Thermodynamics of Einstein-Maxwell-Dilaton Black Hole in (2+1) Dimensions

M.Sc. Thesis

by

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Discipline of Physics

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by

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"Thermodynamics of Einstein-Maxwell-Dilaton Black Hole in (2+1) Dimensions"** in the partial fulfilment of the requirements for the award of the degree of MASTER OF SCIENCE and submitted in the **DISCIPLINE OF PHYSICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from JULY 2018 to JUNE 2019 under the supervision of Dr. Manavendra Mahato, Associate Professor, Indian Institute of Technology Indore. The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Swati Malhotra

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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(PSPC Member 1.) (PSPC Member 2.)

Date: Date:

Dedicated

to

My Family

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Abstract

Curvature in space time fabric arises by taking stress energy tensor as some source field. Motivation for using matter fields are their abundance in nature and Schwarszchild black hole is one of the most symmetric static massive solution of Einstein field equation. Many other solutions of Einstein field equation are also found by modifying Einstein's action with some gauge fields and other theories. Effective theory can be arise if one takes higher energy theory such as string theory. Einstein's action is naturally modified with some scalar-tensor superstring terms at sufficiently high energy scale. On taking low energy limit Einstein's action is recovered with a dilaton scalar field which is coupled to gravity.

In this thesis, we work on Einstein-Maxwell-Scalar theory in (2+1) dimensions that contains a scalar field coupled minimally to gravity and a potential that depends solely on scalar field. We obtained an exact solution containing a black hole with a regular horizon. Further, we also work on the thermodynamic properties of black hole and try to prove the First law of Thermodynamics for our solutions.

- The first chapter give elementary details of Einstein's equation and its solution. The brief description of Black hole with three characteristic properties: mass, charge and angular momentum is given in this chapter.

- In the second chapter, the motivation for the project work, modification of Einstein's action with Maxwell-Scalar terms and the equation of motions are given.

- In the third chapter we work on the anastz in (2+1) dimensions and discussed various techniques used to obtained the exact solution of Einstein-Maxwell-Scalar gravity. The komar integral are used to prove first law of Black hole thermodynamics.

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Chapter 1

Introduction

Objects moved when they are pushed but what does freely falling object experience? the unsatisfactory answer gravitational force gave Einstein the inspiration to developed a theory of ages, renowned as General Theory of Relativity. Einstein's tensorial theory of Gravitation is the achievement of modern physics which explains universe keeping time and space on equal footing. The intricate mathematics of general relativity deals with the geometry of space-time and Einstein's equation finally coined that matter and curvature work together. One of the beautiful solution of Einstein's equation makes this theory more significant. Black Hole, a region which is so curved because of its massiveness that nothing not even light can escape from it. Before stepping into the essence of this thesis, we briefly define the terms used in it.

1.1 Einstein's Equation

The visualization of space being like a globe where you parallel transport a vector on a curve end up crossing each other gives intuitive idea of space being curved and what the force is. According to Einstein, space and time are inextricably linked with each other and can get distorted by massive objects. Einstein Field Equation describes the fundamental relationship of space-time being curved by mass and energy.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi g}{c^4}T_{\mu\nu} \tag{1.1}$$

It is a symmetric 4 × 4 tensor equation in which each tensor has 10 independent components giving 10 non linear coupled differential equation which reduces to 6 using Bianchi Identities. The EFE can also be written in compact form as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi g}{c^4}T_{\mu\nu} \tag{1.2}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{1.3}$$

is the symmetric second-rank Einstein tensor which describes the curvature of space. The term with Λ tells about the energy density of space and the term on the right of EFE is stress-energy-tensor that tells us about mass, energy, momentum, pressure etc.

1.1.1 Metric and Christoffel symbol

Metric is a symmetric, second rank, covariant tensor that computes the distance between two points on a differential manifold. It represent geometrical structure of any spacetime in a bilinear form that takes two tangent vectors X at any given point on manifold to produce a real number. The metric tensor is given as:

$$G : X \times X \to R$$

Christoffel symbol measures the misalignment of coordinate system and geodesics motion of a vector. It shows the deviation of covariant vector on parallel transporting it on a curved spacetime. It is given in terms of coordintes as: [9]

$$\Gamma^k_{ij} = \frac{g^{kl}}{2}(g_{il,j} + g_{jl,i} - g_{ij,l})$$

Since Christoffel symbol depends upon the coordinate system so it can be removed with the proper choice of a frame in the coordinate system. The metric connection of Riemannian Geometry in terms of coordinates of a manifold is represented by the Christoffel symbols.

1.1.2 Riemann Curvature Tensor

Riemann curvature tensor is used to measure the curvature of spacetime on Riemannian manifold. It assigns a tensor to each point on a Reimannian manifold and measures the curvature by parallel transporting the vector from one direction to other on manifold or vice versa. The magnitude of curvature can be determined by:[9][10]

$$R^{\rho}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

1.1.3 Ricci Tensor and Ricci Scalar

On contracting Reimannian curvature tensor we get Ricci tensor given as:

$$R^{\rho}_{\mu\rho\nu} = R_{\mu\nu}$$

Ricci tensor measures the degree of convergence and divergence of geodesic from the volume at a given point on a Reimannian manifold.
On further contracting the Ricci tensor we get Ricci scalar. It measures the amount of deviation of volume of any object from the volume taken in Euclidean space. [9]

- If Ricci scalar is positive, volume will increase and we can have de-Sitter space.

- If its value is negative, volume will decrease and we get anti-de Sitter space.

- If its value is zero, volume remain same and we get flat spacetime.

and Ricci scalar is given as:

$$R = g^{\mu\nu} R_{\mu\nu}.$$

Ricci scalar are invariant under coordinate transformations as they are scalar, thus are more useful in analysing a metric. They directly provide coordinate-invariant information about a given metric.

1.1.4 Stress Energy Tensor

Stress energy tensor is a symmetric, rank-2 tensor and a source of gravitational field in space time fabric. It is an attribute of matter fields, electromagnetic field and other non-gravitation force fields.

From mathmatical point of view, a scalar can be associated to an element of n-dimensional volume with the help of current density. If normal four vector n gives the orientation of surface in space time then element of three volume is $\mathbf{n\Delta V}$. Mass density transforms as a 00-component of rank 2 tensor. To associate a four vector Δp^{α} to nΔV, we need two indices $T^{\alpha\beta}$, such that [9][10]

$$\Delta p^{\alpha} = T^{\alpha\beta} n_{\beta} \Delta V$$

The second rank tensor $T^{\alpha\beta}$ is called stress energy tensor.

In order to have a physical interpretation, lets consider an inertial frame in flat space time and three dimensional volume ΔV, along time like slice.

If n_{α} = (1,0,0,0) is the normal along three dimensional volume which is at rest then components of stress energy tensor are:

T^{00} is energy density

T^{0i} is the energy flux in the i-th-direction;

T^{i0} is the momentum density in the i-th-direction;

T^{ij} is i-th component of force per unit area along a surface with normal in j-th direction.

So, Einstein's equation tells mass warps the space and warped space acts like a moving mass. The left hand side of Einstein equation tells how the curvature is changing while travelling along a vector and term containing $g_{\mu\nu}$ tells how measurements are affected. Right hand side of the EFE tells what physical quantities govern these changes.

1.2 Killing vector and Conserved Charges:

We are used to the fact that symmetries lead to conserved quantities (Noether's theorem). For example, in classical mechanics, the angular momentum of a particle moving in a rotationally symmetric gravitational field is conserved. In the present context, the concept of 'symmetries of a gravitational field' is replaced by 'symmetries of the metric', and we therefore expect conserved charges associated with the presence of Killing vectors. Two important class of this phenomena is

Killing Vectors, Geodesics and Conserved Charges Let Killing vector field be K^μ, and geodesic be $x^\mu(\tau)$, then the quantity

$$Q_K = K^\mu \dot{x}^\mu(\tau)$$

is constant along geodesic. Indeed,

$$\frac{d}{d\tau}Q_K = \frac{d}{d\tau}(K^\mu \dot{x}^\mu(\tau)) = \frac{\partial K_\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} + K_\mu \frac{d^2x^\mu}{d\tau^2}$$

$$= \left(K_{\alpha;\nu} + K_\alpha \Gamma^\alpha_{\mu\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau}\right) + K_\mu \frac{d^2x^\mu}{d\tau^2}$$

$$= K_\mu \left(\Gamma^\alpha_{\mu\nu}\frac{dx^\nu}{d\tau}\frac{dx^\alpha}{d\tau} + \frac{d^2x^\mu}{d\tau^2}\right) = 0,$$

deduced from Nother's theorem, Q_k is a conserved quantity.

Conserved Currents from the Energy-Momentum Tensor: Let K^μ be a Killing vector field, and $T^{\mu\nu}$ the covariantly conserved symmetric energy-momentum tensor satisfying, $\nabla_\mu T^{\mu\nu} = 0$. Then the current

$$J^\mu_k = T^{\mu\nu}K_\nu$$

is covariantly conserved. Indeed,

$$\nabla_\mu J_k^\mu = (\nabla_\mu T^{\mu\nu})K_\nu + T^{\mu\nu}\nabla_\mu K_\nu$$

$$= 0 + \frac{1}{2}T^{\mu\nu}(\nabla_\nu K_\mu + \nabla_\mu K_\nu) = 0$$

here also we have conserved current. Thus, taking symmetries of space time metric, charges related to corresponding Killing vectors as:

- charge generated by timelike Killing vector is mass of black hole.
- charge generated by electromagnetic vector forces are electric and magnetic charges of black hole.
- charge generated by azimuthal-symmetry Killing vector is angular momentum of black hole.

1.2.1 Komar Currents

In the above section, we discussed the symmetries and corresponding conserved currents related to it. The conserved current when written as hypersurface integral of components of divergence of some tensor associated to conserved charge. These are then defined as Komar current associated to the symmetry of the metric. The currents associated with Killing vectors turn out to play a privileged role. Komar currents can be used to find characteristic properties of a system as:

Komar Mass: Komar current for a timelike Killing vector is the total mass of an isolated (asymptotically) static system, given as:

$$M_{Komar} = \frac{1}{4\pi}\int_{\partial\sigma} d^{d-2}\sqrt{\gamma}\eta_\mu\sigma_\nu\triangle^\mu K^\nu \tag{1.4}$$

where $\partial\sigma$ is the boundary of σ at spatial infinity. γ_{ij} is the induced metric on $\partial\sigma$ and σ_ν is the outward pointing unit normal to $\partial\sigma$.

Komar Charge: Conserved current associated to the electromagnetic tensor $F_{\mu\nu}$ gives the total Komar charge contained in a system. The components like F_{tr} are related to electric current and $F_{\theta\phi}$ are for magnetic current. We can associate a Komar charge to spacelike surface σ by:

$$Q_{Komar} = -\int_{\partial\sigma} d^{d-2}x\sqrt{\gamma}\eta_\mu\sigma_\nu F^{\mu\nu} \tag{1.5}$$

We have used Komar integrals in later sections.

1.3 Solution of Einstein's Field Equation

All physically possible solutions of Einstein's equation provide a description of gravitation and space time geometry. The solutions of Einstein's equation depend upon type of source fields like matter field, scalar field, electromagnetic fields etc. We can plug the type of field into EFE and can get prediction for the phenomena of interest. Unfortunately, the Einstein's equations are highly coupled non-linear partial differential equations. Moreover, in (3+1) dimensions there are 20 unknowns that need to be calculated from 14 equations which is bit tricky to solve. One can work on simple solutions by increasing the number of symmetries and degrees of freedom. There are many solutions of Einstein's equation in which Schwarszchild metric is one of the most spherically symmetric solution to explain space time geometry. Further, different solutions were found by coupling Einstein's action with Maxwell terms such as charged Reissner Nordstrom black hole. On reducing some symmetries, we can find different solution with different properties like Kerr Newmann Black Hole which is a massive charged rotating black hole.

1.3.1 Symmetrical (static) Solutions

Schwarzschild Black Hole:

It is most symmetrical vacuum solution of Einstein's equation with the assumption that electric charge, angular momentum, and cosmological constant are all zero and have only one characteristic of mass. Schwarzschild metric is given by:[12][10][11]

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \frac{1}{\left(1 - \frac{2GM}{c^2 r}\right)} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

Properties of Schwarzschild Black Hole:

- The metric coefficients are independent of time, which corresponds to a Killing vector field leaving local geometry of spacetime along the integral of that vector field unchanged.

- The metric is diagonal, making spacetime to be hypersurface orthogonal.

- The last two terms are just standard metric of 2-sphere, which makes spacetime to be spherically symmetrical. The 2-sphere is labelled by the coordinates t and r, which makes the tangent vector of 2-sphere to be orthogonal to $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial r}$ vector fields. Thus metric is independent of θ and ϕ coordinates. For 2-sphere, it has three Killing vectors which preserves the angular momentum.

- The M appearing in the metric is the property of space time geometry, i.e. it is the mass of source of curvature discussed above. Variable r in the last part of metric is related to the area of 2-sphere and it is not related to the radius of black hole.

$$r = \sqrt{\frac{A}{4\pi}}$$

- The metric blows at $r = 0$ and $r = \frac{2GM}{c^2}$, which can be considered as singular points. But to find true singularity one can use scalar quantities that are independent of the choice of coordinates. Kretschmann scalar is one such important quantity and in Schwarszchild metric it is given as:

$$R_{abcd}R^{abcd} = \frac{48G^2M^2}{r^6}$$

which shows $r = 0$ is the true singularity of Schwarszchild black hole.

Reissner Nordstrom Black Hole

RN black hole is the static, sypherically symmetric, charged, non-rotating solution of Einstein's equation with a source constituted by Maxwell's field, given as:

$$T_{\mu\nu} = \frac{1}{4\pi}\left(g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)$$

Equation of motion will be

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}$$

$$\nabla_{\mu}F^{\mu\nu} = 0$$

The metric of the given system is given by:

$$ds^2 = -f(r)c^2dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right)$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

and the radial component of electric field will be:

$$E_r = F_{rt} = \frac{Q}{r^2}$$

RN solution depends on the relation of charge and mass. From the roots of f(r) given as:[10][12]

$$r\pm = M \pm \sqrt{M^2 - Q^2}$$

- if $|Q| < M$, then the value of f(r) is positive, with two horizon values r_+ and r_- interpreted as outer and inner horizons respectively. The exterior horizon look quantitaively similar to Schwarzschild metric.

- if $\mid Q \mid \; > M$, everything is well-behaved for positive r as $f(r)$ is also positive and at r=0 we get naked singularity.

- at $Q = 0$ limit, we get the Schwarzschild spacetime with radius $r_s = 2M$.

1.3.2 Symmetrical (Nonstatic) Solution

Kerr Newmann Black Hole

It is generalized spinning case of Schwarschild Black Hole. Kerr Newmann metric represents the special case of rotating, charged massive black hole with most asymptotically spherically symmetric solution of Einstein-Maxwell equation. There are two coordinate system use to describe the geometry of rotating black hole: Boyer-Lindquist coordinates and Kerr-Schild coordinates. Kerr Newmann metric in Boyer-Lindquist coordinate system is given as:[12][11]

$$c^2 d\tau^2 = -\left(\frac{dr^2}{\triangle} + d\theta^2\right)\rho^2 + (cdt - asin^2\theta d\phi)^2 \frac{\triangle}{\rho^2} - ((r^2+a^2)d\phi - acdt)^2 \frac{sin^2\theta}{\rho^2}$$

where

$$a = \frac{J}{Mc}$$
$$\rho^2 = r^2 + a^2 cos^2\theta$$
$$\triangle = r^2 - r_s r + a^2 + r_Q^2$$
$$r_s = \frac{2GM}{c^2}$$
$$r_Q^2 = \frac{Q^2 G}{4\pi\varepsilon_0 c^4}$$

Thus, black hole solutions of Einstein' equation for different source field give characteristics properties of black hole as massive, charged and rotating to give no-hair theorem. According to no-hair theorem, black hole solution of Einstein-Maxwell equation can be characterized by only three properties i.e. mass, spin and rotation. Therefore, on furthur modifying Einstein' action with some scalar-tensor terms we can observe different properties of black hole. In low energy limit, Einstein' action is modified with some dilatonic scalar field which is coupled to gravity. In the next chapter we discuss about Einstein-Maxwell-Scalar Theory.

Chapter 2

Einstein-Maxwell-Scalar Theory

2.1 Introduction

The attempt to combine the fundamental forces and elementary particles is an open area of research since long. Einstein's equation gave much information about geometry of space time but effective gravity action can emerge by modifying Einstein's action with some other source field in a coupled way. EMD theory generated by coupling scalar field with electromagnetic field minimally or nonminimally in which the sign of kinetic energy term of scalar or Maxwell is opposite to protect the violation of no-hair theorem. The asymptotically flat static black hole solutions for Einstein-Maxwell-dilaton system enriched the physics of the solution of EFE compared to Reissner-Nordström solution, which is the charged black hole solution of the Einstein-Maxwell theory.

2.2 General form of Einstein-Maxwell-Dilaton Action:

Einstein-Maxwell-Dilaton gravity in (d+1) dimensions in which scalar field coupled minimally to gauge field (the Maxwell field) or gravity. The action contains some kinetic terms of fields and few other coupling terms. The general form of Einstein-Maxwell-Dilaton action for d dimensions is given as:[14]

$$S = \int \sqrt{-g} d^d x (R - 2\partial_\mu(\phi)\partial^\nu(\phi) - W(\phi)F_{\mu\nu}F^{\mu\nu} - V(\phi)), \tag{2.1}$$

which is for some dilaton ϕ. $W(\phi)$ is usually an exponential function of the dilaton field and $V(\phi)$ is self-interacting potential. The form of the field strength is given as,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.2}$$

The equation of motion obtained after perturbing the action are as follows

for metric:

On perturbing it with respect to $g_{\mu\nu}$, and using time reversal technique we get,

$$R_{\mu\nu} = 2\partial_\mu(\phi)\partial_\nu(\phi) - \frac{1}{2}g_{\mu\nu}W(\phi)F_{\rho\sigma}F^{\rho\sigma} + 2W(\phi)F^{\mu\rho}F^{\rho}_{\nu} + \frac{1}{2}g_{\mu\nu}V(\phi) = 0. \tag{2.3}$$

for dilaton:

On perturbing it with respect to (ϕ), we get,

$$\nabla_\mu(\partial^\mu \phi) - \frac{1}{4}\frac{W(\phi)}{\phi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\frac{\partial V}{\partial \phi} = 0. \tag{2.4}$$

for gauge field:

On perturbing it with respect to A_μ, we obtain,

$$\nabla_\mu(W\phi F^{\mu\nu}) = 0. \tag{2.5}$$

2.3 (2+1) dimensional space time

Gravity in $(2+1)$ dimensional spacetime has been a fascinating area of theoretical investigations during the last few decades. There are two reasons for it. First, the technical difficulties present in a wide range of problems in $(3+1)$ dimensional gravitation become significantly simpler in lower dimensions. Moreover the study of gravity in (2 + 1) dimensions is also expected to shed some light on the understanding of more realistic or complicated cases of four and higher dimensional gravities. Second, (2+1)-dimensional gravity with a matter source has attracted considerable interest including standard Maxwell term, the inclusion of extra scalar field(s), higher rank tensor fields, higher curvature terms which are intensively studied.

2.3.1 BTZ Black Hole

The first study of three dimensional black hole, as a result of Einstein theory of relativity was done by Bana-dos, Teitelboim and Zanelli, and the solution is commonly known as BTZ black hole. After the discovery of first BTZ black hole, there is a flood of different types of black holes in (2+1) dimensions. The BTZ black holes are different from Schwarzschild and Kerr black hole as they are asyptotically anti-de Sitter rather than asymptotically flat and has no curvature singularity at the origin. It comes out to be asymptotically anti-de Sitter because with zero cosmological constant, Ricci tensor and simultaneously Riemann tensor vanish and there would be no black hole in (2+1) dimensions.[1]

Nonetheless, it is a black hole with properties much similar to (3+1) dimensional black hole as:

- it has characteristic properties of ADM mass, charge and angular momentum and admits no-hair theorm.
- thermodynamical properties are much similar to (3+1) dimensional black hole.
- it has event horizon, inner and outer horizon like Kerr black hole
- it appears as a final state of collapsing matter

BTZ Black hole in Schwarzschild coordinates The BTZ black hole in Schwarzschild coordinates is given by the metric

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2 \tag{2.6}$$

where

$$N = \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{\frac{1}{2}},$$

$$N^{\phi} = \frac{-J}{2r^2}, \qquad\qquad (|J| \leq Ml).$$

- Coordinate singularity of the metric is at points

$$r_{\pm}^2 = \frac{Ml^2}{2}\left(1 \pm \left(1 - \left(\frac{J}{Ml}\right)^2\right)^{\frac{1}{2}}\right).$$

- Value of mass and angular momentum measured at the boundary is

$$M = \frac{r_+^2 + r_-^2}{l^2}, \qquad\qquad J = \frac{2r_+r_-}{l}.$$

2.4 AdS space time

Scalar coupled theory in different space time has gained much interest since they often arise as effective theories in low energy limits of theories suitable at higher energies. Dilaton field coupled with different fields like gauge fields can change the structure of space time geometry and we get the solution of dilaton coupled action with gravity or gauge field in some de Sitter or anti de Sitter space time. Gao and Zhang obtained the first asymptotically non flat charged dilaton black hole solution in de-Sitter and anti de Sitter space time using Schwarzschild coordinate system. For super symmetric theory in various dimensions, solutions were obtained with anti de Sitter space time. Thus the accompanying vacuum state and black hole solution in such a space time is an important area to explore.

