Phenomenology of the 3-3-1 Model

M.Sc. Thesis

By

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **Phenomenology of the 3-3-1 Model** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF PHYSICS**, Indian Institute of Technology Indore, is an authentic record of my own work carried out during the time period from July 2023 to May 2025 under the supervision of Dr.Dipankar Das and Affiliated with Indian Institute of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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21/05/25

Signature of the student with date
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This is to certify that the above statement made by the candidate is correct to the best of my/our knowledge.

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Puspendu Biswas has successfully given his M.Sc. Oral Examination held on 13th May 2025.

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Abstract

In this thesis, I have presented the so-called '331 model' which is based on $\mathrm{SU(3)_C} \times \mathrm{SU(3)_L} \times \mathrm{U(1)_X}$. Firstly, I have talked about some of the basics that are needed to study the '331 model'. Among the basics, I have presented symmetries in a Lagrangian. Then, I also provided an overview of spontaneous symmetry breaking and the Higgs mechanism, from which the gauge boson acquires their masses, which are the carriers of the weak interaction. Then, I delved into the 331 model and why it was introduced, where I also showed the transformation of the fermion fields. Then, I calculated the masses of the gauge bosons that come in this theory. This thesis also contains the Scalar Potential of this model, from which we land upon the 331-EFT, which relates 2HDM parameters to the 331 parameters. And last but not least, special attention has been given to the Yukawa sector for the 331 model, which also reduces to the Yukawa sector of the 2HDM in the 331-EFT case. From where I calculated the FCNC couplings for this model.

Contents

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Nomenclature

$[A, B]$	Commutator AB-BA
β	Parameter of 331 model
γ^μ	γ matrices
$\mathcal{L}$	Lagrangian density
$\mathcal{T}^a$	Generators in adjoint representation
$\mu, \nu, \ldots$	Lorentz indices
ϕ, η, ρ, χ	Complex Scalar field
ψ	Fermion field
θ^a, α, γ	Rotation angle
θ_W	Weinberg angle
$a, b, c, \ldots$	Gauge indices
$A_\mu, W_\mu, Z_\mu, V_\mu, U_\mu$	Gauge field
C	Charge Conjugation operator
D_μ	Covariant derivative
e, g, g'	Coupling constant
$F_{\mu\nu}$	Electromagnetic Field Tensor
f_{abc}	Structure constants of a Lie group
h, H^0, H'	Higgs field
$i, j, k, \ldots$	Flavour indices
$I_\pm, U_\pm, V_\pm$	Ladder operator
J^μ	Conserved current
P	Projection operator
Q	Charge
T^a	Generator SU(N) group
U, V	Unitary matrices
v	VEV

Chapter 1

Symmetries of the Lagrangian

Symmetries play a fundamental role in theoretical physics, particularly in the formulation and interpretation of physical laws. Symmetries of the Lagrangian correspond to transformations that leave the Lagrangian invariant. These symmetries have profound implications through Noether's theorem, which establishes a direct correspondence between continuous symmetries and conserved quantities. This chapter provides an overview of the concept of global symmetries, local or gauge symmetries and non-abelian gauge symmetries of the Lagrangian.

1.1 Global Symmetry

Let us consider the Lagrangian that satisfies the Klein-Gordon equation,

$$\mathcal{L}_{KG} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2(\phi^\dagger \phi)\,. \tag{1.1}$$

Now if ϕ changes from $\phi \to \phi'$. Where

$$\phi' = e^{-i\alpha}\phi\,, \tag{1.2}$$

and α did not depend on space-time. So $\phi^\dagger$ and $\partial_\mu \phi$ will be,

$$\begin{aligned} \phi^{\dagger\prime} &= \phi^\dagger e^{i\alpha} \\ \partial_\mu \phi' &= e^{-i\alpha}(\partial_\mu \phi)\,. \end{aligned} \tag{1.3}$$

So we can clearly see that the Lagrangian is invariant under the transformation Eq. (1.2) as the derivatives do not act on the $e^{-i\alpha}$ term. So this is a symmetry of the Lagrangian, and we call it 'Global symmetry' as it does not depend on the space-time.

Let us analyze the mass dimension of the field. The mass dimension of the Lagrangian is +4, and the mass dimension of ∂_μ is equal to +1. So the mass dimension of the field, $i.e.$, ϕ is also +1. In the second term of Eq. (1.1) we see that the constant term has a mass dimension of +2, so we can safely say that it is a mass square term.

Now, if we consider the symmetry first and then find out the general Lagrangian with mass dimension $+4$, then the Lagrangian would be,

$$\mathcal{L}_{KG} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2(\phi^\dagger \phi) + \lambda(\phi^\dagger \phi)^2\,, \tag{1.4}$$

where the mass dimension of λ is 0, $i.e.$ dimensionless, and it is the interaction strength between fields. So we see, if we consider the symmetry first, then find the Lagrangian, we automatically find the interaction terms between fields. As Eq. (1.2) is a symmetry transformation, according to Noether's theorem it is associated with a conserved current. The infinitesimal transformation is,

$$\begin{aligned}\delta\phi(x) &= \phi'(x) - \phi(x) = -i\alpha\phi(x) \\ \delta\phi^\dagger(x) &= \phi'^\dagger(x) - \phi^\dagger(x) = i\alpha\phi^\dagger(x)\,,\end{aligned} \tag{1.5}$$

from Eq. (A.10), we find the conserved current to be

$$\begin{aligned}J^\mu_\alpha &= \delta\phi\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} + \delta\phi^\dagger\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\dagger} \\ &= -i\alpha\phi(x)\partial^\mu\phi^\dagger(x) + i\alpha\phi^\dagger(x)\partial^\mu\phi(x) \\ &= i\alpha\left(\phi^\dagger(x)\partial^\mu\phi(x) - \phi(x)\partial^\mu\phi^\dagger(x)\right)\,,\end{aligned} \tag{1.6}$$

The parameter independent current would then be,

$$J^\mu = i\left(\phi^\dagger(x)\partial^\mu\phi(x) - \phi(x)\partial^\mu\phi^\dagger(x)\right)\,, \tag{1.7}$$

and the conserved charge is going to be,

$$\begin{aligned}Q &= \int d^3x J^0(x) \\ &= i\int d^3x\left(\phi^\dagger(x)\dot{\phi}(x) - \dot{\phi}^\dagger(x)\phi(x)\right)\,.\end{aligned} \tag{1.8}$$

These types of symmetries which are characterized by the space-time independent parameters α either in Eq. (1.2) or in Eq. (1.5), are called global symmetries [4]. The field $\phi(x)$ transforms exactly the same way for all space-time points.

1.2 Gauge Symmetry

We now consider theories where the symmetry transformations are space-time dependent, $i.e.$ $\alpha = \alpha(x)$, where $x = (\vec{x}, t)$. They are called **local symmetries** or **gauge symmetries**. We will derive the gauge theory by requiring the Dirac free electron theory to be gauge invariant. Let us consider the Lagrangian for the free electron field $\psi(x)$,

$$\mathcal{L}_D = \overline{\psi}(i\partial\!\!\!/ - m)\psi\,, \tag{1.9}$$

where $\partial\!\!\!/ = \gamma^\mu\partial_\mu$. Now, under the local phase transformation, the field transforms as

$$\psi' = e^{-ie\theta(x)}\psi\,, \tag{1.10}$$

where we consider $\alpha(x) = e\theta(x)$, 'e' is a constant that will be later identified as the electric charge. Then $\overline{\psi}$ will transform as,

$$\overline{\psi}' = \overline{\psi}e^{ie\theta(x)}\,, \tag{1.11}$$

and the derivative will transform as,

$$\begin{aligned}\partial_\mu\psi \to \partial_\mu\psi' &= \partial_\mu\left(e^{-ie\theta(x)}\psi\right) \\ &= (\partial_\mu\psi)e^{-ie\theta(x)} + \partial_\mu(e^{-ie\theta(x)})\psi \\ &= e^{-ie\theta(x)}\left[\partial_\mu - ie\partial_\mu\theta(x)\right]\psi\,.\end{aligned} \tag{1.12}$$

We can clearly see that the derivative does not transform as ψ does. So

$$\begin{aligned}\overline{\psi}'\partial_\mu\psi' &= \overline{\psi}e^{ie\theta(x)}e^{-ie\theta(x)}\left[\partial_\mu - ie\partial_\mu\theta(x)\right]\psi \\ &= \overline{\psi}\left[\partial_\mu - ie\partial_\mu\theta(x)\right]\psi \\ &= \overline{\psi}\partial_\mu\psi - ie\overline{\psi}\partial_\mu\theta(x)\psi\,,\end{aligned} \tag{1.13}$$

due to the second term in Eq. (1.13), the Lagrangian is not invariant under this local transformation. To make it invariant we need to form a *gauge-covariant derivative* D_μ , to replace ∂_μ, which transform like ψ does, $i.e.$,

$$D_\mu\psi \rightarrow [D_\mu\psi]' = e^{-ie\theta(x)}[D_\mu\psi]\,, \tag{1.14}$$

so that the combination $\overline{\psi}D_\mu\psi$ is gauge invariant. we replace ∂_μ to D_μ by adding a new field A_μ, The **Gauge field**, to the derivative ∂_μ, so the form of the covariant derivative would be [10],

$$\boxed{D_\mu = \partial_\mu + ieA_\mu} \tag{1.15}$$

Let us see the transformation law for the covariant derivative,

$$\begin{aligned}D_\mu\psi \rightarrow D'_\mu\psi' &= (\partial_\mu + ieA'_\mu)\psi' \\ &= (\partial_\mu + ieA'_\mu)e^{-ie\theta(x)}\psi \\ &= \left(\partial_\mu\psi - ie(\partial_\mu\theta)\psi + ieA'_\mu\psi\right)e^{-ie\theta(x)} \\ &= e^{-ie\theta(x)}\left(\partial_\mu - ie(\partial_\mu\theta) + ieA'_\mu\right)\psi\end{aligned} \tag{1.16}$$

Now, our theory to be invariant under the local transformation as mentioned in Eq. (1.10), has to satisfy Eq. (1.14). So on comparing Eq. (1.14) and Eq. (1.16), we have,

$$\begin{aligned}ieA'_\mu - ie(\partial_\mu\theta) &= ieA_\mu \\ \Rightarrow \boxed{A'_\mu = A_\mu + \partial_\mu\theta(x)}\,.\end{aligned} \tag{1.17}$$

So A_μ should transform as Eq. (1.17) to make the covariant derivative transform as ψ, $i.e.$ to make the Lagrangian invariant. We can identify it as the **Lorenz Gauge** condition from electromagnetism. From Eq. (1.9) we now have,

$$\begin{aligned}\mathcal{L}_D &= i\overline{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\overline{\psi}\psi \\ \mathcal{L}_D &= \overline{\psi}(i\not{D} - m)\psi\end{aligned} \tag{1.18}$$

To make the gauge field a true dynamical variable, we need to add a kinetic term (which describes the equation of motion of the A_μ) to the Lagrangian involving its derivatives. For that, let's first analyze the mass dimension of A_μ. The mass dimension of both ψ and $\overline{\psi}$ is $\frac{3}{2}$, so A_μ must have a mass dimension of 1 and 'e' is a dimensionless quantity. The commutation relation between two covariant derivatives is,

$$\begin{aligned}&[D_\mu, D_\nu]\,\psi \\ &= (D_\mu D_\nu - D_\nu D_\mu)\psi \\ &= (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu)\psi - (\partial_\nu + ieA_\nu)(\partial_\mu + ieA_\mu)\psi \\ &= ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi \\ &= ieF_{\mu\nu}\psi\,,\end{aligned} \tag{1.19}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and we can identify it as the electromagnetic stress-energy tensor. This can be a suitable candidate for the kinetic term for the A_μ field because it contains terms with derivatives of A_μ. But in order to set it as a kinetic term, we need to check whether it is a gauge-invariant quantity or not. Let us check how $F_{\mu\nu}$ transform if A_μ transform as Eq. (1.17),

$$\begin{aligned} F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu \\ \Rightarrow F'_{\mu\nu} &= \partial_\mu(A_\nu + \partial_\nu\theta(x)) - \partial_\nu(A_\mu + \partial_\mu\theta(x)) \\ \Rightarrow F'_{\mu\nu} &= \partial_\mu A_\nu + \partial_\mu\partial_\nu\theta(x) - \partial_\nu A_\mu - \partial_\nu\partial_\mu\theta(x) \\ \Rightarrow F'_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ \Rightarrow F'_{\mu\nu} &= F_{\mu\nu}\,, \end{aligned} \tag{1.20}$$

so $F_{\mu\nu}$ is invariant under local gauge transformation. Now, the simplest kinetic term that is both gauge-invariant and Lorentz-invariant with mass dimension four (with a conventional normalization) is,

$$\mathcal{L}_k = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \tag{1.21}$$

So the total Lagrangian, invariant under local gauge transformation, would be,

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - e\overline{\psi}\gamma^\mu\psi A_\mu - m\overline{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{1.22}$$

where the second term in the above equation is the interaction term between the electron field and the gauge field. and we also identify that $e\overline{\psi}\gamma^\mu\psi$ to be the Noether's current density. The zeroth component of which gives us the total charge, so

$$\begin{aligned} Q &= \int J^0 d^3x \\ \Rightarrow Q &= e\int \overline{\psi}\gamma^0\psi d^3x = e\int \psi^\dagger\psi d^3x \\ \Rightarrow Q &= e\,, \end{aligned} \tag{1.23}$$

where we take $\int \psi^\dagger\psi d^3x$ to be normalized, $i.e.$ equal to 1. Now we can identify e as the total electric charge.

1.3 Non-Abelian Gauge Theory

In the previous two sections, we have shown the simplest of the gauge theories, namely, the Abelian gauge group $\mathrm{U}(1)$. We will now show the gauge theories based on more complicated symmetries such as $\mathrm{SU(N)}$. Such theories belong to non-Abelian (non-commutative) gauge groups, which are commonly known as **Yang-Mills** theory, and they are the fundamental building blocks in the construction of physical theories.

Let us now generalize the previous cases by considering a free Dirac theory described by the Lagrangian density (containing several complex fields),

$$\mathcal{L}_D = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi\,, \tag{1.24}$$

Where $\Psi = [\psi_1, \psi_2, \ldots, \psi_n]^T$ and the ψ_n are different fermionic fields. Taking an analogy from the previous case, we want our theory to be invariant under the transformation,

$$\Psi \to \Psi' = U\Psi\,, \tag{1.25}$$

Where,

$$U = U(x) = e^{-igT^a\theta^a(x)} \,. \tag{1.26}$$

Where T^a's are the generators of the group SU(N),θ^a's are the parameters of rotations and g is the gauge coupling constant analogous to the factor e appearing in Eq. (1.10) and an implicit sum is considered over 'a'. There are $N^2 - 1$ no. of generators for an SU(N) group. That's why there are $N^2 - 1$ no. of parameters, in other words, $a = 1, 2, \ldots, N^2 - 1$. The generators of the group satisfy an algebra, called a Lie algebra, which is given as follows

$$[T^a, T^b] = if^{abc}T^c \,. \tag{1.27}$$

Where f^{abc}'s are the structure constants of the SU(N) algebra. One of the properties that these generators follow is given as

$$\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab} \,. \tag{1.28}$$

Now $\overline{\Psi}$ will transform as,

$$\overline{\Psi} \rightarrow \overline{\Psi'} = \overline{\Psi}U^\dagger \,, \tag{1.29}$$

as long as U is global, Eq. (1.24) is invariant under that transformation because θ^a's do not depend on space-time coordinate. But when we consider the case written in Eq. (1.26), from Eq. (1.12) we see that when the parameters depend on the space-time coordinate the derivative does not transform as ψ does and there is no term in the Lagrangian to cancel the extra term arises in Eq. (1.12). So to make the Lagrangian invariant, $i.e.$, to cancel the extra term, we introduce a new field and replace the normal derivative with the covariant derivative. Taking analogies from the previous case, we will define the covariant derivative as,

$$D_\mu = \partial_\mu + igT^aA^a_\mu \,. \tag{1.30}$$

Notice, however, that A^a_μ has a gauge index 'a', which indicates that there are N^2-1 number of A_μ. The transformation for D_μ will be,

$$\begin{aligned} D_\mu\Psi \rightarrow D'_\mu\Psi' &= (\partial_\mu + igT^aA'^{\,a}_\mu)\Psi' \\ &= (\partial_\mu + igT^aA'^{\,a}_\mu)U\Psi \\ &= (\partial_\mu U)\Psi + U(\partial_\mu\Psi) + UigT^aA'^{\,a}_\mu\Psi \,. \end{aligned} \tag{1.31}$$

Now the Lagrangian to be invariant under the local gauge transformation, $D_\mu\Psi$ should transform as mentioned in Eq. (1.14), so we have,

$$\begin{aligned} &(\partial_\mu U)\Psi + U(\partial_\mu\Psi) + igT^aA'^{\,a}_\mu U\Psi = U(\partial_\mu\Psi) + igUT^aA^a_\mu\Psi \\ \Rightarrow &(\partial_\mu U) + igT^aA^{a\prime}_\mu U = igUT^aA^a_\mu \\ \Rightarrow &T^aA^{a\prime}_\mu U = -\frac{1}{ig}(\partial_\mu U) + UT^aA^a_\mu \\ \Rightarrow &\boxed{T^aA^{a\prime}_\mu = \frac{i}{g}(\partial_\mu U)U^{-1} + UT^aA^a_\mu U^{-1}} \,. \end{aligned} \tag{1.32}$$

So the gauge fields should transform like Eq. (1.32) so that the extra term that arises due to the derivative term, cancels out each other to make the Lagrangian covariant. Now let us see how A^a_μ transforms infinitesimally. For an infinitesimal transformation, we have,

$$U(x) \equiv e^{-igT^a\theta^a(x)} \approx 1 - igT^a\theta^a(x) \,, \tag{1.33}$$

and from Eq. (1.32) we have (considering the term linear in $\theta^a(x)$),

$$\begin{aligned}
T^a A'^{\,a}_\mu &= \frac{i}{g}\left\{\partial_\mu(1 - igT^b\theta^b)\right\}\left\{1 + igT^b\theta^b\right\} + \left(1 - igT^b\theta^b\right)T^a A^a_\mu\left(1 + igT^b\theta^b\right) \\
&= (T^b\partial_\mu\theta^b)(1 + igT^b\theta^b) + T^a A^a_\mu - igT^bT^aA^a_\mu\theta^b + igT^aT^bA^a_\mu\theta^b \\
&= T^a\partial_\mu\theta^a + T^aA^a_\mu + ig\left[T^a, T^b\right]A^a_\mu\theta^b \\
&= T^a\partial_\mu\theta^a + T^aA^a_\mu - gf^{abc}T^cA^a_\mu\theta^b \\
&= T^a\partial_\mu\theta^a + T^aA^a_\mu - gf^{cba}T^aA^c_\mu\theta^b \\
&= T^a\partial_\mu\theta^a + T^aA^a_\mu + gf^{abc}T^aA^c_\mu\theta^b \\
&\Rightarrow \boxed{A'^{\,a}_\mu = A^a_\mu + \partial_\mu\theta^a + gf^{abc}A^c_\mu\theta^b}\,. \qquad (1.34)
\end{aligned}$$

In the above transformation equation for the gauge fields, the term $\partial_\mu\theta^a$ resembles the $\mathrm{U}(1)$ gauge transformation. But there is another extra term that could not have appeared in a $\mathrm{U}(1)$ theory because, in an Abelian theory, the structure constants are zero[2].

