QUANTUM COSMOLOGY

M.Sc. Report

By

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DISCIPLINE OF PHYSICS

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **Quantum Cosmology** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DISCIPLINE OF PHYSICS**, Indian Institute of Technology Indore, is an authentic record of my own work carried out during the time period from **July 2023** to **May 2025** under the supervision of **Dr.Mritunjay Kumar Verma, Assistant Professor Department of Physics IIT Indore**.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Mobashshir Mahmood 20/05/2025

Signature of the student with date
(Mobashshir Mahmood)

....

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Mobashshir Mahmood has successfully given his M.Sc. Oral Examination held on ____________________.

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Signature of Supervisor of MSc thesis
Date: **21 May 2025**

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Date: 21-05-25

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Abstract

This thesis investigates how the universe could have originated from a quantum state, following the Hartle–Hawking no-boundary proposal. We begin by examining the breakdown of classical general relativity near singularities such as the Big Bang. To move beyond these limitations, we turn to quantum cosmology and study the Wheeler–DeWitt equation, which governs the wave function of the universe. We explore this wave function using the minisuperspace approximation, a simplified model that reduces the complex dynamics to a few key variables. Within this framework, a Euclidean path integral is used to describe how the universe may have "tunneled" into existence from a quantum gravitational state. Focusing on the semiclassical limit, where quantum behavior transitions to classical dynamics, we analyze how a smooth, expanding universe like ours could emerge from these quantum origins.

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Chapter 1

Introduction

Introduction

Historically, space and time were thought to be distinct and absolute entities. Time was assumed to flow uniformly for all observers, and distances were considered the same regardless of the observer's position or motion. This classical view changed fundamentally with the advent of Einstein's special theory of relativity and the Lorentz transformations. These developments showed that the passage of time and the measurement of space depend on the observer's frame of reference. In this unified framework, space and time became inseparable and formed what is now known as **spacetime**.

Spacetime is the modern stage on which all physical events occur. Each point in spacetime represents an event, characterized by both spatial position and time. While there are many models of spacetime consistent with physical observations, any such model must satisfy specific mathematical and physical conditions — known as spacetime axioms. In Section 2.2, we will examine these axioms in detail and explore the important concept of **global hyperbolicity**, which ensures that spacetime admits a well-defined causal structure and predictive dynamics.

From a physics standpoint, causality — the principle that an effect cannot precede its cause — is essential. In flat Minkowski spacetime, this idea is represented by light cones: an event can only be influenced by events inside its past light cone and can affect only those inside its future light cone. In curved Lorentzian manifolds, which general relativity uses to model spacetime, the causal structure becomes more complex, especially on a global scale. While local causality mimics that of flat spacetime, global features such as topology can alter causal relationships. Section 2.1 will explore these global causal structures.

To understand how spacetime behaves under extreme conditions, we turn to the **Raychaudhuri equation**, which describes how nearby geodesics (paths followed by freely falling particles) converge or diverge. This equation plays a central role in the **Hawking–Penrose singularity theorems**, which show that under physically reasonable conditions, classical general relativity predicts the formation of singularities — points where curvature becomes infinite. These singularities arise both in cosmology (as in the Big Bang) and in gravitational collapse (as in black holes). In Section 3, we will see how these results demonstrate the breakdown of classical spacetime and the need for a more fundamental theory.

This brings us to one of the deepest questions in physics: **how did the universe begin?** While general relativity has been extraordinarily successful in describing the large-scale evolution of the universe, it breaks down at the so-called Big Bang singularity. At that point, spacetime curvature becomes infinite, and the theory loses its predictive power. This clearly indicates that **classical general relativity is incomplete** when describing the universe's earliest moments. To go further, we must turn to a quantum theory of gravity.

Quantum cosmology is the field that seeks to apply the principles of quantum mechanics to the universe as a whole. One of its central goals is to define the *wave function of the universe*, which captures all possible configurations of spacetime and matter. A foundational framework here is the **Wheeler–DeWitt equation**, which generalizes the Schrödinger equation for gravitational systems. However, this equation is timeless and admits an infinite number of solutions. To identify the solution that actually describes our universe, additional boundary conditions are necessary.

A major step in this direction came from the **Hartle–Hawking no-**

boundary proposal. This idea suggests that the universe did not begin from a singular point in time, but from a smooth, compact Euclidean geometry — essentially a finite, boundaryless space that transitions into our expanding universe. The proposal interprets the universe as having "tunneled" into existence from a quantum gravitational state — a concept often referred to as the universe emerging from "nothing." This is formalized using a **Euclidean path integral**, which sums over all compact four-dimensional geometries matching a given final configuration.

To explore these ideas more concretely, we use the **minisuperspace approximation**, which simplifies the problem by assuming the universe is homogeneous and isotropic. This reduces the infinite degrees of freedom in general relativity to a manageable few — typically the *scale factor* $a(\tau)$ and a spatially homogeneous scalar field $\phi(\tau)$. Within this framework, the wave function of the universe can be expressed as a path integral over these variables, weighted by the Euclidean action.

A major focus of this thesis is on the **semiclassical approximation** of this path integral. In this regime, the path integral is dominated by *saddle points* — classical solutions of the Euclidean equations of motion. These saddle points provide insight into how the quantum universe transitions to a classical spacetime. Depending on the values of the scale factor and scalar potential $V(\phi)$, the wave function shows two distinct behaviors:

- In the **classically forbidden region** ($a^2V < 1$), the wave function is real and exponentially decaying or growing — indicating a tunneling process from a non-classical state.

- In the **classically allowed region** ($a^2V > 1$), the wave function becomes oscillatory, corresponding to the classical evolution of the universe.

This thesis aims to investigate how these regions connect and how a classical universe like ours could have emerged from a fundamentally quantum origin. We focus particularly on the no-boundary wave function in the minisuperspace model and analyze its physical interpretation in different parts of configuration space.

Chapter 2

Causality

2.1 Basic Definitions

In this section section we will see the causal properties of general spacetime.

2.1.1 Space-Time

Definition: A space-time $\mathcal{M}$ is defined as a D-dimensional differentiable manifold equipped with a smooth metric $g_{\mu\nu}(x)$, expressed in coordinates x^μ. This metric has Lorentzian (pseudo-Riemannian) signature. The manifold $\mathcal{M}$ is assumed to be time-orientable, meaning that a consistent distinction between "future" and "past" directions in time can be made throughout $\mathcal{M}$, and this distinction varies smoothly and remains invariant under coordinate transformations. Additionally, $\mathcal{M}$ is required to satisfy the Hausdorff condition and to be paracompact.

• The metric on a spacetime manifold has a Lorentzian signature, typically written as $(- + + \cdots +)$, where one coordinate represents time and the remaining $D-1$ coordinates represent space in a D-dimensional manifold. At any point p on the manifold, the tangent space $T_p(\mathcal{M})$ admits a basis—called a normal coordinate basis—in which the metric takes the form $\text{diag}(-1, +1, +1, \ldots, +1)$. The light cone at a point characterizes the possible instantaneous directions of light rays through that point. In flat Minkowski spacetime with $D = 2$, the light cone consists of two straight lines at 45°; for $D = 3$, it becomes a two-dimensional cone. In a general curved spacetime $\mathcal{M}$, however, the light cone cannot be visualized as part of the manifold itself; instead, it resides in the tangent space $T_p(\mathcal{M})$ at each point $p \in \mathcal{M}$.

A spacetime $\mathcal{M}$ is said to be *time-orientable* if and only if there exists a continuous, nowhere-vanishing timelike vector field defined on $\mathcal{M}$.

2.1.2 Path

Definition: A path $x^\mu(\lambda)$ is **time-like** if the tangent vector $\frac{dx^\mu(\lambda)}{d\lambda}$ is everywhere(at all point or events) time-like, and **causal** if the tangent vector is everywhere(at all point or events) time-like or null.
Consider a path(also called world-lines) $x^\mu(\lambda)$ in a spacetime. Here the λ is an arbitrary parameter that labels points on the path. A tangent vector $\mathbf{X} \in T_p(M)$ can be thought of as the derivative of a curve on the spacetime at a specific point is:

$$\mathbf{X} = \frac{dx^\mu(\lambda)}{d\lambda} \tag{2.1}$$

• A tangent vector $\mathbf{X} \in T_p(M)$ is either spacelike, timelike, or null, according to whether its square norm g(X,X) is positive, negative, or zero

$$g_{\mu\nu}(x(\lambda))\frac{dx^\mu(\lambda)}{d\lambda}\frac{dx^\nu(\lambda)}{d\lambda} < 0 \quad ;\text{Then } \mathbf{X} \text{ is Timelike tangent vector} \tag{2.2}$$

$$g_{\mu\nu}(x(\lambda))\frac{dx^\mu(\lambda)}{d\lambda}\frac{dx^\nu(\lambda)}{d\lambda} > 0 \quad ;\text{Then}\mathbf{X} \text{ is spacelike tangent vector} \tag{2.3}$$

$$g_{\mu\nu}(x(\lambda))\frac{dx^\mu(\lambda)}{d\lambda}\frac{dx^\nu(\lambda)}{d\lambda} = 0 \quad ;\text{Then}\mathbf{X} \text{ is lightlike or null tangent vector} \tag{2.4}$$

Definition: A path is future/past directed if the tangent vector on the path lies in the future/past half of light cone

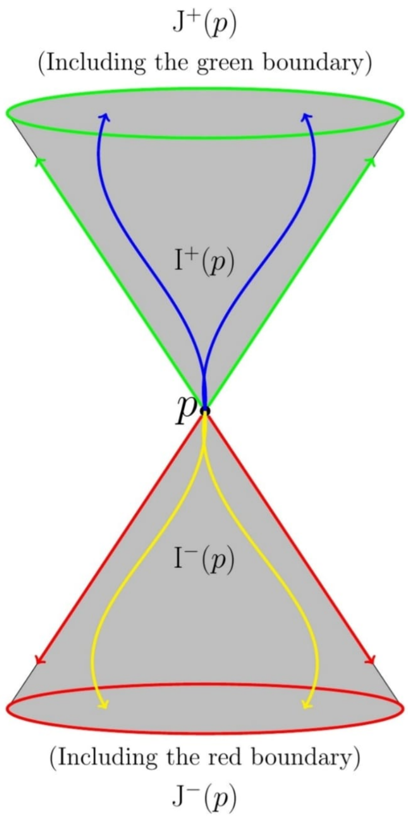

Figure 2.1: chronological and causal past future of a point *(image from Rohan Kulkarni notes)*

2.1.3 Future and Past

Chronological future/past of an event and set of event:

Definition: For a given event $p \in \mathcal{M}$, the *chronological future* (or past) is defined as the set of all points that can be reached from p by a future-directed (or past-directed) timelike curve. These sets are denoted by $I^+(p)$ and $I^-(p)$, respectively (see Fig. 2.1).

More generally, for a subset $S \subset \mathcal{M}$, the chronological future is defined as the union of the chronological futures of all points in S:

$$I^+(S) \equiv \bigcup_{p \in S} I^+(p),$$

and similarly, the chronological past is

$$I^-(S) \equiv \bigcup_{p \in S} I^-(p).$$

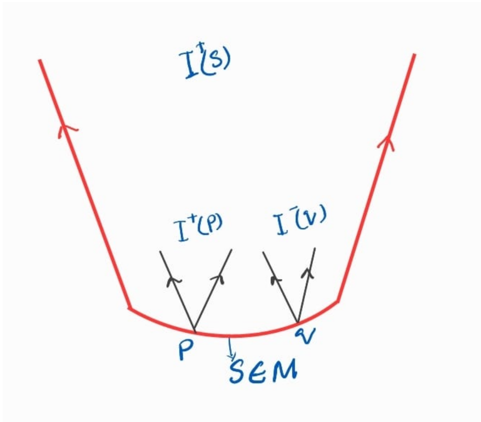

Figure 2.2: Chronological future of a set

Causal future/past of an event and set of event:

Definition: For a given event $p \in \mathcal{M}$, the *causal future* (or *causal past*) is defined as the set of all points that can be reached from p by a future-directed (or past-directed) causal curve—that is, a curve that is either timelike or lightlike (null). These sets are denoted by $J^+(p)$ and $J^-(p)$, respectively (see Fig. 2.2).

More generally, for a subset $S \subset \mathcal{M}$, the causal future is defined as the union of the causal futures of all points in S:

$$J^+(S) \equiv \bigcup_{p\in S} J^+(p),$$

and likewise, the causal past is

$$J^-(S) \equiv \bigcup_{p\in S} J^-(p).$$

.

