omega\rangle}\mathrm{d}\omega \right|^2 \mathrm{d}x\mathrm{d}t. \\
&\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_T(t) \int_\Omega |e^{2\pi i\langle x-t,\omega\rangle}|^2 \mathrm{d}\omega\mathrm{d}x\mathrm{d}t. \\
&\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_T(t) \int_\Omega 1\cdot \mathrm{d}\omega\mathrm{d}x\mathrm{d}t. \\
&\leq |T|\cdot|\Omega| < \infty.
\end{aligned}$$

$\implies K \in L^2(\mathbb{R}^d)$, then by the theorem 2.8 $Q_\Omega P_T$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^d)$. Then the Hilbert-Schmidt norm of $Q_\Omega P_T$ is given by

$$\|Q_\Omega P_T\|_{H.S.}^2 = \|K\|_2^2 = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |K(x,t)|^2 \mathrm{d}x\mathrm{d}t.$$

Now since we know that the translation operator T_t and the Fourier transform $\mathcal{F}$ are unitary operator. Now for a fixed t, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} |K(x,t)|^2 \mathrm{d}x &= \int_{\mathbb{R}^d} \left|\chi_T(t) T_t(\mathcal{F}^{-1}\chi_\Omega)(x)\right|^2 \mathrm{d}x. \\
&= \chi_T(t) \int_{\mathbb{R}^d} \left|T_t(\mathcal{F}^{-1}\chi_\Omega)(x)\right|^2 \mathrm{d}x. \\
&= \chi_T(t) \|T_t(\mathcal{F}^{-1}\chi_\Omega)\|_2^2. \\
&= \chi_T(t)\|\chi_\omega\|_2^2. \\
&= |\Omega|\chi_T(t).
\end{aligned}$$

and therefore

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |K(x,t)|^2 \mathrm{d}x\mathrm{d}t = |\Omega||T|. \tag{4.10}$$

Finally by equation 4.9 ,4.10 and the fact that the operator norm $\|Q_\Omega P_T\|_{op}$ is dominated by the Hilbert-Schmidt norm, we obtain

$$\begin{aligned}(1-\epsilon_\Omega-\epsilon_T)\|f\|_2 &\le \|Q_\Omega P_T f\|_2.\\ &\le \|Q_\Omega P_T\|_{op}^2\|f\|_2^2.\\ &\le \|Q_\Omega P_T\|_{H.S.}^2\|f\|_2^2.\\ &= |T||\Omega|\|f\|^2.\\ \implies |T||\Omega| &\ge (1-\epsilon_\Omega-\epsilon_T).\end{aligned}$$

□

Corollary 4.5. *If $f \in L^2(\mathbb{R}^d)$ with $f \neq 0$ and supp $f \subseteq T$ and supp $\hat{f} \subseteq \Omega$, then*

$$|T||\Omega| \ge 1. \tag{4.11}$$

Proof Choose $\epsilon_T = \epsilon_\Omega = 0$ in theorem 4.4

Theorem 4.6. *Suppose that $f \in L^1(\mathbb{R}^d)$, $suppf \subseteq T$, and $supp\hat{f} \subseteq \Omega$. If $|T||\omega| < \infty$, then $f \neq 0$.*

Proof. Since $|T||\omega| < \infty$ and by replacing $f(x)$ by $f(ax)$ for some $a > 0$, we may assume without loss of generality that $|T| < 1$. Now by applying lemma 3.12 to χ_T and χ_Ω, we obtain

$$\int_{[0,1]^d}\sum_{n\in\mathbb{Z}^d}\chi_T(x+n)\mathrm{d}x = \int_{\mathbb{R}^d}\chi_T(x)\mathrm{d}x = |T| < 1. \tag{4.12}$$

and

$$\int_{[0,1]^d}\sum_{n\in\mathbb{Z}^d}\chi_\Omega(\omega+n)\mathrm{d}\omega = \int_{\mathbb{R}^d}\chi_\Omega(x)\mathrm{d}\omega = |\Omega| < \infty. \tag{4.13}$$

Now by equation 4.13 we can observe that the set $\{n \in \mathbb{Z}^d : \hat{f}(\omega+n) \neq 0\}$ is finite for almost all $\omega \in [0,1]^d$.

Let $\omega \in [0,1]^d$ and $n \in \mathbb{Z}^d$ s.t. $\hat{f}(\omega+n) \neq 0$, then $\omega+n \in supp\hat{f} \subseteq \Omega$. Now if no of those $n \in \mathbb{Z}^d$ s.t. $\omega+n \in \Omega$ is infinite then summation in equation 4.13 will diverge and $|\Omega| > \infty$, which is not possible as we are assuming that $|\Omega| < \infty$.

And we can observe by equation 4.12 that on a set $T \subseteq [0,1]^d$ of positive measure, $f(x+n) = 0$ $\forall n \in \mathbb{Z}^d$.

Now consider the periodization of $M_b f$, that is

$$\psi_b(x) = \sum_{n\in\mathbb{Z}^d} M_b f(x+n) = \sum_{n\in\mathbb{Z}^d} f(x+n)e^{2\pi i\langle b,x+n\rangle}. \tag{4.14}$$

we can clearly see that ψ_b is $\mathbb{Z}^d$-periodic, so by the poisson summation formula

$$\psi_b(x) = \sum_{n\in\mathbb{Z}^d} M_b f^{\wedge}(n) e^{2\pi i\langle x,n\rangle} = \sum_{n\in\mathbb{Z}^d} T_{-b}\hat{f}(n) e^{2\pi i\langle x,n\rangle} = \sum_{n\in\mathbb{Z}^d} \hat{f}(n-b) e^{2\pi i\langle n,x\rangle}. \tag{4.15}$$

□

CHAPTER 5

The Short-Time Fourier Transform

The idea of the short-time Fourier tranform is to obtain information about local properties of f, in particular about some "local frequncy spectrum," we restrict f to an interval and take the Fourier transform of this restriction. We choose a smooth cut-off function as a "window."

5.1 Elementary Properties of Short-Time Fourier Transform

Definition 5.1. ***(Short-Time Fourier Transform**[1])* *Let $f, g : \mathbb{R}^d \to \mathbb{C}$ and $g \neq 0$, then the short time Fourier transform (STFT) of f with respect to g is defined as*

$$V_g f(x,\omega) := \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i\langle t,\omega\rangle}\mathrm{d}t. \tag{5.1}$$

Here g is known as window function.

Note : STFT is a sesquilinear map which is linear in first cordinate and conjugate linear in second coordinate.

Proof. Let f_1, f_2 be two complex valued functions defined on $\mathbb{R}^d$ and $\alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned} V_g(\alpha f_1 + \beta f_2)(x,\omega) &= \int_{\mathbb{R}^d} (\alpha f_1 + \beta f_2)(t)\overline{g(t-x)}e^{-2\pi i\langle t,\omega\rangle}\mathrm{d}t. \\ &= \int_{\mathbb{R}^d} (\alpha f_1(t) + \beta f_2(t))\overline{g(t-x)}e^{-2\pi i\langle t,\omega\rangle}\mathrm{d}t. \\ &= \alpha \int_{\mathbb{R}^d} f_1(t)\overline{g(t-x)}e^{-2\pi i\langle t,\omega\rangle}\mathrm{d}t + \beta \int_{\mathbb{R}^d} f_2(t)\overline{g(t-x)}e^{-2\pi i\langle t,\omega\rangle}\mathrm{d}t. \\ &= \alpha V_g f_1(x,\omega) + \beta V_g f_2(x,\omega). \end{aligned}$$

Now let $g_1, g_2 \in \mathbb{C}$ be two complex valued functions defined on $\mathbb{R}^d$.

$$\begin{aligned} V_{(g_1+g_2)}f(x,\omega) &= \int_{\mathbb{R}^d} f(t)\overline{(\alpha g_1 + \beta g_2)}(t-x)e^{-2\pi i\langle t,\omega\rangle}\mathrm{d}t. \\ &= \int_{\mathbb{R}^d} f(t)\overline{(\alpha g_1(t-x) + \beta g_2(t-x))}e^{-2\pi i\langle t,\omega\rangle}\mathrm{d}t. \\ &= \overline{\alpha} \int_{\mathbb{R}^d} f(t)\overline{g_1(t-x)}e^{-2\pi i\langle t,\omega\rangle}\mathrm{d}t + \overline{\beta} \int_{\mathbb{R}^d} f(t)\overline{g_2(t-x)}e^{-2\pi i\langle t,\omega\rangle}\mathrm{d}t. \\ &= \overline{\alpha}V_{g_1}f(x,\omega) + \overline{\beta}V_{g_2}f(x,\omega). \end{aligned}$$

□

Lemma 5.1. *If $f, g \in L^2(\mathbb{R}^d)$, then $V_g f$ is uniformly continuous on $\mathbb{R}^d$.*

Lemma 5.2. *Let $f, g \in L^2(\mathbb{R}^d)$. Then*

$$\begin{aligned} V_g f(x,\omega) &= (f \cdot T_x\overline{g})\,\hat{}\,(\omega). \\ &= \langle f, M_\omega T_x g\rangle. \\ &= \langle \hat{f}, T_\omega M_{-x}\hat{g}\rangle \\ &= e^{-2\pi i\langle x,\omega\rangle}(\hat{f} \cdot T_\omega\overline{\hat{g}})\,\hat{}\,(-x). \\ &= e^{-2\pi i\langle x,\omega\rangle}V_{\hat{g}}\hat{f}(\omega,-x) \\ &= e^{-2\pi i\langle x,\omega\rangle}(f * M_\omega g^*)(x), \text{ where } g^*(x) := \overline{g(-x)} \\ &= (\hat{f} * M_{-x}\hat{g}^*)(\omega). \end{aligned}$$

Definition 5.2. *(**Partial Fourier Transform**) Let f be a function defined on $\mathbb{R}^{2d}$. Then partial Fourier transform in the second variable of f is a function $\mathcal{F}_2 f$ defined on $\mathbb{R}^{2d}$ and is given by*

$$\mathcal{F}_2 f(x,\omega) := \int_{\mathbb{R}^d} f(x,t)e^{-2\pi i\langle t,\omega\rangle}\mathrm{d}t.$$

Definition 5.3. *(**Tensor Product**) Let f, g be two functions defined on $\mathbb{R}^d$. Then tensor product of f and g is a function $f \otimes g$ defined on $\mathbb{R}^d$ given by*

$$f \otimes g(x,t) := f(x) \cdot g(t).$$

Definition 5.4. *(**Asymmetric Coordinate Transform**) Let f be a function defined on $\mathbb{R}^{2d}$. Then asymmetric coordinate transform of f is a function $\mathcal{T}_a f$ defined on $\mathbb{R}^{2d}$ given by*

$$\mathcal{T}_a f(x,t) := f(x, t-x).$$

Lemma 5.3. *If $f, g \in L^2(\mathbb{R}^d)$, then*

$$V_g f = \mathcal{F}_2 \mathcal{T}_a (f \otimes \overline{g}). \tag{5.2}$$

Proof. Let $f, g \in L^2(\mathbb{R}^d)$. Then

$$\begin{aligned} \mathcal{F}_2 \mathcal{T}_a (f \otimes \overline{g})(x,\omega) &= \int_{\mathbb{R}^d} \mathcal{T}_a (f \otimes \overline{g})(x,t) e^{-2\pi i \langle t, \omega \rangle} \mathrm{d}t. \\ &= \int_{\mathbb{R}^d} (f \otimes \overline{g})(x, t-x) e^{-2\pi i \langle t, \omega \rangle} \mathrm{d}t. \\ &= \int_{\mathbb{R}^d} f(x) \overline{g(t-x)} e^{-2\pi i \langle t, \omega \rangle} \mathrm{d}t. \\ &= V_g f(x,\omega). \end{aligned}$$

□

5.2 Orthogonality Relation of STFT

Theorem 5.4. *Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$, then $V_{g_j} f_j \in L^2(\mathbb{R}^{2d})$ for $j = 1, 2$, and*

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle = \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}. \tag{5.3}$$

Proof. First we prove it for a dense subspace of $L^2(\mathbb{R}^d)$.

Let $g_j \in L^1 \cap L^\infty(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$. Then by the following calculation $f_j \cdot T_x \overline{g_j} \in L^2(\mathbb{R}^d)\ \forall x \in \mathbb{R}^d$.

$$\begin{aligned} \int_{\mathbb{R}^d} |f_j(t) \cdot T_x \overline{g_j}(t)|^2 \mathrm{d}t &= \int_{\mathbb{R}^d} |f_j(t)|^2 |T_x g_j(t)|^2 \mathrm{d}t \\ &\leq \|T_x g\|_\infty^2 \int_{\mathbb{R}^d} |f_j(t)|^2 \mathrm{d}t. \\ &= \|T_x g\|_\infty^2 \|f\|_2^2 < \infty. \end{aligned}$$

Since $V_{g_j}f_j(x,\cdot) = (f_j \cdot T_x\overline{g_j})\,\hat{}\in L^2(\mathbb{R}^d)$. Now, using the Parseval's identity, we get

$$\begin{aligned}
\langle V_{g_1}f_1, V_{g_2}f_2\rangle &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V_{g_1}f_1(x,\omega)\overline{V_{g_2}f_2(x,\omega)}\mathrm{d}\omega\mathrm{d}x.\\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f_1T_x\overline{g_1})\hat{}\,(\omega)\overline{(f_2T_x\overline{g_2})\hat{}\,(\omega)}\,\mathrm{d}\omega\,\mathrm{d}x,\\
&= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f_1(t)T_x\overline{g_1}(t)\overline{f_2(t)T_x\overline{g_2}(t)}\,\mathrm{d}t\right)\mathrm{d}x,\\
&= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f_1(t)\overline{f_2(t)}\overline{g_1}(t-x)g_2(t-x)\,\mathrm{d}t\right)\mathrm{d}x.\\
&= \langle f_1, f_2\rangle\overline{\langle g_1, g_2\rangle}.\ \text{(By Fubini's theorem)}
\end{aligned}$$

Now if we fixed $g_2 \in L^1\cap L^\infty(\mathbb{R}^d)$ then the map $T : L^1\cap L^\infty(\mathbb{R}^d) \to \mathbb{C}$ defined as

$$T(g_2) := \langle V_{g_1}f_1, V_{g_2}f_2\rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2\rangle\overline{\langle g_1, g_2\rangle},$$

is a linear functional and by Cauchy-Schwartz inequality

$$|T(g_2)| \leq |\langle f_1, f_2\rangle\overline{\langle g_1, g_2\rangle}| \leq \|f_1\|_2\|f_2\|_2\|g_1\|_2\|g_2\|_2.$$

Now since $L^1\cap L^\infty(\mathbb{R}^d)$ is dense subspace of $L^2(\mathbb{R}^d)$. So T can be extend to $L^2(\mathbb{R}^d)$. That is $\langle V_{g_1}f_1, V_{g_2}f_2\rangle = \langle f_1, f_2\rangle\overline{\langle g_1, g_2\rangle}\ \forall f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$. □

Corollary 5.5. *Let $f, g \in L^2(\mathbb{R}^d)$, then $\|V_gf\|_{L^2(\mathbb{R}^{2d})} = \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}$. In particular if $\|g\|_{L^2(\mathbb{R}^d)} = 1$, then*

$$\|f\|_{L^2(\mathbb{R}^d)} = \|V_gf\|_{L^2(\mathbb{R}^{2d})}, \forall f \in L^2(\mathbb{R}^d).$$

Thus, STFT is an isometry from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$.

Definition 5.5. *(**Vector-Valued integral**)[1] Let $\mathbb{B}$ be a Banach space and $g : \mathbb{R}^d \to \mathbb{B}$ be a function. Then*

$$f = \int_{\mathbb{R}^d} g(x)\mathrm{d}x. \tag{5.4}$$

Means that,

$$\langle f, h\rangle = \int_{\mathbb{R}^d}\langle g(x), h\rangle\mathrm{d}x, \forall h \in B'. \tag{5.5}$$

Remark 5.1. *If the mapping $l : B' \to \mathbb{C}$, given by*

$$l(h) := \int_{\mathbb{R}}\langle g(x), h\rangle\mathrm{d}x,$$

is a bounded conjugate linear function, then l defines a unique element $f \in B''$. In general we can only say that the vector-valued integral is in the bidualB''.

Note : The most important vector-valued integrals in time-frequency analysis are superpositions of time-frequency shifts of the form

$$f = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x,\omega) M_\omega T_x g \mathrm{d}x\mathrm{d}\omega.$$

For example let us consider $F \in L^2(\mathbb{R}^{2d})$, then the conjugate-linear functional

$$l(h) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x,\omega)\overline{\langle h, M_\omega T_x g\rangle}\mathrm{d}x\mathrm{d}\omega. \tag{5.6}$$

By applying Cauchy-Schwartz inequality and by corollary 5.5

$$|l(h)| \leq \|F\|_2\|V_g h\|_2 = \|F\|_2\|g\|_2\|h\|_2. \tag{5.7}$$

This means that l defines a unique function $f = \int_{\mathbb{R}^{2d}} F(x,\omega)M_\omega T_x g \mathrm{d}(x,\omega) \in L^2(\mathbb{R}^d)$ with the norm $\|f\|_2 \leq \|F\|_2\|g\|_2$ and satisfying $l(h) = \langle f, h\rangle$.

Corollary 5.6. ***(Inversion Formula for the STFT)*** *Suppose that $g, \gamma \in L^2(\mathbb{R}^d)$ and $\langle g, \gamma\rangle \neq 0$. Then $\forall f \in L^2(\mathbb{R}^d)$, we have*

$$f = \frac{1}{\langle \gamma, g\rangle}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V_g f(x,\omega)M_\omega T_x\gamma \ \mathrm{d}\omega\mathrm{d}x. \tag{5.8}$$

Proof. Since $V_g f \in L^2(\mathbb{R}^{2d})$ by the corollary 5.5, the vector-valued integral

$$\tilde{f} = \frac{1}{\langle \gamma, g\rangle}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V_g f(x,\omega)M_\omega T_x\gamma \, dx\, d\omega$$

is a well-defined function in $L^2(\mathbb{R}^d)$. Consider inner the product

$$\langle \tilde{f}, h\rangle = \frac{1}{\langle \gamma, g\rangle}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V_g f(x,\omega)\overline{\langle h, M_\omega T_x\gamma\rangle}\, dx\, d\omega$$
$$= \frac{1}{\langle \gamma, g\rangle}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V_g f(x,\omega)\overline{V_\gamma h}(x.\omega)\, dx\, d\omega.$$

Thus $\tilde{f} = f$, and the inversion formula is proved. □

Theorem 5.7. *Fix $g, \gamma \in L^2(\mathbb{R}^d)$ with $\langle \gamma, g\rangle \neq 0$ and let $K_n \subseteq \mathbb{R}^{2d}$ be a nested exhausting sequence of compact sets. Define*

$$f_n := \frac{1}{\langle \gamma, g\rangle}\int_{K_n} V_g f(x,\omega)M_\omega T_x\gamma \ \mathrm{d}(x,\omega). \tag{5.9}$$

Then $\lim_{n\to\infty}\|f - f_n\|_2 = 0$.

