A Study of Bessel Functions and Regular Coulomb Wave Functions

M.Sc. Thesis

by

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DEPARTMENT OF MATHEMATICS

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A Study of Bessel Functions and Regular Coulomb Wave Functions

A THESIS

Submitted in partial fulfillment of the requirements for the award of the degree of

Master of Science

by

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Under the guidance of

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"A Study of Bessel Functions and Regular Coulomb Wave Functions"** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF MATHEMATICS**, **INDIAN INSTITUTE OF TECHNOLOGY INDORE**, is an authentic record of my own work carried out during the time period from July 2024 to May 2025 under the supervision of **Dr. Sanjeev Singh**, Associate Professor, Department of Mathematics, IIT Indore. The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

Prince Vishwakarma
28/05/2025

Signature of the Student with Date

(**Prince Vishwakarma**)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

S. Singh 28 May 2025

Signature of Thesis Supervisor with date

(**Dr. Sanjeev Singh**)

PrinceVishwakarma has successfully given his M.Sc. Oral Examination held on 28th May, 2025.

S. Singh

Signature of Supervisor of M.Sc. Thesis

Date: 28 May 2025

V. K. Tiwari

Signature of Convener, DPGC

Date: 28 May 2025

Dedicated to my

Family

“ Those times when you get up early and you work hard, those times when you stay up late and you work hard, those times when you don’t feel like working, you’re too tired, you don’t want to push yourself, but you do it anyway. That is actually the dream. That’s the dream. It’s not the destination, it’s the journey. And if you guys can understand that, then what you’ll see happen is you won’t accomplish your dreams, your dreams won’t come true; something greater will.”

– ***Kobe Bryant***

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Abstract

This thesis presents a comprehensive study of the Bessel function and the regular Coulomb wave function, with a focus on their analytic properties, the distribution of their zeros, and the interrelations explored in the literature. The work begins with a detailed derivation of Bessel's differential equation and systematically explores its solutions, $J_\nu(z)$ and $Y_\nu(z)$, including their series representations, recurrence relations, and linear independence. Rigorous proofs are provided for the reality, simplicity, and interlacing properties of the zeros of Bessel functions and their derivatives, employing advanced tools such as the Weierstrass factorization theorem, Mittag-Leffler's theorem, and Laguerre's separation theorem. The generating function for Bessel functions is derived, and its implications for solution structures are discussed.

The study extends to the regular Coulomb wave function $F_L(\eta, \rho)$, defining it via confluent hypergeometric functions and demonstrating its role as a one-parameter generalization of the Bessel function. The thesis investigates the reality and distribution of zeros of the Coulomb wave function, leveraging determinantal criteria (Grommer and Chebotarev theorems) and moment Hankel matrices. Special attention is given to recursive computations of sums over zeros and their connection to Rayleigh functions. The results yield explicit criteria for the number and nature (real or complex) of zeros.

Key contributions include:

- **Analytic Properties:** Derivation of recurrence relations, Wronskians, and infinite product representations for Bessel functions.
- **Zero Distributions:** Proofs of the reality, simplicity, and interlacing of zeros for Bessel functions and their derivatives, with extensions to Coulomb wave functions.
- **Generalizations:** Demonstration of the Coulomb wave function as a generalization of the Bessel function, with analogous results for its zeros.
- **Inequalities and Expansions:** Establishment of Turán-type inequalities and Mittag-Leffler expansions for Coulomb wave functions, providing deeper insights into their behavior.

The findings are supported by rigorous mathematical analysis and references to foundational works in the field, including Watson's treatise on Bessel functions [8], Abramowitz and Stegun's handbook [11], and recent research by Baricz and Stampach[[7], [1] and [2]]. This work not only consolidates classical results but also advances the understanding of these special functions, offering new perspectives and tools for future research in mathematical physics and applied mathematics.

Contents

CHAPTER 1

Introduction

Introduction

Bessel functions and Coulomb wave functions are among the most fundamental and widely studied special functions in mathematical physics and applied mathematics. These functions arise naturally in the solution of partial differential equations in cylindrical and spherical coordinates, making them indispensable tools in a variety of scientific and engineering disciplines, including quantum mechanics, electromagnetism, heat conduction, and wave propagation.

The Bessel function, named after the German mathematician Friedrich Wilhelm Bessel, is a solution to Bessel's differential equation, which appears when separating variables in the Laplace, Helmholtz, and Schrödinger equations in cylindrical or spherical symmetry. The theory of Bessel functions is rich and well-developed, with deep connections to complex analysis, number theory, and orthogonal polynomials. In particular, the distribution of zeros of Bessel functions and their derivatives has been a subject of extensive investigation, leading to classical results such as the reality, simplicity, and interlacing properties of these zeros, as well as to the development of powerful techniques such as the Weierstrass factorization theorem and Laguerre's separation theorem.

The regular Coulomb wave function, which generalizes the Bessel function of the first kind and appears in the theory of quantum scattering, atomic physics, and in the solution of the Schrödinger equation for the Coulomb potential. The regular Coulomb wave function, like the Bessel function, is an entire function of its argument and can be expressed in terms of confluent hypergeometric functions. Recent research has uncovered new properties of these functions, including their zero distributions, recurrence relations, and the development of Hurwitz-type and Turán-type inequalities, which extend and unify classical results known for Bessel functions.

This thesis is devoted to a comprehensive study of the analytic properties, zero distributions, and interrelations of Bessel functions and regular Coulomb wave functions. We begin with a detailed derivation of Bessel's differential equation and systematically explore the solutions $J_\nu(z)$ and $Y_\nu(z)$, including their series representations, recurrence relations, and theorems on linear independence. Rigorous proofs are given for the reality, simplicity, and interlacing properties of the zeros of Bessel functions and their derivatives, utilizing advanced tools such as the Weierstrass factorization theorem, Mittag-Leffler's theorem, and Laguerre's separation theorem.

The study is then extended to the regular Coulomb wave function $F_L(\eta, \rho)$. We investigate the reality and distribution of zeros of the Coulomb wave function, leveraging determinantal criteria from Grommer and Chebotarev and the analysis of moment Hankel matrices. Special attention is given to the recursive computation of sums over zeros and the connection to Rayleigh functions, yielding explicit criteria for the number and nature (real or complex) of zeros and unifying classical theorems such as Hurwitz's theorem for Bessel functions.

CHAPTER 2

Bessel functions

2.1 Bessel's Equation and Bessel Function of the first kind

A second-order linear differential equation of the form

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2) = 0 \tag{2.1}$$

is called Bessel's equation of order ν and $z \in \mathbb{C}$. The differential equation has a regular singularity at $z = 0$ and irrregular singularity at $z = \infty$.

Using method of Fröbenius to solve this differential equation, we get

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{k!\,\Gamma(\nu + k + 1)} \tag{2.2}$$

and

$$J_{-\nu}(z) = \left(\frac{z}{2}\right)^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{k!\,\Gamma(-\nu + k + 1)}, \tag{2.3}$$

which are the independent solutions of the above differential equation for $\nu \notin \mathbb{Z}$. The solution $J_\nu(z)$ is analytic in $z \in \mathbb{C}$ except for a branch point at $z = 0$ when $\nu \notin \mathbb{Z}$. The principal branch of $J_\nu(z)$, defined via the principal value of $\left(\frac{1}{2}z\right)^\nu$, is analytic in the z-plane cut along

$(-\infty, 0]$. For $\nu \in \mathbb{Z}$, $J_\nu(z)$ is entire in z. Moreover, for fixed $z \neq 0$, each branch of $J_\nu(z)$ is entire in ν.

Theorem 2.1. *For $\nu = n \in \mathbb{Z}$*

$$J_{-n}(z) = (-1)^n J_n(z).$$

Proof. For $\nu = n \in \mathbb{Z}$. **Case(I)** If $n > 0$, From equation (2.3),

$$J_{-n}(z) = \left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{k!\,\Gamma(-n+k+1)}.$$

Since, $\frac{1}{\Gamma(-n+k+1)} = 0$ for $k = 0, 1, 2, ...n-1$. we have

$$\begin{aligned} J_{-n}(z) &= \left(\frac{z}{2}\right)^{-n} \sum_{k=n}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{k!\,\Gamma(-n+k+1)} \\ &= \left(\frac{z}{2}\right)^{-n} \sum_{m=0}^{\infty} \frac{(-1)^{m+n} \left(\frac{z}{2}\right)^{2(m+n)}}{(m+n)!\,\Gamma(m+1)} \\ &= (-1)^n \left(\frac{z}{2}\right)^{n} \sum_{m=0}^{\infty} \frac{(-1)^{m} \left(\frac{z}{2}\right)^{2m}}{(m+n)!\,\Gamma(m+1)}. \end{aligned}$$

Since, $(m+n)!\,\Gamma(m+1) = m!\,\Gamma(m+n+1)$, we get

$$\begin{aligned} J_{-n}(z) &= (-1)^n \left(\frac{z}{2}\right)^{n} \sum_{m=0}^{\infty} \frac{(-1)^{m} \left(\frac{z}{2}\right)^{2m}}{m!\,\Gamma(m+n+1)} \\ &= (-1)^n\, J_n(z). \end{aligned}$$

Case(II) If $n < 0$, then let $n = -p$, where $p > 0$

. From case(I), we obtain

$$J_p(z) = (-1)^p J_{-p}(z).$$

which implies

$$J_{-n}(z) = (-1)^n J_n(z).$$

Thus, for all $n \in \mathbb{Z}$

$$J_{-n}(z) = (-1)^n J_n(z).$$

□

Theorem 2.2. *Let $\nu \notin \mathbb{Z}$. Then the Wronskian of the Bessel functions $J_\nu(x)$ and $J_{-\nu}(x)$ is given by*

$$W[J_\nu(x), J_{-\nu}(x)] = J_\nu(x)J'_{-\nu}(x) - J'_\nu(x)J_{-\nu}(x) = -\frac{2\sin(\nu\pi)}{\pi x}.$$

