Stochastic Asymptotic Analysis of an SIRS Epidemic Model with Lévy Jumps

M.Sc. Thesis

by

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DEPARTMENT OF MATHEMATICS

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Submitted in partial fulfillment of the requirements for the award of the degree

of

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by

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"Stochastic Asymptotic Analysis of an SIRS Epidemic Model with Lévy Jumps"** in the partial fulfillment of the requirements for the award of the degree of **Master of Science** and submitted in the **Department of Mathematics**, **Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2024 to May 2025 under the supervision of **Dr. Debopriya Mukherjee**, Assistant Professor, Department of Mathematics, IIT Indore. The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

Roopanshi Mann 23 May,2025

Signature of the student with date

(**Roopanshi Mann**)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Signature of Convener, DPGC

Date: 23/05/2025

"Dedicated to my Parents"

"Nothing in this world can take the place of persistence. Talent will not; nothing is more common than unsuccessful men with talent. Genius will not; unrewarded genius is almost a proverb. Education will not; the world is full of educated derelicts. Persistence and determination alone are omnipotent. The slogan 'Press On' has solved and always will solve the problems of the human race."

–***Calvin Coolidge***

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Publications

[1] **Roopanshi Mann**, Debopriya Mukherjee, *"Stochastic Analysis of an SIRS Epidemic Model with Lévy Jumps"*- **Under Review**

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Abstract

In this thesis, we analyze an SIRS epidemic model where the stochasticity comes from the Brownian motion and Poisson-type jumps. This analysis is divided into two parts. The first part concerns the stochastic analysis in which we prove the well-posedness of the system of SDEs associated with the SIRS epidemic model, globally. Then stochastic boundedness for each population is proved under certain conditions. Eventually, sufficient conditions are derived under which the epidemic disease goes extinct almost surely. All these results are also numerically validated using the Euler-Maruyama method in Python. The second part presents the Asymptotic analysis for the stochastic SIRS epidemic model in which the Laplace Principle (equivalently Large Deviation Principle on a suitable Polish space; the Skorokhod space) for the system of SDEs is verified. The method of proof uses the weak-convergence technique of Budhiraja-Dupuis, along with the celebrated results of Varadhan and Bryc. Our work will undoubtedly assist biologists and government agencies in protecting populations from significant outbreaks.

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CHAPTER 1

Introduction

"A deterministic model is a projection of idealized certainty; a stochastic model is a reflection of reality."

The presence of randomness and uncertainty is a defining feature of many real-world systems. Whether in financial markets, physical systems, biological processes, or engineering applications, unpredictable fluctuations often play a crucial role in shaping the evolution and behavior of such systems. Traditional deterministic models, while useful in certain contexts, are often inadequate for capturing the complexity introduced by these random influences. This reality gives rise to the need to incorporate stochastic noise in mathematical models.

This chapter aims to provide the foundational motivation for the research presented in this thesis. We will also outline our research problem in brief, which will be discussed in detail in the subsequent chapters.

1.1 Motivation

Consider the Drunkard's walk model:

$$\frac{dX(t)}{dt} = v, \quad X(0) = X_0, \tag{1.1}$$

where $X(t)$ is the position at time t, and v is the constant speed. Now, let's introduce randomness to model a more realistic scenario where the person takes steps in random directions. The associated differential equation introducing noise will be given as:

$$dX(t) = vdt + \text{“Noise”}. \tag{1.2}$$

In real-world systems, movement or change is rarely purely deterministic. While a model like (1.1) describes a particle moving at constant velocity, it fails to account for unpredictable factors be it wind in physics, market noise in finance, or random gene expression in biology. Introducing a noise term, as in (1.2), transforms a simple path into a stochastic process, capturing both the predictable trend and the random fluctuations inherent in real phenomena. This combination reflects how systems evolve not only under fixed rules, but also under the influence of uncertainty. Studying stochastic differential equations is thus essential for modeling, analyzing, and understanding dynamical systems where randomness is not an exception but a fundamental feature. Always remember- *"A random walk is a drunkard's path, but it still obeys the laws of probability."*

1.2 Our Research Problem

The spread of infectious diseases continues to be a major area of study in epidemiology, especially when accounting for the randomness that naturally occurs in real-world scenarios. While deterministic models offer useful insights, they often fall short in capturing the unpredictable nature of disease transmission, recovery, and external influences. To address this limitation, our research focuses on a stochastic SIRS (Susceptible–Infectious–Recovered–Susceptible) epidemic model that incorporates both continuous random fluctuations, modeled by Brownian motion, and sudden, discrete changes, represented by Poisson-type jumps. These two types of noise reflect different sources of uncertainty: the former accounts for gradual variations such as daily contact rates or recovery times, while the latter captures abrupt events like super-spreader incidents or sudden changes in public health policy.

The central aim of this research is to analyze the behavior of this stochastic model, both from a probabilistic and asymptotic perspective. We are particularly interested in understanding how randomness affects the disease dynamics and under what conditions the disease may die out over time. To this end, the analysis is divided into two main parts.

1. **Stochastic Analysis**: In this part, we analyzed the system of SDEs and got a few results and findings. First and foremost, we established the well-posedness of the solution for the system. Secondly, we derived specific conditions that demonstrate the stochastic boundedness of each population. Finally, we focused on the primary objective of the research, which is the probability of extinction. We identified sufficient conditions under which the disease could be considered extinct almost surely.

2. **Asymptotic Analysis**: In this part, we aim to establish the Large Deviation Principle (LDP) for a class of Stochastic Differential Equations (SDEs). The approach is grounded in the variational framework developed by Budhiraja and Dupuis [7], supplemented by the use of Varadhan's and Bryc's lemma. We will demonstrate the Laplace principle, which, under suitable topological settings, specifically within the Skorokhod space, a type of Polish space, is known to be equivalent to the LDP.

1.3 Thesis Organisation

The thesis is structured as follows. In chapter 2, we discuss some definitions, results, and key observations made in the field of stochastics. Chapter 3 introduces the stochastic SIRS epidemic model with Lévy jumps, detailing its formulation and underlying assumptions. Chapter 4 provides a rigorous analysis of the system of SDEs associated with the stochastic SIRS epidemic model. Chapter 5 is devoted to establishing a Large Deviation Principle (LDP) for the proposed model, including key theoretical findings and their proofs. Chapter 6 concludes the thesis with a summary of findings and outlines possible directions for future research, along with some open questions.

CHAPTER 2

Preliminaries

This chapter presents essential definitions, prior findings, theoretical perspectives, and significant observations made by scholars in this field. These preliminaries establish a foundation for the formal discussions in the following chapters, making it easier for readers to follow. Key concepts such as stochastic calculus and the theory of Lévy processes will be included. We will also derive the numerical schemes for stochastic differential equations (SDEs), which will be utilized in later chapters. Lastly, we will discuss the theory of Large deviations, which will be necessary to prove the Large deviation principle for the solution of our model. Kindly note that this chapter is all about the preliminary material drawn from well-established books and existing literature, which will be properly cited wherever needed. This does not include our original research at all.

2.1 Stochastic Calculus

Stochastic calculus is a branch of mathematics that extends the tools of classical calculus to stochastic processes. It plays a fundamental role in the modeling and analysis of systems influenced by noise, such as financial markets, physical systems, and various natural phenomena. For a thorough and rigorous study of stochastic processes, we refer [6, 23].

Let us define a probability space, by $(\Omega, \mathcal{G}, \mathbb{P})$.

Definition 2.1. *A measurable function $X(t, \omega)$ defined on a product space $[0, \infty) \times \Omega$ is said to be a stochastic process, if*

1. *$X(t, .)$ is a random variable for each t, and*

2. *$X(., \omega)$ is a measurable function for each ω (called a sample path).*

For the sake of notations, we will write the random variable $X(t, .)$ as $X(t)$ or X_t.

Definition 2.2. *Consider a sequence of σ-algebras $\mathcal{G}_1, \mathcal{G}_2, \dots$ defined on a sample space Ω, such that they form an increasing sequence: $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \dots \subset \mathcal{G}$. This sequence is referred to as a filtration. A sequence of random variables $X_1, X_2, \dots$ is said to be adapted to the filtration $\mathcal{G}_1, \mathcal{G}_2, \dots$ if, for each index $n \geq 1$, the variable X_n is measurable with respect to $\mathcal{G}_n$.*

Definition 2.3. *A random variable $T : \Omega \to \{1, 2, \dots\} \cup \{\infty\}$ is said to be a stopping time with respect to a filtration $\mathcal{G}_n$, if*

$$T^{-1}((-\infty, n]) \in \mathcal{G}_n \quad \forall n = 1, 2, \dots$$

Definition 2.4. *A stochastic process $W(\omega, t)$ is said to be a Brownian motion (or Wiener process in infinite dimension) if it satisfies the following conditions:*

1. *Starts at zero: The process begins at the origin almost surely, i.e.,*
   $\mathbb{P}(W(0, \omega) = 0) = 1.$

2. *Increment independence: For any sequence of times* $0 \leq t_1 < t_2 < \ldots < t_n$, *the increments*

$$W(t_1),\ W(t_2) - W(t_1),\ \ldots,\ W(t_n) - W(t_{n-1})$$

are mutually independent random variables.

3. *Gaussian increments: For all* $0 \leq t_0 < t$, *the increment* $W(t) - W(t_0)$ *follows a normal distribution with mean zero and variance* $t - t_0$, *i.e.,*

$$W(t) - W(t_0) \sim \mathcal{N}(0, t - t_0).$$

4. *Continuity of paths: With probability one, the function* $t \mapsto W(t, \omega)$ *is continuous for almost every* ω *in the sample space; that is, the process has continuous sample paths almost surely.*

Why Itô's Lemma ?

In stochastic calculus, Itô's lemma plays a fundamental role analogous to the chain rule in classical calculus. When working with stochastic differential equations, especially those involving Brownian motion, standard differentiation rules no longer apply due to the non-differentiable nature of sample paths and the presence of quadratic variation. Let's explore what Itô's lemma states, but first, we'll introduce some foundational concepts of Itô's calculus from the book by Brzeźniak et al [6].

Definition 2.5. *Let* X_t *be a process adapted to a filtration and* $[0, T]$ *can be partitioned as* $t_0, t_1, \ldots$. *Let us define by* W_t, *a Brownian motion, then*

1. *the Itô's integral is defined as:*

$$\int_0^T X_t \, dW_t = \lim_{n \to \infty} \sum_{i=0}^{n-1} X_{t_i} \left(W_{t_{i+1}} - W_{t_i} \right),$$

2. *the Stratonovich integral is defined as:*

$$\int_0^T X_t \circ dW_t = \lim_{n \to \infty} \sum_{i=0}^{n-1} \frac{X_{t_i} + X_{t_{i+1}}}{2} \left(W_{t_{i+1}} - W_{t_i} \right).$$

Remark 2.1. *It is interesting to note that the method Itô used to define a stochastic integral is somewhat similar to the Riemann–Stieltjes integral. This is not a coincidence at all. The intuition for Itô's integral comes from the limitations of the Riemann–Stieltjes integral when applied to stochastic processes like Brownian motion.*

Definition 2.6. *A stochastic process $\{X_t : t \geq 0\}$ is called an Itô's process if it admits the following decomposition:*

$$X_t = X_0 + \int_0^t a(s, X_s)\, ds + \int_0^t b(s, X_s)\, dW_s,$$

where X_0 denotes the initial condition, the function $a(s, X_s)$ represents the drift term, $b(s, X_s)$ denotes the diffusion coefficient, and W_t is a standard Brownian motion.

Definition 2.7. *Let $\{X_t : t \geq 0\}$ be an Itô's process. We say that X_t solves the stochastic differential equation*

$$dX_t = a(t, X_t)\, dt + b(t, X_t)\, dW_t,$$

with initial condition X_0, an $\mathcal{F}_0$-measurable random variable, if the following integral equation holds almost surely for all $t \geq 0$:

$$X_t = X_0 + \int_0^t a(s, X_s)\, ds + \int_0^t b(s, X_s)\, dW_s.$$

Now let us introduce the most important result of Itô's calculus, that is, Itô's Lemma.

Lemma 2.1. *Let $\{X_t\}$ be an Itô's process satisfying the stochastic differential equation:*

$$dX_t = a(t, X_t)\, dt + b(t, X_t)\, dW_t.$$

Assume that $F : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a function that is continuously differentiable with respect to t and twice continuously differentiable with respect to x. Then the process $F(t, X_t)$ is also an Itô process and satisfies the following stochastic *differential:*

$$dF(t, X_t) = \left(\frac{\partial F}{\partial t} + a(t, X_t) \frac{\partial F}{\partial x} + \frac{1}{2} b^2(t, X_t) \frac{\partial^2 F}{\partial x^2} \right) dt + b(t, X_t) \frac{\partial F}{\partial x} \, dW_t.$$

The proof of Itô's lemma uses Taylor's series expansion and the non-zero quadratic variation property of Brownian motion. For details, one can refer to [6, 23].

The following theorem outlines the conditions under which a stochastic differential equation (SDE) admits a unique solution.

Theorem 2.2. *Consider the stochastic differential equation*

$$dX_t = a(t, X_t) \, dt + b(t, X_t) \, dW_t, \quad t \in [0, T],$$

with initial condition X_0. Suppose the coefficient functions a and b satisfy the following assumptions:

1. *Lipschitz continuity: There exists a constant $c > 0$ such that for all $t \in [0, T]$ and all $x, y \in \mathbb{R}$,*

$$|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leq c|x - y|.$$

2. *Linear growth bound: There exists a constant $k > 0$ such that for all $t \in [0, T]$ and all $x \in \mathbb{R}$,*

$$|a(t, x)| + |b(t, x)| \leq k(1 + |x|).$$

Then, there exists a unique solution process $\{X_t : t \geq 0\}$ to the above SDE, and this solution lies in the class of Itô's processes. Furthermore, the solution is pathwise unique, in the sense that if another process $\{Y_t : t \geq 0\}$ also satisfies the same SDE, then

$$\mathbb{P}\left(X_t = Y_t \text{ for all } t \geq 0\right) = 1.$$

2.2 Lévy Processes

Lévy processes serve as a powerful generalization of Brownian motion, allowing for the modeling of stochastic systems that exhibit both continuous fluctuations and sudden, discontinuous changes. These processes are characterized by stationary and independent increments, but unlike Brownian motion, they permit jumps, making them well-suited for describing more complex, real-world dynamics.