- The boundary of $I^+(p)$ is denoted by $\partial I^+(p)$. we can say $I^+(p) \cup \partial I^+(p) = J^+(p)$. But it is not always true, when we delete one point from the boundary, let suppose $Q \notin \partial I^+(p)$ then $I^+(p) \cup \partial I^+(p) \neq J^+(p)$(see fig 2.4)

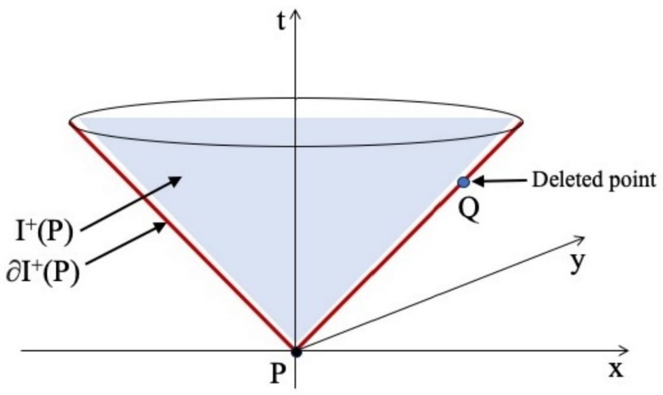

Figure 2.3: An Example of Minkowski spacetime: The causal future $J^+(P)$ does not contain the future of the deleted point Q along the lightcone, but $I^+(P) \cup \partial I^+(P)$ does contain it [1].

2.1.4 Causal Diamond

Definition: The intersection of the causal future of P and causal past of Q, called causal diamond point P and Q, $D_P^Q = J^+(p) \cap J^-(Q)$ (see fig 2.4)

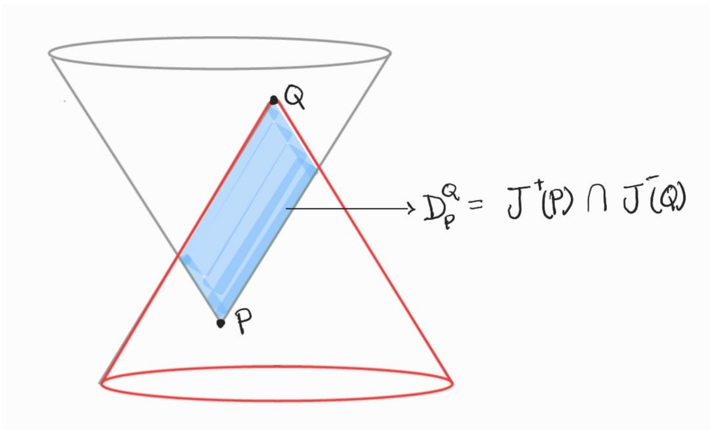

Figure 2.4: example of causal diamond in Minkowski spacetime

• Causal diamond tells about location of the path of two connected points. An another way we can say the causal path which connects the two point P and Q, must lies inside the causal diamond of P and Q. But not every path that lies in the causal diamond is a causal path.
Two events always connected by causal path. If causal diamond does not form between two points(or events) then it will separate by space like path.(see fig 2.5)

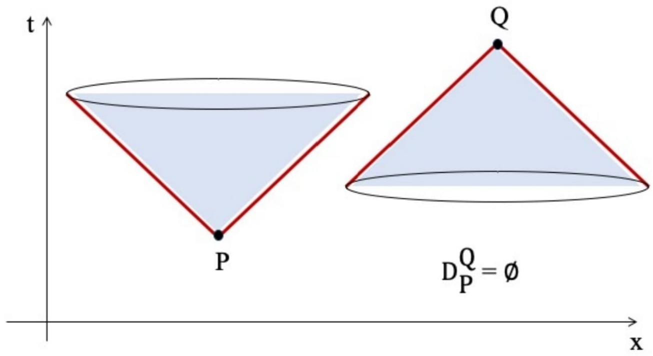

Figure 2.5: Example of empty causal diamond in Minkowski space [1]

2.2 Global Hyperbolic Space-Time

• All spacetime manifold with a smooth metric may not be good we need extra condition to do physics on it i.e we want to make a spacetime which will physically acceptable. we will see the extra condition that global hyperbolicity imposes on spacetime, so that I can define initial data and predict the future.
• Global hyperbolic spacetime refers to a specific type of spacetime in general relativity characterized by certain geometric properties.

2.2.1 Achronal Surface

Definition: A subset $S \subset \mathcal{M}$ is said to be *achronal* (or an *achronal surface*) if no two points $P, Q \in S$ can be connected by a timelike curve. Equivalently, this means that the intersection

$$I^+(P) \cap I^-(Q)$$

is empty for all $P, Q \in S$.
• In an achronal set S, there are no pairs of points that are timelike-separated. This means that every pair of points in S is either spacelike-separated or null-separated. Consequently, within an achronal surface, no event can causally influence another through a timelike path.

• however, the surface that is locally space-like or null at each point may still fail to be achronal [1]
• $\partial I^+(S)$ is example of achronal surface *(for proof see Wald theorem 8.1.3)*
• Achronal sets S are subsets of spacetime $\mathcal{M}$ that hold the property $S \cap I^{\pm}(S) = \phi$.

2.2.2 Extendible and Inextendible Curve

Lets understand a curve in spacetime can extend forever or it has any end point.

Definition: Let $x^\mu(\lambda)$ be a future-directed causal curve in M. A point $P \in M$ is called a future endpoint of the curve $x^\mu(\lambda)$ if, for every open neighborhood O of P, there exists a parameter value λ_O such that for all $\lambda > \lambda_O$, the curve lies entirely within O; that is,

$$x^\mu(\lambda) \in O \quad \text{for all } \lambda > \lambda_O.$$

Thus by Hausdorff property of M we can have at most one future endpoint.

Definition: let $x^\mu(\lambda)$ be a future directed causal curve in M. We say that point $P \in M$ is a future endpoint of $x^\mu(\lambda)$ if for every open neighbourhood O around P completely contains the curve beyond some value λ_0 parameter. such that $x^\mu(\lambda) \in O$ for all $\lambda < \lambda_O$. Thus by Hausdorff property of M we can have at most one future endpoint.

Definition: A causal curve $x^\mu(\lambda)$ with future/past end point is called future/past extendible curve otherwise future/past inextendible curve.

2.2.3 Cauchy Hypersurface and Global Hyperbolic Spacetime

Definition: A Cauchy hypersurface in M is an **closed** achronal (S) space-like surface $\sum$ such that every inextendible causal path through a point P$\in$M but P$\notin \sum$, passes through $\sum$. **A spacetime with a Cauchy hy-**

persurface is said to be globally hyperbolic spacetime(See fig 2.6)

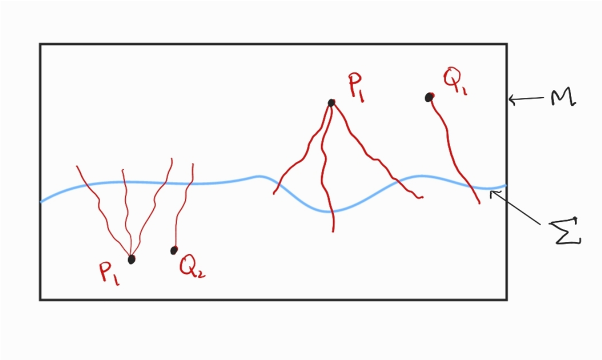

Figure 2.6: Globally hyperbolic Spacetime

- $\sum$ divides the spacetime into a past and future.
- Physically, this means that if the physical state is fully known on a Cauchy hypersurface Σ (i.e., initial conditions are specified along with suitable evolution laws, such as Einstein's equations), then the entire evolution of the spacetime M — both its past and future — is determined. In other words, the data on Σ uniquely determines what happens throughout the entire manifold M.
- Since a Cauchy surface is Achronal , it can be viewed as an instant in time. (As no two events are connected causally on an Achronal set)
- An inxtendible causal path from P can only intersect $\sum$ exactly once. more then one intersection not possible, because a Cauchy surface is achronal which means no causal path connect any two point on the surface, so it contradicts achronal.(see Fig 2.7)

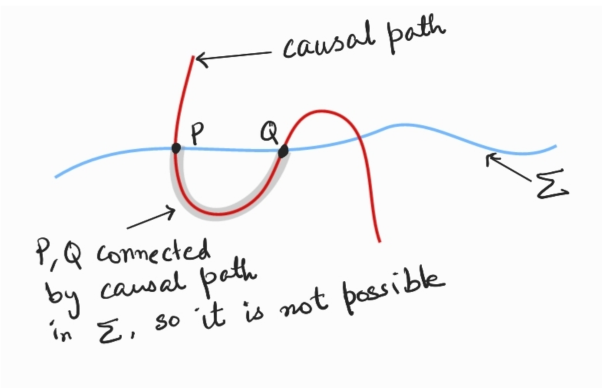

Figure 2.7: No causal path connect any two point on the Σ

2.2.4 Domain of Dependence

The future domain of dependence of a given region in spacetime includes all points that can be influenced by events occurring in that region. If you have an initial set of data or an event, the future domain of dependence contains all the points that can be affected by this initial condition.
In causal structure we need initial value or condition for future/past domain of dependence of an event or set of events.

Definition: The future domain of dependence $D^{+}(S)$ of an achronal (S) spacelike set, is the set of all points $\{P_i\}$ where $P_i \in M$ such that all past inextendible causal paths through P_i pass through S. Similarly past domain of dependence$D^{-}(s)$ in same way but for future inextendible causal path instead of past inextendible. The domain of dependence: $D(S) = D^{+}(S) \cup D^{-}(S)$.

- the Domain of dependence of Cauchy hypersurface $\sum$ is the whole space-time i.e, $D(\sum) = M$.
- Domain of dependence need not to be closed.
- If S is achronal but not Cauchy hypersurface $\sum$, then $D(S) \neq M$.

2.2.5 Cauchy Horizon

Cauchy horizon is the boundary of the domain of dependence of a Cauchy surface. Beyond this horizon, the future evolution is no longer uniquely determined by the initial value. The Cauchy horizon indicates where causality can break down. points on one side of the horizon may not be able to influence points(or events) on the other side.

Definition: the boundary of the closure of the domain of dependence of an achronal spacelike hypersurface S, namely $\partial\overline{D(S)}$, is called Cauchy horizon H(S). where $\overline{D(S)}$ is the Closure of the domain of dependence.

Cauchy horizon H(S) also defined as future/past Cauchy horizon by H^+S and H^-S respectively, in term of $\partial\overline{D^{\pm}(S)}$.

$$H^+(S) = \overline{D^+(S)} - I^-(D^+(S)) \tag{2.5}$$
$$H^-(S) = \overline{D^-(S)} - I^+(D^-(S)) \tag{2.6}$$

• Where $I^-(D^+(S))$ is Chronological past of the future domain of the dependence of S. In the example of (Fig: 2.8), we can see that the Cauchy horizon of the past hyperboloid is the past light-cone.

• Here $H^{\pm}(S)$ is achronal. Because no two events in $H^{\pm}(S)$ are causally

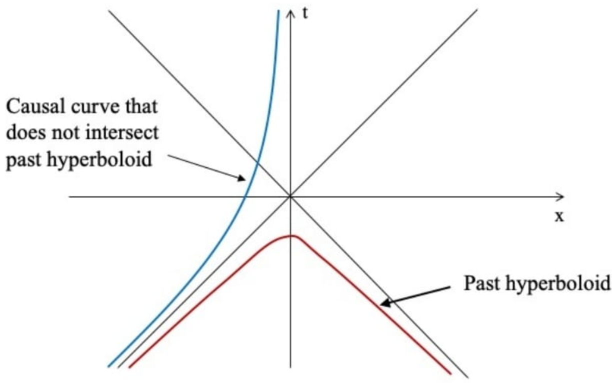

Figure 2.8: The past hyperboloid (red) is not Cauchy because the time-like curve (blue) does not intersect it [1]

connected to each other

2.2.6 Causal Spacetime

A spacetime is said to be causal if it contains no closed causal curves, that is, no causal curve that returns to its starting point. To distinguish this basic level of causality from stronger conditions, we may refer to such a spacetime as simply causal.

Chapter 3

Geodesic and Focal point

If a family of geodesics converges at a specific point, that point is called a focal point (or conjugate point) of the family. A focal point has the Important property that the geodesic may not minimize the path length when we go beyond it.

3.1 The Raychaudhuri Equation.

To prove a singularity theorem, we need a good way to predict the occurrence of focal points on the geodesics. Here, the Raychaudhuri equation tells us the condition under which focal points exist on the geodesics, this equation is only for timelike geodesics.

Definition: Let M be a manifold and let a subset O $\subset$ M be open. A congruence in O is a family of curves such that through each point P $\in$ O there passes precisely one curve in this family. No two trajectories within the family can intersect each other over time evolution. If they do, then the definition of congruence breaks down.

• Let in D=d+1 dimensions spacetime M with initial value surface $\sum$, $\vec{x} = (x^1, x^2, ..., x^d)$ is the co-ordinate on surface $\sum$. Let the time like geodesics pass through a point at $\vec{x}$ on surface $\sum$ orthogonally. Now take any point P on the timelike geodesic from the surface $\sum$ and label that point as t, where t is proper time measured along the geodesic from $\vec{x}$ at surface $\sum$. We are constructing a coordinate system and assigning to P of coordinate (t, $\vec{x}$), if P is to the future of $\sum$, or $(-t, \vec{x})$ if it is to the past.