Mathematically, AdS_n is an n dimensional solution of Einstein-Hilbert action with negative cosmological constant, signature (p, q) isometrically embedded in the space $R^{p,q+1}$ with coordinates $x_1, ...x_p, t_1, ..., t_{q+1}$ and metric

$$ds^2 = \sum_{i=1}^{p} dx_i^2 - \sum_{j=1}^{q+1} dt_j^2$$

and three-dimensional anti-de sitter space obtained from coordinated (X_1, X_2, T_1, T_2) and metric is

$$ds^2 = dX_1^2 + dX_2^2 - dT_1^2 - dT_2^2$$

The hypersurface may be defined, through the algebraic constraint,

$$-X_1^2 - X_2^2 + \sum_{i=3}^{n+1} X_i^2 = \frac{(n-1)(n-2)}{2\Lambda}$$

involving $n+1$ coordinates X_1^{n+1} in $n+1$-dimensional flat space with signature $(n-1, 2)$. In order to show the hypersurface satisfies the Einstien equations with a negative cosmological constant, we need to solve the constraints for X in terms of n coordinates.

Denoting the n coordinates on the submanifold σ, the induced metric on the submanifold can be expressed as,

$$\gamma_{ab} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b}$$

which is related by a projection operator to the first fundamental form, $h_{\mu\nu}$. From this metric, one can now directly compute the curvature tensors and arrive at the Einstein equations.

2.5 Black Hole Thermodynamics

The dicussion of classical Black hole thermodynamics appears to be same as ordinary laws of thermodynamics applied to a system containing a black hole. The analogy of thermodynamic behavior where the horizon area playing the role of entropy is striking. This gave the boost to the theory with more terms coined for black hole thermodynamics as

surface gravity: For a Killing vector field ξ, however for all Killing horizon, we only have the freedom to rescale ξ by a constant, and if we have a preferred normalisation for ξ, then κ_ξ is uniquely determined and is known as the surface gravity of the Killing horizon or of the corresponding black hole. An equivalent definition of κ is the the magnitude of the acceleration, with respect to Killing time, of a stationary zero angular momentum particle just outside the horizon.

Temperature: Temperature of the black hole is connected to the whole concept of Hawking radiation. The idea that black hole generate virtual particles at the edge of event horizon and these particles recombine and disappear in a puff of annihilation near event horizon. Thus temperature of the black hole relate to the surface gravity at horizon.

By its striking behaviour, there must be some relationship between properties of black hole and classical thermodynamic laws, which are then formulated as

- surface gravity remains constant on event horizon for a stationary black hole
- total mass of the black hole is related to charge, angular momentum, entropy of the black hole as

  $$\partial M = \frac{\kappa}{8\pi} dA + \Phi dQ + \Omega dJ$$

  where, κ is the surface gravity at horizon
- total area of the black hole is increased in every process
- $\kappa = 0$ is not possible in any process

For studying black hole thermodynamics and other black hole related event always consider;[10]

$$M > Q > 0$$

Two reason related to this choice are;

- In order to follow the Cosmic Censorship principle given by Penrose which states that all singularities need to be hidden from an observer at infinity by the event horizon of black hole.

- Not much is known about the naked singularity.

We can divide the metric in three regions;

$$Region1 : r_+ < r < \infty$$

$$Region2 : r_- < r < r_+$$

$$Region3 : 0 < r < r_-$$

For a particle crossing the event horizon from region 1 to region 2, an observer far away from the geometry of black hole would think him to be redshifted. However the falling observer takes finite time to reach the horizon.

Inside region 2, particles move in the direction of decreasing r. On reaching the region 3, r switches back to spacelike coordinate and hence we are saved from being hit by the singularity. One can continue moving in the direction of singularity or can move in the direction of increasing r. Again on reaching region 2, r swaps its nature but with reversed direction.

AdS Black Hole Thermodynamics

AdS black hole Thermodynamics is much extended as it considers cosmological constant as one of the thermodynamical parameter. Thus, the first law of thermodynamics is modified by VdP term, where $P = -\frac{\Lambda}{8\pi}$ and in this context mass is considered as enthalpy of the system rather then internal energy.[2]

Chapter 3

Einstein-Maxwell-Dilaton Theory in (2+1) dimensions

3.1 Introduction

We aim to study black hole solution of an Einstein-Maxwell gravity with minimally coupled scalar field in $(2+1)$ dimensions. Black hole solution comprising gravity coupled to a scalar field is known as hairy black hole. The general form of Einstein Maxwell dilaton action in $(2+1)$ dimensions is given as,

$$S = \int \sqrt{-g} d^3x (R - 2\partial_\mu(\phi)\partial^\nu(\phi) - W(\phi)F_{\mu\nu}F^{\mu\nu} - V(\phi)) \tag{3.1}$$

which contains some dilaton ϕ. $W(\phi)$ is usually an exponential function of the dilaton field and $V(\phi)$ is self-interacting potential. The form of the field strength is given as,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.2}$$

Here, we start with an action in which scalar couples to gravity in a minimal way and it also couples to itself via a self interacting potential $V(\phi)$.

$$S = \int \sqrt{-g} d^3x (R - g^{\mu\nu}\nabla_\mu(\phi)\nabla_\nu(\phi) - 2V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}) \tag{3.3}$$

where,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.4}$$

The equations of motion obtained after perturbing the action are as follows:

for metric:

On perturbing it with respect to $g_{\mu\nu}$, and using trace reversal technique we get,

$$R_{\mu\nu} - \partial_\mu\phi\partial_\nu\phi - 2g_{\mu\nu}V - \frac{1}{2}F_{\mu\rho}F^{\rho}_{\nu} + \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} = 0. \tag{3.5}$$

for dilaton:

On perturbing it with respect to (ϕ) we get,

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) - \frac{\partial V}{\partial\phi} = 0. \tag{3.6}$$

for gauge field:

On perturbing it with respect to A_μ we get,

$$\partial_\nu(\sqrt{-g}F^{\mu\nu)} = 0. \tag{3.7}$$

We work on the ansatz of the following form,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + y^2(r)dz^2, \tag{3.8}$$

where coordinate ranges from $-\infty \leqslant \mathrm{t} \leqslant \infty$, $\mathrm{r} \geqslant 0$ and $-\pi \leqslant \mathrm{z} \leqslant \pi$, Let us consider that A_μ and ϕ are functions of radial coordinates only. Using Eq. 3.7 and above assumption, Maxwell's field can be given by chossing another ansatz,

$$A_\mu = F(r)dt, \tag{3.9}$$

such that,

$$F_{tr} = -\frac{Q}{y(r)}, \tag{3.10}$$

where Q is a constant.

Using above ansatz we can write below the equation of motion by substituting the value of $R_{\mu\nu}$,

$$ff'y' + fyf'' + 4fyV = 0, \tag{3.11}$$

$$-\frac{f'y'}{2fy} - \frac{f''}{2f} - \frac{y''}{y} - (\phi')^2 - \frac{2V}{f} = 0, \tag{3.12}$$

$$-yf'y' - yfy'' - 2y^2V - Q^2 = 0, \tag{3.13}$$

$$\frac{y'f\phi'}{y} + f'\phi' + f\phi'' - \frac{dV}{d\phi} = 0. \tag{3.14}$$

On multiplying Eq. 3.12 by $(2yf^2)$ and then add to Eq. 3.11, we get a relation between $y(r)$ and ϕ, given as

$$\frac{y''}{y} = -(\phi')^2. \tag{3.15}$$

In Eq. 3.12 on putting the value of ϕ from Eq. 3.15, we can get the equation for potential $V(\phi)$ as:

$$V = -\frac{f'y'}{4y} - \frac{f''}{4}. \tag{3.16}$$

In Eq 3.13 on putting the value of potential, we get second order non-homogenous differential equation as:

$$-\frac{f'y'}{2fy} - \frac{y''}{y} + \frac{f''}{2f} = \frac{Q^2}{fy^2}. \tag{3.17}$$

Eq. 3.15, 3.16 and 3.17 can be used to determine the expression for scalar field, potential, $f(r)$ and $y(r)$.

We can solve differential Eq. 3.15 by assuming a form of $y(r)$, which we choose to be,

$$y(r) = g(\phi(r)) \tag{3.18}$$

Solving for ϕ then gives,

$$\phi = c_2 - \frac{\log(r\alpha + c_1)}{\alpha} \tag{3.19}$$

where α is some constant from calculation. By rescaling r, we can take $c_1 = 0$ and then ϕ becomes

$$\phi = \phi_0 - \frac{1}{\alpha} ln(r). \tag{3.20}$$

Here, $\phi_0 = c_2 - \frac{1}{\alpha} ln(\alpha)$. On doing further suitable calculations, we get the expression for $y(\phi)$ of the form

$$y(\phi) = e^{\frac{1}{2}(-\alpha-\sqrt{-4+\alpha^2})\phi} c_3 + e^{\frac{1}{2}(-\alpha+\sqrt{-4+\alpha^2}\phi)} c_4. \tag{3.21}$$

Thus we obtained exact expression for scalar field and $y(r)$. To obtain the form of $f(r)$, we use equation Eq. 3.17 which is bit tedious to solve by ordinary differential equation solving method so we use numerical methods to solve the differential equation. We solved $f(r)$ in steps i.e. for small value of r.

3.2 Asymptotic Solution for small r

To ease our problem, we use series solution method for small r and assume the solution of $f(r)$ as

$$f(r) = \sum_{n=0}^{6} a_n r^n. \tag{3.22}$$

Expanding Eq. 3.19 for small r up to fifth order for Eq. 3.17, we get,

$$\phi(r) = \left(c_2 - \frac{\log(c_1)}{\alpha}\right) - \frac{r}{c_1} + \frac{\alpha r^2}{2c_1^2} - \frac{\alpha^2 r^3}{3c_1^3} + \frac{\alpha^3 r^4}{4c_1^4} - \frac{\alpha^4 r^5}{5c_1^5} + O(r)^6. \tag{3.23}$$

Using DSolve technique in mathematica and truncating the result upto sixth order, we get the series solution of differential Eq. 3.17. The expression for $f(r)$ for small value of r is given in Appendix [4.1]
To get a value of event horizon, we solved the problem numerically using NDSolve technique in mathematica for small value of r. NDSolve technique is based on Runge Kutta Method, that gives an iterative solution when initial condition is known at one end but has to be evaluated at the other boundary.

Since the range of r is from 0 to infinity, we make our range finite by reparameterized r to $u=\frac{1}{1+r}$ such that, u lies in finite range between 0 and 1. We confined our calculation to small r and upper range of u was set to be 1. From the graph we got the numerical value of event horizon at 0.880812 for $f(r)$ to be zero in new range of r, as shown in Fig.[3.1]

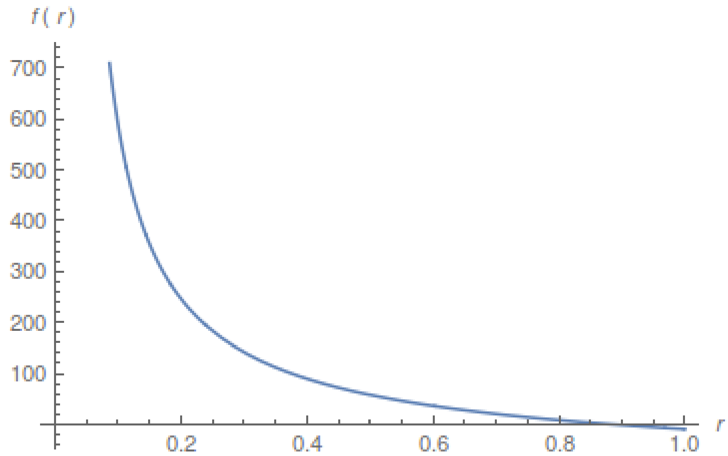

Figure 3.1: Graph between $f(r)$ vs r for $a_0 = a_1 = Q = c_2 = c_1 = c_3 = c_4 = 1$ and $\alpha = 3$

Here, the coordinates of t and r change their sign on the other end of the boundary signifing a black hole region. Root of $f(r)$ for which curvature invariants viz., Ricci scalar, $R_{ab}R^{ab}$ and Kretschmann scalar $R_{abcd}R^{abcd}$, give finite result is interpreted as horizon. The equation of curvature invariants are given respectively

$$R = -\frac{2f'y' + yf'' + 2fy''}{y}, \tag{3.24}$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{(3f'^2y'^2 + y^2f''^2 + 2fyf''y'' + 4f^2y''^2 + 2f'y'(yf'' + 3fy''))}{2y^2}, \tag{3.25}$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{2f'^2y'^2 + y^2f''^2 + 4ff'y'y'' + 4f^2y''^2}{y^2}. \tag{3.26}$$

To check whether the horizon is coordinate, naked or a real singularity, we find numerically the values of curvature invariants at event horizon and putting the numerical value of $f(r)$ and $y(r)$ in above equations, they are calculated to be -143.974, 14193.1 and 36043.9 respectively. For our case they are finite values which means we have coordinate singularity.

The value of potential $V(\phi(r))$ is found by substituting the value of $f(r)$, $\phi(r)$ and $y(r)$ in Eq. 3.16 to get the expression given in Appendix[4.1]. The graph between $V(r)$ and r is shown in Fig.[3.2]

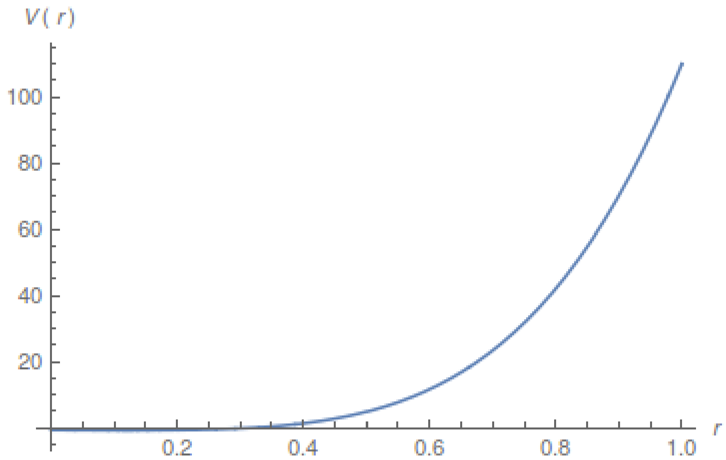

Figure 3.2: Graph between $V(r)$ and r for $a_0 = a_1 = Q = c_2 = c_1 = c_3 = c_4 = 1$ and $\alpha = 3$

Since the value of potential increases at other boundary where $u = 1$ and also the space time metric swap their signs, so we can signify the black hole region near $r \to 0$ for r to be in its whole infinite range.

Kinetic energy of scalar field: The kinetic term of the scalar field at boundary is given as

$$K.E. = \partial_\mu \phi \partial^\nu \phi, \tag{3.27}$$

$$K.E. = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \tag{3.28}$$

For, $\mu = \nu = r$,

$$K.E. = g^{rr} \partial_r \phi \partial_r \phi. \tag{3.29}$$

$$K.E.(r \to \infty) = f(r) \left(\frac{1}{r\alpha + c_2} \right)^2 = 0 \tag{3.30}$$

$$K.E.(r \to 0) = f(r) \left(\frac{1}{r\alpha + c_2} \right)^2 = \frac{a_0}{c_2^2} \tag{3.31}$$

Thus, kinetic term of scalar field become zero at $r \to \infty$ and have a contact value at $r \to 0$ i.e. near black hole region.

Komar charge: The electric current associated to the electromagnetic field tensor $F_{\mu\nu}$ is conserved, so we can associate an electric charge to spacelike surface σ by,

$$Q_{Komar} = -\int_{\partial\sigma} d^{d-2}x\sqrt{\gamma}\eta_\mu\sigma_\nu F^{\mu\nu}, \tag{3.32}$$

where $\partial\sigma$ is the boundary of σ at spatial infinity. γ_{ij} is the induced metric on $\partial\sigma$ and σ_ν is the outward pointing unit normal to $\partial\sigma$. For our case:

$$\eta^\mu = (f(r)^{\frac{-1}{2}}, 0, 0),$$

$$\sigma^\nu = (0, f(r)^{\frac{1}{2}}, 0),$$

$$\sqrt{\gamma} = y(r),$$

which gives us the value of Komar charge as,

$$Q_{Komar} = 2\pi Q$$

Komar mass: Using the above analogy we can define Komar Mass since we have timelike Killing vector which can give rise to energy/mass :

$$M_{Komar} = \frac{1}{4\pi}\int_{\partial\sigma} d^{d-2}\sqrt{\gamma}\eta_\mu\sigma_\nu\triangle^\mu K^\nu \tag{3.33}$$

$$M_{Komar} = \frac{a_1}{4}e^{-\frac{(\alpha+\sqrt{-4+\alpha^2})\tilde{\phi}}{2}}\left(c_3 + e^{\sqrt{-4+\alpha^2}\tilde{\phi}}c_4\right) \tag{3.34}$$

for $r \to 0$ and $\tilde{\phi} = \frac{c_2\alpha - \log c_1}{\alpha}$. Thus mass can be interpreted in terms of a_1 and parameters of $y(r)$ and scalar field at boundary.

From expression of ϕ at $r \to 0$, we get

$$\phi = \frac{c_2\alpha - \log c_1}{\alpha}$$

3.3 Special case: $Q = 0$

Now we consider a case for $Q = 0$, and solve second order homogenous differential equation given as,

$$-\frac{f'y'}{2fy} - \frac{y''}{y} + \frac{f''}{2f} = 0. \tag{3.35}$$

We adopt the same method as $Q \neq 0$ to solve this differential equation for small value of r. The expression for $f(r)$ given in Appendix[4.2].

To get the value of event horizon, we used NDSolve technique in mathematica based on Runge Kutta Method in this case also. On reparameterizing r to $u = \frac{1}{1+r}$ finite our range such that u lies between 0 and 1.

We confined our calculation to small r and upper range of u was set to be 1. From the graph we got the numerical value of event horizon at 0.479657 for $f(r)$ to be zero in new range of r, as shown in Fig.[3.3]

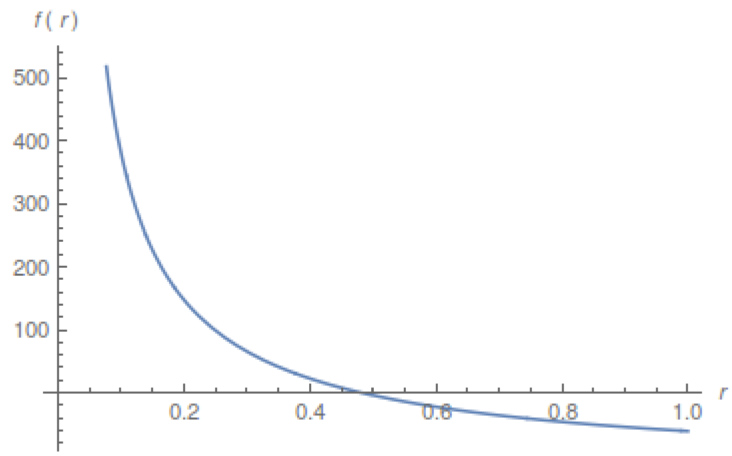

Figure 3.3: Graph between $f(r)$ vs r for $a_0 = a_1 = Q = c_2 = c_1 = c_3 = c_4 = 1$ and $\alpha = 3$

Here also the sign of coordinates t and r change on the other end of the boundary. Same as previous case, we find numerically the values of curvature invariant at horizon and they are calculated to be -760.218, 353483. and 836002. respectively. Their finite values shows coordinate singularity in this case also.