Proof. By Cauchy-Schwartz inequality and by corollary5.5, we have the following estimate for $h \in L^2(\mathbb{R}^d)$.

$$\begin{aligned}
\langle f_n, h\rangle &= \frac{1}{\langle \gamma, g\rangle} \int_{K_n} V_g f(x,\omega)\overline{\langle h, M_\omega T_x \gamma\rangle}\ \mathrm{d}(x,\omega) \\
\implies |\langle f_n, h\rangle| &= \frac{1}{|\langle \gamma, g\rangle|} |\langle V_g f, V_\gamma h\rangle| \\
&\leq \frac{1}{|\langle \gamma, g\rangle|} \|V_g f\|_{L^2(K_n)} \|V_\gamma h\|_{L^2(K_n)}. \\
&= \|f\|_2 \|g\|_2 \|h\|_2 \|\gamma\|_2
\end{aligned}$$

Thus, $|\langle f_n, h\rangle|$ is bounded, and f_n is well defined for each n. Furthermore, $\|f_n\| \leq |\langle \gamma, g\rangle^{-1} \|g\|_2 \|f\|_2 \|f\|_2$. Now consider,

$$\begin{aligned}
\langle f - f_n, h\rangle &= \frac{1}{|\langle \gamma, g\rangle|} \left[\left(\int_{\mathbb{R}^{2d}} - \int_{K_n} \right) V_g f(x,\omega) \overline{V_\gamma h(x,\omega)}\ \ \mathrm{d}(x,\omega) \right] \\
&= \frac{1}{|\langle \gamma, g\rangle|} \left[\left(\int_{\mathbb{R}^{2d}} - \int_{K_n} \right) V_g f(x,\omega) \overline{\langle h, M_\omega T_x \gamma\rangle}\ \ \mathrm{d}(x,\omega) \right] \\
&= \frac{1}{|\langle \gamma, g\rangle|} \int_{K_n^c} V_g f(x,\omega) \overline{\langle h, M_\omega T_x \gamma\rangle}\ \ \mathrm{d}(x,\omega). \\
\therefore |\langle f - f_n, h\rangle| &= \frac{1}{|\langle \gamma, g\rangle|} |\langle V_g f, V_\gamma h\rangle|, \\
\implies |\langle f - f_n, h\rangle| &\leq \frac{1}{|\langle \gamma, g\rangle|} \|\gamma\| \|h\| \int_{K_n^c} |V_g f(x,\omega)|^2\ \mathrm{d}(x,\omega) \\
\implies \|f_n - f\| &\leq \frac{1}{|\langle \gamma, g\rangle|} \|\gamma\| \int_{K_n^c} |V_g f(x,\omega)|^2\ \mathrm{d}(x,\omega).
\end{aligned}$$

Since K_n is a nested exhausting sequence of sets, we have $\lim\limits_{n\to\infty} \|f_n - f\|_2 = 0$. □

5.3 Lieb's Uncertainty Principle

We saw how the time-frequency resolution of the STFT depends on the choice of the window function g. In particular, the time-frequency resolution of $V_g f$ is limited by the size of the essential supports of g and $\hat{g}$. The calssical uncertainty principle (Theorem 4.2) for g thus implies an uncertainty principle for $V_g f$. In this section we present uncertainty principles that directly apply to the STFT.

Theorem 5.8 (**Weak Uncertainty Principle for the STFT**). *Suppose that* $\|f\|_2 = \|g\|_2 = 1$ *and that* $\mathbb{U} \subseteq \mathbb{R}^{2d}$ *and* $\epsilon \geq 0$ *are such that*

$$\int_{\mathbb{U}} |V_g f(x,\omega)|^2\ \mathrm{d}(x,\omega) \geq 1 - \epsilon. \tag{5.10}$$

then $|\mathbb{U}| \geq 1 - \epsilon$.

Proof. By Cauchy-Schwartz inequality

$$|V_g f(x,\omega)| = |\langle f, M_\omega T_x g\rangle| \le \|f\|_2\|g\|_2 = 1, \forall (x,\omega) \in \mathbb{R}^{2d}.$$

$\implies V_g f \in L^\infty(\mathbb{R}^{2d})$, with $\|V_g f\|_\infty \le \|f\|_2\|g\|_2 = 1$.

Therefore,

$$1-\epsilon \le \int_{\mathbb{U}} |V_g f(x,\omega)|^2 \, \mathrm{d}(x,\omega) \le \|V_g f\|_\infty^2 |U| \le |U|.$$

□

Theorem 5.9. *If $f, g \in L^2(\mathbb{R}^d)$ and $2 \le p < \infty$, then*

$$\int_{\mathbb{R}^{2d}} |V_g f(x,\omega)|^2 \, \mathrm{d}(x,\omega) \le \left(\frac{2}{p}\right)^d (\|f\|_2\|g\|_2)^p. \tag{5.11}$$

Proof. Let p' be the conjugate of p i.e. $\frac{1}{p'} + \frac{1}{p} = 1$. Since $2 \le p < \infty$, we have $1 < p' \le 2$. Note that by the Holder's inequality $f \cdot T_x\overline{g} \in L^1(\mathbb{R}^d)$. And since by the theorem 5.4 we have $V_g f(x,\omega) = (f \cdot T_x\overline{g})\hat{}\,(\omega) \in L^2(\mathbb{R}^{2d})$, then $f \cdot T_x\overline{g} \in L^1 \cap L^2(\mathbb{R}^d)$. for almost all $x \in \mathbb{R}^d$ which implies that $f \cdot T_x\overline{g} \in L^{p'}(\mathbb{R}^d)$ for a.e. $x \in \mathbb{R}^d$. Now consider,

$$\left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p \, \mathrm{d}\omega\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^d} |(f \cdot T_x\overline{g})\hat{}\,(\omega)|^p \, \mathrm{d}\omega\right)^{\frac{1}{p}}$$

By Hausdorff-Young inequality

$$\le A_{p'}^d \left(\int_{\mathbb{R}^d} |f \cdot T_x\overline{g}(y)|^{p'} \mathrm{d}y\right)^{\frac{1}{p'}}.$$

$$\le A_{p'}^d \left(|f|^{p'} * |g^*|^{p'}(x)\right)^{\frac{1}{p'}}.$$

$$\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p \mathrm{d}x\mathrm{d}\omega\right)^{\frac{1}{p}} \le A_{p'}^d \left(\int_{\mathbb{R}^d} (|f|^{p'} * |g^*|^{p'}(x))^{\frac{p}{p'}} \mathrm{d}x\right)^{\frac{1}{p}}$$

Since $f, g \in L^2(\mathbb{R}^d), |f|^{p'}, |g|^{p'} \in L^{\frac{2}{p'}}(\mathbb{R}^d)$

$$\text{take } s = \frac{2}{p'}, r = \frac{p}{p'} \implies \frac{1}{s} + \frac{1}{s} = 1 + \frac{1}{r}.$$

□

Theorem 5.10. *Suppose that $\|f\|_2 = \|g\|_2 = 1$. If $\mathbb{U} \subseteq \mathbb{R}^{2d}$ and $\epsilon \ge 0$ are such that*

$$\int_{\mathbb{U}} |V_g f(x,\omega)|^2 \mathrm{d}(x,\omega) \ge 1-\epsilon, \tag{5.12}$$

then

$$|\mathbb{U}| \ge (1-\epsilon)^{\frac{p}{p-2}} \left(\frac{p}{2}\right)^{\frac{2d}{p-2}} \ge (1-\epsilon)^2 2d \quad \forall p \ge 2.$$

Proof. By Holder's inequality and by theorem 5.9

$$\int_{\mathbb{U}} |V_g f(x,\omega)|^2 \mathrm{d}(x,\omega) \leq \left(\int_{\mathbb{U}} |V_g f(x,\omega)|^{2\cdot\frac{p}{2}} \mathrm{d}(x,\omega)\right)^{\frac{2}{p}} \left(\int_{\mathbb{U}} |\chi_{\mathbb{U}}(x,\omega)|^{\frac{p}{p-2}} \mathrm{d}(x,\omega)\right)^{\frac{p-2}{p}}$$
$$\leq \left(\frac{2}{p}\right)^{\frac{2d}{p}} (\|f\|_2\|g\|_2)^2 |U|^{\frac{p-2}{p}}.$$

Since $\int_{\mathbb{U}} |V_g f(x,\omega)|^2 \mathrm{d}(x,\omega) \geq 1-\epsilon$, then

$$1-\epsilon \leq \left(\frac{2}{p}\right)^{\frac{2d}{p}} (\|f\|_2\|g\|_2)^2 |U|^{\frac{p-2}{p}}.$$
$$|U| \geq (1-\epsilon)^{\frac{p}{p-2}} \left(\frac{p}{2}\right)^{\frac{2d}{p-2}}.$$

Take $p=4$, then above inequality becomes

$$|U| \geq 2^d(1-\epsilon)^2.$$

□

5.4 The Bargmann Transform

Since we know Gaussian functions minimize the uncertainty principle, it is of special case to study the STFT with respect to a Gaussian window. Let $\psi(x) = 2^{\frac{d}{2}} e^{-\pi\|x\|^2}$ be the Gaussian on $\mathbb{R}^d$ s.t. $\|\psi\|_2 = 1$. Then

$$V_\varphi f(x,\omega) = 2^{d/4} \int_{\mathbb{R}^d} f(t)\, e^{-\pi\|t-x\|^2} e^{-2\pi i\langle \omega t\rangle} \mathrm{d}t$$
$$= 2^{d/4} \int_{\mathbb{R}^d} f(t)\, e^{-\pi\|t\|^2} e^{2\pi\langle x,t\rangle} e^{-\pi\|x\|^2} e^{-2\pi i\langle \omega,t\rangle} \mathrm{d}t$$
$$= 2^{d/4} e^{-\pi i\langle x,\omega\rangle} e^{-\frac{\pi}{2}(\|x\|^2+\|\omega\|^2)} \int_{\mathbb{R}^d} f(t)\, e^{-\pi t^2} e^{2\pi\langle t,x\rangle} e^{2\pi i\langle t,\omega\rangle)} e^{-\frac{\pi}{2}(x-i\omega)^2} \mathrm{d}t.$$

Let us convert $(x,\omega) \in \mathbb{R}^{2d}$ into a complex vector $z = x + i\omega \in \mathbb{C}^d$. We will keep the notation consistent with $\mathbb{R}^d$, that is, we will write $z^2 = (x+i\omega)\cdot(x+i\omega)$ and $|z|^2 = z\cdot\bar{z} = (x+i\omega)\cdot(x-i\omega) = x^2+\omega^2$. Further, dz denotes the Lebesgue measure on $\mathbb{C}^d$.Then

$$V_\varphi f(x,\omega) = 2^{d/4} e^{-\pi i\langle x,\omega\rangle} e^{-\frac{\pi}{2}(\|x\|^2+\|\omega\|^2)} \int_{\mathbb{R}^d} f(t)\, e^{-\pi t^2} e^{2\pi\langle t,x\rangle} e^{2\pi i\langle t,\omega\rangle)} e^{-\frac{\pi}{2}(x-i\omega)^2} \mathrm{d}t. \tag{5.13}$$

Definition 5.6 (**Bargmann-Fock Space**[1],[10])**.** *Bargmann-Fock space denoted by $\mathcal{F}^2(\mathbb{C}^d)$ is collection of all entire function F that satisfy*

$$\int_{\mathbb{C}^d} |F(z)|^2 e^{-\pi|z|^2} \mathrm{d}z < \infty. \tag{5.14}$$

- $\mathcal{F}^2(\mathbb{C}^d)$ is a Hilbert space w.r.t. the inner product given by

$$\langle F, G\rangle_{\mathcal{F}} = \int_{\mathbb{C}^d} F(z)\overline{G(z)} e^{-\pi|z|^2} \mathrm{d}z.$$

- The norm on $\mathcal{F}^2(\mathbb{C}^d)$ corresponding to the inner product is given by

$$\|F\|_{\mathcal{F}}^2 = \int_{\mathbb{C}^d} |F(z)|^2 e^{-\pi|z|^2} \mathrm{d}z.$$

Definition 5.7. *(**Bargman transform**[1],[10]) The Bergmann transform a function f defined on $\mathbb{R}^d$ is a function Bf on $\mathbb{R}^d$, and it is given by*

$$Bf(z) := 2^{\frac{d}{4}} \int_{\mathbb{R}^d} f(t) e^{2\pi t\cdot z - \pi t^2 - \frac{\pi}{2} z^2} \mathrm{d}t. \tag{5.15}$$

We can observe that we can obtain equation 5.15 by substitution $z = x + i\omega$ in equation 5.13.

Theorem 5.11. *1. If we write $z = x + i\omega$, then*

$$V_\psi f(x, -\omega) = e^{\pi i x\cdot\omega} Bf(z) e^{-\pi\frac{|z|^2}{2}}. \tag{5.16}$$

2. If $f \in L^2(\mathbb{R}^d)$, then

$$\|f\|_2 = \left(\int_{\mathbb{C}^d} |f(z)|^2 e^{-\pi|z|^2} \mathrm{d}z\right)^{\frac{1}{2}} = \|Bf\|_{\mathcal{F}}. \tag{5.17}$$

Theorem 5.12. *The collection of all monomials of the form*

$$e_\alpha(z) = \left(\frac{\pi^{|\alpha|}}{\alpha!}\right)^{\frac{1}{2}} z^\alpha = \prod_{j=1}^{d} \left(\frac{\pi^{\alpha_j}}{\alpha_j!}\right)^{\frac{1}{2}} z_j^{\alpha_j},$$

for $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$ with $\alpha_j \in \mathbb{Z}, \alpha_j \geq 0$, forms an orthonormal basis for $\mathcal{F}^2(\mathbb{C}^d)$.

Proof. Let $z = (z_1, z_2, \cdot, \cdot, \cdot, z_d)$. Now consider each variable $z_j = r_j e^{i\theta_j}$. First let us compute the inner product of z^α and z^β restricted to the disk

$P_R = \{z = (z_1, z_2, \cdot, \cdot\cdot, z_d) \in \mathbb{C}^d : |z_j| \leq R\}$.

$$\langle z^\alpha, z^\beta\rangle = \int_{\mathbb{C}^d} z^\alpha \overline{z^\beta} e^{-\pi|z|^2} \mathrm{d}z.$$

If $\alpha \neq \beta$, then

$$\begin{aligned}
\langle z^\alpha, z^\beta\rangle_{\mathcal{F}} &= \lim_{R\to 0} \int_{P_R} z^\alpha \overline{z^\beta} e^{-\pi|z|^2} \mathrm{d}z. \\
&= \prod_{j=1}^{d} \int_{|z_j|\leq R} z_j^\alpha \overline{z_j^\beta} e^{-\pi|z_j|^2} \mathrm{d}z_j. \\
&= \prod_{j=1}^{d} \int_0^{2\pi} \int_0^R r_j^\alpha r_J^\beta e^{i\alpha\theta_j} e^{i\beta\theta_j} \mathrm{d}r_j \mathrm{d}\theta_j. \\
&= \prod_{j=1}^{d} \int_0^{2\pi} \int_0^R r_j^{\alpha+\beta} e^{i(\alpha+\beta)\theta_j} \mathrm{d}r_j \mathrm{d}\theta_j. \\
&= \prod_{j=1}^{d} \frac{R^{(\alpha+\beta+1)}}{\alpha+\beta+1} \int_0^{2\pi} e^{i(\alpha+\beta)\theta_j} \mathrm{d}\theta_j = 0
\end{aligned}$$

If $\alpha = \beta$ then,

$$\langle z^\alpha, z^\alpha \rangle_{\mathcal{F}} = \|z^\alpha\|_{\mathcal{F}} = \int_{P_R} |z^\alpha|^2 e^{-\pi|z|^2} \mathrm{d}z = \prod_{j=1}^{d} \left(2\pi \int_0^R r_j^{2\alpha_j+1} e^{-\pi r_j^2} \mathrm{d}r_j \right) = \mu\alpha, R.$$

For $R = \infty$, by making the change of variables $s_j = \pi r_2^2$, then $\mathrm{d}s_j = 2\pi r \mathrm{d}r$, we can continue as follows :

$$\mu\alpha, \infty = \prod_{j=1}^{d} \left(\int_0^\infty \left(\frac{s}{\pi} \right)^{\alpha_j} e^{-s} \mathrm{d}s \right) = \prod_{j=1}^{d} \frac{\Gamma(\alpha_j + 1)}{\pi^{\alpha_j}} = \prod_{j=1}^{d} \frac{\alpha_j i}{\pi^{\alpha_j}} = \frac{\alpha i}{\pi^{|\alpha|}} \tag{5.18}$$

Consequently $\{\mu_{\alpha,R}^{-\frac{1}{2}} : \alpha \geq 0\}$ is an orthonormal system in $L^2(P_R, e^{-\pi|z|^2}\mathrm{d}z)$. In particular $\{e_\alpha, \alpha \geq 0\}$ is an orthonormal system in $\mathcal{F}^2(\mathbb{C}^d)$. Now to show that $\{e_\alpha, \alpha \geq 0\}$ is an orthonormal basis for $\mathcal{F}^2(\mathbb{C}^d)$ we show that it is a maximal orthonormal set for $\mathcal{F}^2(\mathbb{C}^d)$. Now let $F \in \mathcal{F}^2(\mathbb{C}^d)$ then since F is analytic function we can express it as a power series expansion i.e

$$F(z) = \sum_{\alpha \geq 0} c_\alpha \cdot z^\alpha.$$

Now Let us assume that F is also orthogonal to all z^α i.e.$\langle F, e_\beta \rangle = 0$ for all $\beta \geq 0$. Then

$$\langle F, e_\beta \rangle = \lim_{R\to\infty} \left(\frac{\pi^{|\beta|}}{\beta!} \right)^{\frac{1}{2}} \int_{P_R} \left(\sum_{\alpha \geq 0} c_\alpha \cdot z^\alpha \right) \overline{z^\beta} e^{-\pi|z|^2} \mathrm{d}z$$

Since the power series inside the integral is absolutely convergent on a compact set So we can interchange the sum and integral.

$$= \sum_{\alpha \geq 0} c_\alpha \int_{P_R} z^\alpha \overline{z^\beta} e^{-\pi|z|^2} \mathrm{d}z = c_\beta \mu_{\beta,R}$$

Thus $\langle \mathcal{F}, e_\beta \rangle \mathcal{F} = \left(\frac{\pi^{|\beta|}}{\beta!} \right)^{\frac{1}{2}} \lim_{R\to\infty} c_\beta \mu_{\beta,R} = 0$. This implies that $c_\beta = 0\ \forall\beta$ and thus $F = 0$. So $\{e_\alpha, \alpha \geq 0\}$ is the maximal orthonormal set of $\mathcal{F}^2(\mathbb{C}^d)$. Thus it is an orthonormal basis for $\mathcal{F}^2(\mathbb{C}^d)$. ☐

Theorem 5.13. $\mathcal{F}^2(\mathbb{C}^d)$ *is a reproducing kernel Hilbert space, i.e.*

$$|F(z)| \leq \|F\|_{\mathcal{F}} e^{\pi \frac{|z|^2}{2}}, \forall z \in \mathbb{C}^d.$$

The reproducing kernel is $K_\omega(z) = e^{\pi\overline{\omega}z}; F(\omega) = \langle F, K_\omega \rangle$.

Proof. By theorem 5.12 we can express $F \in \mathcal{F}^2(\mathbb{C}^d)$ as $F(z) = \sum_{\alpha\geq 0} \langle F, e_\alpha \rangle_{\mathcal{F}} e_\alpha(z)$. Now by using Cauchy-Schwartz inequality

$$|F(z)| \leq \left(\sum_{\alpha \geq 0} |\langle F, e_\alpha \rangle_{\mathcal{F}}|^2 \right)^{\frac{1}{2}} \left(\sum_{\alpha \geq 0} \frac{\pi^\alpha}{\alpha!} |z^\alpha|^2 \right)^{\frac{1}{2}} = \|F\|_{\mathcal{F}} \cdot e^{\pi \frac{|z|^2}{2}}. \tag{5.19}$$

Now consider the map $T_\omega : \mathcal{F}^2(\mathbb{C}^d) \to \mathbb{C}$, defined as

$$T_\omega(f) := f(\omega), \tag{5.20}$$

then T_ω is a linear map by equation 5.19 T_ω is bounded for each $\omega \in \mathbb{C}^d$. So by Reize representation theorem $\exists K_\omega \in \mathcal{F}^2(\mathbb{C}^d)$ s.t.

$$T_\omega(f) = f(\omega) = \langle f, K_\omega \rangle.$$

Now since $K_\omega \in \mathcal{F}^2(\mathbb{C}^d)$, again by 5.19

$$\begin{aligned} K_\omega(z) &= \sum_{\alpha \geq 0} \langle K_\omega, e_\alpha \rangle e_\alpha(z). \\ &= \sum_{\alpha \geq 0} \overline{e_\alpha(\omega)} e_\alpha(z). \\ &= \sum_{\alpha \geq 0} \frac{\pi^{|\alpha|}}{\alpha!} \omega^\alpha \cdot z^\alpha \\ &= e^{\pi \langle z, \omega \rangle} \end{aligned}$$

□

Theorem 5.14. *The Bargmann transform is a unitary operator from $L^2(\mathbb{R}^d)$ onto $\mathcal{F}^2(\mathbb{C}^d)$*

Proof. We have already seen in theorem 5.11 that B is an isometry. Thus its range is a closed subspace of $\mathcal{F}^2(\mathbb{C}^d)$. Therefore, if we show that $B(L^2(\mathbb{R}^d))$ is dense in $\mathcal{F}^2(\mathbb{C}^d)$, then it follows that $B(L^2(\mathbb{R}^d)) = \mathcal{F}^2(\mathbb{C}^d)$, so B is surjective and the proof is complete.