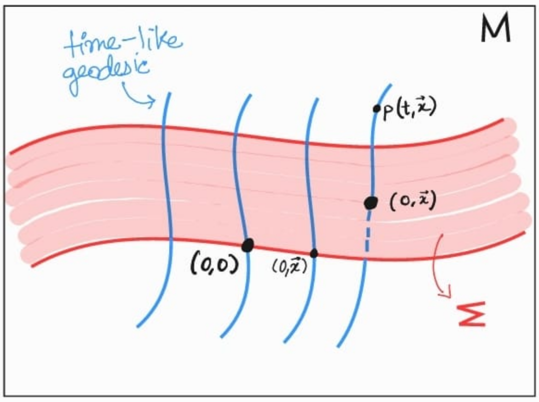

Figure 3.1: Spacetime co-ordinate system

in this coordinate system, the line element is

$$ds^2 = -dt^2 + g_{ij}(t, \vec{x})dx^i dx^j \tag{3.1}$$

here

$$g_{tt} = -1 \text{ and } g_{ti} = 0 \tag{3.2}$$

for all t(because in coordinate system geodesics are orthogonal to surface $\sum$ at t=0).

Some points From [2],

- *advantage of the orthogonal geodesics is that this will help us understand how the coordinate system can break down.*

- *Even if M remains non-singular, our coordinate system breaks down if orthogonal geodesics originating at different points on S meet at the same point $p \in M$. In this case, we do not know what x value to assign to P. A related statement is that the coordinate system breaks down at focal points. then the starting points of the orthogonal geodesics will not be part of a good coordinate system near P (see fig:)*

- $$detg_{ij} = 0 \tag{3.3}$$

  is necessary as well as a sufficient condition for a focal point (Since $g_{ij}(t, \vec{x})$ *measures the distance between two nearby orthogonal geodesics in a manifold* M*). This implies that focal points can be located by finding where this determinant vanishes.*

- *Raychaudhuri's equation gives a useful criterion for predicting that* $detg_{ij}$ *will go to 0 within a known time. In general, this will represent only a breakdown of the coordinate system, not a true spacetime singularity, but we will see that the criterion provided by Raychaudhuri's equation is a useful starting point for predicting spacetime singularities.*

The trace-reversed form of the Einstein Field Equations is (see A.1)

$$R_{tt} = 8\pi G\left(T_{tt} - \frac{1}{(D-2)}Tg_{tt}\right) + \frac{2}{(D-2)}\Lambda g_{tt} \tag{3.4}$$

$$R_{tt} = 8\pi G\hat{T}_{tt} \tag{3.5}$$

where

$$\hat{T}_{tt} = T_{tt} - \frac{1}{(D-2)}Tg_{tt} + \frac{1}{4\pi G(D-2)}\Lambda g_{tt} \tag{3.6}$$

we know,

$$R_{tt} = \partial_\lambda \Gamma^\lambda_{tt} - \partial_t \Gamma^\lambda_{t\lambda} + \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{tt} - \Gamma^\lambda_{t\sigma}\Gamma^\sigma_{\lambda t} \tag{3.7}$$

where $\Gamma^\lambda_{tt} = \Gamma^\sigma_{tt} = 0$, $\Gamma^\lambda_{t\sigma} = \frac{1}{2}g^{\lambda i}\partial_t g_{\sigma i}$ (see A.2 and A.3)

$$R_{tt} = -\frac{1}{2}\partial_t(g^{\lambda i}\partial_t g_{\lambda i}) - \frac{1}{4}(g^{\lambda i}\partial_t g_{\sigma i})(g^{\sigma j}\partial_t g_{\lambda j}) \tag{3.8}$$

$$R_{tt} = -\frac{1}{2}\partial_t(g^{\lambda i}\dot{g}_{\lambda i}) - \frac{1}{4}(g^{\lambda i}\dot{g}_{\sigma i})(g^{\sigma j}\dot{g}_{\lambda j}) \tag{3.9}$$

for convenience let us define three quantity:

$$V(\vec{x}, t) = \sqrt{detg_{ij}(\vec{x}, t)} \tag{3.10}$$

this quantity measures the volume occupied by a little bundle of geodesics.

The others are

$$\text{Expansion: } \theta = \frac{\dot{V}}{V} = \frac{1}{2} g^{\lambda i} \dot{g}_{\lambda i} \tag{3.11}$$

$$\text{Shear: } \sigma^{\lambda}_{\sigma} = \frac{1}{2} \left(g^{\lambda i} \dot{g}_{\sigma i} - \frac{1}{D-1} \delta^{\lambda}_{\sigma} g^{ij} \dot{g}_{ij} \right) \tag{3.12}$$

From equation (A.5), we can write equation (3.10) as,

$$tr\sigma^2 = \frac{1}{4} (g^{\lambda i} \dot{g}_{\sigma i} g^{\sigma j} \dot{g}_{\lambda j}) - \frac{1}{(D-1)} \theta^2 \tag{3.13}$$

From equations (3.9), (3.11), and (3.7) we can write

$$R_{tt} = -\dot{\theta} - \frac{1}{(D-1)} \theta^2 - tr\sigma^2 \tag{3.14}$$

combining equations (3.5) and (3.14) we get,

$$\dot{\theta} + \frac{1}{(D-1)} \theta^2 = -tr\sigma^2 - 8\pi G \hat{T}_{tt} \tag{3.15}$$

This equation is known as the Einstein-Raychaudhuri equation and equation (3.14) is known as the Raychaudhuri equation.

Taking assumptions,

$$\text{Strong Energy Condition: } \quad \hat{T}_{tt} \geq 0$$
$$\text{and } \quad \Lambda < 0$$

σ^2 and $\hat{T}_{tt}$ is positive so we get inequality in (3.14)

$$\dot{\theta} + \frac{1}{(D-1)} \theta^2 \leq 0 \tag{3.16}$$

$$\partial_t \left(\frac{1}{\theta} \right) = \partial_t \left(\frac{1}{\dot{V}/V} \right) \leq \frac{1}{D-1} \tag{3.17}$$

Integrating the above eqn, we get (where D=d+1)

$$\frac{1}{\theta(\vec{x},t)} - \frac{1}{\theta_0(\vec{x},0)} \geq \frac{t}{d} \tag{3.18}$$

$$\theta \leq \left(\frac{1}{\theta_0} + \frac{t}{d}\right)^{-1} \tag{3.19}$$

here θ_0 is the initial value on the initial value surface Σ. Let at any point on Σ, $\theta_0 < 0$(this means geodesics will focus at future. where $\theta_0 > 0$ is for past), say $\theta_0 = -w$ where $w > 0$.

$$\theta = \frac{\dot{V}}{V} = \frac{d(logV)}{dt} \leq -\left(\frac{1}{w} - \frac{t}{d}\right)^{-1} \tag{3.20}$$

integrating the above eqn.

$$logV - logV_0 \leq d\left[log\left(\frac{1}{w} - \frac{t}{d}\right) - log\left(\frac{1}{w}\right)\right] \tag{3.21}$$

here $V = V(\vec{x},t)$, $V_0 = V_0(\vec{x},0)$. If $logV \to -\infty$ then $V \to 0$ so,

$$t \leq \frac{d}{w} \tag{3.22}$$

$$\leq \frac{D-1}{-\theta_0} = -\frac{D-1}{\frac{\dot{V}(\vec{x},0)}{V(\vec{x},0)}} \tag{3.23}$$

$$t \leq -\frac{D-1}{\frac{\dot{V}_0}{V_0}} \tag{3.24}$$

In eqn (3.3), we have already discussed that vanishing V gave us a focal point, or possibly a space-time Singularity. So assuming the strong energy condition, an orthogonal geodesic that departs from initial value surface Σ at $t = 0$ at a point at which $\theta_0 = -w < 0$ will reach a focal point, or possibly a singularity, after a proper time $t \leq d/w$. (see fig:3.2)

Conclusions

- *Generally the vanishing of V in the Raychaudhuri equation tells about a focal point of time-like geodesics, not space-time singularity.*
- *The coordinate system breaks down at the focal point.*
- *To predict a spacetime singularity requires a more precise argument, with some input beyond Raychaudhuri's equation.*

3.2 Hawking's Singularity Theorem:

Hawking used the Raychaudhuri equation to derive a singularity theorem. He assumed the universe is globally hyperbolic with Cauchy hypersurface Σ. He also assumed the strong energy condition $T_{tt} \geq 0$
If the universe is perfectly homogeneous and isotropic, it is described by the Friedmann- Lemaître-Robertson-Walker (FLRW) metric.

$$ds^2 = -dt^2 + a(t)^2 dx^i dx^i \tag{3.25}$$

where a is independent of $\vec{x}$, this assumption is for homogeneit.
Here Hubble parameter H is defined as,

$$H(t) = \frac{\dot{a}(t)}{a(t)} \tag{3.26}$$

If we compare eqn(3.1) and (3.23), we get

$$g_{ij}(t, \vec{x}) \to g_{ij}(t) = \delta_{ij} a(t)^2 \tag{3.27}$$

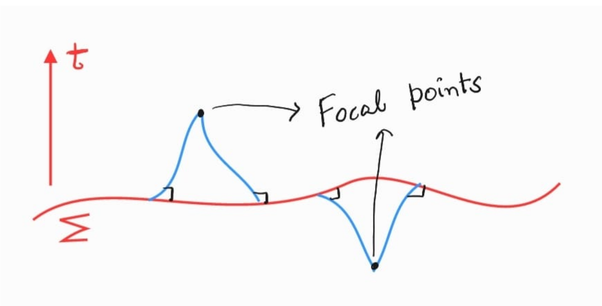

Figure 3.2: Focal points

We can write from the previous section as

$$V \equiv \sqrt{detg_{ij}} \; = \; a^{D-1}a \; = \; V^{\frac{1}{D-1}} \tag{3.28}$$

Now local Hubble parameter dependent on $(\vec{x}, t)$ is defined as,

$$H(\vec{x}, t) \; = \; \frac{\dot{a}(\vec{x}, t)}{a(\vec{x}, t)} \; = \; \frac{1}{D-1} \frac{\dot{V}(\vec{x}, t)}{V(\vec{x}, t)} \tag{3.29}$$

Let us consider a globally hyperbolic spacetime with Cauchy hypersurface Σ. let us choose $H(\vec{x}, 0)$ on Σ where

$$H(\vec{x}, 0) = h_{min} > 0 \tag{3.30}$$

from equations 3.24 and 3.29, we can write,

$$H(\vec{x}, 0) = h_{min} \; = \; \frac{1}{D-1} \frac{\dot{V}(\vec{x}, 0)}{V(\vec{x}, 0)} = -\frac{1}{t} \tag{3.31}$$

$$t \; = \; -\frac{1}{h_{min}} \tag{3.32}$$

We know from the Raychaudhuri equation that all past-directed timelike geodesics from Σ must reach a focal point before $t = -\frac{1}{h_{\min}}$. This means there are no timelike geodesics beyond this time. Therefore, we can conclude that there is a spacetime singularity at a time in the past. A singularity is simply a geodesic incompleteness. Beyond the time $t = -\frac{1}{h_{\min}}$, we cannot extend our timelike geodesic. This implies that there is no point P beyond this time that connects Σ orthogonally with a timelike geodesic. However, if such a point P existed, it would not have a future-directed timelike geodesic connecting to Σ, which would contradict the global hyperbolicity of the spacetime. Thus, the spacetime is no longer globally hyperbolic.

Singularity, focal point, and geodesic incompleteness in spacetime tell that there is something wrong in spacetime.

Conclusions

We see that there is no point in space-time in the past of Σ beyond the time $-1/h_{min}$, along any causal path. The minimum value of the local Hubble parameter gives an upper bound on how long anything in the universe could have existed in the past. This is Hawking's theorem about the Big Bang.

Chapter 4

Hamiltonian Formulation of GR

4.1 Introduction

Since ancient times, the origin of the universe has been a big question. People have tried to understand this question using Einstein's General Theory of Relativity, but it has failed to provide a satisfactory answer. So, people approached another way of looking for the answer: they combined quantum mechanics with the General Theory of Relativity to develop a new theory of gravity, known as the quantum theory of gravity.

The basic principles and machinery of Quantum gravity with which we shall explore the wave functions for a closed universe[1]. The quantum state of the universe is described by wave function and it obeys the Wheeler-DeWitt Equation[2]. The main object of which is a wave function of the Universe that can be defined as some space geometry and some matter distribution.

In order to understand the universe by using quantum mechanics, first we have to understand the wave function of the universe and this wave function satisfy an equation, known as Wheeler-DeWitt equation. In this section we are going to derive the Wheeler-DeWitt equation.
First, we will obtain the Hamiltonian from the Hilbert-Einstein action. Then we will use Dirac's method to obtain the equation.