This section introduces the fundamental concepts of Lévy processes, including their definition, and properties. The motivation for using Lévy processes arises from the limitations of traditional models based solely on Brownian motion, particularly in capturing abrupt events. By incorporating jump components, Lévy processes provide a more flexible and realistic framework for stochastic modeling, which is essential for the aims of this research. This section has been studied from the book by D. Applebaum, refer [2, 37] for detailed explanations and proofs.

Definition 2.8. *Consider a stochastic process $X = (X_t : t \geq 0)$ defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. The process X is called a Lévy process if it satisfies the following properties:*

1. *Starting point: $X_0 = 0$ almost surely.*

2. *Independent increments: For any integer $n \geq 1$ and any collection of times $0 \leq t_1 < t_2 < \cdots < t_{n+1} < \infty$, the increments*

$$X_{t_2} - X_{t_1}, \quad X_{t_3} - X_{t_2}, \quad \ldots, \quad X_{t_{n+1}} - X_{t_n}$$

are mutually independent random variables.

3. *Stationarity of increments: $X_{t_{j+1}} - X_{t_j} = X_{t_{j+1}-t_j} - X_0$ for each j.*

4. *Stochastic continuity: For every $t_0 \geq 0$ and every $\varepsilon > 0$,*

$$\lim_{t \to t_0} \mathbb{P}\big(|X_t - X_{t_0}| > \varepsilon\big) = 0.$$

To name a few, Brownian motion and Poisson process are examples of Lévy process. We already discussed Brownian motion, now let us define Poisson process.

Definition 2.9. *A Lévy process $N = (N_t : t \geq 0)$ with values in the set $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ is called a Poisson process with rate (or intensity) $\lambda > 0$ if, for every* *fixed time $T \geq 0$, the random variable $N(T)$ follows a Poisson distribution with parameter λT. That is,*

$$\mathbb{P}(N(T) = n) = \frac{(\lambda T)^n}{n!} e^{-\lambda T}, \quad n = 0, 1, 2, \ldots$$

It is interesting to note that the Poisson process follows cádlág paths, which is defined as follows.

Definition 2.10. *Let $I = [a, b]$ be a subset of $\mathbb{R}^+$. A function $f : I \to \mathbb{R}^d$ is called càdlàg if for every point $t \in (a, b]$ the following conditions hold:*

1. *The left-hand limit of f at t exists,*

2. *The function f is continuous from the right at t.*

Let us now introduce the concept of Poisson random measure, which is fundamental to the theory of Lévy processes.

Definition 2.11. *Consider a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and a measurable space $(D, \mathcal{D})$. A function $M : \Omega \times \mathcal{D} \to \mathbb{R}$ is called a random measure if it satisfies the following:*

1. *For each fixed $\omega \in \Omega$, the mapping $A \mapsto M(\omega, A)$ defines a measure on $(D, \mathcal{D})$.*

2. *For every $A \in \mathcal{D}$, the mapping $\omega \mapsto M(\omega, A)$ is measurable.*

Definition 2.12. *Let $(\Omega, \mathcal{G}, \mathbb{P})$ denote a probability space and $(D, \mathcal{D})$ a measurable space. A mapping $M : \Omega \times \mathcal{D} \to \mathbb{R}$ is referred to as a Poisson random measure with intensity measure μ if the following hold:*

1. *For any $A \in \mathcal{D}$ such that $\mu(A) < \infty$, the random variable $M(A)$ follows a Poisson distribution with mean $\mu(A)$; that is, $\mathbb{E}[M(A)] = \mu(A)$.*

2. *If $A_1, A_2, \ldots, A_n$ are pairwise disjoint sets in $\mathcal{D}$, then the collection $\{M(A_i)\}_{i=1}^n$ consists of independent random variables.*

Definition 2.13. *Consider a Lévy process X taking values in $\mathbb{R}^d$. Define the measure ν on Borel sets $A \in \mathcal{B}(\mathbb{R}^d)$ by*

$$\nu(A) := \mathbb{E}\left[\#\left\{t \in [0,1] : \Delta X_t \in A \setminus \{0\}\right\}\right],$$

where the jump at time t is given by $\Delta X_t = X_t - X_{t-}$. This measure ν is referred to as the Lévy measure associated with X.

We will use the following generalized version of Itô's lemma in further chapters.

Lemma 2.3. *For a Lévy process of the form,*

$$dX_t = \mu dt + \sigma dB_t + \int_A \gamma(u)\widetilde{\mathcal{N}}(dt,du)$$

where B_t is the Brownian motion, $\widetilde{\mathcal{N}}(dt,du)$ is the compensated Poisson random measure over measurable set A. It is defined as $\widetilde{\mathcal{N}}(dt,du) = \mathcal{N}(dt,du) - \lambda(du)dt$, with the feature measure λ such that $\lambda(A) < \infty$. Let $\gamma(u)$ be a deterministic bounded and continuous function and $f \in C^{1,2}([0,\infty);\mathbb{R})$, then we have

$$\begin{aligned} df(t,X_t) = &\left(f_t + \mu f_x + \frac{1}{2}\sigma^2 f_{xx} + \int_A \left[f(t,X_t+\gamma(u)) - f(t,X_t)\right]\lambda(du)\right) dt \\ &+ \sigma f_x dB_t + \int_A \left[f(t,X_{t-}+\gamma(u)) - f(t,X_{t-})\right]\widetilde{\mathcal{N}}(dt,du). \end{aligned}$$

2.3 Numerical methods for SDEs

When dealing with stochastic differential equations (SDEs), analytical solutions are often unattainable due to the complexity of the models or the nature of the stochastic terms involved. In such cases, numerical approximation methods become indispensable for studying and simulating the behavior of stochastic systems.

This section presents a literature review of the research conducted by Mustafa et al. [3] where two fundamental numerical schemes for solving SDEs: the Euler–Maruyama method and the Milstein method, are deduced using the truncated Itô-Taylor expansion.

2.3.1 Itô–Taylor Expansion

Let us consider the stochastic differential equation (SDE)

$$\begin{cases} dX_t = f(X_t)\,dt + g(X_t)\,dW_t, \\ X_{t_0} = X_0, \end{cases}$$

where the drift f and diffusion coefficient g satisfy standard conditions such as Lipschitz continuity and linear growth. Let F be a function that is twice continuously differentiable with respect to X_t.

Applying Itô's formula to $F(X_t)$, we obtain:

$$dF(X_t) = \left(f(X_t)\frac{\partial F}{\partial X}(X_t) + \frac{1}{2}g^2(X_t)\frac{\partial^2 F}{\partial X^2}(X_t) \right) dt + g(X_t)\frac{\partial F}{\partial X}(X_t)\,dW_t. \tag{2.1}$$

Define the following differential operators:

$$L^0 := f(X_t)\frac{\partial}{\partial X} + \frac{1}{2}g^2(X_t)\frac{\partial^2}{\partial X^2},$$
$$L^1 := g(X_t)\frac{\partial}{\partial X}.$$

Then equation (2.1) may be expressed more compactly as:

$$F(X_t) = F(X_0) + \int_{t_0}^{t} L^0 F(X_\tau)\,d\tau + \int_{t_0}^{t} L^1 F(X_\tau)\,dW_\tau.$$

By selecting $F(x) = x$, $f(x)$, and $g(x)$, we get:

$$X_t = X_0 + \int_{t_0}^{t} f(X_\tau)\,d\tau + \int_{t_0}^{t} g(X_\tau)\,dW_\tau, \tag{2.2}$$

$$f(X_t) = f(X_0) + \int_{t_0}^{t} L^0 f(X_\tau)\,d\tau + \int_{t_0}^{t} L^1 f(X_\tau)\,dW_\tau, \tag{2.3}$$

$$g(X_t) = g(X_0) + \int_{t_0}^{t} L^0 g(X_\tau)\,d\tau + \int_{t_0}^{t} L^1 g(X_\tau)\,dW_\tau. \tag{2.4}$$

Substituting (2.3) and (2.4) into (2.2), we obtain:

$$X_t = X_0 + f(X_0)\int_{t_0}^{t} d\tau + g(X_0)\int_{t_0}^{t} dW_\tau + \mathcal{R}, \tag{2.5}$$

where the remainder term $\mathcal{R}$ is given by:

$$\begin{aligned} \mathcal{R} = &\int_{t_0}^{t} \left(\int_{t_0}^{\tau_1} L^0 f(X_{\tau_2})\,d\tau_2 + \int_{t_0}^{\tau_1} L^1 f(X_{\tau_2})\,dW_{\tau_2} \right) d\tau_1 \\ &+ \int_{t_0}^{t} \left(\int_{t_0}^{\tau_1} L^0 g(X_{\tau_2})\,d\tau_2 + \int_{t_0}^{\tau_1} L^1 g(X_{\tau_2})\,dW_{\tau_2} \right) dW_{\tau_1}. \end{aligned}$$

Now choose $F(x) = L^1 g(x)$, and integrating twice:

$$\int_{t_0}^{t}\int_{t_0}^{\tau_1} L^1 g(X_{\tau_2})\, dW_{\tau_2}\, dW_{\tau_1} = \int_{t_0}^{t}\int_{t_0}^{\tau_1} L^1 g(X_0)\, dW_{\tau_2}\, dW_{\tau_1}$$

$$+ \int_{t_0}^{t}\int_{t_0}^{\tau_1} \left(\int_{t_0}^{\tau_2} L^0 L^1 g(X_{\tau_3})\, d\tau_3 + \int_{t_0}^{\tau_2} L^1 L^1 g(X_{\tau_3})\, dW_{\tau_3}\right) dW_{\tau_2}\, dW_{\tau_1}. \tag{2.6}$$

Inserting (2.6) into (2.5), we derive:

$$X_t = X_0 + f(X_0)(t - t_0) + g(X_0)(W_t - W_{t_0})$$
$$+ g(X_0)g'(X_0) \int_{t_0}^{t}\int_{t_0}^{\tau_1} dW_{\tau_2}\, dW_{\tau_1} + \text{higher order terms.}$$

Using the known identity from Itô's calculus:

$$\int_{t_0}^{t}\int_{t_0}^{\tau_1} dW_{\tau_2}\, dW_{\tau_1} = \frac{1}{2}(W_t - W_{t_0})^2 - \frac{1}{2}(t - t_0),$$

and applying a time partition $0 = t_0 < t_1 < \cdots < t_N = T$, we obtain the Itô–Taylor expansion:

$$\begin{aligned} X_{t_{i+1}} &= X_{t_i} + f(X_{t_i})\Delta t_i + g(X_{t_i})\Delta W_i \\ &\quad + \frac{1}{2} g(X_{t_i})g'(X_{t_i}) \left((\Delta W_i)^2 - \Delta t_i\right) + \text{higher order terms,} \end{aligned} \tag{2.7}$$

where $\Delta t_i = t_{i+1} - t_i$, $\Delta W_i = W_{t_{i+1}} - W_{t_i}$, and $X_{t_0} = X_0$.

Euler–Maruyama Scheme

The Euler–Maruyama method is a first-order numerical approximation for SDEs, derived by truncating the Itô–Taylor expansion (2.7) to include only the drift and diffusion terms:

$$X_{t_{i+1}} = X_{t_i} + f(X_{t_i})\Delta t_i + g(X_{t_i})\Delta W_i.$$

Milstein Scheme

To achieve improved accuracy, the Milstein method includes the next-order correction term from the Itô–Taylor expansion (2.7):

$$X_{t_{i+1}} = X_{t_i} + f(X_{t_i})\Delta t_i + g(X_{t_i})\Delta W_i + \frac{1}{2} g(X_{t_i})g'(X_{t_i}) \left((\Delta W_i)^2 - \Delta t_i\right).$$

2.4 Large Deviation Theory

The Large Deviation Principle (LDP) is a powerful theoretical framework in Probability Theory used to quantify the probabilities of rare events in stochastic systems. Unlike the Central limit theorem or the Law of large numbers, which describe typical or average behavior, large deviation theory focuses on the exponential decay of probabilities associated with significant deviations from expected outcomes.

This section introduces the fundamental definitions and key results of Large Deviation theory, including rate functions, the formal definition of the Large Deviation Principle (LDP), and the Laplace Principle.

Definition 2.14. *A metric space is called Polish if it is complete and separable.*

Remark 2.2. *An important example of a Polish space is the Skorokhod space, which is widely used in the theory of stochastic processes with Lévy jumps. For a detailed analysis, see [20].*

We denote any Polish space by $\mathcal{P}$.

Definition 2.15. *A function $I : \mathcal{P} \to [0, \infty]$ is called a rate function if it is lower semicontinuous, i.e., for every $x \in \mathcal{P}$,*

$$\liminf_{y \to x} I(y) \geq I(x).$$

Definition 2.16. *A rate function I is said to be a good rate function if all its level sets $\{x \in \mathcal{P} : I(x) \leq a\}$ are compact for every $a \geq 0$.*

Definition 2.17. *Let I be a rate function on a Polish space $\mathcal{P}$. A sequence of random variables $\{X^n\}_{n \in \mathbb{N}}$ satisfies the Large Deviation Principle with rate function I if the following hold:*

1. *Large Deviation lower bound: For every open set $G \subset \mathcal{P}$,*
$$\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X^n \in G) \geq -I(G),$$
where $I(G) := \inf_{x \in G} I(x)$.

2. *Large Deviation upper bound: For every closed set* $F \subset \mathcal{P}$*,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X^n \in F) \le -I(F),$$

where $I(F) := \inf_{x\in F} I(x)$*.*

Definition 2.18. *Let* I *be a rate function on a Polish space* $\mathcal{P}$*. The sequence* $\{X^n\}_{n\in\mathbb{N}}$ *satisfies the Laplace Principle with rate function* I *if for every bounded continuous function* $h : \mathcal{P} \to \mathbb{R}$*,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[\exp\left(-nh(X^n)\right)\right] = -\inf_{x\in\mathcal{P}} \left\{h(x) + I(x)\right\}.$$

The following theorem establishes the equivalence between the Large Deviation Principle and the Laplace Principle in the setting of Polish spaces. This result comes as an observation from Varadhan's seminal work [39] and Bryc's Lemma [14].