Potential $V(\phi)$ is given by substituting the value of $f(r)$, $\phi(r)$ and $y(r)$ in

Eq. 3.16 to get the expression given in Appendix[4.2] The graph between $V(r)$ and r is shown in Fig.[3.4], the value of potential increases at boundary where space time swap their signs.

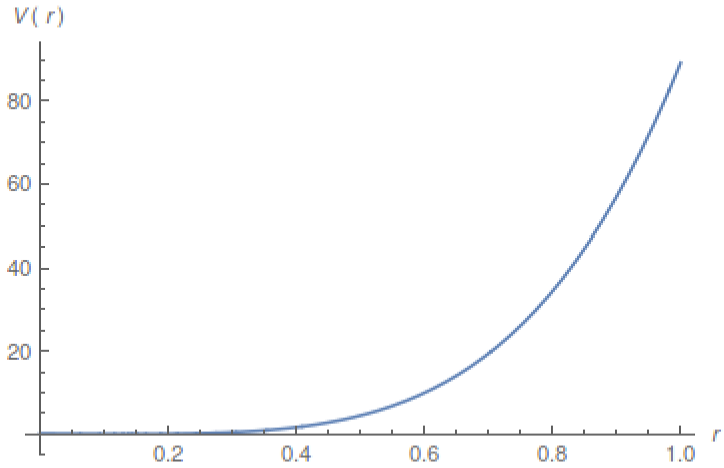

Figure 3.4: Graph between $V(r)$ and r for $a_0 = a_1 = Q = c_2 = c_1 = c_3 = c_4 = 1$ and $\alpha = 3$

Kinetic energy of scalar field and Komar mass have the same values in this case also.

We have expression for ϕ and $y(\phi(r))$. We will consider $\alpha = \frac{5}{2}$ for the special case. Implementing the value of α. our expression for $y(\phi)$ will be,

$$y(\phi) = c_1 e^{-\frac{\phi}{2}} + c_2 e^{-2\phi}, \tag{3.36}$$

and ϕ will be modified accordingly

$$\phi = \phi_0 - \frac{2}{5} ln(r). \tag{3.37}$$

Substituting the value of ϕ from Eq. 3.37 into Eq. 3.36, we get the expression as,

$$y(\phi) = c_1 r^{\frac{1}{5}} e^{-\frac{\phi_0}{2}} + c_2 r^{\frac{4}{5}} e^{-2\phi_0}. \tag{3.38}$$

Further we get from Eq. 3.17 the following equation for Q=0 case as,

$$-\frac{f'y'}{2fy} - \frac{y''}{y} + \frac{f''}{2f} = 0. \tag{3.39}$$

On solving Eq. 3.12, we get the expression for potential for $Q = 0$ case, which is given as,

$$V = -\frac{f'y'}{4y} - \frac{f''}{4}. \tag{3.40}$$

3.3.1 Solution for $y(r) = r^{\frac{1}{5}}$

To solve differential Eq. 3.39, we consider c_1=1,c_2=0 and ϕ_0=0 in Eq. 3.36 to get $y(r) = r^{\frac{1}{5}}$. On putting the values of ϕ and y in Eq. 3.39, we obtain the final differential equation as,

$$5rf'' + \frac{8f}{5r} - f' = 0. \tag{3.41}$$

The solution for the above differential Eq. 3.41 is,

$$f(r) = r^{\frac{2}{5}}c_1 + r^{\frac{4}{5}}c_2. \tag{3.42}$$

Let $c_1 = -r_0^{\frac{2}{5}}$ and $c_2 = 1$ we get,

$$f(r) = r^{\frac{2}{5}}[-r_0^{\frac{2}{5}} + r^{\frac{2}{5}}]. \tag{3.43}$$

Our motive here is to get the event horizon for which we substitute f(r)=0 in Eq. 3.43. We obtained event horizon at $r = r_0$ and curvature singularity at $r = r_0$ and $r = 0$ for which spacial coordinate blow up.
The graph between $f(r)$ and r is shown in Fig[3.5] which shows $r \to 0$ be the region of black hole in this case as sign of $f(r)$ is negative near this boundary.

For the curvature singularities Ricci scalar, Ricci tensor squared and

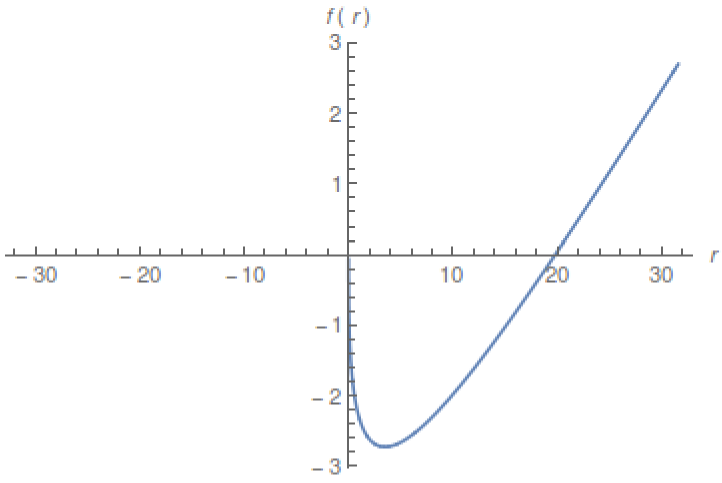

Figure 3.5: Graph between $f(r)$ and r for $y(r) = r^{\frac{1}{5}}$

Kretschmann scalar can be written as,

$$R = \frac{2(5a^{\frac{8}{5}}r^{\frac{4}{5}} + 21a^{\frac{4}{5}}r^{\frac{8}{5}} - 11a^{\frac{2}{5}}r^2 + 2r^{\frac{12}{5}} - 17(ar)^{\frac{6}{5}})}{25(-a^{\frac{2}{5}}r + r^{\frac{7}{5}})^2(r^{\frac{4}{5}} - (ar)^{\frac{2}{5}})}, \tag{3.44}$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{4(11a^{\frac{4}{5}} + 4(r^{\frac{4}{5}} - 3(ar)^{\frac{2}{5}}))}{625r^{\frac{16}{5}}}, \tag{3.45}$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{4(a^{\frac{2}{5}} - r^{\frac{2}{5}})^4(19a^{\frac{4}{5}} + 12r^{\frac{4}{5}} - 28(ar)^{\frac{2}{5}})}{625r^{\frac{12}{5}}(a^{\frac{4}{5}} + r^{\frac{4}{5}} - 2(ar)^{\frac{2}{5}})(r^{\frac{4}{5}} - (ar)^{\frac{2}{5}})^2}, \tag{3.46}$$

respectively. The value of Ricci scalar, norm of Ricci tensor and Kretschmann scalar was found to be finite for finite values of the radial component r.

On substituting Eq. 3.42 into Eq. 3.40 we obtain the value of potential as,

$$V = -\frac{r_0^{\frac{2}{5}}}{25r^{\frac{8}{5}}}, \tag{3.47}$$

which at $r \to \infty$ is zero shows asymptotically flat space time in (2+1) dimensions. The graph between $V(r)$ vs r is shown in Fig.[3.6]

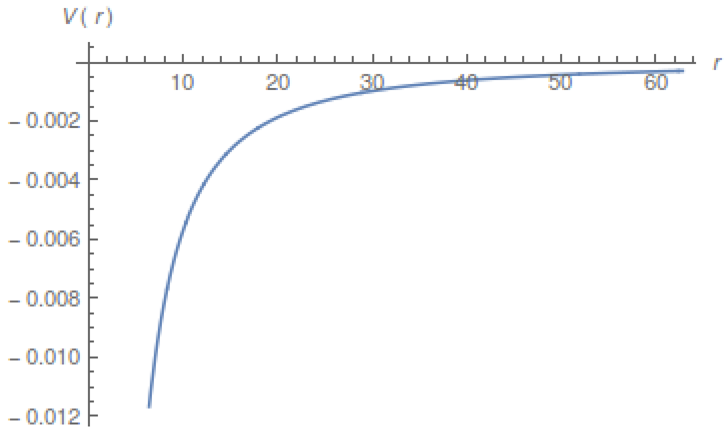

Figure 3.6: Graph between $V(r)$ and r for $y(r) = r^{\frac{1}{5}}$

Scalar field at boundary: At $r \to 0$, the value of scalar field comes out to be constant i.e. ϕ_0 and kinetic term of the scalar field is given as,

$$K.E. = \partial_\mu \phi \partial^\nu \phi, \tag{3.48}$$

$$K.E. = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \tag{3.49}$$

For, $\mu = \nu = r$,

$$K.E. = g^{rr} \partial_r \phi \partial_r \phi, \tag{3.50}$$

$$K.E. = r^{\frac{2}{5}}(-r_0^{\frac{2}{5}} + r^{\frac{2}{5}}) \left(\frac{4}{25r^2} \right), \tag{3.51}$$

which shows that scalar field decays at large value of r.

Komar Mass

Using the above analogy we can define Komar Mass since we have timelike Killing vector allows us to define associated conserved charges as energy/-mass as follows,[12][11]

$$M_{Komar} = \frac{1}{4\pi} \int_{\partial\sigma} d^{d-2} \sqrt{\gamma} \eta_\mu \sigma_\nu \triangle^\mu K^\nu \tag{3.52}$$

In this case,

$$\eta^\mu = (f(r)^{\frac{-1}{2}}, 0, 0),$$

$$\sigma^\nu = (0, f(r)^{\frac{1}{2}}, 0),$$

$$\sqrt{\gamma} = y(r),$$

$$K^{\nu} = (-1, 0, 0).$$

It gives Komar mass as,

$$M_{Komar} = \frac{1}{5} - \frac{{r_0}^{\frac{2}{5}}}{10r^{\frac{2}{5}}}. \tag{3.53}$$

First law of thermodynamics

Before we proceed further, let us examine the generalized first law of thermodynamics for this particular solution. The metric is then given as

$$f(r) = 10r^{\frac{4}{5}}\left(M - \frac{1}{10}\right) \tag{3.54}$$

The parameter M in eq 3.54 represents the Komar mass of the black hole. The event horizon is located at $r = r_0$ for which $f(r_0) = 0$, gives the value of mass as,

$$M = \frac{1}{10} \tag{3.55}$$

The surface gravity, temperature and entropy at event horizon (derivation is given in Appendix E [4.5]) is given as

$$\kappa = \frac{1}{2}\frac{\partial f}{\partial r} \qquad T = \frac{1}{4\pi}f'(r_0) = 0 \qquad S = \frac{\pi}{2} \times y(r_0)$$

Taking variation of following quantites, we get

$$\partial M = 0 \qquad \partial S = \frac{\pi}{10}{r_0}^{\frac{-4}{5}} \qquad T\partial S = 0$$

such that

$$\partial M = T\partial S$$

which satisfies first law of thermodynamics for asymptotically flat uncharged black hole solution in (2+1) dimensions.

3.3.2 Solution for $y(r) = r^{\frac{4}{5}}$

Since we are working on different conditions to obtain the exact solution of Einstein-Maxwell-Dilaton coupled theory, and already investigate one solution for which we prove first law of thermodynamics.

For a different solution we consider second case, for which we assume c_1=0, c_2=1 and ϕ_0=0 in Eq. 3.36 to get $y(r) = r^{\frac{4}{5}}$. On solving differential Eq. 3.39 for new values of ϕ and $y(r)$, we obtain the final differential equation as,

$$5rf'' + \frac{8f}{5r} - 4f' = 0. \tag{3.56}$$

The solution for the above differential Eq. 3.56 is,

$$f(r) = r^{\frac{1}{5}}c_3 + r^{\frac{8}{5}}c_4. \tag{3.57}$$

Let $c_3 = -r_0^{\frac{7}{5}}$ and $c_4 = 1$ we get,

$$f(r) = r^{\frac{1}{5}}[-r_0^{\frac{7}{5}} + r^{\frac{7}{5}}]. \tag{3.58}$$

The event horizon is obtained at $r = r_0$. Since, metric has a singularity at $r = 0$, which is a curvature singularity. It also seems to have a singularity at event horizon same as in Schwarszchild case but in both the cases, only spacial coordinate blow up.

In this case, value of Ricci scalar, norm of Ricci tensor and Kretschmann scalar are found to be,

$$R = -\frac{4(r_0^{\frac{7}{5}} + 20r^{\frac{7}{5}})}{20r^{\frac{9}{5}}}, \tag{3.59}$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{16}{625}\left(\frac{98}{r^{\frac{4}{5}}} + \frac{(r_0^{\frac{7}{5}} + 6r^{\frac{7}{5}})^2(r^{\frac{8}{5}} - (r_0^7 r)^{\frac{1}{5}})^2}{r^4(r_0^{\frac{7}{5}} - r^{\frac{7}{5}})^2}\right), \tag{3.60}$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{16(-r_0^{\frac{7}{5}} + r^{\frac{7}{5}})^3(3r_0^{\frac{14}{5}} + 136r^{\frac{14}{5}} + 8(r_0 r)^{\frac{7}{5}})}{625r^{\frac{16}{5}}(r^{\frac{8}{5}} - (r_0^7 r)^{\frac{1}{5}})(-2r_0^{\frac{7}{5}}r^{\frac{8}{5}} + r^3 + (r_0^1 4r)^{\frac{1}{5}})}. \tag{3.61}$$

Curvature invariants come out to be finite for finite value of r and at $r = r_0$ in this case also which saved us from violating the Cosmic Censorship principle.

The graph between $f(r)$ and r is shown in Fig.[3.7]

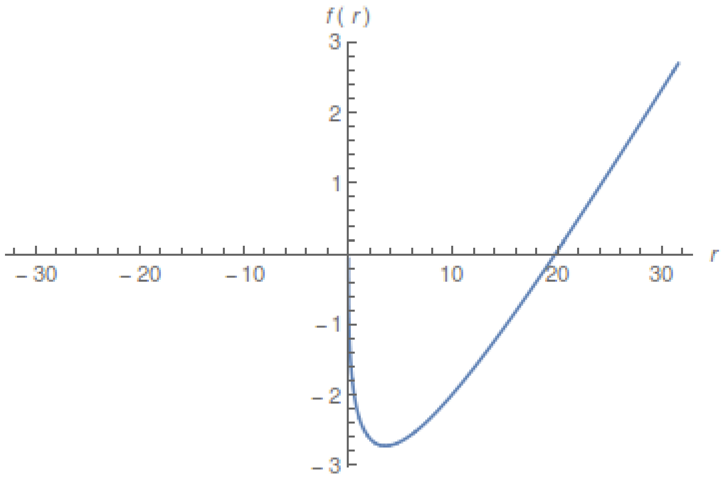

Figure 3.7: Graph between $f(r)$ and r for $y(r) = r^{\frac{4}{5}}$

The value of potential is obtained on substituting Eq. 3.58 in Eq. 3.16 as,

$$V = \frac{-14}{25r^{\frac{2}{5}}}, \tag{3.62}$$

which also shows asymptotically flat space time for $r \to \infty$. The graph between $V(r)$ and r is shown in Fig.[3.8]

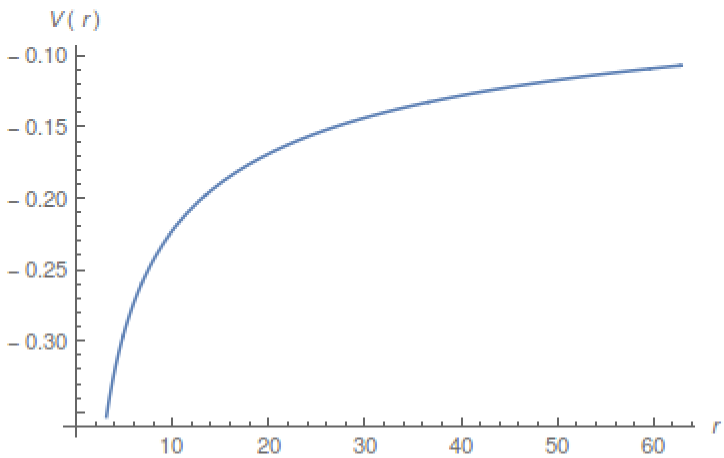

Figure 3.8: Graph between $V(r)$ and r for $y(r) = r^{\frac{4}{5}}$

Scalar field at boundary: At $r \to 0$, the value of scalar field comes to be constant i.e. ϕ_0 and kinetic term of the scalar field is given as,

$$K.E. = \partial_\mu \phi \partial^\nu \phi, \tag{3.63}$$

$$K.E. = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \tag{3.64}$$

For, $\mu = \nu = r$,

$$K.E. = g^{rr} \partial_r \phi \partial_r \phi. \tag{3.65}$$

$$K.E. = r^{\frac{1}{5}} (-r_0^{\frac{7}{5}} + r^{\frac{7}{5}}) \left(\frac{-2}{5r}\right)^2, \tag{3.66}$$

which shows that scalar field decays at much larger values of r as compared to previous case.

Komar mass

For this solution, we can find Komar mass like above taking Killing vector to be $K^\nu = (2, 0, 0)$ and the value is calculated to be

$$M_{Komar} = -\frac{4r^{\frac{7}{5}}}{5} + \frac{r_0{}^{\frac{7}{5}}}{10}. \tag{3.67}$$

First law of thermodynamics

In this case, metric is given as

$$f(r) = 10r^{\frac{1}{5}} \left(-M - \frac{7}{10} r^{\frac{7}{5}}\right) \tag{3.68}$$

and thermodynamic quanties can be expressed in terms of r_0 as,

$$M = -\frac{7}{10} r_0{}^{\frac{7}{5}} \qquad T = \frac{f'(r_0)}{4\pi} = \frac{-49 r_0{}^{\frac{3}{5}}}{20\pi} \qquad S = \frac{\pi}{2} r_0{}^{\frac{4}{5}}$$

variation of following quantities with respect to r_0 is

$$\partial M = \frac{-49}{50} r_0{}^{\frac{2}{5}} \partial r_0 \qquad \partial S = \frac{4\pi}{10} r_0{}^{\frac{-1}{5}} \partial r_0 \qquad T\partial S = \frac{-49}{50} r_0{}^{\frac{2}{5}} \partial r_0$$

which proves first law of thermodynamics for this case also by renormalizing Killing vector.

3.4 Results and Discussion

In this thesis, we have obtained and analysed a solution for Einstein-Maxwell-Dilaton gravity in (2+1) dimensions. The solution is a static charged black hole with a regular horizon. Komar integrals are used to get total mass and charge of black hole. For a generalised case in which we considered charge to be zero and took specific solution of $y(r)$, we futher analysed the thermodynamic property of (2+1)-dimensional black hole which comes out to be much similar as (3+1)-dimensional black hole. From the above calculations, following results are obtained as follows:

- For both the cases, we obtained event horizon at $r = 0$ and $r = r_0$, thus curvature singulaity is obtained at the origin which is similar to charged black hole in literature. Kretschmann Scalar, Ricci Scalar and $R_{\mu\nu}R^{ab}$ were calculated and their finite result saved us from violating the Cosmic Censorship principle.

- In both the case for $r \to \infty$ we obtained asymptotically flat space time from the expression of potential. In general, (2+1) dimensional black hole are found in de-Sitter or anti-de Sitter space time.

- Mass of the black hole is calculated using Komar integrals and other thermodynamical quantities are calculated which shows similar results as in Schwarszchild case.

- First law of black hole thermodynamics is further proved in both the cases.

3.5 Conclusion and Outlook

We started with coupled action and found a solution for scalar field and an exact expression of $y(r)$. We obtained and analysed solution for Einstein-

Maxwell-Dilaton gravity which is charged black hole in anti de-Sitter space time with a regular horizon. We have also found the exact solution of Einstein- Maxwell-Dilaton gravity which is a massive uncharged black hole for some specific values of α, c_1 and c_2, considering charge to be zero. The value of potential found to be zero at $r \to \infty$ in both the cases which shows asymptotically flat space time for general case. We have demonstrated the first law of thermodynamics for both the cases.