4.2 1+3 Decomposition of space time

Let us consider a globally hyperbolic space-time M, which can be expressed as a product $M = \Sigma \times \mathbb{R}$, where $\mathbb{R}$ represents the time dimension and Σ is a space-like Cauchy hypersurface. In this framework, space-time is foliated into a family of non-intersecting, three-dimensional space-like hypersurfaces, each labeled by a uniquely time coordinate. This decomposition of space-time into space and time is commonly referred to as the $(1 + 3)$ formalism. (Fig: 4.1-I)

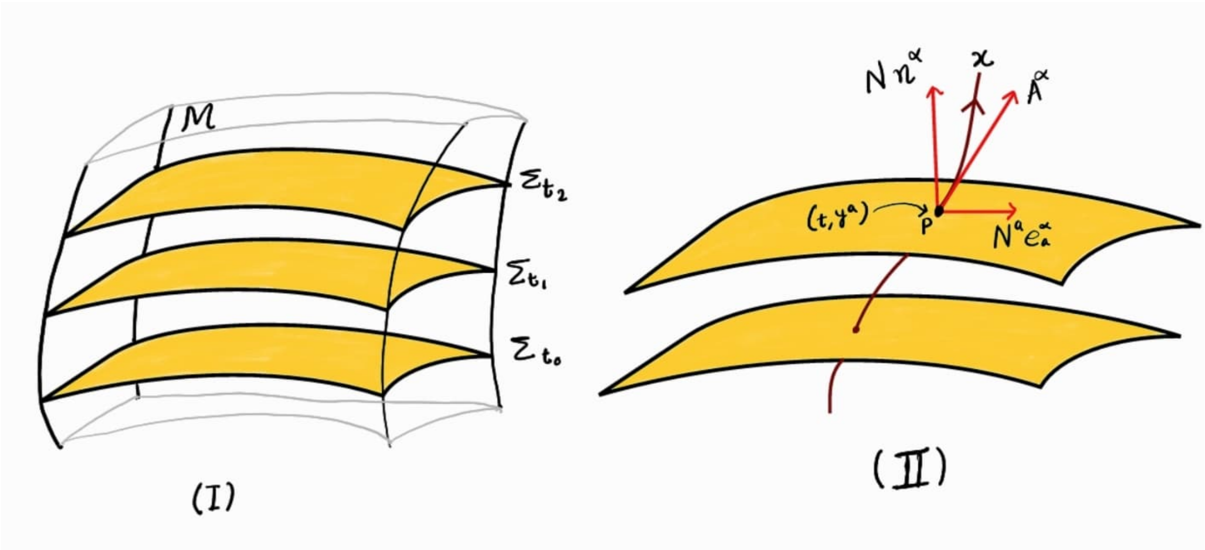

Figure 4.1: spacetime in 1+3

4.2.1 Metric in the (1+3) Decomposition of Spacetime

In general relativity, spacetime is a 4-dimensional manifold with a metric $g_{\mu\nu}$. To study dynamics or formulate initial value problems, we often perform a **(1+3) decomposition** of spacetime, splitting it into space and time. This is known as the ADM (Arnowitt-Deser-Misner) decomposition.

Foliation of Spacetime

We consider spacetime as being foliated into a family of spacelike hypersurfaces Σ_t, each labeled by a scalar function $f(x^\mu)$, such that

$$f(x^\mu) = \text{constant}.$$

We define a coordinate system (t, y^a), where:

- t is a time coordinate labeling the hypersurfaces,
- y^a $(a = 1, 2, 3)$ are spatial coordinates on each Σ_t.

The embedding of spacetime is given by $x^\mu = x^\mu(t, y^a)$.

Tangent and Normal Vectors

We define the following vectors:

$$A^\mu = \frac{\partial x^\mu}{\partial t}, \quad \text{(tangent to the flow of time)} \tag{4.1}$$

$$e_a^\mu = \frac{\partial x^\mu}{\partial y^a}, \quad \text{(tangent to the hypersurface).} \tag{4.2}$$

We introduce a unit normal vector to the hypersurface:

$$n^\mu = \frac{\nabla^\mu f}{\sqrt{|\nabla^\nu f \nabla_\nu f|}}, \quad \text{with} \quad n^\mu n_\mu = -1.$$

It satisfies the orthogonality condition:

$$n_\mu e_a^\mu = 0.$$

Lapse and Shift

The evolution vector A^μ is generally not orthogonal to the hypersurface. Instead, it is decomposed along normal unit vector n^μ and tangent basis vector e_a^μ of hypersurfce as (Fig: 4.1-II)

$$A^\mu = N n^\mu + N^a e_a^\mu,$$

where:

- N is the **lapse function**, measuring the proper time between slices,
- N^a is the **shift vector**, measuring the displacement of spatial coordinates.
- we can also say N and N^a are components of A^μ

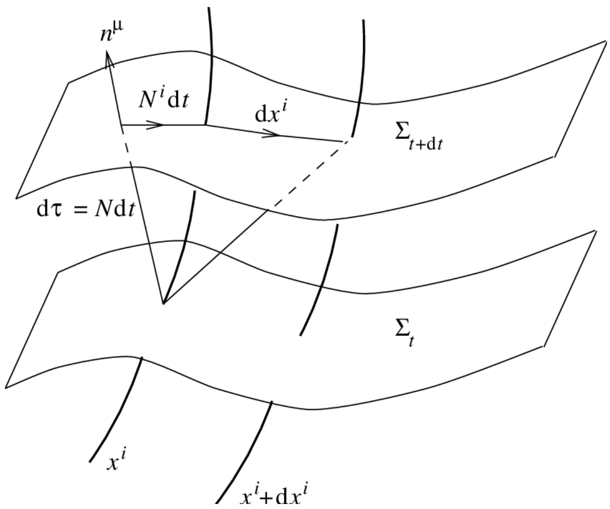

Figure 4.2: lapse-function-N-and-shift-vector-$N^a, a \to i$ (Fig uploaded by David L. Wiltshire)

Spacetime Line Element

A differential displacement is given by $x^\mu = x^\mu(t, y^a)$:

$$dx^\mu = \frac{\partial x^\mu}{\partial t}dt + \frac{\partial x^\mu}{\partial y^a}dy^a = A^\mu dt + e^\mu_a dy^a.$$

Substituting the decomposition of A^μ, we get:

$$dx^\mu = Nn^\mu dt + e^\mu_a(dy^a + N^a dt).$$

The spacetime line element becomes:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu.$$

Expanding this, we obtain:

$$\begin{aligned}
ds^2 &= (Ndt\, n^\mu + (N^a dt + dy^a)e^\mu_a)\left(Ndt\, n_\mu + (N^b dt + dy^b)e_{\mu b}\right) \\
&= \left(N^2 n^\mu n_\mu + NN^a n_\mu e^\mu_a + NN^b n^\mu e_{\mu b} + N^a N^b e^\mu_a e_{\mu b}\right) dt^2 \\
&\quad + \left(Nn_\mu e^\mu_a + N^b e^\mu_a e_{\mu b}\right) dt\, dy^a + \left(Nn^\mu e_{\mu b} + N^a e^\mu_a e_{\mu b}\right) dt\, dy^b + e^\mu_a e_{\mu b}\, dy^a dy^b \\
&= \left(-N^2 + N^a N^b g_{\mu\nu} e^\mu_a e^\nu_b\right) dt^2 + N^b g_{\mu\nu} e^\mu_a e^\nu_b dt dy^a + N^a g_{\mu\nu} e^\mu_a e^\nu_b dt dy^b \\
&+ g_{\mu\nu} e^\mu_a e^\nu_b dy^a dy^b \\
&= (-N^2 + N^a N^b h_{ab})dt^2 + 2N_a dt dy^a + h_{ab} dy^a dy^b
\end{aligned} \tag{4.3}$$

where the induced 3-metric is defined by:

$$h_{ab} = g_{\mu\nu} e^{\mu}_{a} e^{\nu}_{b}.$$

Using $N_a = h_{ab}N^b$, the metric becomes:

$$ds^2 = -(N^2 - N^a N_a)dt^2 + 2N_a dt dy^a + h_{ab} dy^a dy^b.$$

from above Eq. we can write metric $g_{\alpha\beta}$ as

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^a N_a & N_1 & N_2 & N_3 \\ N_1 & h_{11} & h_{12} & h_{13} \\ N_2 & h_{21} & h_{22} & h_{23} \\ N_3 & h_{31} & h_{32} & h_{33} \end{pmatrix} \tag{4.4}$$

for simplification we can write as

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^a N_a & N_a \\ N_b & h_{ab} \end{pmatrix} \tag{4.5}$$

the inverse of the metric $g_{\alpha\beta}$ is $[Appendix\ B]$

$$g^{\mu\nu} = \begin{pmatrix} -N^{-2} & N_b N^{-2} \\ N_a N^{-2} & h^{ab} - N^a N^b N^{-2} \end{pmatrix} \tag{4.6}$$

4.2.2 Value of $g \rightarrow (\mathbf{det} g_{\mu\nu})$:

$$g^{00} = \frac{cofactor(g_{00})}{g} = \frac{det\ h_{ab}}{g} = \frac{h}{g} \tag{4.7}$$

$$g = -\frac{h}{N^2} \tag{4.8}$$

$$\boxed{\sqrt{-g} = N\sqrt{h}} \tag{4.9}$$

4.2.3 Value of R :

• We have a 4D metric $g^{\alpha\beta}$ that describes the geometry of spacetime. We can decompose this metric into spatial part(tangent to the hypersurface) and normal part(in time evolution direction) to the hypersurface as [3]

$$g^{\alpha\beta} = h^{\alpha\beta} + (-n^{\alpha}n^{\beta}) \tag{4.10}$$
$$h^{\alpha\beta} = h^{ab}e_a^{\alpha}e_b^{\beta} = g^{\alpha\beta} + n^{\alpha}n^{\beta} \tag{4.11}$$

where $h^{\alpha\beta}$ is the tangential component of $g^{\alpha\beta}$ or we can say projection of 4D metric $g^{\alpha\beta}$ onto 3D hypersurface. It is also called, induced 3-metric which tells about the intrinsic geometry of the hypersurface.
$n^{\mu}n^{\nu}$ is the projection of 4D metric $g^{\alpha\beta}$ along normal the hypersurface.
• Let an arbitrary tensor $T^{\alpha\beta\cdots}$ in $4D$
→ projection of $T^{\alpha\beta\cdots}$ to the hypersurface is

$$\begin{aligned} \tilde{T}^{\mu\nu} &= h^{\mu}_{\alpha}h^{\nu}_{\beta}T^{\alpha\beta} \\ (T^{\alpha\beta\ldots})_{tangent} &= \tilde{T}^{\mu\nu} \end{aligned} \tag{4.12}$$

where $(T^{\alpha\beta\cdots})_{tangent}$ is the $4D$ tensor in the $3D$ hypersurface, it is a tangent tensor of the hypersurface.
→ $4D$ tangent tensor in $3D$ intrinsic coordinate of the hypersurface:

$$\begin{aligned} \tilde{T}^{\mu\nu} &= e^{\mu}_{a}e^{\nu}_{b}\tilde{T}_{ab} \\ \tilde{T}_{ab} &= e^{\mu}_{a}e^{\nu}_{b}\ \tilde{T}_{\mu\nu} \\ \tilde{T}_{ab} &= h_{ia}h_{jb}\tilde{T}^{ij} \\ \tilde{T}^{ab} &= h^{ia}h^{jb}\tilde{T}_{ij} \end{aligned} \tag{4.13}$$

• Projection of covariant derivative of a tangent 4-vector $\tilde{A}_\alpha \rightarrow D_\beta \tilde{A}_\alpha$ is defined as covariant derivative of of a 3-vector $\tilde{A}_a \rightarrow D_b \tilde{A}_a$

$$\begin{aligned} D_b \tilde{A}_a &= e_a^\alpha e_b^\beta D_\beta \tilde{A}_\alpha \\ &= e_b^\beta D_\beta (e_a^\alpha \tilde{A}_\alpha) - \tilde{A}_\alpha e_b^\beta D_\beta e_a^\alpha \\ &= D_b(\tilde{A}_a) - \tilde{A}_\alpha e_b^\beta D_\beta (g^{\mu\alpha} e_{\mu a}) \\ &= \partial_b \tilde{A}_a - \tilde{A}_\alpha g^{\mu\alpha} e_b^\beta D_\beta e_{\mu a} \\ &= \partial_b \tilde{A}_a - e^{\mu c} e_b^\beta (D_\beta e_{\mu a}) \tilde{A}_c \end{aligned} \tag{4.14}$$

Christoffel symbol in 3-vector is defined as

$$\Gamma^c_{ba} = e^{\mu c} e_b^\beta D_\beta e_{\mu a} \tag{4.15}$$

$$\text{and} \quad \Gamma^c_{ba} = \frac{1}{2} h^{ci} (\partial_a h_{bi} + \partial_b h_{ia} - \partial_i h_{ba}) \tag{4.16}$$

we can write

$$D_b \tilde{A}_a = \partial_b \tilde{A}_a - \Gamma^c_{bc} \tilde{A}_c \tag{4.17}$$

• If $D_k h_{ab} = 0$ then we can say Christoffel symbol is symmetric $\Gamma^c_{ab} = \Gamma^c_{ba}$

$$\begin{aligned} D_k h_{ab} &= e_k^\alpha e_a^\beta e^\lambda D_\alpha h_{\beta\lambda} \\ &= e_k^\alpha e_a^\beta e^\lambda D_\alpha (g_{\beta\lambda} + n_\beta n_\lambda) \\ &= e_k^\alpha e_a^\beta e_b^\lambda (D_\alpha g_{\beta\lambda} + n_\lambda D_\alpha n_\beta + n_\beta D_\alpha n_\lambda) \\ &\text{Here } D_\alpha g_{\beta\lambda} = 0 \quad \& \quad e_b^\lambda n_\lambda = 0 = e_a^\beta n_\beta \\ &= 0 \end{aligned} \tag{4.18}$$

4-D Ricci scalar R

•

$$\nabla_\alpha \nabla_\beta A^\gamma - \nabla_\beta \nabla_\alpha A^\gamma = R^\gamma_{\lambda\alpha\beta} A^\lambda \tag{4.19}$$