Now back to the global transformation, for global transformation, $\partial_\mu\theta^a = 0$, but the second extra term does not vanish, so from Eq. (1.34) we have,

$$\begin{aligned}
A'^{\,a}_\mu &= A^a_\mu + ig(-if^{abc})A^c_\mu\theta^b \\
&= A^a_\mu + ig\left[\mathcal{T}^b\right]_{ac}A^c_\mu\theta^b\,, \qquad (1.35)
\end{aligned}$$

where $\mathcal{T}^b$ denotes the generators in the adjoint representation. We conclude that the gauge fields transform in the adjoint representation of the gauge group.

From Eq. (1.17), we see that for global transformation, A_μ transforms trivially, $i.e.$ does not transform under the gauge group. So, from Noether's theorem, the field does not possess the charge of the mother group. That is why the photon field is chargeless. But for the non-Abelian gauge group, they transform non-trivially under the gauge group. Which means that they carry charges corresponding to the gauge currents. For example, $\mathrm{SU}(3)_\mathrm{C}$ and the charge associated with this group is color charges, and the gauge field corresponding to $\mathrm{SU}(3)_\mathrm{C}$, which we called the gluon fields, also possesses color charges.

Let us calculate the commutation between two covariant derivatives,

$$\begin{aligned}
&[D_\mu, D_\nu]\,\Psi \\
&= (D_\mu D_\nu - D_\nu D_\mu)\Psi \\
&= (\partial_\mu + igT^aA^a_\mu)(\partial_\nu + igT^bA^b_\nu)\Psi - (\partial_\nu + igT^bA^b_\nu)(\partial_\mu + igT^aA^a_\mu)\Psi \\
&= \left(igT^b(\partial_\mu A^b_\nu) - igT^a(\partial_\nu A^a_\mu) - g^2\left[T^a, T^b\right]A^a_\mu A^b_\nu\right)\Psi \\
&= \left(igT^a(\partial_\mu A^a_\nu) - igT^a(\partial_\nu A^a_\mu) - ig^2f^{abc}T^cA^a_\mu A^b_\nu\right)\Psi \\
&= igT^a\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^c_\mu A^b_\nu\right)\Psi \\
&= igT^a\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu\right)\Psi = igT^aF^a_{\mu\nu}\Psi\,, \qquad (1.36)
\end{aligned}$$

where we take

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu\,. \qquad (1.37)$$

As $D_\mu\Psi$ transforms like Ψ, we can show that $D_\mu D_\nu\Psi$ also transforms like Ψ, using this fact, we have,

$$\begin{aligned}
&[D_\mu, D_\nu]'\,\Psi' = U\,[D_\mu, D_\nu]\,\Psi \\
&\Rightarrow igT^aF'^{\,a}_{\mu\nu}U\Psi = igUT^aF^a_{\mu\nu}\Psi \\
&\Rightarrow \boxed{T^aF'^{\,a}_{\mu\nu} = UT^aF^a_{\mu\nu}U^{-1}}\,. \qquad (1.38)
\end{aligned}$$

Now, to construct a kinetic term for the gauge fields, we need a quantity that is both Lorentz and gauge invariant. Let's take a Lorentz invariant quantity, $T^b F^{b,\mu\nu} T^a F^a_{\mu\nu}$, now let us check whether this is a gauge invariant quantity or not. we have

$$\begin{aligned} T^b F'^{b,\mu\nu} T^a F'^a_{\mu\nu} &= (UT^b F^{b,\mu\nu} U^{-1})(UT^a F^a_{\mu\nu} U^{-1}) \\ &= UT^b F'^{b,\mu\nu} T^a F'^a_{\mu\nu} U^{-1} \,. \end{aligned} \tag{1.39}$$

So $T^b F^{b,\mu\nu} T^a F^a_{\mu\nu}$, is not a gauge invariant quantity but[2],

$$\begin{aligned} \mathrm{Tr}\left(T^b F'^{b,\mu\nu} T^a F'^a_{\mu\nu}\right) &= \mathrm{Tr}\left(UT^b F^{b,\mu\nu} T^a F^a_{\mu\nu} U^{-1}\right) \\ &= \mathrm{Tr}\left(U^{-1} UT^b F^{b,\mu\nu} T^a F^a_{\mu\nu}\right) \\ &= \mathrm{Tr}\left(T^b F^{b,\mu\nu} T^a F^a_{\mu\nu}\right) \\ &= \mathrm{Tr}\left(T^a T^b\right) F^{b,\mu\nu} F^a_{\mu\nu} \\ &= \frac{1}{2}\delta^{ab} F^{b,\mu\nu} F^a_{\mu\nu} \\ &= \frac{1}{2} F^{a,\mu\nu} F^a_{\mu\nu} \,. \end{aligned} \tag{1.40}$$

Where in the intermediate step we use the cyclic property of the trace and Eq. (1.28). From Eq. (1.37) we see that $F_{\mu\nu}$ contain an extra term other than just momentum term without any derivative, so when we multiply two such $F^{a,\mu\nu}$, we get a term with three fields and a term with four fields. From those terms, we conclude that the gauge fields self-interact. We finally arrive at the pure Yang-Mills Lagrangian, $i.e.$, the kinetic term for the A^a_μ fields.

$$\mathcal{L}_{YM} = -\frac{1}{4} F^{a,\mu\nu} F^a_{\mu\nu} \,, \tag{1.41}$$

and the total Lagrangian will be,

$$\mathcal{L} = \overline{\psi}(i\not{D} - m)\psi - \frac{1}{4} F^{a,\mu\nu} F^a_{\mu\nu} \,. \tag{1.42}$$

Chapter 2

Symmetry Breaking and Higgs mechanism

Symmetry breaking in physics can occur in two distinct ways: explicit and spontaneous. Explicit symmetry breaking occurs when the equations or Lagrangian of a system contain terms that directly violate a particular symmetry. This breaking is imposed by external parameters or interactions.

In contrast, spontaneous symmetry breaking (SSB) arises when the system remains symmetric, but the ground state does not reflect this symmetry. In other words, the symmetry is not explicitly broken by the extra terms, but rather by the choice of a particular state. SSB is central to many physical phenomena, including the Higgs mechanism in particle physics. A detailed discussion on the spontaneous symmetry breaking of various symmetries has been presented in this chapter.

2.1 Spontaneous Symmetry Breaking of Global U(1) Gauge Symmetry

Consider a complex scalar field $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, where ϕ_1 and ϕ_2 are two real fields. The Lagrangian describing the complex field is given by,

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - \mu^2(\phi^*\phi) - \lambda(\phi^*\phi)^2 \,, \tag{2.1}$$

which is invariant under the transformation $\phi \to e^{i\alpha}\phi$. That is, $\mathcal{L}$ possesses a U(1) global gauge symmetry. For the case when $\lambda > 0$ and $\mu^2 < 0$. we rewrite Eq. (2.1) as,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \lambda\left(\phi_1^2 + \phi_2^2\right)^2 \,. \tag{2.2}$$

Now to get the minima of the potential $V(\phi)$ we have to differentiate the potential with respect to ϕ_1 and ϕ_2 both. After differentiating we have a circle of minima in the ϕ_1, ϕ_2 plane of radius v such that,

$$\phi_1^2 + \phi_2^2 = v^2; \qquad v^2 = -\frac{\mu^2}{\lambda} \,. \tag{2.3}$$

The extremum $\phi = 0$ does not correspond to the minimum energy state. So we need to translate ϕ to one of the minima points, which without loss of generality we can take as the point $\phi_1 = v$ and $\phi_2 = 0$. we expand $\mathcal{L}$ about this vacuum in term of fields η, ξ by substituting,

$$\phi(x) = \frac{1}{\sqrt{2}}\left[v + \eta(x) + i\xi(x)\right] , \tag{2.4}$$

into Eq. (2.1), we obtain,

$$\mathcal{L}' = \frac{1}{2}(\partial_\mu\eta)^2 + \frac{1}{2}(\partial_\mu\xi)^2 + \mu^2\eta^2 + \text{constant} + \text{cubic and quartic terms in } \eta, \xi\,. \tag{2.5}$$

The third term has the form of a mass term $\left(-\frac{1}{2}m_\eta^2\eta^2\right)$ for the field η. Thus the mass of the η-field is $m_\eta = \sqrt{-2\mu^2}$, where μ^2 itself is negative so m_η is positive. The first term in the $\mathcal{L}'$ represents the kinetic term for the η-fields, and the second term corresponds to the kinetic term for the ξ-fields, but there is no mass term for ξ. Therefore, the theory also contains a massless scalar, which is known as a **Goldstone boson**.

The Lagrangian in Eq. (2.5) is a simple example of the Goldstone theorem, which states that "**whenever a continuous symmetry of a physical system is spontaneously broken, there occurs a massless scalar**".

2.2 Spontaneous Symmetry Breaking of local U(1) Gauge Symmetry

Again, consider the complex scalar fields ϕ but now the Lagrangian is invariant under a local U(1) gauge symmetry, $\phi \to e^{i\alpha(x)}\phi$. So the gauge invariant Lagrangian is thus,

$$\mathcal{L} = (\partial^\mu + ieA^\mu)\phi^*(\partial_\mu - ieA_\mu)\phi - \mu^2(\phi^*\phi) - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \tag{2.6}$$

where A_μ is the gauge field and the term is the kinetic term for the gauge fields. Now same as the previous case if $\mu^2 < 0$, the Lagrangian would be after symmetry breaking ($i.e.$ the field after acquiring a VEV(vacuum expectation value) or in other words shifting the field to one of the minima), $i.e.$ on substituting Eq. (2.4) in Eq. (2.6), we have,

$$\mathcal{L}' = \frac{1}{2}(\partial_\mu\eta)^2 + \frac{1}{2}(\partial_\mu\xi)^2 + \mu^2\eta^2 - \frac{1}{2}e^2v^2A_\mu A^\mu + evA_\mu\partial^\mu\xi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \text{interaction terms}\,. \tag{2.7}$$

The particle spectrum of $\mathcal{L}'$ appears to be a massive scalar η, a massless Goldstone boson ξ, and more importantly a massive vector gauge boson A_μ, so we have,

$$m_\eta = \sqrt{-2\mu^2} = \sqrt{2\lambda v^2}, \quad m_\xi = 0, \quad m_A = ev\,. \tag{2.8}$$

We dynamically generate mass for the gauge field. But the presence of the off-diagonal term in the field $A_\mu\partial^\mu\xi$ means that we are mistaken in interpreting the Lagrangian. Indeed, by giving mass to A_μ, we have clearly raised polarization degrees of freedom by 2 to 3. Because it can now have a longitudinal polarization, but simply translating the field variables, as in Eq. (2.4), does not create a new degree of freedom. So let's expand the field in spherical polar coordinates instead of Cartesian coordinates as

$$\phi = \frac{1}{\sqrt{2}}\left(v + h(x)\right)e^{i\theta(x)/v}\,, \tag{2.9}$$

and make a gauge transformation of the gauge field as [4]

$$A_\mu \to B_\mu = A_\mu + \frac{1}{ev}\partial_\mu \theta\,. \tag{2.10}$$

On substituting Eq. (2.9) and Eq. (2.10) into the original Lagrangian of Eq. (2.6), we obtain,

$$\mathcal{L}'' = \frac{1}{2}(\partial_\mu h)^2 + \mu^2 h^2 + \frac{1}{2}e^2 v^2 B_\mu B^\mu - \lambda v h^3 - \frac{1}{4}\lambda h^4 + \frac{1}{2}e^2 B_\mu^2 h^2 + e^2 v B_\mu^2 h - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \tag{2.11}$$

From this Lagrangian, we see that the Goldstone boson does not appear in the theory. The Lagrangian just describes two interacting massive particles, a vector boson B_μ and a massive scalar field h, which is called a Higgs boson. The massless Goldstone boson has been turned into the needed longitudinal polarization of the massive gauge particle. This is called the "**Higgs Mechanism**".[1]

2.3 Spontaneous Symmetry Breaking of local SU(2) Gauge Symmetry

Let us repeat the same procedure for the SU(2) gauge symmetry. Here ϕ is a SU(2) doublet of complex scalar fields,

$$\phi = \begin{pmatrix} \phi_\alpha \\ \phi_\beta \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}\,. \tag{2.12}$$

Now, the Lagrangian is invariant under the transformation $\phi \to e^{i\alpha_a \tau_a}\phi$. Where τ_a's are the generators for the SU(2) algebra, α_a's are the rotation parameters, and there is an implicit sum over 'a'. The gauge invariant Lagrangian is then,

$$\mathcal{L} = (\partial^\mu \phi + ig\frac{\tau}{2}.W^\mu\phi)^\dagger(\partial_\mu\phi + ig\frac{\tau}{2}.W_\mu\phi) - \mu^2(\phi^\dagger\phi) - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}W_{\mu\nu}W^{\mu\nu}\,, \tag{2.13}$$

where W_μ's are the gauge fields, and it transforms as $W_\mu \to W_\mu - \frac{1}{g}\partial_\mu\alpha - \overrightarrow{\alpha}\times\overrightarrow{W_\mu}$.

As before we are interested in the case where $\mu^2 < 0$, in this case, the potential has its minimum at a finite value of '$\mid \phi \mid$' where the manifolds of minima are given by

$$\frac{1}{2}\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right) = -\frac{\mu^2}{2\lambda}\,. \tag{2.14}$$

Let us choose one of the minima for the doublet,

$$\phi_1 = \phi_2 = \phi_4 = 0, \quad \text{and} \quad \phi_3^2 = -\frac{\mu^2}{\lambda} \equiv v^2\,. \tag{2.15}$$

This effect is equivalent to the spontaneous breaking of the SU(2) symmetry; in other words, the symmetry in Eq. (2.14) has become hidden. Then ϕ becomes

$$\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} e^{i\tau_a\theta_a/v}\,. \tag{2.16}$$

And if we expand the expression in the Eq. (2.16) in lowest order, we have[1]

$$\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} \theta_2 + i\theta_1 \\ v + h - i\theta_3 \end{pmatrix}\,. \tag{2.17}$$

We see that the four fields are indeed independent and fully parameterized the deviations from the vacuum mentioned in Eq. (2.15), $i.e.$, $\phi_0 = \frac{1}{\sqrt{2}}(0 \quad v)^{\mathrm{T}}$. Now, as the Lagrangian is locally SU(2) invariant, we can gauge the three (massless goldstone boson) fields $\theta(x)$ of Eq. (2.16). The Lagrangian then contains no trace of the $\theta(x)$'s.