Expressions obtained in the theory

Solution of scalar field and $y(r)$ are given as,

$$\text{Scalar Field}(\phi) = c_2 - \frac{\log(r\alpha + c_1)}{\alpha},$$
$$y(\phi) = e^{\frac{1}{2}(-\alpha-\sqrt{-4+\alpha^2})\phi}c_3 + e^{\frac{1}{2}(-\alpha+\sqrt{-4+\alpha^2}\phi)}c_4.$$

For $Q \neq 0$, expression of $f(r)$ and potential $V(r)$ are given in Appendix[4.1]
For $Q = 0$, expression of $f(r)$ and potential $V(r)$ are given in Appendix[4.2]
For $y(r) = r^{\frac{1}{5}}$ and $\phi = \phi_0 - \frac{2}{5}ln(r)$. Expression of $f(r)$ and potential $V(r)$ are given as,

$$f(r) = r^{\frac{2}{5}}[-r_0^{\frac{2}{5}} + r^{\frac{2}{5}}],$$
$$V = -\frac{r_0^{\frac{2}{5}}}{25r^{\frac{8}{5}}}.$$

For $y(r) = r^{\frac{4}{5}}$ and $\phi = \phi_0 - \frac{2}{5}ln(r)$. Expression of $f(r)$ and potential $V(r)$ are given as,

$$f(r) = r^{\frac{1}{5}}[-r_0^{\frac{7}{5}} + r^{\frac{7}{5}}],$$
$$V = \frac{-14}{25r^{\frac{2}{5}}}.$$

It was shown that curvature singularity comes at $r = 0$ and at event horizon r_0 where in both the cases spacial coordinate blow up. Further, scalar field decays at large value of r but in second case it deacys for much larger value of r. In future, we can work on more generalised cases of ϕ and $y(r)$ and considered non zero component of charge. We can also work on thermodynamic properties and can also try to prove first law of thermodynamics for more general cases.

We can extend our work for higher dimensions and can study the dynamics of de Sitter space or anti de Sitter space time and study its vacuum properties and properties of wave propagating in it. We can also analyse the stability of Einstein Maxwell black hole using the heat capacity and can find whether it undergoes any phase transition.

Chapter 4

Appendix

4.1 Appendix A

The expression of $f(r)$ and potential $V(r)$ for the most general case where $Q \neq 0$ is given in this appendix. Since we have calculated the expression of $f(r)$ in mathematica we need to change our constants with some different variables to run the code in mathematica file. The expression given below for $f(r)$ and $V(r)$ holds such changes given as,

$c = c_2, b = c_1$ and $p = c_3, q = c_4$

Expression for f(r)
f[r] =

$$a[0]+a[1]*r+\left(-\frac{1}{4b^2\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}p+e^{c\sqrt{-4+\alpha^2}}\,q\right)^2}\left(4b^{\frac{2\sqrt{-4+\alpha^2}}{\alpha}}p^2a[0]+8b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}e^{c\sqrt{-4+\alpha^2}}\,p\,q\,a[0]+\right.\right.$$

$$4e^{2c\sqrt{-4+\alpha^2}}\,q^2a[0]+b\,e^{2c\sqrt{-4+\alpha^2}}\,q^2\left(-\alpha+\sqrt{-4+\alpha^2}\right)a[1]-$$

$$\left.\left.b^{1+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}p^2\left(\alpha+\sqrt{-4+\alpha^2}\right)a[1]-2b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}}e^{c\sqrt{-4+\alpha^2}}\,\left(2e^{c\alpha}Q^2+p\,q\,\alpha\,a[1]\right)\right)\right)*r^2+$$

$$\left(\frac{1}{6b^{7/2}\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}p+e^{c\sqrt{-4+\alpha^2}}\,q\right)^5}\left(-2b^{\frac{1}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}e^{2c\sqrt{-4+\alpha^2}}\,p^3q^2\left(-15\alpha+\sqrt{-4+\alpha^2}\right)a[0]-\right.\right.$$

$$3b^{\frac{1}{2}+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}e^{c\sqrt{-4+\alpha^2}}\,p^4q\left(-5\alpha+\sqrt{-4+\alpha^2}\right)a[0]-$$

$$b^{\frac{1}{2}+\frac{5\sqrt{-4+\alpha^2}}{\alpha}}p^5\left(-3\alpha+\sqrt{-4+\alpha^2}\right)a[0]+\sqrt{b}\,e^{5c\sqrt{-4+\alpha^2}}\,q^5\left(3\alpha+\sqrt{-4+\alpha^2}\right)a[0]+$$

$$3b^{\frac{1}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}e^{4c\sqrt{-4+\alpha^2}}\,p\,q^4\left(5\alpha+\sqrt{-4+\alpha^2}\right)a[0]+2b^{\frac{1}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}e^{3c\sqrt{-4+\alpha^2}}\,p^2q^3$$

$$\left(15\alpha+\sqrt{-4+\alpha^2}\right)a[0]-3b^{\frac{3}{2}+\frac{5\sqrt{-4+\alpha^2}}{\alpha}}p^5a[1]-3b^{3/2}e^{5c\sqrt{-4+\alpha^2}}\,q^5a[1]+$$

$$b^{\frac{3}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}e^{3c\sqrt{-4+\alpha^2}}\,p\,q^2\left(e^{c\alpha}Q^2\left(-3\alpha+\sqrt{-4+\alpha^2}\right)-30\,p\,q\,a[1]\right)+$$

$$b^{\frac{3}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}e^{4c\sqrt{-4+\alpha^2}}\,q^3\left(e^{c\alpha}Q^2\left(-\alpha+\sqrt{-4+\alpha^2}\right)-15\,p\,q\,a[1]\right)-$$

$$b^{\frac{3}{2}+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}e^{c\sqrt{-4+\alpha^2}}\,p^3\left(e^{c\alpha}Q^2\left(\alpha+\sqrt{-4+\alpha^2}\right)+15\,p\,q\,a[1]\right)-$$

$$\left.\left.b^{\frac{3}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}e^{2c\sqrt{-4+\alpha^2}}\,p^2q\left(e^{c\alpha}Q^2\left(3\alpha+\sqrt{-4+\alpha^2}\right)+30\,p\,q\,a[1]\right)\right)\right)*r^3+$$

$$\left(\frac{1}{48b^4\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}p+e^{c\sqrt{-4+\alpha^2}}\,q\right)^4}\left(-4b^{\frac{2\sqrt{-4+\alpha^2}}{\alpha}}e^{2c\sqrt{-4+\alpha^2}}\,p^2q^2\left(-26+29\alpha^2\right)a[0]+\right.\right.$$

$$6\,b^{\frac{4\sqrt{-4+\alpha^2}}{\alpha}}p^4\left(2-3\alpha^2+\alpha\sqrt{-4+\alpha^2}\right)a[0]-6\,e^{4c\sqrt{-4+\alpha^2}}\,q^4\left(-2+3\alpha^2+\alpha\sqrt{-4+\alpha^2}\right)a[0]+$$
$$4\,b^{\frac{3\sqrt{-4+\alpha^2}}{\alpha}}e^{c\sqrt{-4+\alpha^2}}\,p^3\,q\left(16-19\alpha^2+3\alpha\sqrt{-4+\alpha^2}\right)a[0]-$$
$$4\,b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}e^{3c\sqrt{-4+\alpha^2}}\,p\,q^3\left(-16+19\alpha^2+3\alpha\sqrt{-4+\alpha^2}\right)a[0]-$$
$$5\,b^{1+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}p^4\left(-3\alpha+\sqrt{-4+\alpha^2}\right)a[1]+5\,b\,e^{4c\sqrt{-4+\alpha^2}}\,q^4\left(3\alpha+\sqrt{-4+\alpha^2}\right)a[1]+$$
$$2\,b^{1+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}e^{2c\sqrt{-4+\alpha^2}}\,p\,q\left(4\,e^{c\alpha}Q^2+45\,p\,q\,\alpha\,a[1]\right)+$$
$$2\,b^{1+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}e^{c\sqrt{-4+\alpha^2}}\,p^2\left(e^{c\alpha}Q^2\left(-10+3\alpha^2+3\alpha\sqrt{-4+\alpha^2}\right)-\right.$$
$$\left.5\,p\,q\left(-6\alpha+\sqrt{-4+\alpha^2}\right)a[1]\right)+2\,b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}}e^{3c\sqrt{-4+\alpha^2}}\,q^2$$
$$\left.\left(-e^{c\alpha}Q^2\left(10-3\alpha^2+3\alpha\sqrt{-4+\alpha^2}\right)+5\,p\,q\left(6\alpha+\sqrt{-4+\alpha^2}\right)a[1]\right)\right)\right)*r^4+$$

$$\left(\frac{1}{120\,b^{17/2}\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}p+e^{c\sqrt{-4+\alpha^2}}\,q\right)^7}\left(-b^{\frac{7}{2}+\frac{7\sqrt{-4+\alpha^2}}{\alpha}}p^7\left(39\alpha-36\alpha^3-5\sqrt{-4+\alpha^2}+12\alpha^2\sqrt{-4+\alpha^2}\right)\right.\right.$$
$$a[0]+b^{7/2}e^{7c\sqrt{-4+\alpha^2}}\,q^7\left(-39\alpha+36\alpha^3-5\sqrt{-4+\alpha^2}+12\alpha^2\sqrt{-4+\alpha^2}\right)a[0]-$$
$$b^{\frac{7}{2}+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}e^{3c\sqrt{-4+\alpha^2}}\,p^4\,q^3\left(1925\alpha-1400\alpha^3-9\sqrt{-4+\alpha^2}+56\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+$$
$$b^{\frac{7}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}e^{4c\sqrt{-4+\alpha^2}}\,p^3\,q^4\left(-1925\alpha+1400\alpha^3-9\sqrt{-4+\alpha^2}+56\alpha^2\sqrt{-4+\alpha^2}\right)a[0]-$$
$$b^{\frac{7}{2}+\frac{6\sqrt{-4+\alpha^2}}{\alpha}}e^{c\sqrt{-4+\alpha^2}}\,p^6\,q\left(329\alpha-266\alpha^3-17\sqrt{-4+\alpha^2}+58\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+$$
$$b^{\frac{7}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}e^{6c\sqrt{-4+\alpha^2}}\,p\,q^6\left(-329\alpha+266\alpha^3-17\sqrt{-4+\alpha^2}+58\alpha^2\sqrt{-4+\alpha^2}\right)a[0]-$$
$$b^{\frac{7}{2}+\frac{5\sqrt{-4+\alpha^2}}{\alpha}}e^{2c\sqrt{-4+\alpha^2}}\,p^5\,q^2\left(1099\alpha-826\alpha^3-21\sqrt{-4+\alpha^2}+102\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+$$
$$b^{\frac{7}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}e^{5c\sqrt{-4+\alpha^2}}\,p^2\,q^5\left(-1099\alpha+826\alpha^3-21\sqrt{-4+\alpha^2}+102\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+$$
$$b^{\frac{9}{2}+\frac{7\sqrt{-4+\alpha^2}}{\alpha}}p^7\left(11-29\alpha^2+13\alpha\sqrt{-4+\alpha^2}\right)a[1]-$$
$$b^{9/2}e^{7c\sqrt{-4+\alpha^2}}\,q^7\left(-11+29\alpha^2+13\alpha\sqrt{-4+\alpha^2}\right)a[1]+$$
$$b^{\frac{9}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}e^{5c\sqrt{-4+\alpha^2}}\,p\,q^4\left(3\,e^{c\alpha}Q^2\left(47\alpha-8\alpha^3+11\sqrt{-4+\alpha^2}+8\alpha^2\sqrt{-4+\alpha^2}\right)+\right.$$

$$p\,q\left(271-619\,\alpha^2-117\,\alpha\sqrt{-4+\alpha^2}\right)a[1]\Bigg)-$$

$$b^{\frac{9}{2}+\frac{5\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2\,c\sqrt{-4+\alpha^2}}\,p^4\,q\left(3\,e^{c\,\alpha}\,Q^2\left(-47\,\alpha+8\,\alpha^3+11\sqrt{-4+\alpha^2}+8\,\alpha^2\sqrt{-4+\alpha^2}\right)+\right.$$

$$\left.p\,q\left(-271+619\,\alpha^2-117\,\alpha\sqrt{-4+\alpha^2}\right)a[1]\right)+$$

$$b^{\frac{9}{2}+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}\,e^{3\,c\sqrt{-4+\alpha^2}}\,p^3\,q^2\left(-2\,e^{c\,\alpha}\,Q^2\left(-69\,\alpha+6\,\alpha^3+19\sqrt{-4+\alpha^2}+6\,\alpha^2\sqrt{-4+\alpha^2}\right)+\right.$$

$$\left.5\,p\,q\left(93-207\,\alpha^2+13\,\alpha\sqrt{-4+\alpha^2}\right)a[1]\right)+$$

$$b^{\frac{9}{2}+\frac{6\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p^5\left(-e^{c\,\alpha}\,Q^2\left(-57\,\alpha+12\,\alpha^3-5\sqrt{-4+\alpha^2}+12\,\alpha^2\sqrt{-4+\alpha^2}\right)+\right.$$

$$\left.5\,p\,q\left(17-41\,\alpha^2+13\,\alpha\sqrt{-4+\alpha^2}\right)a[1]\right)+$$

$$b^{\frac{9}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{6\,c\sqrt{-4+\alpha^2}}\,q^5\left(e^{c\,\alpha}\,Q^2\left(57\,\alpha-12\,\alpha^3-5\sqrt{-4+\alpha^2}+12\,\alpha^2\sqrt{-4+\alpha^2}\right)-\right.$$

$$\left.5\,p\,q\left(-17+41\,\alpha^2+13\,\alpha\sqrt{-4+\alpha^2}\right)a[1]\right)+$$

$$b^{\frac{9}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,e^{4\,c\sqrt{-4+\alpha^2}}\,p^2\,q^3\left(2\,e^{c\,\alpha}\,Q^2\left(69\,\alpha-6\,\alpha^3+19\sqrt{-4+\alpha^2}+6\,\alpha^2\sqrt{-4+\alpha^2}\right)-\right.$$

$$\left.\left.5\,p\,q\left(-93+207\,\alpha^2+13\,\alpha\sqrt{-4+\alpha^2}\right)a[1]\right)\right)\Bigg)*r^5+$$

$$\left(-\frac{1}{1440\,b^6\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p+e^{c\sqrt{-4+\alpha^2}}\,q\right)^6}\left(16\,b^{\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,e^{3\,c\sqrt{-4+\alpha^2}}\,p^3\,q^3\left(73-913\,\alpha^2+519\,\alpha^4\right)a[0]-\right.\right.$$

$$8\,b^{\frac{5\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p^5\,q\left(-55+481\,\alpha^2-294\,\alpha^4-27\,\alpha\sqrt{-4+\alpha^2}+54\,\alpha^3\sqrt{-4+\alpha^2}\right)a[0]+$$

$$8\,b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{5\,c\sqrt{-4+\alpha^2}}\,p\,q^5\left(55-481\,\alpha^2+294\,\alpha^4-27\,\alpha\sqrt{-4+\alpha^2}+54\,\alpha^3\sqrt{-4+\alpha^2}\right)a[0]-$$

$$2\,b^{\frac{6\sqrt{-4+\alpha^2}}{\alpha}}\,p^6\left(-22+253\,\alpha^2-180\,\alpha^4-51\,\alpha\sqrt{-4+\alpha^2}+60\,\alpha^3\sqrt{-4+\alpha^2}\right)a[0]+$$

$$2\,e^{6\,c\sqrt{-4+\alpha^2}}\,q^6\left(22-253\,\alpha^2+180\,\alpha^4-51\,\alpha\sqrt{-4+\alpha^2}+60\,\alpha^3\sqrt{-4+\alpha^2}\right)a[0]-2\,b^{\frac{4\sqrt{-4+\alpha^2}}{\alpha}}$$

$$e^{2\,c\sqrt{-4+\alpha^2}}\,p^4\,q^2\left(-490+5323\,\alpha^2-3072\,\alpha^4-63\,\alpha\sqrt{-4+\alpha^2}+252\,\alpha^3\sqrt{-4+\alpha^2}\right)a[0]+$$

$$2\,b^{\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{4\,c\sqrt{-4+\alpha^2}}\,p^2\,q^4\left(490-5323\,\alpha^2+3072\,\alpha^4-63\,\alpha\sqrt{-4+\alpha^2}+252\,\alpha^3\sqrt{-4+\alpha^2}\right)$$

$$a[0]+3\,b^{1+\frac{6\sqrt{-4+\alpha^2}}{\alpha}}\,p^6\left(69\,\alpha-96\,\alpha^3-7\sqrt{-4+\alpha^2}+48\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]-$$

$$3\,b\,e^{6c\sqrt{-4+\alpha^2}}\,q^6\left(-69\,\alpha+96\,\alpha^3-7\sqrt{-4+\alpha^2}+48\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]+$$

$$4\,b^{1+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,e^{3c\sqrt{-4+\alpha^2}}\,p^2\,q^2\left(e^{c\,\alpha}\,Q^2\left(-598+415\,\alpha^2\right)+9\,p\,q\,\alpha\left(143-167\,\alpha^2\right)a[1]\right)+$$

$$b^{1+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{4c\sqrt{-4+\alpha^2}}\,p\,q^3\left(4\,e^{c\,\alpha}\,Q^2\left(160+137\,\alpha^2+189\,\alpha\sqrt{-4+\alpha^2}\right)+\right.$$

$$\left.3\,p\,q\left(1259\,\alpha-1496\,\alpha^3+19\sqrt{-4+\alpha^2}-236\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]\right)-$$

$$b^{1+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2c\sqrt{-4+\alpha^2}}\,p^3\,q\left(4\,e^{c\,\alpha}\,Q^2\left(-160-137\,\alpha^2+189\,\alpha\sqrt{-4+\alpha^2}\right)+\right.$$

$$\left.3\,p\,q\left(-1259\,\alpha+1496\,\alpha^3+19\sqrt{-4+\alpha^2}-236\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]\right)+$$

$$6\,b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}}\,e^{5c\sqrt{-4+\alpha^2}}\,q^4\left(e^{c\,\alpha}\,Q^2\left(-14+113\,\alpha^2-20\,\alpha^4-17\,\alpha\sqrt{-4+\alpha^2}+20\,\alpha^3\sqrt{-4+\alpha^2}\right)+\right.$$

$$\left.5\,p\,q\left(47\,\alpha-59\,\alpha^3+2\sqrt{-4+\alpha^2}-19\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]\right)-$$

$$6\,b^{1+\frac{5\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p^4\left(e^{c\,\alpha}\,Q^2\left(14-113\,\alpha^2+20\,\alpha^4-17\,\alpha\sqrt{-4+\alpha^2}+20\,\alpha^3\sqrt{-4+\alpha^2}\right)+\right.$$

$$\left.\left.\left.5\,p\,q\left(-47\,\alpha+59\,\alpha^3+2\sqrt{-4+\alpha^2}-19\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]\right)\right)\right)*r^6$$