Theorem 2.4. *A sequence* $\{X^n\}_{n\in\mathbb{N}}$ *satisfies the Laplace Principle with a good rate function* I *on a Polish space* $\mathcal{P}$ *if and only if it satisfies the Large Deviation Principle with the same rate function* I*.*

The proof of this equivalence can be found in [39, 15].

CHAPTER 3

Stochastic SIRS Epidemic Model

"The mathematical method of treatment is, in its way, as powerful as the microscope."

–Ronald Ross

This chapter is completely dedicated to highlighting, if not all, some research work done in the field of mathematical epidemiology, emphasizing the importance of incorporating stochastic elements into the model. In addition, we have explained the SIRS epidemic model using a system of stochastic differential equations derived from its deterministic framework.

3.1 Introduction and Novelty of our Work

Mathematical epidemiology modeling has proven to be quite useful for preventing extreme outbreaks. In recent years, there has been humongous research conducted in disease modeling, particularly epidemiology. One of the most studied model in epidemiology is the SIRS (Susceptible-Infected-Recovered-Susceptible) model, where the recovered individuals can loose immunity and revert back to the susceptible case. We are grateful to Kermack and McKendrick[21] for their contribution to the mathematical theory of epidemics. Since their work, the growth in the literature concerned with mathematical epidemiology has seen a spike. We will not be discussing all the work in detail here, but to mention a few works worth mentioning, we would highlight the works by Roy Anderson and Robert May[1], Herbert Hethcote [18], and Bryan Grenfell and Ottar Bjørnstad [17] where the dynamics of SIRS type disease models has been studied.

When a disease starts to spread, the most thoughtful thing anyone can do is to make their peers aware of the disease and the steps to take in order to prevent it. Numerous studies and papers have demonstrated that mass media also play a huge role in preventing the disease from significant outbreaks by spreading the necessary and important information[12, 29, 42, 43]. Intervention policies such as quarantine, mass wearing, border screening, vaccination, and isolation have slowed down the spread rate of infectious disease, as pointed out by Cui et al.[12], and Tang and Xiao[36]. Many thanks to Wang[40] who formulated an SIRS model which incorporates the effect of intervention policies.

Apart from the above factors, what really can change the spread rate significantly are the environmental factors such as temperature, humidity, precipitation, seasonal variability, social interactions, variations in human mobility, and large-scale population migrations[9]. These factors influence key parameters like recruitment rate, recovery rate, and death rate, thereby introducing stochastic elements into the system of ordinary differential equations (ODEs). Hence, deterministic models are inadequate for capturing the complexities of infectious diseases due to above mentioned sources of uncertainty. Incorporating Gaussian white noise into the system enables us to simulate environmental fluctuations that can significantly impact small-scale epidemics or the early stages of a major outbreak.

On the other hand, the population may be subject to rare events during the outbreak, such as natural disasters like earthquakes, hurricanes, or floods, which can trigger the spread of epidemic diseases by potentially increasing mass gatherings or a decrease in susceptible population because of sudden deaths. All these factors are rare, but if they occur, they affect the dynamics of the disease significantly. So, it is important to consider them, and Gaussian noise is not suitable to capture these discontinuous fluctuations. Therefore, we need to incorporate another type of stochastic noise, specifically Lévy noise.

Many researchers have worked on the epidemic models with Gaussian noise [24, 38, 44]. Only a limited amount of work has been done on epidemic models that incorporate Lévy jumps [4, 11, 33, 45], but they used the same jump component for each population. Assigning unique jump components to each population, instead of using a single shared noise term, enables us to simulate population-specific disturbances more accurately. Based on the information available to us, there is no work on the SIRS model with a different jump component for each equation in the system. This makes our work novel and closer to reality.

3.2 Model Description

In this section, we will discuss how the system of stochastic differential equations which models an SIRS-type epidemic disease, gets formulated from its deterministic framework referencing the works of Wang[40] and Cai et al.[10]. We will consider two types of noises, one is the Brownian motion, which captures the continuous disturbances, and the second one is the Lévy jumps, which are responsible for the discontinuous stochastic events.

The main components of the SIRS epidemic model are the S: Susceptible population, I: Infected population, and R: Recovered population. Therefore, the total population, that is, $N = S + I + R$, surely. The table below describes the physical meaning of the constants that will affect the dynamics of the disease.

Table 3.1: The description of the constants

Constants	**Description**
$\wedge$	Recruitment rate
μ	Natural death rate
γ	Recovery rate of infected population
ν	Rate of recovered individuals that become susceptible again.
δ	Death rate caused by disease.

The progression of the disease is described by the following system of ordinary differential equations (cf. [10]):

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \mu S - H(I)S + \gamma R, \\ \dfrac{dI}{dt} = H(I)S - (\mu + \nu + \delta)I, \\ \dfrac{dR}{dt} = \nu I - (\mu + \gamma)R. \end{cases} \tag{3.1}$$

Infectious force $H(I)$: This function characterizes the influence of infected individuals on disease spread and is a critical factor in transmission dynamics. Notably, as the infected population grows, $H(I)$ tends to decrease. This reflects behavioral changes such as reduced contact rates in response to intervention measures.

From a modeling perspective, $H(I)$ is taken to be inversely related to the number of infective individuals. Hence, it can be formulated as

$$H(I) = \frac{\beta I}{f(I)},$$

where the term $1/f(I)$ quantifies the effectiveness of control strategies on limiting contacts, and β represents the transmission coefficient (see [40] for a comprehensive discussion).

The following assumptions are essential to achieve a non-monotonic infection force:

1. $f(0) > 0$ and $f'(I) > 0$ for $I > 0$.
2. There is a $\xi > 0$ such that $\left(\frac{I}{f(I)}\right)' > 0$ for $0 < I \leq \xi$ and $\left(\frac{I}{f(I)}\right)' < 0$ for $I > \xi$.

These assumptions reflect the impact of intervention policies on the disease dynamics. The incidence rate, given by $\frac{I}{f(I)}$, is assumed to be increasing for $0 < I \leq \xi$ and decreasing for $I > \xi$.

Taking into account the factorization of $H(I)$, we consider the following deterministic SIRS model adapted from [10]:

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \mu S - \dfrac{\beta I}{f(I)} S + \gamma R, \\ \dfrac{dI}{dt} = \dfrac{\beta I}{f(I)} S - (\mu + \nu + \delta) I, \\ \dfrac{dR}{dt} = \nu I - (\mu + \gamma) R, \end{cases} \tag{3.2}$$

where the state space is defined as $\mathbb{X} := \mathbb{R}^3_+ = \{(S, I, R) : S > 0, I \geq 0, R \geq 0\}$.

To incorporate the effects of environmental variability and intervention strategies, we introduce multiplicative stochastic perturbations into the susceptible and infectious compartments. This yields the following system of stochastic differential equations driven by Brownian motions, following the framework in [10]:

$$\begin{cases} dS_t = \left(\Lambda - \mu S_t - \dfrac{\beta I_t}{f(I_t)} S_t + \gamma R_t\right) dt - \dfrac{\sigma I_t}{f(I_t)} S_t \, d\mathcal{B}^1_t, \\ dI_t = \left(\dfrac{\beta I_t}{f(I_t)} S_t - (\mu + \nu + \delta) I_t\right) dt + \dfrac{\sigma I_t}{f(I_t)} S_t \, d\mathcal{B}^2_t, \\ dR_t = (\nu I_t - (\mu + \gamma) R_t) \, dt, \end{cases} \tag{3.3}$$

where $\{\mathcal{B}^1_t, \mathcal{B}^2_t\}_{t \geq 0}$ are Brownian motions defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ satisfying the following properties:

1. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right-continuous.

2. The probability space is complete, meaning any subset of a null set is measurable.

As discussed previously, the disease dynamics may be subject to sudden and unpredictable shifts, such as sharp spikes in infections triggered by external events (e.g., mass gatherings) or rapid recoveries due to effective interventions. Although such events are infrequent, their impact on the epidemic progression can be significant. To model these abrupt, large-scale fluctuations, we incorporate Lévy jump terms into the stochastic system (3.3) as follows:

$$\begin{cases} dS_t = \left(\Lambda - \mu S_t - \dfrac{\beta I_t}{f(I_t)} S_t + \gamma R_t\right) dt - \dfrac{\sigma I_t}{f(I_t)} S_t \, d\mathcal{B}_t^1 + \displaystyle\int_Z S_{t^-} \gamma_1(u) \, \widetilde{\mathcal{N}}_1(dt, du), \\ dI_t = \left(\dfrac{\beta I_t}{f(I_t)} S_t - (\mu + \nu + \delta) I_t\right) dt + \dfrac{\sigma I_t}{f(I_t)} S_t \, d\mathcal{B}_t^2 + \displaystyle\int_Z I_{t^-} \gamma_2(u) \, \widetilde{\mathcal{N}}_2(dt, du), \\ dR_t = (\nu I_t - (\mu + \gamma) R_t) \, dt + \displaystyle\int_Z R_{t^-} \gamma_3(u) \, \widetilde{\mathcal{N}}_3(dt, du), \end{cases} \tag{3.4}$$

where $S_{t^-}, I_{t^-}, R_{t^-}$ denote the left-hand limits of S_t, I_t, and R_t, respectively. Here, $\mathcal{N}_i(dt, du)$ is a Poisson random measure counting the number of jumps of size du occurring within the time interval dt on a measurable subset Z equipped with a finite intensity measure λ satisfying $\lambda(Z) < +\infty$.

The compensated Poisson random measure is defined as

$$\widetilde{\mathcal{N}}_i(dt, du) = \mathcal{N}_i(dt, du) - \lambda_i(du)dt,$$

and the functions $\gamma_i : Z \times \Omega \to \mathbb{R}$ for $i = 1, 2, 3$ are bounded and continuous.

Note that the product σ-algebra $\mathbb{B}(Z) \times \mathcal{F}_t$ is measurable, where $\mathbb{B}(Z)$ denotes the Borel σ-algebra on Z, and $\{\mathcal{F}_t\}_{t\geq 0}$ is the filtration.

Remark 3.1. *For brevity, we will omit the explicit time subscript in notation and write S_t as S (similarly, I_t as I and R_t as R), as the time dependence is clear from context.*

CHAPTER 4

Stochastic Analysis of an SIRS Epidemic Model with Lévy Jumps

In this chapter, we present the main results obtained from the analysis of the stochastic SIRS epidemic model, accompanied by their detailed proofs. The results cover the fundamental existence and uniqueness of the model's solution, the stochastic boundedness of the solution under appropriate conditions, and the probability of disease extinction.

4.1 Existence and Uniqueness

A global positive solution is required in order for our problem to make sense. It gives us a way for further analysis. In this section, we will demonstrate that the system described in (3.4) has indeed a unique global solution.

Theorem 4.1. *System (3.4) has a unique solution* $(S_t, I_t, R_t) \in \mathbb{R}^3_+$ *for* $t \geq 0$ *for any initial value* $(S_0, I_0, R_0) \in \mathbb{R}^3_+$*. Furthermore, it is guaranteed that* (S_t, I_t, R_t)

will always remain within $\mathbb{R}^3_+$ with a probability of one i.e.

$$\mathbb{P}\{(S_t, I_t, R_t) \in \mathbb{R}^3_+ : \forall t \geq 0\} = 1.$$

Proof. Because the coefficients of system (3.4) satisfy the local Lipschitz and linear growth conditions, the Existence and Uniqueness Theorem for SDEs guarantees that system (3.4) admits a unique local solution $(S_t, I_t, R_t) \in \mathbb{R}^3_+$ for any initial value $(S_0, I_0, R_0) \in \mathbb{R}^3_+$ on the interval $t \in [0, \tau_e)$ almost surely, where τ_e denotes the explosion time. To complete the proof, it suffices to show that $\tau_e = \infty$ a.s.

Suppose n_0 is a sufficiently large positive number such that

$$(S_0, I_0, R_0) \in \left[\frac{1}{n_0}, n_0\right]^3.$$

For any $n \geq n_0$, define the stopping time

$$\tau_n = \inf\left\{t \in [0, \tau_e) : S_t \notin \left(\frac{1}{n}, n\right) \text{ or } I_t \notin \left(\frac{1}{n}, n\right) \text{ or } R_t \notin \left(\frac{1}{n}, n\right)\right\}.$$

Note that the sequence $\{\tau_n\}$ is monotonically increasing and bounded above by τ_e. Define $\tau_\infty = \lim_{n\to\infty} \tau_n$. It is clear that $\tau_\infty \leq \tau_e$ a.s. Thus, if $\tau_\infty = \infty$ a.s., it follows that $\tau_e = \infty$ and hence $(S_t, I_t, R_t) \in \mathbb{R}^3_+ \quad \text{for all } t \geq 0 \quad$ a.s. Therefore, it suffices to prove $\tau_\infty = \infty$ a.s.