We start by rewriting Lemma in terms of the Bargmann transform. Using Lemma 1.5.2 with $a = 1$ and taking the normalization of φ into account, we compute

$$\begin{aligned} V_\varphi(T_u M_{-\eta}\varphi)(x, -\omega) &= \langle T_u M_{-\eta}\varphi, M_{-\omega} T_x \varphi \rangle \\ &= e^{2\pi i x \cdot \omega} \langle T_u M_{-\eta}\varphi, T_x M_{-\omega}\varphi \rangle \\ &= e^{\pi i (\eta - \omega)\cdot(-\eta - \omega)} e^{2\pi i x \cdot \omega} e^{-\pi[(x-u)^2 + (\eta-\omega)^2]/2}. \end{aligned}$$

On the other hand, writing $z = x + i\omega$ and $w = u + i\eta$, we obtain after some bookkeeping in the exponents and after applying theorem 5.11 that

$$\begin{aligned} B(T_u M_{-\eta}\varphi)(z) &= e^{-\pi i x \cdot \omega} e^{\pi |z|^2/2} V_\varphi(T_u M_{-\eta}\varphi)(x, -\omega) \\ &= e^{\pi i u \cdot \eta} e^{-\pi(u^2 + \eta^2)/2} e^{\pi(x \cdot u + \eta \cdot \omega + i(\omega \cdot u - x \cdot \eta))} \\ &= e^{\pi i u \cdot \eta} e^{-\pi |w|^2/2} e^{\pi \bar{w} \cdot z}. \end{aligned}$$

We can rewrite this in short as

$$B(T_u M_{-\eta}\varphi)(z) = e^{\pi i u\cdot\eta} e^{-\pi|w|^2/2} K_w(z). \tag{5.21}$$

This shows that the reproducing kernel of $\mathcal{F}^2(\mathbb{C}^d)$ is in the range of B. Now suppose that for some $F \in \mathcal{F}^2(\mathbb{C}^d)$ we have $\langle F, Bf\rangle_{\mathcal{F}} = 0$ for all $f \in L^2(\mathbb{R}^d)$. In particular, using equation 5.21 we have for all $w \in \mathbb{C}^d$ that

$$\begin{aligned} 0 &= \langle F, B(T_u M_{-\eta}\varphi)\rangle \\ &= e^{-\pi i u\cdot\eta} e^{-\pi|w|^2/2}\langle F, K_w\rangle \\ &= e^{-\pi i u\cdot\eta} e^{-\pi|w|^2/2} F(w). \end{aligned}$$

Therefore $F \equiv 0$, and consequently the range of B is dense in $\mathcal{F}^2(\mathbb{C}^d)$. □

Definition 5.8 (Hermite Function). *Let $B : L^2(\mathbb{R}^d) \to \mathcal{F}^2(\mathbb{C}^d)$ be a Bargmann transform and $\{e_\alpha\}$ be the orthonormal basis of monomials in $\mathcal{F}^2(\mathbb{R}^d)$ Then $H = B^{-1}e_\alpha$ is known as Hermite function.*
Since Bargamann transform is unitary map from $L^2(\mathbb{R}^d)$ to $\mathcal{F}^2(\mathbb{R}^d)$ $\{H_\alpha\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$.

Proposition 5.15. *The Hermite functions are eigenfunctions of the Fourier transform, particularly, $\forall \alpha \geq 0$, we have*

$$\hat{H}_\alpha = (-i^{|\alpha|})H_\alpha.$$

Proof. By theorem 5.11 writing $z = x + i\omega \in \mathbb{C}^d$, we have and using the property that $\psi = \hat{\psi}$.

$$\begin{aligned} V_\varphi H_\alpha(x, -\omega) &= e^{\pi i x\cdot\omega} BH_\alpha(z) e^{-\pi|z|^2/2} \\ &= e^{\pi i x\cdot\omega} e^{-\pi|z|^2/2} e_\alpha. \\ &= e^{2\pi i x\cdot\omega} V_\varphi H_\alpha(\omega, x) \\ &= e^{2\pi i x\cdot\omega} e^{-\pi i x\cdot\omega} BH_\alpha(\omega - ix) e^{-\pi|z|^2/2} \\ &= e^{\pi i x\cdot\omega} e^{-\pi|z|^2/2} \left(\frac{\pi^{|\alpha|}}{\alpha!}\right)^{1/2} (\omega - ix)^\alpha \\ &= e^{\pi i x\cdot\omega} e^{-\pi|z|^2/2} \left(\frac{\pi^{|\alpha|}}{\alpha!}\right)^{1/2} (-iz)^\alpha \\ &= (-i)^{|\alpha|} e^{\pi i x\cdot\omega} e^{-\pi|z|^2/2} e_\alpha(z). \end{aligned}$$

Since V_φ is one-to-one, it follows that $\widehat{H}\alpha = (-i)^{|\alpha|} H\alpha$.

□

CHAPTER 6

Quadratic Time-Frequency Representations

6.1 Introduction

In this section we will discuss about some quadratic forms. Till now we have seen short time Fourier transform, which is a sesquilinear form. We know that STFT of a function depends on the window function. To avoid this ambiguity we will see more general time-frequency representations.

6.2 Spectrogram

Definition 6.1. *(**Spectrogram**[1]) Let $g \in L^2(\mathbb{R}^d)$ be s.t. $\|g\|_2 = 1$ and f be a function defined on $\mathbb{R}^d$. The spectrogram of f w.r.t window function g is a function $SPEC_g f$ defined on $\mathbb{R}^{2d}$ and given by*

$$SPEC_g f(x, \omega) := |V_g f(x, \omega)|^2, \forall x, \omega \in \mathbb{R}^{2d}. \tag{6.1}$$

- $SPEC$ is a quadratic form.
- $SPEC_g f(x, \omega) \geq 0, \forall x, \omega \in \mathbb{R}^d$. (**Non-negativity**)
- $SPEC_g(T_\mu M_\eta f)(x, \omega) = SPEC_g f(x-\mu, \omega-\eta) \forall x, \omega, \mu, \eta \in \mathbb{R}^d$. By Lemma (**Covariant**)

- If $f \in L^2(\mathbb{R}^d)$ then $SPEC_g f \in L^1(\mathbb{R}^d)$ and

$$\|SPEC_g f\|_1 = \|V_g f\|_2^2 = \|f\|_2^2 \|g\|_2^2. \qquad \textbf{(Energy-preserving)}$$

The interpretation of spectrograme is slightly ambiguous because it depends on window function. For this reason spectrogram sometimes discarded as genuinely quadratic time-frequency representations.

6.3 The Ambiguity Function

Definition 6.2. *(**The cross-ambiguity function**[1]) Let $f, g \in L^2(\mathbb{R}^d)$, the cross-ambiguity function of f and g is a function $A(f,g)$ defined on $\mathbb{R}^{2d}$ and given by*

$$A(f,g)(x,\omega) := \int_{\mathbb{R}^d} f(x+t/2)\overline{g(x-t/2)}e^{-2\pi i\langle t,\omega\rangle}\mathrm{d}t. \tag{6.2}$$

If $f = g$ then we write $A(f,f) = Af$ and Af is known as ambiguity function of f.

- $A(f,g)(x,\omega) = e^{\pi i\langle x,\omega\rangle}V_g f(x,\omega) \forall x,\omega \in \mathbb{R}^d$.

  Proof.

$$A(f,g)(x,\omega) := \int_{\mathbb{R}^d} f(x+t/2)\overline{g(x-t/2)}e^{-2\pi i\langle x,\omega\rangle}\mathrm{d}t.$$

  Substituting $t \to u$

$$= e^{\pi i\langle x,\omega\rangle}\int_{\mathbb{R}^d} f(u)\overline{g(u-x)}e^{-2\pi i\langle u,\omega\rangle}\mathrm{d}u.$$

$$= e^{\pi i\langle x,\omega\rangle}V_g f(x,\omega).$$

  □

- $A(f,g) \in L^2(\mathbb{R}^d) \forall f,g \in L^2(\mathbb{R}^d)$ and

$$\|A(f,g)\|_2^2 = \|f\|_2^2\|g\|_2^2 \forall f,g \in L^2(\mathbb{R}^d).$$

  Proof.

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |A(f,g)(x,\omega)|^2 \mathrm{d}x\mathrm{d}\omega := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |e^{\pi i\langle x,\omega\rangle}V_g f(x,\omega)|^2\mathrm{d}x\mathrm{d}\omega.$$

$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |V_g f(x,\omega)|^2\mathrm{d}x\mathrm{d}\omega.$$

$$= \|f\|_2^2\|g\|_2^2. \text{ (By orthogonality relation of STFT)}$$

  □

Note : $Af(x,\omega) = A(c.f)(x,\omega) = \int_{\mathbb{R}^d} |c|^2 f(t+x/2)\overline{f(t-x/2)} e^{-2\pi i\langle x,\omega\rangle} \mathrm{d}t$ when $|c|=1$. The ambiguity function Af determines f only upto phase factor.

Theorem 6.1. ***(Inversion Formula for Ambiguity Function)*** *Let $f \in L^2(\mathbb{R}^d)$ s.t. $f(0) \neq 0$ and Af it's ambiguity function then*

$$f(x) = \frac{1}{\overline{f(0)}} \int_{\mathbb{R}^d} Af(x,\omega) e^{\pi i\langle x,\omega\rangle} \mathrm{d}\omega. \tag{6.3}$$

And the other solutions of above integral equation are cf when $|c| = 1$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^d)$, now observe that for a fixed $x \in \mathbb{R}^d$, $Af(x,\cdot)$ is Fourier transform of g where $g(t) := f(x+t/2)\overline{f(x-t/2)}$.
Now by the inversion formula of Fourier transform

$$g(t) = f(x+t/2)\overline{f(x-t/2)} = \int_{\mathbb{R}^d} Af(x,\omega) e^{2\pi i\langle t,\omega\rangle} \mathrm{d}\omega.$$

put $x = t/2$ then

$$f(x) = \frac{1}{\overline{f(0)}} \int_{\mathbb{R}^d} Af(x,\omega) e^{\pi i\langle x,\omega\rangle} \mathrm{d}\omega.$$

□

Note : Let $f \in L^2(\mathbb{R}^d)$ then $Af(0,0) = \int_{\mathbb{R}^d} |f(t)|^2 \mathrm{d}t = \|f\|_2^2$.
Suppose that $f \in L^2(\mathbb{R}^d)$ and $f \neq 0$. Then $|Af(x,\omega)| < \|f\|_2^2, \forall (x,\omega) \neq (0,0)$.

Proof. We know that

$$\begin{aligned} |Af(x,\omega)| &= |V_f f(x,\omega)| = |\langle f, M_\omega T_x f\rangle| \\ &\leq \|f\|_2 \|M_\omega T_x f\|_2 = \|f\|_2^2. \text{ (By Cauchy-Schwartz inequality)} \end{aligned}$$

and equality holds when $M_\omega T_x f = cf$ for some $(x,\omega) \in \mathbb{R}^{2d}$ and $|c| = 1$.
Case 1: When $x \neq 0$ then $T_x|f| =$

6.4 The Wigner Distribution

Definition(The Cross Wigner Distribution)[1] Let $f, g \in L^2(\mathbb{R}^d)$,Then the cross-Wigner distribution of f and g is a function $W(f,g)$ defined on $\mathbb{R}^{2d}$ and given by

$$W(f,g)(x,\omega) := \int_{\mathbb{R}^d} f(x+t/2)\overline{g(x-t/2)} e^{-2\pi i\langle t,\omega\rangle} \mathrm{d}t. \tag{6.4}$$

If $f = g$ then we write $W(f,f) = Wf$ and Wf is known as Wigner distribution of f. □

Lemma 6.2. *For all $f, g \in L^2(\mathbb{R}^d)$,*

$$W(f,g)(x,\omega) = 2^d e^{4\pi i\langle x,\omega\rangle} V_{\mathcal{I}g} f(2x, 2\omega), \tag{6.5}$$

where $\mathcal{I}g(x) = g(-x) \forall x \in \mathbb{R}^d$ is the reflection operator.

Proof.

$$W(f,g)(x,\omega) = \int_{\mathbb{R}^d} f(x + t/2)\overline{g(x - t/2)} e^{-2\pi i\langle t,\omega\rangle} \mathrm{d}t.$$

Substitute $x + t/2 \to u$ then $\mathrm{d}t = 2^d \mathrm{d}u$

$$\begin{aligned}
&= 2^d \int_{\mathbb{R}^d} f(u)\overline{g(2x-u)} e^{-2\pi i\langle \omega, 2u-2x\rangle} \mathrm{d}u. \\
&= 2^d \int_{\mathbb{R}^d} f(u)\overline{g(2x-u)} e^{4\pi i\langle x,\omega\rangle} e^{-2\pi i\langle 2\omega, u\rangle} \mathrm{d}u. \\
&= 2^d e^{4\pi i\langle x,\omega\rangle} \int_{\mathbb{R}^d} f(u)\overline{\mathcal{I}_g(u-2x)} e^{-2\pi i\langle 2\omega, u\rangle} \mathrm{d}u. \\
&= 2^d e^{4\pi i\langle x,\omega\rangle} V_{\mathcal{I}g} f(2x, 2\omega).
\end{aligned}$$

□

Lemma 6.3. *Let $f, g \in L^2(\mathbb{R}^d)$, then*

(a) $W(f,g)$ is uniformly continuous on $\mathbb{R}^{2d}$, and

$$\|W(f,g)\|_\infty \leq 2^d \|f\|_2 \|g\|_2.$$

(b) $W(f,g) = \overline{W(g,f)}$. In particular, Wf is real-valued.

(c) For $u, u, \eta, \gamma \in \mathbb{R}^d$, we have

$$W(T_\mu M\eta f, T_\nu M_\gamma g)(x.\omega) = e^{\pi i\langle u+v, \gamma-\eta\rangle} e^{2\pi i\langle x, \eta-\gamma\rangle} e^{-2\pi i\langle \omega, u-v\rangle} \cdot W(f,g)(x - \frac{u+v}{2}, \omega - \frac{\eta+\gamma}{2}).$$

In particular Wf is covariant, that it

$$W(T_\mu M\eta f)(x,\omega) = Wf(x-u, \omega-\eta). \tag{6.6}$$

(d) $W(\hat{f}, \hat{g})(x,\omega) = W(f,g)(-\omega, x)$.

(e) ***Moyal's Formula*** *Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$ then*

$$\langle W(f_1, g_1), W(f_2, g_2)\rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2\rangle \overline{\langle g_1, g_2\rangle}. \tag{6.7}$$

Proof. *(a)* Since $W(f,g)(x,\omega) = 2^d e^{4\pi i\langle x,\omega\rangle} V_{\mathcal{I}g} f(2x, 2\omega)$ and we know STFT is uniformly continuous on $\mathbb{R}^{2d}$, so $W(f,g)$ is uniformly continuous on $\mathbb{R}^{2d}$.

Now

$$
\begin{aligned}
|W(f,g)(x,\omega)| = |2^d V_{\mathcal{I}g} f(2x,2\omega)| &\leq 2^d |\langle f, M_{2\omega} T_{2x} \mathcal{I}_g \rangle| \\
&\leq 2^d \|f\|_2 \|M_{2\omega} T_{2x} \mathcal{I}_g\|_2 \text{ (By Cauchy-Swartz inequality)} \\
&= 2^d \|f\|_2 \|g\|_2.
\end{aligned}
$$

(b)

$$
\begin{aligned}
W(f,g)(x,\omega) &= \int_{\mathbb{R}^d} f(x+t/2)\overline{g(x-t/2)} e^{-2\pi i \langle t,\omega\rangle} \mathrm{d}t. \\
&= \overline{\int_{\mathbb{R}^d} \overline{f(x+t/2)} g(x-t/2) e^{2\pi i \langle t,\omega\rangle} \mathrm{d}t}.
\end{aligned}
$$

Substitute $t \to -t$

$$
\begin{aligned}
&= \overline{\int_{\mathbb{R}^d} \overline{f(x-t/2)} g(x+t/2) e^{-2\pi i \langle t,\omega\rangle} \mathrm{d}t}. \\
&= \overline{W(g,f)}.
\end{aligned}
$$

(d)

$$
\begin{aligned}
W(\hat{f},\hat{g})(x,\omega) &= 2^d e^{4\pi i \langle x,\omega\rangle} V_{\mathcal{I}_{\hat{g}}} \hat{f}(2x,2\omega) \text{(By Lemma 4.4.1)} \\
&= 2^d e^{4\pi i \langle x,\omega\rangle} \langle \hat{f}, M_{2\omega} T_{2x} \mathcal{I}_{\hat{g}} \rangle \\
&= 2^d e^{4\pi i \langle x,\omega\rangle} \langle \hat{f}, M_{2\omega} T_{2x} \hat{\mathcal{I}}_g \rangle . (\text{ as } \hat{\mathcal{I}}_g = \mathcal{I}_{\hat{g}}) \\
&= 2^d e^{4\pi i \langle x,\omega\rangle} \langle \hat{f}, e^{2\pi i \langle 2x,2\omega\rangle} T_{2x} M_{2\omega} \hat{\mathcal{I}}_g \rangle \\
&= 2^d e^{-4\pi i \langle x,\omega\rangle} \langle f, M_{-2x} T_{2\omega} \mathcal{I}_g \rangle \\
&= 2^d e^{4\pi i \langle x,-\omega\rangle} V_{\mathcal{I}_g} f(-2\omega, 2x) \\
&= W(f,g)(-\omega, x).
\end{aligned}
$$

(e)

$$
\begin{aligned}
W(\hat{f},\hat{g})(x,\omega) &= 2^d e^{4\pi i \langle x,\omega\rangle} V_{\mathcal{I}_{\hat{g}}} \hat{f}(2x,2\omega) \text{(By Lemma 4.4.1)} \\
&= 2^d e^{4\pi i \langle x,\omega\rangle} \langle \hat{f}, M_{2\omega} T_{2x} \mathcal{I}_{\hat{g}} \rangle \\
&= 2^d e^{4\pi i \langle x,\omega\rangle} \langle \hat{f}, M_{2\omega} T_{2x} \hat{\mathcal{I}}_g \rangle . (\text{ as } \hat{\mathcal{I}}_g = \mathcal{I}_{\hat{g}}) \\
&= 2^d e^{4\pi i \langle x,\omega\rangle} \langle \hat{f}, e^{2\pi i \langle 2x,2\omega\rangle} T_{2x} M_{2\omega} \hat{\mathcal{I}}_g \rangle \\
&= 2^d e^{-4\pi i \langle x,\omega\rangle} \langle f, M_{-2x} T_{2\omega} \mathcal{I}_g \rangle \\
&= 2^d e^{4\pi i \langle x,-\omega\rangle} V_{\mathcal{I}_g} f(-2\omega, 2x) \\
&= W(f,g)(-\omega, x).
\end{aligned}
$$

□

Note : $\mathcal{F}_1 f, \mathcal{F}_2 f$ are the partial Fourier transform in first and second variable respectively of the function f definied on $\mathbb{R}^{2d}$ and we can define by

$$\mathcal{F}_1 f(\omega, t) := \int_{\mathbb{R}^d} f(x,t) e^{-2\pi i \langle x, \omega \rangle} \mathrm{d}x.$$
$$\mathcal{F}_2 f(x, \omega) := \int_{\mathbb{R}^d} f(x,t) e^{-2\pi i \langle t, \omega \rangle} \mathrm{d}t.$$

Fourier transform and of f is given by $\mathcal{F} f := \mathcal{F}_1 \mathcal{F}_2 f$, and $\mathcal{F}^{-1} f := \mathcal{F}_1^{-1} \mathcal{F}_2^{-1} f$.

Lemma 6.4. *Let $f, g \in L^2(\mathbb{R}^d)$. Then,*

$$W(f,g) = \mathcal{F}_2 \mathcal{T}_s (f \otimes \overline{g}), \tag{6.8}$$

where $\mathcal{T}_s$ is symetric cordinate transform of a function defined on $\mathbb{R}^{2d}$ and defined as $\mathcal{T}_s f(x,t) := f(x+t/2, x-t/2)$ and $\mathcal{F}_2$ is the partial Fourier transform in second variable, and $(f \otimes g)(x,t) := f(x)g(t)$.