This is the Ricci Identity in $4D$. Where $\boldsymbol{\nabla}$ is $4D$ covariant derivative of $4D$ vector A^γ & A^λ

•

$$D_\alpha D_\beta \tilde{A}^\gamma - D_\beta D_\alpha \tilde{A}^\gamma = R^\gamma_{\sigma\alpha\beta} \tilde{A}^\sigma \tag{4.20}$$

This is the $4D$ Ricci Identity on the $3D$ hypersurface. Where $\boldsymbol{D}$ is the $4D$ covariant derivative of the tangent vector $\tilde{A}^\gamma$ & $\tilde{A}^\sigma$ of hypersurface. From Eq. (4.20) we can write

$$\begin{aligned} D_\alpha(D_\beta\tilde{A}^\gamma) &= h^\mu_\alpha h^\nu_\beta h^\gamma_\rho \nabla_\mu\ (D_\nu\tilde{A}^\rho) \\ &= h^\mu_\alpha h^\nu_\beta h^\gamma_\rho \nabla_\mu\ (h^\rho_\sigma h^\lambda_\nu D_\lambda\tilde{A}^\sigma) \\ &= h^\mu_\alpha h^\nu_\beta h^\gamma_\rho\ [(\nabla_\mu h^\rho_\sigma)h^\lambda_\nu\nabla_\lambda\tilde{A}^\sigma + h^\rho_\sigma(\nabla_\mu h^\lambda_\nu)\nabla_\lambda\tilde{A}^\sigma + h^\rho_\sigma h^\lambda_\nu(\nabla_\mu\nabla_\lambda\tilde{A}^\sigma)] \end{aligned} \tag{4.21}$$

Here $\nabla_\mu h^\lambda_\nu = \nabla_\mu(g^\lambda_\nu + n^\lambda n_\nu) = (\nabla_\mu n^\lambda)n_\nu + n^\lambda(\nabla_\mu n_\nu)$ where $\nabla_\mu g^\lambda_\nu = 0$

$$\begin{aligned} &= h^\mu_\alpha h^\nu_\beta h^\gamma_\rho\ \{[(\nabla_\mu n^\rho)n_\sigma + n^\rho(\nabla_\mu n_\sigma)]h^\lambda_\nu\nabla_\lambda\tilde{A}^\sigma + h^\rho_\sigma[(\nabla_\mu n^\lambda)n_\nu \\ &+ n^\lambda(\nabla_\mu n_\nu)]\nabla_\lambda\tilde{A}^\sigma + h^\rho_\sigma h^\lambda_\nu(\nabla_\mu\nabla_\lambda\tilde{A}^\sigma)\} \end{aligned} \tag{4.22}$$

2^{nd} & 3^{rd} term will be zero because $n^\rho h^\gamma_\rho = 0 = n_\nu h^\nu_\beta$ (projection of normal component into tangent component is zero)

$$\begin{aligned} &= h^\mu_\alpha h^\nu_\beta h^\gamma_\rho\ [(\nabla_\mu n^\rho)n_\sigma h^\lambda_\nu\nabla_\lambda\tilde{A}^\sigma + h^\rho_\sigma n^\lambda(\nabla_\mu n_\nu)\nabla_\lambda\tilde{A}^\sigma \\ &+ h^\rho_\sigma h^\lambda_\nu(\nabla_\mu\nabla_\lambda\tilde{A}^\sigma)] \end{aligned} \tag{4.23}$$

In 1^{st} term, $h^\nu_\beta h^\lambda_\nu = h^\lambda_\beta$ & $n_\sigma\nabla_\lambda\tilde{A}^\sigma = \nabla_\lambda(n_\sigma\tilde{A}^\sigma) - \tilde{A}^\sigma\nabla_\lambda n_\sigma = -\tilde{A}^\sigma\nabla_\lambda n_\sigma$ and $K^\gamma_\alpha = -h^\mu_\alpha h^\gamma_\rho\nabla_\mu n^\rho,\ K_{\alpha\beta} = -h^\mu_\alpha h^\nu_\beta\nabla_\mu n_\nu$ are curvature tensors of Σ

$$\begin{aligned} &= -h^\mu_\alpha h^\gamma_\rho(\nabla_\mu n^\rho)\tilde{A}^\sigma(h^\lambda_\sigma h^\sigma_\beta)\nabla_\lambda n_\sigma + h^\mu_\alpha h^\nu_\beta(\nabla_\mu n_\nu)n^\lambda h^\gamma_\sigma\nabla_\lambda\tilde{A}^\sigma \\ &+ h^\mu_\alpha h^\lambda_\beta h^\gamma_\sigma(\nabla_\mu\nabla_\lambda\tilde{A}^\sigma) \\ &= -K^\gamma_\alpha K_{\sigma\beta}\tilde{A}^\sigma - K_{\alpha\beta}\ n^\lambda h^\gamma_\sigma\nabla_\lambda\tilde{A}^\sigma + h^\mu_\alpha h^\lambda_\beta h^\gamma_\sigma(\nabla_\mu\nabla_\lambda\tilde{A}^\sigma) \\ &= -K^\gamma_\alpha K_{\nu\beta}\tilde{A}^\nu - K_{\beta\alpha}\ n^\lambda h^\gamma_\sigma\nabla_\lambda\tilde{A}^\sigma + h^\lambda_\alpha h^\rho_\beta h^\gamma_\sigma\nabla_\lambda\nabla_\rho\tilde{A}^\sigma \end{aligned} \tag{4.24}$$

Similarly we can find

$$D_\beta D_\alpha\tilde{A}^\gamma = -K^\gamma_\beta K_{\nu\alpha}\tilde{A}^\nu - K_{\beta\alpha}\ n^\lambda h^\gamma_\sigma\nabla_\lambda\tilde{A}^\sigma + h^\lambda_\alpha h^\rho_\beta h^\gamma_\sigma\nabla_\rho\nabla_\lambda\tilde{A}^\sigma \tag{4.25}$$

Now Eq. (4.20) can written as

$$\begin{aligned} D_\alpha D_\beta\tilde{A}^\gamma - D_\beta D_\alpha\tilde{A}^\gamma &= (-K^\gamma_\alpha K_{\nu\beta} + K^\gamma_\beta K_{\nu\alpha})\tilde{A}^\nu + h^\lambda_\alpha h^\rho_\beta h^\gamma_\sigma(\nabla_\lambda\nabla_\rho - \nabla_\rho\nabla_\lambda)\tilde{A}^\sigma \\ R^\gamma_{\mu\alpha\beta}\tilde{A}^\mu &= (-K^\gamma_\alpha K_{\nu\beta} + K^\gamma_\beta K_{\nu\alpha})\tilde{A}^\nu + h^\lambda_\alpha h^\rho_\beta h^\gamma_\sigma\ R^\sigma_{\nu\lambda\rho}\tilde{A}^\nu \end{aligned} \tag{4.26}$$

$\tilde{A}^\nu = h^\nu_\mu \tilde{A}^\mu \rightarrow$ projection of tangent vector again onto hypersurface is same

$$\begin{aligned} R^\gamma_{\mu\alpha\beta}\tilde{A}^\mu &= (-K^\gamma_\alpha K_{\nu\beta} + K^\gamma_\beta K_{\nu\alpha})h^\nu_\mu \tilde{A}^\mu + h^\lambda_\alpha h^\rho_\beta h^\gamma_\sigma \; R^\sigma_{\nu\lambda\rho} h^\nu_\mu \tilde{A}^\mu \\ R^\gamma_{\mu\alpha\beta} &= (-K^\gamma_\alpha K_{\mu\beta} + K^\gamma_\beta K_{\mu\alpha}) + h^\lambda_\alpha h^\rho_\beta h^\gamma_\sigma h^\nu_\mu \; R^\sigma_{\nu\lambda\rho} \end{aligned} \tag{4.27}$$

contracting the indices γ, α, we may obtain an expression for the intrinsic 3-curvature scalar ${}^3R$ as

$$\begin{aligned} {}^3R_{\mu\beta} &= (-K^\alpha_\alpha K_{\mu\beta} + K^\alpha_\beta K_{\mu\alpha}) + h^\lambda_\alpha h^\rho_\beta h^\alpha_\sigma h^\nu_\mu \; R^\sigma_{\nu\lambda\rho} \\ {}^3R &= g^{\mu\beta} \;\; {}^3R_{\mu\beta} = (-KK_{\mu\beta} + K^\alpha_\beta K_{\mu\alpha})g^{\mu\beta} + h^\lambda_\alpha h^\alpha_\sigma h^\rho_\beta h^\nu_\mu \; R^\sigma_{\nu\lambda\rho} \; g^{\mu\beta} \\ &= (-K^2 + K^{\mu\alpha}K_{\mu\alpha}) + h^\lambda_\sigma h^{\rho\mu} h^\nu_\mu \; R^\sigma_{\nu\lambda\rho} \\ &= (-K^2 + K^{\mu\alpha}K_{\mu\alpha}) + \underline{h^\lambda_\sigma h^{\rho\nu} \; R^\sigma_{\nu\lambda\rho}} \end{aligned} \tag{4.28}$$

Here,

$$\begin{aligned} \underline{h^\lambda_\sigma h^{\rho\nu} \; R^\sigma_{\nu\lambda\rho}} &= h^\lambda_\sigma (g^{\rho\nu} + n^\rho n^\nu) \; R^\sigma_{\nu\lambda\rho} \\ &= (h^\lambda_\sigma \; R^\sigma_\lambda) + (h^\lambda_\sigma n^\rho n^\nu \; R^\sigma_{\nu\lambda\rho}) \\ &= (R + n^\lambda n_\sigma \; R^\sigma_\lambda) + (n^\rho n^\nu R_{\nu\rho} + n^\lambda n_\sigma \; R^\sigma_{\nu\lambda\rho}) \\ &= (R + n^\lambda n^\sigma \; R_{\sigma\lambda}) + (n^\rho n^\nu \; R_{\nu\rho} + n^\lambda n_\sigma \; R^\sigma_{\nu\lambda\rho}) \\ &= R + 2n^\lambda n^\sigma \; R_{\sigma\lambda} \end{aligned}$$

And $K^\mu_\mu = K^a_b = K$ & $K_{\alpha\beta}K^{\alpha\beta} = K_{ab}K^{ab}$

we can write

$$R = ({}^3R + K^2 - K^{ab}K_{ab}) - 2n^\alpha n^\beta R_{\alpha\beta} \tag{4.29}$$

- From Eq. (4.19), we can write

$$\begin{aligned} R_{\alpha\beta} \; n^\alpha n^\beta &= (\nabla_\alpha \nabla_\beta n^\alpha - \nabla_\beta \nabla_\alpha n^\alpha) n^\beta \\ &= \nabla_\alpha (n^\beta \nabla_\beta n^\alpha) - \nabla_\alpha n^\beta \; \nabla_\beta n^\alpha - \nabla_\beta (n^\beta \nabla_\alpha n^\alpha) + \nabla_\beta n^\beta \; \nabla_\alpha n^\alpha \\ &= \nabla_\alpha (n^\beta \nabla_\beta n^\alpha) - K^{\alpha\beta} K_{\alpha\beta} - \nabla_\beta (n^\beta \nabla_\alpha n^\alpha) + K^2 \\ -2R_{\alpha\beta} \; n^\alpha n^\beta &= -2K^2 + 2K^{ab}K_{ab} + 2\nabla_\alpha (n^\beta \nabla_\beta n^\alpha - n^\alpha \nabla_\beta n^\beta) \end{aligned} \tag{4.30}$$

from Eq. (2.43) and (2.44)

$$\boxed{R = \left({}^{3}R - K^2 + K_{ab}K^{ab}\right) + 2\nabla_\mu(n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu)} \tag{4.31}$$

4.3 The Action

The Einstein-Hilbert action S_{EH}, which, with presence of boundary, the Action S_B and the action S_m in the presence of matter, is coupled to the corresponding matter field ϕ is defined as

$$S = S_{EH} \; + \; S_B \; + \; S_m$$

$$S = \frac{1}{16\pi G}\int_M (R - 2\Lambda)\sqrt{-g}\, d^4x \; + \; \frac{1}{8\pi G}\int_{\partial M} K\sqrt{-h}\, d^3x + S_m \tag{4.32}$$

where,

$$S_m \; = \; -\frac{1}{2}\int_M d^4x\sqrt{-g}\; \left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 2V(\phi)\right] \tag{4.33}$$

Where K is the trace of extrinsic curvature tensor K_{ab} of a spatial hypersurface and it is defined as

$$K_{ab} = \frac{1}{2}\left[-\frac{\partial h_{ab}}{\partial t} + D_a N_a + D_b N_a\right] \tag{4.34}$$

Now Putting the value of R & g in total action we will get

$$\begin{aligned} S &= \frac{1}{16\pi G}\int_M \left[\left({}^{3}R - K^2 + K_{ab}K^{ab}\right) - 2\Lambda\right] N\sqrt{h}\, t\, d^3x + S_m \\ &= \int_M L dt \end{aligned} \tag{4.35}$$

where [4]

$$-\int_{\partial M} \nabla_\mu(n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu) N\sqrt{h}\; d^3y = \int_{\partial M} K\sqrt{h}\; d^3y$$