We proceed by contradiction. Suppose, on the contrary, that $\tau_\infty = \infty$ a.s. is not true. Then there exist a positive constant $T > 0$ and an arbitrary $\epsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} \geq \epsilon.$$

Hence, there exists an integer $\tilde{n} \geq n_0$ such that

$$\mathbb{P}\{\tau_n \leq T\} \geq \epsilon \quad \text{for all } n \geq \tilde{n}.$$

Now, consider the twice continuously differentiable Lyapunov function $V : \mathbb{R}^3_+ \to \overline{\mathbb{R}}_+$ defined by

$$V(S, I, R) = (S - 1 - \ln S) + (I - 1 - \ln I) + (R - 1 - \ln R).$$

Applying Itô's lemma to the function V, we obtain

$$\begin{aligned} dV(S,I,R) = \mathscr{L}V(S,I,R)\,dt &- \left(1-\frac{1}{S}\right)\frac{\sigma I}{f(I)}S\,d\mathcal{B}_t^1 + \left(1-\frac{1}{I}\right)\frac{\sigma I}{f(I)}S\,d\mathcal{B}_t^2 \\ &+ \int_Z \big[\gamma_1(u)S - \ln(1+\gamma_1(u))\big]\widetilde{\mathcal{N}}_1(dt,du) \\ &+ \int_Z \big[\gamma_2(u)I - \ln(1+\gamma_2(u))\big]\widetilde{\mathcal{N}}_2(dt,du) \\ &+ \int_Z \big[\gamma_3(u)R - \ln(1+\gamma_3(u))\big]\widetilde{\mathcal{N}}_3(dt,du), \end{aligned} \tag{4.1}$$

where the generator $\mathscr{L}V$ is given by

$$\begin{aligned} \mathscr{L}V(S,I,R) = \left(1-\frac{1}{S}\right)\left[\Lambda - \mu S - \frac{\beta I}{f(I)}S + \gamma R\right] + \frac{1}{2}\frac{1}{S^2}\frac{\sigma^2 I^2 S^2}{f^2(I)} \\ + \left(1-\frac{1}{I}\right)\left[\frac{\beta I}{f(I)}S - (\mu+\nu+\delta)I\right] + \frac{1}{2}\frac{1}{I^2}\frac{\sigma^2 I^2 S^2}{f^2(I)} \\ + \left(1-\frac{1}{R}\right)[\nu I - (\mu+\gamma)R] \\ + \int_Z [\gamma_1(u) - \ln(1+\gamma_1(u))]\,\lambda_1(du) \\ + \int_Z [\gamma_2(u) - \ln(1+\gamma_2(u))]\,\lambda_2(du) \\ + \int_Z [\gamma_3(u) - \ln(1+\gamma_3(u))]\,\lambda_3(du). \end{aligned}$$

By expanding and rearranging terms, we obtain the following upper bound estimate:

$$\begin{aligned} \mathscr{L}V(S,I,R) \le \Lambda + 3\mu + \nu + \delta + \gamma + \frac{\beta\xi}{f(\xi)} + \frac{\sigma^2\xi^2}{2f^2(\xi)} + \frac{\sigma^2\Lambda^2}{2\mu^2 f^2(0)} \\ + \int_Z [\gamma_1(u) - \ln(1+\gamma_1(u))]\,\lambda_1(du) \\ + \int_Z [\gamma_2(u) - \ln(1+\gamma_2(u))]\,\lambda_2(du) \\ + \int_Z [\gamma_3(u) - \ln(1+\gamma_3(u))]\,\lambda_3(du). \end{aligned} \tag{4.2}$$

Define constants

$$K := \Lambda + 3\mu + \nu + \delta + \gamma + \frac{\beta\xi}{f(\xi)} + \frac{\sigma^2\xi^2}{2f^2(\xi)} + \frac{\sigma^2\Lambda^2}{2\mu^2 f^2(0)},$$

and

$$C' := \max_{i=1,2,3}\left\{\int_Z [\gamma_i(u) - \ln(1+\gamma_i(u))]\,\lambda_i(du)\right\}.$$

Then, from (4.2), it follows that

$$\mathscr{L}V(S,I,R) \le K + 3C' := C.$$

Hence, the differential inequality

$$\begin{aligned} dV(S,I,R) \le C\,dt &- (S-1)\frac{\sigma I}{f(I)}d\mathcal{B}_t^1 + (I-1)\frac{\sigma S}{f(I)}d\mathcal{B}_t^2 \\ &+ \int_Z \left[\gamma_1(u)S - \ln(1+\gamma_1(u))\right]\widetilde{\mathcal{N}}_1(dt,du) \\ &+ \int_Z \left[\gamma_2(u)I - \ln(1+\gamma_2(u))\right]\widetilde{\mathcal{N}}_2(dt,du) \\ &+ \int_Z \left[\gamma_3(u)R - \ln(1+\gamma_3(u))\right]\widetilde{\mathcal{N}}_3(dt,du) \end{aligned} \tag{4.3}$$

holds. Integrating (4.3) from 0 to $\tau_n \wedge T$, we get

$$\begin{aligned} V(S_{\tau_n\wedge T}, I_{\tau_n\wedge T}, R_{\tau_n\wedge T}) &\le V(S_0,I_0,R_0) + CT \\ &- \int_0^{\tau_n\wedge T}(S_s-1)\frac{\sigma I_s}{f(I_s)}d\mathcal{B}_s^1 \\ &+ \int_0^{\tau_n\wedge T}(I_s-1)\frac{\sigma S_s}{f(I_s)}d\mathcal{B}_s^2 \\ &+ \int_0^{\tau_n\wedge T}\int_Z \left[\gamma_1(u)S_s - \ln(1+\gamma_1(u))\right]\widetilde{\mathcal{N}}_1(ds,du) \\ &+ \int_0^{\tau_n\wedge T}\int_Z \left[\gamma_2(u)I_s - \ln(1+\gamma_2(u))\right]\widetilde{\mathcal{N}}_2(ds,du) \\ &+ \int_0^{\tau_n\wedge T}\int_Z \left[\gamma_3(u)R_s - \ln(1+\gamma_3(u))\right]\widetilde{\mathcal{N}}_3(ds,du). \end{aligned}$$

Taking expectations on both sides and using the martingale property of Brownian motion and compensated Poisson random measure in stochastic integrals, we get

$$\mathbb{E}\left[V(S_{\tau_n\wedge T}, I_{\tau_n\wedge T}, R_{\tau_n\wedge T})\right] \le V(S_0,I_0,R_0) + CT.$$

Now, define the event $\Omega_n := \{\tau_n \le T\}$ for $n > n_1$. From the previous contradiction assumption, $\mathbb{P}(\Omega_n) \ge \epsilon > 0$. On Ω_n, at least one of $S_{\tau_n}(\omega)$, $I_{\tau_n}(\omega)$, or $R_{\tau_n}(\omega)$ equals either $\frac{1}{n}$ or n. Thus,

$$V(S_{\tau_n}(\omega), I_{\tau_n}(\omega), R_{\tau_n}(\omega)) \ge \min\left\{n - 1 - \ln n,\ \frac{1}{n} - 1 - \ln\frac{1}{n}\right\}.$$

Hence,

$$V(S_0,I_0,R_0) + CT \ge \mathbb{E}\left[\mathbf{1}_{\Omega_n}V(S_{\tau_n}, I_{\tau_n}, R_{\tau_n})\right] \ge \epsilon\min\left\{n - 1 - \ln n,\ \frac{1}{n} - 1 - \ln\frac{1}{n}\right\}.$$

Letting $n \to \infty$, the right-hand side tends to infinity, contradicting the finiteness of the left-hand side. Therefore, the assumption was false, and we conclude that $\tau_\infty = \infty$ a.s. Hence, the system (3.4) admits a unique global positive solution. $\square$

4.2 Stochastic Boundedness

This section demonstrates that, under certain conditions, the susceptible and infected populations remain stochastically bounded, ensuring that the model dynamics do not exhibit an explosion due to random fluctuations.

Theorem 4.2. *Assume there exists a constant* $K \geq \dfrac{I}{f(I)}$ *such that*

$$\mathbb{E}\left[p(\Lambda+\gamma)+\frac{1}{2}p(p-1)\sigma^2K^2\right]<0,$$

for some $p > 0$. *Then the susceptible component satisfies*

$$\limsup_{t\to\infty}\mathbb{E}[S^p(t)]=0.$$

Proof. From the system (3.4),

$$dS=\Big(\wedge-\mu S-\frac{\beta I}{f(I)}S+\gamma R\Big)dt-\frac{\sigma I}{f(I)}Sd\mathcal{B}_t^1+\int_Z S_{t^-}\gamma_1(u)\widetilde{\mathcal{N}_1}(dt,du).$$

Applying Itô's Lemma,

$$\begin{aligned}dS^p=&\Big(pS^{p-1}\Big(\wedge-\mu S-\frac{\beta I}{f(I)}S+\gamma R\Big)+\frac{1}{2}p(p-1)S^{p-2}\left(\frac{\sigma^2I^2S^2}{f^2(I)}\right)\\&-\int_Z(\gamma_1-\ln(1+\gamma_1))\lambda_1(du)\Big)dt+pS^{p-1}\left(-\frac{\sigma IS}{f(I)}\right)d\mathcal{B}_t^1\\&+\int_Z\ln(1+\gamma_1)\widetilde{\mathcal{N}_1}(dt,du).\end{aligned}$$

Integrating both sides on $[0,t]$, we will get

$$\begin{aligned}S^p(t)\leq&S^p(0)+p\int_0^t(\wedge+\gamma R)S^{p-1}\,dt+\frac{1}{2}p(p-1)\sigma^2\int_0^t\frac{I^2}{f^2(I)}S^{p-2}S^2dt\\&-\int_0^t\int_Z(\gamma_1-\ln(1+\gamma_1))\lambda_1(du)dt+p\int_0^tS^{p-1}\left(-\frac{\sigma IS}{f(I)}\right)d\mathcal{B}_t^1\\&+\int_0^t\int_Z\ln(1+\gamma_1)\widetilde{\mathcal{N}_1}(dt,du).\end{aligned}$$

Taking expectations on both sides and then using Fubini's theorem,

$$\begin{aligned}\mathbb{E}[S^p(t)] \leq & S^p(0) + \mathbb{E}\left[\int_0^t S^p\left\{p\left(\frac{\wedge+\gamma R}{S}\right)+\left(\frac{1}{2}p(p-1)\frac{\sigma^2 I^2}{f^2(I)}\right)\right\}dt\right] - A \\ \leq & S^p(0) + \int_0^t \mathbb{E}[S^p(t)]\mathbb{E}\left[p\left(\frac{\wedge+\gamma R}{S}\right)+\left(\frac{1}{2}p(p-1)\frac{\sigma^2 I^2}{f^2(I)}\right)\right]dt - A\end{aligned}$$

where

$$A = \int_0^t \int_Z (\gamma_1 - \ln(1+\gamma_1))\lambda_1(du)dt.$$

Applying Grönwall's Inequality,

$$\mathbb{E}[S^p(t)] \leq (S^p(0) - A)\exp\int_0^t \mathbb{E}\left[p\left(\frac{\wedge+\gamma R}{S}\right)+\left(\frac{1}{2}p(p-1)\frac{\sigma^2 I^2}{f^2(I)}\right)\right]dt.$$

Now, observe that $\dfrac{I}{f(I)}$ is a decreasing function after a certain stage, so we can bound it by some number, and $\dfrac{R}{S}$ is the number of recovered individuals upon the number of susceptible individuals. So, $R < S$ and $\dfrac{R}{S} < 1$. Denote $\dfrac{I}{f(I)} \leq K$. Now we have,

$$\begin{aligned}\mathbb{E}[S^p(t)] \leq & (S^p(0) - A)\exp\int_0^t \mathbb{E}\left[p\left(\frac{\wedge}{S}+\frac{\gamma R}{S}\right)+\left(\frac{1}{2}p(p-1)\frac{\sigma^2 I^2}{f^2(I)}\right)\right]dt \\ \leq & (S^p(0) - A)\exp\int_0^t \mathbb{E}\left[\left(p(\wedge+\gamma)+\frac{1}{2}p(p-1)\sigma^2 K^2\right)\right]dt \\ \leq & (S^p(0) - A)\exp\left(\mathbb{E}\left[p(\wedge+\gamma)+\frac{1}{2}p(p-1)\sigma^2 K^2\right]\right)t\end{aligned}$$

and

$$\limsup_{t\to\infty} \mathbb{E}[S^p(t)] = 0.$$

Hence, the result follows. □

Theorem 4.3. *If the following condition is satisfied*

$$p\beta + \frac{1}{2}p(p-1)\sigma^2 < 0,$$

then

$$\limsup_{t\to\infty} \mathbb{E}[I^p(t)] = 0.$$

Proof. From the system (3.4),

$$dI = \left(\frac{\beta I}{f(I)}S - (\mu+\nu+\delta)I\right)dt + \frac{\sigma I}{f(I)}S d\mathcal{B}_t^2 + \int_Z I_{t^-}\gamma_2(u)\widetilde{\mathcal{N}}_2(dt, du).$$

Applying Itô's Lemma,

$$
\begin{aligned}
dI^p(t) \leq & \Bigg(pI^{p-1}\left(\frac{\beta IS}{f(I)}\right) + \frac{1}{2}p(p-1)I^{p-2}\left(\frac{\sigma^2 I^2 S^2}{f^2(I)}\right) \\
& - \int_Z (\gamma_2 - \ln(1+\gamma_2))\lambda_2(du) \Bigg) dt \\
& + pI^{p-1}\left(\frac{\sigma IS}{f(I)}\right) d\mathcal{B}_t^2 + \int_Z \ln(1+\gamma_2)\widetilde{\mathcal{N}}_2(dt, du).
\end{aligned}
$$

Integrating both sides,

$$
\begin{aligned}
I^p(t) \leq & I^p(0) + \int_0^t I^p\left(p\frac{\beta S}{f(I)} + \frac{1}{2}p(p-1)\frac{\sigma^2 S^2}{f^2(I)}\right) dt \\
& - \int_0^t \int_Z (\gamma_2 - \ln(1+\gamma_2))\lambda_2(du)dt \\
& + p\int_0^t I^{p-1}\frac{\sigma IS}{f(I)} d\mathcal{B}_s^2 + \int_0^t \int_Z \ln(1+\gamma_2)\widetilde{\mathcal{N}}_2(dt, du).
\end{aligned}
$$

Taking expectation on both sides and then applying Grönwall's inequality, we will get

$$
\begin{aligned}
\mathbb{E}[I^p(t)] \leq & I^p(0) - B + \int_0^t \mathbb{E}[I^p]\mathbb{E}\left[p\frac{\beta S}{f(I)} + \frac{1}{2}p(p-1)\frac{\sigma^2 S^2}{f^2(I)}\right] dt \\
\leq & (I^p(0) - B)\exp\int_0^t \mathbb{E}\left[p\frac{\beta S}{f(I)} + \frac{1}{2}p(p-1)\frac{\sigma^2 S^2}{f^2(I)}\right] dt \\
\leq & (I^p(0) - B)\exp\int_0^t \mathbb{E}\left[p\beta + \frac{1}{2}p(p-1)\sigma^2\right] dt \\
= & (I^p(0) - B)\exp\left(\mathbb{E}\left[p\beta + \frac{1}{2}p(p-1)\sigma^2\right]\right) t
\end{aligned}
$$

where

$$
B = \int_0^t \int_Z (\gamma_2 - \ln(1+\gamma_2))\lambda_2(du)dt
$$

which implies

$$
\limsup_{t\to\infty} \mathbb{E}[I^p(t)] = 0.
$$

Hence, the result follows. $\square$

4.3 Extinction Criteria

In this section, we derive a condition under which the infectious disease becomes extinct with probability one.