Proof. $\mathcal{F}_2 f(x,\omega) := \int_{\mathbb{R}^d} f(x,t) e^{-2\pi i \langle t, \omega \rangle} \mathrm{d}t$

$$\begin{aligned}
\mathcal{F}_2 \mathcal{T}_s (f \otimes \overline{g})(x,\omega) &= \int_{\mathbb{R}^d} \mathcal{T}_s (f \otimes \overline{g})(x,t) e^{-2\pi i \langle t, \omega \rangle} \mathrm{d}t \\
&= \int_{\mathbb{R}^d} (f \otimes \overline{g})(x + t/2, x - t/2) e^{-2\pi i \langle t, \omega \rangle} \mathrm{d}t \\
&= \int_{\mathbb{R}^d} f(x + t/2) \overline{g(x - t/2)} e^{-2\pi i \langle t, \omega \rangle} \mathrm{d}t. \\
&= W(f,g)(x,\omega).
\end{aligned}$$

□

Lemma 6.5. *Let $f, g \in L^2(\mathbb{R}^d)$. Then*

$$W(f,g) = \mathcal{F} \mathcal{U} A(f,g),$$

where $\mathcal{U} f(x,\omega) = f(\omega, -x)$ is the rotation operator.

Proof.

$$\begin{aligned}
\mathcal{F}^{-1} W(f,g)(x,\omega) &= \mathcal{F}_1^{-1} \mathcal{F}_2^{-1} W(f,g)(x,\omega) \\
&= \mathcal{F}_1^{-1} \mathcal{F}_2^{-1} \mathcal{F}_2 \mathcal{T}_s (f \otimes \overline{g})(x,\omega) \\
&= \int_{\mathbb{R}^d} \mathcal{T}_s (f \otimes \overline{g})(t,\omega) e^{2\pi i \langle t, x \rangle} \mathrm{d}t \\
&= \int_{\mathbb{R}^d} (f \otimes \overline{g}) \left(t + \frac{\omega}{2}, t - \frac{\omega}{2} \right) e^{2\pi i \langle t, x \rangle} \mathrm{d}t \\
&= \int_{\mathbb{R}^d} f \left(t + \frac{\omega}{2} \right) \overline{g} \left(t - \frac{\omega}{2} \right) e^{-2\pi i \langle t, -x \rangle} \mathrm{d}t \\
&= A(f,g)(\omega, -x) = \mathcal{U} A(f,g)(x,\omega).
\end{aligned}$$

□

Lemma 6.6. *Let $f \in L^2(\mathbb{R})$. If supp$f \subseteq [a,b]$, then $Wf(x,\omega) = 0$ for $x \notin [a,b]$.If supp$\hat{f} \subseteq [\alpha,\beta]$, then $Wf(x,\omega) = 0$ for $\omega \notin [\alpha,\beta]$.*

Proof.

$$\text{Let } Wf(x,\omega) = \int_{\mathbb{R}^d} f(x+t/2)\overline{f(x-t/2)}e^{-2\pi i\langle t,\omega\rangle}\mathrm{d}t \neq 0,$$
$$\implies x+t/2, x-t/2 \in \textit{supp} f \subseteq [a,b]$$
$$\implies \frac{1}{2}(x+t/2) + \frac{1}{2}(x-t/2) = x \in [a,b]. \text{ (As } [a,b] \text{ is a convex set.)}$$

Since $W\hat{f}(x,\omega) = Wf(-\omega,x)$

$$\text{Let } Wf(-\omega,x) = \int_{\mathbb{R}^d} f(-\omega+t/2)\overline{f(-\omega-t/2)}e^{-2\pi i\langle t,x\rangle}\mathrm{d}t \neq 0$$
$$\implies -\omega+t/2, -\omega-t/2 \in \textit{supp}$$

□

Lemma 6.7. *If $f, \hat{f} \in L^1 \cap L^2(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} Wf(x,\omega)\mathrm{d}\omega = |f(x)|^2, \tag{6.9}$$
$$\int_{\mathbb{R}^d} Wf(x,\omega)\mathrm{d}x = |\hat{f}(x)|^2. \tag{6.10}$$

In particular,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Wf(x,\omega)\mathrm{d}x\mathrm{d}\omega = \|f\|_2^2. \tag{6.11}$$

Proof. By Lemma 4.4.1 and Lemma $|Wf(x,\omega)| = 2^d|V_f f()$ □

Note : Let $f \in L^2(\mathbb{R}^d)$ be a odd function s.t $f \neq 0$ then

$$Wf(0,0) = \int_{\mathbb{R}^d} f(t/2)\overline{f(-t/2)}\mathrm{d}t$$
$$= -\int_{\mathbb{R}^d} f(t/2)\overline{f(t/2)}\mathrm{d}t.\text{(as } f \text{ odd function)}$$

Substitute $t/2 \to u$ then $\mathrm{d}t = 2^d\mathrm{d}u$

$$= -2^d\int_{\mathbb{R}^d} |f(u)|^2\mathrm{d}u$$
$$= -2^d\|f\|_2^2 < 0.$$

Therefore the Wigner distribution is not non-negative for all $f \in L^2(\mathbb{R}^d)$, So it is also not a ideal quadratic time-frequency representation. Now we will try to see that for which kind of functions Wigner distribution is non-negative and the possible other ideal quadratic time-frequency representations.

6.5 Positivity of Wigner Distribution

Definition 6.3. *[1] Let B be an invertible $d \times d$-matrix over $\mathbb{R}$ and C be a symmetric $d \times d$-matrix over $\mathbb{R}$. Then we define*

$$\mathcal{U}_B f(x) = |\det B|^{1/2} f(Bx), \tag{6.12}$$

and

$$\mathcal{N}_C f(x) = e^{-\pi i \langle x \cdot Cx \rangle} f(x). \tag{6.13}$$

These are unitary operators of coordinate change and multiplication.

Lemma 6.8. *Let f be a generalized Gaussian function of the form*

$$f(x) = e^{-\pi \langle x, Ax \rangle + 2\pi \langle b, x \rangle + c} \tag{6.14}$$

where $A \in GL(d, \mathbb{C})$ s.t. $A = B + iC$, B is real positive definite symmetric matrix and C is a real symmetric matrix, also $b = b_1 + ib_2, b_1, b_2 \in \mathbb{R}^d$. Then
(a)

$$f = k M_{b_2 C B^{-1} b_1} T_{B^{-1} b_1} \mathcal{N}_C \mathcal{U}_{B^{\frac{1}{2}}} \phi_1, \tag{6.15}$$

where $k \in \mathbb{C}$ and $\phi_1(x) = e^{-\pi \|x\|^2}$.
(b) The Fourier transform of f is again a Gaussian, specially

$$\hat{f} = (det A)^{-\frac{1}{2}} k T_{b_2 - CB^{-1} b_1} M_{-B^{-1} b_1} (e^{-\pi \langle \omega, A^{-1} \omega \rangle}). \tag{6.16}$$

Proof. *(a)* We write the exponent of f as

$$-\pi \langle x, Ax \rangle + 2\pi \langle b, x \rangle + c = 2\pi i \langle b_2 - CB^{-1} b_1, x \rangle - \pi i \langle x - B^{-1} b_1, C(x - B^{-1} b_1 \rangle$$
$$- \pi \langle x - B^{-1} b_1, B(x - B^{-1} b_1) \rangle + c + \pi \langle B^{-1} b_1, AB^{-1} b_1 \rangle.$$

This yields $f = k M_{b_2 - CB^{-1} b_1} T_{B^{-1} b_1} \mathcal{N} C(e^{-\pi \langle x, Bx \rangle})$ for some $k \in \mathbb{C}$. Since B is positive definite, we can take its square root and write $\langle x, Bx \rangle = \langle B^{\frac{1}{2}} x, B^{\frac{1}{2}} x \rangle$.
Then $e^{-\pi \langle x, Bx \rangle} = |\det B|^{-\frac{1}{4}} \mathcal{U} B^{\frac{1}{2}} \varphi_1$, 6.14 follows.

(b) In view of equation 3.3 and 3.4, we only need to compute the Fourier transform of $e^{-\pi \langle x, Ax \rangle}$. Assume first that $C = 0$ and that $A = B$ is positive definite. Since $\hat{\varphi}_1 = \varphi_1$ and since

$$(\mathcal{U}_A f)^\wedge = \mathcal{U} A^{-\top} \hat{f},$$

we obtain

$$\int_{\mathbb{R}^d} e^{-\pi\langle x,Bx\rangle} e^{2\pi i\langle x,\omega\rangle} \mathrm{d}x = (\det B)^{-\frac{1}{4}} (\mathcal{U}_{B^{\frac{1}{2}}} \varphi_1)^\wedge(\omega)$$
$$= (\det B)^{-\frac{1}{4}} \mathcal{U}_{B^{-\frac{1}{2}}} \hat{\varphi}_1(\omega)$$

$$\int_{\mathbb{R}^d} e^{-\pi\langle x,Bx\rangle} e^{2\pi i\langle x,\omega\rangle} \mathrm{d}x = (\det B)^{-\frac{1}{2}} e^{-\pi\langle \omega,B^{-1}\omega\rangle}. \tag{6.17}$$

Both sides of this equality are analytic functions in the variables b_{jk} of B. By analytic continuation, 6.17 can be extended to complex-valued symmetric invertible matrices $A = B + iC$, as long as B remains positive definite (so that the left-hand side of 6.17 makes sense). The square root $(\det A)^{-1/2}$ is well-defined on $\{A \in \mathrm{GL}(d,\mathbb{C}),\ A = A^\top,\ \mathrm{Re}\, A > 0\}$ since this region is simply connected. Now we take the branch that extends the root of positive definite matrices. Consequently,

$$(e^{-\pi\langle x,Ax\rangle})^\wedge(\omega) = (\det A)^{-\frac{1}{2}} e^{-\pi\langle \omega,A^{-1}\omega\rangle}$$

for all $A = A^\top \in \mathrm{GL}(d,\mathbb{C})$ with positive definite real part, and therefore 6.16 follows.

The ugly formulas 6.17 and 6.16 say that every Gaussian and its Fourier transform can be obtained from the standard Gaussian $e^{-\pi\|x\|^2}$ by applying time-frequency shifts, a coordinate transform, and a multiplication by a chirp. $\square$

Lemma 6.9. *For $f, g \in L^2(\mathbb{R}^d)$, $B \in \mathrm{GL}(d,\mathbb{R})$, and C symmetric we have*

$$W(\mathcal{U}_B f, \mathcal{U}_B g)(x,\omega) = W(f,g)(Bx, (B^\top)^{-1}\omega)$$

and

$$W(\mathcal{N}_C f, \mathcal{N}_C g)(x,\omega) = W(f,g)(x, \omega + Cx).$$

Proof. The proof is by computation

$$W(\mathcal{U}_B f, \mathcal{U}_B g)(x,\omega) = |\det B| \int_{\mathbb{R}^d} f(Bx + \tfrac{1}{2}Bt)\, \overline{g(Bx - \tfrac{1}{2}Bt)}\, e^{-2\pi i\langle t,\omega\rangle} \mathrm{d}t$$
$$= \int_{\mathbb{R}^d} f(Bx + \tfrac{1}{2}t)\, \overline{g(Bx - \tfrac{1}{2}t)}\, e^{-2\pi i\langle B^{-1}t,\omega\rangle}\, \mathrm{d}\mathrm{d}t$$
$$= W(f,g)(Bx, (B^\top)^{-1}\omega).$$

Since C is symmetric, the exponent of the chirp simplifies as

$$\langle x + \tfrac{t}{2}, C\left(x + \tfrac{t}{2}\right)\rangle - \langle x - \tfrac{t}{2}, C\left(x - \tfrac{t}{2}\right)\rangle = \langle 2t, Cx\rangle.$$

Then we find

$$\begin{aligned} W(\mathcal{N}_C f, \mathcal{N}_C g)(x,\omega) &= \int_{\mathbb{R}^d} e^{-2\pi i\langle t, Cx\rangle} f\left(x + \tfrac{t}{2}\right) \overline{g\left(x - \tfrac{t}{2}\right)} e^{-2\pi i\langle t,\omega\rangle} \mathrm{d}t \\ &= W(f,g)(x, \omega + Cx). \end{aligned}$$

□

Theorem 6.10. ***(Hudson)*** *Let* $f \in L^2(\mathbb{R}^d)$*. Then* $Wf(x,\omega) > 0, \forall (x,\omega) \in \mathbb{R}^{2d}$*, iff* f *is generalized Gaussian of the form as equation 6.14.*

Proof. Proof of Hudson's Theorem. In view of Lemma 6.8 and 6.9, the sufficiency is now easy to see. Since the action of the operators $T, M, \mathcal{N}$, and $\mathcal{U}$ on f amounts to a coordinate transformation of Wf, we see that $Wf > 0$ if and only if $W\varphi_1 > 0$. Precisely, if f is a generalized Gaussian of the form 6.14, then

$$Wf(x,\omega) = W\varphi_1\left(B^{1/2}(x - B^{-1}b_1), B^{-1/2}(\omega - b_2 + Cx)\right).$$

Thus all we need to show is that $W\varphi_1 > 0$. We calculate $W\varphi_a$ for arbitrary $a > 0$:

$$\begin{aligned} W\varphi_a(x,\omega) &= \int_{\mathbb{R}^d} e^{-\pi a^{-1}\left[\left\|x+\frac{1}{2}t\right\|^2 + \left\|x-\frac{1}{2}t\right\|^2\right]} e^{-2\pi i\langle t,\omega\rangle} \mathrm{d}t && (6.18) \\ &= e^{-2\pi\|x\|^2} \int_{\mathbb{R}^d} e^{-\pi\frac{\|t\|^2}{2a}} e^{-2\pi i\langle t,\omega\rangle} \mathrm{d}t. && (6.19) \\ &= \varphi_{\frac{a}{2}}(x)\, \widehat{\varphi_{2a}}(\omega) \\ &= (2a)^{d/2} \varphi_{\frac{a}{2}}(x)\, \varphi_{\frac{1}{2a}}(\omega) > 0. && (6.20) \end{aligned}$$

To establish the necessity in Hudson's theorem, we use properties of the Bargmann transform and some complex analysis. Assume that $f \in L^2(\mathbb{R}^d)$ and $Wf \geq 0$. We take the inner product of Wf with the Wigner distribution of the normalized Gaussian $\varphi = 2^{d/4}\varphi_1$. By equation 6.6 $W(M_{-\omega}T_x\varphi) > 0$ for all $(x,\omega) \in \mathbb{R}^{2d}$; therefore

$$\langle Wf, W(M_{-\omega}T_x\varphi)\rangle_{L^2(\mathbb{R}^{2d})} = \iint_{\mathbb{R}^{2d}} Wf(u,\eta) W(M_{-\omega}T_x\varphi)(u,\eta)\, du\, d\eta > 0. \qquad (6.21)$$

for all $(x,\omega) \in \mathbb{R}^{2d}$. We apply Moyal's formula a identify the inner product $\langle Wf, W(M_{-\omega}T_x\varphi)\rangle$ as the Bargmann transform of f. Writing $z = x + i\omega \in \mathbb{C}^d$, we obtain:

$$\begin{aligned}\langle Wf, W(M_{-\omega}T_x\varphi)\rangle &= |\langle f, M_{-\omega}T_x\varphi\rangle|^2\\ &= |\mathcal{V}_\varphi f(x,-\omega)|^2\\ &= |Bf(z)|^2 e^{-\pi|z|^2}.\end{aligned}$$

Since the entire function Bf does not vanish by (4.21), there exists an entire function $q(z)$ such that

$$Bf(z) = e^{q(z)}.$$

Furthermore, since

$$|Bf(z)|e^{-\pi\frac{|z|^2}{2}} \leq \|\mathcal{V}\varphi f\|\infty \leq \|\varphi\|_2\|f\|_2 = \|f\|_2,$$

Bf satisfies the growth estimate

$$|Bf(z)| \leq \|f\|_2 e^{\pi|z|^2/2}.$$

By taking the logarithm, we obtain the estimate

$$|\mathrm{Re}\, q(z)| \leq c + \frac{\pi}{2}|z|^2.$$

It follows from Carathéodory's inequality of complex analysis that q itself satisfies a similar estimate, namely,

$$|q(z)| \leq C_1 + C_2|z|^2.$$

Therefore q must be a quadratic polynomial of the formTherefore q must be a quadratic polynomial of the form

$$q(z) = \pi z \cdot A'z + 2\pi b' \cdot z + c', \tag{6.22}$$

The restriction of Bf to vectors in $i\mathbb{R}^d$ is

$$Bf(i\omega)e^{-\pi\omega^2/2} = e^{q(i\omega)}e^{-\pi\omega^2/2} = e^{-\pi\omega\cdot(A'+1/2)\omega+2\pi ib'\cdot\omega+c'}, \tag{6.23}$$

in other words, a generalized Gaussian. Next, we express the restriction of $\mathcal{V}_\varphi f$ to $\{0\}\times\mathbb{R}^d$ in two different ways. On one hand,

$$\mathcal{V}\varphi f(0,-\omega) = \langle f, M{-}\omega\varphi\rangle = (f\cdot\varphi)^\wedge(-\omega).$$

On the other hand

$$\mathcal{V}_\varphi f(0,-\omega) = Bf(i\omega)e^{-\pi\omega^2/2}.$$

Therefore $(f\cdot\varphi)^\wedge$ is in $L^2(\mathbb{R}^d)$ and is a generalized Gaussian. By Lemma 6.14, $f\cdot\varphi$ is again a generalized Gaussian and consequently f is also a generalized Gaussian of the form

$$f(x) = e^{-\pi\langle x,Ax\rangle+2\pi\langle b,x\rangle+c}, \quad b\in\mathbb{C}^d,\ A\in \mathrm{GL}(d,\mathbb{C}),\ A = A^T.$$

Since $f \in L^2(\mathbb{R}^d)$, A must have positive definite real part, and Hudson's theorem is proved completely. $\square$

Note : Since the Wigner distribution behaves exemplarily in most aspects, its lack of positivity is quite bothersome. As a remedy for its negative values, one might take average at each point. The standard averaging procedure in mathematics is the convolution of Wf with a smoothing function 4σ which is centered at $(0,0)$. Then the convolution

$$(Wf * \sigma)(x,\omega) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Wf(t,\eta)\sigma(x-t,\omega-\eta)\mathrm{d}t\mathrm{d}\eta. \tag{6.24}$$

Theorem 6.11. *Let* $\sigma_{a,b}(x,\omega) = e^{-2\pi(\frac{\|x\|^2}{2}+\frac{\|\omega\|^2}{b})} = \psi_{\frac{a}{2}}(x)\psi_{\frac{b}{2}}(\omega)$.