Proof: Let us consider the Bessel differential equation:

$$z^2y''(z) + zy'(z) + (z^2 - \nu^2)y(z) = 0.$$

This can be rewritten as:

$$\frac{d}{dz}\left(zy'(z)\right) + \left(z - \frac{\nu^2}{z}\right)y(z) = 0.$$

Applying this to $J_\nu(z)$ and $J_{-\nu}(z)$, we get two identities:

$$\frac{d}{dz}\left(zJ'_\nu(z)\right) + \left(z - \frac{\nu^2}{z}\right)J_\nu(z) = 0 \tag{2.4}$$

$$\frac{d}{dz}\left(zJ'_{-\nu}(z)\right) + \left(z - \frac{\nu^2}{z}\right)J_{-\nu}(z) = 0 \tag{2.5}$$

Now multiply equation (2.3) by $J_{-\nu}(z)$ and equation (2.3) by $J_\nu(z)$, and subtract,we get

$$J_{-\nu}(z)\frac{d}{dz}\left(zJ'_\nu(z)\right) - J_\nu(z)\frac{d}{dz}\left(zJ'_{-\nu}(z)\right) = 0,$$
$$\frac{d}{dz}\left(zJ'_\nu(z)J_{-\nu}(z) - zJ'_{-\nu}(z)J_\nu(z)\right) = 0.$$

On Integrating,

$$z\left(J'_\nu(z)J_{-\nu}(z) - J'_{-\nu}(z)J_\nu(z)\right) = C,$$

for some constant C. To find C, take the limit as $z \to 0$,

$$\lim_{z\to 0} J_{-\nu}(z)J'_\nu(z) = \frac{\nu}{\Gamma(1-\nu)\Gamma(\nu+1)}\frac{1}{z}$$
$$\lim_{z\to 0} J_\nu(z)J'_{-\nu}(z) = \frac{-\nu}{\Gamma(1-\nu)\Gamma(\nu+1)}\frac{1}{z}$$

we get,

$$C = \frac{2\nu}{\Gamma(\nu+1)\Gamma(1-\nu)}$$

Using the identity:

$$\Gamma(\nu+1)\Gamma(1-\nu) = \frac{\pi\nu}{\sin(\pi\nu)},$$

we get,

$$z\left(J'_\nu(z)J_{-\nu}(z) - J'_{-\nu}(z)J_\nu(z)\right) = -\frac{2\sin(\pi\nu)}{\pi}$$

Hence, the Wronskian is:

$$\mathcal{W}(J_\nu(z), J_{-\nu}(z)) = -\frac{2\sin(\pi\nu)}{\pi z}.$$

Theorem 2.3. *The following recurrence relations hold for the Bessel function $J_\nu(z)$:*

$$\begin{aligned}
&\text{(i)} && \frac{d}{dz}\left\{z^\nu J_\nu(z)\right\} = z^\nu J_{\nu-1}(z),\\
&\text{(ii)} && \frac{d}{dz}\left\{z^{-\nu} J_\nu(z)\right\} = -z^{-\nu} J_{\nu+1}(z),\\
&\text{(iii)} && J'_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z} J_\nu(z),\\
&\text{(iv)} && J'_\nu(z) = \frac{\nu}{z} J_\nu(z) - J_{\nu+1}(z),\\
&\text{(v)} && J'_\nu(z) = \frac{1}{2}\left\{J_{\nu-1}(z) - J_{\nu+1}(z)\right\},\\
&\text{(vi)} && J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z).
\end{aligned}$$

Proof. **(i)** From the series definition of the Bessel function,

$$J_\nu(z) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(\nu+r+1)} \left(\frac{z}{2}\right)^{2r+\nu},$$

we multiply both sides by z^ν to get

$$z^\nu J_\nu(z) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(\nu+r+1)} \frac{z^{2r+2\nu}}{2^{2r+\nu}}.$$

Differentiating term-by-term:

$$\frac{d}{dz}\left(z^\nu J_\nu(z)\right) = \sum_{r=0}^{\infty} \frac{(-1)^r(2r+2\nu)}{r!\,\Gamma(\nu+r+1)} \left(\frac{z}{2}\right)^{2r+2\nu-1}.$$

This expression matches the series expansion of $z^\nu J_{\nu-1}(z)$, hence

$$\frac{d}{dz}\left\{z^\nu J_\nu(z)\right\} = z^\nu J_{\nu-1}(z).$$

(ii) We have $J_\nu(z)$,

$$J_\nu(z) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(\nu+r+1)} \left(\frac{z}{2}\right)^{2r+\nu},$$

multiplying both sides by $z^{-\nu}$, we get:

$$z^{-\nu} J_\nu(z) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(\nu+r+1)} \frac{z^{2r}}{2^{2r+\nu}}.$$

Differentiating term-by-term:

$$\frac{d}{dz}\left(z^{-\nu} J_\nu(z)\right) = \sum_{r=1}^{\infty} \frac{(-1)^r}{r!\,\Gamma(\nu+r+1)} \cdot \frac{2r z^{2r-1}}{2^{2r+\nu}}.$$

Change index by setting $s = r-1$, then

$$= \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{(s+1)!\,\Gamma(\nu+s+2)} \cdot \frac{2(s+1) z^{2s+1}}{2^{2s+2+\nu}}.$$

Simplifying,

$$= -\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(\nu+s+2)} \left(\frac{z}{2}\right)^{2s+\nu+2},$$

which is equal to $-z^{-\nu}J_{\nu+1}(z)$. Hence,

$$\frac{d}{dz}\left\{z^{-\nu}J_\nu(z)\right\} = -z^{-\nu}J_{\nu+1}(z).$$

(iii) Using the identity

$$\frac{d}{dz}\{z^\nu J_\nu(z)\} = z^\nu J_{\nu-1}(z),$$

we apply the product rule:

$$\frac{d}{dz}\left(z^\nu J_\nu(z)\right) = \nu z^{\nu-1}J_\nu(z) + z^\nu J_\nu'(z).$$

Therefore,

$$\nu z^{\nu-1}J_\nu(z) + z^\nu J_\nu'(z) = z^\nu J_{\nu-1}(z).$$

Dividing both sides by z^ν, we obtain:

$$J_\nu'(z) = J_{\nu-1}(z) - \frac{\nu}{z}J_\nu(z).$$

(iv) Using the identity

$$\frac{d}{dz}\left\{z^{-\nu}J_\nu(z)\right\} = -z^{-\nu}J_{\nu+1}(z),$$

we differentiate using the product rule:

$$\frac{d}{dz}\left(z^{-\nu}J_\nu(z)\right) = -\nu z^{-\nu-1}J_\nu(z) + z^{-\nu}J_\nu'(z).$$

Therefore,

$$-\nu z^{-\nu-1}J_\nu(z) + z^{-\nu}J_\nu'(z) = -z^{-\nu}J_{\nu+1}(z).$$

Multiplying both sides by z^ν, we obtain:

$$-\nu z^{-1}J_\nu(z) + J_\nu'(z) = -J_{\nu+1}(z),$$

which simplifies to:

$$J_\nu'(z) = -J_{\nu+1}(z) + \frac{\nu}{z}J_\nu(z).$$

(v) We add them to eliminate $J_\nu'(z)$:

$$J_\nu'(z) + J_\nu'(z) = J_{\nu-1}(z) - \frac{\nu}{z}J_\nu(z) - J_{\nu+1}(z) + \frac{\nu}{z}J_\nu(z).$$

Simplifying,

$$2J_\nu'(z) = J_{\nu-1}(z) - J_{\nu+1}(z),$$

which implies

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z}J_\nu(z).$$

(vi) Starting from the two derivative identities:

$$J'_\nu(z) = \tfrac{1}{2}\big[J_{\nu-1}(z) - J_{\nu+1}(z)\big], \quad J'_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z}J_\nu(z),$$

we equate them to eliminate $J'_\nu(z)$:

$$\tfrac{1}{2}\big[J_{\nu-1}(z) - J_{\nu+1}(z)\big] = J_{\nu-1}(z) - \frac{\nu}{z}J_\nu(z).$$

Multiply both sides by 2:

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J_{\nu-1}(z) - \frac{2\nu}{z}J_\nu(z).$$

Rearranging,

$$J_{\nu+1}(z) = J_{\nu-1}(z) - 2J_{\nu-1}(z) + \frac{2\nu}{z}J_\nu(z) = -J_{\nu-1}(z) + \frac{2\nu}{z}J_\nu(z).$$

Hence

$$J_{\nu-1}(z) + J_{\nu+1}(z) = J_{\nu-1}(z) - J_{\nu-1}(z) + \frac{2\nu}{z}J_\nu(z) = \frac{2\nu}{z}J_\nu(z),$$

as required. □

2.2 Bessel Function of the second kind

Now, for the second linearly independent solution of the Bessel differential equation, we introduce, for $\nu \notin \mathbb{Z}$

$$Y_\nu(z) = \frac{cos(\nu z)J_\nu(z) - J_{-\nu}(z)}{sin(\nu z)} \tag{2.6}$$

and for $\nu = n \in \mathbb{Z}$

$$Y_n(z) = \lim_{\nu \to n} Y_\nu(z) \tag{2.7}$$

$Y_\nu(z)$ has a branch point at $z = 0$. The principal branch corresponds to the principal branches of $J_{\pm\nu}(z)$. $Y_\nu(z)$ is called the Bessel Function of the second kind.