Expression for V(r)

$$V[r]=\frac{1}{4}\left(-\frac{1}{b^{7/2}\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p+e^{c\sqrt{-4+\alpha^2}}\,q\right)^5}\,r\left(-2\,b^{\frac{1}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2c\sqrt{-4+\alpha^2}}\,p^3\,q^2\left(-15\,\alpha+\sqrt{-4+\alpha^2}\right)a[0]-\right.\right.$$

$$3\,b^{\frac{1}{2}+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p^4\,q\left(-5\,\alpha+\sqrt{-4+\alpha^2}\right)a[0]-$$

$$b^{\frac{1}{2}+\frac{5\sqrt{-4+\alpha^2}}{\alpha}}\,p^5\left(-3\,\alpha+\sqrt{-4+\alpha^2}\right)a[0]+\sqrt{b}\,e^{5c\sqrt{-4+\alpha^2}}\,q^5\left(3\,\alpha+\sqrt{-4+\alpha^2}\right)a[0]+$$

$$3\,b^{\frac{1}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{4c\sqrt{-4+\alpha^2}}\,p\,q^4\left(5\,\alpha+\sqrt{-4+\alpha^2}\right)a[0]+2\,b^{\frac{1}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{3c\sqrt{-4+\alpha^2}}\,p^2\,q^3$$

$$\left(15\,\alpha+\sqrt{-4+\alpha^2}\right)a[0]-3\,b^{\frac{3}{2}+\frac{5\sqrt{-4+\alpha^2}}{\alpha}}\,p^5\,a[1]-3\,b^{3/2}\,e^{5c\sqrt{-4+\alpha^2}}\,q^5\,a[1]+$$

$$b^{\frac{3}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{3c\sqrt{-4+\alpha^2}}\,p\,q^2\left(e^{c\,\alpha}\,Q^2\left(-3\,\alpha+\sqrt{-4+\alpha^2}\right)-30\,p\,q\,a[1]\right)+$$

$$b^{\frac{3}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{4c\sqrt{-4+\alpha^2}}\,q^3\left(e^{c\,\alpha}\,Q^2\left(-\alpha+\sqrt{-4+\alpha^2}\right)-15\,p\,q\,a[1]\right)-$$

$$b^{\frac{3}{2}+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p^3\left(e^{c\,\alpha}\,Q^2\left(\alpha+\sqrt{-4+\alpha^2}\right)+15\,p\,q\,a[1]\right)-$$

$$b^{\frac{3}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\, e^{2c\sqrt{-4+\alpha^2}}\, p^2\, q\left(e^{c\alpha}\, Q^2\left(3\alpha+\sqrt{-4+\alpha^2}\right)+30\, p\, q\, a[1]\right)\Bigg)+$$

$$\frac{1}{2b^2\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}p+e^{c\sqrt{-4+\alpha^2}}\, q\right)^2}\Bigg(4\, b^{\frac{2\sqrt{-4+\alpha^2}}{\alpha}}p^2\, a[0]+8\, b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\, e^{c\sqrt{-4+\alpha^2}}\, p\, q\, a[0]+$$

$$4\, e^{2c\sqrt{-4+\alpha^2}}\, q^2\, a[0]+b\, e^{2c\sqrt{-4+\alpha^2}}\, q^2\left(-\alpha+\sqrt{-4+\alpha^2}\right)a[1]-$$

$$b^{1+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}p^2\left(\alpha+\sqrt{-4+\alpha^2}\right)a[1]-2\, b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}}\, e^{c\sqrt{-4+\alpha^2}}\left(2\, e^{c\alpha}\, Q^2+p\, q\, \alpha\, a[1]\right)\Bigg)-$$

$$\frac{1}{4b^4\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}p+e^{c\sqrt{-4+\alpha^2}}\, q\right)^4}\, r^2\Bigg(-4\, b^{\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\, e^{2c\sqrt{-4+\alpha^2}}\, p^2\, q^2\left(-26+29\alpha^2\right)a[0]+$$

$$6\, b^{\frac{4\sqrt{-4+\alpha^2}}{\alpha}}p^4\left(2-3\alpha^2+\alpha\sqrt{-4+\alpha^2}\right)a[0]-6\, e^{4c\sqrt{-4+\alpha^2}}\, q^4\left(-2+3\alpha^2+\alpha\sqrt{-4+\alpha^2}\right)$$

$$a[0]+4\, b^{\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\, e^{c\sqrt{-4+\alpha^2}}\, p^3\, q\left(16-19\alpha^2+3\alpha\sqrt{-4+\alpha^2}\right)a[0]-$$

$$4\, b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\, e^{3c\sqrt{-4+\alpha^2}}\, p\, q^3\left(-16+19\alpha^2+3\alpha\sqrt{-4+\alpha^2}\right)a[0]-$$

$$5\, b^{1+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}p^4\left(-3\alpha+\sqrt{-4+\alpha^2}\right)a[1]+5\, b\, e^{4c\sqrt{-4+\alpha^2}}\, q^4\left(3\alpha+\sqrt{-4+\alpha^2}\right)a[1]+$$

$$2\, b^{1+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\, e^{2c\sqrt{-4+\alpha^2}}\, p\, q\left(4\, e^{c\alpha}\, Q^2+45\, p\, q\, \alpha\, a[1]\right)+2\, b^{1+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\, e^{c\sqrt{-4+\alpha^2}}\, p^2$$

$$\left(e^{c\alpha}\, Q^2\left(-10+3\alpha^2+3\alpha\sqrt{-4+\alpha^2}\right)-5\, p\, q\left(-6\alpha+\sqrt{-4+\alpha^2}\right)a[1]\right)+2\, b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}}$$

$$e^{3c\sqrt{-4+\alpha^2}}\, q^2\left(-e^{c\alpha}\, Q^2\left(10-3\alpha^2+3\alpha\sqrt{-4+\alpha^2}\right)+5\, p\, q\left(6\alpha+\sqrt{-4+\alpha^2}\right)a[1]\right)\Bigg)-$$

$$\frac{1}{6\, b^{17/2}\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}p+e^{c\sqrt{-4+\alpha^2}}\, q\right)^7}\, r^3$$

$$\Bigg(-b^{\frac{7}{2}+\frac{7\sqrt{-4+\alpha^2}}{\alpha}}p^7\left(39\alpha-36\alpha^3-5\sqrt{-4+\alpha^2}+12\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+$$

$$b^{7/2}\, e^{7c\sqrt{-4+\alpha^2}}\, q^7\left(-39\alpha+36\alpha^3-5\sqrt{-4+\alpha^2}+12\alpha^2\sqrt{-4+\alpha^2}\right)a[0]-$$

$$b^{\frac{7}{2}+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}\, e^{3c\sqrt{-4+\alpha^2}}\, p^4\, q^3\left(1925\alpha-1400\alpha^3-9\sqrt{-4+\alpha^2}+56\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+$$

$$b^{\frac{7}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\, e^{4c\sqrt{-4+\alpha^2}}\, p^3\, q^4\left(-1925\alpha+1400\alpha^3-9\sqrt{-4+\alpha^2}+56\alpha^2\sqrt{-4+\alpha^2}\right)a[0]-$$

$$b^{\frac{7}{2}+\frac{6\sqrt{-4+\alpha^2}}{\alpha}}\, e^{c\sqrt{-4+\alpha^2}}\, p^6\, q\left(329\alpha-266\alpha^3-17\sqrt{-4+\alpha^2}+58\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+$$

$$b^{\frac{7}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}\, e^{6c\sqrt{-4+\alpha^2}}\, p\, q^6\left(-329\alpha+266\alpha^3-17\sqrt{-4+\alpha^2}+58\alpha^2\sqrt{-4+\alpha^2}\right)a[0]-$$

$$b^{\frac{7}{2}+\frac{5\sqrt{-4+\alpha^2}}{\alpha}} e^{2c\sqrt{-4+\alpha^2}} p^5 q^2 \left(1099\alpha-826\alpha^3-21\sqrt{-4+\alpha^2}+102\alpha^2\sqrt{-4+\alpha^2}\right) a[0]+$$

$$b^{\frac{7}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}} e^{5c\sqrt{-4+\alpha^2}} p^2 q^5 \left(-1099\alpha+826\alpha^3-21\sqrt{-4+\alpha^2}+102\alpha^2\sqrt{-4+\alpha^2}\right) a[0]+$$

$$b^{\frac{9}{2}+\frac{7\sqrt{-4+\alpha^2}}{\alpha}} p^7 \left(11-29\alpha^2+13\alpha\sqrt{-4+\alpha^2}\right) a[1]-$$

$$b^{9/2} e^{7c\sqrt{-4+\alpha^2}} q^7 \left(-11+29\alpha^2+13\alpha\sqrt{-4+\alpha^2}\right) a[1]+$$

$$b^{\frac{9}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}} e^{5c\sqrt{-4+\alpha^2}} p q^4 \left(3 e^{c\alpha} Q^2 \left(47\alpha-8\alpha^3+11\sqrt{-4+\alpha^2}+8\alpha^2\sqrt{-4+\alpha^2}\right)+\right.$$
$$\left. p q \left(271-619\alpha^2-117\alpha\sqrt{-4+\alpha^2}\right) a[1]\right)-$$

$$b^{\frac{9}{2}+\frac{5\sqrt{-4+\alpha^2}}{\alpha}} e^{2c\sqrt{-4+\alpha^2}} p^4 q \left(3 e^{c\alpha} Q^2 \left(-47\alpha+8\alpha^3+11\sqrt{-4+\alpha^2}+8\alpha^2\sqrt{-4+\alpha^2}\right)+\right.$$
$$\left. p q \left(-271+619\alpha^2-117\alpha\sqrt{-4+\alpha^2}\right) a[1]\right)+$$

$$b^{\frac{9}{2}+\frac{4\sqrt{-4+\alpha^2}}{\alpha}} e^{3c\sqrt{-4+\alpha^2}} p^3 q^2 \left(-2 e^{c\alpha} Q^2 \left(-69\alpha+6\alpha^3+19\sqrt{-4+\alpha^2}+6\alpha^2\sqrt{-4+\alpha^2}\right)+\right.$$
$$\left. 5 p q \left(93-207\alpha^2+13\alpha\sqrt{-4+\alpha^2}\right) a[1]\right)+$$

$$b^{\frac{9}{2}+\frac{6\sqrt{-4+\alpha^2}}{\alpha}} e^{c\sqrt{-4+\alpha^2}} p^5 \left(-e^{c\alpha} Q^2 \left(-57\alpha+12\alpha^3-5\sqrt{-4+\alpha^2}+12\alpha^2\sqrt{-4+\alpha^2}\right)+\right.$$
$$\left. 5 p q \left(17-41\alpha^2+13\alpha\sqrt{-4+\alpha^2}\right) a[1]\right)+$$

$$b^{\frac{9}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}} e^{6c\sqrt{-4+\alpha^2}} q^5 \left(e^{c\alpha} Q^2 \left(57\alpha-12\alpha^3-5\sqrt{-4+\alpha^2}+12\alpha^2\sqrt{-4+\alpha^2}\right)-\right.$$
$$\left. 5 p q \left(-17+41\alpha^2+13\alpha\sqrt{-4+\alpha^2}\right) a[1]\right)+$$

$$b^{\frac{9}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}} e^{4c\sqrt{-4+\alpha^2}} p^2 q^3 \left(2 e^{c\alpha} Q^2 \left(69\alpha-6\alpha^3+19\sqrt{-4+\alpha^2}+6\alpha^2\sqrt{-4+\alpha^2}\right)-\right.$$
$$\left.\left. 5 p q \left(-93+207\alpha^2+13\alpha\sqrt{-4+\alpha^2}\right) a[1]\right)\right)+\frac{1}{48 b^6 \left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}} p+e^{c\sqrt{-4+\alpha^2}} q\right)^6}$$

$$r^4 \left(16 b^{\frac{3\sqrt{-4+\alpha^2}}{\alpha}} e^{3c\sqrt{-4+\alpha^2}} p^3 q^3 \left(73-913\alpha^2+519\alpha^4\right) a[0]-8 b^{\frac{5\sqrt{-4+\alpha^2}}{\alpha}} e^{c\sqrt{-4+\alpha^2}}\right.$$
$$p^5 q \left(-55+481\alpha^2-294\alpha^4-27\alpha\sqrt{-4+\alpha^2}+54\alpha^3\sqrt{-4+\alpha^2}\right) a[0]+$$
$$8 b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}} e^{5c\sqrt{-4+\alpha^2}} p q^5 \left(55-481\alpha^2+294\alpha^4-27\alpha\sqrt{-4+\alpha^2}+54\alpha^3\sqrt{-4+\alpha^2}\right) a[0]-$$
$$2 b^{\frac{6\sqrt{-4+\alpha^2}}{\alpha}} p^6 \left(-22+253\alpha^2-180\alpha^4-51\alpha\sqrt{-4+\alpha^2}+60\alpha^3\sqrt{-4+\alpha^2}\right) a[0]+$$
$$2 e^{6c\sqrt{-4+\alpha^2}} q^6 \left(22-253\alpha^2+180\alpha^4-51\alpha\sqrt{-4+\alpha^2}+60\alpha^3\sqrt{-4+\alpha^2}\right) a[0]-$$

$$2\,b^{\frac{4\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2c\sqrt{-4+\alpha^2}}\,p^4\,q^2\left(-490+5323\,\alpha^2-3072\,\alpha^4-63\,\alpha\sqrt{-4+\alpha^2}+252\,\alpha^3\sqrt{-4+\alpha^2}\right)a[0]+2\,b^{\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{4c\sqrt{-4+\alpha^2}}\,p^2\,q^4$$

$$\left(490-5323\,\alpha^2+3072\,\alpha^4-63\,\alpha\sqrt{-4+\alpha^2}+252\,\alpha^3\sqrt{-4+\alpha^2}\right)a[0]+$$

$$3\,b^{1+\frac{6\sqrt{-4+\alpha^2}}{\alpha}}\,p^6\left(69\,\alpha-96\,\alpha^3-7\sqrt{-4+\alpha^2}+48\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]-$$

$$3\,b\,e^{6c\sqrt{-4+\alpha^2}}\,q^6\left(-69\,\alpha+96\,\alpha^3-7\sqrt{-4+\alpha^2}+48\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]+$$

$$4\,b^{1+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,e^{3c\sqrt{-4+\alpha^2}}\,p^2\,q^2\left(e^{c\alpha}\,Q^2\left(-598+415\,\alpha^2\right)+9\,p\,q\,\alpha\left(143-167\,\alpha^2\right)a[1]\right)+$$

$$b^{1+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{4c\sqrt{-4+\alpha^2}}\,p\,q^3\left(4\,e^{c\alpha}\,Q^2\left(160+137\,\alpha^2+189\,\alpha\sqrt{-4+\alpha^2}\right)+3\,p\,q\left(1259\,\alpha-1496\,\alpha^3+19\sqrt{-4+\alpha^2}-236\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]\right)-$$

$$b^{1+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2c\sqrt{-4+\alpha^2}}\,p^3\,q\left(4\,e^{c\alpha}\,Q^2\left(-160-137\,\alpha^2+189\,\alpha\sqrt{-4+\alpha^2}\right)+3\,p\,q\left(-1259\,\alpha+1496\,\alpha^3+19\sqrt{-4+\alpha^2}-236\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]\right)+6\,b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}}$$

$$e^{5c\sqrt{-4+\alpha^2}}\,q^4\left(e^{c\alpha}\,Q^2\left(-14+113\,\alpha^2-20\,\alpha^4-17\,\alpha\sqrt{-4+\alpha^2}+20\,\alpha^3\sqrt{-4+\alpha^2}\right)+5\,p\,q\left(47\,\alpha-59\,\alpha^3+2\sqrt{-4+\alpha^2}-19\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]\right)-6\,b^{1+\frac{5\sqrt{-4+\alpha^2}}{\alpha}}$$

$$e^{c\sqrt{-4+\alpha^2}}\,p^4\left(e^{c\alpha}\,Q^2\left(14-113\,\alpha^2+20\,\alpha^4-17\,\alpha\sqrt{-4+\alpha^2}+20\,\alpha^3\sqrt{-4+\alpha^2}\right)+5\,p\,q\left(-47\,\alpha+59\,\alpha^3+2\sqrt{-4+\alpha^2}-19\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]\right)\Bigg)\Bigg)-$$

$$\left(\left(-\frac{e^{\frac{1}{2}\left(-\alpha-\sqrt{-4+\alpha^2}\right)\left(c-\frac{\mathrm{Log}[b+r\alpha]}{\alpha}\right)}\,p\left(-\alpha-\sqrt{-4+\alpha^2}\right)}{2\,(b+r\,\alpha)}-\frac{e^{\frac{1}{2}\left(-\alpha+\sqrt{-4+\alpha^2}\right)\left(c-\frac{\mathrm{Log}[b+r\alpha]}{\alpha}\right)}\,q\left(-\alpha+\sqrt{-4+\alpha^2}\right)}{2\,(b+r\,\alpha)}\right)\right.$$

$$\left(a[1]+\frac{1}{2\,b^{7/2}\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p+e^{c\sqrt{-4+\alpha^2}}\,q\right)^5}\,r^2\right.$$

$$\left(-2\,b^{\frac{1}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2c\sqrt{-4+\alpha^2}}\,p^3\,q^2\left(-15\,\alpha+\sqrt{-4+\alpha^2}\right)a[0]-\right.$$

$$3\,b^{\frac{1}{2}+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p^4\,q\,\left(-5\,\alpha+\sqrt{-4+\alpha^2}\right)\,a[0]\,-$$

$$b^{\frac{1}{2}+\frac{5\sqrt{-4+\alpha^2}}{\alpha}}\,p^5\,\left(-3\,\alpha+\sqrt{-4+\alpha^2}\right)\,a[0]+\sqrt{b}\,e^{5\,c\sqrt{-4+\alpha^2}}\,q^5\,\left(3\,\alpha+\sqrt{-4+\alpha^2}\right)\,a[0]\,+$$

$$3\,b^{\frac{1}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{4\,c\sqrt{-4+\alpha^2}}\,p\,q^4\,\left(5\,\alpha+\sqrt{-4+\alpha^2}\right)\,a[0]+2\,b^{\frac{1}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{3\,c\sqrt{-4+\alpha^2}}\,p^2\,q^3$$

$$\left(15\,\alpha+\sqrt{-4+\alpha^2}\right)\,a[0]-3\,b^{\frac{3}{2}+\frac{5\sqrt{-4+\alpha^2}}{\alpha}}\,p^5\,a[1]-3\,b^{3/2}\,e^{5\,c\sqrt{-4+\alpha^2}}\,q^5\,a[1]\,+$$

$$b^{\frac{3}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{3\,c\sqrt{-4+\alpha^2}}\,p\,q^2\,\left(e^{c\,\alpha}\,Q^2\,\left(-3\,\alpha+\sqrt{-4+\alpha^2}\right)-30\,p\,q\,a[1]\right)\,+$$

$$b^{\frac{3}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{4\,c\sqrt{-4+\alpha^2}}\,q^3\,\left(e^{c\,\alpha}\,Q^2\,\left(-\alpha+\sqrt{-4+\alpha^2}\right)-15\,p\,q\,a[1]\right)\,-$$

$$b^{\frac{3}{2}+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p^3\,\left(e^{c\,\alpha}\,Q^2\,\left(\alpha+\sqrt{-4+\alpha^2}\right)+15\,p\,q\,a[1]\right)\,-$$

$$b^{\frac{3}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2\,c\sqrt{-4+\alpha^2}}\,p^2\,q\,\left(e^{c\,\alpha}\,Q^2\,\left(3\,\alpha+\sqrt{-4+\alpha^2}\right)+30\,p\,q\,a[1]\right)\Big)\,-$$

$$\frac{1}{2\,b^2\,\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p+e^{c\sqrt{-4+\alpha^2}}\,q\right)^2}\,r\,\Big(4\,b^{\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,p^2\,a[0]+8\,b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p\,q\,a[0]\,+$$

$$4\,e^{2\,c\sqrt{-4+\alpha^2}}\,q^2\,a[0]+b\,e^{2\,c\sqrt{-4+\alpha^2}}\,q^2\,\left(-\alpha+\sqrt{-4+\alpha^2}\right)\,a[1]\,-$$

$$b^{1+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,p^2\,\left(\alpha+\sqrt{-4+\alpha^2}\right)\,a[1]-2\,b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,\left(2\,e^{c\,\alpha}\,Q^2+p\,q\,\alpha\,a[1]\right)\Big)\,+$$

$$\frac{1}{12\,b^4\,\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p+e^{c\sqrt{-4+\alpha^2}}\,q\right)^4}\,r^3\,\Big(-4\,b^{\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2\,c\sqrt{-4+\alpha^2}}\,p^2\,q^2\,\left(-26+29\,\alpha^2\right)\,a[0]\,+$$

$$6\,b^{\frac{4\sqrt{-4+\alpha^2}}{\alpha}}\,p^4\,\left(2-3\,\alpha^2+\alpha\sqrt{-4+\alpha^2}\right)\,a[0]\,-$$

$$6\,e^{4\,c\sqrt{-4+\alpha^2}}\,q^4\,\left(-2+3\,\alpha^2+\alpha\sqrt{-4+\alpha^2}\right)\,a[0]\,+$$

$$4\,b^{\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p^3\,q\,\left(16-19\,\alpha^2+3\,\alpha\sqrt{-4+\alpha^2}\right)\,a[0]\,-$$

$$4\,b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{3\,c\sqrt{-4+\alpha^2}}\,p\,q^3\,\left(-16+19\,\alpha^2+3\,\alpha\sqrt{-4+\alpha^2}\right)\,a[0]-5\,b^{1+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}$$

$$p^4\,\left(-3\,\alpha+\sqrt{-4+\alpha^2}\right)\,a[1]+5\,b\,e^{4\,c\sqrt{-4+\alpha^2}}\,q^4\,\left(3\,\alpha+\sqrt{-4+\alpha^2}\right)\,a[1]\,+$$

$$2\,b^{1+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2\,c\sqrt{-4+\alpha^2}}\,p\,q\,\left(4\,e^{c\,\alpha}\,Q^2+45\,p\,q\,\alpha\,a[1]\right)+2\,b^{1+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}$$

$$p^2\,\left(e^{c\,\alpha}\,Q^2\,\left(-10+3\,\alpha^2+3\,\alpha\sqrt{-4+\alpha^2}\right)-5\,p\,q\,\left(-6\,\alpha+\sqrt{-4+\alpha^2}\right)\,a[1]\right)\,+$$

$$2\,b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}}\,e^{3\,c\sqrt{-4+\alpha^2}}\,q^2\,\Big(-e^{c\,\alpha}\,Q^2\,\left(10-3\,\alpha^2+3\,\alpha\sqrt{-4+\alpha^2}\right)\,+$$