(a) *If* $ab = 1$, *then* $Wf * \sigma_{a,b} \geq 0, \forall f \in L^2(\mathbb{R}^d)$

(b) *If* $ab > 1$, *then* $Wf * \sigma_{a,b} > 0, \forall f \in L^2(\mathbb{R}^d)$

(c) *If* $ab < 1$, *then* $Wf * \sigma_{a,b}$ *may assume negative values.*

Lemma 4.5.3. *For* $a, b > 0$ *we have*

$$\psi_a * \psi_b = \left(\frac{ab}{a+b}\right)^{d/2} \psi_{a+b}. \tag{6.25}$$

Proof. By Lemma

$$\begin{aligned}
(\psi_a * \psi_b)\hat{}(\omega) &= \hat{\psi}_a(\omega)\hat{\psi}_b(\omega)(\text{ as } (f*g)\hat{} = \hat{f} * \hat{g}) \\
&= (ab)^{d/2}\psi_{\frac{1}{a}}(\omega)\psi_{\frac{1}{b}}(\omega)(\text{as } \hat{\psi}_a(\omega) = a^{d/2}\psi_{\frac{1}{a}}\hat{}(\omega)) \\
&= (ab)^{d/2}e^{-\pi(a+b)\|\omega\|^2} \\
&= \left(\frac{ab}{a+b}\right)^{d/2}(a+b)^{d/2}\psi_{\frac{1}{a+b}}(\omega). \\
&= \left(\frac{ab}{a+b}\right)^{d/2}\psi_{a+b}\hat{}(\omega).
\end{aligned}$$

$\square$

Proof. Claim : $\mathcal{I}(Wf) = W(\mathcal{I}f)$ where $\mathcal{I}f(x) = f(-x)$

$$\mathcal{I}(Wf)(x,\omega) = Wf(-x,-\omega) = \int_{\mathbb{R}^d} f(-x+t/2)\overline{f(-x-t/2)}e^{-2\pi i\langle t,-\omega\rangle}\mathrm{d}t.$$

substitute $t \to -t$

$$\begin{aligned}
&= \int_{\mathbb{R}^d} f(-x-t/2)\overline{f(-x+t/2)}e^{-2\pi i\langle t,\omega\rangle}\mathrm{d}t. \\
&= \int_{\mathbb{R}^d} \mathcal{I}f(x+t/2)\overline{\mathcal{I}f(x-t/2)}e^{-2\pi i\langle t,\omega\rangle}\mathrm{d}t. \\
&= W(\mathcal{I}f)(x,\omega).
\end{aligned}$$

Now let $ab = 1$ then by equation (17)

$$\sigma_{a,b} = \psi_{\frac{a}{2}}(x)\psi_{\frac{1}{2a}}(\omega) = (2a)^{\frac{-d}{2}} W\psi_a(x, \omega).$$

Now by using covariance and Moyal's formula of Wf, we have

$$(Wf * \sigma_{a,b})(x, \omega) = \int_{\mathbb{R}} \int_{\mathbb{R}} Wf(t, \eta)\sigma_{a,\frac{1}{a}}(x - t, \omega - \eta)\mathrm{d}t\mathrm{d}\eta$$

Substitute $x - t \to u$ and $\omega - \eta \to v$ then

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} Wf(x - u, \omega - v)\sigma_{a,\frac{1}{a}}(u, v)\mathrm{d}u\mathrm{d}v$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{I}(Wf)(u - x, v - \omega)\sigma_{a,\frac{1}{a}}(u, v)\mathrm{d}u\mathrm{d}v$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} W(\mathcal{I}f)(u - x, v - \omega)\sigma_{a,\frac{1}{a}}(u, v)\mathrm{d}u\mathrm{d}v$$

$$= (2a)^{\frac{-d}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} W(\mathcal{I}f)(u - x, v - \omega)W\psi_a(u, v)\mathrm{d}u\mathrm{d}v$$

$$= (2a)^{\frac{-d}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} W(\mathcal{I}f)(u - x, v - \omega)\overline{W\psi_a(u, v)}\mathrm{d}u\mathrm{d}v$$

(as $W\psi_a \geq 0$)

By the covariance of Wf

$$= (2a)^{\frac{-d}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} W(M_\omega T_x \mathcal{I}f)(u, v)\overline{W\psi_a(u, v)}\mathrm{d}u\mathrm{d}v$$

$$= (2a)^{\frac{-d}{2}} \langle W(M_\omega T_x \mathcal{I}f), W\psi_a\rangle$$

By the Moyal's formula

$$= (2a)^{\frac{-d}{2}} |\langle M_\omega T_x \mathcal{I}f, \psi_a\rangle|^2 \geq 0$$

Now let $ab > 1$, then we can find c, d s.t. $0 < c < a$, $0 < d < b$ and $cd = 1$.

Claim : $\sigma_{a,b} = \sigma_{a,b} * \sigma_{a-c,b-d}$.

$$\sigma_{a,b}(x, \omega) = \psi_{\frac{a}{2}}(x)\psi_{\frac{b}{2}}(\omega),$$

$$\sigma_{a-c,b-d}(x, \omega) = \psi_{\frac{a-c}{2}}(x)\psi_{\frac{b-d}{2}}(\omega)$$

$$
\begin{aligned}
(\sigma_{c,d} * \sigma_{a-c,b-d})(x,\omega) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma_{c,d}(t,\eta)\sigma_{a-c,b-d}(x-t,\omega-\eta)\mathrm{d}t\mathrm{d}\eta \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi_{\frac{c}{2}}(t)\psi_{\frac{d}{2}}(\eta)\psi_{\frac{a-c}{2}}(x-t)\psi_{\frac{b-d}{2}}(\omega-\eta)\mathrm{d}t\mathrm{d}\eta \\
&= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \psi_{\frac{c}{2}}(t)\psi_{\frac{a-c}{2}}(x-t)\mathrm{d}t\right)\psi_{\frac{d}{2}}(\eta)\psi_{\frac{b-d}{2}}(\omega-\eta)\mathrm{d}\eta \\
&= \left(\psi_{\frac{c}{2}} * \psi_{\frac{a-c}{2}}\right)(x)\left(\psi_{\frac{d}{2}} * \psi_{\frac{b-d}{2}}\right)(\omega) \\
&= \psi_{\frac{a}{2}}(x)\psi_{\frac{b}{2}}(\omega) \\
&= \sigma_{a,b}(x,\omega).
\end{aligned}
$$

□

Now $Wf * \sigma_{a,b} = (Wf * \sigma_{c,d}) * \sigma_{a-c,b-d}$ (Convolution is associative in $L^2(\mathbb{R}^d)$)
Since $Wf * \sigma_{c,d} \geq 0$ and $\sigma_{a-c,b-d} > 0$, so $Wf * \sigma_{a,b} > 0$ as we know that the convolution of a non-negative function with a strictly positive function is always strictly positive.

6.6 Cohen's Class

Definition 6.4. *[1] A quadratic time-frequency representation Qf belongs to Cohen's class, if it is of the form*

$$Qf = Q_\sigma f = Wf * \sigma, \tag{6.26}$$

for some $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$.

Lemma 6.12. *For instance by equation, we have*

$$W(T_x M_\omega f) = T_{(x,\omega)} Wf, \text{ by (7)}$$

are therefore

$$Q_\sigma(T_x M_\omega f) = T_{(x,\omega)} Q_\sigma f. \tag{6.27}$$

Thus, all time-frequency representations in Cohen's class are covariant.

Proof. For instance, since $W(T_x M_\omega f) = T_{(x,\omega)} Wf$ by equation 6.6, we have

$$
\begin{aligned}
Q_\sigma(T_x M_\omega f) &= W(T_x M_\omega f) * \sigma \\
&= (T_{(x,\omega)} Wf) * \sigma \\
&= T_{(x,\omega)}(Wf * \sigma) \\
&= T_{(x,\omega)} Q_\sigma f.
\end{aligned}
$$

□

Lemma 6.13.

$$\iint_{\mathbb{R}^{2d}} Q_\sigma f(x,\omega)\, dx\, d\omega = \|f\|_2^2 \tag{6.28}$$

holds if and only if $\iint_{\mathbb{R}^{2d}} \sigma(x,\omega)\, \mathrm{d}x\, \mathrm{d}\omega = 1$.

Proof.

$$\begin{aligned}
\iint_{\mathbb{R}^{2d}} Q_\sigma f(x,\omega)\, dx\, d\omega &= \iint_{\mathbb{R}^{2d}} Wf * \sigma(x,\omega)\, dx\, d\omega \\
&= \left(\iint_{\mathbb{R}^{2d}} Wf(x,\omega)\, dx\, d\omega\right)\left(\iint_{\mathbb{R}^{2d}} \sigma(x,\omega)\, dx\, d\omega\right) \\
&= \|f\|_2^2 \iint_{\mathbb{R}^{2d}} \sigma(x,\omega)\, dx\, d\omega.
\end{aligned}$$

To make this computation work, we need $\sigma \in L^1(\mathbb{R}^{2d})$ and to restrict f to the subspace where $Wf \in L^1(\mathbb{R}^{2d})$. The same argument also works if σ is a bounded measure on $\mathbb{R}^{2d}$ with $\sigma(\mathbb{R}^{2d}) = 1$. □

Lemma 6.14. *The analogue of Moyal's formula takes the form*

$$\langle Q_\sigma f, Q_\sigma g\rangle_{L^2(\mathbb{R}^{2d})} = |\langle f, g\rangle|^2 \tag{6.29}$$

for $f, g \in L^2(\mathbb{R}^d)$*. It holds if and only if* $|\hat{\sigma}(x,\omega)| = 1$ *almost everywhere.*

Proof. we note first that for $\hat{\sigma} \in L^\infty(\mathbb{R}^{2d})$ and $f \in L^2(\mathbb{R}^d)$, the convolution $Wf * \sigma$ is well-defined in the sense that $Wf * \sigma = \mathcal{F}^{-1}(\widehat{Wf} \cdot \hat{\sigma})$. Now use Parseval's and Moyal's formulas and Moyal's formula several times:

$$\begin{aligned}
\langle Q_\sigma f, Q_\sigma g\rangle_{L^2(\mathbb{R}^{2d})} &= \langle Wf * \sigma, Wg * \sigma\rangle \\
&= \left\langle \mathcal{F}^{-1}(\widehat{Wf} \cdot \hat{\sigma}), \mathcal{F}^{-1}(\widehat{Wg} \cdot \hat{\sigma})\right\rangle \\
&= \langle \widehat{Wf}, \widehat{Wg} \cdot |\hat{\sigma}|^2\rangle \\
&= \langle \widehat{Wf}, \widehat{Wg}\rangle \\
&= \langle Wf, Wg\rangle \quad \text{(by (4.8))} \\
&= |\langle f, g\rangle|^2.
\end{aligned}$$

The converse is similar. □

CHAPTER 7

Discrete Time-Frequency Representations : Gabor Frame

7.1 Frame Theory

Definition 7.1. *(**Frames**) Let $\mathcal{H}$ be a seperable Hilbert space. A sequence $\{e_j : j \in J\} \subseteq \mathcal{H}$ is a frame if $\exists A, B > 0$ s.t.*

$$A\|f\|^2 \leq \sum_{j\in J} |\langle f, e_j\rangle|^2 \leq B\|f\|^2, \forall f \in \mathcal{H}. \tag{7.1}$$

Here A and B are known as frame bounds.

Note : If $A = B$ then we say $\{e_J : j \in J\}$ is a tight frame.

Example 7.1. *Orthonormal basis is a tight frame with frame bounds $A = B = 1$.*

Proof. Let $\{e_j : j \in J\} \subseteq \mathcal{H}$ be an orthonormal basis for $\mathcal{H}$, and $f \in L^2(\mathbb{R}^d)$, then by the Parseval relation $\|f\|^2 = \sum_{j\in J} |\langle f, e_j\rangle|^2$.

$$\|f\|^2 \leq \sum_{j\in J} |\langle f, e_j\rangle|^2 \leq \|f\|^2.$$

Clearly, $\{e_j : j \in J\} \subseteq \mathcal{H}$ is a tight frame with frame bound $A = B = 1$. □

Example 7.2. *The union of two orthonormal basis is a tight frame with frame bounds $A = B = 2$.*

Proof. Let $\{e_j : j \in J\} \subseteq \mathcal{H}$ and $\{c_j : j \in J\} \subseteq \mathcal{H}$ be two orthonormal basis for $\mathcal{H}$, and $f \in L^2(\mathbb{R}^d)$ then by the Parseval relation

$$\|f\|^2 = \sum_{j\in J} |\langle f, e_j\rangle|^2$$

$$\|f\|^2 = \sum_{j\in J} |\langle f, c_j\rangle|^2$$

Now consider, $\{e_j : j \in J\} \cup \{c_j : j \in J\} \subseteq \mathcal{H}$. Let Then

$$\sum_{j\in J} |\langle f, e_j\rangle|^2 + \sum_{j\in J} |\langle f, c_j\rangle|^2 = 2\|f\|^2.$$

$$2\|f\|^2 \leq \sum_{j\in J} |\langle f, e_j\rangle|^2 + \sum_{j\in J} |\langle f, c_j\rangle|^2 \leq 2\|f\|^2.$$

Clearly, $\{e_j : j \in J\} \cup \{c_j : j \in J\} \subseteq \mathcal{H}$ is a tight frame with frame bound $A = B = 2$. □

Example 7.3. *The union of orthonormal basis with L arbitary unit vectors is a frame with frame bounds $A = 1, B = L + 1$.*

Proof. Let $\{e_j : j \in J\} \subseteq \mathcal{H}$ be an orthonormal basis for $\mathcal{H}$ and $\{c_1, c_2, \cdot, \cdot, \cdot, c_L\} \subseteq \mathcal{H}$ s.t. $\|c_j\|_{\mathcal{H}} = 1$. for $j = 1, 2, \cdot, \cdot, \cdot, L$. Now consider $f \in \mathcal{H}$ and since $\mathcal{H}$ is seperable space we can write f as

$$\|f\|^2_{\mathcal{H}} = \sum_{j\in J} |\langle f, e_j\rangle|^2.$$

Now, by Cauchy-Schwartz inequality

$$\|f\|^2_{\mathcal{H}} \leq \sum_{j\in J} |\langle f, e_j\rangle|^2 + \sum_{j=1}^{L} |\langle f, c_j\rangle|^2 \leq \|f\|^2_{\mathcal{H}} + \sum_{j=1}^{L} \|f\|^2_{\mathcal{H}} \|c_j\|^2_{\mathcal{H}} \leq (1 + L)\|f\|^2_{\mathcal{H}}.$$

Clearly,$\{e_j : j \in J\} \cup \{c_1, c_2, \cdot, \cdot, \cdot, c_L\}$ is a frame for $\mathcal{H}$ with frame bounds $A = 1, B = L$. □

Remark 7.1. *Frames generalize orthonormal basis. However, these trivial examples already show that in general the fram elements are neither orthogonal nor linearly independent. To understand frames and reconstruction methods better, we study some important associated operators*

Definition 7.2 (Coefficient or Analysis Operator). *Let $\{e_j : j \in J\} \subseteq \mathcal{H}$, the coefficient operator C is a map defined $\mathcal{H}$ given by*

$$Cf := \{\langle f, e_j\rangle : j \in J\}. \tag{7.2}$$

Definition 7.3 (**Synthesis Or Reconstruction Operator**). *Synthesis operator D is a map from C_{00} to $\mathcal{H}$ given by*

$$Dc := \sum_{j\in J} c_j e_j. \tag{7.3}$$

Definition 7.4 (**Frame Operator**). *Frame operator S is a map defined on $\mathcal{H}$ and S given by*

$$Sf := \sum_{j\in J} \langle f, e_j\rangle e_j. \tag{7.4}$$

Proposition 7.1. *Suppose that $\{e_j : j \in J\}$ is a frame for $\mathcal{H}$.*

(a) C is a bounded operator from $\mathcal{H}$ into $l^2(J)$ with closed range.

(b) The operators C and D are adjoint to each other, that is, $D = C^*$. Consequently, D extends to a bounded operator from $l^2(J)$ into $\mathcal{H}$ and satisfies

$$\left\|\sum_{j\in J} c_j e_j\right\| \leq B^{1/2}\|c\|_2. \tag{7.5}$$

(c) The frame operator $S = C^*C = DD^*$ maps $\mathcal{H}$ onto $\mathcal{H}$ and is a positive invertible operator satisfying $AI_{\mathcal{H}} \leq S \leq BI_{\mathcal{H}}$ and $B^{-1}I_{\mathcal{H}} \leq S^{-1} \leq A^{-1}I_{\mathcal{H}}$.

In particular, $\{e_j : j \in J\}$ is a tight frame if and only if $S = AI_{\mathcal{H}}$.

(d) The optimal frame bounds are $B_{opt} = \|S\|op$ and $Aopt = \|S^{-1}\|op^{-1}$, where $\|\cdot\|op$ is the operator norm of S.

Proof. *(b)* Let $c = (c_j)_{j\in J}$ be a finite sequence. Then

$$\langle C^*c, f\rangle = \langle c, Cf\rangle = \sum_{j\in J} c_j \overline{\langle f, e_j\rangle} = \left\langle \sum_{j\in J} c_j e_j, f\right\rangle = \langle Dc, f\rangle. \tag{5.9}$$

Since C is bounded on $\mathcal{H}$ and has operator norm $\|C\|_{\text{op}} \leq B^{1/2}$ by (5.4), it follows that $D = C^* : \ell^2(J) \longrightarrow \mathcal{H}$ is also bounded with the same operator norm. Thus (b) follows.

(c) Obviously the frame operator is $S = C^*C = DD^*$ and consequently S is self-adjoint and positive. Since

$$\langle Sf, f\rangle = \sum_{j\in J} |\langle f, e_j\rangle|^2,$$

the operator inequality $AI \leq S \leq BI$ is just (5.4) rewritten. S is invertible on $\mathcal{H}$ because $A > 0$. Inequalities are preserved under multiplication with positive commuting operators, therefore

$$AS^{-1} \leq SS^{-1} \leq BS^{-1},$$

as desired.

(d) follows from the frame inequalities (5.4) and the fact that the operator norm of a positive operator is determined by

$$\|S\|_{\text{op}} = \sup\{\langle Sf, f\rangle : \|f\| \le 1\}.$$

The argument for A_{opt} is similar. □

Corollary 7.2. *Let $\{e_j : j \in J\}$ be a frame for $\mathcal{H}$. If $f = \sum_{j\in J} c_j e_j$ for some $c \in \ell^2(J)$, then for every $\epsilon > 0$ there exists a finite subset $F_0 = F_0(\epsilon) \subseteq J$ such that*

$$\left\| f - \sum_{j\in F} c_j e_j \right\| < \epsilon \quad \text{for all finite subsets } F \supseteq F_0. \tag{7.6}$$

We say that the series $\sum_{j\in J} c_j e_j$ converges unconditionally to $f \in \mathcal{H}$.

Proof. Choose $F_0 \subseteq J$ such that $\sum_{n\notin F} |c_j|^2 < \epsilon/B^{1/2}$ for $F \supseteq F_0$. Let $c_F = c \cdot \chi_F \in \ell^2(J)$ be the finite sequence with terms $c_{F,j} = c_j$ if $j \in F$ and $c_{F,j} = 0$ if $j \notin F$.

Then $\sum_{j\in F} c_j e_j = Dc_F$ and

$$\left\| f - \sum_{j\in F} c_j e_j \right\| = \|Dc - Dc_F\| = \|D(c - c_F)\| \le B^{1/2}\|c - c_F\|_2 < \epsilon.$$

□

Corollary 7.3. *If $\{e_j : j \in J\}$ is a frame with frame bounds $A, B > 0$, then $\{S^{-1}e_j : j \in J\}$ is a frame with frame bounds $B^{-1}, A^{-1} > 0$, the so-called dual frame. Every $f \in \mathcal{H}$ has non-orthogonal expansions*

$$f = \sum_{j\in J} \langle f, S^{-1}e_j\rangle e_j \tag{7.7}$$

and

$$f = \sum_{j\in J} \langle f, e_j\rangle S^{-1}e_j, \tag{7.8}$$

where both sums converge unconditionally in $\mathcal{H}$.

Proof. First observe that

$$\sum_{j\in J} |\langle f, S^{-1}e_j\rangle|^2 = \sum_{j\in J} |\langle S^{-1}f, e_j\rangle|^2 = \langle S(S^{-1}f), S^{-1}f\rangle = \langle S^{-1}f, f\rangle.$$

Therefore Proposition 5.1.1(c) implies that

$$B^{-1}\|f\|^2 \le \langle S^{-1}f, f\rangle = \sum_{j\in J} |\langle f, S^{-1}e_j\rangle|^2 \le A^{-1}\|f\|^2.$$

Thus the collection $\{S^{-1}e_j : j \in J\}$ is a frame with frame bounds B^{-1} and A^{-1}.

Using the factorizations $I_{\mathcal{H}} = S^{-1}S = SS^{-1}$, we obtain the series expansions

$$f = S(S^{-1}f) = \sum_{j\in J}\langle S^{-1}f, e_j\rangle e_j = \sum_{j\in J}\langle f, S^{-1}e_j\rangle e_j$$

and

$$f = S^{-1}Sf = \sum_{j\in J}\langle f, e_j\rangle S^{-1}e_j.$$

Because both $\{\langle f, e_j\rangle\}$ and $\{\langle f, S^{-1}e_j\rangle\}$ are in $\ell^2(J)$, both series converge unconditionally by corollary 7.2. □

Proposition 7.4. *If $\{e_j : j \in J\}$ is a frame for $\mathcal{H}$ and $f = \sum_{j\in J} c_j e_j$ for some coefficients $c \in l^2(J)$, then*

$$\sum_{j\in J} |c_j|^2 \geq \sum_{j\in J} |\langle f, S^{-1}e_j\rangle|^2 \tag{7.9}$$

with equality only if $c_j = \langle f, S^{-1}e_j\rangle$ for all $j \in J$.