Theorem 2.4. *$J_\nu(z)$ and $Y_\nu(z)$ are the two linearly independent solutions of the differential equation* (2.1) *for all values of ν.*

Proof. **Case(I)** When $\nu \notin \mathbb{Z}$, Here $sin(\nu\pi) \neq 0$, so that $Y_\nu(z)$ is just a linear combination of $J_\nu(z)$ and $J_{-\nu}(z)$. From the previous theorem we know $J_\nu(z)$ and $J_{-\nu}(z)$ are the linearly independent solutions for $\nu \notin \mathbb{Z}$, so that $J_\nu(z)$ and linear combination of $J_\nu(z)$ and $J_{-\nu}(z)$ must be linearly independent solutions of Bessel's equation.

Case(II) When $\nu \in \mathbb{Z}$,

$$\begin{aligned} Y_n(x) &= \lim_{\nu \to n} Y_\nu(x) \\ &= \lim_{\nu \to n} \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi} \\ &= \frac{[(\partial/\partial\nu)(\cos \nu\pi J_\nu(x) - J_{-\nu}(x))]_{\nu=n}}{[(\partial/\partial\nu)\sin \nu\pi]_{\nu=n}} \end{aligned}$$

(by L'Hôpital's rule)

$$\begin{aligned} &= \frac{-\pi \sin \nu\pi J_\nu(x) + \cos \nu\pi (\partial/\partial\nu) J_\nu(x) - (\partial/\partial\nu) J_{-\nu}(x)\big|_{\nu=n}}{[\pi \cos \nu\pi]_{\nu=n}} \\ &= \frac{\cos n\pi[(\partial/\partial\nu) J_\nu(x)]_{\nu=n} - [(\partial/\partial\nu) J_{-\nu}(x)]_{\nu=n}}{\pi \cos n\pi} \\ &= \frac{1}{\pi}\left[(\partial/\partial\nu) J_\nu(x) - (-1)^n (\partial/\partial\nu) J_{-\nu}(x)\right]_{\nu=n} \end{aligned} \tag{2.8}$$

We must now prove two things; firstly that $Y_n(x)$ as defined by equation (2.8) is in fact a solution of Bessel's equation and, secondly, that it is a solution independent from $J_n(x)$. To accomplish the first of these we note that $J_\nu(x)$ obeys Bessel's equation of order ν:

$$x^2 \frac{d^2 J_\nu}{dx^2} + x \frac{dJ_\nu}{dx} + (x^2 - \nu^2) J_\nu = 0$$

differentiating with respect to ν, we get

$$x^2 \frac{d^2}{dx^2}\frac{\partial J_\nu}{\partial\nu} + x \frac{d}{dx}\frac{\partial J_\nu}{\partial\nu} + (x^2 - \nu^2)\frac{\partial J_\nu}{\partial\nu} - 2\nu J_\nu = 0 \tag{2.9}$$

Also, of course, $J_{-\nu}(x)$ satisfies Bessel's equation of order ν, so that we have in exactly the same way

$$x^2 \frac{d^2}{dx^2}\frac{\partial J_{-\nu}}{\partial\nu} + x \frac{d}{dx}\frac{\partial J_{-\nu}}{\partial\nu} + (x^2 - \nu^2)\frac{\partial J_{-\nu}}{\partial\nu} - 2\nu J_{-\nu} = 0 \tag{2.10}$$

Multiplying equation (2.10) by $(-1)^\nu$ and subtracting from equation (2.9) gives

$$\begin{aligned} x^2 \frac{d^2}{dx^2}\left\{\frac{\partial J_\nu}{\partial\nu} - (-1)^\nu \frac{\partial J_{-\nu}}{\partial\nu}\right\} + x \frac{d}{dx}\left\{\frac{\partial J_\nu}{\partial\nu} - (-1)^\nu \frac{\partial J_{-\nu}}{\partial\nu}\right\} \\ + (x^2 - \nu^2)\left\{\frac{\partial J_\nu}{\partial\nu} - (-1)^\nu \frac{\partial J_{-\nu}}{\partial\nu}\right\} - 2\nu\left\{J_\nu - (-1)^\nu J_{-\nu}\right\} = 0. \end{aligned}$$

for $\nu = n \in \mathbb{Z}$,

$$x^2 \frac{d^2}{dx^2} Y_n(x) + x \frac{d}{dx} Y_n(x) + (x^2 - n^2) Y_n(x) - \frac{2n}{\pi}\left(J_n(x) - (-1)^n J_{-n}(x)\right) = 0.$$

Thus,

$$x^2 \frac{d^2}{dx^2} Y_n(x) + x \frac{d}{dx} Y_n(x) + (x^2 - n^2) Y_n(x) = 0$$

which just states that $Y_n(x)$ satisfies Bessel's equation of order n. □

Theorem 2.5. *When n is integral, $Y_{-n}(x) = (-1)^n Y_n(x)$.*

Proof. From equation (4.9) we have

$$\begin{aligned}
Y_{-n}(x) &= \frac{1}{\pi}\left[\frac{\partial}{\partial \nu}J_\nu(x) - (-1)^{-n}\frac{\partial}{\partial \nu}J_{-\nu}(x)\right]_{\nu=n} \\
&= \frac{1}{\pi}\left[\frac{\partial}{\partial(-\nu)}J_{-\nu}(x) - (-1)^{-n}\frac{\partial}{\partial \nu}J_{\nu}(x)\right]_{\nu=n} \\
&= \frac{1}{\pi}\left[-\frac{\partial}{\partial \nu}J_{-\nu}(x) + (-1)^{n}\frac{\partial}{\partial \nu}J_{\nu}(x)\right]_{\nu=n} \\
&= (-1)^n\frac{1}{\pi}\left[\frac{\partial}{\partial \nu}J_{\nu}(x) - (-1)^{n}\frac{\partial}{\partial \nu}J_{-\nu}(x)\right]_{\nu=n} \\
&= (-1)^n Y_n(x).
\end{aligned}$$

□

Theorem 2.6. *For $\nu \notin \mathbb{Z}$ and $z \neq 0$, the Wronskian of the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ satisfies*

$$W(J_\nu, Y_\nu)(z) = J_\nu(z)Y_\nu'(z) - J_\nu'(z)Y_\nu(z) = \frac{2}{\pi z}.$$

Proof. For non-integer ν,

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}.$$

Using the linearity of the Wronskian,

$$W(J_\nu, Y_\nu)(z) = W\left(J_\nu, \frac{J_\nu\cos(\nu\pi) - J_{-\nu}}{\sin(\nu\pi)}\right)(z)$$

$$W(J_\nu, Y_\nu)(z) = \frac{\cos(\nu\pi)}{\sin(\nu\pi)}W(J_\nu, J_\nu)(z) - \frac{1}{\sin(\nu\pi)}W(J_\nu, J_{-\nu})(z).$$

Since $W(J_\nu, J_\nu)(z) = 0$, it follows that

$$W(J_\nu, Y_\nu)(z) = -\frac{1}{\sin(\nu\pi)}W(J_\nu, J_{-\nu})(z).$$

From theorem (2.2),

$$W(J_\nu, J_{-\nu})(z) = -\frac{2\sin(\nu\pi)}{\pi z}.$$

Therefore,

$$W(J_\nu, Y_\nu)(z) = -\frac{1}{\sin(\nu\pi)}\left(-\frac{2\sin(\nu\pi)}{\pi z}\right) = \frac{2}{\pi z}.$$

By analytic continuation, this result extends to all ν.

□

2.3 Infinite Product representation of $J_\nu(z)$

Theorem 2.7 ([8],15.5). *For $\nu > -1$, the zeros of $J_\nu(z)$ are real. Let $\pm j_{\nu,1}$, $\pm j_{\nu,2}$, $\pm j_{\nu,3}$, ... are the real zeros of $J_n(z)$ such that $0 < j_{\nu,1} < j_{\nu,2} < j_{\nu,3}$.... Then $J_n(z)$ can be written as*

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,k}^2}\right)$$

Proof. Since $J_n(z)$ is an entire function, by the Weierstrass factorization theorem, it is possible to express $J_n(z)$ as a product of simple factors, where each factor vanishing at one of zeros of $J_n(z)$. To express $J_n(z)$ in this form, we will first express the logarithmic derivative of $z^{-\nu}J_n(z)$as a rational function by Mittag-Leffler's theorem.

Consider a (large) rectangle D, whose vertices are $\pm A \pm iB$, where $A, B > 0$. Suppose that $\pm j_{\nu,m}$ are the zeros with the maximum value of m inside the rectangle.

Let z_0 be any non-zero point inside the rectangle D, other than zeros of $J_\nu(z)$.

Let

$$f(z) = \frac{z_0}{z(z-z_0)} \frac{J_{\nu+1}(z)}{J_\nu(z)} \tag{2.11}$$

$f(z)$ is an meromorphic function, the only singularties of $f(z)$ inside D are z_0 and $\pm j_{\nu,n}$ for $n = 1, 2, 3, ...m$.