We define the basic reproduction number R_0, a key indicator of disease spread, as follows:

$$R_0 := \frac{\Lambda\beta}{\mu f(0)(\mu+\nu+\delta)}, \tag{4.4}$$

where R_0 determines whether the infection will persist or fade out over time.

Denote

$$R_0^s := R_0 - \frac{\sigma^2 \wedge^2}{2\mu^2 f^2(0)(\mu+\nu+\delta)}. \tag{4.5}$$

We assume the following:

$$\beta_i = \int_Z (\gamma_i - ln(1+\gamma_i))\lambda_i(du). \tag{4.6}$$

Theorem 4.4. *Let the solution of* (3.4) *be denoted by* (S_t, I_t, R_t) *with initial value* (S_0, I_0, R_0).

1. *If*

$$R_0^s < 1 + \frac{\beta_2}{(\mu+\nu+\delta)} \qquad \text{and} \qquad \sigma^2 \leq \frac{\beta\mu f(0)}{\wedge}$$

or

$$\frac{\beta^2}{2\sigma^2} < (\mu+\nu+\delta) + \beta_2$$

then

$$\limsup_{t\to\infty} \frac{\ln I_t}{t} \leq (\mu+\nu+\delta)(R_0^s - 1) - \beta_2 < 0 \qquad a.s.$$

2. *If*

$$(\mu+\gamma) + \beta_3 > 0$$

then

$$\limsup_{t\to\infty} \frac{\ln R_t}{t} \leq -(\mu+\gamma) - \beta_3 < 0 \qquad a.s.$$

In other words, the disease dies out with probability one.

Proof. 1. From the system (3.4)

$$dI = \left(\frac{\beta I}{f(I)}S - (\mu+\nu+\delta)I\right)dt + \frac{\sigma I}{f(I)}S d\mathcal{B}_t^2 + \int_Z I_{t-}\gamma_2(u)\widetilde{\mathcal{N}}_2(dt,du).$$

Using Itô's Lemma,

$$\begin{aligned}
d\ln I =&\Big(\frac{1}{I}\left(\frac{\beta I}{f(I)}S - (\mu+\nu+\delta)I\right) + \frac{1}{2}\left(-\frac{1}{I^2}\right)\frac{\sigma^2 I^2 S^2}{f^2(I)} \\
&- \int_Z (\gamma_2 - \ln(1+\gamma_2))\lambda_2(du)\Big)dt + \left(\frac{1}{I}\right)\left(\frac{\sigma I S}{f(I)}\right)d\mathcal{B}_t^2 \\
&+ \int_Z \ln(1+\gamma_2)\widetilde{\mathcal{N}}_2(dt,du), \\
d\ln I =&\Big(\underbrace{\frac{\beta S}{f(I)} - (\mu+\nu+\delta) - \frac{1}{2}\frac{\sigma^2 S^2}{f^2(I)}}_{:=\phi(S,I)} - \int_Z (\gamma_2 - \ln(1+\gamma_2))\lambda_2(du)\Big)dt \\
&+ \left(\frac{\sigma S}{f(I)}\right)d\mathcal{B}_t^2 + \int_Z \ln(1+\gamma_2)\widetilde{\mathcal{N}}_2(dt,du).
\end{aligned} \tag{4.7}$$

Consider $\phi(S,I)$ and then by elementary calculations, we get

$$\begin{aligned}
\phi(S,I) :=& \frac{\beta S}{f(I)} - (\mu+\nu+\delta) - \frac{1}{2}\frac{\sigma^2 S^2}{f^2(I)} \\
=& \left(\frac{\beta S}{f(I)} - \frac{1}{2}\frac{\sigma^2 S^2}{f^2(I)} - \frac{\beta^2}{2\sigma^2}\right) + \frac{\beta^2}{2\sigma^2} - (\mu+\nu+\delta) \\
=& -\frac{\sigma^2}{2}\left(\frac{S^2}{f^2(I)} + \frac{\beta^2}{\sigma^4} - \frac{2\beta S}{\sigma^2 f(I)}\right) + \frac{\beta^2}{2\sigma^2} - (\mu+\nu+\delta) \\
=& -\frac{\sigma^2}{2}\left(\frac{S}{f(I)} - \frac{\beta}{\sigma^2}\right)^2 + \frac{\beta^2}{2\sigma^2} - (\mu+\nu+\delta) \\
\leq& \frac{\beta^2}{2\sigma^2} - (\mu+\nu+\delta).
\end{aligned}$$

Then equation (4.7) becomes

$$d\ln I_t \leq \left(\frac{\beta^2}{2\sigma^2} - (\mu+\nu+\delta) - \beta_2\right)dt + \frac{\sigma S}{f(I)}d\mathcal{B}_t^2 + \int_Z \ln(1+\gamma_2)\widetilde{\mathcal{N}}_2(dt,du),$$

$$\begin{aligned}
\ln I_t \leq& \ln I_0 + \int_0^t \left(\frac{\beta^2}{2\sigma^2} - (\mu+\nu+\delta) - \beta_2\right)ds + \sigma' \int_0^t d\mathcal{B}_s^2 \\
&+ \int_0^t \left(\int_Z \ln(1+\gamma_2)\widetilde{\mathcal{N}}_2(dt,du)\right)ds
\end{aligned}$$

where $\sigma S \leq \sigma'$ under the boundedness condition of S.

Dividing by t both sides and then letting $t \to \infty$, we get

$$\limsup_{t\to\infty} \frac{\ln I_t}{t} \leq \frac{\beta^2}{2\sigma^2} - (\mu+\nu+\delta) - \beta_2 < 0 \quad a.s.$$

for the case when $\sigma^2 > \dfrac{\beta^2}{2(\mu+\nu+\delta)}$ since

$$\lim_{t\to\infty} \frac{\mathcal{B}_t^2}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t \int_Z \ln(1+\gamma_2)\widetilde{\mathcal{N}}_2(dt,du)ds}{t} = 0 \quad a.s.$$

Now, consider the case that $\sigma^2 \leq \dfrac{\beta\mu f(0)}{\wedge}$. Here,

$$\begin{aligned}
\phi(S,I) =& -\frac{\sigma^2}{2}\left(\frac{S}{f(I)} - \frac{\beta}{\sigma^2}\right)^2 + \frac{\beta^2}{2\sigma^2} - (\mu+\nu+\delta) \\
\leq& -\frac{\sigma^2}{2}\left(\frac{\wedge}{\mu f(0)} - \frac{\beta}{\sigma^2}\right)^2 + \frac{\beta^2}{2\sigma^2} - (\mu+\nu+\delta) \\
=& \frac{\wedge\beta}{\mu f(0)} - \frac{\sigma^2\wedge^2}{2\mu^2 f^2(0)} - (\mu+\nu+\delta) \\
=& (\mu+\nu+\delta)\left(R_0 - \frac{\sigma^2\wedge^2}{2\mu^2 f^2(0)(\mu+\nu+\delta)} - 1\right) \\
=& (\mu+\nu+\delta)(R_0^s - 1).
\end{aligned}$$

It then follows that

$$\begin{aligned}
d\ln I \leq& \Big((\mu+\nu+\delta)(R_0^s-1) - \beta_2\Big)dt + \frac{\sigma S}{f(I)} d\mathcal{B}_t^2 \\
&+ \int_Z \ln(1+\gamma_2)\widetilde{\mathcal{N}}_2(dt,du), \\
\ln I_t \leq& \ln I_0 + \int_0^t \Big((\mu+\nu+\delta)(R_0^s-1) - \beta_2\Big)ds + \int_0^t \frac{\sigma S}{f(I)} d\mathcal{B}_s^2 \\
&+ \int_0^t \left(\int_Z \ln(1+\gamma_2)\widetilde{\mathcal{N}}_2(dt,du)\right) ds \quad a.s.
\end{aligned}$$

This implies,

$$\limsup_{t\to\infty} \frac{\ln I_t}{t} \leq (\mu+\nu+\delta)(R_0^s-1) - \beta_2 < 0 \quad a.s. \tag{4.8}$$

and

$$\lim_{t\to\infty} I_t = 0 \quad a.s.$$

Remark 4.1. *Based on equation (4.8), there exists a null set N_1 with $\mathbb{P}(N_1) = 0$ such that for every $\omega \in \Omega \setminus N_1$, the following holds:*

$$\limsup_{t\to\infty} \frac{\ln I_t(\omega)}{t} < -c \quad \textit{for some constant } c > 0.$$

This implies that for any arbitrarily small $\epsilon > 0$, there exists a time $T_1 = T_1(\omega)$ such that

$$I_t(\omega) \leq \exp\big((-c+\epsilon)t\big), \quad \textit{for all } t \geq T_1.$$

2. From the system (3.4),

$$dR_t = \Big(\nu I_t - (\mu+\gamma)R_t\Big)dt + \int_Z R_{t-}\gamma_3(u)\widetilde{\mathcal{N}_3}(dt,du).$$

Using Itô's Lemma,

$$\begin{aligned} d\ln R &= \left(\frac{1}{R}(\nu I - (\mu+\gamma)R) - \int_Z (\gamma_3 - \ln(1+\gamma_3))\lambda_3(du)\right) dt \\ &\quad + \int_Z \ln(1+\gamma_3)\widetilde{\mathcal{N}_3}(dt,du) \\ &\leq \left(\nu\frac{I}{R} - (\mu+\gamma) - \beta_3\right) dt + \int_Z \ln(1+\gamma_3)\widetilde{\mathcal{N}_3}(dt,du) \\ &\leq (\nu I - (\mu+\gamma) - \beta_3)dt + \int_Z \ln(1+\gamma_3)\widetilde{\mathcal{N}_3}(dt,du). \end{aligned}$$

From the Remark 4.1, we will have

$$\begin{aligned} d\ln R &\leq \big(\nu e^{(-c+\epsilon)t} - (\mu+\gamma) - \beta_3\big)\, dt + \int_Z \ln(1+\gamma_3)\widetilde{\mathcal{N}_3}(dt,du), \\ \ln R_t &\leq \ln R_0 + \int_0^t \big(\nu e^{(-c+\epsilon)s} - (\mu+\gamma) - \beta_3\big)\, ds \\ &\quad + \int_0^t \int_Z \ln(1+\gamma_3)\widetilde{\mathcal{N}_3}(dt,du)ds, \\ &= \ln R_0 + \frac{\nu e^{(-c+\epsilon)t}}{(-c+\epsilon)} - (\mu+\gamma)t - \beta_3 t + \int_0^t \int_Z \ln(1+\gamma_3)\widetilde{\mathcal{N}_3}(dt,du)ds. \end{aligned}$$

This implies

$$\limsup_{t\to\infty} \frac{\ln R_t}{t} \leq -(\mu+\gamma) - \beta_3 < 0 \quad a.s. \tag{4.9}$$

and

$$\lim_{t\to\infty} R_t = 0 \quad a.s.$$

Remark 4.2. *According to equation (4.9), there exists a null set N_2 with $\mathbb{P}(N_2) = 0$, such that for every $\omega \in \Omega \setminus N_2$, the inequality*

$$\limsup_{t\to\infty} \frac{\ln R_t(\omega)}{t} < -\tilde{c}$$

holds for some constant $\tilde{c} > 0$. Therefore, for any chosen $\epsilon > 0$, however *small, one can find a time $T_2 = T_2(\omega)$ such that*

$$R_t(\omega) \le \exp\big((-\tilde{c} + \epsilon)t\big), \quad \textit{for all } t \ge T_2.$$

$\square$

4.4 Numerical Simulations

This section validates our previous analytic results numerically using the Euler-Maruyama method in Python. In the figures that follow, note the color representations:

- **Green Line:** Represents the model (3.4) with Lévy Jumps.
- **Red Line:** Represents the model (3.3) with only Gaussian white noise.
- **Blue Line:** Represents the deterministic model (3.2).

Example 4.1. *In this case we take the following values:* $\wedge = 0.1, \mu = 0.05, \beta = 0.3, \gamma = 0.05, \nu = 0.1, \delta = 0.1, \sigma = 0.1, \gamma_1 = 0.05, \gamma_2 = 0.05, \gamma_3 = 0.05$, *then, Theorem 1 can be validated by the following graphs. It is clearly visible from Figure 4.1 that S, I, and R have a solution given some initial conditions.*

Example 4.2. *In this case we take the following values:*
$\wedge = 0.05, \mu = 0.1, \beta = 0.5, \gamma = 0.05, \nu = 0.05, \delta = 0.01, \sigma = 0.05, \gamma_1 = 0.05, \gamma_2 = 0.05, \gamma_3 = 0.05$, *then it is shown by the Figure 4.2 that S and I have a bounded solution.*

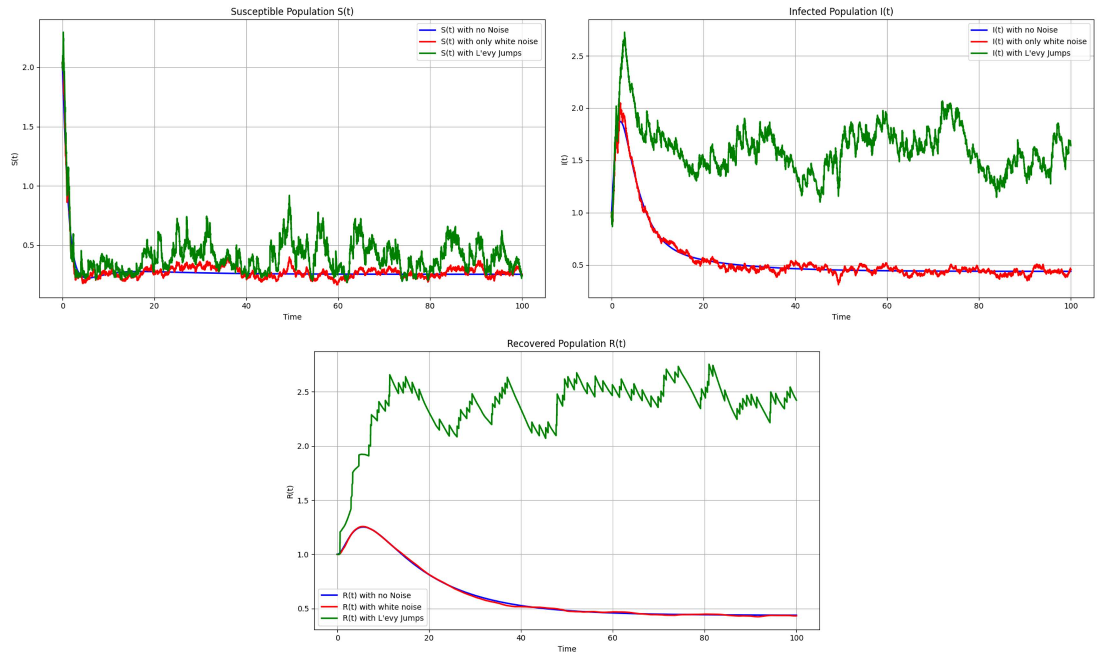

Figure 4.1: Simulation results validating the existence of the solution.