Proof. Set $a_j = \langle f, S^{-1}e_j\rangle$. Then $f = \sum_j a_j e_j$ and

$$\langle f, S^{-1}f\rangle = \sum_{j\in J} a_j\langle e_j, S^{-1}f\rangle = \sum_{j\in J} |a_j|^2.$$

On the other hand,

$$\langle f, S^{-1}f\rangle = \sum_{j} c_j\langle e_j, S^{-1}f\rangle = \sum_{j} c_j\overline{a_j} = \langle c, a\rangle.$$

Therefore $\|a\|_2^2 = \langle c, a\rangle$, and we see that

$$\|c\|_2^2 = \|c - a + a\|_2^2$$

$$= \|c - a\|_2^2 + \|a\|_2^2 + \langle c - a, a\rangle + \langle a, c - a\rangle$$

$$= \|c - a\|_2^2 + \|a\|_2^2 \geq \|a\|_2^2,$$

with equality only if $c = a$. □

Proposition 7.5. *Suppose that $\{e_j : j \in J\}$ is a frame for $\mathcal{H}$. Then the following conditions are equivalent.*

(i) The coefficients $c \in \ell^2(J)$ in the series expansion (5.12) are unique.

(ii) The analysis operator C maps onto $\ell^2(J)$.

(iii) There exist constants $A', B' > 0$ such that the inequalities

$$A'\|c\|2 \leq \left\|\sum j \in J c_j e_j\right\| \leq B'\|c\|_2 \tag{7.10}$$

hold for all finite sequences $c = (c_j)_{j\in J}$.

(iv) $\{e_j : j \in J\}$ *is the image of an orthonormal basis* $\{g_j : j \in J\}$ *under an invertible operator* $T \in \mathcal{B}(\mathcal{H})$.

(v) The Gram matrix G*, given by* $G_{jm} = \langle e_m, e_j \rangle$*,* $m, j \in J$*, defines a positive invertible operator on* $\ell^2(J)$.

Definition 7.5. *A frame that satisfies the conditions of Proposition 7.5 is called a Riesz basis of* $\mathcal{H}$.

Proof. Since the omission of one element results in an incomplete set, as can be seen from (iv), Riesz bases are sometimes referred to as exact frames. The conditions are just different ways of saying that the operators C and D are bijections. The assumptions that $\{e_j\}$ is a frame implies that C is one-to-one with closed range and that D is onto (Proposition 7.1 and Corollary 7.3 . Recall that a bounded operator is one-to-one if and only if its adjoint operator has dense range.

(i) $\Leftrightarrow$ (ii) The coefficients are unique if and only if D is one-to-one if and only if $D^* = C$ is onto (its range is closed and dense).

(i) $\Rightarrow$ (iii) The continuity of D implies the existence of a constant B' in (5.14) by Proposition 5.1.1(b). Since D is bijective, D^{-1} is continuous by the open mapping theorem (Appendix A.5), from which the lower estimate follows.

(iii) $\Rightarrow$ (iv) Let $\{f_j : j \in J\}$ be an orthonormal basis of $\mathcal{H}$. For $f = \sum_{j \in J} c_j f_j$, define Tf by $Tf = \sum_{j \in J} c_j e_j$. Then $\|f\| = \|c\|_2$ and

$$\|Tf\| = \left\| \sum_{j \in J} c_j e_j \right\| \geq A\|c\|_2 = A\|f\|,$$

and similarly, $\|Tf\| \leq B\|f\|$ for all $f \in \mathcal{H}$. Thus T is a well defined, invertible operator on $\mathcal{H}$ and $Tf_j = e_j$, as desired.

(iv) $\Rightarrow$ (i) If $Tf_j = e_j$, $j \in J$, for an orthonormal basis $\{f_j\}$ and an invertible operator $T \in \mathcal{B}(\mathcal{H})$, then $\sum_{j \in J} c_j e_j = T\left(\sum_{j \in J} c_j f_j\right) = 0$ if and only if $\sum_{j \in J} c_j f_j = 0$ if and only if $c_j = 0$ for all $j \in J$.

(iii) $\Leftrightarrow$ (v) For any finite sequence $c = (c_j)_{j \in J}$,

$$\langle Gc, c \rangle = \sum_{m,j \in J} \langle e_m, e_j \rangle c_m \overline{c_j} = \left\| \sum_{m \in J} c_m e_m \right\|^2 .$$

Therefore equation 7.10 is equivalent to saying that G is a positive invertible operator on $l^2(J)$. □

Proposition 7.6. *Given a relaxation parameter* $0 < \lambda < \frac{2}{B}$*, set*

$$\delta = \max\{|1-\lambda A|, |1-\lambda B|\} < 1.$$

Let $f_0 = 0$ *and define recursively*

$$f_{n+1} = f_n + \lambda S(f - f_n). \tag{7.11}$$

Then $\lim_{n\to\infty} f_n = f$ *with a geometric rate of convergence, that is,*

$$\|f - f_n\| \leq \delta^n \|f\|. \tag{7.12}$$

Observe that $f_1 = \lambda S f = \lambda \sum_j \langle f, e_j\rangle e_j$ *contains the frame coefficients as input. This suffices to compute the further approximations* f_n *and to reconstruct* f *completely.*

Proof. Since $AI \leq S \leq BI$, we obtain

$$(1-\lambda B)I \leq I - \lambda S \leq (1-\lambda A)I.$$

Therefore

$$\|I - \lambda S\|_{\text{op}} \leq \max\{|1-\lambda A|, |1-\lambda B|\} = \delta < 1, \tag{7.13}$$

because $\lambda < \frac{2}{B}$. Assume that the error estimate (5.16) is true for $k = 1, \ldots, n$ (there is nothing to show for $n = 0$). Then

$$\begin{aligned}\|f - f_{n+1}\| &= \|f - f_n - \lambda S(f - f_n)\| \\ &= \|(I - \lambda S)(f - f_n)\| \\ &\leq \|I - \lambda S\|_{\text{op}} \|f - f_n\| \\ &\leq \delta\delta^n \|f\| = \delta^{n+1}\|f\|;\end{aligned}$$

so we are done. □

Lemma 7.7. *(a) If* $\{e_j : j \in J\}$ *is a tight frame of* $\mathcal{H}$ *with frame bounds* $A = B = 1$ *and if* $\|e_j\| = 1$ *for all* $j \in J$*, then* $\{e_j\}$ *is an orthonormal basis.*

(b) If $\{e_j\}$ *is a frame, then* $\{S^{-1/2}e_j\}$ *is a tight frame with frame bounds* $A = B = 1$*.*

(c) If $\{e_j\}$ *is a frame, then the inverse frame operator* S^{-1} *is given by*

$$S^{-1}f = \sum_{j\in J} \langle f, S^{-1}e_j\rangle S^{-1}e_j. \tag{5.19}$$

Thus S^{-1} *is the frame operator with respect to the dual frame* $\{S^{-1}e_j : j \in J\}$

Proof.

$$1 = \|e_m\|^2 = \sum_{j\in J} |\langle e_m, e_j\rangle|^2 = 1 + \sum_{j\neq m} |\langle e_m, e_j\rangle|^2,$$

and consequently $\langle e_m, e_j\rangle = \delta_{jm}$.

(b) Note that since S is a positive operator, the operator $S^{-1/2}$ is well defined and positive by the spectral theorem. Writing f as

$$f = S^{-1/2}S(S^{-1/2}f) = \sum_{j\in J} \left\langle f, S^{-1/2}e_j\right\rangle S^{-1/2}e_j,$$

we obtain that

$$\langle f, f\rangle = \sum_{j\in J} \left|\left\langle f, S^{-1/2}e_j\right\rangle\right|^2.$$

Therefore $\{S^{-1/2}e_j : j \in J\}$ is a tight frame. The vectors $S^{-1/2}e_j$ are not normalized in general; therefore $\{S^{-1/2}e_j\}$ need not be an orthonormal basis.

(c) follows from

$$S^{-1}f = S^{-1}S(S^{-1}f) = \sum_{j\in J}\langle f, S^{-1}e_j\rangle S^{-1}e_j.$$

□

7.2 Gabor Frames

Definition 7.6 (Gabor Frame). *Let $g \in L^2(\mathbb{R}^d)$ be a window function s.t. $g \neq 0$ and lattice parameters $\alpha, \beta > 0$, the set of time-frequency shifts*

$$\mathcal{G}(g, \alpha, \beta) = \{T_{\alpha k}M_{\beta n}g : k, n \in \mathbb{Z}^d\}, \tag{7.14}$$

is known as Gabor system. If $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d)$, then it is called as Gabor frame.

Note : The frame operator is w.r.t Gabor frame is given by

$$Sf = \sum_{k,n\in\mathbb{Z}^d} \langle f, T_{\alpha k}M_{\beta n}g\rangle T_{\alpha k}M_{\beta n}g = \sum_{k,n\in\mathbb{Z}^d} V_g f(\alpha k, \beta n)M_{\beta n}T_{\alpha k}g. \tag{7.15}$$

Proposition 7.8. *If $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d)$, then there exists a dual window $\gamma \in L^2(\mathbb{R}^d)$, such that the dual frame of $\mathcal{G}(g, \alpha, \beta)$ is $\mathcal{G}(\gamma, \alpha, \beta)$. Consequently, every $f \in L^2(\mathbb{R}^d)$*

possesses the expansions

$$f = \sum_{k,n\in\mathbb{Z}^d} \langle f, T_{\alpha k}M_{\beta n}g\rangle T_{\alpha k}M_{\beta n}\gamma \tag{5.22}$$

$$= \sum_{k,n\in\mathbb{Z}^d} \langle f, T_{\alpha k}M_{\beta n}\gamma\rangle T_{\alpha k}M_{\beta n}g \tag{5.23}$$

with unconditional convergence in $L^2(\mathbb{R}^d)$. Further, the following norm equivalences hold:

$$A\|f\|^2 \le \sum_{k,n\in\mathbb{Z}^d} |V_g f(\alpha k, \beta n)|^2 \le B\|f\|^2, \tag{7.16}$$

$$B^{-1}\|f\|^2 \le \sum_{k,n\in\mathbb{Z}^d} |\langle f, T_{\alpha k}M_{\beta n}\gamma\rangle|^2 \le A^{-1}\|f\|^2. \tag{7.17}$$

Proof. We show first that the Gabor frame operator $S = S_{g,g}^{\alpha,\beta}$ commutes with time-frequency shifts $T_{\alpha r}M_{\beta n}$. Given $f \in L^2(\mathbb{R}^d)$ and $r, s \in \mathbb{Z}^d$,

$$(T_{\alpha r}M_{\beta s})^{-1} S T_{\alpha r}M_{\beta s} f = \sum_{k,n\in\mathbb{Z}^d} \langle T_{\alpha r}M_{\beta s}f, T_{\alpha k}M_{\beta n}g\rangle (T_{\alpha r}M_{\beta s})^{-1} T_{\alpha k}M_{\beta n}g. \tag{7.18}$$

By (1.7) we have

$$(T_{\alpha r}M_{\beta s})^{-1}(T_{\alpha k}M_{\beta n}) = e^{-2\pi i\alpha\beta(k-r)\cdot s} T_{\alpha(k-r)}M_{\beta(n-s)}.$$

The phase factor $e^{2\pi i\alpha\beta(k-r)\cdot s}$ cancels in (5.24), and we obtain

$$(T_{\alpha r}M_{\beta s})^{-1} S T_{\alpha r}M_{\beta s} f = \sum_{k,n\in\mathbb{Z}^d} \langle f, T_{\alpha(k-r)}M_{\beta(n-s)}g\rangle T_{\alpha(k-r)}M_{\beta(n-s)}g = Sf. \tag{7.19}$$

After renaming the indices. Consequently S^{-1} also commutes with $T_{\alpha r}M_{\beta s}$, and the dual frame consists of the functions

$$S^{-1}(T_{\alpha k}M_{\beta n}g) = T_{\alpha k}M_{\beta n}S^{-1}g.$$

Thus we may take $\gamma = S^{-1}g$ as the dual window. The other assertions have already been proved in Corollaries 7.2 and 7.3 □

7.3 Unconditional Convergence

Definition 7.7. *(**Unconditional Convergence**) Let $\{f_j : j \in J\}$ be a countable set in a Banach space B. The series $\sum_{j\in J} f_j$ is said to converge unconditionally to $f \in B$ if for every $\epsilon > 0$ there exists a finite set $F_0 \subseteq J$ such that*

$$\left\| f - \sum_{j\in F} f_j \right\|_B < \epsilon \quad \textit{for all finite sets } F \supseteq F_0.$$

In technical terminology one says that the net of partial sums defined by $s_F = \sum_{j\in F} f_j$ converges to f.

Remark 7.2. *Since the index set J is countable, one might enumerate J by choosing a bijective map $\pi : \mathbb{N} \to J$ and then define the convergence of $\sum_{j\in J} f_j$ by the convergence of the partial sums $\sum_{n=1}^{N} f_{\pi(n)}$; that is,*

$$f = \lim_{N\to\infty} \sum_{n=1}^{N} f_{\pi(n)}.$$

This approach encounters two problems: (a) For unstructured index sets there is no natural enumeration π and thus no natural sequence of partial sums. (b) In general it is not clear whether the limit in (5.28) is independent of the enumeration π. However, if the series converges unconditionally, then these problems cannot arise, as is shown by the following proposition.

Proposition 7.9. *Let $\{f_j : j \in J\}$ be a countable set in the Banach space B. Then the following are equivalent:*

(i) $f = \sum_{j\in J} f_j$ converges unconditionally to $f \in B$.

(ii) For every enumeration $\pi : \mathbb{N} \to J$ the sequence of partial sums $\sum_{n=1}^{N} f_{\pi(n)}$ converges to $f \in B$, that is,

$$\lim_{N\to\infty} \left\| f - \sum_{n=1}^{N} f_{\pi(n)} \right\|_B = 0.$$

In particular, the limit f is independent of the enumeration π.

Proof. (i) $\Rightarrow$ (ii) Let $\pi : \mathbb{N} \to J$ be an enumeration of J and let $\epsilon > 0$. Since $\sum_{j\in J} f_j$ converges unconditionally, there is a finite set $F_0 \subseteq J$ such that $\left\| f - \sum_{j\in F} f_j \right\|_B < \epsilon$ for $F \supseteq F_0$. Now choose N_0 large enough such that $F_0 \subseteq \{\pi(1), \pi(2), \dots, \pi(N_0)\}$. Then

$$\left\| f - \sum_{n=1}^{N} f_{\pi(n)} \right\|_B < \epsilon \quad \text{for } N \geq N_0.$$

(ii) $\Rightarrow$ (i) Assume that every rearrangement of $\sum_{j\in J} f_j$ converges to f, but that $\sum_{j\in J} f_j$ does not converge unconditionally. Then there exists $\epsilon > 0$ such that for every finite set $F \subseteq J$ there is $F' \supseteq F$ with $\left\| f - \sum_{j\in F'} f_j \right\| B \geq \epsilon$. Fix an enumeration $\pi : \mathbb{N} \to J$. Since $\sum n = 1^{\infty} f_{\pi(n)}$ converges, there is an index $N_0 \in \mathbb{N}$ such that

$$\left\| f - \sum_{n=1}^{N} f_{\pi(n)} \right\|_B < \epsilon/2 \quad \text{for all } N \geq N_0.$$

By induction we can therefore construct a sequence of finite sets $F_n \subseteq J$ of cardinality N_n with the following properties:

(a) $F_n \subseteq F_{n+1}$ for $n \in \mathbb{N}$,

(b) $\left\| f - \sum_{j \in F_{2n}} f_j \right\|_B > \epsilon$ for the sets with even index, and

(c) F_{2n+1} is of the form $\{\pi(1), \pi(2), \ldots, \pi(N_{2n+1})\}$ where N_{2n+1} is chosen large enough so that $F_{2n+1} \supseteq F_{2n}$. Then

$$\left\| f - \sum_{j \in F_{2n+1}} f_j \right\| B = \left\| f - \sum n = 1^{N_{2n+1}} f_{\pi(n)} \right\|_B < \epsilon/2.$$

Now we define a new rearrangement $\sigma : \mathbb{N} \to J$ by enumerating the elements in the finite sets $F_1, F_2 \setminus F_1, \ldots, F_{n+1} \setminus F_n, \ldots$ consecutively. Then we have

$$\left\| \sum_{n=N_{2n}+1}^{N_{2n+1}} f_{\sigma(n)} \right\| B = \left\| \sum j \in F_{2n+1} f_j - \sum_{j \in F_{2n}} f_j \right\|_B$$

$$\geq \left\| f - \sum_{j \in F_{2n}} f_j \right\| B - \left\| f - \sum j \in F_{2n+1} f_j \right\|_B > \epsilon - \epsilon/2 = \epsilon/2.$$

Therefore $\sum_{n=1}^{\infty} f_{\sigma(n)}$ does not converge, which contradicts the assumption. □

CHAPTER 8

Existence of Gabor Frames

8.1 Wiener Space

Definition 8.1. *(**Wiener Space**[1]) A function $g \in L^\infty(\mathbb{R}^d)$ belongs to the Wiener space $W = W(\mathbb{R}^d)$ if*

$$\|g\|_W = \sum_{n\in\mathbb{Z}^d} ess \sup_{x\in[0,1]^d} |g(x+n)| < \infty. \tag{8.1}$$

Note : Observe that $\|g\|_W$ is the sum of all $ess \sup$ of g on the shifted cubes. We can rewrite $\|g\|_W$ as

$$\|g\|_W = \sum_{n\in\mathbb{Z}^d} \|g \cdot T_n \chi_[0,1]^d\|_\infty. \tag{8.2}$$

Remark 8.1. *$W(\mathbb{R}^d)$ ia a Banach space over the norm $\|\cdot\|_W$.*

Lemma 8.1. *If $f \in W(\mathbb{R}^d)$ and $\gamma > 0$, then*

$$ess \sup_{x\in\mathbb{R}^d} \sum_{n\in\mathbb{Z}^d} |g(x-\gamma n)| \leq \left(\frac{1}{\gamma}+1\right)^d \|g\|_W. \tag{8.3}$$

Proof. Since any interval of length one contains at most $\gamma^{-1}+1$ distinct points of minimum distance γ between each other, the cube $k + Q = \prod_{j=1}^d [k_j, k_j+1]$ contains at most $(\gamma^{-1}+1)^d$ points of the form $x + \gamma n$, $n \in \mathbb{Z}^d$, independently of $x \in \mathbb{R}^d$. Therefore,

$$\sum_{n\in\mathbb{Z}^d} |g(x-\gamma n)| \leq \sum_{k\in\mathbb{Z}^d} \left(\frac{1}{\gamma}+1\right)^d \sup_{\{n: x-\gamma n\in k+Q\}} |g(x-\gamma n)|$$

$$\leq \left(\frac{1}{\gamma}+1\right)^d \sum_{k\in\mathbb{Z}^d} \|g\cdot T_k\chi_Q\|_\infty$$
$$= \left(\frac{1}{\gamma}+1\right)^d \|g\|_W.$$

□

8.2 Boundedness of Gabor Frame Operator

Lemma 8.2. ***(Sachur's Test)***

(a) Let $(a_{jk})_{j,k\in J}$ be an infinite matrix over the index set J such that

$$\sup_{j\in J}\sum_{k\in J}|a_{jk}| \leq K_1, \tag{8.4}$$
$$\sup_{k\in J}\sum_{j\in J}|a_{jk}| \leq K_2. \tag{8.5}$$

Then the operator A defined by the matrix-vector multiplication $(Ac)j = \sum_{k\in J} a_{jk}c_k$ is bounded from $l^p(J)$ to $l^p(J)$ for $1\leq p\leq\infty$. The operator norm of A is bounded by

$$\|A\|_{l^p\to l^p} \leq K_1^{1/p'}K_2^{1/p}. \tag{8.6}$$

(b) Let $K(x,y)$ be a (measurable) function on $\mathbb{R}^{2d}$ that satisfies the conditions

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|K(x,y)|\,\mathrm{d}y \leq K_1 \quad \textit{and} \quad \sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}|K(x,y)|\mathrm{d}x \leq K_2.$$

Then the integral operator A defined by $Af(x) = \int_{\mathbb{R}^d}K(x,y)f(y)\mathrm{d}y$ is bounded from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for $1\leq p\leq\infty$ with the same bound for the operator norm as in equation 8.6.