To determine the masses generated for the gauge bosons W_μ^a's, it is sufficient to substitute ϕ_0 into the Lagrangian. The relevant term for the masses is

$$\begin{aligned} \left| ig\frac{1}{2}\vec{\tau}.\vec{W}_\mu \phi \right|^2 &= \frac{g^2}{8} \left| \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2 \\ &= \frac{g^2 v^2}{8} \left[\left(W_\mu^1\right)^2 + \left(W_\mu^2\right)^2 + \left(W_\mu^3\right)^2 \right] . \end{aligned} \tag{2.18}$$

Now, if we compare these terms with the mass term of a gauge boson, $\frac{1}{2}M^2 W_\mu^2$, we find $M_W = \frac{1}{2}gv$. So the Lagrangian describes three gauge fields and one massive scalar Higgs h. We interpret it as the gauge fields have "eaten up" the Goldstone bosons and become massive. The scalar degrees of freedom become the longitudinal polarizations of the massive vector bosons.[1][4]

2.4 Electroweak Symmetry Breaking

For Electroweak theory, the gauge group is $\mathrm{SU(2)_L} \times \mathrm{U(1)_Y}$. The gauge invariant Lagrangian under this group is given by [1]

$$\begin{aligned} \mathcal{L}_1 = \bar{\psi}_L \gamma^\mu \left[i\partial_\mu - g\frac{1}{2}\vec{\tau}.\vec{W}_\mu - g'\frac{Y}{2}B_\mu \right] \psi_L + \bar{\psi}_R \gamma^\mu \left[i\partial_\mu - g'\frac{Y}{2}B_\mu \right] \psi_R \\ - \frac{1}{4}\vec{W}_{\mu\nu}\vec{W}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} , \end{aligned} \tag{2.19}$$

Where ψ_L is the left-handed fields, ψ_R,s are the right-handed fields and the operator $\frac{1}{2}\vec{\tau}$ and Y are the generators of the $\mathrm{SU(2)_L}$ and $\mathrm{U(1)_Y}$ groups of the gauge transformations, respectively.

In Electroweak theory, we want to formulate the Higgs mechanism so that the $W^\pm$ and Z^0 become massive and the photon remains massless. To do this, we introduce a scalar doublet which consists of four real scalar fields ϕ_i. we have to add to $\mathcal{L}_1$ an $\mathrm{SU(2)_L} \times \mathrm{U(1)_Y}$ gauge invariant Lagrangian for the scalar fields

$$\mathcal{L}_2 = \left| \left(i\partial_\mu - gT^a W_\mu^a - g'\frac{Y}{2}B_\mu \right) \phi \right|^2 - V(\phi) , \tag{2.20}$$

where $|\ \ |^2 \equiv (\ \ )^\dagger(\ \ )$. To keep $\mathcal{L}_2$ gauge invariant, the ϕ_i must belong to $\mathrm{SU(2)_L} \times \mathrm{U(1)_Y}$ multiplets. The simplest choice is to arrange four fields in an isospin doublet with weak hypercharge $Y = 1$:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} . \tag{2.21}$$

To generate the boson masses, we will do the same procedure as in the previous section. Due to the charge conservation after spontaneous symmetry breaking, only ϕ^0 will acquire the vacuum expectation value,

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} . \tag{2.22}$$

Then the masses of the gauge boson will come from $\mathcal{L}_2$, and is given by putting $Y = 1$ and $T^a = \frac{\sigma^a}{2}$; (σ^a are the Pauli matrices),

$$\begin{aligned}
&\left|\left(-g\frac{\sigma^a}{2}W^a_\mu - g'\frac{1}{2}B_\mu\right)\phi\right|^2 \\
&= \frac{1}{8}\left|\begin{pmatrix} gW^3_\mu + g'B_\mu & g(W^1_\mu - iW^2_\mu) \\ g(W^1_\mu + iW^2_\mu) & -gW^3_\mu + g'B_\mu \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix}\right|^2 \\
&= \frac{1}{8}v^2g^2\left[(W^1_\mu)^2 + (W^2_\mu)^2\right] + \frac{1}{8}v^2(-gW^3_\mu + g'B_\mu)(-gW^{3\mu} + g'B^\mu) \\
&= \left(\frac{1}{2}vg\right)^2 W^+_\mu W^{\mu-} + \frac{1}{8}v^2(W^3_\mu \quad B_\mu)\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}\begin{pmatrix} W^{3\mu} \\ B^\mu \end{pmatrix}, \qquad (2.23)
\end{aligned}$$

since $W^\pm = (W^1_\mu \mp iW^2_\mu)/\sqrt{2}$. Comparing the first term in the above equation, we get the mass of the $W^\pm$ bosons, which are

$$\boxed{M_W = \frac{1}{2}vg}\,. \qquad (2.24)$$

In the second term, the matrix has non-diagonal elements. So the W^3_μ and B_μ are non-physical fields. To get the physical fields, we need to diagonalise the matrix. After diagonalization, the physical fields and their masses would be,

$$A_\mu = \frac{g'W^3_\mu + gB_\mu}{\sqrt{g^2 + g'^2}} \quad \text{with } M_A = 0 \qquad (2.25)$$

$$Z_\mu = \frac{gW^3_\mu - g'B_\mu}{\sqrt{g^2 + g'^2}} \quad \text{with } M_Z = \frac{1}{2}v\sqrt{g^2 + g'^2}\,. \qquad (2.26)$$

So we see that the W^3_μ and B_μ fields mix up to create the physical fields A_μ and Z_μ. And the mixing angle θ_W (θ_W is called Weinberg or weak mixing angle) is given by,

$$\frac{g'}{g} = \tan\theta_{\rm W} \qquad (2.27)$$

We also infer from Eqs. (2.26) and (2.24),

$$\frac{M_W}{M_Z} = \cos\theta_{\rm W}\,. \qquad (2.28)$$

On comparing with QED, we have a relation with the electric charge as

$$\boxed{e = g\sin\theta_W = g'\cos\theta_W} \qquad (2.29)$$

Chapter 3

331 Model

3.1 Introduction

These models were initially introduced by Steven Weinberg in 1977 [7]. Models based on the $\mathrm{SU}(2) \times \mathrm{U}(1)$ gauge group of the SM stand out for the natural way in which they account for observed general features of the weak interactions. The presence of neutral currents strongly indicates that any viable model must incorporate the $\mathrm{SU}(2) \times \mathrm{U}(1)$ gauge group. At that time, the new observations of high-energy trimuons required that the $\mathrm{SU}(2) \times \mathrm{U}(1)$ must be embedded in a larger gauge group. The SM can be extended in various ways, with the simplest approach involving the introduction of new particles organized into irreducible representations of the SM gauge group. This method is primarily driven by phenomenology, where certain new phenomena, such as dark matter, neutrino masses, etc., are explained through the interactions of these new particles.

Another approach involves modifying the SM gauge group. Models in this category typically replace the SM gauge group with a single, larger group in an effort to unify the SM gauge couplings. Alternatively, some models only modify one of the SM gauge subgroups or introduce additional subgroups. The 331 model falls within this latter category, replacing the SM gauge group with $\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(3)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{X}$. To reproduce the particles and interactions of the SM, a series of symmetry breaking mechanisms are proposed. These models are interesting because they provide a partial answer to the problem of the origin of the three generations. These models are free of anomalies only if the number of generations is three or a multiple of three. These models include particles that also appear in other extensions of the SM, for instance, several scalar fields in doublets and triplets of $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$, new neutral and charged vector fields, and new vector-like fermions (in some cases with an exotic electric charge) that are singlets of the SM symmetries. These models are defined not only by the gauge symmetries but also by their representation content. [12]

We introduced a parameter β to define its electric charge operator[1], and the electric charge operator of these models are given by[12]

$$Q_\beta(X) = \mathrm{T}_3 + \beta \mathrm{T}_8 + \mathrm{XI}\,, \tag{3.1}$$

where T^3 and T^8 are the Generators of the $\mathrm{SU}(3)$ group, which are proportional to the Gell-

[1]Later we will see that β only takes discrete values.

Mann matrices. The matrix form of the Operator is given by,

$$Q_\beta(X) = \begin{pmatrix} \frac{1}{2} + \frac{\beta}{2\sqrt{3}} + X & 0 & 0 \\ 0 & -\frac{1}{2} + \frac{\beta}{2\sqrt{3}} + X & 0 \\ 0 & 0 & -\frac{\beta}{\sqrt{3}} + X \end{pmatrix} . \tag{3.2}$$

We have some freedom in choosing the fermion fields that will transform as triplets or antitriplets under the 3-3-1 gauge group. In this study, we choose the Leptons and third quark family to transform as a triplet under $\mathrm{SU}(3)$. The transformation of the Left-handed Lepton fields is shown below [12][3]

Leptons:

$$L_i = \begin{pmatrix} \nu_i \\ e_i \\ E_i \end{pmatrix}_{\mathrm{L}} \sim \left(1, 3, -\frac{1}{2} - \frac{\beta}{2\sqrt{3}}\right) , \tag{3.3}$$

where 'i' runs from 1 to 3. The 1st term in the parenthesis, $i.e.$ '1', denotes that it is a singlet under $\mathrm{SU}(3)_{\mathrm{C}}$, the 2nd term denotes it transforms as a triplet under $\mathrm{SU}(3)_{\mathrm{L}}$, and the last term is the 'X' charges of the triplet. Similarly, the quark families can be written as

Quarks:

$$q_1 = \begin{pmatrix} n_1 \\ -p_1 \\ D \end{pmatrix}_{\mathrm{L}} \sim \left(3, \bar{3}, \frac{1}{6} + \frac{\beta}{2\sqrt{3}}\right) , \qquad q_2 = \begin{pmatrix} n_2 \\ -p_2 \\ S \end{pmatrix}_{\mathrm{L}} \sim \left(3, \bar{3}, \frac{1}{6} + \frac{\beta}{2\sqrt{3}}\right) , \tag{3.4}$$

$$q_3 = \begin{pmatrix} p_3 \\ n_3 \\ T \end{pmatrix}_{\mathrm{L}} \sim \left(3, 3, \frac{1}{6} - \frac{\beta}{2\sqrt{3}}\right) . \tag{3.5}$$

The p and n are the positively and negatively charged quark fields, respectively. The transformation of the corresponding Right-handed fields is given by,

$$p_{1R},\, p_{2R},\, p_{3R} \sim (3, 1, 2/3); \quad n_{1R}\,, n_{2R}\,, n_{3R} \sim (3, 1, -1/3); \quad e_{iR} \sim (1, 1, -1), \tag{3.6}$$

$$D_R, S_R \sim \left(3, 1, \frac{1}{6} + \frac{\sqrt{3}\beta}{2}\right) ; \; T_R \sim \left(3, 1, \frac{1}{6} - \frac{\sqrt{3}\beta}{2}\right) ; \; E_{iR} \left(1, 1, -\frac{1}{2} - \frac{\sqrt{3}\beta}{2}\right) . \tag{3.7}$$

The electric charges of the heavy fermion fields[2], $i.e.$, E_i, D, S, T fields (both left and right-handed) are given by

$$Q_{E_i} = -\frac{1}{2} - \frac{\sqrt{3}\beta}{2} \qquad Q_{D,S} = \frac{1}{6} + \frac{\sqrt{3}\beta}{2} \qquad Q_T = \frac{1}{6} - \frac{\sqrt{3}\beta}{2} . \tag{3.8}$$

We will only consider the 3-3-1 model, where three scalar triplets η, ρ, and χ are introduced, and after symmetry breaking, their VEVs will give rise to the masses of gauge bosons as well

[2]we will see in 3.4 why these fields are heavy.

as charged fermions. Complying with standard notation, the first breaking will be due to the VEV of χ, while the second breaking comes from the VEVs of both η, ρ. Schematically,[3]

$$\mathrm{SU(3)_C \times SU(3)_L \times U(1)_X} \xrightarrow{v_\chi} \mathrm{SU(3)_C \times SU(2)_L \times U(1)_Y} \xrightarrow{v_\eta, v_\rho} \mathrm{SU(3)_C \times U(1)_{em}}$$

In order to set up our notation, we will choose the scalar fields to have the following transformation under the irreducible representation of the $\mathrm{SU(3)_C \times SU(3)_L \times U(1)_X}$ gauge group[3]

$$\eta = \begin{pmatrix} \eta^0 \\ \eta^- \\ \eta^{-Q^A} \end{pmatrix} \sim (1, \mathbf{3}, X_\eta), \quad \rho = \begin{pmatrix} \rho^+ \\ \rho^0 \\ \rho^{-Q^B} \end{pmatrix} \sim (1, \mathbf{3}, X_\rho), \quad \chi = \begin{pmatrix} \chi^{Q^A} \\ \chi^{Q^B} \\ \chi^0 \end{pmatrix} \sim (1, \mathbf{3}, X_\chi), \tag{3.9}$$

Let us calculate the $\mathrm{U(1)_X}$ charges for the scalar fields along with Q^A and Q^B, from the Eq. (3.1)

$$\begin{aligned} Q^{\eta^0} &= \mathrm{T_3} + \beta \mathrm{T_8} + \mathrm{X_\eta} & Q^{\rho^0} &= \mathrm{T_3} + \beta \mathrm{T_8} + \mathrm{X_\rho} & Q^{\chi^0} &= \mathrm{T_3} + \beta \mathrm{T_8} + \mathrm{X_\chi} \\ \Rightarrow 0 &= \frac{1}{2} + \frac{\beta}{2\sqrt{3}} + X_\eta & \Rightarrow 0 &= -\frac{1}{2} + \frac{\beta}{2\sqrt{3}} + X_\rho & \Rightarrow 0 &= -\frac{\beta}{\sqrt{3}} + X_\chi \\ \Rightarrow X_\eta &= -\frac{1}{2} - \frac{\beta}{2\sqrt{3}} & \Rightarrow X_\rho &= \frac{1}{2} - \frac{\beta}{2\sqrt{3}} & \Rightarrow X_\chi &= \frac{\beta}{\sqrt{3}}, \end{aligned} \tag{3.10}$$

and the electric charges Q^A, Q^B are In terms of the parameter β given by,

$$\begin{aligned} Q^A &= -\mathrm{T_3} - \beta \mathrm{T_8} - \mathrm{X_\eta} & Q^B &= -\mathrm{T_3} - \beta \mathrm{T_8} - \mathrm{X_\rho} \\ \Rightarrow Q^A &= 0 + \frac{\beta}{\sqrt{3}} + \frac{1}{2} + \frac{\beta}{2\sqrt{3}} & \Rightarrow Q^B &= 0 + \frac{\beta}{\sqrt{3}} - \frac{1}{2} + \frac{\beta}{2\sqrt{3}} \\ \Rightarrow Q^A &= \frac{1}{2} + \frac{\sqrt{3}\beta}{2} & \Rightarrow Q^B &= -\frac{1}{2} + \frac{\sqrt{3}\beta}{2}. \end{aligned} \tag{3.11}$$

At this stage, we can figure out why β always takes discrete values. To see that, let us, for example, take the Lepton triplet. We know that the difference between the two fields should be an integer. Using this fact, we have,

$$\begin{aligned} Q_\nu - Q_E &= N \\ \Rightarrow \frac{1}{2} + \frac{\sqrt{3}\beta}{2} &= N \\ \Rightarrow \beta &= \frac{2N-1}{\sqrt{3}}, \end{aligned} \tag{3.12}$$

where 'N' is an integer. Let's make a table for the values of β[iii],

So from this table, we see that the values for β are $\pm\sqrt{3}$, $\pm\frac{1}{\sqrt{3}}$, which are discrete values.

3.2 Masses of the Gauge Bosons

The Gauge bosons of this theory consist of an octet W_μ^a associated with $\mathrm{SU(3)_L}$ and a singlet X_μ associated with $\mathrm{U(1)_X}$. The masses of the gauge bosons come from the kinetic terms for

[iii] we will see the constraint on β in this model in the section 3.2.2.

N	β	β^2
-2	$-\frac{5}{\sqrt{3}}$	$\frac{25}{3} \gg 3.35$
-1	$-\sqrt{3}$	$3 < 3.35$
0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{3} < 3.35$
1	$\frac{1}{\sqrt{3}}$	$\frac{1}{3} < 3.35$
2	$\sqrt{3}$	$3 < 3.35$
3	$\frac{5}{\sqrt{3}}$	$\frac{25}{3} \gg 3.35$

Table 3.1: Table for values of β

the scalar fields, which contain covariant derivatives, and the covariant derivative is defined as,

$$\begin{aligned} D_\mu &= \partial_\mu + ig\left(\overrightarrow{W_\mu}.\frac{\overrightarrow{\lambda}}{2}\right) + ig'\mathrm{X}\mathrm{X}_\mu \\ &= \partial_\mu + \frac{ig}{2}\begin{pmatrix} W^3_\mu + \frac{1}{\sqrt{3}}W^8_\mu + 2\frac{g'}{g}\mathrm{X}\mathrm{X}_\mu & W^1_\mu - iW^2_\mu & W^4_\mu - iW^5_\mu \\ W^1_\mu + iW^2_\mu & W^3_\mu - \frac{1}{\sqrt{3}}W^8_\mu + 2\frac{g'}{g}\mathrm{X}\mathrm{X}_\mu & W^6_\mu - iW^7_\mu \\ W^4_\mu + iW^5_\mu & W^6_\mu + iW^7_\mu & -\frac{2}{\sqrt{3}}W^8_\mu + 2\frac{g'}{g}\mathrm{X}\mathrm{X}_\mu \end{pmatrix}. \end{aligned} \tag{3.13}$$

Where λ's are the usual Gell-Mann matrices and X denotes the X charges for the Higgs multiplets. The masses of the Gauge Bosons come from the kinetic terms for the scalar fields, $i.e.$,

$$\mathcal{L}_{mass} = (D_\mu\eta)^\dagger(D^\mu\eta) + (D_\mu\rho)^\dagger(D^\mu\rho) + (D_\mu\chi)^\dagger(D^\mu\chi)\,. \tag{3.14}$$

We can write the scalar fields in the following manner,

$$\eta = \begin{pmatrix} \frac{1}{\sqrt{2}}(v_\eta + h_\eta + iz_\eta) \\ \eta^- \\ \eta^{-Q^A} \end{pmatrix}, \quad \rho = \begin{pmatrix} \rho^+ \\ \frac{1}{\sqrt{2}}(v_\rho + h_\rho + iz_\rho) \\ \rho^{-Q^B} \end{pmatrix}, \quad \chi = \begin{pmatrix} \chi^{Q^A} \\ \chi^{Q^B} \\ \frac{1}{\sqrt{2}}(v_\chi + h_\chi + iz_\chi) \end{pmatrix}, \tag{3.15}$$

where $v_\phi,\ h_\phi,$ and z_ϕ are the VEV, CP-even, and CP-odd scalar for respective fields.