Integrating $f(z)$ around the rectangle D and applying residue theorem we get,

$$\frac{1}{2\pi i}\int_D f(z)\,dz = \text{Res}(f, z_0) + \sum_{n=1}^{m} \text{Res}(f, j_{\nu,n}) + \sum_{n=1}^{m} \text{Res}(f, -j_{\nu,n}). \tag{2.12}$$

Residue at $z = z_0$,

$$\text{Res}(f, z_0) = \lim_{z\to z_0} (z - z_0)\, f(z).$$

Using equation(2.11), we obtain

$$\text{Res}(f, z_0) = \frac{J_{\nu+1}(z_0)}{J_\nu(z_0)}. \tag{2.13}$$

Residue at $z = j_{\nu,n}$, for any fixed $n = 1, 2, 3, ...m$.

$$\text{Res}(f, j_{\nu,n}) = \lim_{z\to j_{\nu,n}} (z - j_{\nu,n})\, \frac{z_0}{z(z-z_0)} \frac{J_{\nu+1}(z)}{J_\nu(z)}.$$

Since, $j_{\nu,n}$ is zeros of $J_\nu(z)$ by Taylor expansion, we can write

$$J_\nu(z) = J'_\nu(j_{\nu,n})(z - j_{\nu,n}) + \frac{1}{2!}J''_\nu(j_{\nu,n})(z - j_{\nu,n})^2 + \frac{1}{3!}J'''_\nu(j_{\nu,n})(z - j_{\nu,n})^3 + ..$$

Rearranging,

$$\frac{z - j_{\nu,n}}{J_\nu(z)} = \frac{1}{J'_\nu(j_{\nu,n}) + \frac{1}{2!}J''_\nu(j_{\nu,n})(z - j_{\nu,n}) + \frac{1}{3!}J'''_\nu(j_{\nu,n})(z - j_{\nu,n})^2 + \cdots}$$

Taking limits on both sides, we get

$$\lim_{z\to j_{\nu,n}} \frac{z-j_{\nu,n}}{J_\nu(z)} = \frac{1}{J'_\nu(j_{\nu,n})}$$

Since, $J'_\nu(z) = \frac{\nu}{z}J_\nu(z) - J_{\nu+1}(z)$, at $z = j_{\nu,n}$, we have $J'_\nu(j_{\nu,n}) = -J_{\nu+1}(j_{\nu,n})$

Thus,

$$\lim_{z\to j_{\nu,n}} \frac{z-j_{\nu,n}}{J_\nu(z)} = -\frac{1}{J_{\nu+1}(j_{\nu,n})}.$$

Thus, we get

$$\text{Res}(f, j_{\nu,n}) = -\frac{z_0}{j_{\nu,n}(j_{\nu,n}-z_0)},$$

$$\text{Res}(f, j_{\nu,n}) = \frac{1}{z_0-j_{\nu,n}} + \frac{1}{j_{\nu,n}}. \tag{2.14}$$

Similarly,

$$\text{Res}(f, -j_{\nu,n}) = \frac{1}{z_0+j_{\nu,n}} - \frac{1}{j_{\nu,n}} \tag{2.15}$$

using equations (2.13), (2.14) and (2.15) in equation (2.12), we get

$$\frac{1}{2\pi i}\int_D f(z)\,dz = \frac{J_{\nu+1}(z_0)}{J_\nu(z_0)} + \sum_{n=1}^{m}\left(\frac{1}{z_0-j_{\nu,n}} + \frac{1}{j_{\nu,n}}\right) + \sum_{n=1}^{m}\left(\frac{1}{z_0+j_{\nu,n}} - \frac{1}{j_{\nu,n}}\right) \tag{2.16}$$

as A and B tends to infinity with suitable sequences, $f(z)$ is remain bounded [[8],15.4] and

$$\frac{1}{2\pi i}\int_D f(z)\,dz \longrightarrow 0$$

From (2.16), we get

$$\frac{J_{\nu+1}(z_0)}{J_\nu(z_0)} + \sum_{n=1}^{\infty}\left(\frac{1}{z_0-j_{\nu,n}} + \frac{1}{j_{\nu,n}}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{z_0+j_{\nu,n}} - \frac{1}{j_{\nu,n}}\right) = 0$$

$$\implies -\frac{J_{\nu+1}(z_0)}{J_\nu(z_0)} = \sum_{n=1}^{\infty}\left(\frac{2z_0}{z_0^2-j_{\nu,n}^2}\right)$$

On integrating both sides from 0 to z, we get

$$-\int_0^z \frac{J_{\nu+1}(t)}{J_\nu(t)}dt = \int_0^z \left(\sum_{n=1}^{\infty}\frac{2t}{t^2-j_{\nu,n}^2}\right)dt$$

Since $\frac{d}{dz}\left(\log(z^{-\nu}J_\nu(z))\right) = -\frac{J_{\nu+1}(zf)}{J_\nu(z)}$, we get

$$\log(z^{-\nu}J_\nu(z)) - \log\frac{1}{2^\nu\Gamma(\nu+1)} = \sum_{n=0}^{\infty}\left(\log(z^2-j_{\nu,n}^2) - \log(-j_{\nu.n}^2)\right),$$

Which implies,

$$\log\left(\left(\frac{2}{z}\right)^\nu \Gamma(\nu+1)J_\nu(z)\right) = \sum_{n=0}^{\infty}\log\left(1-\frac{z^2}{j_{\nu,n}^2}\right)$$

Taking the exponential on both sides, we get

$$\left(\frac{2}{z}\right)^{\nu}\Gamma(\nu+1)J_\nu(z)=\prod_{k=1}^{\infty}\left(1-\frac{z^2}{j_{\nu,k}^2}\right)$$

Hence,

$$J_\nu(z)=\frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu+1)}\prod_{k=1}^{\infty}\left(1-\frac{z^2}{j_{\nu,k}^2}\right),\text{for}\quad \nu>-1.$$

□

Theorem 2.8 ([18])**.** *For $n\in\{0,1,2,3...\}$,*

(i) If $\nu>n-1$, then $J_\nu^{(n)}(z)$ has infinitely many zeros, which are all real and simple, except at $z=0$.

(ii) If $\nu\geq n$, then between any two consecutive positive zeros of $J_\nu^{(n)}(z)$, there lies exactly one positive zero of $J_\nu^{(n+1)}(z)$.

(iii) For $\nu>n-1$, all the zeros of $(n-\nu)J_\nu^{(n)}(z)+xJ_\nu^{(n+1)}(z)$ are real, and they lie strictly between the successive zeros of $J_\nu^{(n)}(z)$.

Proof. **(i)** We will use **mathematical induction** to prove that the zeros are real.

- For $n=\{0,1,2,3\}$ and $\nu>n-1$, it is known that the zeros of $J_\nu^{(n)}(x)$ are all real ([12], [13]).
- Assume for some fixed $n\in\{4,5,\dots\}$ and $\nu>n-1$, the function $J_\nu^{(n)}(x)$ has only real zeros.
- We aim to show that for $\nu>n$, the function $J_\nu^{(n+1)}(x)$ also has only real zeros.

Let $j_{\nu,m}^{(n)}$ denote the m^{th} positive zero of $J_\nu^{(n)}(x)$, where $m\in\mathbb{N}$, $n\in\mathbb{N}_0$. From Skelton [[6]] we know that the Weierstrass product representation for $J_\nu^{(n)}(x)$ is given by

$$J_\nu^{(n)}(x)=\frac{x^{\nu-n}}{2^\nu\Gamma(\nu+1-n)}\prod_{m\geq1}\left(1-\frac{x^2}{\left(j_{\nu,m}^{(n)}\right)^2}\right),\tag{2.17}$$

which holds for all $\nu\geq n$, and the product converges uniformly on compact subsets of the complex plane. For $\nu>n-1$, we have

$$J_\nu^{(n)}(x)=\frac{x^{\nu-n}}{2^\nu\Gamma(\nu+1-n)}\sum_{m\geq0}\frac{(-1)^m\Gamma(\nu+2m+1)\Gamma(\nu+1-n)x^{2m}}{m!2^{2m}\Gamma(\nu+2m-n+1)\Gamma(\nu+m+1)}.$$

Define the rescaled entire function:

$$\mathbb{J}_{\nu,n}(x) = 2^{\nu}\Gamma(\nu+1-n)x^{n-\nu}J_{\nu}^{(n)}(x) = \sum_{m\geq 0}\frac{(-1)^m\Gamma(\nu+2m+1)\Gamma(\nu+1-n)x^{2m}}{m!2^{2m}\Gamma(\nu+2m-n+1)\Gamma(\nu+m+1)}.$$

We compute

$$\lim_{m\to\infty}\frac{m\log m}{\log\left(\frac{2^{2m}}{\Gamma(\nu+1-n)}\right)+\log\Gamma(m+1)+\log\Gamma(\nu+m+1)+\Delta_{n,m}(\nu)} = \frac{1}{2},$$

where, $\Delta_{n,m}(\nu) = \log\Gamma(\nu+2m-n+1) - \log\Gamma(\nu+2m+1)$.

Hence, the growth order of the entire function $x^{n-\nu}J_{\nu}^{(n)}(x)$ is $\frac{1}{2}$, and by Hadamard factorization theorem [[5]], the product representation holds for $\nu > n-1$. Therefore, $J_{\nu}^{(n)}(x)$ has infinitely many zeros for $\nu > n-1$.

From the product formula, we obtain

$$J_{\nu}^{(n+1)}(x) = \frac{x^{\nu-n}\prod_{m\geq 1}\left(1-\frac{x^2}{(j_{\nu,m}^{(n)})^2}\right)}{2^{\nu}\Gamma(\nu+1-n)}\left(\frac{\nu-n}{x} - \sum_{m\geq 1}\frac{2x}{(j_{\nu,m}^{(n)})^2-x^2}\right),$$

so that

$$\frac{J_{\nu}^{(n+1)}(x)}{J_{\nu}^{(n)}(x)} = \frac{\nu-n}{x} - \sum_{m\geq 1}\frac{2x}{(j_{\nu,m}^{(n)})^2-x^2}. \tag{2.18}$$

Now, assume for contradiction that $J_{\nu}^{(n+1)}(iy) = 0$ for some real $y \neq 0$. Then

$$-i\left((\nu-n)+\sum_{m\geq 1}\frac{2y^2}{(j_{\nu,m}^{(n)})^2+y^2}\right) = 0,$$

a contradiction since the imaginary part is nonzero.