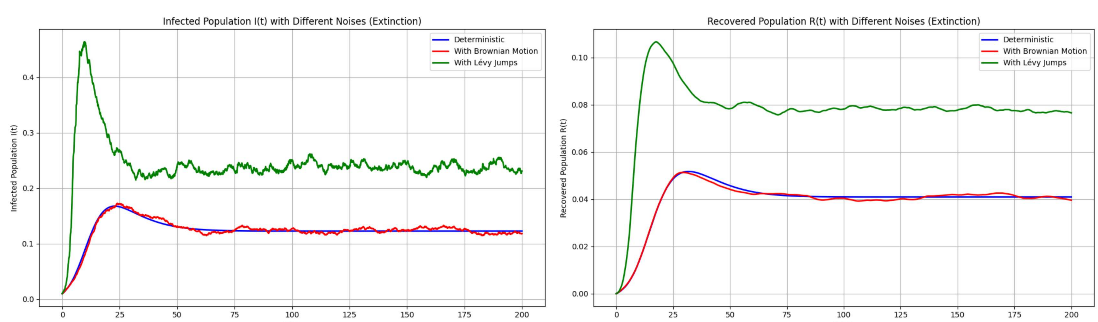

Figure 4.2: Paths of $I(t)$ and $R(t)$ for extinction as time increases to infinity.

CHAPTER 5

Large Deviations for Stochastic SIRS Epidemic Model with Lévy Jumps

This chapter is dedicated to presenting all the theorems and results obtained while establishing the Large Deviation Principle for the solution of the stochastic SIRS epidemic model, which is perturbed by an arbitrary small parameter $\epsilon > 0$. In order to prove LDP, we will use the weak convergence approach along with the celebrated results of Varadhan[39] and Bryc[14]. This theory was developed by Budhiraja, Dupuis, and Maroulas[8]. We would also like to acknowledge the following works which helped us in reviewing the literature on LDP for different models [22, 26, 30, 31, 34, 35].

We will consider system (3.4) with its abstract formulation as follows:

$$\begin{cases} dX_t = b(t,X_t)dt + \Sigma(t,X_t)d\mathcal{B}_t + \displaystyle\int_Z g(X_t,u)\widetilde{\mathcal{N}}(dt,du), \quad t \in (0,T], \\ X_0 = X(0) \end{cases} \tag{5.1}$$

where

$$X_t = \begin{pmatrix} S_t \\ I_t \\ R_t \end{pmatrix}, \quad b(t, X_t) = \begin{pmatrix} \wedge - \mu S_t - \dfrac{\beta I_t}{f(I_t)} S_t + \gamma R_t \\ \dfrac{\beta I_t}{f(I_t)} S_t - (\mu + \nu + \delta) I_t \\ \nu I_t - (\mu + \gamma) R_t \end{pmatrix},$$

$$\Sigma(t, X_t) = \begin{pmatrix} -\sigma \dfrac{I_t}{f(I_t)} S_t \\ \sigma \dfrac{I_t}{f(I_t)} S_t \\ 0 \end{pmatrix}, \quad g(X_t, u) = \begin{pmatrix} S_{t-}\gamma_1(u) \\ I_{t-}\gamma_2(u) \\ R_{t-}\gamma_3(u) \end{pmatrix}.$$

where $\mathcal{B}_t$ and $\widetilde{\mathcal{N}}$ are vector-valued Brownian motion and vector-valued compensated Poisson random measure, respectively. Also, $b(t, X_t), \Sigma(t, X_t)$, and $g(X_t, u)$ are measurable functions satisfying the linear growth and Lipschitz continuity conditions. Studying in an abstract setting allows us to analyze a system of multiple equations as a single differential equation. For $X_t^\epsilon = (S_t^\epsilon, I_t^\epsilon, R_t^\epsilon)$, let us define the abstract formulation of stochastic SIRS epidemic model driven by Lévy noise (3.4) which is perturbed by a small parameter $\epsilon > 0$ as

$$\begin{cases} dX_t^\epsilon = b(t, X_t^\epsilon)dt + \sqrt{\epsilon}\Sigma(t, X_t^\epsilon)d\mathcal{B}_t + \epsilon \displaystyle\int_Z g(X_t^\epsilon, u)\widetilde{\mathcal{N}}(dt, du), \quad t \in (0, T], \\ X_t^\epsilon(0) = X_0^\epsilon, \end{cases} \tag{5.2}$$

5.1 Moment Bounds

Moment bounds ensure that in a physical system with small random shocks, the system does not explode uncontrollably, making the system physically meaningful and mathematically tractable. In this section, bound on the solution of SDE (5.2).

Theorem 5.1. *For any $p \geq 2$, there exists a constant $C_p > 0, k > 0$, such that the solution of* (5.2) *satisfies,*

$$\mathbb{E}\left[\sup_{s\in[0,t]} |X_s^\epsilon|^p\right] \leq (k + C_p t) \exp(C_p t).$$

Proof. Applying Itô's Lemma to $f(x) = |x|^p$, we will get,

$$
\begin{aligned}
d|X_t^\epsilon|^p &= p|X_t^\epsilon|^{p-2}(X_t^\epsilon)^{\mathbf{T}} b(t, X_t^\epsilon)dt \\
&\quad + \frac{\epsilon}{2}\left[p(p-2)|X_t^\epsilon|^{p-4}((X_t^\epsilon)^{\mathbf{T}}\Sigma)^2 + p|X_t^\epsilon|^{p-2}||\Sigma||^2\right]dt \\
&\quad + \epsilon \int_Z \left[|X_{t-}^\epsilon + g(X_{t-}^\epsilon, u)|^p - |X_{t-}^\epsilon|^p - p|X_t^\epsilon|^{p-2}(X_t^\epsilon)^{\mathbf{T}} g(X_t^\epsilon, u)\right]\lambda(du)dt \\
&\quad + \sqrt{\epsilon} p|X_t^\epsilon|^{p-2}(X_t^\epsilon)^{\mathbf{T}}\Sigma d\mathcal{B}_t + \epsilon \int_Z \left[|X_{t-}^\epsilon + g(X_{t-}^\epsilon, u)|^p - |X_{t-}^\epsilon|^p\right]\widetilde{\mathcal{N}}(dt, du).
\end{aligned}
$$

Upon integrating, we get,

$$
\begin{aligned}
|X_t^\epsilon|^p &= |X_0^\epsilon|^p + \underbrace{\int_0^t p|X_s^\epsilon|^{p-2}(X_s^\epsilon)^{\mathbf{T}} b(s, X_s^\epsilon)ds}_{:=I_1} \\
&\quad + \underbrace{\frac{\epsilon}{2}\int_0^t \left[p(p-2)|X_s^\epsilon|^{p-4}((X_s^\epsilon)^{\mathbf{T}}\Sigma)^2 + p|X_s^\epsilon|^{p-2}||\Sigma||^2\right]ds}_{:=I_2} \\
&\quad + \underbrace{\epsilon \int_0^t \int_Z \left[|X_{s-}^\epsilon + g(X_{s-}^\epsilon, u)|^p - |X_{s-}^\epsilon|^p - p|X_s^\epsilon|^{p-2}(X_s^\epsilon)^{\mathbf{T}} g(X_s^\epsilon, u)\right]\lambda(du)ds}_{:=I_3} \\
&\quad + \sqrt{\epsilon}\int_0^t p|X_s^\epsilon|^{p-2}(X_s^\epsilon)^{\mathbf{T}}\Sigma d\mathcal{B}_s \\
&\quad + \epsilon \int_0^t \int_Z \left[|X_{s-}^\epsilon + g(X_{s-}^\epsilon, u)|^p - |X_{s-}^\epsilon|^p\right]\widetilde{\mathcal{N}}(ds, du).
\end{aligned}
\tag{5.3}
$$

Consider each integral separately. For I_1, using the Linear growth condition, we get

$$
I_1 \leq C_p^1 \int_0^t |X_s^\epsilon|^{p-1}(1 + |X_s^\epsilon|)ds \leq C_p^1 \int_0^t (1 + |X_s^\epsilon|^p)ds
$$

For I_2, using Linear growth condition for Σ and the inequality $(a+b)^2 \leq 2(a^2+b^2)$,

$$
\begin{aligned}
I_2 &= \frac{\epsilon}{2}\int_0^t \left[p(p-2)|X_s^\epsilon|^{p-4}((X_s^\epsilon)^{\mathbf{T}}\Sigma)^2 + p|X_s^\epsilon|^{p-2}\|\Sigma\|^2\right]ds \\
&= \frac{\epsilon}{2}\int_0^t \left[p(p-2)|X_s^\epsilon|^{p-2}\|\Sigma\|^2 + p|X_s^\epsilon|^{p-2}\|\Sigma\|^2\right]ds \\
&\leq \frac{\epsilon}{2}\int_0^t \left[p(p-2)|X_s^\epsilon|^{p-2}(1 + |X_s^\epsilon|^2) + p|X_s^\epsilon|^{p-2}(1 + |X_s^\epsilon|^2)\right]ds \\
&\leq C_p^2 \int_0^t (1 + |X_s^\epsilon|^p)ds
\end{aligned}
$$

By using Taylor's expansion of second-order, and since λ is $\sigma-$finite measure, for

I_3, we will get,

$$I_3 = \epsilon \int_0^t \int_Z \left[|X^\epsilon_{s-} + g(X^\epsilon_{s-}, u)|^p - |X^\epsilon_{s-}|^p - p|X^\epsilon_s|^{p-2}(X^\epsilon_s)^{\mathbf{T}} g(X^\epsilon_s, u) \right] \lambda(du)ds$$
$$= \epsilon \int_0^t \int_Z \frac{1}{2} \left[p(p-2)|X^\epsilon_s|^{p-4}((X^\epsilon_s)^{\mathbf{T}} g(X^\epsilon_{s-}, u))^2 + p|X^\epsilon_s|^{p-2} \|g(X^\epsilon_{s-}, u))\|^2 \right] \lambda(du)ds$$
$$\leq C^3_p \int_0^t (1 + |X^\epsilon_s|^p) ds$$

Now for the integral involving brownian motion, Bürkholder's-Davis-Gundy inequality, and Young's inequality yields,

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq s\leq T} \left| \int_0^s p|X^\epsilon_r|^{p-2}(X^\epsilon_r)^{\mathbf{T}} \Sigma d\mathcal{B}_r \right| \right] &\leq C^4_p \left(\int_0^t \mathbb{E}\left[|X^\epsilon_r|^{2p-2} \|\Sigma\|^2 \right] dr \right)^{1/2} \\
&\leq C^4_p \left(\int_0^t \mathbb{E}\left[1 + |X^\epsilon_r|^p \right] dr \right)^{1/2} \\
&\leq k_1 \int_0^t \mathbb{E}\left[1 + |X^\epsilon_r|^p \right] dr + \frac{(C^4_p)^2}{4k_1}
\end{aligned}$$

Similarly, for the last integral, we will use Bürkholder's-Davis-Gundy inequality, and Young's inequality to get

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{s\leq t} \Big| \int_0^t \int_Z \left[|X^\epsilon_{s-} + g(X^\epsilon_{s-}, u)|^p - |X^\epsilon_{s-}|^p \right] \widetilde{\mathcal{N}}(ds, du) \Big| \Big] \\
&\qquad\qquad \leq C^5_p \left(\int_0^t \mathbb{E}\left[1 + |X^\epsilon_r|^p \right] dr \right)^{1/2} \\
&\qquad\qquad \leq k_2 \int_0^t \mathbb{E}\left[1 + |X^\epsilon_r|^p \right] dr + \frac{(C^5_p)^2}{k_2}
\end{aligned}$$

Now that we have gotten estimates for all the integrals in the (5.3), let us combine all of them, using Fubini's theorem and then apply Grönwall's inequality,

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |X^\epsilon_s|^p \right] \leq k + C_p \int_0^t \mathbb{E}\left(1 + |X^\epsilon_s|^p \right) ds$$

where $k \geq \dfrac{(C^4_p)^2}{4k_1} + \dfrac{(C^5_p)^2}{4k_2} + |X^\epsilon_0|^p$ and $C_p \geq C^1_p + C^2_p + C^3_p + k_1 + k_2$.

On simplifying, we get

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq s\leq t} |X^\epsilon_s|^p \right] &\leq k + C_p \int_0^t \mathbb{E}\left(1 + |X^\epsilon_s|^p \right) ds \\
&\leq k + C_p t + C_p \int_0^t \mathbb{E}\left[\sup_{0\leq u\leq r} |X^\epsilon_u|^p \right] dr
\end{aligned}$$

Applying Grönwall's inequality, we get,

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq s\leq t}|X_s^\epsilon|^p\right] \leq & (k+C_pt)\exp\left(\int_0^t C_p ds\right)\\ \leq & (k+C_pt)\exp\left(C_pt\right).\end{aligned}$$

□

5.2 Large Deviation Principle

In this section, asymptotic analysis of the system (3.4) is done and the Large deviation principle is established for the perturbed SDE (5.2). We are interested in seeing how the solution of perturbed SDE (5.2) deviates from its deterministic counterpart as $\epsilon \to 0$. But before that let us define some function spaces based on the theory developed by BUDHIRAJA, DUPUIS, AND MAROULAS. Refer [8] for detailed analysis.