3.2.1 Masses of the Charged Gauge Bosons

Now with considering $\sqrt{2}W^{\pm Q_1}_\mu \equiv (W^1_\mu \mp iW^2_\mu)$,$\sqrt{2}V^{\pm Q_2}_\mu \equiv (W^4_\mu \mp iW^5_\mu)$ and $\sqrt{2}U^{\pm Q_3}_\mu \equiv (W^6_\mu \mp iW^7_\mu)$, and after symmetry breaking the masses of the gauge bosons can be obtained as done before in section 2.4 from Eq. (3.13) and the masses are,

$$M^2_{W^{\pm Q_1}_\mu} = \frac{1}{4}g^2(v^2_\eta + v^2_\rho); \quad M^2_{V^{\pm Q_2}_\mu} = \frac{1}{4}g^2(v^2_\eta + v^2_\chi); \quad M^2_{U^{\pm Q_3}_\mu} = \frac{1}{4}g^2(v^2_\rho + v^2_\chi)\,. \tag{3.16}$$

The charges of the Gauge Bosons can be obtained by simultaneously diagonalising the T^3 and T^8, which are in the adjoint representation. The charges are then given by,

$$Q_1 = 1; \quad Q_2 = \frac{1}{2} + \frac{\sqrt{3}\beta}{2}; \quad Q_3 = \frac{1}{2} - \frac{\sqrt{3}\beta}{2}\,. \tag{3.17}$$

Now notice that if $v_\eta^2 + v_\rho^2 = v^2$, v being the usual vacuum expectation value of the Higgs in the standard model, we can identify $W_\mu^\pm$ as the SM Gauge bosons. And even if $v_\eta^2 + v_\rho^2 = v^2$, the v_χ must be large enough to keep the new gauge bosons sufficiently heavy in order to have consistency with low energy phenomenology.

3.2.2 Mass of the Neutral Gauge Bosons

The neutral gauge bosons have the following mass matrix in the unphysical bases, $i.e.$ in $\begin{pmatrix} W_\mu^3 & W_\mu^8 & \mathrm{X}_\mu \end{pmatrix}$ basis,[11]

$$\mathcal{L}_{NG} = \frac{1}{2}\begin{pmatrix} W_\mu^3 & W_\mu^8 & \mathrm{X}_\mu \end{pmatrix} M^2 \begin{pmatrix} W^{3\mu} \\ W^{8\mu} \\ \mathrm{X}^\mu \end{pmatrix}, \tag{3.18}$$

where

$$M^2 = \frac{1}{4}g^2 \begin{pmatrix} v_\eta^2 + v_\rho^2 & \frac{1}{\sqrt{3}}(v_\eta^2 - v_\rho^2) & \mathcal{M}_{3X}^2 \\ \frac{1}{\sqrt{3}}(v_\eta^2 - v_\rho^2) & \frac{1}{3}(v_\eta^2 + v_\rho^2 + 4v_\chi^2) & \mathcal{M}_{8X}^2 \\ \mathcal{M}_{3X}^2 & \mathcal{M}_{8X}^2 & \mathcal{M}_{XX}^2 \end{pmatrix}, \tag{3.19}$$

where,

$$\begin{aligned} \mathcal{M}_{3X}^2 &= -\frac{t_{\theta_X}}{\sqrt{3}}\left((\beta + \sqrt{3})v_\eta^2 - (\beta - \sqrt{3})v_\rho^2\right) \\ \mathcal{M}_{8X}^2 &= -\frac{t_{\theta_X}}{3}\left((\beta + \sqrt{3})v_\eta^2 - (\beta - \sqrt{3})v_\rho^2 + 4\beta v_\chi^2\right) \\ \mathcal{M}_{XX}^2 &= \frac{t_{\theta_X}^2}{3}\left((\beta + \sqrt{3})^2 v_\eta^2 + (\beta - \sqrt{3})^2 v_\rho^2 + 4\beta v_\chi^2\right). \end{aligned} \tag{3.20}$$

Where we consider $t_{\theta_X} \equiv \tan\theta_X = \frac{g'}{g}$

Now, to get the masses of the physical fields, we need to diagonalize the mass matrix, and we will do it in three steps.

- First, we will rotate from $\begin{pmatrix} W_\mu^3 & W_\mu^8 & \mathrm{X}_\mu \end{pmatrix}$ basis to $\begin{pmatrix} W_\mu^3 & B_\mu & \mathrm{X}'_\mu \end{pmatrix}$ basis, Where B_μ is the standard hypercharge gauge boson. To do this, we will use a rotation matrix, and the change of basis is given by,

$$\begin{pmatrix} W_\mu^3 \\ B_\mu \\ X'_\mu \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\beta t_{\theta_X}}{\sqrt{1+\beta^2 t_{\theta_X}^2}} & \frac{1}{\sqrt{1+\beta^2 t_{\theta_X}^2}} \\ 0 & -\frac{1}{\sqrt{1+\beta^2 t_{\theta_X}^2}} & \frac{\beta t_{\theta_X}}{\sqrt{1+\beta^2 t_{\theta_X}^2}} \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ W_\mu^8 \\ X_\mu \end{pmatrix}. \tag{3.21}$$

- Secondly, we will rotate the $\begin{pmatrix} W_\mu^3 & B_\mu & \mathrm{X}'_\mu \end{pmatrix}$ basis to $\begin{pmatrix} A_\mu & \widetilde{Z}_\mu & \mathrm{X}'_\mu \end{pmatrix}$ basis. Where A_μ is the Photon field. We will use a second rotation matrix to do this. After these two rotations, the mass matrix will be block diagonalised. The change of basis is given by,

$$\begin{pmatrix} A_\mu \\ \widetilde{Z}_\mu \\ X'_\mu \end{pmatrix} = \begin{pmatrix} \frac{t_{\theta_X}}{\sqrt{1+(1+\beta^2)t_{\theta_X}^2}} & \frac{\sqrt{1+\beta^2 t_{\theta_X}^2}}{\sqrt{1+(1+\beta^2)t_{\theta_X}^2}} & 0 \\ -\frac{\sqrt{1+\beta^2 t_{\theta_X}^2}}{\sqrt{1+(1+\beta^2)t_{\theta_X}^2}} & \frac{t_{\theta_X}}{\sqrt{1+(1+\beta^2)t_{\theta_X}^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \\ X'_\mu \end{pmatrix}. \tag{3.22}$$

The block diagonalized matrix is given by,

$$\mathcal{L}_N = \frac{1}{2}\begin{pmatrix} A_\mu & \widetilde{Z}_\mu & X'_\mu \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & M_{11}^2 & M_{12}^2 \\ 0 & M_{12}^2 & M_{22}^2 \end{pmatrix} \begin{pmatrix} A^\mu \\ \widetilde{Z}^\mu \\ X'^\mu \end{pmatrix}, \tag{3.23}$$

here M_{ij}^2 can be written as(in terms of $M_{W,U,V}$and t_{θ_X}):

$$\begin{aligned} M_{11}^2 &= \frac{1+(1+\beta^2)t_{\theta_X}^2}{1+\beta^2 t_{\theta_X}^2} M_W^2 \,, \\ M_{12}^2 &= -\frac{\sqrt{1+(1+\beta^2)t_{\theta_X}^2}}{\sqrt{3}(1+\beta^2 t_{\theta_X}^2)} \left[\sqrt{3}\beta t_{\theta_X}^2 M_W^2 + (1+\beta^2 t_{\theta_X}^2)(M_U^2 - M_V^2)\right] , \\ M_{22}^2 &= \frac{1}{3(1+\beta^2 t_{\theta_X}^2)} \Big[- M_W^2 (1 + 2\beta^2 t_{\theta_X}^2 + \beta^2(\beta^2-3)t_{\theta_X}^4) \\ &\quad + 2M_V^2(1+\beta^2 t_{\theta_X}^2)(1+\beta(\beta-\sqrt{3})t_{\theta_X}^2) + 2M_U^2(1+\beta^2 t_{\theta_X}^2)(1+\beta(\beta+\sqrt{3})t_{\theta_X}^2)\Big] . \end{aligned} \tag{3.24}$$

- Lastly, we will rotate $\begin{pmatrix} A_\mu & \widetilde{Z}_\mu & \mathrm{X}'_\mu \end{pmatrix}$ basis to $\begin{pmatrix} A_\mu & Z_\mu & Z'_\mu \end{pmatrix}$ basis. Where Z_μ is the standard massive neutral gauge boson, and Z'_μ is the new heavy neutral gauge boson. To find out the masses of $Z_\mu, \;\; Z'_\mu$ and their mixing angle α_z, using the following conditions,

$$\begin{aligned} M_{11}^2 &= M_Z^2 \cos^2\alpha_z + M_{Z'}^2 \sin^2\alpha_z \,, \\ M_{22}^2 &= M_Z^2 \sin^2\alpha_z + M_{Z'}^2 \cos^2\alpha_z \,, \\ 2M_{12}^2 &= (M_Z^2 - M_{Z'}^2)\sin 2\alpha_z \,. \end{aligned} \tag{3.25}$$

where the rotation is given as,

$$\begin{pmatrix} Z_\mu \\ Z'_\mu \end{pmatrix} = \begin{pmatrix} \cos\alpha_z & \sin\alpha_z \\ -\sin\alpha_z & \cos\alpha_z \end{pmatrix} \begin{pmatrix} \widetilde{Z}_\mu \\ X'_\mu \end{pmatrix} . \tag{3.26}$$

After solving for $M_Z^2, \; M_{Z'}^2$, and α_z in Eq. (3.25), we have,

$$\begin{aligned} M_Z &= \frac{1}{2}\left(M_{11}^2 + M_{22}^2 - \sqrt{(M_{11}^2 - M_{22}^2)^2 + 4M_{12}^4}\right) \\ M_{Z'} &= \frac{1}{2}\left(M_{11}^2 + M_{22}^2 + \sqrt{(M_{11}^2 - M_{22}^2)^2 + 4M_{12}^4}\right) \\ \tan\alpha_z &= \frac{2M_{12}^2}{M_{11}^2 - M_{22}^2} \,, \end{aligned} \tag{3.27}$$

and after simplifying Eq. (3.27), the masses of the neutral gauge bosons are,

$$M_A^2 = 0, \quad M_Z^2 \simeq \frac{1+(1+\beta^2)t_{\theta_X}^2}{1+\beta^2 t_{\theta_X}^2} M_W^2 \,, \quad M_{Z'}^2 \simeq \frac{2}{3}(1+\beta^2 t_{\theta_X}^2)\left(M_U^2 + M_V^2 - M_W^2\right) . \tag{3.28}$$

From the above equation, *i.e.* Eq. (3.28), we see that

$$\begin{aligned}\frac{M_Z^2}{M_W^2} &= \frac{1+(1+\beta^2)t_{\theta_X}^2}{1+\beta^2 t_{\theta_X}^2} = \frac{1}{cos^2\theta_W}\\ \Rightarrow \sin\theta_W &= \frac{t_{\theta_X}}{\sqrt{1+(1+\beta^2)t_{\theta_X}^2}}\\ \Rightarrow t_{\theta_X} &= \frac{\sin\theta_W}{\sqrt{1-(1+\beta^2)\sin^2\theta_W}}\\ \Rightarrow g' \to \infty \ \text{ as } &\quad \sin^2\theta_W \to \frac{1}{1+\beta^2}\,. \end{aligned} \tag{3.29}$$

So from the above equation, we get a condition for β^2, which is given by,

$$\boxed{\beta^2 < \cot^2\theta_W}\,. \tag{3.30}$$

After all these three steps, the physical states of the neutral gauge bosons are given by,

$$\begin{aligned} A_\mu &= \frac{1}{\sqrt{1+(1+\beta^2)t_{\theta_X}^2}}\left[(W_\mu^3+\beta W_\mu^8)t_{\theta_X}+B_\mu\right],\\ Z_\mu &\simeq -\frac{1}{\sqrt{1+(1+\beta^2)t_{\theta_X}^2}}\left[\left(\sqrt{1+\beta^2 t_{\theta_X}^2}\right)W_\mu^3+\frac{\beta t_{\theta_X}^2}{\sqrt{1+\beta^2 t_{\theta_X}^2}}W_\mu^8-\frac{t_{\theta_X}}{\sqrt{1+\beta^2 t_{\theta_x}^2}}B_\mu\right],\\ Z'_\mu &\simeq \frac{1}{\sqrt{1+\beta^2 t^2}}(W_\mu^8-\beta t_{\theta_X}B_\mu)\,. \end{aligned} \tag{3.31}$$

3.3 Scalar Potential of the 331 model

In terms of the scalar triplets, the scalar potential takes the form[3][8]

$$\begin{aligned} V(\eta,\rho,\chi) =& \mu_1^2\rho^\dagger\rho+\mu_2^2\eta^\dagger\eta+\mu_3^2\chi^\dagger\chi+\lambda_1\left(\rho^\dagger\rho\right)^2+\lambda_2\left(\eta^\dagger\eta\right)^2+\lambda_3\left(\chi^\dagger\chi\right)^2\\ &+\lambda_{12}\left(\rho^\dagger\rho\right)\left(\eta^\dagger\eta\right)+\lambda_{13}\left(\chi^\dagger\chi\right)\left(\rho^\dagger\rho\right)+\lambda_{23}\left(\eta^\dagger\eta\right)\left(\chi^\dagger\chi\right)\\ &+\zeta_{12}\left(\rho^\dagger\eta\right)\left(\eta^\dagger\rho\right)+\zeta_{13}\left(\rho^\dagger\chi\right)\left(\chi^\dagger\rho\right)+\zeta_{23}\left(\eta^\dagger\chi\right)\left(\chi^\dagger\eta\right)\\ &-\sqrt{2}f\epsilon_{ijk}\eta_i\rho_j\chi_k+\text{h.c.}, \end{aligned} \tag{3.32}$$

where 'f' is a dimensionfull parameter with mass dimension '1'. After χ^0 recieves a VEV, the remaining symmetry is $SU(3)_c\times SU(2)_L\times U(1)_Y$. At this stage, we can group the scalar fields in the following representations under the SM gauge group[12][3]

$$\begin{aligned} \Phi_\eta = \begin{pmatrix}\eta^+\\ \eta^0\end{pmatrix} \sim (1,2,1/2),\quad \Phi_\rho = \begin{pmatrix}\rho^+\\ \rho^0\end{pmatrix} \sim (1,2,1/2),\quad \Phi_\chi = \begin{pmatrix}\chi^{Q^A}\\ \chi^{Q^B}\end{pmatrix} \sim (1,2,\sqrt{3}\beta/2)\\ \eta^{-Q^A} \sim (1,1,-Q^A),\quad \rho^{-Q^B} \sim (1,1,-Q^B),\quad \chi^0 \sim (1,1,0), \end{aligned} \tag{3.33}$$

Where '$1/2, 1/2, \sqrt{3}\beta/2, -Q^A, -Q^B$ and 0' are the hypercharge of the respective doublet and singlet. while the scalar potential can be rewritten as [3]

$$\begin{aligned}
V(\eta,\rho,\chi) \;=\;& \mu_1^2\left(\Phi_\rho^\dagger\Phi_\rho+\rho^{Q^B}\rho^{-Q^B}\right)+\mu_2^2\left(\Phi_\eta^\dagger\Phi_\eta+\eta^{Q^A}\eta^{-Q^A}\right)+\mu_3^2\left(\Phi_\chi^\dagger\Phi_\chi+\chi^{0 2}\right)\\
&+\lambda_1\left(\Phi_\rho^\dagger\Phi_\rho+\rho^{Q^B}\rho^{-Q^B}\right)^2+\lambda_2\left(\Phi_\eta^\dagger\Phi_\eta+\eta^{Q^A}\eta^{-Q^A}\right)^2+\lambda_3\left(\Phi_\chi^\dagger\Phi_\chi+\chi^{0 2}\right)^2\\
&+\lambda_{12}\left(\Phi_\rho^\dagger\Phi_\rho+\rho^{Q^B}\rho^{-Q^B}\right)\left(\Phi_\eta^\dagger\Phi_\eta+\eta^{Q^A}\eta^{-Q^A}\right)\\
&+\lambda_{13}\left(\Phi_\chi^\dagger\Phi_\chi+\chi^{0 2}\right)\left(\Phi_\rho^\dagger\Phi_\rho+\rho^{Q^B}\rho^{-Q^B}\right)\\
&+\lambda_{23}\left(\Phi_\chi^\dagger\Phi_\chi+\chi^{0 2}\right)\left(\Phi_\eta^\dagger\Phi_\eta+\eta^{Q^A}\eta^{-Q^A}\right)\\
&+\zeta_{12}\left(\Phi_\rho^\dagger\Phi_\eta+\rho^{Q^B}\eta^{-Q^A}\right)\left(\Phi_\eta^\dagger\Phi_\rho+\eta^{Q^A}\rho^{-Q^B}\right)\\
&+\zeta_{13}\left(\Phi_\rho^\dagger\Phi_\chi+\rho^{Q^B}\chi^0\right)\left(\Phi_\chi^\dagger\Phi_\rho+\chi^{0*}\rho^{-Q^B}\right)\\
&+\zeta_{23}\left(\Phi_\eta^\dagger\Phi_\chi+\eta^{Q^A}\chi^0\right)\left(\Phi_\chi^\dagger\Phi_\eta+\chi^{0*}\eta^{-Q^A}\right)\\
&-\sqrt{2}f\left(\Phi_\eta^\dagger\Phi_\rho\chi^0+\Phi_\eta^\dagger\Phi_\chi\rho^{-Q^B}+\Phi_\rho^\dagger\Phi_\chi\eta^{-Q^A}\right)+\text{h.c.}\,. \qquad (3.34)
\end{aligned}$$