Proof. *(a)* Let p and p' are exponent conjugate of each other i.e $\frac{1}{p}+\frac{1}{p'}=1$. Consider

$$(Ac)_j = \sum_{k\in J} a_{jk}c_k = \sum_{k\in J} a_{jk}^{\frac{1}{p'}} a_{jk}^{\frac{1}{p}} c_k.$$

By Holder's inequaliy

$$\implies |(Ac)_j| \leq \left(\sum_{k\in J}|a_{jk}|\right)^{\frac{1}{p'}} \left(\sum_{k\in J}|a_{jk}||c_k|^{\frac{1}{p}}\right)^{\frac{1}{p}}.$$

By taking sum over j

$$\sum_{j\in J}|(Ac)_j|^p \leq \sum_{j\in J}\left(\sum_{k\in J}|a_{jk}|\right)^{\frac{p}{p'}} \left(\sum_{k\in J}|a_{jk}||c_k|^{p}\right).$$

By Fubini's theorem

$$K_1^{\frac{p}{p'}} \leq \sum_{j\in J}\sum_{k\in J}|a_{jk}||c_k|^p$$

□

Proposition 8.3. *If $g \in W(\mathbb{R}^d)$ and $\alpha,\beta > 0$, then $D_{g,\alpha,\beta}$ is bounded from $\ell^2(\mathbb{Z}^{2d})$ into $L^2(\mathbb{R}^d)$, and its operator norm satisfies*

$$\|D_{g,\alpha,\beta}\|_{op} \leq \left(\frac{1}{\alpha}+1\right)^{d/2}\left(\frac{1}{\beta}+1\right)^{d/2}\|g\|_W. \tag{8.7}$$

Proof. Suppose that $c \in \ell^2(\mathbb{Z}^{2d})$ has finite support and consider the trigonometric polynomials

$$m_k(x) = \sum_{n\in\mathbb{Z}^d} c_{kn} e^{2\pi i\beta n\cdot(x-\alpha k)}.$$

These m_k are $\frac{1}{\beta}\mathbb{Z}^d$-periodic and by their L^2-norm is

$$\|m_k\|L^2(Q1/\beta)^2 = \int_{Q_{1/\beta}} |m_k(x)|^2\,dx = \beta^{-d}\sum_{n\in\mathbb{Z}^d}|c_{kn}|^2. \tag{8.8}$$

Further, only finitely many m_k's are non-zero. We write

$$f = \sum_{k,n\in\mathbb{Z}^d} c_{kn}T_{\alpha k}M_{\beta n}g = \sum_{k\in\mathbb{Z}^d} m_k\cdot T_{\alpha k}g$$

and use the periodization trick to calculate the L^2-norm of f

$$\|f\|2^2 = \sum j,k\in\mathbb{Z}^d \int_{\mathbb{R}^d} m_k(x)\overline{m_j(x)}g(x-\alpha k)\overline{g(x-\alpha j)}\,dx$$

$$= \sum_{j,k\in\mathbb{Z}^d}\int_{Q_{1/\beta}} m_k(x)\overline{m_j(x)}\sum_{r\in\mathbb{Z}^d} g\left(x-\alpha k-\frac{r}{\beta}\right)\overline{g\left(x-\alpha j-\frac{r}{\beta}\right)}\,dx.$$

Everything is well defined since $T_{\alpha k}g\cdot T_{\alpha j}\bar{g} \in L^1(\mathbb{R}^d)$ and thus its periodization is in $L^1(Q_{1/\beta})$.

For each $x \in \mathbb{R}^d$ define the matrix $\Gamma(x) = (\Gamma(x)jk)j, k \in \mathbb{Z}^d$ by

$$\Gamma_{jk}(x) = \sum_{r \in \mathbb{Z}^d} \overline{g\left(x - \alpha j - \frac{r}{\beta}\right)} g\left(x - \alpha k - \frac{r}{\beta}\right). \tag{8.9}$$

The corresponding operators $\Gamma(x)$ acts on finite sequences $a = (a_k)_{k \in \mathbb{Z}^d}$ by

$$(\Gamma(x)a)j = \sum k \in \mathbb{Z}^d \Gamma_{jk}(x) a_k.$$

We wish to apply Schur's test to verify that almost all $\Gamma(x)$ are bounded operators on $\ell^2(\mathbb{Z}^d)$.

Claim: If $g \in W(\mathbb{R}^d)$, then for almost all x

$$\sum_{j \in \mathbb{Z}^d} |\Gamma_{jk}(x)| \leq \left(\frac{1}{\alpha} + 1\right)^d (\beta + 1)^d \|g\|_W^2. \tag{8.10}$$

The same estimate holds for $\sum_{k \in \mathbb{Z}^d} |\Gamma_{jk}(x)|$. To see this, we use Fubini's theorem and 8.3

$$\begin{aligned}
\sum_{j \in \mathbb{Z}^d} |\Gamma_{jk}(x)| &\leq \sum_{j \in \mathbb{Z}^d} \sum_{r \in \mathbb{Z}^d} |g(x - \alpha j - \tfrac{r}{\beta})|\, |g(x - \alpha k - \tfrac{r}{\beta})| \\
&= \sum_{r \in \mathbb{Z}^d} \left(\sum_{j \in \mathbb{Z}^d} |g(x - \alpha j - \tfrac{r}{\beta})| \right) |g(x - \alpha k - \tfrac{r}{\beta})| \qquad \text{by lemma 8.3} \\
&\leq \left(\frac{1}{\alpha} + 1\right)^d \|g\|W \cdot \sum r \in \mathbb{Z}^d |g(x - \alpha k - \tfrac{r}{\beta})| \qquad \text{by lemma 8.3} \\
&\leq \left(\frac{1}{\alpha} + 1\right)^d (\beta + 1)^d \|g\|_W^2.
\end{aligned}$$

Since $\Gamma_{jk}(x) = \overline{\Gamma_{kj}(x)}$, the sum over k is estimated identically.

Now Schur's test implies that, for almost all x, $\Gamma(x)$ extends to a bounded operator on $\ell^p(\mathbb{Z}^d), 1 \leq p \leq \infty$, with bound

$$\|\Gamma(x)\|_{op} \leq \left(\frac{1}{\alpha} + 1\right)^d (\beta + 1)^d \|g\|_W^2 := k(\alpha, \beta, g).$$

Therefore for almost all x,

$$0 \leq \sum_{j,k \in \mathbb{Z}^d} m_k(x) \overline{m_j(x)}\, \Gamma_{jk}(x) \leq k(\alpha, \beta, g) \sum_{k \in \mathbb{Z}^d} |m_k(x)|^2.$$

After integrating over $Q_{1/\beta}$ and using equation 8.8, we obtain

$$\begin{aligned}
\|f\|2^2 &= \int_{Q_{1/\beta}} \sum_{j,k \in \mathbb{Z}^d} m_k(x) \overline{m_j(x)}\, \Gamma_{jk}(x) \mathrm{d}x \\
&\leq k(\alpha, \beta, g) \sum_{k \in \mathbb{Z}^d} \int_{Q_{1/\beta}} |m_k(x)|^2 \mathrm{d}x \\
&= \left(\frac{1}{\alpha} + 1\right)^d (\beta + 1)^d \beta^{-d} \|g\|_W^2 \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} |c_{kn}|^2,
\end{aligned}$$

and the proposition is proved. □

Corollary 8.4. *If $g \in W(\mathbb{R}^d)$, then $C_{g,\alpha,\beta} : L^2(\mathbb{R}^d) \to \ell^2(\mathbb{Z}^{2d})$ and $S_{g,g} = D_g C_g : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ are bounded operators with operator norms*

$$\|C_{g,\alpha,\beta}\|_{\text{op}} \leq \left(\frac{1}{\alpha}+1\right)^{d/2} \left(\frac{1}{\beta}+1\right)^{d/2} \|g\|_W$$

and

$$\|S_{g,g}\|_{\text{op}} \leq \left(\frac{1}{\alpha}+1\right)^{d} \left(\frac{1}{\beta}+1\right)^{d} \|g\|_W^2. \tag{8.11}$$

Corollary 8.5. *If $\hat{g} \in W(\mathbb{R}^d)$, then C_g, D_g, and $S_{g,g}$ are bounded operators, and*

$$\|D_{g,\alpha,\beta}\|_{\text{op}} \leq \left(\frac{1}{\alpha}+1\right)^{d/2} \left(\frac{1}{\beta}+1\right)^{d/2} \|\hat{g}\|_W.$$

Proof. Taking the Fourier transform of Dc and re-indexing, we obtain

$$(D_{g,\alpha,\beta}c)\hat{} = \sum_{k,n\in\mathbb{Z}^d} c_{kn}\, M_{-\alpha k} T_{\beta n}\hat{g} = D_{\hat{g},\beta,\alpha}\, \tilde{c},$$

where $\tilde{c}_{kn} = c_{-n,k} e^{2\pi i\langle \alpha\beta k,n\rangle}$. Thus

$$\|D_{g,\alpha,\beta}\, c\|2 = \|D\hat{g},\beta,\alpha\,\tilde{c}\|_2 \leq \left(\frac{1}{\alpha}+1\right)^{d/2} \left(\frac{1}{\beta}+1\right)^{d/2} \|\hat{g}\|_W\, \|\tilde{c}\|_2$$

and $\|\tilde{c}\|_2 = \|c\|_2$. □

8.3 Walnut's Representation of the Gabor Frame Operator

The Gabor frame operator w.r.t function g and γ is given by

$$S_{g,\gamma} f = D_\gamma C_g f = \sum_{k,n\in\mathbb{Z}^d} \langle f, T_{\alpha k} M_{\beta n} g\rangle T_{\alpha k} M_{\beta n}\gamma. \tag{8.12}$$

Definition 8.2. *(**Correlation functions**[1]) Given $g,\gamma \in L^2(\mathbb{R}^d)$ and $\alpha,\beta > 0$, the* corre-lation functions *of the pair (g,γ) are defined to be*

$$G_n(x) = \sum_{k\in\mathbb{Z}^d} \overline{g\left(x - \frac{n}{\beta} - \alpha k\right)}\, \gamma(x-\alpha k) \tag{8.13}$$

for $n \in \mathbb{Z}^d$.

By definition, the G_n's are the periodizations of $T_{n/\beta}\overline{g}\cdot\gamma$ with period $\alpha\mathbb{Z}^d$, and by Lemma , we have $G_n \in L^1(Q_\alpha)$.

Lemma 8.6. *If $g, \gamma \in W(\mathbb{R}^d)$, then $G_n \in L^\infty(\mathbb{R}^d)$, and*

$$\sum_{n\in\mathbb{Z}^d} \|G_n\|_\infty \le \left(\frac{1}{\alpha}+1\right)^d (2\beta+2)^d \, \|g\|_W \, \|\gamma\|_W. \tag{8.14}$$

Proof. Since $\|T_{\frac{n}{\beta}}\overline{g}\cdot\gamma\|W \le \|g\|\infty\|\gamma\|W$, we have $T\frac{n}{\beta}\overline{g}\cdot\gamma \in W(\mathbb{R}^d)$. Lemma 8.3 applied to $T_{\frac{n}{\beta}}\overline{g}\cdot\gamma$ implies that

$$\|G_n\|\infty \le \left(\frac{1}{\alpha}+1\right)^d \|T\frac{n}{\beta}\overline{g}\cdot\gamma\|_W.$$

Consequently,

$$\sum_{n\in\mathbb{Z}^d} \|G_n\|_\infty \le \left(\frac{1}{\alpha}+1\right)^d \sum_{n\in\mathbb{Z}^d} \|T_{\frac{n}{\beta}}\overline{g}\cdot\gamma\|_W \tag{8.15}$$

$$\begin{aligned}
\sum_{n\in\mathbb{Z}^d} \|G_n\|_\infty &\le \left(\frac{1}{\alpha}+1\right)^d \sum_{n\in\mathbb{Z}^d} \|T_{\frac{n}{\beta}}\overline{g}\cdot\gamma\|_W \\
&= \left(\frac{1}{\alpha}+1\right)^d \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d} \|(T_{\frac{n}{\beta}}\overline{g}\cdot T_k\chi_Q)\cdot(\gamma\cdot T_k\chi_Q)\|_\infty \\
&\le \left(\frac{1}{\alpha}+1\right)^d \sum_{k\in\mathbb{Z}^d} \left(\sum_{n\in\mathbb{Z}^d} \|T_{\frac{n}{\beta}}\overline{g}\cdot T_k\chi_Q\|\infty\right) \|\gamma\cdot T_k\chi_Q\|\infty.
\end{aligned} \tag{8.16}$$

We estimate each term in the inner sum by

$$\|T_{\frac{n}{\beta}}\overline{g}\cdot T_k\chi_Q\|\infty = ess\sup x \in -\frac{n}{\beta}+k+Q|g(x)| \le \sum_{\ell\in I_n} \|g\cdot T_\ell\chi_Q\|_\infty,$$

where $I_n = \{\ell\in\mathbb{Z}^d : -\frac{n}{\beta}+k+Q\cap\ell+Q \ne \emptyset\}$. Since each $\ell\in\mathbb{Z}^d$ occurs in at most $(2\beta+2)^d$ of the I_n's,

$$\sum_{n\in\mathbb{Z}^d} \|T_{\frac{n}{\beta}}\overline{g}\cdot T_k\chi_Q\|\infty \le (2\beta+2)^d \sum_{\ell\in\mathbb{Z}^d} \|g\cdot T_\ell\chi_Q\|_\infty = (2\beta+2)^d\|g\|_W,$$

independently of $k\in\mathbb{Z}^d$. The outer sum over k in equation 8.15 becomes $\|\gamma\|_W$, and by combining these estimates, we obtain

$$\sum_{n\in\mathbb{Z}^d} \|G_n\|_\infty \le \left(\frac{1}{\alpha}+1\right)^d (2\beta+2)^d\|g\|_W\|\gamma\|_W.$$

□

Theorem 8.7. *(**Walnut's representation**[1]) Let $g,\gamma\in W(\mathbb{R}^d)$ and let $\alpha,\beta>0$. Then the operator*

$$S_{g,\gamma}f = \sum_{k,n\in\mathbb{Z}^d} \langle f, T_{\alpha k}M_{\beta n}g\rangle \, T_{\alpha k}M_{\beta n}\gamma \tag{8.17}$$

can be written as

$$S_{g,\gamma}f = \beta^{-d} \sum_{n\in\mathbb{Z}^d} G_n\cdot T_{\frac{n}{\beta}}f. \tag{8.18}$$

Moreover, $S_{g,\gamma}$ is bounded on all L^p-spaces, $1 \le p \le \infty$ with operator norm

$$\|S_{g,\gamma}\|_{L^p \to L^p} \le 2^d \left(\frac{1}{\alpha}+1\right)^d \left(\frac{1}{\beta}+1\right)^d \|g\|_W \|\gamma\|_W.$$

Proof. It is more convenient to write $S_{g,\gamma}$ in the form

$$S_{g,\gamma} f = \sum_{k,n} \langle f, M_{\beta n} T_{\alpha k} g\rangle \, M_{\beta n} T_{\alpha k} \gamma.$$

We already know from Corollary 8.4 that, for all $f \in L^2(\mathbb{R}^d)$,

$$\left\{\langle f, M_{\beta n} T_{\alpha k} g\rangle : k, n \in \mathbb{Z}^d\right\} \in \ell^2(\mathbb{Z}^{2d}).$$

Therefore the Fourier series

$$m_k(x) = \sum_{n\in\mathbb{Z}^d} (f \cdot T_{\alpha k}\overline{g})\,(\beta n)\, e^{2\pi i \beta n \cdot x} = \sum_{n\in\mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} g\rangle \, e^{2\pi i \beta n\cdot x}$$

are in $L^2(Q_{\frac{1}{\beta}})$ and are $\frac{1}{\beta}\mathbb{Z}^d$-periodic with L^2-norm

$$\|m_k\| L^2(Q_{\frac{1}{\beta}})^2 = \beta^{-d} \sum_{n\in\mathbb{Z}^d} |\langle f, M_{\beta n} T_{\alpha k} g\rangle|^2.$$

We would like to employ the Poisson summation formula and write m_k as

$$m_k(x) = \beta^{-d} \sum_{n\in\mathbb{Z}^d} (f \cdot T_{\alpha k}\overline{g}) \left(x - \frac{n}{\beta}\right). \tag{8.19}$$

This is certainly a correct identity almost everywhere on $Q_{1/\beta}$, whenever f is bounded with compact support. With this assumption, the right-hand side of equation 8.19 converges absolutely almost everywhere and has the Fourier coefficients

$$\beta^d \int_{Q_{1/\beta}} \left(\beta^{-d} \sum_{n\in\mathbb{Z}^d} (f \cdot T_{\alpha k}\bar{g}) \left(x - \frac{n}{\beta}\right)\right) e^{-2\pi i \beta l\cdot x}\, dx = \int_{\mathbb{R}^d} (f \cdot T_{\alpha k}\bar{g})(x)\, e^{-2\pi i\beta l \cdot x}\, dx$$

$$= \langle f, M_{\beta l} T_{\alpha k} g\rangle.$$

Writing $S_{g,\gamma} f$ as an iterated sum and substituting in equation 8.19, we obtain

$$S_{g,\gamma} f(x) = \sum_{k\in\mathbb{Z}^d} \left(\beta^{-d} \sum_{n\in\mathbb{Z}^d} f\left(x - \frac{n}{\beta}\right) \bar{g}(x - \alpha k - \frac{n}{\beta})\right) \gamma(x-\alpha k).$$

If f has compact support, then for fixed $x \in \mathbb{R}^d$ the sum over n is finite, and the order of summation can be interchanged. Thus

$$S_{g,\gamma} f(x) = \sum_{n\in\mathbb{Z}^d} \left(\beta^{-d} \sum_{k\in\mathbb{Z}^d} \bar{g}\left(x - \frac{n}{\beta} - \alpha k\right) \gamma(x-\alpha k)\right) f\left(x - \frac{n}{\beta}\right),$$

or in short $S_{g,\gamma} f = \beta^{-d} \sum_{n\in\mathbb{Z}^d} G_n \cdot T_{\frac{n}{\beta}} f$.

The boundedness on $L^p(\mathbb{R}^d)$ then follows via the triangle inequality and Lemma 8.6:

$$\begin{aligned}\|S_{g,\gamma}f\|p &\leq \beta^{-d}\sum n\in\mathbb{Z}^d\|G_n\cdot T_{\frac{n}{\beta}}f\|_p\\ &\leq \beta^{-d}\sum_{n\in\mathbb{Z}^d}\|G_n\|\infty\|T\frac{n}{\beta}f\|_p\\ &= \left(\beta^{-d}\sum_{n\in\mathbb{Z}^d}\|G_n\|_\infty\right)\|f\|_p.\end{aligned}$$

So far we have proved this representation of $S_{g,\gamma}$ for bounded functions with compact support. By density it extends to the full L^p-space . □

Corollary 8.8. *Under the assumptions of Theorem 8.7 we have*

$$\langle S_{g,\gamma}f,h\rangle = \sum_{j,l\in\mathbb{Z}^d}\beta^{-d}\int_{Q_{1/\beta}} G_{jl}(x)\,T_{\frac{j}{\beta}}f(x)\,\overline{T_{\frac{l}{\beta}}h(x)}\,dx \tag{8.20}$$

for all $f,h\in L^2(\mathbb{R}^d)$.

Proof. Once again we assume that f and h are bounded and have compact support. Then the sequences $\{T_{\frac{j}{\beta}}f(x) : j\in\mathbb{Z}^d\}$ and $\{T_{\frac{l}{\beta}}h(x) : j\in\mathbb{Z}^d\}$ have finite support. We substitute 8.18 for $S_{g,\gamma}f$ and use the periodization trick with period $\frac{1}{\beta}\mathbb{Z}^d$ and obtain

$$\langle S_{g,\gamma}f,h\rangle = \beta^{-d}\int_{\mathbb{R}^d}\left(\sum_{n\in\mathbb{Z}^d}G_n(x)\,f(x-\tfrac{n}{\beta})\right)h(x)\,dx \tag{8.21}$$

$$= \beta^{-d}\int_{Q_{1/\beta}}\sum_{j\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}G_n(x-\tfrac{j}{\beta})\,f(x-\tfrac{j+n}{\beta})\,h(x-\tfrac{j}{\beta})\,dx. \tag{8.22}$$

The sum over j and n is finite, so the interchange of the order of summation and integration need not be justified.

$$\langle S_{g,\gamma}f,h\rangle = \beta^{-d}\int_{Q_{1/\beta}}\sum_{j,l\in\mathbb{Z}^d}G_{jl}(x)\,T_{\frac{j}{\beta}}f(x)\,\overline{T_{\frac{l}{\beta}}h(x)}\,dx, \tag{8.23}$$

as desired.