5.2.1 Function spaces and Controlled Equations

Consider a space $\mathcal{P}$ that is both locally compact and Polish. Let $\mathcal{M}(\mathcal{P})$ denote the set of all Borel measures ν on $(\mathcal{P}, \mathcal{B}(\mathcal{P}))$ for which $\nu(G) < \infty$ holds for any compact subset $G \subset \mathcal{P}$. Equip $\mathcal{M}(\mathcal{P})$ with the coarsest topology that ensures the mapping $\nu \mapsto \int_{\mathcal{P}} f(z)\,\nu(dz)$ is continuous for each $f \in C_c(\mathcal{P})$, where $C_c(\mathcal{P})$ is the space of continuous functions with compact support. This topology renders $\mathcal{M}(\mathcal{P})$ a Polish space, and it can be metrized accordingly, as demonstrated in [8]. Let us define the extended space $\mathcal{P}_T := [0,T]\times\mathcal{P}$, and let $\mathbb{P} := \mathcal{M}(\mathcal{P}_T)$. We introduce the product space $\mathbb{L}(\mathbb{R}^3) := C([0,T];\mathbb{R}^3)\times\mathbb{P}$. Furthermore, let $\{\mathcal{G}_t\}_{t\in[0,T]}$ denote the filtration given by

$$\mathcal{G}_t := \sigma\big(\{\widetilde{\mathcal{N}}(s,Z), \mathcal{B}_s : 0 \leq s \leq t, Z \in \mathcal{B}(\mathcal{P}_T)\}\big).$$

For a fixed constant $\theta > 0$, we define $\mathbb{P}_\theta$ to be the unique probability measure on the measurable space $(\mathbb{L}(\mathbb{R}^3), \mathcal{B}(\mathbb{L}(\mathbb{R}^3)))$ such that, under $\mathbb{P}_\theta$, both the

Brownian motion $\{\mathcal{B}_t\}$ and the compensated Poisson random measure $\{\widetilde{\mathcal{N}}(t,Z)\}$ are $\{\mathcal{G}_t\}$-martingales for every measurable set $Z \subset \mathcal{P}_T$.

Let us define $\{\mathcal{F}_t\}_{t\in[0,T]}$ as the $\mathbb{P}$-augmentation of the filtration $\{\mathcal{G}_t\}$. Consider P to be the σ-algebra on the product space $[0,T] \times \mathbb{L}(\mathbb{R}^3)$ associated with the filtration $\{\mathcal{F}_t\}$ on the measurable space $(\mathbb{L}(\mathbb{R}^3), \mathcal{B}(\mathbb{L}(\mathbb{R}^3)))$.

Now, let $\mathcal{A}$ denote the collection of all non-negative measurable mappings $\phi : \mathcal{P}_T \times \mathbb{L}(\mathbb{R}^3) \to [0,\infty)$ that are $(P \otimes \mathcal{B}(\mathcal{P}))/\mathcal{B}([0,\infty))$-measurable. For any $\phi \in \mathcal{A}$, we define the corresponding counting process $\mathcal{N}^\phi$ on $\mathcal{P}_T$ by:

$$\mathcal{N}^\phi(t,Z) = \int_{(0,t]\times Z} \int_0^\infty \mathbf{1}_{[0,\phi(s,z)]}(r)\, \widetilde{\mathcal{N}}(ds,dz)\, dr, \quad t \in [0,T],\ Z \in \mathcal{B}(\mathcal{P}).$$

Define the function $\ell : [0,\infty) \to [0,\infty)$ by:

$$\ell(r) := r \log r - r + 1, \quad \forall r \in [0,\infty).$$

Given any $\phi \in \mathcal{A}$, let the functional $L_T(\phi)$ be defined by:

$$L_T(\phi) := \int_0^T \int_Z \ell(\phi(t,u,\omega))\, \lambda(du)\, dt.$$

Let $\mathcal{P}_2$ denote the set of all predictable processes $\psi : [0,T] \to \mathbb{R}^3$ such that:

$$\int_0^T \|\psi(s)\|^2 ds < \infty \quad \text{a.s.}$$

We define the set of admissible control pairs by:

$$\mathcal{U}(\mathbb{R}^3) := \mathcal{P}_2 \times \mathcal{A}.$$

For any $\psi \in \mathcal{P}_2$, define the associated energy functional:

$$\widetilde{L}_T(\psi) := \frac{1}{2} \int_0^T \|\psi(s)\|^2 ds.$$

Given $u = (\psi,\phi) \in \mathcal{U}(\mathbb{R}^3)$, we define the total cost as:

$$\widetilde{L}_T(u) := L_T(\phi) + \widetilde{L}_T(\psi).$$

Let us define the controlled Brownian motion by:

$$\mathcal{B}_t^\psi := \mathcal{B}_t + \int_0^t \psi(s) ds, \quad t \in [0,T].$$

For a fixed $N \in \mathbb{N}$, we define:

$$\widetilde{S}^N(\mathbb{R}^3) := \left\{ \psi \in L^2([0,\infty);\mathbb{R}^3) : \widetilde{L}_T(\psi) \leq N \right\},$$

$$S^N := \{\phi : \mathcal{P}_T \to [0,\infty) : L_T(\phi) \leq N\}.$$

Let us now define:

$$\lambda_T^g := \int_0^T \int_Z g(s,u)\lambda(du)ds, \quad Z \in \mathcal{B}(\mathcal{P}_T).$$

Then, the family $\{\lambda_T^g : g \in S^N\}$ is relatively compact in $\mathbb{P}$. Equip S^N with the topology that renders it compact. The product space $\widetilde{S}^N := \widetilde{S}^N(\mathbb{R}^3) \times S^N$ inherits the product topology.

Finally, let

$$\begin{aligned} \mathcal{U} &:= \mathcal{P}_2(\mathbb{R}^3) \times \mathcal{A}, \\ \mathbb{S} &:= \bigcup_{N \geq 1} \widetilde{S}^N, \\ \mathcal{U}^N &:= \left\{ z = (\psi, \phi) \in \mathcal{U} : z(\omega) \in \widetilde{S}^N \text{ a.s.} \right\}. \end{aligned}$$

That is, $\mathcal{U}^N$ consists of all admissible controls taking values in $\widetilde{S}^N$ almost surely.

The solution X^ϵ to the perturbed stochastic differential equation given in (5.2) can be expressed in terms of a measurable mapping as follows:

$$X^\epsilon = \mathcal{G}^\epsilon\left(\sqrt{\epsilon}\mathcal{B}(\cdot), \, \epsilon\mathcal{N}^{\epsilon^{-1}}\right),$$

where $\mathcal{G}^\epsilon : \mathcal{P} \times \mathbb{L}(\mathbb{R}^3) \to \mathcal{P}$ is a Borel-measurable transformation. We will prove the Laplace principle for X^ϵ using the Budhiraja-Dupuis principle whose statement is as given below.

Theorem 5.2. *Let us suppose there exists a measurable map $\mathcal{G}^0 : \mathcal{P} \times \mathbb{L}(\mathbb{R}^3) \to \mathcal{P}$ such that it satisfies the following two conditions:*

(i) Weak Convergence: Consider $N < \infty$ and a family

$$\left\{\theta^\epsilon = (\psi^\epsilon, \phi^\epsilon) \in \mathcal{U} : \theta^\epsilon(\omega) \in \widetilde{S}_N \quad a.s.\right\} \subset \mathcal{U}_N.$$

If $\theta^\epsilon = (\psi^\epsilon, \phi^\epsilon)$ converges to $\theta = (\psi, \phi)$ in distribution as $\widetilde{S}_N$-valued random elements, then

$$\mathcal{G}^\epsilon\left(\sqrt{\epsilon}\mathcal{B}(.) + \int_0^\cdot \psi^\epsilon(s)ds, \epsilon\mathcal{N}^{\epsilon^{-1}\phi^\epsilon}\right) \to \mathcal{G}^0\left(\int_0^\cdot \psi(s)ds, \lambda_T^\phi\right)$$

in distribution as $\epsilon \to 0$.

(ii) Compactness: For every $N < \infty$, the set

$$K_N := \left\{ \mathcal{G}^0 \left(\int_0^{\cdot} \psi(s) ds, \lambda_T^{\phi} \right) : (\phi, \psi) \in \mathcal{U}_N \right\}$$

is a compact subset of $\mathcal{P}$.

Then the family $\{X^\epsilon : \epsilon > 0\}$ satisfies the Laplace principle in $\mathcal{P}$ with the good rate function I given by

$$I(f) := \inf_{(\phi,\psi)\in S_g} \left\{ \int_0^T \int_Z \ell(\phi(s,u))\lambda(du)ds + \frac{1}{2}\int_0^T \|\psi(s)\|_0^2 ds \right\}, \tag{5.4}$$

where $S_g := \{(\phi, \psi) \in \mathbb{S} : g = \mathcal{G}^0(\int_0^{\cdot} \psi(s)ds, \lambda_T^{\phi})\}$ and $I(\phi) := \infty$.

Remark 5.1. *The considered stochastic SIRS epidemic model which is given as*

$$\begin{cases} dX_t^\epsilon = b(t, X_t^\epsilon)dt + \sqrt{\epsilon}\Sigma(t, X_t^\epsilon)d\mathcal{B}_t + \epsilon \displaystyle\int_Z g(X_t^\epsilon, u)\widetilde{\mathcal{N}}(dt, du), \quad t \in (0, T], \\ X_t^\epsilon(0) = X_0^\epsilon, \end{cases}$$

has a unique solution in the Polish space $\mathcal{P} = \mathcal{D}([0,T];\mathbb{R}_3) \cap L^2([0,T];\mathbb{R}_3)$. We will denote the solution of above to be X^ϵ, which can be written as $\mathcal{G}^\epsilon\left(\sqrt{\epsilon}\mathcal{B}(.), \epsilon\mathcal{N}^{\epsilon^{-1}}\right)$ for a Borel-measurable function $\mathcal{G}^\epsilon : \mathcal{D}([0,T];\mathbb{R}_3) \to \mathcal{P}$.

Observe that the $\mathcal{D}([0,T];\mathbb{R}_3)$ is the Skorokhod space, which is a Polish space (cf. [20]). But before going any further, we need the well posedness of the controlled equations associated with system (5.2) as well. The deterministic controlled equation (or skeleton equation) will be as follows:

$$\begin{cases} dz_\theta(t) = b(t, z_\theta)dt + \Sigma(t, z_\theta)\psi(t)dt + \displaystyle\int_Z g(z_\theta, u)\ell(\phi(t,u))\lambda(du)dt, \quad t \in (0, T]; \\ z_\theta(0) = z_0, \end{cases} \tag{5.5}$$

The existence and uniqueness follows readily since the coefficients b and Σ satisfy the Lipschitz continuity and linear growth conditions. The solution of the above equation can be expressed as $\mathcal{G}^0(\int_0^{\cdot} \psi(s)ds, \lambda_T^{\phi})$.

Now let us present this lemma which ensures the well-posedness of the solution of the stochastic controlled equation associated with (5.2).

Lemma 5.3. *Consider $N < \infty$ and a family* $\Big\{\theta^\epsilon = (\psi^\epsilon, \phi^\epsilon) \in \mathcal{U} : \theta^\epsilon(\omega) \in \tilde{S}_N a.s.\Big\}$ $\subset \mathcal{U}_N$. *For $\epsilon > 0$, define*

$$X^\epsilon_{\theta^\epsilon}(t) = G^\epsilon\Big(\sqrt{\epsilon}B(.) + \int_0^{\cdot} \psi^\epsilon(s)ds, \epsilon\mathcal{N}^{\epsilon^{-1}\phi^\epsilon}\Big).$$

Then $X^\epsilon_{\theta^\epsilon}$ is the unique solution of the following stochastic controlled equation:

$$\begin{cases} dX^\epsilon_{\theta^\epsilon}(t) = b(t, X^\epsilon_{\theta^\epsilon}(t))dt + \Sigma(t, X^\epsilon_{\theta^\epsilon}(t))\psi^\epsilon(t)dt + \int_Z g(X^\epsilon_{\theta^\epsilon}, u)\ell(\phi^\epsilon)\lambda(du)dt \\ \qquad\qquad + \sqrt{\epsilon}\Sigma(t, X^\epsilon_{\theta^\epsilon}(t))dB(t) + \epsilon\int_Z g(X^\epsilon_{\theta^\epsilon}, u)\widetilde{\mathcal{N}}(dt, du), \quad t \in (0, T], \\ X^\epsilon_{\theta^\epsilon}(0) = X_0^{\theta^\epsilon} \end{cases} \tag{5.6}$$

The proof is a consequence of Girsanov's theorem and the Existence-Uniqueness theorem for SDEs. Also refer to [13].

Now, that the necessary conditions are satisfied, let us prove the sufficient conditions as stated in Theorem 5.2, that is, compactness and weak convergence.

5.2.2 Compactness

Theorem 5.4. *For every $N < \infty$, the set*

$$K_N := \big\{z_\theta \in \mathcal{D}([0, T]; \mathbb{R}_3) \cap L^2([0, T]; \mathbb{R}^3) : \theta \in \mathcal{U}^N\big\}$$

is a compact subset of $\mathcal{D}([0, T]; \mathbb{R}^3) \cap L^2([0, T]; \mathbb{R}^3)$ where z_θ is the unique solution of the deterministic controlled equation (5.5) .

Proof. Consider a sequence $\{z_{\theta_n}\}$ of K_N where $\theta_n \in \mathcal{U}_N$ is the control. Then $\{z_{\theta_n}\}$ satisfies (5.5), we get,

$$dz_{\theta_n}(t) = b(t, z_{\theta_n})dt + \Sigma(t, z_{\theta_n})\psi_n(t)dt + \int_Z g(z_{\theta_n}, u)\ell(\phi_n)\lambda(du)dt$$

and $z_{\theta_n}(0) = z_0$. Since the set $\mathcal{U}_N$ is weakly compact, so there exists a subsequence of $\{z_{\theta_n}\}$, also denoted by $\{z_{\theta_n}\}$ (for the sake of less notations). This subsequence converges weakly to $\theta \in \mathcal{U}_N$ in $\mathcal{U}$. All we need to prove is $z_{\theta_n} \to z_\theta$ in $\mathcal{D}([0, T]; \mathbb{R}^3) \cap L^2([0, T]; \mathbb{R}^3)$ as $n \to \infty$.