For finding the VEV for χ we have,

$$\begin{aligned}
&\frac{\partial V}{\partial \chi^0}=0\\
&\Rightarrow 2\mu_3^2 v_\chi+4\lambda_3 v_\chi^3=0\\
&\Rightarrow \mu_3^2=-2\lambda_3 v_\chi^2\,. \qquad (3.35)
\end{aligned}$$

To get the mass of any field, we need to look at the bilinear terms of that field in the potential, and that can be done by differentiating the potential twice with respect to the field. After χ acquires the VEV, the masses for the heavy fields would be,

$$m_{\chi^0}^2=\left.\frac{\partial^2 V}{\partial \chi^{0 2}}\right|_{\chi^0=v_\chi} = 2\mu_3^2+12\lambda_3 v_\chi^2=8\lambda_3 v_\chi^2\,, \qquad (3.36)$$

$$m_{\rho^{Q^B}}^2=\left.\frac{\partial^2 V}{\partial \rho^{Q^B}\partial\rho^{-Q^B}}\right|_{\chi^0=v_\chi} = \mu_1^2+(\zeta_{13}+\lambda_{13})v_\chi^2\,, \qquad (3.37)$$

$$m_{\eta^{Q^A}}^2=\left.\frac{\partial^2 V}{\partial \eta^{Q^A}\partial\eta^{-Q^A}}\right|_{\chi^0=v_\chi} = \mu_2^2+(\zeta_{23}+\lambda_{23})v_\chi^2\,, \qquad (3.38)$$

After the electroweak symmetry breaking, we can trade off the parameters $\mu_{1,2}^2$ with the respective VEVs as in Eq. (3.35). From Eq. (3.32) we have,

$$\begin{aligned}
&\frac{\partial V}{\partial \rho^0}=0 && \frac{\partial V}{\partial \eta^0}=0\\
&\Rightarrow 2\mu_1^2 v_\rho+4\lambda_1 v_\rho^3=0 && \Rightarrow 2\mu_2^2 v_\eta+4\lambda_2 v_\eta^3=0\\
&\Rightarrow \mu_1^2=-2\lambda_1 v_\rho^2\,, && \Rightarrow \mu_2^2=-2\lambda_2 v_\eta^2\,, \qquad (3.39)
\end{aligned}$$

as $\mu_{1,2}^2\sim v_{\rho,\eta}^2$, implying that all heavy masses are proportional to v_χ in the limit $v_\chi \gg v_\rho, v_\eta$, as previously stated.

In order to define an effective field theory, we will integrate out the heavy fields $\chi,\ \rho^{Q^B},\ \eta^{Q^A}$. Tree-level matching can be conveniently done by solving the equation of motion (E.O.M.)

for the heavy fields and defining our EFT Lagrangian. Notice that the fields Φ_χ, ρ^{Q^B} and η^{Q^A} appear only in pairs. Thus, considering tree-level matching, they can be integrated out without any consequence at low energy. The only exception is for terms containing a single field, χ^0, for example $\left(\Phi_\eta^\dagger \Phi_\rho \chi^0\right)$. Thus, as long as we consider only tree-level matching, terms of this kind are only relevant. The Lagrangian for the χ fields then, would be

$$\begin{aligned}
\mathcal{L}_{\chi^0} &= \left(\partial_\mu \chi^\dagger\right)\left(\partial^\mu \chi\right) + V\left(\eta, \rho, \chi\right) \\
&= \left(\partial_\mu \chi^\dagger\right)\left(\partial^\mu \chi\right) + 2\mu_3^2 \chi^0 + 8\lambda_3 v_\chi^2 \chi^0 + 4\lambda_3 v_\chi^2 \chi^0 \\
&+ 2\lambda_{13}(\Phi_\rho^\dagger \Phi_\rho)\chi^0 + 2\lambda_{23}(\Phi_\eta^\dagger \Phi_\eta)\chi^0 - \sqrt{2} f(\Phi_\eta^\dagger \Phi_\rho + \Phi_\rho^\dagger \Phi_\eta)\chi^0 \\
&= \left(\partial_\mu \chi^\dagger\right)\left(\partial^\mu \chi\right) - 4\mu_3^2 \chi^0 + 2\lambda_{13}(\Phi_\rho^\dagger \Phi_\rho)\chi^0 \\
&+ 2\lambda_{23}(\Phi_\eta^\dagger \Phi_\eta)\chi^0 - \sqrt{2} f(\Phi_\eta^\dagger \Phi_\rho + \Phi_\rho^\dagger \Phi_\eta)\chi^0 \,.
\end{aligned} \tag{3.40}$$

Let us solve the EOM for χ^0 after integrating out the heavy fields using the Euler-Lagrange equation, namely [9],

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi} \,. \tag{3.41}$$

Let us calculate the left-hand side of Eq. (3.41) for $\chi^\dagger$,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\sigma \chi^\dagger)} &= \frac{\partial}{\partial(\partial_\sigma \chi^\dagger)} \left[\left(\partial_\mu \chi^\dagger\right)\left(\partial^\mu \chi\right)\right] \\
&= \left(\partial^\mu \chi\right) \frac{\partial \left(\partial_\mu \chi^\dagger\right)}{\partial(\partial_\sigma \chi^\dagger)} \\
&= \left(\partial^\mu \chi\right) \delta^\mu_\sigma \\
&= \partial^\sigma \chi \,.
\end{aligned} \tag{3.42}$$

So for the χ^0 field, the Eq. (3.42) would be $\partial^\mu \chi^0$. The right-hand side of the Eq. (3.41) becomes,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \chi^0} \equiv \frac{\partial V}{\partial \chi^0} &= m_{\chi^0}^2 \chi^0 + \lambda_{13} v_\chi (\Phi_\rho^\dagger \Phi_\rho) \\
&\quad + \lambda_{23} v_\chi (\Phi_\eta^\dagger \Phi_\eta) - \sqrt{2} f(\Phi_\eta^\dagger \Phi_\rho + \Phi_\rho^\dagger \Phi_\eta) \,.
\end{aligned} \tag{3.43}$$

Now, for effective field theory, the kinetic term can be ignored with respect to the mass of the heavy particle, so from Eqs. (3.41) and (3.43) we have,

$$\begin{aligned}
\Rightarrow 0 &= \frac{\partial V}{\partial \chi^0} \\
\Rightarrow 0 &= m_{\chi^0}^2 \chi^0 + \lambda_{13} v_\chi (\Phi_\rho^\dagger \Phi_\rho) + \lambda_{23} v_\chi (\Phi_\eta^\dagger \Phi_\eta) - \sqrt{2} f(\Phi_\eta^\dagger \Phi_\rho + \Phi_\rho^\dagger \Phi_\eta) \\
\Rightarrow \chi^0 &= \frac{\sqrt{2} f(\Phi_\eta^\dagger \Phi_\rho + \text{h.c.})}{m_{\chi^0}^2} - \frac{\lambda_{13} v_\chi (\Phi_\rho^\dagger \Phi_\rho)}{m_{\chi^0}^2} - \frac{\lambda_{23} v_\chi (\Phi_\eta^\dagger \Phi_\eta)}{m_{\chi^0}^2} \,.
\end{aligned} \tag{3.44}$$

Finally, we will get the 331 EFT by putting Eq. (3.44) in Eq. (3.34) and rearranging terms and integrating out the heavy fields, we have,[3]

$$\begin{aligned} V(\eta,\rho,\chi) &= (\mu_1^2+\lambda_{13}v_\chi^2)(\Phi_\rho^\dagger\Phi_\rho)+(\mu_2^2+\lambda_{23}v_\chi^2)(\Phi_\eta^\dagger\Phi_\eta)-\sqrt{2}fv_\chi(\Phi_\eta^\dagger\Phi_\rho) \\ &+\left[\lambda_1-\frac{2v_\chi^2\lambda_{13}^2}{m_{\chi_0}^2}\right](\Phi_\rho^\dagger\Phi_\rho)^2+\left[\lambda_2-\frac{2v_\chi^2\lambda_{23}^2}{m_\chi^2}\right](\Phi_\eta^\dagger\Phi_\eta)^2 \\ &+\left[\lambda_{12}-\frac{4v_\chi^2\lambda_{13}\lambda_{23}}{m_{\chi_0}^2}\right](\Phi_\rho^\dagger\Phi_\rho)(\Phi_\eta^\dagger\Phi_\eta)+\left[\zeta_{12}-\left(\frac{2f^2}{m_{\chi_0}^2}\right)\right](\Phi_\eta^\dagger\Phi_\rho)^\dagger(\Phi_\eta^\dagger\Phi_\rho) \\ &+\Bigg[-\left(\frac{f^2}{m_{\chi_0}^2}\right)(\Phi_\eta^\dagger\Phi_\rho)^2+\left(\frac{2\sqrt{2}fv_\chi\lambda_{13}}{m_{\chi_0}^2}\right)\left(\Phi_\eta^\dagger\Phi_\rho\right)\left(\Phi_\rho^\dagger\Phi_\rho\right) \\ &+\left(\frac{2\sqrt{2}fv_\chi\lambda_{23}}{m_{\chi_0}^2}\right)(\Phi_\eta^\dagger\Phi_\eta)\left(\Phi_\eta^\dagger\Phi_\rho\right)+\text{h.c.}\Bigg]. \end{aligned} \tag{3.45}$$

Now, if we rename the fields to connect the 331-EFT to the conventional 2HDM(Two Higgs Doublet Model)

$$\Phi_1 \equiv \Phi_\rho \quad \text{and} \quad \Phi_2 \equiv \Phi_\eta\,. \tag{3.46}$$

The form of the potential reduces to

$$\begin{aligned} V(\Phi_1,\Phi_2) =\ & m_{11}^2\Phi_1^\dagger\Phi_1+m_{22}^2\Phi_2^\dagger\Phi_2+\Lambda_1\left(\Phi_1^\dagger\Phi_1\right)^2+\Lambda_2\left(\Phi_2^\dagger\Phi_2\right)^2 \\ &+\Lambda_3\left(\Phi_1^\dagger\Phi_1\right)\left(\Phi_2^\dagger\Phi_2\right)+\Lambda_4\left(\Phi_1^\dagger\Phi_2\right)\left(\Phi_2^\dagger\Phi_1\right) \\ &+\Bigg[-m_{12}^2\Phi_1^\dagger\Phi_2+\Lambda_5\left(\Phi_1^\dagger\Phi_2\right)^2+\Lambda_6\left(\Phi_1^\dagger\Phi_1\right)\left(\Phi_1^\dagger\Phi_2\right) \\ &+\Lambda_7\left(\Phi_2^\dagger\Phi_2\right)\left(\Phi_1^\dagger\Phi_2\right)+\text{h.c.}\Bigg]. \end{aligned} \tag{3.47}$$

Where the 2HDM parameters are related to the 331 model parameters as

$$\begin{aligned} m_{11}^2 &= \mu_1^2+\lambda_{13}v_\chi^2\,, & \Lambda_3 &= \left[\lambda_{12}-\frac{4v_\chi^2\lambda_{13}\lambda_{23}}{m_{\chi_0}^2}\right], \\ m_{22}^2 &= \mu_2^2+\lambda_{23}v_\chi^2\,, & \Lambda_4 &= \left[\zeta_{12}-\left(\frac{2f^2}{m_{\chi_0}^2}\right)\right], \\ m_{12}^2 &= \sqrt{2}fv_\chi\,, & \Lambda_5 &= -\left(\frac{f^2}{m_{\chi_0}^2}\right), \\ \Lambda_1 &= \left[\lambda_1-\frac{2v_\chi^2\lambda_{13}^2}{m_{\chi_0}^2}\right], & \Lambda_6 &= \left(\frac{2\sqrt{2}fv_\chi\lambda_{13}}{m_{\chi_0}^2}\right), \\ \Lambda_2 &= \left[\lambda_2-\frac{2v_\chi^2\lambda_{23}^2}{m_\chi^2}\right], & \Lambda_7 &= \left(\frac{2\sqrt{2}fv_\chi\lambda_{23}}{m_{\chi_0}^2}\right). \end{aligned} \tag{3.48}$$

3.4 Yukawa Sector of 331 model

The general gauge invariance Yukawa Lagrangian for the quarks and leptons in the 3-3-1 model is written below [3]

$$\begin{aligned} -\mathcal{L}^q_{\text{Yuk}} &= y^u_{ij}\bar{q}_{iL}\rho^* p_{jR} + y^u_{3j}\bar{q}_{3L}\eta p_{jR} + y^d_{ij}\bar{q}_{iL}\eta^* n_{jR} + y^d_{3j}\bar{q}_{3L}\rho n_{jR} \\ &\quad + y^D_i\bar{q}_{iL}\chi^* D_R + y^S_i\bar{q}_{iL}\chi^* S_R + y^T\bar{q}_{3L}\chi T_R + \text{h.c.}\,, \end{aligned} \tag{3.49}$$

$$-\mathcal{L}^l_{\text{Yuk}} = y^e_{mn}\bar{L}_{mL}\rho e_{nR} + y^E_{mn}\bar{L}_{mL}\chi E_{nR} + \text{h.c.}\,, \tag{3.50}$$

where 'i' runs from 1 to 2, while the other indices run from 1 to 3. Let us see why this Lagrangian is gauge invariant. The transformation of the fields is given by,

$$\begin{aligned} q_{iL} \to q'_{iL} &= e^{-i\vec{\alpha}\cdot\vec{T}^* + i\gamma X} q_{iL} \\ \Phi \to \Phi' &= e^{i\vec{\alpha}\cdot\vec{T} + i\gamma X}\Phi \\ q^k_{jR} \to q'^k_{jR} &= e^{i\gamma X_j} q^k_{jR}\,, \end{aligned} \tag{3.51}$$

where $\alpha \equiv \alpha(x)$ and $\gamma \equiv \gamma(x)$ are arbitrary parameters. T_i are the generator of the $SU(3)_L$ group, 'X' is the X-charges of the fields, $i = \{1, 2\}$, $\Phi = \{\eta, \rho, \chi\}$, $k = \{u, d\}$ and lastly $j = \{1, 2, 3\}$. We take $i = 1, j = 1, k = p_1$ for the first term to see the gauge invariance,

$$\begin{aligned} y^u_{11}\bar{q}_{1L}\rho^* p_{1R} \to y^u_{11}\bar{q}'_{1L}\rho'^* p'_{1R} &= y^u_{11}\bar{q}_{1L} e^{i\vec{\alpha}\cdot\vec{T}^* - i\left(\frac{1}{6} + \frac{\beta}{2\sqrt{3}}\right)\gamma} e^{-i\vec{\alpha}\cdot\vec{T}^* - i\left(\frac{1}{2} - \frac{\beta}{2\sqrt{3}}\right)\gamma}\rho^* e^{2i\gamma/3} p_{1R} \\ &= y^u_{11}\bar{q}_{1L}\rho^* p_{1R}\,. \end{aligned} \tag{3.52}$$