$$5\,p\,q\left(6\alpha+\sqrt{-4+\alpha^2}\right)a[1]\Big)\Big)+\frac{1}{24\,b^{17/2}\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}p+e^{c\sqrt{-4+\alpha^2}}\,q\right)^7}r^4$$

$$\Big(-b^{\frac{7}{2}+\frac{7\sqrt{-4+\alpha^2}}{\alpha}}p^7\left(39\alpha-36\alpha^3-5\sqrt{-4+\alpha^2}+12\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+b^{7/2}\,e^{7c\sqrt{-4+\alpha^2}}$$

$$q^7\left(-39\alpha+36\alpha^3-5\sqrt{-4+\alpha^2}+12\alpha^2\sqrt{-4+\alpha^2}\right)a[0]-b^{\frac{7}{2}+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}e^{3c\sqrt{-4+\alpha^2}}$$

$$p^4\,q^3\left(1925\alpha-1400\alpha^3-9\sqrt{-4+\alpha^2}+56\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+b^{\frac{7}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}$$

$$e^{4c\sqrt{-4+\alpha^2}}\,p^3\,q^4\left(-1925\alpha+1400\alpha^3-9\sqrt{-4+\alpha^2}+56\alpha^2\sqrt{-4+\alpha^2}\right)a[0]-$$

$$b^{\frac{7}{2}+\frac{6\sqrt{-4+\alpha^2}}{\alpha}}e^{c\sqrt{-4+\alpha^2}}\,p^6\,q\left(329\alpha-266\alpha^3-17\sqrt{-4+\alpha^2}+58\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+$$

$$b^{\frac{7}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}e^{6c\sqrt{-4+\alpha^2}}\,p\,q^6\left(-329\alpha+266\alpha^3-17\sqrt{-4+\alpha^2}+58\alpha^2\sqrt{-4+\alpha^2}\right)$$

$$a[0]-b^{\frac{7}{2}+\frac{5\sqrt{-4+\alpha^2}}{\alpha}}e^{2c\sqrt{-4+\alpha^2}}\,p^5\,q^2$$

$$\left(1099\alpha-826\alpha^3-21\sqrt{-4+\alpha^2}+102\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+$$

$$b^{\frac{7}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}e^{5c\sqrt{-4+\alpha^2}}\,p^2\,q^5\left(-1099\alpha+826\alpha^3-21\sqrt{-4+\alpha^2}+102\alpha^2\sqrt{-4+\alpha^2}\right)$$

$$a[0]+b^{\frac{9}{2}+\frac{7\sqrt{-4+\alpha^2}}{\alpha}}p^7\left(11-29\alpha^2+13\alpha\sqrt{-4+\alpha^2}\right)a[1]-$$

$$b^{9/2}\,e^{7c\sqrt{-4+\alpha^2}}\,q^7\left(-11+29\alpha^2+13\alpha\sqrt{-4+\alpha^2}\right)a[1]+$$

$$b^{\frac{9}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}e^{5c\sqrt{-4+\alpha^2}}\,p\,q^4\Big(3\,e^{c\alpha}Q^2\left(47\alpha-8\alpha^3+11\sqrt{-4+\alpha^2}+8\alpha^2\sqrt{-4+\alpha^2}\right)+$$

$$p\,q\left(271-619\alpha^2-117\alpha\sqrt{-4+\alpha^2}\right)a[1]\Big)-$$

$$b^{\frac{9}{2}+\frac{5\sqrt{-4+\alpha^2}}{\alpha}}e^{2c\sqrt{-4+\alpha^2}}\,p^4\,q\Big(3\,e^{c\alpha}Q^2\left(-47\alpha+8\alpha^3+11\sqrt{-4+\alpha^2}+8\alpha^2\sqrt{-4+\alpha^2}\right)+$$

$$p\,q\left(-271+619\alpha^2-117\alpha\sqrt{-4+\alpha^2}\right)a[1]\Big)+b^{\frac{9}{2}+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}e^{3c\sqrt{-4+\alpha^2}}$$

$$p^3\,q^2\Big(-2\,e^{c\alpha}Q^2\left(-69\alpha+6\alpha^3+19\sqrt{-4+\alpha^2}+6\alpha^2\sqrt{-4+\alpha^2}\right)+$$

$$5\,p\,q\left(93-207\alpha^2+13\alpha\sqrt{-4+\alpha^2}\right)a[1]\Big)+$$

$$b^{\frac{9}{2}+\frac{6\sqrt{-4+\alpha^2}}{\alpha}}e^{c\sqrt{-4+\alpha^2}}\,p^5\Big(-e^{c\alpha}Q^2\left(-57\alpha+12\alpha^3-5\sqrt{-4+\alpha^2}+12\alpha^2\sqrt{-4+\alpha^2}\right)+$$

$$5\,p\,q\left(17-41\alpha^2+13\alpha\sqrt{-4+\alpha^2}\right)a[1]\Big)+$$

$$b^{\frac{9}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}e^{6c\sqrt{-4+\alpha^2}}\,q^5\Big(e^{c\alpha}Q^2\left(57\alpha-12\alpha^3-5\sqrt{-4+\alpha^2}+12\alpha^2\sqrt{-4+\alpha^2}\right)-$$

$$5\,p\,q\left(-17+41\,\alpha^2+13\,\alpha\sqrt{-4+\alpha^2}\right)a[1]\Bigg)+$$

$$b^{\frac{9}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,e^{4\,c\,\sqrt{-4+\alpha^2}}\,p^2\,q^3\left(2\,e^{c\,\alpha}\,Q^2\left(69\,\alpha-6\,\alpha^3+19\sqrt{-4+\alpha^2}+6\,\alpha^2\sqrt{-4+\alpha^2}\right)-\right.$$

$$\left.5\,p\,q\left(-93+207\,\alpha^2+13\,\alpha\sqrt{-4+\alpha^2}\right)a[1]\right)\Bigg)-$$

$$\frac{1}{240\,b^6\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p+e^{c\,\sqrt{-4+\alpha^2}}\,q\right)^6}\,r^5\left(16\,b^{\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,e^{3\,c\,\sqrt{-4+\alpha^2}}\,p^3\,q^3\right.$$

$$\left(73-913\,\alpha^2+519\,\alpha^4\right)a[0]-8\,b^{\frac{5\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\,\sqrt{-4+\alpha^2}}\,p^5\,q$$

$$\left(-55+481\,\alpha^2-294\,\alpha^4-27\,\alpha\sqrt{-4+\alpha^2}+54\,\alpha^3\sqrt{-4+\alpha^2}\right)a[0]+8\,b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}$$

$$e^{5\,c\,\sqrt{-4+\alpha^2}}\,p\,q^5\left(55-481\,\alpha^2+294\,\alpha^4-27\,\alpha\sqrt{-4+\alpha^2}+54\,\alpha^3\sqrt{-4+\alpha^2}\right)a[0]-$$

$$2\,b^{\frac{6\sqrt{-4+\alpha^2}}{\alpha}}\,p^6\left(-22+253\,\alpha^2-180\,\alpha^4-51\,\alpha\sqrt{-4+\alpha^2}+60\,\alpha^3\sqrt{-4+\alpha^2}\right)a[0]+$$

$$2\,e^{6\,c\,\sqrt{-4+\alpha^2}}\,q^6\left(22-253\,\alpha^2+180\,\alpha^4-51\,\alpha\sqrt{-4+\alpha^2}+60\,\alpha^3\sqrt{-4+\alpha^2}\right)a[0]-$$

$$2\,b^{\frac{4\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2\,c\,\sqrt{-4+\alpha^2}}\,p^4\,q^2\left(-490+5323\,\alpha^2-3072\,\alpha^4-63\,\alpha\sqrt{-4+\alpha^2}+\right.$$

$$\left.252\,\alpha^3\sqrt{-4+\alpha^2}\right)a[0]+2\,b^{\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{4\,c\,\sqrt{-4+\alpha^2}}\,p^2\,q^4$$

$$\left(490-5323\,\alpha^2+3072\,\alpha^4-63\,\alpha\sqrt{-4+\alpha^2}+252\,\alpha^3\sqrt{-4+\alpha^2}\right)a[0]+$$

$$3\,b^{1+\frac{6\sqrt{-4+\alpha^2}}{\alpha}}\,p^6\left(69\,\alpha-96\,\alpha^3-7\sqrt{-4+\alpha^2}+48\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]-3\,b$$

$$e^{6\,c\,\sqrt{-4+\alpha^2}}\,q^6\left(-69\,\alpha+96\,\alpha^3-7\sqrt{-4+\alpha^2}+48\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]+4\,b^{1+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}$$

$$e^{3\,c\,\sqrt{-4+\alpha^2}}\,p^2\,q^2\left(e^{c\,\alpha}\,Q^2\left(-598+415\,\alpha^2\right)+9\,p\,q\,\alpha\left(143-167\,\alpha^2\right)a[1]\right)+$$

$$b^{1+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{4\,c\,\sqrt{-4+\alpha^2}}\,p\,q^3\left(4\,e^{c\,\alpha}\,Q^2\left(160+137\,\alpha^2+189\,\alpha\sqrt{-4+\alpha^2}\right)+\right.$$

$$\left.3\,p\,q\left(1259\,\alpha-1496\,\alpha^3+19\sqrt{-4+\alpha^2}-236\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]\right)-$$

$$b^{1+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2\,c\,\sqrt{-4+\alpha^2}}\,p^3\,q\left(4\,e^{c\,\alpha}\,Q^2\left(-160-137\,\alpha^2+189\,\alpha\sqrt{-4+\alpha^2}\right)+\right.$$

$$\left.3\,p\,q\left(-1259\,\alpha+1496\,\alpha^3+19\sqrt{-4+\alpha^2}-236\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]\right)+$$

$$6\,b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}}\,e^{5\,c\,\sqrt{-4+\alpha^2}}\,q^4\left(e^{c\,\alpha}\,Q^2\left(-14+113\,\alpha^2-20\,\alpha^4-\right.\right.$$

$$\left.17\,\alpha\sqrt{-4+\alpha^2}+20\,\alpha^3\sqrt{-4+\alpha^2}\right)+$$

$$5\,p\,q\left(47\,\alpha-59\,\alpha^{3}+2\,\sqrt{-4+\alpha^{2}}-19\,\alpha^{2}\,\sqrt{-4+\alpha^{2}}\right)a[1]\Bigg)-6\,b^{1+\frac{5\sqrt{-4+\alpha^{2}}}{\alpha}}$$

$$e^{c\sqrt{-4+\alpha^{2}}}\,p^{4}\left(e^{c\,\alpha}\,Q^{2}\left(14-113\,\alpha^{2}+20\,\alpha^{4}-17\,\alpha\,\sqrt{-4+\alpha^{2}}+20\,\alpha^{3}\,\sqrt{-4+\alpha^{2}}\right)+\right.$$

$$\left.\left.5\,p\,q\left(-47\,\alpha+59\,\alpha^{3}+2\,\sqrt{-4+\alpha^{2}}-19\,\alpha^{2}\,\sqrt{-4+\alpha^{2}}\right)a[1]\right)\right)\Bigg)\Bigg/$$

$$\left(4\left(e^{\frac{1}{2}\left(-\alpha-\sqrt{-4+\alpha^{2}}\right)\left(c-\frac{\text{Log}[b+r\,\alpha]}{\alpha}\right)}\,p+e^{\frac{1}{2}\left(-\alpha+\sqrt{-4+\alpha^{2}}\right)\left(c-\frac{\text{Log}[b+r\,\alpha]}{\alpha}\right)}\,q\right)\right)$$

4.2 Appendix B

The expression of $f(r)$ and potential $V(r)$ for the most general case where $Q = 0$ is given in this appendix. Since we have calculated the expression of $f(r)$ in mathematica we need to change our constants with some different variables to run the code in mathematica file. The expression given below for $f(r)$ and $V(r)$ holds such changes given as,

$c = c_2, b = c_1$ and $p = c_3, q = c_4$

Expression for f(r)

f[r] =

$$a[0]+a[1]*r+\Big(-\Big(\Big(4\,b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p\,a[0]+4\,e^{c\sqrt{-4+\alpha^2}}\,q\,a[0]+b\,e^{c\sqrt{-4+\alpha^2}}\,q\,\big(-\alpha+\sqrt{-4+\alpha^2}\big)\,a[1]-b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}}$$

$$p\,\big(\alpha+\sqrt{-4+\alpha^2}\big)\,a[1]\Big)\Big/\Big(4\,b^2\,\Big(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p+e^{c\sqrt{-4+\alpha^2}}\,q\Big)\Big)\Big)\Big)*r^2+$$

$$\Bigg(\frac{1}{6\,b^{9/2}\,\Big(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p+e^{c\sqrt{-4+\alpha^2}}\,q\Big)^3}\Big(-b^{\frac{3}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p^2\,q\,\big(-9\,\alpha+\sqrt{-4+\alpha^2}\big)\,a[0]-$$

$$b^{\frac{3}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,p^3\,\big(-3\,\alpha+\sqrt{-4+\alpha^2}\big)\,a[0]+b^{3/2}\,e^{3\,c\sqrt{-4+\alpha^2}}\,q^3\,\big(3\,\alpha+\sqrt{-4+\alpha^2}\big)\,a[0]+$$

$$b^{\frac{3}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2\,c\sqrt{-4+\alpha^2}}\,p\,q^2\,\big(9\,\alpha+\sqrt{-4+\alpha^2}\big)\,a[0]-3\,b^{\frac{5}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,p^3\,a[1]-9\,b^{\frac{5}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}$$

$$e^{c\sqrt{-4+\alpha^2}}\,p^2\,q\,a[1]-9\,b^{\frac{5}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2\,c\sqrt{-4+\alpha^2}}\,p\,q^2\,a[1]-3\,b^{5/2}\,e^{3\,c\sqrt{-4+\alpha^2}}\,q^3\,a[1]\Big)\Bigg)*$$

$$r^3+\Bigg(\frac{1}{48\,b^4\,\Big(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p+e^{c\sqrt{-4+\alpha^2}}\,q\Big)^2}\Big(-40\,b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p\,q\,\big(-1+\alpha^2\big)\,a[0]+$$

$$6\,b^{\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,p^2\,\big(2-3\,\alpha^2+\alpha\sqrt{-4+\alpha^2}\big)\,a[0]-6\,e^{2\,c\sqrt{-4+\alpha^2}}\,q^2\,\big(-2+3\,\alpha^2+\alpha\sqrt{-4+\alpha^2}\big)\,a[0]+$$

$$30\,b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p\,q\,\alpha\,a[1]-5\,b^{1+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,p^2\,\big(-3\,\alpha+\sqrt{-4+\alpha^2}\big)\,a[1]+$$

$$5\,b\,e^{2\,c\sqrt{-4+\alpha^2}}\,q^2\,\big(3\,\alpha+\sqrt{-4+\alpha^2}\big)\,a[1]\Big)\Bigg)*r^4+\Bigg(\frac{1}{120\,b^{15/2}\,\Big(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p+e^{c\sqrt{-4+\alpha^2}}\,q\Big)^5}$$

$$\Big(-2\,b^{\frac{5}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2\,c\sqrt{-4+\alpha^2}}\,p^3\,q^2\,\big(279\,\alpha-201\,\alpha^3-\sqrt{-4+\alpha^2}+11\,\alpha^2\sqrt{-4+\alpha^2}\big)\,a[0]+$$

$$2\,b^{\frac{5}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{3\,c\sqrt{-4+\alpha^2}}\,p^2\,q^3\,\big(-279\,\alpha+201\,\alpha^3-\sqrt{-4+\alpha^2}+11\,\alpha^2\sqrt{-4+\alpha^2}\big)\,a[0]-$$

$$b^{\frac{5}{2}+\frac{5\sqrt{-4+\alpha^2}}{\alpha}}\,p^5\,\big(39\,\alpha-36\,\alpha^3-5\sqrt{-4+\alpha^2}+12\,\alpha^2\sqrt{-4+\alpha^2}\big)\,a[0]+$$

$$b^{5/2}\,e^{5\,c\sqrt{-4+\alpha^2}}\,q^5\,\big(-39\,\alpha+36\,\alpha^3-5\sqrt{-4+\alpha^2}+12\,\alpha^2\sqrt{-4+\alpha^2}\big)\,a[0]-$$

$$b^{\frac{5}{2}+\frac{4\sqrt{-4+\alpha^2}}{\alpha}} e^{c\sqrt{-4+\alpha^2}}\, p^4\, q \left(251\,\alpha - 194\,\alpha^3 - 7\sqrt{-4+\alpha^2} + 34\,\alpha^2\sqrt{-4+\alpha^2}\right) a[0] +$$

$$b^{\frac{5}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}} e^{4c\sqrt{-4+\alpha^2}}\, p\, q^4 \left(-251\,\alpha + 194\,\alpha^3 - 7\sqrt{-4+\alpha^2} + 34\,\alpha^2\sqrt{-4+\alpha^2}\right) a[0] +$$

$$2\,b^{\frac{7}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}} e^{2c\sqrt{-4+\alpha^2}}\, p^3\, q^2 \left(67 - 148\,\alpha^2 + 13\,\alpha\sqrt{-4+\alpha^2}\right) a[1] +$$

$$3\,b^{\frac{7}{2}+\frac{4\sqrt{-4+\alpha^2}}{\alpha}} e^{c\sqrt{-4+\alpha^2}}\, p^4\, q \left(21 - 49\,\alpha^2 + 13\,\alpha\sqrt{-4+\alpha^2}\right) a[1] +$$

$$b^{\frac{7}{2}+\frac{5\sqrt{-4+\alpha^2}}{\alpha}} p^5 \left(11 - 29\,\alpha^2 + 13\,\alpha\sqrt{-4+\alpha^2}\right) a[1] -$$

$$b^{7/2} e^{5c\sqrt{-4+\alpha^2}}\, q^5 \left(-11 + 29\,\alpha^2 + 13\,\alpha\sqrt{-4+\alpha^2}\right) a[1] -$$

$$3\,b^{\frac{7}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}} e^{4c\sqrt{-4+\alpha^2}}\, p\, q^4 \left(-21 + 49\,\alpha^2 + 13\,\alpha\sqrt{-4+\alpha^2}\right) a[1] -$$

$$\left.\left. 2\,b^{\frac{7}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}} e^{3c\sqrt{-4+\alpha^2}}\, p^2\, q^3 \left(-67 + 148\,\alpha^2 + 13\,\alpha\sqrt{-4+\alpha^2}\right) a[1]\right)\right) * r^5 +$$

$$\left(\frac{1}{1440\,b^6 \left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}} p + e^{c\sqrt{-4+\alpha^2}}\, q\right)^4} \left(-4\,b^{\frac{2\sqrt{-4+\alpha^2}}{\alpha}} e^{2c\sqrt{-4+\alpha^2}}\, p^2\, q^2 \left(58 - 1117\,\alpha^2 + 630\,\alpha^4\right) a[0] +\right.\right.$$

$$4\,b^{\frac{3\sqrt{-4+\alpha^2}}{\alpha}} e^{c\sqrt{-4+\alpha^2}}\, p^3\, q \left(-88 + 709\,\alpha^2 - 408\,\alpha^4 - 3\,\alpha\sqrt{-4+\alpha^2} + 48\,\alpha^3\sqrt{-4+\alpha^2}\right) a[0] -$$

$$4\,b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}} e^{3c\sqrt{-4+\alpha^2}}\, p\, q^3 \left(88 - 709\,\alpha^2 + 408\,\alpha^4 - 3\,\alpha\sqrt{-4+\alpha^2} + 48\,\alpha^3\sqrt{-4+\alpha^2}\right) a[0] +$$

$$2\,b^{\frac{4\sqrt{-4+\alpha^2}}{\alpha}} p^4 \left(-22 + 253\,\alpha^2 - 180\,\alpha^4 - 51\,\alpha\sqrt{-4+\alpha^2} + 60\,\alpha^3\sqrt{-4+\alpha^2}\right) a[0] -$$

$$2\,e^{4c\sqrt{-4+\alpha^2}}\, q^4 \left(22 - 253\,\alpha^2 + 180\,\alpha^4 - 51\,\alpha\sqrt{-4+\alpha^2} + 60\,\alpha^3\sqrt{-4+\alpha^2}\right) a[0] +$$

$$6\,b^{1+\frac{2\sqrt{-4+\alpha^2}}{\alpha}} e^{2c\sqrt{-4+\alpha^2}}\, p^2\, q^2\, \alpha \left(-263 + 302\,\alpha^2\right) a[1] -$$

$$6\,b^{1+\frac{3\sqrt{-4+\alpha^2}}{\alpha}} e^{c\sqrt{-4+\alpha^2}}\, p^3\, q \left(166\,\alpha - 199\,\alpha^3 - 3\sqrt{-4+\alpha^2} + 47\,\alpha^2\sqrt{-4+\alpha^2}\right) a[1] +$$

$$6\,b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}} e^{3c\sqrt{-4+\alpha^2}}\, p\, q^3 \left(-166\,\alpha + 199\,\alpha^3 - 3\sqrt{-4+\alpha^2} + 47\,\alpha^2\sqrt{-4+\alpha^2}\right) a[1] -$$

$$3\,b^{1+\frac{4\sqrt{-4+\alpha^2}}{\alpha}} p^4 \left(69\,\alpha - 96\,\alpha^3 - 7\sqrt{-4+\alpha^2} + 48\,\alpha^2\sqrt{-4+\alpha^2}\right) a[1] +$$

$$3\,b\,e^{4c\sqrt{-4+\alpha^2}}\,q^4\left(-69\,\alpha+96\,\alpha^3-7\sqrt{-4+\alpha^2}+48\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]\Bigg)\Bigg)*r^6$$