Our next step is to find the conjugate momenta, which are essential for obtaining the Hamiltonian

$$\begin{aligned} \Pi^{ab} &\equiv \frac{\delta L}{\delta \dot{h}_{ab}} = -\frac{\sqrt{h}}{16\pi G}(K^{ab} - h^{ab}K) \\ \Pi_{\phi} &\equiv \frac{\delta L}{\delta \dot{\phi}} = \frac{\sqrt{h}}{N}(\dot{\phi} - N^{a}\ \partial_a \phi) \end{aligned} \tag{4.36}$$

The conjugate momenta corresponding to the lapse function and the shift vector vanish, leading to the imposition of primary constraints.

$$\Pi^0 \equiv \frac{\delta L}{\delta \dot{N}} = 0, \qquad \Pi^a \equiv \frac{\delta L}{\delta \dot{N}} = 0 \tag{4.37}$$

from the conjugate momenta we can obtained Hamiltonian [5]

$$\begin{aligned} H &= \int d^3x\ [\dot{h}_{ab}\ \Pi^{ab}\ + \dot{\phi}\ \Pi_{\phi}\ + \Pi^0 \dot{N} + \Pi^i \dot{N}_a] - L \\ &= \int d^3x [\Pi^0 \dot{N} + \Pi^i \dot{N}_a +\ N\mathcal{H}\ + N^a \mathcal{H}_a] \end{aligned} \tag{4.38}$$

and Action

$$\begin{aligned} S &= \int d^3x\ dt\ [\dot{h}_{ab}\ \Pi^{ab}\ + \dot{\phi}\ \Pi_{\phi}\ -\ N\mathcal{H}\ - N^a \mathcal{H}_a] \\ &= \int d^3x\ dt [\Pi^0 \dot{N} + \Pi^i \dot{N}_a -\ N\mathcal{H}\ - N^a \mathcal{H}_a] \end{aligned} \tag{4.39}$$

where

$$\begin{aligned} \mathcal{H} &= 16\pi G\ \mathcal{G}_{abcd}\ \Pi^{ab}\ \Pi^{cd} - \frac{\sqrt{h}}{16\pi G}({}^3R - 2\Lambda) + \mathcal{H}_m \\ \mathcal{H}^a &= -2D_a\ \Pi^{ab} +\ \mathcal{H}^i_m \end{aligned} \tag{4.40}$$

and

$$\mathcal{G}_{abcd} = \frac{1}{2\sqrt{2}}(h_{ac}h_{bd} + h_{ad}h_{bc} - h_{ab}h_{cd}) \tag{4.41}$$

is the DeWitt metric of signature $(-,+,+,+,+,+)$ can be be viewed as a metric on the superspace.
Varying equation (4.39) with respect to Π^{ab} and Π_{Φ} reproduces the definitions given in (4.36) and variation with respect to N imposes the Hamil-

tonian constraint

$$\mathcal{H} = 0 \tag{4.42}$$

while variation with respect to N_a enforces the momentum constraints

$$\mathcal{H}^a = 0 \tag{4.43}$$

4.4 Canonical Quantization

One of the main difficulties in developing a theory of quantum gravity comes from how the Hamiltonian behaves. When the constraints are fully applied, the Hamiltonian ends up being zero. This is surprising because, in quantum mechanics, the Hamiltonian is what drives changes in time — it tells us how a system evolves. If the Hamiltonian is zero, it suggests there is no change or flow of time, giving a "frozen" view of the universe. This problem points to a deeper conflict between quantum mechanics and general relativity, known as the "Problem of Time" [6].
The root of the problem is that these two theories have very different ideas about time. Quantum mechanics treats time as a fixed, absolute background that moves forward uniformly. General relativity, on the other hand, treats time as part of spacetime that can stretch and bend depending on matter and energy. To create a theory that combines both, we need to solve this conflict, especially for situations like the early universe or inside black holes where both quantum and gravitational effects are important.

To derive the Wheeler–DeWitt equation, we start with the Hamiltonian constraint (4.42) and apply Dirac's quantization method for constrained systems. This involves promoting the classical momenta to quantum operators:

$$\Pi^{ab} \to -i\frac{\delta}{\delta h_{ab}}, \quad \Pi_\phi \to -i\frac{\delta}{\delta\phi}. \tag{4.44}$$

Substituting these into the Hamiltonian constraint gives the Wheeler–DeWitt equation:

$$\hat{H}\Psi[h_{ab},\phi] = \left[-16\pi G^2 \mathcal{G}_{abcd}\frac{\delta^2}{\delta h_{ab}\delta h_{cd}} - \frac{1}{16\pi G}h^{1/2}\left({}^{(3)}R - 2\Lambda\right) + \hat{H}_m\right]\Psi = 0. \tag{4.45}$$

This equation is static; it contains no explicit time. Unlike in quantum mechanics, where time is an external parameter driving evolution, here time must emerge internally from the wave function itself. An idea is to define an internal clock $t(h_{ab},\phi)$. Here Ψ is the wave function of the universe, we will see it later and it must obey the Wheeler–DeWitt equation [7].

Chapter 5

Wavefunction of the Universe

5.1 Introduction

The Wheeler–DeWitt equation and the path integral approach offer two distinct but complementary perspectives on the problem of defining the quantum state of the universe. The Wheeler–DeWitt equation arises from applying the principles of quantum mechanics to general relativity and takes the form of a constraint equation [8]. It does not describe evolution in time — in fact, time itself is absent from the formalism — but instead imposes a condition that any candidate for the universe's quantum state must satisfy [9]. However, this equation alone is not sufficient to specify a unique solution. Since it lacks a notion of external time, there is no clear way to impose initial conditions as in conventional quantum mechanics. As a result, there exist infinitely many mathematically valid solutions, with no obvious principle to single out the one corresponding to our universe [10]. This is where the path integral approach becomes essential. Rather than postulating the wave function directly, it provides a method for constructing it by summing over possible histories of spacetime, subject to a chosen boundary condition. In particular, the Hartle–Hawking no-boundary proposal suggests summing over smooth, compact geometries without an initial boundary [7]. This offers a concrete criterion for selecting a specific quantum state that is consistent with the Wheeler–DeWitt equation. Thus, while the Wheeler–DeWitt equation defines the set of allowable quantum states, the path integral approach offers a way to choose among them. Taken together, these frameworks represent two parts of a broader effort to understand the origin and structure of the universe from a quantum perspective.

5.2 Path Integral Approach to the Wave Function

In quantum physics, the wave function provides a mathematical description of the state of a quantum system. To construct a wave function, one typically needs the quantum amplitude for a complete history of the system. For example, in single-particle quantum mechanics, the amplitude for a particle to propagate from an initial position (x_1, t_1) to a final position (x_2, t_2) is given by the path integral (*see fig 5.1*):

$$\Psi = \langle x_2, t_2 | x_1, t_1 \rangle = N \int \mathcal{D}x(t)\, e^{iS[x(t)]} \tag{5.1}$$

Here, the integration is performed over all possible paths $x(t)$ that the

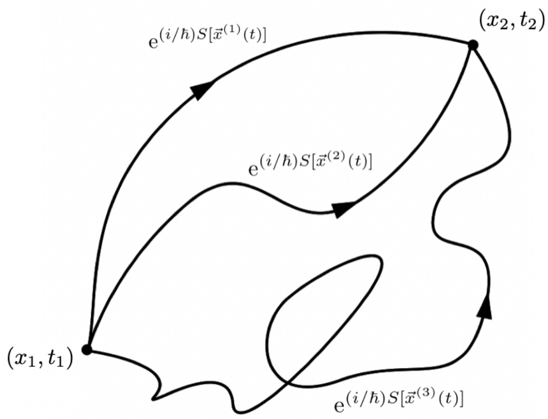

Figure 5.1: Path's of a quantum particle from x_1, t_1 to x_2, t_2

particle can take between the two spacetime points, and each path contributes a phase factor $e^{iS[x(t)]}$, where $S[x(t)]$ is the classical Lorentzian action associated with the path. N is a normalization constant.

By analogy, in quantum cosmology, one can express the ground-state wave function of a closed universe using a similar path integral formulation:

$$\Psi[h_{ij}, \phi] = \mathcal{N} \int_C \mathcal{D}g_{\mu\nu}\, \mathcal{D}\phi\, e^{iS[g_{\mu\nu},\phi]} \tag{5.2}$$

Here, $\Psi[h_{ij}, \phi]$ is the wave function of the universe corresponding to a given three-metric h_{ij} and matter field configuration ϕ on a spatial hypersurface. The integral is taken over all possible four-dimensional geometries $g_{\mu\nu}$ and matter field configurations ϕ.

One performs a Wick rotation to *Euclidean time* by substituting $t \to i\tau$, which transforms the Lorentzian action S into the *Euclidean action* I. The wave function then becomes:

$$\Psi[h_{ij}, \phi] = \mathcal{N} \int_C \mathcal{D}g_{\mu\nu}\, \mathcal{D}\phi\, e^{-I[g_{\mu\nu},\phi]} \tag{5.3}$$

In this Euclidean version, the path integral is over a class C of all *compact four-geometries* and matter field configurations that have a single three-dimensional boundary on which the geometry induces the metric h_{ij} and the field values match ϕ. This approach is central to the ***no-boundary proposal***, *where the universe is imagined to emerge from a smooth, compact geometry with no initial boundary*—a concept that offers a well-defined prescription for computing the wave function of the universe.

5.3 Minisuperspace Model with FRW Metric

The full path integral formulation of the wave function of the universe involves integration over all four-geometries and matter field configurations, which is computationally intractable due to the infinite degrees of freedom. To make progress, one typically imposes symmetry assumptions such as homogeneity and isotropy that lead to the minisuperspace approximation. This reduces the path integral to a finite-dimensional integral over a small set of variables, such as the scale factor and homogeneous matter fields.

To make the problem tractable, we simplify it to a *minisuperspace* model using a closed FRW universe metric:

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 d\Omega_3^2 \tag{5.4}$$

We let $t \to -i\tau$ (Wick rotation), converting this to Euclidean signature:

$$ds^2 = N(\tau)^2 d\tau^2 + a(\tau)^2 d\Omega_3^2 \tag{5.5}$$

Here:

- $a(\tau)$ is the scale factor,
- $N(\tau)$ is the lapse function,
- $\Phi(\tau)$ is the homogeneous scalar field.
- $d\Omega_3^2$ is the metric of a unit 3-sphere, indicating the universe is spatially closed (like the surface of a 4D ball).

In this case, the wave function depends on a, and the value of a scalar field ϕ, and is given by a path integral:

$$\Psi[a, \phi] = \int \mathcal{D}N \, \mathcal{D}a(\tau) \, \mathcal{D}\phi(\tau) \; e^{-I[N,a,\phi]} \tag{5.6}$$

This wavefunction has only two degrees of freedom. Now the Euclidean action for this model is:

$$I = \frac{1}{2} \int_0^1 d\tau \, N \left[-\frac{a}{N^2} \left(\frac{da}{d\tau} \right)^2 + \frac{a^3}{N^2} \left(\frac{d\Phi}{d\tau} \right)^2 - a + a^3 V(\Phi) \right] \tag{5.7}$$

want to evaluate $I[a(\tau)]$ for a **closed geometry** that starts from "nothing" $a(0) = 0$ and evolves to a given scale factor $\tilde{a} = a(1)$ at $\tau = 1$.

5.3.1 Semiclassical Approximation

Since the path integral involves integration over infinitely many possible field configurations, it is a functional integral over infinitely many degrees of freedom. Even after simplifying the problem using the minisuperspace, the resulting integral remains highly nontrivial due to the complexity of the action. As a result, solving it exactly by analytical means is generally not possible. So we use the saddle point approximation (or steepest descent method) to obtained the semiclassical approximation of the wave function.