So we can see it is a gauge invariant quantity. Similarly, the other terms are also gauge invariant quantities. We are distinguishing the third quark family, choosing them to be a SU(3) triplet. Let us explore the Yukawa sector more thoroughly, the 1st term in Eq. (3.49) is given by,

$$\begin{aligned} y^u_{ij}\bar{q}_{iL}\rho^* p_{jR} &= y^u_{11}\bar{q}_{1L}\rho^* p_{1R} + y^u_{12}\bar{q}_{1L}\rho^* p_{2R} + y^u_{13}\bar{q}_{1L}\rho^* p_{3R} \\ &\quad + y^u_{21}\bar{q}_{2L}\rho^* p_{1R} + y^u_{22}\bar{q}_{2L}\rho^* p_{2R} + y^u_{23}\bar{q}_{3L}\rho^* p_{3R} \\ &= y^u_{11}\begin{pmatrix}\bar{n}_1 & -\bar{p}_1 & \bar{D}\end{pmatrix}_L \begin{pmatrix}\rho^- \\ \rho^{0*} \\ \rho^{+Q^B}\end{pmatrix} p_{1R} + y^u_{12}\begin{pmatrix}\bar{n}_1 & -\bar{p}_1 & \bar{D}\end{pmatrix}_L \begin{pmatrix}\rho^- \\ \rho^{0*} \\ \rho^{+Q^B}\end{pmatrix} p_{2R} \\ &\quad + y^u_{13}\begin{pmatrix}\bar{n}_1 & -\bar{p}_1 & \bar{D}\end{pmatrix}_L \begin{pmatrix}\rho^- \\ \rho^{0*} \\ \rho^{+Q^B}\end{pmatrix} p_{3R} + y^u_{21}\begin{pmatrix}\bar{n}_2 & -\bar{p}_2 & \bar{S}\end{pmatrix}_L \begin{pmatrix}\rho^- \\ \rho^{0*} \\ \rho^{+Q^B}\end{pmatrix} p_{1R} \\ &\quad + y^u_{22}\begin{pmatrix}\bar{n}_2 & -\bar{p}_2 & \bar{S}\end{pmatrix}_L \begin{pmatrix}\rho^- \\ \rho^{0*} \\ \rho^{+Q^B}\end{pmatrix} p_{2R} + y^u_{23}\begin{pmatrix}\bar{n}_2 & -\bar{p}_2 & \bar{S}\end{pmatrix}_L \begin{pmatrix}\rho^- \\ \rho^{0*} \\ \rho^{+Q^B}\end{pmatrix} p_{3R} \\ &= \boxed{y^u_{ij}\bar{Q}_{iL}\widetilde{\Phi}_\rho p_{jR} + y^u_{1j}\bar{D}_L\rho^{+Q^B} p_{jR} + y^u_{2j}\bar{S}_L\rho^{+Q^B} p_{jR}}\,, \end{aligned} \tag{3.53}$$

where the Q_{iL} is a doublet formed by the positive and the negative charged quarks, $i.e.,$ $Q_{iL} = \begin{pmatrix}p_i & n_i\end{pmatrix}^{\text{T}}_L$, and here $i = \{1, 2\}$ and 'j' runs from 1 to 3. The mass the 't' quark will be proportional to v_η, the VEV of the η multiplets, which come from the 2nd term of Eq. (3.49) and is given by,

$$\begin{aligned} y^u_{3j}\bar{q}_{3L}\eta p_{jR} &= y^u_{31}\begin{pmatrix}\bar{p}_3 & \bar{n}_3 & \bar{T}\end{pmatrix}_L \begin{pmatrix}\eta^0\\ \eta^-\\ \eta^{-Q^A}\end{pmatrix} p_{1R} + y^u_{32}\begin{pmatrix}\bar{p}_3 & \bar{n}_3 & \bar{T}\end{pmatrix}_L \begin{pmatrix}\eta^0\\ \eta^-\\ \eta^{-Q^A}\end{pmatrix} p_{2R} \\ &+ y^u_{33}\begin{pmatrix}\bar{p}_3 & \bar{n}_3 & \bar{T}\end{pmatrix}_L \begin{pmatrix}\eta^0\\ \eta^-\\ \eta^{-Q^A}\end{pmatrix} p_{3R} \\ &= \boxed{y^u_{3j}\bar{Q}_{3L}\widetilde{\Phi}_\eta p_{jR} + y^u_{3j}\bar{T}_L\eta^{-Q^A}u_{jR}}\,, \end{aligned} \tag{3.54}$$

where 'j' goes from 1 to 3.

Similarly, for the down type quarks, we can expand the 3rd and 4th terms in Eq. (3.49). After χ receives a VEV, the 5th and 7th terms will be given by,

$$\begin{aligned} y^D_i\bar{q}_{iL}\chi^* D_R = & \ y^D_1\begin{pmatrix}\bar{n}_1 & -\bar{p}_1 & \bar{D}\end{pmatrix}_L \begin{pmatrix}\chi^{-Q^A}\\ \chi^{-Q^B}\\ \chi^{0*}\end{pmatrix} D_R + y^D_2\begin{pmatrix}\bar{n}_2 & -\bar{p}_2 & \bar{S}\end{pmatrix}_L \begin{pmatrix}\chi^{-Q^A}\\ \chi^{-Q^B}\\ \chi^{0*}\end{pmatrix} D_R \\ = & \ \frac{y^D_1}{\sqrt{2}}\begin{pmatrix}\bar{n}_1 & -\bar{p}_1 & \bar{D}\end{pmatrix}_L \begin{pmatrix}0\\ 0\\ h_\chi + v_\chi\end{pmatrix} D_R + \frac{y^D_2}{\sqrt{2}}\begin{pmatrix}\bar{n}_2 & -\bar{p}_2 & \bar{S}\end{pmatrix}_L \begin{pmatrix}0\\ 0\\ h_\chi + v_\chi\end{pmatrix} D_R \\ = & \ \frac{y^D_1}{\sqrt{2}}\bar{D}_L h_\chi D_R + \frac{y^D_1 v_\chi}{\sqrt{2}}\bar{D}_L D_R + \frac{y^D_2}{\sqrt{2}}\bar{S}_L h_\chi D_R + \frac{y^D_2 v_\chi}{\sqrt{2}}\bar{S}_L D_R\,, \end{aligned} \tag{3.55}$$

and

$$\begin{aligned} y^T\bar{q}_{3L}\chi T_R = & \ y^T\begin{pmatrix}\bar{p}_3 & \bar{n}_3 & \bar{T}\end{pmatrix}_L \begin{pmatrix}\chi^{Q^A}\\ \chi^{Q^B}\\ \chi^0\end{pmatrix} T_R \\ = & \ \frac{y^T}{\sqrt{2}}\begin{pmatrix}\bar{p}_3 & \bar{n}_3 & \bar{T}\end{pmatrix}_L \begin{pmatrix}0\\ 0\\ h_\chi + v_\chi\end{pmatrix} T_R \\ = & \ \frac{1}{\sqrt{2}}\left(y^T\bar{T}_L h_\chi T_R + y^T v_\chi \bar{T}_L T_R\right)\,. \end{aligned} \tag{3.56}$$

Similarly, we will get the 6th term as shown in Eq. (3.55). Let's delve into the Lepton sector, from Eq. (3.50) we have,

$$\begin{aligned} y^e_{mn}\bar{L}_{mL}\rho e_{nR} = & \ y^e_{11}\bar{L}_{1L}\rho e_{1R} + y^e_{12}\bar{L}_{1L}\rho e_{2R} + y^e_{13}\bar{L}_{1L}\rho e_{3R} \\ & + y^e_{21}\bar{L}_{2L}\rho e_{1R} + y^e_{22}\bar{L}_{2L}\rho e_{2R} + y^e_{23}\bar{L}_{2L}\rho e_{3R} \\ & + y^e_{31}\bar{L}_{3L}\rho e_{1R} + y^e_{32}\bar{L}_{3L}\rho e_{2R} + y^e_{33}\bar{L}_{3L}\rho e_{3R} \\ = & \ y^e_{mn}\bar{l}_{mL}\Phi_\rho e_{nR} + y^e_{mn}\bar{E}_{mL}\rho^{-B}e_{nR}\,, \end{aligned} \tag{3.57}$$

and

$$\begin{aligned} y^E_{mn}\bar{L}_{mL}\chi E_{nR} = & \ y^E_{mn}\begin{pmatrix}\bar{\nu} & \bar{e} & \bar{E}\end{pmatrix}_{mL} \begin{pmatrix}\chi^{Q^A}\\ \chi^{Q^B}\\ \chi^0\end{pmatrix} E_{nR} \\ = & \ \frac{y^E_{mn}}{\sqrt{2}}\begin{pmatrix}\bar{\nu} & \bar{e} & \bar{E}\end{pmatrix}_{mL} \begin{pmatrix}0\\ 0\\ h_\chi + v_\chi\end{pmatrix} E_{nR} \\ = & \ \frac{1}{\sqrt{2}}\left(y^E_{mn}\bar{E}_{mL}h_\chi E_{nR} + y^E_{mn}v_\chi\bar{E}_{mL}E_{nR}\right)\,. \end{aligned} \tag{3.58}$$

After χ receives a VEV, the full Yukawa Lagrangian is given by [3]

$$\begin{aligned}
-\mathcal{L}^q_{\text{Yuk}} &= y^u_{ij}\bar{Q}_{iL}\widetilde{\Phi}_\rho p_{jR} + y^u_{1j}\bar{D}_L\rho^{Q^B} p_{jR} + y^u_{2j}\bar{S}_L\rho^{Q^B} p_{jR} + y^u_{3j}\bar{Q}_{3L}\widetilde{\Phi}_\eta p_{jR} + y^u_{3j}\bar{T}_L\eta^{-Q^A} p_{jR} \\
&+ y^d_{ij}\bar{Q}_{iL}\Phi_\eta n_{jR} + y^d_{1j}\bar{D}_L\eta^{Q^A} n_{jR} + y^d_{2j}\bar{S}_L\eta^{Q^A} n_{jR} + y^d_{3j}\bar{Q}_{3L}\Phi_\rho n_{jR} + y^d_{3j}\bar{T}_L\rho^{-Q^B} n_{jR} \\
&+ \frac{1}{\sqrt{2}}(y^D_1\bar{D}_L h_\chi D_R + y^D_2\bar{S}_L h_\chi D_R + y^S_1\bar{D}_L h_\chi S_R + y^S_2\bar{S}_L h_\chi S_R + y^T\bar{T}_L h_\chi T_R \\
&+ y^D_1 v_\chi\bar{D}_L D_R + y^D_2 v_\chi\bar{S}_L D_R + y^S_1 v_\chi\bar{D}_L S_R + y^S_2 v_\chi\bar{S}_L S_R + y^T v_\chi\bar{T}_L T_R) + \text{h.c.}\,,
\end{aligned} \tag{3.59}$$

$$-\mathcal{L}^l_{\text{Yuk}} = y^e_{mn}\bar{l}_{mL}\Phi_\rho e_{nR} + y^e_{mn}\bar{E}_{mL}\rho^{-B}e_{nR} + \frac{1}{\sqrt{2}}\left(y^E_{mn}\bar{E}_{mL}h_\chi E_{nR} + y^E_{mn}v_\chi\bar{E}_{mL}E_{nR}\right) + \text{h.c.}\,. \tag{3.60}$$

The masses of the extra fermions will be at the scale v_χ, $i.e.$, heavy if we consider that the Yukawa coupling constants (y^D, y^S, y^T, y^E) are of the order of 1. Consequently, they can be integrated out of the theory. So the 331 EFT Yukawa sector is simply given by [3]

$$\begin{aligned}
-\mathcal{L}^q_{\text{Yuk}} &= y^u_{1j}\left(\bar{p}_{1L}\rho^{0*} - \bar{n}_{1L}\rho^-\right)p_{jR} + y^u_{2j}\left(\bar{p}_{2L}\rho^{0*} - \bar{n}_{2L}\rho^-\right)p_{jR} \\
&+ y^u_{3j}\left(\bar{p}_{3L}\eta^{0*} - \bar{n}_{3L}\eta^-\right)p_{jR} + y^d_{1j}\left(\bar{p}_{1L}\eta^+ + \bar{n}_{1L}\eta^0\right)n_{jR} \\
&+ y^d_{2j}\left(\bar{p}_{2L}\eta^+ + \bar{n}_{2L}\eta^0\right)n_{jR} + y^d_{3j}\left(\bar{p}_{3L}\rho^+ + \bar{n}_{3L}\rho^0\right)n_{jR} + \text{h.c.}
\end{aligned} \tag{3.61}$$

$$-\mathcal{L}^l_{\text{Yuk}} = y^e_{mn}\left(\bar{e}_{mL}\rho^0 + \bar{\nu}_{mL}\rho^+\right)e_{nR} + \text{h.c.} \tag{3.62}$$

For simplicity, considering only the third family case, we obtain [3]

$$-\mathcal{L}^q_{\text{Yuk}} \supset y^u_{33}\left(\bar{p}_{3L}\eta^{0*} - \bar{n}_{3L}\eta^-\right)p_{3R} + y^d_{33}\left(\bar{p}_{3L}\rho^+ + \bar{n}_{3L}\rho^0\right)n_{3R} + \text{h.c.} \tag{3.63}$$

$$-\mathcal{L}^l_{\text{Yuk}} \supset y^e_{33}\left(\bar{e}_{3L}\rho^0 + \bar{\nu}_{3L}\rho^+\right)e_{3R} + \text{h.c.} \tag{3.64}$$

Now, if we consider $\Phi_1 \equiv \Phi_\rho$ and $\Phi_2 \equiv \Phi_\eta$, then we can compactify Eq. (3.61), so that it can be connected to the conventional 2HDM, as

$$-\mathcal{L}^q_y = \sum_{m=1}^{2}\left[\sum_{i,j=1}^{3}\left\{\bar{Q}_{iL}\left(\Gamma_m\right)_{ij}\left(n_R\right)_j\Phi_m + \bar{Q}_{iL}\left(\Delta_m\right)_{ij}\left(p_R\right)_j\widetilde{\Phi}_m\right\}\right]\,, \tag{3.65}$$

where Γ_m and Δ_m are the Yukawa matrices, and the explicit form of these matrices is given below,

$$\Gamma_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y^d_{31} & y^d_{32} & y^d_{33} \end{bmatrix} \qquad \Gamma_2 = \begin{bmatrix} y^d_{11} & y^d_{12} & y^d_{13} \\ y^d_{21} & y^d_{22} & y^d_{23} \\ 0 & 0 & 0 \end{bmatrix} \tag{3.66}$$

and

$$\Delta_1 = \begin{bmatrix} y^u_{11} & y^u_{12} & y^u_{13} \\ y^u_{21} & y^u_{22} & y^u_{23} \\ 0 & 0 & 0 \end{bmatrix} \qquad \Delta_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y^u_{31} & y^u_{32} & y^u_{33} \end{bmatrix} \tag{3.67}$$

After spontaneous symmetry breaking, all the doublets acquire real VEVs and are expanded as [5]

$$\phi_k = \begin{pmatrix} w^+_k \\ \frac{1}{\sqrt{2}}(v_k + h_k + iz_k) \end{pmatrix}\,, \qquad (k = 1, 2), \tag{3.68}$$

where v_k and h_k represent the VEVs of each doublet, which satisfy $v_1^2 + v_2^2 = v^2$, v is the usual SM VEV, and CP-even scalar fields, respectively. The up and down quark mass matrices are given by

$$M_p = \frac{1}{\sqrt{2}}(\Delta_1 v_1 + \Delta_2 v_2)\,, \quad M_n = \frac{1}{\sqrt{2}}(\Gamma_1 v_1 + \Gamma_2 v_2)\,, \tag{3.69}$$

the eigenvalues of which will be the physical quark masses. These mass matrices can be bidiagonalized to obtain the masses of the usual SM physical quarks. And is given by

$$D_u = V_L^\dagger M_p V_R = \text{diag}\{m_u, m_c, m_t\}, \quad D_d = U_L^\dagger M_n U_R = \text{diag}\{m_d, m_s, m_b\}\,, \tag{3.70}$$

where m_x are the physical quark masses, whereas V and U are $\mathrm{U}(3)$ matrices. These matrices relate the physical quark states u and d to the original p and n states in the following manner,

$$\begin{aligned} p_L &= V_L u_L\,, \qquad p_R = V_R u_R\,, \\ n_L &= U_L d_L\,, \qquad n_R = U_R d_R\,. \end{aligned} \tag{3.71}$$

Then the CKM matrix is obtained as

$$V = V_L^\dagger U_L\,. \tag{3.72}$$

The explicit form of the CKM matrix in the Wolfenstein parameters is given by[14]

$$V = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}. \tag{3.73}$$

We can parameterize the VEVs as

$$v_1 = v\cos\beta\,, \qquad v_2 = v\sin\beta\,, \tag{3.74}$$

and define the following orthogonal matrix, which rotates the gauge eigenstates into the so-called Higgs basis, which will simplify our analysis, as follows:

$$\mathcal{O}(\beta) = \begin{pmatrix} \frac{v_1}{v} & \frac{v_2}{v} \\ -\frac{v_2}{v} & \frac{v_1}{v} \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}, \tag{3.75}$$

and the change of basis is given by

$$\begin{pmatrix} H^0 \\ H' \end{pmatrix} = \mathcal{O}(\beta)\cdot\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}. \tag{3.76}$$

The physical CP even scalars, $h,\ H^1$, are obtained via a different orthogonal rotation,

$$\begin{pmatrix} h \\ H_1 \end{pmatrix} = \mathcal{O}(\alpha)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\cdot\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}. \tag{3.77}$$

Therefore, the mass matrix for the CP-even scalar should be diagonalized via the following transformation,

$$\mathcal{O}(\alpha)\cdot M_{h_k}^2\cdot\mathcal{O}^{\mathrm{T}}(\alpha) \equiv \begin{pmatrix} m_h^2 & 0 \\ 0 & m_{H_1}^2 \end{pmatrix}, \tag{3.78}$$

where the first eigenvalue corresponds to the SM-like Higgs boson state h.

At this point, we can discuss the alignment condition in our model. To study the alignment condition, we need to write down the mass eigenstates in terms of the Higgs basis as

$$\begin{pmatrix} h \\ H_1 \end{pmatrix} = \mathcal{O}(\alpha)\mathcal{O}^{\mathrm{T}}(\beta) \begin{pmatrix} H^0 \\ H' \end{pmatrix} , \tag{3.79}$$

where one can define the matrix

$$\mathcal{O} \equiv \mathcal{O}(\alpha)\mathcal{O}^{\mathrm{T}}(\beta) . \tag{3.80}$$

The alignment limit is achieved once the SM-like Higgs boson completely overlaps with H^0, which in practice, results in the condition

$$\mathcal{O}_{11} = 1 . \tag{3.81}$$

With our VEV parameterization in Eq. (3.74), the alignment condition can be writtten down as

$$\cos(\alpha - \beta) = 1 \Rightarrow \alpha = \beta . \tag{3.82}$$

Let us now carefully analyze the Yukawa couplings between the neutral scalar eigenstates and the physical quarks, with particular attention to any FCNC couplings which may arise.