Expression for V(r)
V[r] =

$$\frac{1}{4}\Bigg(-\frac{1}{b^{9/2}\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}p+e^{c\sqrt{-4+\alpha^2}}\,q\right)^3}\,r\Bigg(-b^{\frac{3}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p^2\,q\left(-9\,\alpha+\sqrt{-4+\alpha^2}\right)a[0]-b^{\frac{3}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}$$

$$p^3\left(-3\,\alpha+\sqrt{-4+\alpha^2}\right)a[0]+b^{3/2}\,e^{3c\sqrt{-4+\alpha^2}}\,q^3\left(3\,\alpha+\sqrt{-4+\alpha^2}\right)a[0]+$$

$$b^{\frac{3}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2c\sqrt{-4+\alpha^2}}\,p\,q^2\left(9\,\alpha+\sqrt{-4+\alpha^2}\right)a[0]-3\,b^{\frac{5}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,p^3\,a[1]-9\,b^{\frac{5}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}$$

$$e^{c\sqrt{-4+\alpha^2}}\,p^2\,q\,a[1]-9\,b^{\frac{5}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2c\sqrt{-4+\alpha^2}}\,p\,q^2\,a[1]-3\,b^{5/2}\,e^{3c\sqrt{-4+\alpha^2}}\,q^3\,a[1]\Bigg)+$$

$$\Bigg(4\,b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p\,a[0]+4\,e^{c\sqrt{-4+\alpha^2}}\,q\,a[0]+b\,e^{c\sqrt{-4+\alpha^2}}\,q\left(-\alpha+\sqrt{-4+\alpha^2}\right)a[1]-$$

$$b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}}\,p\left(\alpha+\sqrt{-4+\alpha^2}\right)a[1]\Bigg)\Bigg/\left(2\,b^2\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}p+e^{c\sqrt{-4+\alpha^2}}\,q\right)\right)-$$

$$\frac{1}{4\,b^4\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}p+e^{c\sqrt{-4+\alpha^2}}\,q\right)^2}\,r^2\Bigg(-40\,b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p\,q\left(-1+\alpha^2\right)a[0]+6\,b^{\frac{2\sqrt{-4+\alpha^2}}{\alpha}}$$

$$p^2\left(2-3\,\alpha^2+\alpha\sqrt{-4+\alpha^2}\right)a[0]-6\,e^{2c\sqrt{-4+\alpha^2}}\,q^2\left(-2+3\,\alpha^2+\alpha\sqrt{-4+\alpha^2}\right)a[0]+$$

$$30\,b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p\,q\,\alpha\,a[1]-5\,b^{1+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,p^2\left(-3\,\alpha+\sqrt{-4+\alpha^2}\right)a[1]+$$

$$5\,b\,e^{2c\sqrt{-4+\alpha^2}}\,q^2\left(3\,\alpha+\sqrt{-4+\alpha^2}\right)a[1]\Bigg)-\frac{1}{6\,b^{15/2}\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}p+e^{c\sqrt{-4+\alpha^2}}\,q\right)^5}\,r^3$$

$$\Bigg(-2\,b^{\frac{5}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2c\sqrt{-4+\alpha^2}}\,p^3\,q^2\left(279\,\alpha-201\,\alpha^3-\sqrt{-4+\alpha^2}+11\,\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+$$

$$2\,b^{\frac{5}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{3c\sqrt{-4+\alpha^2}}\,p^2\,q^3\left(-279\,\alpha+201\,\alpha^3-\sqrt{-4+\alpha^2}+11\,\alpha^2\sqrt{-4+\alpha^2}\right)a[0]-$$

$$b^{\frac{5}{2}+\frac{5\sqrt{-4+\alpha^2}}{\alpha}}\,p^5\left(39\,\alpha-36\,\alpha^3-5\sqrt{-4+\alpha^2}+12\,\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+$$

$$b^{5/2}\,e^{5c\sqrt{-4+\alpha^2}}\,q^5\left(-39\,\alpha+36\,\alpha^3-5\sqrt{-4+\alpha^2}+12\,\alpha^2\sqrt{-4+\alpha^2}\right)a[0]-$$

$$b^{\frac{5}{2}+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p^4\,q\left(251\,\alpha-194\,\alpha^3-7\sqrt{-4+\alpha^2}+34\,\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+$$

$$b^{\frac{5}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}} e^{4 c \sqrt{-4+\alpha^2}}\, p\, q^4 \left(-251 \alpha+194 \alpha^3-7 \sqrt{-4+\alpha^2}+34 \alpha^2 \sqrt{-4+\alpha^2}\right) a[0]+$$

$$2 b^{\frac{7}{2}+\frac{3 \sqrt{-4+\alpha^2}}{\alpha}} e^{2 c \sqrt{-4+\alpha^2}}\, p^3 q^2 \left(67-148 \alpha^2+13 \alpha \sqrt{-4+\alpha^2}\right) a[1]+$$

$$3 b^{\frac{7}{2}+\frac{4 \sqrt{-4+\alpha^2}}{\alpha}} e^{c \sqrt{-4+\alpha^2}}\, p^4 q \left(21-49 \alpha^2+13 \alpha \sqrt{-4+\alpha^2}\right) a[1]+$$

$$b^{\frac{7}{2}+\frac{5 \sqrt{-4+\alpha^2}}{\alpha}} p^5 \left(11-29 \alpha^2+13 \alpha \sqrt{-4+\alpha^2}\right) a[1]-$$

$$b^{7/2} e^{5 c \sqrt{-4+\alpha^2}}\, q^5 \left(-11+29 \alpha^2+13 \alpha \sqrt{-4+\alpha^2}\right) a[1]-$$

$$3 b^{\frac{7}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}} e^{4 c \sqrt{-4+\alpha^2}}\, p\, q^4 \left(-21+49 \alpha^2+13 \alpha \sqrt{-4+\alpha^2}\right) a[1]-$$

$$2 b^{\frac{7}{2}+\frac{2 \sqrt{-4+\alpha^2}}{\alpha}} e^{3 c \sqrt{-4+\alpha^2}}\, p^2 q^3 \left(-67+148 \alpha^2+13 \alpha \sqrt{-4+\alpha^2}\right) a[1]\Big)-$$

$$\frac{1}{48 b^6 \left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}} p+e^{c \sqrt{-4+\alpha^2}}\, q\right)^4} r^4 \Big(-4 b^{\frac{2 \sqrt{-4+\alpha^2}}{\alpha}} e^{2 c \sqrt{-4+\alpha^2}}\, p^2 q^2 \left(58-1117 \alpha^2+630 \alpha^4\right) a[0]+$$

$$4 b^{\frac{3 \sqrt{-4+\alpha^2}}{\alpha}} e^{c \sqrt{-4+\alpha^2}}\, p^3 q \left(-88+709 \alpha^2-408 \alpha^4-3 \alpha \sqrt{-4+\alpha^2}+48 \alpha^3 \sqrt{-4+\alpha^2}\right) a[0]-$$

$$4 b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}} e^{3 c \sqrt{-4+\alpha^2}}\, p\, q^3 \left(88-709 \alpha^2+408 \alpha^4-3 \alpha \sqrt{-4+\alpha^2}+48 \alpha^3 \sqrt{-4+\alpha^2}\right) a[0]+$$

$$2 b^{\frac{4 \sqrt{-4+\alpha^2}}{\alpha}} p^4 \left(-22+253 \alpha^2-180 \alpha^4-51 \alpha \sqrt{-4+\alpha^2}+60 \alpha^3 \sqrt{-4+\alpha^2}\right) a[0]-$$

$$2 e^{4 c \sqrt{-4+\alpha^2}}\, q^4 \left(22-253 \alpha^2+180 \alpha^4-51 \alpha \sqrt{-4+\alpha^2}+60 \alpha^3 \sqrt{-4+\alpha^2}\right) a[0]+$$

$$6 b^{1+\frac{2 \sqrt{-4+\alpha^2}}{\alpha}} e^{2 c \sqrt{-4+\alpha^2}}\, p^2 q^2 \alpha \left(-263+302 \alpha^2\right) a[1]-$$

$$6 b^{1+\frac{3 \sqrt{-4+\alpha^2}}{\alpha}} e^{c \sqrt{-4+\alpha^2}}\, p^3 q \left(166 \alpha-199 \alpha^3-3 \sqrt{-4+\alpha^2}+47 \alpha^2 \sqrt{-4+\alpha^2}\right) a[1]+$$

$$6 b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}} e^{3 c \sqrt{-4+\alpha^2}}\, p\, q^3 \left(-166 \alpha+199 \alpha^3-3 \sqrt{-4+\alpha^2}+47 \alpha^2 \sqrt{-4+\alpha^2}\right) a[1]-$$

$$3 b^{1+\frac{4 \sqrt{-4+\alpha^2}}{\alpha}} p^4 \left(69 \alpha-96 \alpha^3-7 \sqrt{-4+\alpha^2}+48 \alpha^2 \sqrt{-4+\alpha^2}\right) a[1]+$$

$$3 b\, e^{4 c \sqrt{-4+\alpha^2}}\, q^4 \left(-69 \alpha+96 \alpha^3-7 \sqrt{-4+\alpha^2}+48 \alpha^2 \sqrt{-4+\alpha^2}\right) a[1]\Big)\Bigg)-$$

$$\Bigg(\Bigg(-\frac{e^{\frac{1}{2}\left(-\alpha-\sqrt{-4+\alpha^2}\right)\left(c-\frac{\mathrm{Log}[b+r \alpha]}{\alpha}\right)} p \left(-\alpha-\sqrt{-4+\alpha^2}\right)}{2 (b+r \alpha)}-\frac{e^{\frac{1}{2}\left(-\alpha+\sqrt{-4+\alpha^2}\right)\left(c-\frac{\mathrm{Log}[b+r \alpha]}{\alpha}\right)} q \left(-\alpha+\sqrt{-4+\alpha^2}\right)}{2 (b+r \alpha)}\Bigg)$$

$$\Bigg(a[1]+\frac{1}{2\,b^{9/2}\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p+e^{c\sqrt{-4+\alpha^2}}\,q\right)^3}\,r^2$$

$$\Big(-b^{\frac{3}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p^2\,q\left(-9\,\alpha+\sqrt{-4+\alpha^2}\right)a[0]-b^{\frac{3}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,p^3\left(-3\,\alpha+\sqrt{-4+\alpha^2}\right)$$

$$a[0]+b^{3/2}\,e^{3\,c\sqrt{-4+\alpha^2}}\,q^3\left(3\,\alpha+\sqrt{-4+\alpha^2}\right)a[0]+b^{\frac{3}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2\,c\sqrt{-4+\alpha^2}}\,p$$

$$q^2\left(9\,\alpha+\sqrt{-4+\alpha^2}\right)a[0]-3\,b^{\frac{5}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,p^3\,a[1]-9\,b^{\frac{5}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p^2$$

$$q\,a[1]-9\,b^{\frac{5}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2\,c\sqrt{-4+\alpha^2}}\,p\,q^2\,a[1]-3\,b^{5/2}\,e^{3\,c\sqrt{-4+\alpha^2}}\,q^3\,a[1]\Big)-$$

$$\Big(r\,\Big(4\,b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p\,a[0]+4\,e^{c\sqrt{-4+\alpha^2}}\,q\,a[0]+b\,e^{c\sqrt{-4+\alpha^2}}\,q\left(-\alpha+\sqrt{-4+\alpha^2}\right)a[1]-$$

$$b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}}\,p\left(\alpha+\sqrt{-4+\alpha^2}\right)a[1]\Big)\Big)\Big/$$

$$\left(2\,b^2\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p+e^{c\sqrt{-4+\alpha^2}}\,q\right)\right)+\frac{1}{12\,b^4\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p+e^{c\sqrt{-4+\alpha^2}}\,q\right)^2}\,r^3$$

$$\Big(-40\,b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p\,q\left(-1+\alpha^2\right)a[0]+6\,b^{\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,p^2\left(2-3\,\alpha^2+\alpha\sqrt{-4+\alpha^2}\right)a[0]-$$

$$6\,e^{2\,c\sqrt{-4+\alpha^2}}\,q^2\left(-2+3\,\alpha^2+\alpha\sqrt{-4+\alpha^2}\right)a[0]+$$

$$30\,b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p\,q\,\alpha\,a[1]-5\,b^{1+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,p^2\left(-3\,\alpha+\sqrt{-4+\alpha^2}\right)a[1]+$$

$$5\,b\,e^{2\,c\sqrt{-4+\alpha^2}}\,q^2\left(3\,\alpha+\sqrt{-4+\alpha^2}\right)a[1]\Big)+\frac{1}{24\,b^{15/2}\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p+e^{c\sqrt{-4+\alpha^2}}\,q\right)^5}\,r^4$$

$$\Big(-2\,b^{\frac{5}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2\,c\sqrt{-4+\alpha^2}}\,p^3\,q^2\left(279\,\alpha-201\,\alpha^3-\sqrt{-4+\alpha^2}+11\,\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+$$

$$2\,b^{\frac{5}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{3\,c\sqrt{-4+\alpha^2}}\,p^2\,q^3\left(-279\,\alpha+201\,\alpha^3-\sqrt{-4+\alpha^2}+11\,\alpha^2\sqrt{-4+\alpha^2}\right)$$

$$a[0]-b^{\frac{5}{2}+\frac{5\sqrt{-4+\alpha^2}}{\alpha}}\,p^5\left(39\,\alpha-36\,\alpha^3-5\sqrt{-4+\alpha^2}+12\,\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+$$

$$b^{5/2}\,e^{5\,c\sqrt{-4+\alpha^2}}\,q^5\left(-39\,\alpha+36\,\alpha^3-5\sqrt{-4+\alpha^2}+12\,\alpha^2\sqrt{-4+\alpha^2}\right)a[0]-$$

$$b^{\frac{5}{2}+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p^4\,q\left(251\,\alpha-194\,\alpha^3-7\sqrt{-4+\alpha^2}+34\,\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+$$

$$b^{\frac{5}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{4\,c\sqrt{-4+\alpha^2}}\,p\,q^4\left(-251\,\alpha+194\,\alpha^3-7\sqrt{-4+\alpha^2}+34\,\alpha^2\sqrt{-4+\alpha^2}\right)a[0]+$$

$$2\,b^{\frac{7}{2}+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2c\sqrt{-4+\alpha^2}}\,p^3\,q^2\left(67-148\,\alpha^2+13\,\alpha\sqrt{-4+\alpha^2}\right)a[1]+$$

$$3\,b^{\frac{7}{2}+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p^4\,q\left(21-49\,\alpha^2+13\,\alpha\sqrt{-4+\alpha^2}\right)a[1]+$$

$$b^{\frac{7}{2}+\frac{5\sqrt{-4+\alpha^2}}{\alpha}}\,p^5\left(11-29\,\alpha^2+13\,\alpha\sqrt{-4+\alpha^2}\right)a[1]-$$

$$b^{7/2}\,e^{5c\sqrt{-4+\alpha^2}}\,q^5\left(-11+29\,\alpha^2+13\,\alpha\sqrt{-4+\alpha^2}\right)a[1]-3\,b^{\frac{7}{2}+\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,e^{4c\sqrt{-4+\alpha^2}}$$

$$p\,q^4\left(-21+49\,\alpha^2+13\,\alpha\sqrt{-4+\alpha^2}\right)a[1]-2\,b^{\frac{7}{2}+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{3c\sqrt{-4+\alpha^2}}\,p^2\,q^3$$

$$\left.\left(-67+148\,\alpha^2+13\,\alpha\sqrt{-4+\alpha^2}\right)a[1]\right)+\frac{1}{240\,b^6\left(b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}\,p+e^{c\sqrt{-4+\alpha^2}}\,q\right)^4}\,r^5$$

$$\left(-4\,b^{\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2c\sqrt{-4+\alpha^2}}\,p^2\,q^2\left(58-1117\,\alpha^2+630\,\alpha^4\right)a[0]+4\,b^{\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\right.$$

$$p^3\,q\left(-88+709\,\alpha^2-408\,\alpha^4-3\,\alpha\sqrt{-4+\alpha^2}+48\,\alpha^3\sqrt{-4+\alpha^2}\right)a[0]-4\,b^{\frac{\sqrt{-4+\alpha^2}}{\alpha}}$$

$$e^{3c\sqrt{-4+\alpha^2}}\,p\,q^3\left(88-709\,\alpha^2+408\,\alpha^4-3\,\alpha\sqrt{-4+\alpha^2}+48\,\alpha^3\sqrt{-4+\alpha^2}\right)a[0]+$$

$$2\,b^{\frac{4\sqrt{-4+\alpha^2}}{\alpha}}\,p^4\left(-22+253\,\alpha^2-180\,\alpha^4-51\,\alpha\sqrt{-4+\alpha^2}+60\,\alpha^3\sqrt{-4+\alpha^2}\right)a[0]-$$

$$2\,e^{4c\sqrt{-4+\alpha^2}}\,q^4\left(22-253\,\alpha^2+180\,\alpha^4-51\,\alpha\sqrt{-4+\alpha^2}+60\,\alpha^3\sqrt{-4+\alpha^2}\right)a[0]+$$

$$6\,b^{1+\frac{2\sqrt{-4+\alpha^2}}{\alpha}}\,e^{2c\sqrt{-4+\alpha^2}}\,p^2\,q^2\,\alpha\left(-263+302\,\alpha^2\right)a[1]-$$

$$6\,b^{1+\frac{3\sqrt{-4+\alpha^2}}{\alpha}}\,e^{c\sqrt{-4+\alpha^2}}\,p^3\,q\left(166\,\alpha-199\,\alpha^3-3\sqrt{-4+\alpha^2}+47\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]+$$

$$6\,b^{\frac{\alpha+\sqrt{-4+\alpha^2}}{\alpha}}\,e^{3c\sqrt{-4+\alpha^2}}\,p\,q^3\left(-166\,\alpha+199\,\alpha^3-3\sqrt{-4+\alpha^2}+47\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]-$$

$$3\,b^{1+\frac{4\sqrt{-4+\alpha^2}}{\alpha}}\,p^4\left(69\,\alpha-96\,\alpha^3-7\sqrt{-4+\alpha^2}+48\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]+$$

$$\left.\left.\left.3\,b\,e^{4c\sqrt{-4+\alpha^2}}\,q^4\left(-69\,\alpha+96\,\alpha^3-7\sqrt{-4+\alpha^2}+48\,\alpha^2\sqrt{-4+\alpha^2}\right)a[1]\right)\right)\right)\Big/$$

$$\left(4\left(e^{\frac{1}{2}\left(-\alpha-\sqrt{-4+\alpha^2}\right)\left(c-\frac{\mathrm{Log}[b+r\,\alpha]}{\alpha}\right)}\,p+e^{\frac{1}{2}\left(-\alpha+\sqrt{-4+\alpha^2}\right)\left(c-\frac{\mathrm{Log}[b+r\,\alpha]}{\alpha}\right)}\,q\right)\right)$$

4.3 Appendix C

This is the mathematica file use to evaluate christoffel symbols, Ricci and Riemann Tensor and Ricci Scalar, and further calculated curvature invariants for the first case where the value of $y(r) = r^{\frac{1}{5}}$.