Euler–Lagrange Equations

Varying the action with respect to a, ϕ, and N gives the field equations:

- Scalar field equation (from $\delta I/\delta\phi = 0$):

$$\frac{1}{N^2}\frac{d^2\phi}{d\tau^2} + \frac{3}{Na}\frac{da}{d\tau}\frac{d\phi}{d\tau} - \frac{1}{2}\frac{dV}{d\phi} = 0 \tag{3.36}$$

- Scale factor equation (from $\delta I/\delta a = 0$):

$$\frac{1}{N^2 a}\frac{d^2 a}{d\tau^2} + \frac{2}{N^2}\left(\frac{d\phi}{d\tau}\right)^2 + V(\phi) = 0 \tag{3.37}$$

- Hamiltonian constraint (from $\delta I/\delta N = 0$):

$$\frac{1}{N^2}\left(\frac{da}{d\tau}\right)^2 - \frac{a^2}{N^2}\left(\frac{d\phi}{d\tau}\right)^2 - 1 + a^2 V(\phi) = 0 \tag{3.38}$$

Impose the no-boundary condition

We require the 4-geometry to be *smoothly closed at* $\tau = 0$, like a 4-sphere. So we impose:

- $a(0) = 0$: ensures regular origin (no conical singularity),
- $\frac{d\Phi}{d\tau}(0) = 0$

Solving for the Scale Factor $a(\tau)$

The Eq. (3.37) becomes:

$$\frac{1}{N^2 a}\frac{d^2 a}{d\tau^2} + V = 0 \Rightarrow \frac{d^2 a}{d\tau^2} = -N^2 a V$$

This is a harmonic oscillator equation:

$$\Rightarrow a(\tau) = A\sin(\omega\tau) \quad \text{with } \omega = \sqrt{V}N$$

We impose $a(1) = \tilde{a}$ (value on boundary), so:

$$\tilde{a} = A\sin(\sqrt{V}N) \Rightarrow A = \frac{\tilde{a}}{\sin(\sqrt{V}N)} \Rightarrow a(\tau) = \frac{\tilde{a}\sin(\sqrt{V}N\tau)}{\sin(\sqrt{V}N)} \tag{3.41}$$

Applying the Saddle-Point Constraint (Hamiltonian Constraint)

We plug the solution back into the Hamiltonian constraint (3.38) at $\tau = 1$:

$$\begin{aligned} \tilde{a}^2V\cot^2(\sqrt{V}N) + \tilde{a}^2V - 1 = 0 \Rightarrow \tilde{a}^2V(1+\cot^2(\sqrt{V}N)) = 1 \\ \tilde{a}^2V\csc^2(\sqrt{V}N) = 1 \Rightarrow \sin^2(\sqrt{V}N) = \tilde{a}^2V \end{aligned} \tag{5.8}$$

This equation has multiple solutions*(see Appendix D)*:

$$\sqrt{V}N_n^{\pm} = \left(n + \frac{1}{2}\right)\pi \pm \cos^{-1}(\tilde{a}\sqrt{V}) \tag{3.45}$$

We choose the $n = 0$ solution, for simplicity and take $cos^{-1}(a\sqrt{V})$ to lies in the principal value $(0, \pi/2)$ we will get

$$a(\tau) \approx \frac{1}{\sqrt{V}}sin(\sqrt{V}N) = \frac{1}{\sqrt{V}}sin\left[\left(\frac{\pi}{2} \pm cos^{-1}(\tilde{a}V)\right)\tau\right] \tag{5.9}$$

Evaluating the Action

We substitute $a(\tau)$ into the original action (5.7) to evaluate the saddle-point action to compute the saddle-point approximation to the path integral:

$$\Psi(\tilde{a}, \tilde{\Phi}) \approx \sum_{\text{saddles}} e^{-I} \tag{5.10}$$

and obtained two possible solution of I at saddle point *(see Appendix D)*

$$I_{\pm} = -\frac{1}{3V(\tilde{\phi})}\left[1 \pm (1 - \tilde{a}^2V(\tilde{\phi}))^{3/2}\right] \tag{5.11}$$

5.3.2 Wave Function in minisuperspace FRW universe

The dominant contribution to the wave function comes from the I_- *(see fig 5.2)*, so wave function of the universe is:

$$\Psi \propto e^{-I_-} \tag{5.12}$$

this wave function (5.12) has two possible part which depends on $\tilde{a}^2 V$

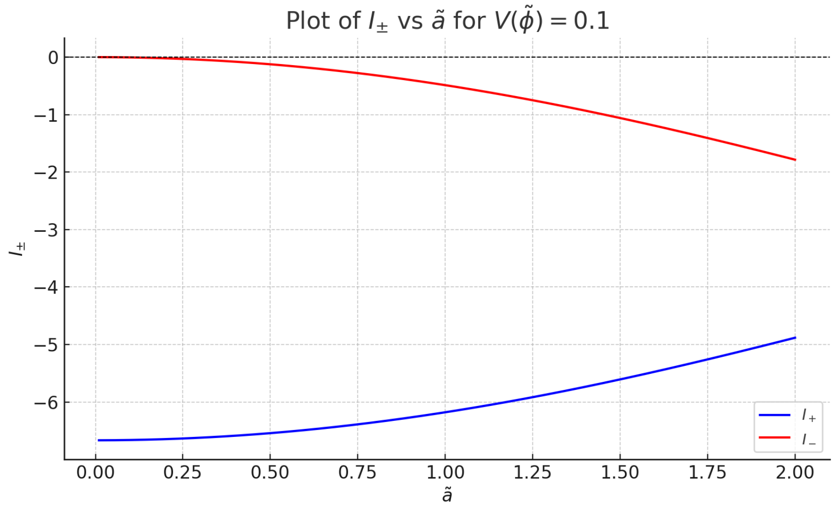

Figure 5.2: $I_\pm$ versus $\tilde{a}$ for a fixed scalar field potential $V(\tilde{\phi}) = 1$

- If $\tilde{a}^2 V < 1$, called *Classically forbidden region (see fig 5.13)*:

$$\Psi \propto \exp\left[\frac{1-(1-\tilde{a}^2 V)^{3/2}}{3V}\right] \tag{5.13}$$

- If $\tilde{a}^2 V > 1$, called *Classically allowed region (see fig 5.13)*:

$$\Psi \propto \exp\left[\frac{1}{3V}\right] \cos\left[\frac{(\tilde{a}^2 V - 1)^{3/2}}{3V} - \frac{\pi}{4}\right] \tag{5.14}$$

Behavior Summary

Condition	Nature of I_-	Wavefunction Ψ	Interpretation	Dominant Contribution
$\tilde{a}^2 V < 1$	Real	Exponential decay	Tunneling region	Ψ_-
$\tilde{a}^2 V > 1$	Complex	Oscillatory (classical)	Lorentzian evolution	Ψ_-

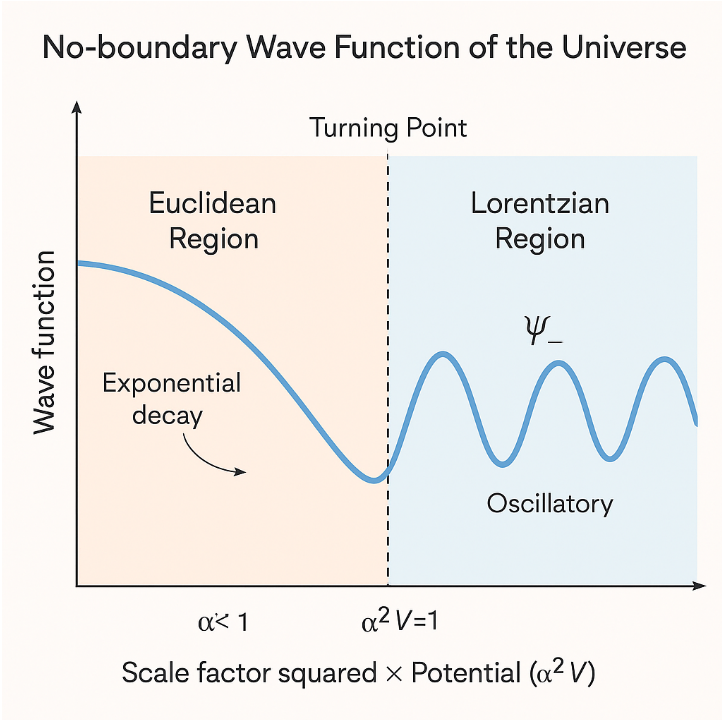

Figure 5.3: No boundary wave function of the FRW universe

5.4 Interpretation of Semiclassical Ψ

The semiclassical approximation of the no-boundary wave function has two distinct behaviors depending on the value of the dimensionless parameter $\tilde{a}^2V$, which determines whether the universe is in a classically forbidden or allowed region [7, 10].

- **Classically Forbidden Region** ($\tilde{a}^2V < 1$):

$$\Psi \propto \exp\left[\frac{1 - (1 - \tilde{a}^2V)^{3/2}}{3V}\right] \tag{5.15}$$

  In this region, which corresponds to small scale factors $\tilde{a}$, the wave function is real and exponentially increasing. This behavior is analogous to quantum tunneling, where the universe can emerge from "nothing" (no classical spacetime). The exponential growth of the wave function suggests that the universe tunnels from a quantum state without classical spacetime into an expanding classical universe. This supports the no-boundary idea that the universe could spontaneously nucleate from nothing [7].

- **Classically Allowed Region** ($\tilde{a}^2 V > 1$):

$$\Psi \propto \exp\left[\frac{1}{3V}\right] \cos\left[\frac{(\tilde{a}^2 V - 1)^{3/2}}{3V} - \frac{\pi}{4}\right] \tag{5.16}$$

For larger values of $\tilde{a}$, the wave function becomes oscillatory, which is a characteristic feature of classical behavior in quantum mechanics. The oscillations imply that the universe behaves classically in this region, following trajectories described by general relativity. Thus, after nucleating from a quantum regime, the universe undergoes classical expansion [10].

These two regimes together describe a cosmological scenario in which the universe originates via a quantum tunneling process from "nothing" and subsequently evolves classically, in agreement with the Hartle–Hawking no-boundary proposal [7].

Region	Condition	Wave Function Behavior	Physical Interpretation
Classically Forbidden	$\tilde{a}^2 V < 1$	Exponential decay	Quantum tunneling from "nothing"
Classically Allowed	$\tilde{a}^2 V > 1$	Oscillatory (cosine)	Classical evolution of the universe

Conclusion and future direction

This thesis examined the quantum origin of the universe through the Hartle–Hawking no-boundary proposal. Starting from the breakdown of classical general relativity at singularities, we turned to quantum cosmology and the Wheeler–DeWitt equation to describe the universe's wave function. Using the minisuperspace approximation, we simplified the dynamics to a few key variables and expressed the wave function via a Euclidean path integral.

In the semiclassical limit, we found that classical saddle points dominate the path integral, leading to two distinct behaviors: exponential in the classically forbidden region (tunneling) and oscillatory in the allowed region (classical evolution). This suggests a natural quantum-to-classical transition where the universe could have "tunneled" into existence.

Appendix A

Appendix

A.1

The Einstein Field Equations in their standard form are:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

R is the Ricci scalar, given by $R = g^{\mu\nu}R_{\mu\nu}$,

To find the trace, contract both sides with $g^{\mu\nu}$:

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}g_{\mu\nu} + \Lambda g^{\mu\nu}g_{\mu\nu} = \frac{8\pi G}{c^4}g^{\mu\nu}T_{\mu\nu}$$

Since $g^{\mu\nu}g_{\mu\nu} = D$ in D-dimensional spacetime, this becomes:

$$R - \frac{D}{2}R + D\Lambda = \frac{8\pi G}{c^4}T$$

where $T = g^{\mu\nu}T_{\mu\nu}$ is the trace of the stress-energy tensor. Simplifying, we get:

$$-R\left(\frac{D-2}{2}\right) + D\Lambda = \frac{8\pi G}{c^4}T$$

or

$$R = -\frac{16\pi G}{c^4(D-2)}T + \frac{2D\Lambda}{(D-2)}$$

Substitute R Back into the Original Equation

$$R_{\mu\nu} - \frac{1}{2}\left(-\frac{16\pi G}{c^4(D-2)}T + \frac{2D\Lambda}{(D-2)}\right)g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

$$R_{\mu\nu} + \frac{8\pi G}{c^4(D-2)} T g_{\mu\nu} - \frac{D\Lambda}{(D-2)} g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{(D-2)} T g_{\mu\nu} \right) + \frac{2}{(D-2)} \Lambda g_{\mu\nu}$$

The trace-reversed form of the Einstein Field Equations is if $c = 1$:

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{(D-2)} T g_{\mu\nu} \right) + \frac{2}{(D-2)} \Lambda g_{\mu\nu} \tag{A.1}$$

This form directly expresses $R_{\mu\nu}$ in terms of $T_{\mu\nu}$, T, and the cosmological constant Λ, without the R term.

A.2

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda i} \left(\partial_\mu g_{\nu i} + \partial_\nu g_{\mu i} - \partial_i g_{\mu\nu} \right)$$

$$\Gamma^{\lambda}_{tt} = \frac{1}{2} g^{\lambda i} \left(\partial_t g_{ti} + \partial_t g_{ti} - \partial_i g_{tt} \right)$$

from equation (3.2) we get

$$\boxed{\Gamma^{\lambda}_{tt} = 0} \tag{A.2}$$

Now,

$$\Gamma^{\lambda}_{t\sigma} = \frac{1}{2} g^{\lambda i} \left(\partial_t g_{\sigma i} + \partial_\sigma g_{ti} - \partial_i g_{t\sigma} \right)$$

$$\boxed{\Gamma^{\lambda}_{t\sigma} = \frac{1}{2} g^{\lambda i} \partial_t g_{\sigma i}} \tag{A.3}$$

A.3

Let $detM_{ij} = |M|$, where M_{ij} is d x d invertible matrix.

$$\begin{aligned} \delta M &= \delta\ exp(log|M|) \\ &= \delta\ exp(tr\ logM) \\ &= (trM^{-1}\delta M)\ exp(tr\ logM) \\ &= (trM^{-1}\delta M)\ |M| \end{aligned}$$

Hence

$$|M|^{-1}\,\delta|M| = M_{ij}\,\delta M_{ji} \tag{A.4}$$

To calculate derivatives, we can replace δ by $\frac{d}{dx}$ where x is some variable on which the matrix depends.