Suppose $z = x+iy$ is a non-real zero. Let $\omega = (j_{\nu,m}^{(n)})^2 - x^2 + y^2$. Then

$$\frac{zJ_{\nu}^{(n+1)}(z)}{J_{\nu}^{(n)}(z)} = (\nu-n) - 2\sum_{m\geq 1}\frac{(x^2-y^2)\omega - 4x^2y^2}{\omega^2+4x^2y^2} - 4ixy\sum_{m\geq 1}\frac{\omega+x^2-y^2}{\omega^2+4x^2y^2} = 0,$$

On simplifying, we get

$$\sum_{m\geq 1}\frac{(j_{\nu,m}^{(n)})^2}{\omega^2+4x^2y^2} = 0,$$

again, a contradiction. Hence, all zeros of $J_{\nu}^{(n+1)}(x)$ are real for $\nu > n$.

To prove simplicity of zeros (except possibly at 0), assume $\rho \neq 0$ is a multiple zero of $J_{\nu}^{(n)}(z)$. Then

$$\frac{d}{dz}\left(\frac{J_{\nu}^{(n)}(z)}{J_{\nu}^{(n-1)}(z)}\right) = -\frac{\nu-n+1}{z^2} - 2\sum_{m\geq 1}\frac{(j_{\nu,m}^{(n-1)})^2+z^2}{\left((j_{\nu,m}^{(n-1)})^2-z^2\right)^2} \neq 0.$$

Thus, all zeros of $J_{\nu}^{(n)}(x)$ are simple except possibly at the origin.

(ii) Since the zeros of $J_\nu^{(n)}(x)$ are all real, it follows that the function

$$\mathbb{J}_{\nu,n}(x) = 2^\nu \Gamma(\nu+1-n) x^{n-\nu} J_\nu^{(n)}(x)$$

belongs to the Laguerre-Pólya class[[9]]. Thus, it satisfies the Laguerre inequality:

$$\left(\mathbb{J}_{\nu,n}^{(k)}(x)\right)^2 - \mathbb{J}_{\nu,n}^{(k-1)}(x)\mathbb{J}_{\nu,n}^{(k+1)}(x) > 0,$$

where $n \in \mathbb{N}_0$, $k \in \mathbb{N}$, $\nu > n-1$, and $x \in \mathbb{R}$.

Choosing $k = 1$ in the above inequality, we get:

$$\left(xJ_\nu^{(n+1)}(x)\right)^2 - x^2 J_\nu^{(n+2)}(x) J_\nu^{(n)}(x) + (n-\nu)\left(J_\nu^{(n)}(x)\right)^2 > 0,$$

which simplifies to

$$\left(J_\nu^{(n+1)}(x)\right)^2 - J_\nu^{(n+2)}(x) J_\nu^{(n)}(x) > \frac{(\nu-n)\left(J_\nu^{(n)}(x)\right)^2}{x^2} > 0,$$

for $\nu > n \geq 0$ and $x \neq 0$.

Therefore, the function $\frac{J_\nu^{(n+1)}(x)}{J_\nu^{(n)}(x)}$ is strictly decreasing on each interval $\left(j_{\nu,m-1}^{(n)}, j_{\nu,m}^{(n)}\right)$, where $m \in \mathbb{N}$, and we define $j_{\nu,0}^{(n)} = 0$.

Moreover, for fixed $m \in \mathbb{N}$, the function $\frac{J_\nu^{(n+1)}(x)}{J_\nu^{(n)}(x)}$ satisfies

$$\lim_{x \to j_{\nu,m-1}^{(n)}{}^+} \left(\frac{J_\nu^{(n+1)}(x)}{J_\nu^{(n)}(x)}\right) = +\infty, \quad \lim_{x \to j_{\nu,m}^{(n)}{}^-} \left(\frac{J_\nu^{(n+1)}(x)}{J_\nu^{(n)}(x)}\right) = -\infty.$$

Hence, for each $m \in \mathbb{N}$, the graph of $\frac{J_\nu^{(n+1)}(x)}{J_\nu^{(n)}(x)}$ on $\left(j_{\nu,m-1}^{(n)}, j_{\nu,m}^{(n)}\right)$ crosses the horizontal axis exactly once, and the x-coordinate of this crossing is precisely $j_{\nu,m}^{(n+1)}$.

Therefore, the positive zeros of $J_\nu^{(n+1)}(x)$ and $J_\nu^{(n)}(x)$ interlace when $\nu > n$.

(iii) We have

$$\mathbb{J}_{\nu,n}(x) = 2^\nu \Gamma(\nu+1-n) x^{n-\nu} J_\nu^{(n)}(x),$$

which implies

$$\frac{d}{dx}\mathbb{J}_{\nu,n}(x) = 2^\nu x^{n-\nu-1}\Gamma(\nu+1-n)\left((n-\nu)J_\nu^{(n)}(x) + xJ_\nu^{(n+1)}(x)\right).$$

As shown in part (a), $\mathbb{J}_{\nu,n}(x)$ is a real entire function of genus zero.

Laguerre's theorem on separation of zeros[[**16**]] If $f(z)$ is a nonconstant entire function of genus 0 or 1, real on $\mathbb{R}$, and has only real zeros, then the zeros of f' are real and interlaced with those of f.

Hence, by Laguerre's separation theorem, the zeros of

$$(n-\nu)J_\nu^{(n)}(x) + xJ_\nu^{(n+1)}(x)$$

are real when $\nu > n-1$, and they interlace with the zeros of $J_\nu^{(n)}(x)$. $\square$

Lemma 1: Let g be an entire function of growth order 0 or 1, which has only real zeros and exactly m positive zeros. Then the function $\sum_{n\geq 0} \frac{(-1)^n g^{(2n)}(z)}{n!} z^{2n}$ has at most $2m$ complex zeros[14].

Theorem 2.9 ([8])**.** *Let $n \in \mathbb{N}_0$. If $\nu > n-1$, then all zeros of $J_\nu^{(n)}(z)$ are real. Moreover, if $\nu \geq 0$, then $2^\nu z^{\nu+n} J_{-\nu}^{(n)}(2z)$ has at most $2[\nu]+2n$ complex zeros. In other words:*

- *If $n-2s-2 < \nu < n-2s-1$, $s \in \mathbb{N}_0$, then $J_\nu^{(n)}(z)$ has at most $4s+2$ complex zeros.*
- *If $n-2s-1 < \nu < n-2s$, $s \in \mathbb{N}$, then $J_\nu^{(n)}(z)$ has at most $4s$ complex zeros.*

Proof. Consider the entire function

$$2^\nu z^{n-\nu} J_\nu^{(n)}(2\sqrt{z}) = \sum_{m\geq 0} \frac{(-1)^m \Gamma(\nu+2m+1) z^m}{m!\Gamma(\nu+2m-n+1)\Gamma(\nu+m+1)}.$$

Let us define

$$g_\nu(2z) = \frac{\Gamma(\nu+2z+1)}{\Gamma(\nu+2z-n+1)\Gamma(\nu+z+1)}.$$

This is an entire function of growth order 1, because the poles of the numerator are canceled by those of the denominator.

The zeros of g_ν are of the form:

$$\tau_k = \frac{k-1-\nu}{2}, \quad k \in \{1,\dots,n\}, \quad \text{and} \quad \zeta_s = -1-\nu-s, \quad s \in \mathbb{N}_0.$$

If $\nu > n-1$, $n \in \mathbb{N}_0$, then all zeros of q_ν are non-positive.

Hence, by Obreschkoff's Lemma, the function $z^{n-\nu} J_\nu^{(n)}(2z)$ has no complex zeros; that is, all zeros are real.

Now consider the function $g_{-\nu}(z)$. For $\nu \geq 0$, this function has $[\nu]+n$ positive zeros, where $[\nu]$ denotes the greatest integer less than or equal to ν. Applying Obreschkoff's Lemma again, we conclude that

$$2^\nu z^{\nu+n} J_\nu^{(n)}(2z) = \sum_{m\geq 0} \frac{(-1)^m g_{-\nu}(2m)}{m!} z^{2m}$$

has at most $2[\nu]+2n$ complex zeros.

As a consequence, we obtain the following estimates on the number of complex zeros of $J_\nu^{(n)}(z)$:

- For $s \in \mathbb{N}_0$, if $\nu \in (n-2s-2,\, n-2s-1)$, then $J_\nu^{(n)}(z)$ has at most $4s+2$ complex zeros.
- For $s \in \mathbb{N}$, if $\nu \in (n-2s-1,\, n-2s)$, then $J_\nu^{(n)}(z)$ has at most $4s$ complex zeros.

$\square$

CHAPTER 3

Regular Coulomb wave function

3.1 Coulomb wave Equation

The second-order differential equation,

$$\frac{d^2u}{d\rho^2} + \left(1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2}\right) u = 0$$

is known as the Coulomb wave equation. Regular and irregular Coulomb wave functions, $F_L(\eta, \rho)$ and $G_L(\eta, \rho)$, are two linearly independent solutions of the Coulomb wave equation.