Now, the terms in the Yukawa Lagrangian containing the interactions between CP-even scalars and quarks are

$$\mathcal{L}_Y^{CP\ even} = -\frac{1}{\sqrt{2}} \left[\bar{n}_L \left(\sum_{k=1}^{2} \Gamma_k h_k \right) n_R + \bar{p}_L \left(\sum_{k=1}^{2} \Delta_k h_k \right) p_R + \text{h.c.} \right] , \tag{3.83}$$

from which, using the rotation matrix of Eq. (3.76) to express the h_k in terms of H^0, we can obtain

$$\begin{aligned} \mathcal{L}_Y^{CP\ even} &= -\frac{H^0}{v} \left[\bar{n}_L \left(\frac{1}{\sqrt{2}} \sum_{k=1}^{2} \Gamma_k h_k \right) n_R + \bar{p}_L \left(\frac{1}{\sqrt{2}} \sum_{k=1}^{2} \Delta_k h_k \right) p_R + \text{h.c.} \right] \\ &= -\frac{H^0}{v} \left[\bar{n}_L M_n n_R + \bar{p}_L M_p p_R + \text{h.c.} \right] \\ &= -\frac{H^0}{v} \left[\bar{d}_L D_d d_R + \bar{u}_L D_u u_R + \text{h.c.} \right] . \end{aligned} \tag{3.84}$$

In writing the last step, we use the Eq. (3.70) and Eq. (3.71). Thus, we see that H^0 possesses a Yukawa coupling similar to the SM at tree level. Similarly, we can write down the Yukawa couplings of H' with the down type and up type quarks as follows:

$$\begin{aligned} \mathcal{L}_Y^{H'} =& -\frac{H'}{v} \left[\bar{n}_L \left(\frac{1}{\sqrt{2}} (\Gamma_1 v_2 - \Gamma_2 v_1) \right) n_R + \bar{p}_L \left(\frac{1}{\sqrt{2}} (\Delta_1 v_2 - \Delta_2 v_1) \right) p_R + h.c. \right] \\ =& -\frac{H'}{v} \left[\bar{d}_L N_d d_R + \bar{u}_L N_u u_R + h.c. \right] , \end{aligned} \tag{3.85}$$

where the matrices N_d and N_u are given by

$$\begin{aligned} N_d &= \frac{1}{\sqrt{2}} U_L^\dagger \left(\Gamma_1 v_2 - \Gamma_2 v_1 \right) U_R , \\ N_u &= \frac{1}{\sqrt{2}} V_L^\dagger \left(\Delta_1 v_2 \Delta_2 v_1 \right) V_R . \end{aligned} \tag{3.86}$$

To simplify further the expressions for N_d and N_u, it is useful to define the following projection matrix

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.87}$$

Thus, given the structures of the Yukawa matrices, we obtain the following relations,

$$\begin{aligned} PM_p &= \frac{1}{\sqrt{2}}\Delta_2 v_2 & PM_n &= \frac{1}{\sqrt{2}}\Gamma_1 v_1 \\ \Rightarrow \Delta_2 &= \frac{\sqrt{2}PM_p}{v_2} & \Rightarrow \Gamma_1 &= \frac{\sqrt{2}PM_n}{v_1} \\ \Rightarrow \Delta_2 &= \frac{\sqrt{2}PV_L D_u V_R^\dagger}{v_2} & \Rightarrow \Gamma_1 &= \frac{\sqrt{2}PU_L D_d U_R^\dagger}{v_1} . \end{aligned} \tag{3.88}$$

Similarly, we find

$$\Delta_1 = \frac{\sqrt{2}(I-P)V_L D_u V_R^\dagger}{v_1} \qquad \Gamma_2 = \frac{\sqrt{2}(I-P)U_L D_d U_R^\dagger}{v_2} . \tag{3.89}$$

Using Eqs. (3.88) and (3.89), the expression for N_u and N_d can be simplified, the element-wise form of N_u and N_d can be written as

$$(N_u)_{AB} = \frac{v_2}{v_1}(D_u)_{BB}\delta_{AB} - \left(\frac{v_2}{v_1} + \frac{v_1}{v_2}\right)(V_L^*)_{3A}(V_L)_{3B}(D_u)_{BB} , \tag{3.90}$$

$$(N_d)_{AB} = -\frac{v_1}{v_2}(D_d)_{BB}\delta_{AB} + \left(\frac{v_1}{v_2} + \frac{v_2}{v_1}\right)(D_d)_{BB}\sum_{i=1}^{3}\sum_{j=1}^{3}(V_L)^*_{3i}(V)^*_{iA}(V_L)_{3j}(V)_{jB} . \tag{3.91}$$

Where, in the last step, we have made use of the Eq. (3.72). The above two equations tell us that the FCNC interaction of H' is present in this model at the tree level.

One of the major challenges in constructing Beyond the Standard Model (BSM) theories with flavor-changing neutral currents (FCNCs) present at the tree level in the Yukawa sector lies in achieving a consistent fit to the observed quark mass spectrum and the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix simultaneously. This task is particularly demanding due to the hierarchical nature of the quark masses, which span several orders of magnitude. Achieving a successful fit to both sectors—quark and scalar—within a given BSM framework thus becomes a highly non-trivial and computationally intensive.

To address this issue, we adopt an inversion procedure, following the approach described in Ref. [5]. In this framework, the strategy is reversed compared to the conventional method: instead of specifying the Yukawa couplings as input parameters, we take the quark masses and CKM matrix elements as our input parameters. The goal is then to express the parameters of the BSM in terms of these physical observables and mixing angles connecting the gauge basis with the physical basis.

To make this procedure clear, we begin with the diagonal quark mass matrices, D_u and D_d, as defined in Eq. (3.70). Utilizing the unitarity of the rotation matrices, these relations can be formally inverted. Furthermore, given the definition of the CKM matrix, which implies

$U_L = V_L V$, we can write,

$$M_p = V_L D_u V_R^\dagger = \begin{pmatrix} (\Delta_1)_{11}v_1 & (\Delta_1)_{12}v_1 & (\Delta_1)_{13}v_1 \\ (\Delta_1)_{21}v_1 & (\Delta_1)_{22}v_1 & (\Delta_1)_{23}v_1 \\ (\Delta_2)_{31}v_2 & (\Delta_2)_{32}v_2 & (\Delta_2)_{33}v_2 \end{pmatrix} ,$$
$$M_n = V_L V D_d U_R^\dagger = \begin{pmatrix} (\Gamma_2)_{11}v_2 & (\Gamma_2)_{12}v_2 & (\Gamma_2)_{13}v_2 \\ (\Gamma_2)_{21}v_2 & (\Gamma_2)_{22}v_2 & (\Gamma_2)_{23}v_2 \\ (\Gamma_1)_{31}v_1 & (\Gamma_1)_{32}v_1 & (\Gamma_1)_{33}v_1 \end{pmatrix} , \tag{3.92}$$

where U_L has been replaced by V_L and the CKM matrix. In Eq. (3.92), the quark mass eigenvalues, the CKM matrix, and the scalar vacuum expectation values (VEVs) are treated as input parameters. The unknowns in this formulation are the rotation matrices V_L, V_R, and U_R, which relate the gauge basis to the physical mass basis. Each of these rotation matrices belongs to the unitary group $U(3)$, and thus can, in general, be parameterized by three mixing angles and six complex phases. This leads to a total of 27 free parameters across the three matrices, which one would typically attempt to fit to Eq. (3.92).

However, within the inversion framework adopted here, no explicit parameter fitting is required. Instead, the Yukawa textures allow us to treat the rotation matrices as generic 3×3 unitary matrices. To facilitate this, we adopt a parametrization of the rotation matrices in terms of nine real mixing angles:

$$\gamma_L^1 \, , \gamma_L^2 \, , \gamma_L^3 \, , \gamma_R^1 \, , \gamma_R^2 \, , \gamma_R^3 \, , \beta_R^1 \, , \beta_R^2 \, , \beta_R^3$$

corresponding to the three rotation angles for each of the left-handed and right-handed up and down sectors. This parametrization is sufficiently general for the systematic reconstruction of the Yukawa matrices from physical observables.

$$V_L = \begin{pmatrix} \cos\gamma_L^1\cos\gamma_L^2 & \sin\gamma_L^1\cos\gamma_L^2 & \sin\gamma_L^2 \\ -\cos\gamma_L^1\sin\gamma_L^2\sin\gamma_L^3 - \sin\gamma_L^1\cos\gamma_L^3 & \cos\gamma_L^1\cos\gamma_L^3 - \sin\gamma_L^1\sin\gamma_L^2\sin\gamma_L^3 & \cos\gamma_L^2\sin\gamma_L^3 \\ -\cos\gamma_L^1\sin\gamma_L^2\cos\gamma_L^3 + \sin\gamma_L^1\sin\gamma_L^3 & -\cos\gamma_L^1\sin\gamma_L^3 - \sin\gamma_L^1\sin\gamma_L^2\cos\gamma_L^3 & \cos\gamma_L^2\cos\gamma_L^3 \end{pmatrix} , \tag{3.93a}$$

$$V_R = \begin{pmatrix} \cos\gamma_R^1\cos\gamma_R^2 & \sin\gamma_R^1\cos\gamma_R^2 & \sin\gamma_R^2 \\ -\cos\gamma_R^1\sin\gamma_R^2\sin\gamma_R^3 - \sin\gamma_R^1\cos\gamma_R^3 & \cos\gamma_R^1\cos\gamma_R^3 - \sin\gamma_R^1\sin\gamma_R^2\sin\gamma_R^3 & \cos\gamma_R^2\sin\gamma_R^3 \\ -\cos\gamma_R^1\sin\gamma_R^2\cos\gamma_R^3 + \sin\gamma_R^1\sin\gamma_R^3 & -\cos\gamma_R^1\sin\gamma_R^3 - \sin\gamma_R^1\sin\gamma_R^2\cos\gamma_R^3 & \cos\gamma_R^2\cos\gamma_R^3 \end{pmatrix} , \tag{3.93b}$$

$$U_R = \begin{pmatrix} \cos\beta_R^1\cos\beta_R^2 & \sin\beta_R^1\cos\beta_R^2 & \sin\beta_R^2 \\ -\cos\beta_R^1\sin\beta_R^2\sin\beta_R^3 - \sin\beta_R^1\cos\beta_R^3 & \cos\beta_R^1\cos\beta_R^3 - \sin\beta_R^1\sin\beta_R^2\sin\beta_R^3 & \cos\beta_R^2\sin\beta_R^3 \\ -\cos\beta_R^1\sin\beta_R^2\cos\beta_R^3 + \sin\beta_R^1\sin\beta_R^3 & -\cos\beta_R^1\sin\beta_R^3 - \sin\beta_R^1\sin\beta_R^2\cos\beta_R^3 & \cos\beta_R^2\cos\beta_R^3 \end{pmatrix} , \tag{3.93c}$$

All in all, given the physical quark masses and CKM mixing (with a complex CP-phase), the nine angles with the VEVs, we can reconstruct the elements of the Γ and Δ matrices.

3.5 Conclusion and Scope of future work

3.5.1 Conclusion

In this thesis, I have calculated the masses of both the charged and neutral gauge bosons. After that, I talked about the Scalar sector of 331-EFT, which reduces to the Scalar sector of 2HDM. From which we connected the 2HDM parameters to 331 parameters. Lastly, I talked about the Yukawa sector of the 331-EFT model, from which I calculated the FCNC couplings.

3.5.2 Future plans

- Finding out the constraint on the parameters from the FCNC couplings.
- Study the more trinification model, which is based on $\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(3)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{R}$.

Appendix A

Noether's Theorem

Noether's theorem states that for every continuous symmetry of a physical system, there is a corresponding conservation law. [4]

Let us consider a dynamical system described by the action

$$S = \int d^4x \, \mathcal{L} \, , \tag{A.1}$$

where we assume,

$$\mathcal{L} = \mathcal{L} \left(\phi(x), \partial_\mu \phi(x) \right) . \tag{A.2}$$

A general transformation can be written in the form

$$\begin{aligned} &x^\mu \to x'^\mu \\ &\phi(x) \to \phi'(x') \\ &\partial_\mu \phi(x) \to \partial'_\mu \phi'(x') \, . \end{aligned} \tag{A.3}$$

If the transformation described in Eq. (A.3) is a symmetry of the system then,

$$\begin{aligned} &\delta S = 0 \\ &\int d^4x' \; \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x')) - \int d^4x \; \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = 0 \\ &\int d^4x \; \mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) - \int d^4x \; \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = 0 \\ &\int d^4x \; \left(\mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) - \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \right) = 0 \, , \end{aligned} \tag{A.4}$$

where we change $x' \to x$ in the 3rd step. Then, we have from Eq. (A.4),

$$\mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) - \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = \partial_\mu K^\mu \, . \tag{A.5}$$

This must hold independent of the use of the equations of motion. For internal symmetry transformation, $K^\mu = 0$.

The infinitesimal change in the field variable is

$$\phi'(x) - \phi(x) = \delta\phi(x), \tag{A.6}$$

so that,

$$\delta \left(\partial_\mu \phi(x) \right) = \partial_\mu \phi'(x) - \partial_\mu \phi(x) = \partial_\mu \partial \phi(x) \, . \tag{A.7}$$

We can calculate Eq. (A.4) explicitly, keeping terms linear in $\delta\phi(x)$,

$$
\begin{aligned}
&\mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) - \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \\
&= \mathcal{L}(\phi(x), \partial_\mu \phi(x)) + \delta\phi(x) \frac{\partial \mathcal{L}}{\partial \phi(x)} + \delta(\partial_\mu \phi(x)) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} - \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \\
&= \delta\phi(x) \frac{\partial \mathcal{L}}{\partial \phi(x)} + \delta(\partial_\mu \phi(x)) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} \\
&= \delta\phi(x) \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} + \delta(\partial_\mu \phi(x)) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} \\
&= \partial_\mu \left(\delta\phi(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} \right) . \qquad \text{(A.8)}
\end{aligned}
$$

Comparing Eqs. (A.4) and (A.8), we have,

$$
\begin{aligned}
&\partial_\mu \left(\delta\phi(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} \right) = \partial_\mu K^\mu \\
&\partial_\mu \left(\delta\phi(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} - K^\mu \right) = 0 \\
&\partial_\mu (J^\mu(x)) = 0 \, . \qquad \text{(A.9)}
\end{aligned}
$$

This shows that for a continuous symmetry associated with a system, we can define a conserved current,

$$
J^\mu(x) = \delta\phi(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} - K^\mu \, , \qquad \text{(A.10)}
$$

and the charge is defined as,

$$
Q = \int d^3x \; J^0(x) \qquad \text{(A.11)}
$$

Appendix B

SU(3)

SU(3) is one of the most important algebras in particle physics. It is the group of unitary matrices of determinant 1(where U stands for 'unitary' and 's' stands for 'special'(meaning determinant 1), whose lowest order irreducible representation is 3×3 matrices.

B.1 Generators of $\mathrm{SU}(3)$

$\mathrm{SU}(3)$ is generated by 3×3 traceless hermitian matrices. We know that the exponent of a hermitian matrix is always a unitary matrix. So we will get the unitary matrices corresponding to $\mathrm{SU}(3)$ by exponentiating the hermitian generators. Let us represent them as [6],

$$U(\theta) = e^{iT^a\theta^a}\,, \tag{B.1}$$

where T^a's are the hermitian generators of the $\mathrm{SU}(3)$ group and θ^a's are the arbitrary parameters of rotation. Now to see the traceless property, we know a matrix property which goes as, if we have a matrix represented as $U = e^A$, then,

$$\det(U) = \det(e^A) = e^{\mathrm{Tr}(A)}\,, \tag{B.2}$$

and the determinant to be 1, from Eq. (B.2), we have $\mathrm{Tr}(A) = 0$, $i.e.$, the generators, namely T^a's, are Traceless.

The standard basis for the hermitian 3×3 matrices in the physics literature is in terms of a generalization of the Pauli matrices, called the Gell-Mann matrices;

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\lambda_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \lambda_6 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}$$

$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\,.$$

The generators of the group are represented as $T_i = \frac{\lambda_i}{2}$. Some of the basic properties of the generators are given below,

$$[T_a, T_b] = i f_{abc} T_c \,, \tag{B.3a}$$

$$\mathrm{Tr}\,(T_a T_b) = \frac{1}{2}\delta_{ab} \,, \tag{B.3b}$$

$$f_{abc} = -2i \mathrm{Tr}\left\{T_c\left[T_a, T_b\right]\right\} \,, \tag{B.3c}$$

$$\{T_a, T_b\} = \frac{1}{3}\delta_{ab} I + d_{abc} T_c \,. \tag{B.3d}$$

From Eq. (B.3c) and from Eq. (B.3d), we find the structure constant, *i.e.*, f^{abc} and the d^{abc}, respectively. For example, let us find out the structure constant, f^{458}, using Eq. (B.3a),

$$\begin{aligned}
[T_4, T_5] &= \frac{1}{4}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{bmatrix} - \frac{1}{4}\begin{bmatrix}0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}\\
&= \frac{1}{4}\begin{bmatrix}i & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -i\end{bmatrix} - \frac{1}{4}\begin{bmatrix}-i & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & i\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix}i & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -i\end{bmatrix} \,,
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
T_8[T_4, T_5] &= \frac{1}{4\sqrt{3}}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}\begin{bmatrix}i & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -i\end{bmatrix}\\
&= \frac{i}{4\sqrt{3}}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2\end{bmatrix} \,.
\end{aligned} \tag{B.5}$$

Now from Eq. (B.3c), we have,

$$\begin{aligned}
f_{458} &= -2i \mathrm{Tr}\left\{T_8\left[T_4, T_5\right]\right\}\\
&= -2i \times \frac{i}{4\sqrt{3}} \times 3\\
&= \frac{\sqrt{3}}{2} \,.
\end{aligned} \tag{B.6}$$

The other structure constants are listed below [13],

$$\begin{gathered}
f_{123} = 1\\
f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2}\\
f_{458} = f_{678} = \frac{\sqrt{3}}{2} \,.
\end{gathered} \tag{B.7}$$

All the other structure constant vanishes. Now lets us calculate the d_{abc}'s,

$$\begin{aligned}
\{T_1, T_1\} &= \frac{1}{3}\delta_{11} I + d_{11c} T_c \\
2T_1^2 &= \frac{1}{3} I + \sum_c d_{11c} T_c \,, \\
\sum_c d_{11c} T_c &= \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} - \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \\
\sum_c d_{11c} T_c &= \frac{2\sqrt{3}}{6} T_8 \,.
\end{aligned} \tag{B.8}$$

On comparing we get $d_{118} = \frac{1}{\sqrt{3}}$ and all the other terms are zero.