Step 2. For the extension of 8.20 to arbitrary $f,h\in L^2(\mathbb{R}^d)$, we consider for each $x\in\mathbb{R}^d$ the operator $G(x)$ defined on finite sequences $c=(c_l)_{l\in\mathbb{Z}^d}$ by the matrix multiplication

$$(G(x)c)j = \sum l\in\mathbb{Z}^d G_{jl}(x)\,c_l.$$

If $g,\gamma\in W(\mathbb{R}^d)$, then we verify as in 8.10 that each $G(x)$ satisfies the conditions of Schur's test. Using Lemma 8.3, we obtain that

$$\sum_{l\in\mathbb{Z}^d}|G_{jl}(x)| \leq \sum_{l\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}|g(x-\tfrac{l}{\beta}-\alpha k)|\,|\gamma(x-\tfrac{j}{\beta}-\alpha k)| \tag{8.24}$$

$$= \sum_{k\in\mathbb{Z}^d}\left(\sum_{l\in\mathbb{Z}^d}|g(x-\tfrac{l}{\beta}-\alpha k)|\right)|\gamma(x-\tfrac{j}{\beta}-\alpha k)| \tag{8.25}$$

$$\leq (\beta+1)^d \left(\tfrac{1}{\alpha}+1\right)^d \|g\|_W \|\gamma\|_W. \tag{8.26}$$

A similar estimate holds for $\sum_{j\in\mathbb{Z}^d} |G_{jl}(x)|$. Thus for almost all $x \in \mathbb{R}^d$, the matrix $G(x)$ defines a bounded operator on $\ell^p(\mathbb{Z}^d), 1 \leq p \leq \infty$. If $f \in L^2(\mathbb{R}^d)$, then the sequences $\{f(x-\frac{l}{\beta}) : l \in \mathbb{Z}^d\}$ and $\{h(x-\frac{l}{\beta}) : l \in \mathbb{Z}^d\}$ are in $\ell^2(\mathbb{Z}^d)$ for almost all $x \in \mathbb{R}^d$, and the matrix representation holds indeed for arbitrary $f, h \in L^2(\mathbb{R}^d)$. □

Proposition 8.9. ***(Ron-Shen)*** *Let $g \in L^2(\mathbb{R}^d)$ and let $\alpha, \beta > 0$.*

(a) Then $S_{g,g}$ is a bounded operator on $L^2(\mathbb{R}^d)$ if and only if

$$G(x) \leq b\, I_{\ell^2} \quad \text{for a.a. } x \in \mathbb{R}^d$$

for some constant $b > 0$. In this case, $\|S_{g,g}\|\text{op} = \beta^{-d} \operatorname{ess\,sup} x \in \mathbb{R}^d \|G(x)\|_{\text{op}}$.

(b) $S_{g,g}$ is invertible on $L^2(\mathbb{R}^d)$ if and only if

$$G(x) \geq a\, I_{\ell^2} \quad \text{for a.a. } x \in \mathbb{R}^d$$

for some constant $a > 0$.

Proof. We show only (b), the proof of (a) is similar.

Assume first that $G(x) \geq a\, I_{\ell^2}$ for a.a. x. Then we have for every $f \in L^\infty(\mathbb{R}^d)$ with compact support that

$$\sum_{j,l\in\mathbb{Z}^d} G_{jl}(x)\, T_{\frac{j}{\beta}} f(x)\, \overline{T_{\frac{l}{\beta}} f(x)} \geq a \sum_{j\in\mathbb{Z}^d} |T_{\frac{j}{\beta}} f(x)|^2.$$

Integrating over $Q_{1/\beta}$ and using 8.20, we obtain

$$\langle Sf, f\rangle \geq \alpha\, \beta^{-d} \int_{Q_{1/\beta}} \sum_{j\in\mathbb{Z}^d} |T_{\frac{j}{\beta}} f(x)|^2 = a\, \beta^{-d} \|f\|_2^2. \tag{8.27}$$

By density this inequality extends to all $f \in L^2(\mathbb{R}^d)$.

Conversely, suppose that S is not invertible. Then there exists a sequence of bounded functions with compact support such that

$$\langle Sf_n, f_n\rangle < \frac{1}{n} \|f_n\|_2^2.$$

Using 8.20, this inequality implies that

$$\int_{Q_{1/\beta}} \left(\frac{1}{n} \sum_{j\in\mathbb{Z}^d} |f(x-\tfrac{j}{\beta})|^2 - \beta^{-d} \sum_{j,l\in\mathbb{Z}^d} G_{jl}(x)\, f(x-\tfrac{l}{\beta})\, \overline{f(x-\tfrac{j}{\beta})} \right) dx > 0.$$

Therefore, there exist sets $E_n \subseteq Q_{1/\beta}$ of positive measure such that

$$\frac{1}{n} \sum_{j\in\mathbb{Z}^d} |f(x-\tfrac{j}{\beta})|^2 - \beta^{-d} \sum_{j,l\in\mathbb{Z}^d} G_{jl}(x)\, f(x-\tfrac{l}{\beta})\, \overline{f(x-\tfrac{j}{\beta})} > 0 \quad \text{for } x \in E_n.$$

Consequently,

$$\inf_{\|c\|_{\ell^2}\leq 1} \langle c, G(x)c\rangle < \frac{\beta^d}{n} \quad \text{for } x \in E_n,$$

and a uniform inequality $a\, I_{\ell^2} \leq G(x)$ cannot hold for almost all x.

□

Theorem 8.10. *Suppose that $g \in L^\infty(\mathbb{R}^d)$ is supported on the cube $Q_L = [0, L]^d$. If $\alpha \leq L$ and $\beta \leq \frac{1}{L}$, then the frame operator $S = S_{g,g}$ is the multiplication operator*

$$Sf(x) = \left(\beta^{-d} \sum_{k\in\mathbb{Z}^d} |g(x-\alpha k)|^2\right) f(x).$$

Consequently, $G(g, \alpha, \beta)$ is a frame with frame bounds $\beta^{-d}a$ and $\beta^{-d}b$ if and only if

$$a \leq \sum_{k\in\mathbb{Z}^d} |g(x-\alpha k)|^2 \leq b \quad a.e. \tag{8.28}$$

Further, $G(g, \alpha, \beta)$ is a tight frame, if and only if $\sum_{k\in\mathbb{Z}^d} |g(x-\alpha k)|^2 = const.$ almost everywhere.

Proof. Since $g \in W(\mathbb{R}^d)$, we only need to check the correlation functions $G_n = \sum_{k\in\mathbb{Z}^d} T_{\alpha k}(T_{n/\beta}\tilde{g}\cdot g)$ in Walnut's representation. If $n \neq 0$, then

$$\text{supp } T_{n/\beta}\tilde{g}\cdot g \subseteq \left(\frac{n}{\beta} + [0, L]^d\right) \cap [0, L]^d$$

is either empty (if $\beta < \frac{1}{L}$) or possibly a set of measure 0 (if $\beta = \frac{1}{L}$). This implies that $G_n = 0$ a.e. for $n \neq 0$, and thus from (6.19),

$$Sf = \beta^{-d} G_0 \cdot f$$

is a multiplication operator. It is obvious that S is bounded and invertible if and only if (6.28) holds, and that S is a multiple of the identity if and only if G_0 is constant. □

8.4 Existence of Gabor Frames

We are now in a position to show that every "reasonable" function g generates a Gabor frame for some value of α and β. This puts our discussion of discrete time-frequency representations in Chapter 7 on firm ground and leads the way to more algorithmic approaches.

An honest but sloppy formulation of a central result on Gabor frames would read as follows:

If $g \in W(\mathbb{R}^d)$ and $\alpha, \beta > 0$ are small enough, then $G(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d)$.

For a precise formulation we need to specify a range of lattice parameters α, β for which the frame operator $S_{g,g}$ is invertible on $L^2(\mathbb{R}^d)$. In view of Walnut's representation we expect that such estimates can be expressed in terms of the correlation functions

$$G_n(x) = G_n^{(\alpha,\beta)}(x) = \sum_{k\in\mathbb{Z}^d} \overline{g}(x - \frac{n}{\beta} - \alpha k)\, g(x - \alpha k).$$

The notation $G_n^{(\alpha,\beta)}$ is ugly, but the dependence on (α, β) is crucial in the following existence theorem.

Theorem 8.11. *(**Walnut**) Suppose that $g \in W(\mathbb{R}^d)$ and that $\alpha > 0$ is chosen such that for constants $a, b > 0$*

$$a \le \sum_{k\in\mathbb{Z}^d} |g(x - \alpha k)|^2 \le b < \infty \quad a.e. \tag{8.29}$$

Then there exists a value $\beta_0 = \beta_0(\alpha) > 0$, such that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for all $\beta \le \beta_0$. Specifically, if $\beta_0 > 0$ is chosen such that

$$\sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \|G_n^{(\alpha,\beta_0)}\|\infty < \operatorname{ess\,inf} x \in \mathbb{R}^d |G_0(x)|, \tag{8.30}$$

then $\mathcal{G}(g, \alpha, \beta)$ is a frame for all $\beta \le \beta_0$ with frame bounds

$$A = \beta^{-d} \left(a - \sum_{n\neq 0} \|G_n^{(\alpha,\beta)}\|_\infty \right) \tag{8.31}$$

and

$$B = \beta^{-d} \sum_{n\in\mathbb{Z}^d} \|G_n^{(\alpha,\beta)}\|_\infty.$$

As before, we wrap the technical details of the proof in a lemma about the correlation functions.

Lemma 8.12. *Let $g \in W(\mathbb{R}^d)$ and let $\alpha > 0$. Then*

$$\lim_{\beta\to 0} \sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \|G_n^{(\alpha,\beta)}\|_\infty = 0.$$

Proof. Given $\varepsilon > 0$, we can choose a finite set $F \subseteq \mathbb{Z}^d$ such that

$$\sum_{k\notin F} \|g \cdot T_k \chi_Q\|_\infty < \varepsilon.$$

Set

$$g_1 = \sum_{k\in F} g \cdot T_k \chi_Q \quad \text{and} \quad g_2 = g - g_1 = \sum_{k\notin F} g \cdot T_k \chi_Q.$$

Then $\|g_2\|W = \sum k \notin F\|g \cdot T_k\chi_Q\|_\infty < \varepsilon$ and $\|g_1\|_W \leq \|g\|_W$. Using the decomposition $g = g_1 + g_2$, we write each G_n as $G_n = H_n + K_n + L_n$, where

$$H_n = \sum_{k\in\mathbb{Z}^d} T_{\alpha k}(T_{\frac{n}{\beta}}\bar{g}_1 \cdot g_1), \quad K_n = \sum_{k\in\mathbb{Z}^d} T_{\alpha k}(T_{\frac{n}{\beta}}\bar{g}_2 \cdot g_1), \quad \text{and} \quad L_n = \sum_{k\in\mathbb{Z}^d} T_{\alpha k}(T_{\frac{n}{\beta}}\bar{g} \cdot g_2).$$

If $\beta < (\text{diam}F + \sqrt{d})^{-1}$, then $\bar{g}_1(x - \frac{n}{\beta} - \alpha k)g_1(x - \alpha k) = 0$ for all $n \neq 0$, and thus $H_n(x) = 0$ almost everywhere. The sums over K_n and L_n are estimated using Lemma 8.6 We obtain for $\beta < (\text{diam}F + \sqrt{d})^{-1}$ that

$$\begin{aligned}\sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \|G_n\|\infty \leq \sum n \in \mathbb{Z}^d \left(\|K_n\|\infty + \|L_n\|\infty\right) \\ \leq \left(\frac{1}{\alpha} + 1\right)^d (2\beta + 2)^d \left(\|g_2\|_W\|g_1\|_W + \|g\|_W\|g_2\|_W\right) \\ \leq 2\left(\frac{1}{\alpha} + 1\right)^d (2\beta + 2)^d\|g\|_W \cdot \varepsilon,\end{aligned}$$

as asserted. □

Proof of Theorem 1.11. By lemma 8.12 there exists a $\beta_0 > 0$, such that

$$\sum_{n\neq 0} \|G_n^{(\alpha,\beta)}\|\infty < \operatorname{ess\,inf} x \in \mathbb{R}^d \sum_{k\in\mathbb{Z}^d} |g(x - \alpha k)|^2 = a$$

for $\beta \leq \beta_0$. Substituting Walnut's representation equation 8.18 for $S_{g,g}$, we obtain for any $f \in L^2(\mathbb{R}^d)$ that

$$\begin{aligned}\langle Sf, f\rangle &= \beta^{-d}\left\langle \sum_{n\in\mathbb{Z}^d} G_n \cdot T_{\frac{n}{\beta}}f, f\right\rangle. \\ &= \beta^{-d}\left(\int_{\mathbb{R}^d} G_0(x)|f(x)|^2\,dx + \sum_{n\neq 0}\left\langle G_n \cdot T_{\frac{n}{\beta}}f, f\right\rangle\right).\end{aligned}$$

(since the sum over n converges absolutely in L^2, the interchange of summation and integration is permitted). Using the trivial estimates $G_0(x) \geq a$ and

$$\left|\left\langle G_n \cdot T_{\frac{n}{\beta}}f, f\right\rangle\right| \leq \|G_n\|\infty\|T\frac{n}{\beta}f\|2\|f\|_2 = \|G_n\|\infty\|f\|_2^2,$$

$$\begin{aligned}\langle Sf, f\rangle &\geq \beta^{-d}\left(a\|f\|2^2 - \sum_{n\neq 0}\|G_n\|_\infty\|f\|_2^2\right) \\ &= \beta^{-d}\left(a - \sum_{n\neq 0}\|G_n^{(\alpha,\beta)}\|_\infty\right)\|f\|_2^2.\end{aligned}$$

This is the lower frame estimate for S. By construction the frame bound

$$A = \beta^{-d} \left(a - \sum_{n \neq 0} \|G_n\|_\infty \right)$$

is positive for all $\beta \leq \beta_0$. The upper frame estimate was already proved in Theorem 8.7.

Corollary 8.13. *Under the assumptions of Theorem 6.5.1, $S_{g,g}$ is invertible on any solid, isometrically translation-invariant Banach space $B \subseteq S'(\mathbb{R}^d)$ in which the bounded functions of compact support are dense. In particular, $S_{g,g}$ is invertible on each $L^p(\mathbb{R}^d), 1 \leq p < \infty$.*

Proof. We write $S = M + R$, where M is the multiplication operator $Mf = \beta^{-d} G_0 \cdot f$ and R is the remainder $Rf = \beta^{-d} \sum_{n \neq 0} G_n \cdot T_{\frac{n}{\beta}} f$. Then $M^{-1} f = \beta^d G_0^{-1} \cdot f$, and its operator norm on B is

$$\|M^{-1}\|B \to B = \sup \|f\|B \leq 1 \|\beta^d G_0^{-1} \cdot f\|_B \leq \beta^d \|G_0^{-1}\| \infty \leq \beta^d a^{-1}$$

as a consequence of equation 8.28.

We will show that

$$\|I - M^{-1} S\|_{B \to B} < 1. \tag{8.32}$$

This estimate implies that $M^{-1}S$ is invertible on B with inverse $(M^{-1}S)^{-1} = \sum_{\ell=0}^{\infty} (I - M^{-1}S)^\ell$. Consequently, S is also invertible on B with inverse $S^{-1} = (M^{-1}S)^{-1} M^{-1}$.

Estimate 8.32 is deduced from equation 8.30 as follows

$$\|f - M^{-1} S f\|_B = \|M^{-1} R f\|_B$$

$$= \left\| \sum_{n \neq 0} G_0^{-1} G_n \cdot T_{\frac{n}{\beta}} f \right\|_B \leq \sum_{n \neq 0} \|G_0^{-1}\| \infty \|G_n\| \infty \|f\|_B.$$

Thus $\|I - M^{-1}S\|B \to B \leq a^{-1} \sum n \neq 0 \|G_n\|_\infty < 1$. □

Lemma 8.14. *Assume that the matrix $(a_{jl})_{j,l \in J}$ defines a bounded, positive operator A on $\ell^2(J)$ and that A is diagonally dominant; that is,*

$$\inf_{j \in J} \left(|a_{jj}| - \sum_{l: l \neq j} |a_{jl}| \right) \geq \delta > 0. \tag{8.33}$$

Then $A \geq \delta I$ in the sense that $\langle Ac, c \rangle \geq \delta \|c\|^2$ for all $c \in \ell^2(J)$. Consequently, A is invertible on $\ell^2(J)$ and $\|A^{-1}\|_{\mathrm{op}} \leq \delta^{-1}$.

Proof. As in the proof of Corollary 8.13, we write $A = M + R$ where M is the diagonal matrix with entries $m_{jl} = a_{jj} \delta_{jl}$ and R is the remainder with entries $r_{jl} = a_{jl}$ if $j \neq k$ and $r_{jj} = 0$.

We estimate the expression $\langle Rc, c\rangle = \sum_{j\in J}\sum_{l:l\neq j} a_{jl}c_l\bar{c}_j$ by applying the Cauchy–Schwarz inequality first to the inner sum over l and then to the outer sum over j

$$\begin{aligned}
|\langle Rc, c\rangle| &\leq \sum_{j\in J}\sum_{l:l\neq j} |a_{jl}|^{1/2}|c_l|\,|a_{jl}|^{1/2}|c_j| \\
&\leq \sum_{j\in J}\left(\sum_{l:l\neq j}|a_{jl}||c_l|^2\right)^{1/2}\left(\sum_{l:l\neq j}|a_{jl}||c_j|^2\right)^{1/2} \\
&\leq \left(\sum_{j\in J}\sum_{l:l\neq j}|a_{jl}||c_l|^2\right)^{1/2}\left(\sum_{j\in J}\sum_{l:l\neq j}|a_{jl}||c_j|^2\right)^{1/2} = \Sigma_1\cdot\Sigma_2.
\end{aligned}$$

Since A is self-adjoint, we have $\Sigma_1 = \Sigma_2$, and hypothesis 8.33 implies that

$$\Sigma_1 = \Sigma_2 = \left(\sum_{j\in J}\sum_{l:l\neq j}|a_{jl}||c_j|^2\right)^{1/2} \leq \left(\sum_{j\in J}(a_{jj}-\delta)|c_j|^2\right)^{1/2}.$$

Therefore we obtain that

$$|\langle Rc, c\rangle| \leq \sum_{j\in J}(a_{jj}-\delta)|c_j|^2 = \langle (M-\delta I)c, c\rangle,$$

or, written as an operator inequality, $R \leq M - \delta I$. The desired estimate for A now follows from

$$\langle Ac, c\rangle = \langle (M+R)c, c\rangle \geq \langle Mc, c\rangle - |\langle Rc, c\rangle| \geq \delta\|c\|^2.$$

Consequently, A is invertible and $\|A^{-1}\|_{\mathrm{op}} \leq \delta^{-1}$. □

CHAPTER 9

Conclusion and Future Work

9.1 Conclusion

In this thesis, we have developed a foundational understanding of time-frequency analysis from a purely mathematical standpoint. Starting from basic concepts in measure theory, L^p spaces, and operator theory, we built up to the essential tools of Fourier analysis and gradually introduced the need for joint time-frequency representations.

We studied the uncertainty principle, which plays a central role in motivating the limitations of classical Fourier analysis, and proceeded to explore the short-time Fourier transform and various quadratic time-frequency distributions such as the Wigner distribution, spectrogram, and the ambiguity function. These constructions demonstrated how time and frequency information can be localized simultaneously, albeit with trade-offs governed by the uncertainty principle.

A significant portion of the thesis was devoted to frame theory, especially Gabor frames, which provide a robust framework for representing functions in $L^2(\mathbb{R}^d)$ using time-frequency shifts of a single window function. We discussed conditions for the existence of such frames and studied Walnut's representation of the frame operator, which offers further insight into the structure and behavior of Gabor systems.

Overall, this work aimed to provide a systematic and rigorous exposition of the key ideas and results in time-frequency analysis, with an emphasis on mathematical clarity and structure.

9.2 Future Work

There remain several interesting directions for further exploration. One natural extension is to study the role of pseudodifferential operators and their connection with time-frequency representations, particularly in the context of partial differential equations. Another promising area is the extension of these concepts to function spaces beyond $L^2(\mathbb{R}^d)$, such as modulation spaces and Besov spaces, which may offer finer control over localization properties.

Moreover, while the thesis focused on theory, time-frequency analysis has a wide range of applications in signal processing, quantum mechanics, and data analysis. Investigating these applied perspectives, possibly in collaboration with numerical methods or machine learning, could be a valuable future endeavor.

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