3.2 Regular Coulomb wave function

The regular Coulomb wave function $F_L(\eta, \rho)$ can be defined as[[11], Chp. 14],

$$F_L(\eta, \rho) = C_L(\eta)\rho^{L+1}\phi_L(\eta, \rho), \tag{3.1}$$

where

$$C_L(\eta) := \frac{2^L \mathrm{e}^{-\pi\eta/2}|\Gamma(L+1+i\eta)|}{\Gamma(2L+2)},$$

$$\phi_L(\eta, \rho) := \mathrm{e}^{-i\rho}\,{}_1F_1(L+1-i\eta; 2L+2; 2i\rho) \tag{3.2}$$

and confluent hypergeometric function ${}_1F_1$ is defined by

$$ {}_1F_1(a;b;z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}, $$

for $a, b, z \in \mathbb{C}$, such that $b \notin -\mathbb{N}_0$, where $(a)_0 = 1$ and $(a)_n = a(a+1)\dots(a+n-1)$, for $n \in \mathbb{N}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

For specific parameter values $L = \nu - \frac{1}{2}, \quad \eta = 0$ [[11], Eqs. 14.6.6 and 13.6.1] The regular Coulomb wave functions simplify to expressions involving the Bessel function of the first kind:

$$ F_{\nu-\frac{1}{2}}(0,\rho) = \sqrt{\frac{\pi\rho}{2}} J_\nu(\rho) $$

$$ \phi_{\nu-\frac{1}{2}}(0,\rho) = e^{-i\rho}\,{}_1F_1\left(\nu + \frac{1}{2}; 2\nu+1; 2i\rho\right) = \Gamma(\nu+1)\left(\frac{2}{\rho}\right)^{\nu} J_\nu(\rho) $$

This result shows that the regular Coulomb wave function is a **one-parameter generalization** of the Bessel function of the first kind.

One can see from equation (1) that, with the possible exception of the origin, the zeros of $F_L(\eta,\cdot)$ coincide with those of $\phi_L(\eta,\cdot)$. If $L \notin -\frac{N+1}{2}$, the function $\phi_L(\eta,\cdot)$ is well-defined for all $\eta \in \mathbb{C}$. Even when $L = -\frac{N+1}{2}$, the function $\phi_L(\eta,\cdot)$ remains well-defined provided that $i\eta \in \mathbb{Z}$ and the condition $L + 2 + i\eta \leq 0$ is satisfied, in which case the confluent hypergeometric series in equation (2) terminates. In general, the function $F_L(\eta,\rho)$ admits analytic continuation to complex values of all its parameters: L, η, and ρ.

Lemma 1. Let f be an entire function of order 1 with real Taylor coefficients. Denote $D_{-1} := 1$ and $D_n := \det\left(s_{i+j}\right)_{i,j=0}^{n-1}$, for $n \in \mathbb{N}_0$
where

$$ s_k := \prod_{j=1}^{\infty} \frac{1}{z_j^{k+2}}, \quad k \in \mathbb{N}_0, $$

and $z_1, z_2, \dots$ are the zeros of f. Then the following statements hold:

(i) (**Grommer**[10]) All the zeros of f are real if and only if $D_n > 0$ for all $n \in \mathbb{N}_0$.

(ii) (**Chebotarev**[15]) If the sequence $\{D_{n-1}D_n\}_{n\in\mathbb{N}_0}$ contains exactly m negative numbers, then the function f has m distinct pairs of complex conjugate zeros and an infinite number of real zeros.

3.3 Hurwitz-type Theorem for The Regular Coulomb Wave Function

Theorem 3.1 ([2]). *Suppose $\eta, L \in \mathbb{R}$. The following statements hold:*

(i) If $L \neq -1$ and $-\frac{3}{2} < L$ for $\eta \neq 0$, and $L > -\frac{3}{2}$ for $\eta = 0$, then all zeros of $F_L(\eta, \cdot)$ are real.

(ii) If $L < -\frac{3}{2}$ and $L \notin -\mathbb{N}/2$ for $\eta \neq 0$, and $L \notin -\mathbb{N} - 1/2$ for $\eta = 0$, then $F_L(\eta, \cdot)$ has $\lfloor -L - \frac{1}{2} \rfloor$ distinct pairs of complex conjugate zeros and an infinite number of real zeros.

Proof. By definition of the regular Coulomb wave function

$$F_L(\eta, \rho) = C_L(\eta)\rho^{L+1}\phi_L(\eta, \rho),$$

Clearly from this equation the zeros of $F_L(\eta, .)$ are the same as the zeros of $\phi_L(\eta, \rho)$ except origin.

$$\phi_L(\eta, \rho) = \mathrm{e}^{-i\rho}\, {}_1F_1(L + 1 - i\eta; 2L + 2; 2i\rho).$$

From this equation using Taylor coefficients of the confluent hypergeometric function we can say $\phi_L(\eta, .)$ is an entire function of order 1 for $L \notin -(\mathbb{N} + 1)/2$ and $\eta \in \mathbb{C}$.
Let us define

$$\zeta_L(k) := \sum_{n=1}^{\infty} \frac{1}{\rho_{L,n}^k}, \quad k \geq 2.$$

where $\rho_{L,1}, \rho_{L,2}, \ldots$ are the zeros of $\varphi_L(\eta, \cdot)$.
This series is **absolutely convergent** for $k \geq 2$, $L \notin -(\mathbb{N} + 1)/2$, and $\eta \in \mathbb{C}$. In addition, $\zeta_L(k)$ can be computed recursively as shown in [[7], Eqs. (78) and (79)] using the following relations:

$$\zeta_L(2) = \frac{1}{2L + 3}\left(1 + \frac{\eta^2}{(L + 1)^2}\right) \tag{3.3}$$

and

$$\zeta_L(k + 1) = \frac{1}{2L + k + 2}\left(\frac{2\eta}{L + 1}\zeta_L(k) + \sum_{l=1}^{k-2} \zeta_L(l + 1)\zeta_L(k - l)\right), \quad k \geq 2. \tag{3.4}$$

Lemma 2.[[2]] For $L \notin -(\mathbb{N}+1)/2, \eta \in \mathbb{C}$, and $n \in \mathbb{N}$, we define the Hankel matrix

$$H_n(L,\eta) := \begin{pmatrix} \zeta_L(2) & \zeta_L(3) & \dots & \zeta_L(n+1) \\ \zeta_L(3) & \zeta_L(4) & \dots & \zeta_L(n+2) \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_L(n+1) & \zeta_L(n+2) & \dots & \zeta_L(2n) \end{pmatrix} \tag{3.5}$$

For , $L \notin -(\mathbb{N}+1)/2, \eta \in \mathbb{C}$, and $n \in \mathbb{N}$, then

$$\det H_n(L,\eta) = \prod_{k=0}^{n-1} \frac{1}{(2L+2n-2k+1)^{2k+1}} \left(1 + \frac{\eta^2}{(L+n-k)^2}\right)^{k+1}. \tag{3.6}$$

Proof. We will use the well-known relation between the recurrence coefficients from the three-term recurrence for a family of orthogonal polynomials and the determinant of the corresponding moment Hankel matrix.

The orthogonal polynomials $p_n(z)$ are generated by the recurrence relation

$$p_{n+1}(z) = (z-b_n)p_n(z) - a_n p_{n-1}(z), \quad n \in \mathbb{N}, \tag{3.7}$$

with the initial conditions

$$p_{-1}(z) = 0, \quad p_0(z) = 1.$$

then

$$\Delta_0 := 1, \quad \Delta_n := \prod_{m=1}^{n-1} \prod_{j=1}^{m} a_j, \quad n \in \mathbb{N}, \tag{3.8}$$

where

$$\Delta_0 := 1, \quad \Delta_n := \det\left(\mathcal{L}\left(z^{i+j}\right)\right)_{i,j=0}^{n-1}, \quad n \in \mathbb{N},$$

and $\mathcal{L}$ is the corresponding normalized moment functional. The result (3.8) follows from [[8], Chp. I, Thm. 4.2(a)].

Now, for

$$a_n = \frac{(n+L+1)^2 + \eta^2}{(n+L+1)^2(2n+2L+1)(2n+2L+3)} \tag{3.9}$$

and

$$b_n = -\frac{\eta}{(n+L+1)(n+L+2)}, \tag{3.10}$$

and the corresponding moment sequence is given by

$$\mathcal{L}(z^n) = \frac{\zeta_L(n+2)}{\zeta_L(2)}, \quad n \in \mathbb{N}_0.$$

From (3.8), we will get

$$\det H_n(L,\eta) = (\zeta_L(2))^n \prod_{m=1}^{n-1}\prod_{j=1}^{m} a_j, \quad n\in\mathbb{N}.$$

using equations (3.3) and (3.9), we will have

$$\det H_n(L,\eta) = \prod_{k=0}^{n-1} \frac{1}{(2L+2n-2k+1)^{2k+1}}\left(1+\frac{\eta^2}{(L+n-k)^2}\right)^{k+1}. \tag{3.11}$$

□

In the special case of $L=\nu-\frac{1}{2}$ and $\eta=0$, we have

$$\zeta_{\nu-\frac{1}{2}}(k) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{j_{\nu,n}^k},$$

where $j_{\nu,1}, j_{\nu,2},\dots$ are the zeros of J_ν.