All the non zero d_{abc}'s are listed below [13],

abc	d_{abc}
118	$1/\sqrt{3}$
146	$1/2$
157	$1/2$
228	$1/\sqrt{3}$
247	$-1/2$
256	$1/2$
338	$1/\sqrt{3}$
344	$1/2$

abc	d_{abc}
355	$1/2$
366	$-1/2$
377	$-1/2$
448	$-1/(2\sqrt{3})$
558	$-1/(2\sqrt{3})$
668	$-1/(2\sqrt{3})$
778	$-1/(2\sqrt{3})$
888	$-1/\sqrt{3}$

Table B.1: Tables containing values of d_{abc} for different combinations of indices abc.

B.2 Ladder Operator

Now let us construct a ladder operator for $\mathrm{SU}(3)$ in the light of flavor physics. Let us recall from the SU(2) case the that the ladder operators are $J_\pm$, which follows the algebra, where we diagonalised J_3,

$$[J_3, J_\pm] = \pm J_\pm \,, \qquad and \qquad [J_+, J_-] = J_3 \,. \tag{B.9}$$

We wrote in the eigenbasis of J_3,

$$J_3 \left|m\right\rangle = m\left|m\right\rangle ; \qquad J_+\left|m\right\rangle = c_{m+1}\left|m+1\right\rangle ; \qquad J_-\left|m\right\rangle = c_{m-1}\left|m-1\right\rangle \,, \tag{B.10}$$

where c_m is the normalization constant. We can then think of the states as the rung of the ladder, and the $J_\pm$ operator enables us to climb up or down the ladder. And acting the $J_\pm$ operator on the states beyond the maximum and minimum values of 'm' will give zero, that is the ladder terminates at these values.

From the explicit form of the λ_3 and λ_8, they are diagonal matrices, so we see that the generators T_3 and T_8 commute, $i.e$ $[T_3, T_8] = 0$. So we can simultaneously diagonalise them. From the flavor physics aspect, we take $T_3 \equiv I_3$, the third component of isospin, and

$T_8 = \frac{\sqrt{3}}{2}Y$, as the hypercharge. We will use the eigenstates of I_3 and Y to represent the algebra of SU(3), i.e. the kets $|i_3, y\rangle$ such that $I_3 |i_3, y\rangle = i_3 |i_3, y\rangle$ and $Y |i_3, y\rangle = y |i_3, y\rangle$. As SU(3) has 8 generators, the other 6 generators other than I_3 and Y are defined as,

$$I_\pm = T_1 \pm T_2 \,, \tag{B.11a}$$
$$U_\pm = T_6 \pm T_7 \,, \tag{B.11b}$$
$$V_\pm = T_4 \pm T_5 \,. \tag{B.11c}$$

In particle physics, we refer to the $\mathrm{SU}(2)$ subalgebra generated by $I_\pm$, $U_\pm$ and $V_\pm$, as I-spin, U-spin and V-spin, respectively. Some commutation relations between the generators are given below [13].

$$[I_3, I_\pm] = \pm I_\pm; \qquad [I_3, U_\pm] = \mp\frac{1}{2}U_\pm; \qquad [I_3, V_\pm] = \pm\frac{1}{2}V_\pm; \tag{B.12a}$$
$$[Y, I_\pm] = 0; \qquad [Y, U_\pm] = \pm U_\pm; \qquad [Y, V_\pm] = \pm V_\pm; \tag{B.12b}$$
$$[I_+, I_-] = 2I_3; \qquad [U_+, U_-] = 2U_3; \qquad [V_+, V_-] = 2V_3; \tag{B.12c}$$

where we defined $U_3 = \frac{\sqrt{3}}{2}T_8 - \frac{1}{2}T_3$ and $V_3 = \frac{\sqrt{3}}{2}T_8 + \frac{1}{2}T_3$. Another set of important commutation relations is as follows,

$$[T_1, T_2] = 2iI_3; \qquad [T_2, I_3] = 2iT_1; \qquad [I_3, T_1] = 2iT_2; \tag{B.13a}$$
$$[T_6, T_7] = 2iU_3; \qquad [T_7, U_3] = 2iT_6; \qquad [U_3, T_6] = 2iT_7; \tag{B.13b}$$
$$[T_4, T_5] = 2iV_3; \qquad [T_5, V_3] = 2iT_4; \qquad [V_3, T_4] = 2iT_5 \,. \tag{B.13c}$$

From the above commutation relations, we can see that they individually follow an SU(2) algebra. T_1, T_2 with I_3 lives in the first two sector, generate the SU(2) algebra. Similarly, T_4, T_5 with V_3 lives in the 1-3 sector, generating an SU(2) algebra and T_6, T_7 with U_3 lives in the 2-3 sector, generating an SU(2) algebra. From this, we conclude that SU(3) contains three overlapping SU(2) subalgebras.

Let us see action of the $I_\pm, U_\pm$ and $V_\pm$ on the state $|i_3, y\rangle$, from SU(2) algebra we know,

$$I_\pm |i_3, y\rangle = c_\pm |i_3 \pm 1, y\rangle \,. \tag{B.14}$$

Now lets see the action of $U_\pm$ on the state $|i_3, y\rangle$, let us consider $U_\pm |i_3, y\rangle = |i_3', y'\rangle$, so we have,

$$\begin{aligned} I_3 U_\pm |i_3, y\rangle &= \left(U_\pm I_3 \mp \frac{1}{2}U_\pm\right)|i_3, y\rangle \\ &= i_3 U_\pm |i_3, y\rangle \mp \frac{1}{2}U_\pm |i_3, y\rangle \\ &= \left(i_3 \mp \frac{1}{2}\right)(U_\pm |i_3, y\rangle) \\ I_3 |i_3', y'\rangle &= \left(i_3 \mp \frac{1}{2}\right)|i_3', y'\rangle \,, \end{aligned} \tag{B.15}$$

so, we find $i_3' = i_3 \mp \frac{1}{2}$, for the y values,

$$\begin{aligned} YU_\pm |i_3, y\rangle &= (U_\pm Y \pm U_\pm)|i_3, y\rangle \\ &= yU_\pm |i_3, y\rangle \pm U_\pm |i_3, y\rangle \\ &= (y \pm 1)(U_\pm |i_3, y\rangle) \\ Y |i_3', y'\rangle &= (y \pm 1)|i_3', y'\rangle \,. \end{aligned} \tag{B.16}$$

So, we conclude that,

$$U_{\pm}\left|i_3, y\right\rangle \propto \left|i_3 \mp 1/2, y \pm 1\right\rangle \ . \tag{B.17}$$

Similarly, we can show that,

$$V_{\pm}\left|i_3, y\right\rangle \propto \left|i_3 \pm 1/2, y \pm 1\right\rangle \ . \tag{B.18}$$

Now we see that the operators $I_{\pm}, U_{\pm}$, and $V_{\pm}$ move us around a 2-D plane with I_3 as the x-axis and Y as the y-axis. But in the I_3 and T_8 plane,

$$I_{\pm} \text{ moves us by } (\pm 1, 0)$$
$$U_{\pm} \text{ moves us by } \left(\mp\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$$
$$V_{\pm} \text{ moves us by } \left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$$

Appendix C

Charge Conjugation

C.1 Charge conjugation operator

The Dirac equation for a charged particle interacting with electromagnetic fields is,

$$(i\gamma^\mu D_\mu - m)\psi = 0\,, \tag{C.1}$$

where D_μ is the covariant derivative given by,

$$D_\mu = \partial_\mu + iqA_\mu\,, \tag{C.2}$$

with q being the charge of the particle. So for electrons $(q = -e)$, represented by 'ψ', Eq. (C.1) becomes,

$$(i\gamma^\mu \partial_\mu - m + e\gamma^\mu A_\mu)\psi = 0\,, \tag{C.3}$$

and for the positron $(q = +e)$, represented by ψ^c becomes,

$$(i\gamma^\mu \partial_\mu - m - e\gamma^\mu A_\mu)\psi^c = 0\,. \tag{C.4}$$

We want to obtain a 'charge conjugation' operator which transforms ψ to ψ^c. For this purpose, we take the complex conjugate of Eq. (C.3) to obtain,

$$\left[-i(\gamma^\mu)^* \partial_\mu - m + e(\gamma^\mu)^* A_\mu\right]\psi^* = 0\,. \tag{C.5}$$

Let us multiply the above equation by C'(combination of γ matrices) in such a way, so that

$$C'(\gamma^\mu)^* = -\gamma^\mu C'\,, \tag{C.6}$$

using this in Eq. (C.5), we get,

$$(i\gamma^\mu \partial_\mu - m - e\gamma^\mu A_\mu)(C'\psi^*) = 0\,. \tag{C.7}$$

So now we can see if ψ satisfies Eq. (C.3) then, $C'\psi^*$ satisfies the positron equation, $i.e.$, Eq. (C.4). Thus we identify,

$$\psi^c = C'\psi^*\,. \tag{C.8}$$

Let us define an operator 'C', which is related to C' as follows:

$$C' = C(\gamma^0)^{\mathrm{T}} = C(\gamma^0)^*\,, \tag{C.9}$$

where we have used the fact that $\gamma^{0\dagger} = \gamma^0 \implies (\gamma^0)^{\mathrm{T}} = (\gamma^0)^*$. Putting Eq. (C.9) in Eq. (C.8), we obtain the important transformation rule:

$$\psi^c = C(\gamma^0)^{\mathrm{T}}\psi^* = C\bar{\psi}^{\mathrm{T}}\,. \tag{C.10}$$

The operator 'C' is called the 'charge-conjugation' operator.

C.2 Some properties of C operator

Using Eq. (C.9) in Eq. (C.6), we get,[9]

$$
\begin{aligned}
&(C\gamma^{0*})(\gamma^\mu)^* = -\gamma^\mu(C\gamma^{0*}) \\
\Longrightarrow\ &\gamma^{0*}\gamma^{\mu*} = -C^{-1}\gamma^\mu C\gamma^{0*} \\
\Longrightarrow\ &\gamma^{0*}\gamma^{\mu*}(\gamma^0)^{\mathrm{T}} = -C^{-1}\gamma^\mu C \\
\Longrightarrow\ &\gamma^{0*}\gamma^{\mu*}\gamma^{0*} = -C^{-1}\gamma^\mu C \\
\Longrightarrow\ &C^{-1}\gamma^\mu C = -(\gamma^\mu)^{\mathrm{T}} .
\end{aligned} \tag{C.11}
$$

Now, let us see the anti-commutation relation of $-(\gamma^\mu)^{\mathrm{T}}$,

$$
\begin{aligned}
&\left\{-(\gamma^\mu)^{\mathrm{T}}, -(\gamma^\nu)^{\mathrm{T}}\right\} \\
&= (\gamma^\mu)^{\mathrm{T}}(\gamma^\nu)^{\mathrm{T}} + (\gamma^\nu)^{\mathrm{T}}(\gamma^\mu)^{\mathrm{T}} \\
&= \{\gamma^\mu, \gamma^\nu\}^{\mathrm{T}} \\
&= 2g^{\mu\nu}\mathrm{I}^{\mathrm{T}} \\
&= 2g^{\mu\nu} .
\end{aligned} \tag{C.12}
$$

So, $-(\gamma^\mu)^{\mathrm{T}}$ satisfies the Clifford algebra as γ^μ does. By Pauli's theorem, they must be connected by a unitary transformation, $i.e.$,

$$
-(\gamma^\mu)^{\mathrm{T}} = U^\dagger\gamma^\mu U , \tag{C.13}
$$

comparing this with Eq. (C.11) we conclude that 'C' is unitary,$i.e.$,

$$
C^\dagger C = 1 . \tag{C.14}
$$

We expect the charge to flip sign under charge conjugation. Thus, we need,

$$
(\psi^c)^\dagger(\psi^c) = -\psi^\dagger\psi , \tag{C.15}
$$

the charge $\psi^\dagger\psi$ comes from the Noether's theorem, and it is the zeroth component of the conserved current. Writing $(\psi^c)^\dagger(\psi^c)$ using Eq. (C.10) in the explicit component form, we get,

$$
\begin{aligned}
(\psi^c)^\dagger(\psi^c) &= \psi^{c*}_{\ \alpha}\psi^c_\alpha \\
&= \left[C^*_{\alpha\sigma}\left(\gamma^{0\mathrm{T}}\right)^*_{\sigma\delta}\psi_\delta\right]\left[C_{\alpha\beta}\left(\gamma^{0\mathrm{T}}\right)_{\beta\gamma}\psi^*_\gamma\right] \\
&= C^*_{\alpha\sigma}\left(\gamma^0\right)^\dagger_{\sigma\delta}C_{\alpha\beta}\left(\gamma^0\right)_{\gamma\beta}\psi_\delta\psi^*_\gamma \\
&= \left(\gamma^0\right)_{\gamma\beta}C^{\mathrm{T}}_{\beta\alpha}C^*_{\alpha\sigma}\left(\gamma^0\right)^\dagger_{\sigma\delta}\left(-\psi^*_\gamma\psi_\delta\right) \\
&= -\left(\gamma^0C^{\mathrm{T}}C^*\gamma^{0\dagger}\right)_{\gamma\delta}\left(\psi^*_\gamma\psi_\delta\right) ,
\end{aligned} \tag{C.16}
$$

comparing the above equation with Eq. (C.15), we have,

$$
\begin{aligned}
&\gamma^0C^{\mathrm{T}}C^*\gamma^{0\dagger} = \mathrm{I} \\
\Longrightarrow\ &C^{\mathrm{T}}C^* = \mathrm{I} \\
\Longrightarrow\ &C^\dagger C = \mathrm{I} ,
\end{aligned} \tag{C.17}
$$

In the last step, we take the complex conjugate. Now, if we apply the charge conjugation twice, we expect to get back the original field, $i.e.$,

$$\begin{aligned} & (\psi^c)^c = \psi \\ \Longrightarrow\ & C\gamma^{0*}\psi^{C*} = \psi \\ \Longrightarrow\ & C\gamma^{0*}(C\gamma^{0*}\psi^*)^* = \psi \\ \Longrightarrow\ & C\gamma^{0*}C^*\gamma^0\psi = \psi\,. \end{aligned} \tag{C.18}$$

From the above equation, we see that

$$\begin{aligned} & C\gamma^{0*}C^*\gamma^0 = \mathrm{I} \\ \Longrightarrow\ & C\gamma^{0\mathrm{T}}C^*\gamma^0 = \mathrm{I} \\ \Longrightarrow\ & -C(C^{-1}\gamma^0 C)C^*\gamma^0 = \mathrm{I} \\ \Longrightarrow\ & -\gamma^0 CC^*\gamma^0 = \mathrm{I} \\ \Longrightarrow\ & CC^* = -\mathrm{I} \\ \Longrightarrow\ & C^{-1} = -C^*\,, \end{aligned} \tag{C.19}$$

combining Eq. (C.17) and Eq. (C.19), we conclude,

$$\begin{aligned} & C^{-1} = C^\dagger = -C^* \\ \Longrightarrow\ & C^{\mathrm{T}} = -C\,. \end{aligned} \tag{C.20}$$

Hence, ‘C’ is an anti-symmetric matrix. Now, since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, one can easily show, using Eq. (C.11),

$$C^{-1}\gamma^5 C = (\gamma^5)^{\mathrm{T}} = \gamma^{5*}\,. \tag{C.21}$$

Now

$$\begin{aligned} P_R\psi^c &= \frac{1}{2}(1+\gamma^5)(C\gamma^{0*}\psi^*) \\ &= \frac{1}{2}C(1+C^{-1}\gamma^5 C)\gamma^{0*}\psi^* \\ &= \frac{1}{2}C(1+\gamma^{5*})\gamma^{0*}\psi^* \\ &= \frac{1}{2}C\gamma^{0*}(1-\gamma^{5*})\psi^* \\ &= C\gamma^{0*}\left[\frac{1}{2}(1-\gamma^5)\psi\right]^* \\ &= C\gamma^{0*}\psi_L^*\,. \end{aligned} \tag{C.22}$$

Thus, we conclude:

$$\boxed{(\psi^c)_R = (\psi_L)^c}\,. \tag{C.23}$$

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