Let $w_{\theta_n} = z_{\theta_n} - z_\theta$. Then w_{θ_n} also satisfies (5.5), and we will get the following differential equation,

$$
\begin{aligned}
dw_{\theta_n} =& \left[b(t, z_{\theta_n}) - b(t, z_\theta)\right] dt + \left[\Sigma(t, z_{\theta_n}) - \Sigma(t, z_\theta)\right] \psi_n(t)dt + \Sigma(t, z_\theta)(\psi_n - \psi)(t)dt \\
&+ \int_Z \left[g(z_{\theta_n}, u) - g(z_\theta, u)\right] \ell(\phi_n)\lambda(du)dt + \int_Z g(z_\theta, u) \left[\ell(\phi_n) - \ell(\phi)\right] \lambda(du)dt \\
\|w_{\theta_n}\| \le& \underbrace{\int_0^t \|b(s, z_{\theta_n}) - b(s, z_\theta)\| \, ds}_{:=I_1} + \underbrace{\int_0^t \|\Sigma(s, z_{\theta_n}) - \Sigma(s, z_\theta)\| \, \|\psi_n(s)\| \, ds}_{:=I_2} \\
&+ \underbrace{\int_0^t \|\Sigma(s, z_\theta)\| \, \|(\psi_n - \psi)(s)\| \, ds}_{:=I_3} + \underbrace{\int_0^t \int_Z \|g(z_\theta, u)\| \, \|\ell(\phi_n) - \ell(\phi)\| \, \lambda(du)dt}_{:=I_4} \\
&+ \underbrace{\int_0^t \int_Z \|g(z_{\theta_n}, u) - g(z_\theta, u)\| \, \|\ell(\phi_n)\| \, \lambda(du)dt}_{:=I_5}
\end{aligned} \tag{5.7}
$$

Consider I_1 and I_2, then by Lipschitz continuity of b and Σ, there exists constant $L_1, L_2 > 0$ such that

$$
I_1 = \int_0^t \|b(s, z_{\theta_n}) - b(s, z_\theta)\| \, ds \le L_1 \int_0^t \|w_{\theta_n}(s)\| \, ds
$$

$$
I_2 = \int_0^t \|\Sigma(s, z_{\theta_n}) - \Sigma(s, z_\theta)\| \, \|\psi_n(s)\| \, ds \le L_2 \int_0^t \|w_{\theta_n}(s)\| \, \|\psi_n(s)\| \, ds
$$

For the integral I_3, using Cauchy-Schwarz inequality, and the fact $\psi_n \to \psi$ weakly as $n \to \infty$, we will get the integral $I_3 \to 0$ as $n \to \infty$.

$$
\begin{aligned}
I_3 =& \int_0^t \|\Sigma(s, z_\theta)\| \, \|(\psi_n - \psi)(s)\| \, ds \\
\le& \left(\int_0^t |\Sigma(s, z_\theta|^2\right)^{\frac{1}{2}} \left(\int_0^t |\psi_n - \psi|^2\right)^{\frac{1}{2}} \to 0
\end{aligned}
$$

Considering integral I_4, and the fact that ℓ is continuous and $\psi_n \to \psi$ weakly as $n \to \infty$, so the integral will tend to 0. Now, let us analyze the last integral I_5. Since g satisfies Lipschitz continuity, we will get a constant $L_3 > 0$, and since λ is a σ-finite measure, we will get the following bound on the

$$
\begin{aligned}
I_5 =& \int_0^t \int_Z \|g(z_{\theta_n}, u) - g(z_\theta, u)\| \, \|\ell(\phi_n)\| \, \lambda(du)dt \\
\le& L_3 \int_0^t \|w_{\theta_n}(s)\| \int_Z \|\ell(\phi_n)\| \, \lambda(du)dt \le C_1 \int_0^t \|w_{\theta_n}(s)\| \, ds
\end{aligned}
$$

Combining all the above integrals, (5.7) becomes,

$$\|w_{\theta_n}(s)\| \leq \int_0^t \|w_{\theta_n}(s)\| \left(L_1 + L_2 \|\psi_n(s)\| + C_1\right) ds$$

Applying Grönwall's inequality, we will get $\|w_{\theta_n}\| \to 0$ as $n \to \infty$. □

5.2.3 Weak Convergence

Theorem 5.5. *Consider $N < \infty$ and a family $\{\theta^\epsilon : \epsilon > 0\} \subset \mathcal{U}_N$. If θ^ϵ converges to θ in distribution with respect to the weak topology defined on $\mathcal{U}$, then*

$$\mathcal{G}^\epsilon\Big(\sqrt{\epsilon}B(.) + \int_0^{\cdot} \psi^\epsilon(s)ds, \epsilon\mathcal{N}^{\epsilon^{-1}\phi^\epsilon}\Big) \to \mathcal{G}^0\Big(\int_0^{\cdot} \psi(s)ds, \lambda_T^\phi\Big)$$

in distribution in $\mathcal{D}([0,T];\mathbb{R}^3) \cap L^2([0,T];\mathbb{R}^3)$ as $\epsilon \to 0$.

Proof. Let $W^\epsilon = X^\epsilon - z_\theta$, where z_θ and X^ϵ are solutions of the equations (5.5) and (5.6) respectively. Then applying Itô's Lemma to $|x|^p$, W^ϵ satisfies the following stochastic equation:

$$\begin{aligned}
|W^\epsilon|^p = W_0^p &+ \underbrace{\int_0^t p|W^\epsilon|^{p-2}W^{\epsilon\mathbf{T}}\Big[b(s,X^\epsilon) - b(s,z_\theta)\Big]ds}_{:=I_1} \\
&+ \underbrace{\int_0^t p|W^\epsilon|^{p-2}W^{\epsilon T}\Big[\Sigma(s,X^\epsilon)\psi^\epsilon - \Sigma(s,z_\theta)\psi\Big]ds}_{:=I_2} \\
&+ \underbrace{\int_0^t p|W^\epsilon|^{p-2}W^{\epsilon T}\int_Z \Big[g(X^\epsilon,u)\ell(\phi^\epsilon) - g(z_\theta,u)\ell(\phi)\Big]\lambda(du)dt}_{:=I_3} \\
&+ \underbrace{\frac{\epsilon}{2}\int_0^t \Big[p(p-2)|W^\epsilon|^{p-4}\big(W^{\epsilon T}\Sigma(s,X^\epsilon)\big)^2 + p|W^\epsilon|^{p-2}\left\|\Sigma(s,X^\epsilon)\right\|^2\Big]ds}_{:=I_4} \\
&+ \underbrace{\epsilon\int_0^t\int_Z \Big[|W^\epsilon_{s-} + g(X^\epsilon,u)|^p - |W^\epsilon_{s-}|^p - p|W^\epsilon_{s-}|^{p-2}W^{\epsilon T}_{s-}g(X^\epsilon,u)\Big]\lambda(du)ds}_{:=I_5} \\
&+ \sqrt{\epsilon}\int_0^t p|W^\epsilon|^{p-2}W^{\epsilon T}\Sigma(s,X^\epsilon)dB(s) \\
&+ \epsilon\int_0^t\int_Z \Big[|W^\epsilon_{s-} + g(X^\epsilon,u)|^p - |W^\epsilon_{s-}|^p\Big]\widetilde{\mathcal{N}}(ds,du)
\end{aligned}$$

We will now consider each integral separately and get the bounds as done before.

$$I_1 := \int_0^t p|W^\epsilon|^{p-2} W^{\epsilon \mathbf{T}} \Big[b(s, X^\epsilon) - b(s, z_\theta)\Big] ds \le L_1 \int_0^t |W^\epsilon|^{p-1} W^\epsilon ds$$

$$I_2 := \int_0^t p|W^\epsilon|^{p-2} W^{\epsilon \mathbf{T}} \Big[\Sigma(s, X^\epsilon)\psi^\epsilon - \Sigma(s, z_\theta)\psi\Big] ds \le L_2 \int_0^t |W^\epsilon|^p |\psi^\epsilon| ds$$

$$I_3 := \int_0^t p|W^\epsilon|^{p-2} W^{\epsilon \mathbf{T}} \int_Z \Big[g(X^\epsilon, u)\ell(\phi^\epsilon) - g(z_\theta, u)\ell(\phi)\Big]\lambda(du)dt \le N \int_0^t |W^\epsilon|^p ds$$

$$\begin{aligned}
I_4 :=& \frac{\epsilon}{2} \int_0^t \Big[p(p-2)|W^\epsilon|^{p-4}\big(W^{\epsilon T}\Sigma(s, X^\epsilon)\big)^2 + p|W^\epsilon|^{p-2}\left\|\Sigma(s, X^\epsilon)\right\|^2\Big] ds \\
=& \frac{\epsilon}{2} \int_0^t \Big[p(p-2)|W^\epsilon|^{p-2}\left\|\Sigma(s, X^\epsilon)\right\|^2 + p|W^\epsilon|^{p-2}\left\|\Sigma(s, X^\epsilon)\right\|^2\Big] ds \\
\le& C_p^1 \int_0^t (1 + |W^\epsilon|^p)\, ds + C_p^2 \int_0^t (1 + |W^\epsilon|^p)\, ds
\end{aligned}$$

As also done before, the following integral can be bounded by,

$$I_5 \le C_5' \int_0^t (1 + |W^\epsilon|^p)\, ds$$

Using Bürkholder's-Davis-Gundy inequality, and Young's inequality, the martingale terms can be bounded as follows,

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\le t\le T}\Big|\int_0^t p|W^\epsilon|^{p-2} W^{\epsilon \mathbf{T}}\Sigma(t, X^\epsilon)dB\Big|\Big] \le& C\Big(\int_0^t \mathbb{E}\big[|W^\epsilon|^{2p-2}\|\Sigma\|^2\big]ds\Big)^{1/2} \\
\le& C\Big(\int_0^t \mathbb{E}\big[1 + |W^\epsilon|^p\big]ds\Big)^{1/2} \\
\le& \delta \int_0^t \mathbb{E}\big[1 + |W^\epsilon|^p\big]ds + \frac{C^2}{4\delta}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\le t\le T}\Big|\int_0^t \int_Z \big(|W^\epsilon_{t-} + g|^p - |W^\epsilon_{t-}|^p\big)\widetilde{\mathcal{N}}(dt, du)\Big|\Big] \le& C'\Big(\int_0^t \mathbb{E}\Big[1 + |W^\epsilon|^p\Big]ds\Big)^{1/2} \\
\le& \delta' \int_0^t \mathbb{E}\Big[1 + |W^\epsilon|^p\Big]ds + \frac{C'}{4\delta'^2}
\end{aligned}$$

Combining all these integrals, and using Grönwall's inequality, we will get $\mathbb{E}\left[\sup_{0\le t\le T}|W^\epsilon|^p\right] \to 0$. Then by using Markov's inequality, for any $a > 0$, we will get

$$\mathbb{P}\left(\sup_{0\le t\le T}|W^\epsilon|^p \ge a\right) \le \frac{1}{a}\mathbb{E}\left[\sup_{0\le t\le T}|W^\epsilon|^p\right] \to 0$$

□

CHAPTER 6

Conclusions and Future Directions

"What we know is a drop, what we don't know is an ocean."

–Sir Isaac Newton

In this final chapter, we draw the thesis to a close while also presenting ideas for potential future developments related to this model. In this chapter, the key conclusions are drawn together to highlight how the research objectives have been met and to reflect on the contributions made by the study. We talk about a few ideas for continuing this research and making it better, showing why it's important to build on what we've started here.

6.1 Conclusions

In this thesis, we worked on a stochastic SIRS epidemic model influenced by both Gaussian noise and the Poisson-type jumps. By incorporating Lévy noise into the classical SIRS framework, we captured more realistic dynamics that reflect abrupt outbreaks, and intervention policies.

Our main contributions are summarized below:

1. **Existence and Uniqueness:** We established the existence and uniqueness of the global positive solution to the stochastic SIRS epidemic model, which ensures the well-posedness of the system, opening the gate for further analysis.

2. **Long-term behavior:** We investigated the dynamics of the system in the long-term, particularly focusing on the probability of disease extinction. This analysis provides insights into the conditions under which the disease will die out or persist in the population over time.

3. **Numerical Simulations:** We carried out numerical simulations in Python to illustrate the theoretical results and to visualize how the inclusion of Lévy noise affects the disease dynamics.

4. **Large Deviation Principle (LDP):** We proved the Large Deviation Principle for the model by employing the weak convergence approach developed by Budhiraja, Dupuis, and Maroulas [8]. This result offers a rigorous probabilistic understanding of rare events, such as sudden large outbreaks, and complements the analysis of typical system behavior.

6.2 Future Directions

This research opens up several promising directions for future investigations. We will put on a few open problems here.

Open Questions:

1. We believe that the results from this work can be generalized to the case when the coefficients of the system do not follow a linearity condition. In

particular, it is more relevant if we can add noise in the model that is invariant under local coordinate transformations. Here, the Wong-Zakai approximation is more suitable for talking about the existence theory in population dynamics. For details, we refer to [5].

2. An immediate question is to study the corresponding Optimal Relaxed Control (ORC) (see [27]) problem for the given system when
   (a) no special condition on the dependence of the non-linear operator with respect to the control variable is assumed,
   (b) the cost functional fails to satisfy the usual convexity condition.
3. Moreover, can we adapt Lyon's rough path theory to study the system? We refer the reader to [27].
4. It is natural to ask whether the solution of system (3.4) defines a Markov process. For a thorough study on Markov processes, see [32].
5. Can stability analysis be done for this stochastic dynamical system? Can a deeper investigation into moment stability, pathwise stability, and possible bifurcations in the system driven by changes in noise intensity be done? Refer [19, 25, 28, 41] for detailed study on stochastic dynamical systems, and bifurcation analysis.

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