Consequently, for $k\in\mathbb{N}$, we get

$$\zeta_{\nu-\frac{1}{2}}(2k+1)=0$$

and

$$\zeta_{\nu-\frac{1}{2}}(2k) = 2\sigma_{2k}(\nu) = 2\sum_{n=1}^{\infty}\frac{1}{j_{\nu,n}^{2k}},$$

where $\sigma_{2k}(\nu)$ is the Rayleigh function of order $2k$. We define

$$H_n^{(\ell)}(\nu) := \begin{pmatrix} \sigma_{2\ell+2}(\nu) & \sigma_{2\ell+4}(\nu) & \dots & \sigma_{2n+2\ell}(\nu) \\ \sigma_{2\ell+4}(\nu) & \sigma_{2\ell+6}(\nu) & \dots & \sigma_{2n+2\ell+2}(\nu) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{2n+2\ell}(\nu) & \sigma_{2n+2\ell+2}(\nu) & \dots & \sigma_{4n+2\ell-2}(\nu) \end{pmatrix}, \tag{3.12}$$

for $n\in\mathbb{N}$, $\ell\in\mathbb{N}_0$, and $\nu\notin-\mathbb{N}$.

corollary: For $n\in\mathbb{N}$, $\nu\notin-\mathbb{N}$, and $\ell\in\{0,1\}$, then

$$\det H_n^{(\ell)}(\nu) = 2^{-2n(n+\ell)} \prod_{k=1}^{2n+\ell-1} (\nu+k)^{k-2n-\ell}. \tag{3.13}$$

Denote $D_n := \det H_{n+1}(L,\eta)$ for $n\in\mathbb{N}_0$. It is clear from the identity (3.11) that for $-1\neq L>-\frac{3}{2}$ and $\eta\neq 0$, we have $D_n>0$ for all $n\in\mathbb{N}_0$.When $\eta=0$, by previous Corollary, where $\nu=L+\frac{1}{2}$, it can be easily verified that $D_n>0$ for all $n\in\mathbb{N}_0$, $L>-\frac{3}{2}$.

Hence, by applying the first part of **Lemma 1**, we obtain the first part of the proof.

From (3.11), we will get

$$D_{n-1}D_n = \frac{1}{(2L+2n+3)}\left(1+\frac{\eta^2}{(L+n+1)^2}\right) \times \prod_{k=0}^{n-1}\frac{1}{(2L+2n-2k+1)^{4k+4}}\left(1+\frac{\eta^2}{(L+n-k)^2}\right)^{2k+3} \tag{3.14}$$

for $\eta \neq 0, L \notin \frac{(\mathbb{N}+1)}{2}$, and $n \in \mathbb{N}_0$. Similarly as above, when $\eta = 0$, $\phi_L(0,\rho)$ is to be understood as the Bessel function. In this case, the formula (3.13) takes the form

$$D_{n-1}D_n = \frac{1}{2L+2n+3}\prod_{k=0}^{n-1}\frac{1}{(2L+2n-2k+1)^{4k+4}} \tag{3.15}$$

for $L \notin -\mathbb{N} - \frac{1}{2}$ and $n \in \mathbb{N}_0$. In any case, it is clear from (3.13) that the sign of $D_{n-1}D_n$ equals the sign of the factor $2L+2n+3$. Consequently, the number of negative elements in $\{D_{n-1}D_n\}_{n\in\mathbb{N}_0}$ coincides with the number of elements of the set $\{n \in \mathbb{N}_0 \mid 2L+2n+3 < 0\}$.Hence the second part of Theorem follows from the second part of **Lemma 1**.

The particular case of Theorem (3.1) with $\eta = 0$ and $L = \nu - \frac{1}{2}$ implies Hurwitz's theorem about the zeros of the Bessel function of the first kind.

1. If $\nu > -1$, then all zeros of J_ν are real.

2. If $-2s-2 < \nu < -2s-1$ for $s \in \mathbb{N}_0$, then J_ν has $4s+2$ complex zeros, of which two are purely imaginary.

3. If $-2s-1 < \nu < -2s$ for $s \in \mathbb{N}$, then J_ν has $4s$ complex zeros.

□

3.4 Mittag–Leffler Expansion for The Regular Coulomb Wave Function

Theorem 3.2 ([1]). *Let $\rho, \eta \in \mathbb{R}$ and let $L > -\frac{3}{2}$, with $L = -1$ if $\eta = 0$ and $L > -\frac{3}{2}$ if $\eta \neq 0$. Then the following Mittag–Leffler expansion is valid:*

$$\frac{F_{L+1}(\eta,\rho)}{F_L(\eta,\rho)} = \frac{L+1}{(L+1)^2+\eta^2} + \sum_{n\geq 1}\left(\frac{\rho}{x_{L,\eta,n}(x_{L,\eta,n}-\rho)} + \frac{\rho}{y_{L,\eta,n}(y_{L,\eta,n}-\rho)}\right), \tag{3.16}$$

where $x_{L,\eta,n}$ and $y_{L,\eta,n}$ are the nth positive and negative zeros, respectively, of the Coulomb wave function $F_L(\eta,\rho)$.

Proof. The function $F_L(\eta,\rho)$, is entire in ρ, admits the Hadamard product representation [[7], Eq.(76)] :

$$F_L(\eta,\rho) = C_L(\eta)\,\rho^{L+1}e^{\pi\eta/2}\prod_{n\geq 1}\left(1-\frac{\rho}{\rho_{L,\eta,n}}\right)\exp\left(\frac{\rho}{\rho_{L,\eta,n}}\right), \tag{3.17}$$

where $\rho_{L,\eta,n}$ runs over the non-zero zeros of $F_L(\eta,\cdot)$.

Taking the logarithmic derivative of the Hadamard product gives:

$$\frac{F_L'(\eta,\rho)}{F_L(\eta,\rho)} = \frac{L+1}{\rho} + \sum_{n\geq 1}\left(\frac{1}{\rho_{L,\eta,n}} - \frac{1}{\rho_{L,\eta,n}-\rho}\right) \tag{3.18}$$

$$= \frac{L+1}{\rho} - \sum_{n\geq 1}\frac{\rho}{\rho_{L,\eta,n}(\rho_{L,\eta,n}-\rho)}. \tag{3.19}$$

The $F_L(\eta,\rho)$ satisfy the following recurrence relation [[11], p-539],

$$(L+1)F_L'(\eta,\rho) = \left(\frac{(L+1)^2}{\rho}+\eta\right)F_L(\eta,\rho) - \sqrt{(L+1)^2+\eta^2}\,F_{L+1}(\eta,\rho). \tag{3.20}$$

On rearranging, we get

$$\frac{F_{L+1}(\eta,\rho)}{F_L(\eta,\rho)} = \frac{1}{\sqrt{(L+1)^2+\eta^2}}\left(\frac{(L+1)^2}{\rho}+\eta-(L+1)\frac{F_L'(\eta,\rho)}{F_L(\eta,\rho)}\right). \tag{3.21}$$

$$\frac{F_{L+1}(\eta,\rho)}{F_L(\eta,\rho)} = \frac{1}{\sqrt{(L+1)^2+\eta^2}}\left(\frac{(L+1)^2}{\rho}+\eta-(L+1)\left[\frac{L+1}{\rho}-\sum_{n\geq 1}\frac{\rho}{\rho_{L,\eta,n}(\rho_{L,\eta,n}-\rho)}\right]\right)$$

$$= \frac{1}{\sqrt{(L+1)^2+\eta^2}}\left(\eta+(L+1)\sum_{n\geq 1}\frac{\rho}{\rho_{L,\eta,n}(\rho_{L,\eta,n}-\rho)}\right).$$

Splitting the sum over all zeros $\rho_{L,\eta,n}$ into the positive and negative zeros $x_{L,\eta,n}$ and $y_{L,\eta,n}$, we get

$$\sum_{n\geq 1}\frac{\rho}{\rho_{L,\eta,n}(\rho_{L,\eta,n}-\rho)} = \sum_{n\geq 1}\left(\frac{\rho}{x_{L,\eta,n}(x_{L,\eta,n}-\rho)}+\frac{\rho}{y_{L,\eta,n}(y_{L,\eta,n}-\rho)}\right).$$

Hence, the Mittag–Leffler expansion:

$$\frac{F_{L+1}(\eta,\rho)}{F_L(\eta,\rho)} = \frac{L+1}{(L+1)^2+\eta^2} + \sum_{n\geq 1}\left(\frac{\rho}{x_{L,\eta,n}(x_{L,\eta,n}-\rho)}+\frac{\rho}{y_{L,\eta,n}(y_{L,\eta,n}-\rho)}\right).$$

□

3.5 The Turán-type inequalities for the regular Coulomb wave function

Theorem 3.3 ([1]). *The following assertions are true:*

(a) If $L,\eta>0$, $0<\rho<L(L+1)/\eta$, $\rho<x_{L,\eta,1}$, or $-3/2<L<-1$, $\eta>0$,

$0<\rho<L(L+1)/\eta$, $\rho<x_{L,\eta,1}$, or $\eta\le 0$, $L\ge 0$ and $0<\rho<x_{L,\eta,1}$, then

$$F_L^2(\eta,\rho)-F_{L-1}(\eta,\rho)F_{L+1}(\eta,\rho)\ge 0.$$

(b) If $L,\eta>0$, $L(L+1)/\eta\le\rho<x_{L-1,\eta,1}$, or $-3/2<L<-1$, $\eta>0$, $L(L+1)/\eta\le\rho<x_{L-1,\eta,1}$, or $-1<L<0$, $\eta<0$, $L(L+1)/\eta\le\rho<x_{L-1,\eta,1}$, then

$$\frac{\sqrt{L^2+\eta^2}}{L}F_L^2(\eta,\rho)-\frac{\sqrt{(L+1)^2+\eta^2}}{L+1}F_{L-1}(\eta,\rho)F_{L+1}(\eta,\rho)\ge 0.$$

(c) If $L>-1$, $\eta\in\mathbb{R}$, $\rho^2\le(L^3+1)/(L^2+\eta^2)$, $\eta/((L(L+1))-1/\rho>0$, and $0<\rho<x_{L-1,\eta,1}$, then

$$F_L^2(\eta,\rho)-\frac{\sqrt{L^2+\eta^2}}{L(L+1)}\frac{\sqrt{(L+1)^2+\eta^2}}{L+1}F_{L-1}(\eta,\rho)F_{L+1}(\eta,\rho)\ge 0.$$