In[57]:=
```
ClearAll["Global`*"]
```

ClearAll::wrsym : Symbol antisymmetricQ is Protected. ≫

ClearAll::wrsym : Symbol antisymmetrize is Protected. ≫

ClearAll::wrsym : Symbol Christoffel is Protected. ≫

General::stop : Further output of ClearAll::wrsym will be suppressed during this calculation. ≫

In[58]:=
```
$Assumptions = And[r ∈ Reals, a ∈ Reals, y ∈ Reals, t ∈ Reals, z ∈ Reals, r > 0, a > 0];

metricsign = -1;
```

In[60]:=
```
coord = {t, r, z};
fr = r^(2/5) (r^(2/5) - a^(2/5)); yr = r^(1/5)
metric = DiagonalMatrix[{-fr, 1/fr, yr^2}];
```

Out[61]= $r^{1/5}$

In[63]:=
```
<< diffgeo.m
```

In[64]:=
```
display[Christoffel]
```

Out[64]=

{z, r, z} {z, z, r}	$\frac{1}{5\,r}$
{r, z, z}	$\frac{a^{2/5}-r^{2/5}}{5\,r^{1/5}}$
{t, t, r} {t, r, t}	$\frac{a^{2/5}-2\,r^{2/5}}{5\,a^{2/5}\,r-5\,r^{7/5}}$
{r, r, r}	$\frac{(a^{2/5}-2\,r^{2/5})\,(r^{4/5}-(a\,r)^{2/5})}{5\,(a^{2/5}-r^{2/5})^2\,r^{7/5}}$
{r, t, t}	$\frac{1}{5}\left(\left(\frac{a^4}{r}\right)^{1/5}+2\,r^{3/5}-3\,(a^2\,r)^{1/5}\right)$

In[65]:= **display[Riemann]**

Out[65]=

{z, r, r, z}	$\frac{-3 a^{2/5}+2 r^{2/5}}{25 \left(a^{2/5}-r^{2/5}\right) r^2}$
{z, t, z, t}	$\frac{a^{2/5}-2 r^{2/5}}{25 r^{6/5}}$
{r, z, z, r}	$\frac{3 a^{2/5}-2 r^{2/5}}{25 r^{6/5}}$
{z, r, z, r}	$\frac{-3 a^{2/5}+2 r^{2/5}}{25 r^{6/5}}$
{t, z, z, t}	$\frac{-a^{2/5}+2 r^{2/5}}{25 r^{6/5}}$
{r, z, r, z}	$\frac{2 \left(3 a^{2/5}-2 r^{2/5}\right)}{50 a^{2/5} r^2-50 r^{12/5}}$
{r, t, r, t}	$\frac{-3 a^{6/5} r^{2/5}-7 a^{2/5} r^{6/5}+2 r^{8/5}+8 (a r)^{4/5}}{25 \left(-a^{2/5}+r^{2/5}\right)^3 r^{12/5}}$
{z, t, t, z}	$-\frac{\left(\frac{a^4}{r}\right)^{1/5}+2 r^{3/5}-3 (a^2 r)^{1/5}}{25 r}$
{t, z, t, z}	$\frac{\left(\frac{a^4}{r}\right)^{1/5}+2 r^{3/5}-3 (a^2 r)^{1/5}}{25 r}$
{t, r, t, r}	$\frac{11 a^{6/5} r^{3/5}-15 a^{4/5} r+9 a^{2/5} r^{7/5}-2 r^{9/5}-3 (a^8 r)^{1/5}}{25 \left(a^{2/5}-r^{2/5}\right)^2 r^{7/5}}$
{r, t, t, r}	$\frac{-11 a^{6/5} r^{3/5}+15 a^{4/5} r-9 a^{2/5} r^{7/5}+2 r^{9/5}+3 (a^8 r)^{1/5}}{25 \left(a^{2/5}-r^{2/5}\right)^2 r^{7/5}}$
{t, r, r, t}	$\frac{3 a^{6/5} r^{2/5}+7 a^{2/5} r^{6/5}-2 \left(r^{8/5}+4 (a r)^{4/5}\right)}{25 \left(-a^{2/5}+r^{2/5}\right)^3 r^{12/5}}$

In[66]:= **display[RicciTensor]**

Out[66]=

{z, z}	$-\frac{2 a^{2/5}}{25 r^{6/5}}$
{r, r}	$\frac{6 a^{2/5}-4 r^{2/5}}{25 a^{2/5} r^2-25 r^{12/5}}$
{t, t}	$\frac{2 \left(3 a^{6/5} r^{3/5}-3 a^{4/5} r+a^{2/5} r^{7/5}-(a^8 r)^{1/5}\right)}{25 (a^{2/5}-r^{2/5})^2 r^{7/5}}$

In[67]:= **RicciScalar**

Out[67]=

$$\frac{2 \left(5 a^{8/5} r^{4/5}+21 a^{4/5} r^{8/5}-11 a^{2/5} r^2+2 r^{12/5}-17 (a r)^{6/5}\right)}{25 \left(-a^{2/5} r+r^{7/5}\right)^2 \left(r^{4/5}-(a r)^{2/5}\right)}$$

In[68]:= **FullSimplify[%]**

Out[68]=

$$\frac{2 \left(5 a^{8/5} r^{4/5}+21 a^{4/5} r^{8/5}-11 a^{2/5} r^2+2 r^{12/5}-17 (a r)^{6/5}\right)}{25 \left(-a^{2/5} r+r^{7/5}\right)^2 \left(r^{4/5}-(a r)^{2/5}\right)}$$

In[69]:= `norm [lower[Riemann , {4}]]`

Out[69]=

$$\frac{2}{625}\left(\frac{2\left(5\,a^{8/5}-16\,a^{2/5}\,r^{6/5}+4\,r^{8/5}+25\,(a\,r)^{4/5}-18\,a\left(a\,r^2\right)^{1/5}\right)}{r^{16/5}\left(a^{4/5}+r^{4/5}-2\,(a\,r)^{2/5}\right)}+\right.$$

$$\frac{1}{\left(a^{2/5}-r^{2/5}\right)^6 r^4}\left(r^{4/5}-(a\,r)^{2/5}\right)\left(\left(3\,a^{2/5}-2\,r^{2/5}\right)^2\left(a^{2/5}-r^{2/5}\right)^4\left(r^{4/5}-(a\,r)^{2/5}\right)+\right.$$

$$\left.\frac{r^{2/5}\left(-11\,a^{6/5}\,r^{3/5}+15\,a^{4/5}\,r-9\,a^{2/5}\,r^{7/5}+2\,r^{9/5}+3\left(a^8\,r\right)^{1/5}\right)^2}{r^{4/5}-(a\,r)^{2/5}}\right)+$$

$$\left.\frac{\frac{\left(a^{4/5}+2\,r^{4/5}-3\,(a\,r)^{2/5}\right)^2}{r^{12/5}\left(-r^{4/5}+(a\,r)^{2/5}\right)}-\frac{\left(r^{4/5}-(a\,r)^{2/5}\right)\left(3\,a^{6/5}\,r^{2/5}+7\,a^{2/5}\,r^{6/5}-2\left(r^{8/5}+4\,(a\,r)^{4/5}\right)\right)^2}{\left(-a^{2/5}\,r+r^{7/5}\right)^4}}{-r^{4/5}+(a\,r)^{2/5}}\right)$$

In[70]:= `FullSimplify [%]`

Out[70]=

$$\frac{4\left(a^{2/5}-r^{2/5}\right)^4\left(19\,a^{4/5}+12\,r^{4/5}-28\,(a\,r)^{2/5}\right)}{625\,r^{12/5}\left(a^{4/5}+r^{4/5}-2\,(a\,r)^{2/5}\right)\left(r^{4/5}-(a\,r)^{2/5}\right)^2}$$

In[71]:= `norm [RicciTensor]`

Out[71]=

$$\frac{4\,a^{4/5}}{625\,r^{16/5}}+\frac{\left(6\,a^{2/5}-4\,r^{2/5}\right)^2\left(r^{4/5}-(a\,r)^{2/5}\right)^2}{\left(25\,a^{2/5}\,r^2-25\,r^{12/5}\right)^2}+\frac{4\left(-3\,a^{6/5}\,r^{3/5}+3\,a^{4/5}\,r-a^{2/5}\,r^{7/5}+\left(a^8\,r\right)^{1/5}\right)^2}{625\left(a^{2/5}-r^{2/5}\right)^4 r^{14/5}\left(r^{4/5}-(a\,r)^{2/5}\right)^2}$$

In[72]:= `FullSimplify [%]`

Out[72]=

$$\frac{4\left(11\,a^{4/5}+4\left(r^{4/5}-3\,(a\,r)^{2/5}\right)\right)}{625\,r^{16/5}}$$

4.4 Appendix D

This is the mathematica file use to evaluate christoffel symbols, Ricci and Riemann Tensor and Ricci Scalar, and further calculated curvature invariants for the second case where the value of $y(r) = r^{\frac{4}{5}}$.

In[40]:= `ClearAll["Global`*"]`

ClearAll::wrsym : Symbol antisymmetricQ is Protected. ≫

ClearAll::wrsym : Symbol antisymmetrize is Protected. ≫

ClearAll::wrsym : Symbol Christoffel is Protected. ≫

General::stop : Further output of ClearAll::wrsym will be suppressed during this calculation. ≫

```
In[41]:= $Assumptions = And[r ∈ Reals, a ∈ Reals, y ∈ Reals, t ∈ Reals, z ∈ Reals, r > 0, a > 0];

metricsign = -1;
```

```
In[43]:= coord = {t, r, z};
fr = r^(1/5) (r^(7/5) - a^(7/5)); yr = r^(4/5)
metric = DiagonalMatrix[{-fr, 1/fr, yr^2}];
```

Out[44]= $r^{4/5}$

```
In[46]:= << diffgeo.m
```

```
In[47]:= display[Christoffel]
```

Out[47]=

{z, r, z} {z, z, r}	$\frac{4}{5\,r}$
{t, t, r} {t, r, t}	$\frac{a^{7/5}-8\,r^{7/5}}{10\,a^{7/5}\,r-10\,r^{12/5}}$
{r, r, r}	$\frac{(a^{7/5}-8\,r^{7/5})\,\left(r^{8/5}-(a^{7}\,r)^{1/5}\right)}{10\,r^{6/5}\,(a^{7/5}-r^{7/5})^{2}}$
{r, t, t}	$\frac{-9\,a^{7/5}\,r^{8/5}+8\,r^{3}+(a^{14}\,r)^{1/5}}{10\,r^{4/5}}$
{r, z, z}	$\frac{4}{5}\left(-r^{11/5}+\left(a^{7}\,r^{4}\right)^{1/5}\right)$

In[48]:= **display[Riemann]**

Out[48]=

{z, r, z, r}	$-\frac{2\left(a^{7/5}+6\,r^{7/5}\right)}{25\,r^{1/5}}$
{r, z, z, r}	$\frac{2\left(a^{7/5}+6\,r^{7/5}\right)}{25\,r^{1/5}}$
{t, r, t, r}	$\frac{2\left(-a^{7/5}+r^{7/5}\right)\left(a^{7/5}+6\,r^{7/5}\right)}{25\,r^{8/5}}$
{z, t, z, t}	$\frac{2\left(a^{14/5}+8\,r^{14/5}-9\,(a\,r)^{7/5}\right)}{25\,r^{1/5}\left(a^{7/5}-r^{7/5}\right)}$
{t, z, z, t}	$\frac{2\left(a^{14/5}+8\,r^{14/5}-9\,(a\,r)^{7/5}\right)}{25\left(r^{8/5}-(a^{7}\,r)^{1/5}\right)}$
{z, r, r, z}	$\frac{2\left(-5\,a^{7/5}\,r^{8/5}+6\,r^{3}-(a^{14}\,r)^{1/5}\right)}{25\,r^{11/5}\left(a^{7/5}-r^{7/5}\right)^{2}}$
{r, z, r, z}	$\frac{2\left(5\,a^{7/5}\,r^{8/5}-6\,r^{3}+(a^{14}\,r)^{1/5}\right)}{25\,r^{11/5}\left(a^{7/5}-r^{7/5}\right)^{2}}$
{z, t, t, z}	$-\frac{2\left(-9\,a^{7/5}\,r^{8/5}+8\,r^{3}+(a^{14}\,r)^{1/5}\right)}{25\,r^{9/5}}$
{t, z, t, z}	$\frac{2\left(-9\,a^{7/5}\,r^{8/5}+8\,r^{3}+(a^{14}\,r)^{1/5}\right)}{25\,r^{9/5}}$
{r, t, r, t}	$-\frac{2\left(4\,a^{14/5}\,r^{8/5}-11\,a^{7/5}\,r^{3}+6\,r^{22/5}+(a^{21}\,r)^{1/5}\right)}{25\,r^{11/5}\left(-a^{7/5}+r^{7/5}\right)^{3}}$
{t, r, r, t}	$\frac{2\left(4\,a^{14/5}\,r^{8/5}-11\,a^{7/5}\,r^{3}+6\,r^{22/5}+(a^{21}\,r)^{1/5}\right)}{25\,r^{11/5}\left(-a^{7/5}+r^{7/5}\right)^{3}}$
{r, t, t, r}	$\frac{2\left(3\,a^{21/5}\,r^{9/5}-15\,a^{14/5}\,r^{16/5}+17\,a^{7/5}\,r^{23/5}-6\,r^{6}+(a^{28}\,r^{2})^{1/5}\right)}{25\,r^{2}\left(a^{7/5}-r^{7/5}\right)^{2}}$

In[49]:= **display[RicciTensor]**

Out[49]=

{z, z}	$-\frac{28\,r^{6/5}}{25}$
{t, t}	$\frac{28\,(-a^{7/5}+r^{7/5})}{25\,r^{1/5}}$
{r, r}	$\frac{4\left(3\,a^{21/5}\,r^{9/5}-15\,a^{14/5}\,r^{16/5}+17\,a^{7/5}\,r^{23/5}-6\,r^{6}+(a^{28}\,r^{2})^{1/5}\right)}{25\,r^{12/5}\,(a^{7/5}-r^{7/5})^{4}}$

In[50]:= **RicciScalar**

Out[50]= $-\frac{4\left(a^{7/5}+20\,r^{7/5}\right)}{25\,r^{9/5}}$

In[51]:= **FullSimplify[%]**

Out[51]= $-\frac{4\left(a^{7/5}+20\,r^{7/5}\right)}{25\,r^{9/5}}$

In[52]:= `norm [lower[Riemann , {4}]]`

Out[52]=

$$\frac{1}{625\, r^{22/5}} 8 \left(\frac{2\, r^{3/5} \left(a^{14/5} + 36\, r^{14/5} + 12\, (a\, r)^{7/5}\right) \left(r^{8/5} - (a^7\, r)^{1/5}\right)}{-a^{7/5} + r^{7/5}} + \frac{\frac{\left(r^{8/5} - (a^7\, r)^{1/5}\right)^3 \left(5\, a^{7/5}\, r^{8/5} - 6\, r^3 + (a^{14}\, r)^{1/5}\right)^2}{\left(a^{7/5} - r^{7/5}\right)^4} + \frac{r^{4/5} \left(-a^{7/5} + r^{7/5}\right) \left(-9\, a^{7/5}\, r^{8/5} + 8\, r^3 + (a^{14}\, r)^{1/5}\right)^2}{-2\, a^{7/5}\, r^{8/5} + r^3 + (a^{14}\, r)^{1/5}}}{r^{8/5} - (a^7\, r)^{1/5}} + \frac{r^{2/5} \left(-\frac{r^{3/5} \left(a^{14/5} + 8\, r^{14/5} - 9\, (a\, r)^{7/5}\right)^2}{-a^{7/5} + r^{7/5}} + \frac{\left(-r^{8/5} + (a^7\, r)^{1/5}\right) \left(4\, a^{14/5}\, r^{8/5} - 11\, a^{7/5}\, r^3 + 6\, r^{22/5} + (a^{21}\, r)^{1/5}\right)^2}{\left(a^{7/5} - r^{7/5}\right)^4}\right)}{-r^{8/5} + (a^7\, r)^{1/5}} \right)$$

In[53]:= `FullSimplify [%]`

Out[53]=

$$\frac{16 \left(-a^{7/5} + r^{7/5}\right)^3 \left(3\, a^{14/5} + 136\, r^{14/5} + 8\, (a\, r)^{7/5}\right)}{625\, r^{16/5} \left(r^{8/5} - (a^7\, r)^{1/5}\right) \left(-2\, a^{7/5}\, r^{8/5} + r^3 + (a^{14}\, r)^{1/5}\right)}$$

In[54]:= `norm [RicciTensor]`

Out[54]=

$$\frac{16}{625} \left(\frac{98}{r^{4/5}} + \frac{\left(a^{7/5} + 6\, r^{7/5}\right)^2 \left(r^{8/5} - (a^7\, r)^{1/5}\right)^2}{r^4 \left(a^{7/5} - r^{7/5}\right)^2} \right)$$

In[55]:= `FullSimplify [%]`

Out[55]=

$$\frac{16}{625} \left(\frac{98}{r^{4/5}} + \frac{\left(a^{7/5} + 6\, r^{7/5}\right)^2 \left(r^{8/5} - (a^7\, r)^{1/5}\right)^2}{r^4 \left(a^{7/5} - r^{7/5}\right)^2} \right)$$

4.5 Appendix E

Derivation of surface gravity and entropy is given here:

Surface gravity:

$$\kappa^2 = -\frac{1}{2}\left((\nabla^\mu \chi^\nu)(\nabla_\mu \chi_\nu)\right) \tag{4.1}$$

$$\kappa^2 = -\frac{1}{2}\left((\nabla^\mu \chi^\nu)(\nabla_\mu \chi_\nu)\right) \tag{4.2}$$

$$\kappa^2 = -\frac{1}{2}\left(g^{tt} g_{rr} (\nabla_t \chi^r)^2 + g^{rr} g_{tt} (\nabla_r \chi^t)^2\right) \tag{4.3}$$

for $\rho = \mu = t, \nu = \lambda = r$

the covariant derivative can be calculated as

$$\nabla_t \chi^r = \partial_t \chi^r + \Gamma^r_{tt} \chi^t = \Gamma^r_{tt} \chi^t$$

$$\nabla_r \chi^t = \partial_r \chi^t + \Gamma^t_{rt} \chi^t = \Gamma^t_{rt} \chi^t$$

partial derivative of χ become zero and metric components are calculated as $g^{tt} g_{rr} = -\frac{1}{f(r)^2}$ and $g^{rr} g_{tt} = -f(r)^2$, the surface gravity then becomes,

$$\kappa^2 = \frac{1}{2}\left(\frac{1}{f(r)^2} (\Gamma^r_{tt})^2 + f(r)^2 (\Gamma^t_{rt})^2\right) \tag{4.4}$$

christoffel symbols can be calculated as

$$\Gamma^r_{tt} = \frac{g^{rr}}{2}\left(\frac{\partial g_{tr}}{\partial x^t} + (\frac{\partial g_{tr}}{\partial x^t} - (\frac{\partial g_{tt}}{\partial x^r}\right) = \frac{g^{rr}}{2}\left(\frac{-\partial g_{tt}}{\partial r}\right)$$

$$\Gamma^r_{rt} = \frac{g^{tt}}{2}\left(\frac{\partial g_{rt}}{\partial x^t} + (\frac{\partial g_{tt}}{\partial x^r} - (\frac{\partial g_{rt}}{\partial x^t}\right) = \frac{g^{tt}}{2}\left(\frac{-\partial g_{tt}}{\partial r}\right)$$

on putting the value of christoffel symbol in Eq. 4.4, we get

$$\kappa^2 = \frac{1}{2}\left(\frac{1}{f(r)^2}\left(\frac{g^{rr}}{2}\left(\frac{-\partial g_{tt}}{\partial r}\right)\right)^2 + f(r)^2 \left(\frac{g^{tt}}{2}\left(\frac{-\partial g_{tt}}{\partial r}\right)\right)^2\right) \tag{4.5}$$

on taking components of metric and inverse metric, we get

$$\kappa^2 = \frac{1}{4}\left(\frac{-\partial f}{\partial r}\right)^2 \tag{4.6}$$

$$\kappa = \frac{1}{2}\left(\frac{\partial f}{\partial r}\right) \tag{4.7}$$

Thus, the value of surface gravity in our case found same as Schwarszchild metric.

Entropy

Since the metric in our case is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + y^2(r)dz^2 \tag{4.8}$$

S be the area element of 1-Sphere, calculated as

$$S = \int_{-\pi}^{\pi} y(r)dz = 2\pi y(r)$$

and according to Bekenstein-Hawking entropy law which states that entropy is one-fourth of the event horizon area, then we can get

$$S = \frac{1}{4} \times 2\pi y(r) \tag{4.9}$$

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