Another use full identity

$$\mathrm{tr}\left(M - \frac{1}{d}\,\mathrm{tr}(M)\right)^2 = \mathrm{tr}(M^2) - \frac{1}{d}\,(\mathrm{tr}\,M)^2 \tag{A.5}$$

proof:

This LHS expression can be expanded as

$$\mathrm{tr}\left(M - \frac{1}{d}\,\mathrm{tr}(M)\right)^2 = \mathrm{tr}\left(M^2 - \frac{2}{d}M\,\mathrm{tr}(M) + \frac{1}{d^2}(\mathrm{tr}(M))^2\right).$$

Use Linearity of the Trace Operator, which allows us to take the trace of each term individually:

$$= \mathrm{tr}(M^2) - \frac{2}{d}\,\mathrm{tr}(M)\,\mathrm{tr}(M) + \frac{1}{d^2}\,(\mathrm{tr}(M))^2.$$

$$= \mathrm{tr}(M^2) - \frac{2}{d}\,(\mathrm{tr}(M))^2 + \frac{1}{d^2}\,(\mathrm{tr}(M))^2.$$

$$= \mathrm{tr}(M^2) - \left(\frac{2}{d} - \frac{1}{d^2}\right)(\mathrm{tr}(M))^2.$$

So, we get:

$$\mathrm{tr}\left(M - \frac{1}{d}\,\mathrm{tr}(M)\right)^2 = \mathrm{tr}(M^2) - \left(\frac{2}{d} - \frac{1}{d^2}\right)(\mathrm{tr}(M))^2.$$

This completes the derivation.

Appendix B

Appendix

Inverse of the metric

We define a matrix $g_{\mu\nu}$ as:

$$g_{\mu\nu} = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$$

where: A is a 1×1 matrix , B is a 3×1 matrix, B^T is a 1×3 matrix, C is a 3×3 matrix, we can write,

$$g_{\mu\nu} g^{\nu\rho} = I$$

where $g^{\nu\rho}$ is the inverse of $g_{\mu\nu}$. The above Eq. expand as:

$$\begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \begin{bmatrix} X & Y^T \\ Y & Z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}$$

$$AX + B^T Y = 1 \tag{B.1}$$

$$AY^T + B^T Z = 0 \tag{B.2}$$

$$BX + CY = 0 \tag{B.3}$$

$$BY^T + CZ = I \tag{B.4}$$

From the equation (B.1):

$$AX + B^T Y = 1$$

$$X = A^{-1} - A^{-1}B^TY \tag{B.5}$$

Solving for Y From (B.3):

$$BX + CY = 0$$

$$Y = -C^{-1}BX$$

Substituting X:

$$Y = -C^{-1}B(A^{-1} - A^{-1}B^TY)$$

$$Y + C^{-1}BA^{-1}B^TY = -C^{-1}BA^{-1}$$

$$(I + C^{-1}BA^{-1}B^T)Y = -C^{-1}BA^{-1}$$

Multiplying both sides by $(I + C^{-1}BA^{-1}B^T)^{-1}$:

$$Y = -(I + C^{-1}BA^{-1}B^T)^{-1}C^{-1}BA^{-1}$$

$$Y = -(C^{-1}BA^{-1} + (B^T)^{-1})$$

So:

$$Y = -C^{-1}B(A - B^TC^{-1}B)^{-1} \tag{B.6}$$

A is a scalar 1×1 matrix. $X = (A-B^TC^{-1}B)^{-1}$ and $X^{-1} = (A-B^TC^{-1}B)$ are also scalar. We know scalars transpose to themselves i.e. $A = A^T, X = X^T$. Since $B^TC^{-1}B$ is 1×1, so we can write $(C^{-1})^T = C^{-1}$. Taking the transpose of Y:

$$Y^T = \left(-C^{-1}B(A - B^TC^{-1}B)^{-1}\right)^T$$

$$Y^T = -(A - B^TC^{-1}B)^{-1}B^T(C^{-1})^T$$

Since C^{-1} is symmetric, $C^{-1} = (C^{-1})^T$, we conclude:

$$Y^T = -(A - B^TC^{-1}B)^{-1}B^TC^{-1} \tag{B.7}$$

Solving for Z From Eq.(B.4):

$$BY^T + CZ = I$$

Substituting Y^T:

$$B(-C^{-1}B(A - B^TC^{-1}B)^{-1})^T + CZ = I$$

$$CZ = I - BC^{-1}B(A - B^TC^{-1}B)^{-1}$$

$$Z = C^{-1} + C^{-1}B(A - B^TC^{-1}B)^{-1}B^TC^{-1} \tag{B.8}$$

Thus, the inverse matrix is:

$$g^{\mu\nu} = \begin{bmatrix} (A - B^TC^{-1}B)^{-1} & -(A - B^TC^{-1}B)^{-1}B^TC^{-1} \\ -C^{-1}B(A - B^TC^{-1}B)^{-1} & C^{-1} + C^{-1}B(A - B^TC^{-1}B)^{-1}B^TC^{-1} \end{bmatrix}$$

To find the inverse of the metric:

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^aN_a & N_1 & N_2 & N_3 \\ N_1 & h_{11} & h_{12} & h_{13} \\ N_2 & h_{21} & h_{22} & h_{23} \\ N_3 & h_{31} & h_{32} & h_{33} \end{pmatrix}$$

where: $A = -N^2 + N^aN_a$, $B^T = (N_1, N_2, N_3)$, $B = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$, $C = h_{ij}$.

Computing $A - B^TC^{-1}B$. We define $C^{-1} = h^{ij}$, which satisfies $h^{ik}h_{kj} = \delta^i_j$ and $N^i = h^{ij}N_j$. Now

$$A - B^TC^{-1}B = (-N^2 + N^aN_a) - N^ih^{ij}N_j$$

$$= (-N^2 + N^aN_a) - N^iN_i = -N^2$$

So,

$$X = (A - B^TC^{-1}B)^{-1} = -\frac{1}{N^2}$$

similarly Computing Y and Z:

$$Y = -C^{-1}B(A - B^TC^{-1}B)^{-1}$$

Substituting,

$$Y = -h^{ij}N_j\left(-\frac{1}{N^2}\right) = \frac{N^i}{N^2}$$

For Z:

$$Z = C^{-1} + C^{-1}B(A - B^TC^{-1}B)^{-1}B^TC^{-1}$$

$$Z = h^{ij} + h^{ik}N_k\left(-\frac{1}{N^2}\right)N_lh^{lj}$$

$$Z = h^{ij} - \frac{N^iN^j}{N^2}$$

Thus, the inverse metric is:

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^1}{N^2} & \frac{N^2}{N^2} & \frac{N^3}{N^2} \\ \frac{N^1}{N^2} & h^{11} - \frac{N^1N^1}{N^2} & h^{12} - \frac{N^1N^2}{N^2} & h^{13} - \frac{N^1N^3}{N^2} \\ \frac{N^2}{N^2} & h^{21} - \frac{N^2N^1}{N^2} & h^{22} - \frac{N^2N^2}{N^2} & h^{23} - \frac{N^2N^3}{N^2} \\ \frac{N^3}{N^2} & h^{31} - \frac{N^3N^1}{N^2} & h^{32} - \frac{N^3N^2}{N^2} & h^{33} - \frac{N^3N^3}{N^2} \end{pmatrix}$$

$$g^{00} = -\frac{1}{N^2}, \quad g^{0a} = g^{a0} = \frac{N^a}{N^2}, \quad g^{ab} = h^{ab} - \frac{N^aN^b}{N^2}$$

This is the standard inverse metric in the (3+1) decomposition of space-time.

Appendix C

Appendix

C.1

We begin with the equation:

$$\sin^2\left(\sqrt{V}N\right) = \tilde{a}^2 V \tag{C.1}$$

To solve for N, define $\theta = \sqrt{V}N$, so the equation becomes:

$$\sin^2(\theta) = \tilde{a}^2 V$$

Taking the square root on both sides:

$$\sin(\theta) = \pm\sqrt{\tilde{a}^2 V} = \pm\tilde{a}\sqrt{V}$$

Thus, the general solution for θ is:

$$\theta = \sin^{-1}(\pm\tilde{a}\sqrt{V}) = \pm\sin^{-1}(\tilde{a}\sqrt{V}) \tag{C.2}$$

However, since the sine function is periodic and symmetric, its general solution includes all branches:

$$\theta = n\pi + (-1)^n \sin^{-1}(\tilde{a}\sqrt{V}), \quad n \in \mathbb{Z}$$

This expression can be equivalently written using the identity:

$$\sin(x) = \cos\left(\frac{\pi}{2} - x\right) \quad \Rightarrow \quad \sin^{-1}(x) = \frac{\pi}{2} - \cos^{-1}(x)$$

Hence, we obtain:

$$\theta = \left(n + \frac{1}{2}\right)\pi \pm \cos^{-1}(\tilde{a}\sqrt{V}) \tag{C.3}$$

Substituting back $\theta = \sqrt{V}N$, we arrive at:

$$\sqrt{V}N_n^{\pm} = \left(n + \frac{1}{2}\right)\pi \pm \cos^{-1}(\tilde{a}\sqrt{V}) \tag{C.4}$$

This expression gives the infinite discrete set of complex or real solutions $N_n^{\pm}$, which correspond to the saddle points in the semiclassical approximation of the no-boundary wave function in Euclidean quantum cosmology.

C.2

Evaluating the Action

The Euclidean Einstein-Hilbert action with a cosmological constant (or scalar potential V) in the simplified minisuperspace is given by:

$$I[a] = \frac{3\pi}{4}\int_0^1 d\tau \left[-\frac{\dot{a}^2}{N}a - Na + Na^3V\right]$$

where:

$$a(\tau) = \frac{1}{\sqrt{V}}\sin(\sqrt{V}N\tau)$$

$$\dot{a}(\tau) = N\cos(\sqrt{V}N\tau)$$

We integrate over $\tau \in [0, 1]$.

Substitute the classical solution into the action:

$$a(\tau) = \frac{1}{\sqrt{V}}\sin(\sqrt{V}N\tau)$$

$$\dot{a}(\tau) = N\cos(\sqrt{V}N\tau)$$

$$\dot{a}^2 = N^2\cos^2(\sqrt{V}N\tau)$$

So the terms in the action become:

$$\frac{\dot{a}^2}{N}a = \frac{N^2\cos^2(\sqrt{V}N\tau)}{N}\cdot\frac{1}{\sqrt{V}}\sin(\sqrt{V}N\tau) = N\cos^2(\sqrt{V}N\tau)\cdot\frac{1}{\sqrt{V}}\sin(\sqrt{V}N\tau)$$

$$Na = N\cdot\frac{1}{\sqrt{V}}\sin(\sqrt{V}N\tau)$$

$$Na^3V = N\left(\frac{1}{\sqrt{V}}\sin(\sqrt{V}N\tau)\right)^3 V = N\frac{\sin^3(\sqrt{V}N\tau)}{V^{3/2}}V = N\frac{\sin^3(\sqrt{V}N\tau)}{\sqrt{V}}$$

Instead of doing the full integral, we use the fact that on-shell (i.e., classical) the action reduces to a boundary term. From the Hamiltonian constraint and integrating by parts, one can derive:

$$I_{\text{on-shell}} = -\frac{1}{2}a\dot{a}\Big|_{\tau=1}$$

At $\tau = 1$:

$$a(1) = \tilde{a}$$

$$\dot{a}(1) = N\cos(\sqrt{V}N)$$

So:

$$I = -\frac{1}{2}\tilde{a}\cdot N\cos(\sqrt{V}N)$$

Recall from earlier:

$$\tilde{a} = \frac{1}{\sqrt{V}}\sin(\sqrt{V}N) \Rightarrow \sin(\sqrt{V}N) = \tilde{a}\sqrt{V}$$

So:

$$\cos(\sqrt{V}N) = \pm\sqrt{1-\tilde{a}^2V}$$

Thus:

$$I = -\frac{1}{2}\tilde{a}\cdot N\cdot\sqrt{1-\tilde{a}^2V}$$

But this is still in terms of N, and we want an expression just in $\tilde{a}$ and V. So we use:

$$\tilde{a} = \frac{1}{\sqrt{V}}\sin(\sqrt{V}N) \Rightarrow \sqrt{V}N = \sin^{-1}(\tilde{a}\sqrt{V})$$

And the two saddle points correspond to:

$$\sqrt{V}N = \left(n + \frac{1}{2}\right)\pi \pm \cos^{-1}(\tilde{a}\sqrt{V}) \Rightarrow \cos(\sqrt{V}N) = \mp\sqrt{1 - \tilde{a}^2 V}$$

So the action becomes:

$$I_\pm = -\frac{1}{2}\tilde{a} \cdot N \cdot (\mp\sqrt{1 - \tilde{a}^2 V}) \Rightarrow I_\pm = \pm\frac{1}{2}\tilde{a}N\sqrt{1 - \tilde{a}^2 V}$$

Now recall $\tilde{a} = \frac{1}{\sqrt{V}}\sin(\sqrt{V}N)$, and so from some algebra (detailed in Hartle-Hawking papers), one arrives at the compact formula:

$$I_\pm = -\frac{1}{3V}\left[1 \pm \left(1 - \tilde{a}^2 V\right)^{3/2}\right]$$

Interpretation

There are two saddle points:

- One corresponds to a geometry that "closes off" before the equator of a 4-sphere (like a small cap).
- The other corresponds to a geometry that "closes off" after the equator of a 4-sphere (a large cap).

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