Proof. **(a)** By using the recurrence relation , we have

$$LF_L'(\eta,\rho)=\sqrt{L^2+\eta^2}F_{L-1}(\eta,\rho)-\left(\frac{L^2}{\rho}+\eta\right)F_L(\eta,\rho). \tag{3.22}$$

Combining this with identity (3.20), we obtain the representation

$$\frac{1}{F_L^2(\eta,\rho)}\Delta_L(\eta,\rho)=a_{L,\eta}(\rho)-b_{L,\eta}(\rho)\frac{F_L'(\eta,\rho)}{F_L(\eta,\rho)}+c_{L,\eta}\left[\frac{F_L'(\eta,\rho)}{F_L(\eta,\rho)}\right]^2, \tag{3.23}$$

where the coefficients are defined as

$$a_{L,\eta}(\rho)=1-\frac{\left(\frac{L^2}{\rho}+\eta\right)\left(\frac{(L+1)^2}{\rho}+\eta\right)}{\sqrt{L^2+\eta^2}\sqrt{(L+1)^2+\eta^2}},$$

$$b_{L,\eta}(\rho)=\frac{L(L+1)/\rho-\eta}{\sqrt{L^2+\eta^2}\sqrt{(L+1)^2+\eta^2}},$$

$$c_{L,\eta}=\frac{L(L+1)}{\sqrt{L^2+\eta^2}\sqrt{(L+1)^2+\eta^2}}.$$

Here, $\Delta_L(\eta,\rho)$ denotes the Turán expression for Coulomb wave functions:

$$\Delta_L(\eta,\rho)=F_L^2(\eta,\rho)-F_{L-1}(\eta,\rho)F_{L+1}(\eta,\rho).$$

Now, recall that the Coulomb wave function is a particular solution of the Coulomb differential equation

$$\frac{d^2u}{d\rho^2}+\left(1-\frac{2\eta}{\rho}-\frac{L(L+1)}{\rho^2}\right)u=0$$

From this, we derive the identity

$$\left[\frac{F_L'(\eta,\rho)}{F_L(\eta,\rho)}\right]^2=\frac{L(L+1)}{\rho^2}+\frac{2\eta}{\rho}-1-\left[\frac{F_L'(\eta,\rho)}{F_L(\eta,\rho)}\right]'. \tag{3.24}$$

Using the recurrence relation and the Mittag–Leffler expansion (2.1), we can express

$$\frac{F_L'(\eta,\rho)}{F_L(\eta,\rho)}=\frac{L+1}{\rho}+\frac{\eta}{L+1}-\sum_{n\ge1}\left[\frac{\rho}{x_{L,\eta,n}(x_{L,\eta,n}-\rho)}+\frac{\rho}{y_{L,\eta,n}(y_{L,\eta,n}-\rho)}\right],$$

and
$$\left[\frac{F_L'(\eta,\rho)}{F_L(\eta,\rho)}\right]' = -\frac{L+1}{\rho^2} - \sum_{n\geq 1}\left[\frac{1}{(x_{L,\eta,n}-\rho)^2} + \frac{1}{(y_{L,\eta,n}-\rho)^2}\right]. \tag{3.25}$$
from (3.24) and (3.25), we get
$$\frac{{}_1\Delta_{L,\eta}(\rho)}{F_L^2(\eta,\rho)} = e_{L,\eta} + b_{L,\eta}(\rho)\sum_{n\geq 1}\left[\frac{\rho}{x_{L,\eta,n}(x_{L,\eta,n}-\rho)} + \frac{\rho}{y_{L,\eta,n}(y_{L,\eta,n}-\rho)}\right]$$
$$+ c_{L,\eta}\sum_{n\geq 1}\left[\frac{1}{(x_{L,\eta,n}-\rho)^2} + \frac{1}{(y_{L,\eta,n}-\rho)^2}\right],$$
where
$$e_{L,\eta} = 1 - \frac{L\sqrt{(L+1)^2+\eta^2}}{(L+1)\sqrt{L^2+\eta^2}}.$$
Note that for all $L \geq 0$ or $-3/2 < L < -1$ and $\eta \in \mathbb{R}$ we have $c_{L,\eta} \geq 0$ and $e_{L,\eta} \geq 0$. Thus ${}_1\Delta_{L,\eta}(\rho)$ is positive if $L, \eta > 0$, $0 < \rho < \frac{L(L+1)}{\eta}$, $\rho < x_{L,\eta,1}$ or if $-3/2 < L < -1$, $\eta > 0$, $0 < \rho < \frac{L(L+1)}{\eta}$, $\rho < x_{L,\eta,1}$, or if $\eta \leq 0$, $L \geq 0$ and $0 < \rho < x_{L,\eta,1}$.

(b) By using the recurrence relations (3.22) and (3.20), we get
$$F_{L+1}'(\eta,\rho)F_L(\eta,\rho) - F_L'(\eta,\rho)F_{L+1}(\eta,\rho) = {}_2\Delta_{L+1,\eta}(\rho) - \left[\frac{\eta}{(L+1)(L+2)} - \frac{1}{\rho}\right]F_L(\eta,\rho)F_{L+1}(\eta,\rho),$$
where
$${}_2\Delta_{L,\eta}(\rho) = \frac{\sqrt{L^2+\eta^2}}{L}F_L^2(\eta,\rho) - \frac{\sqrt{(L+1)^2+\eta^2}}{L+1}F_{L-1}(\eta,\rho)F_{L+1}(\eta,\rho).$$
On the other hand, according to [[19], Lemma2.4], we have
$$\rho^2\frac{\sqrt{(L+1)^2+\eta^2}}{L+1}\left[F_{L+1}'(\eta,\rho)F_L(\eta,\rho) - F_L'(\eta,\rho)F_{L+1}(\eta,\rho)\right] = \sum_{n\geq 1}(2L+2n+1)F_{L+n}^2(\eta,\rho).$$
From this we obtain that
$$\frac{{}_2\Delta_{L,\eta}(\rho)}{F_{L-1}^2(\eta,\rho)} \geq \left[\frac{\eta}{L(L+1)} - \frac{1}{\rho}\right]\frac{F_L(\eta,\rho)}{F_{L-1}(\eta,\rho)}.$$
and by using the Mittag-Leffler expansion (3.16), the right-hand side of the above inequality is positive if $L, \eta > 0$ and $\frac{L(L+1)}{\eta} \leq \rho < x_{L-1,\eta,1}$, or if $-\frac{3}{2} < L < -1$, $\eta > 0$ and $\frac{L(L+1)}{\eta} \leq \rho < x_{L-1,\eta,1}$, or if$-1 < L < 0$, $\eta < 0$, $\frac{L(L+1)}{\eta} \leq \rho < x_{L-1,\eta,1}$.

(c) From (3.25) implies that for all $\eta, \rho \in \mathbb{R}$, $\rho \neq 0$ and $L \geq -1$ we have
$$D_{L,\eta}(\rho) = F_L''(\eta,\rho)F_L(\eta,\rho) - F_L'(\eta,\rho)^2 \leq 0.$$
Now, by using the recurrence relations (3.22) and (3.20) and also the fact that $F_L(\eta,\rho)$ satisfies the Coulomb differential equation, we get
$$D_{L,\eta}(\rho) = f_{L,\eta}(\rho)F_L^2(\eta,\rho) + \frac{1}{c_{L,\eta}}F_{L-1}(\eta,\rho)F_{L+1}(\eta,\rho) + \left[\frac{\eta}{L(L+1)} - \frac{1}{\rho}\right]F_{L-1}(\eta,\rho)F_L(\eta,\rho),$$

where

$$f_{L,\eta}(\rho) = \frac{L}{\rho^2} - 1 - \frac{\eta^2}{L^2}.$$

If $L > -1$, $\eta \in \mathbb{R}$ and $(L^3+1)/(L^2+\eta^2) \geq \rho^2$, then we have that $f_{L,\eta}(\rho) \geq -1$ and consequently we have

$$0 \geq D_{L,\eta}(\rho) \geq -{}_3\Delta_{L,\eta}(\rho) + \left[\frac{\eta}{L(L+1)} - \frac{1}{\rho}\right] F_{L-1}(\eta,\rho)F_L(\eta,\rho),$$

where

$${}_3\Delta_{L,\eta}(\rho) = F_L^2(\eta,\rho) - \frac{\sqrt{L^2+\eta^2}\sqrt{(L+1)^2+\eta^2}}{L(L+1)} F_{L-1}(\eta,\rho)F_{L+1}(\eta,\rho).$$

But the above inequality is equivalent to

$$\frac{{}_3\Delta_{L,\eta}(\rho)}{F_{L-1}^2(\eta,\rho)} \geq \left[\frac{\eta}{L(L+1)} - \frac{1}{\rho}\right] \frac{F_L(\eta,\rho)}{F_{L-1}(\eta,\rho)},$$

and by using again the Mittag–Leffler expansion (3.16), the right-hand side of the above inequality is positive if

$$\frac{\eta}{L(L+1)} - \frac{1}{\rho} > 0 \quad \text{and} \quad 0 < \rho < x_{L-1,\eta,1}.$$

Which completed the proof. □

Theorem 3.4 ([1]). *If $L > -1/2$ and $\eta \in \mathbb{R}$, then the zeros of $\rho \mapsto F_L(\eta,\rho)$ and $\rho \mapsto F_L'(\eta,\rho)$ are interlacing. Moreover, if $L > -1$ and $\eta \in \mathbb{R}$, then the zeros of $\rho \mapsto F_L(\eta,\rho)$ and $\rho \mapsto \rho F_L'(\eta,\rho) - (L+1)F_L(\eta,\rho)$ are interlacing.*

Proof. In view of (3.25), for $L > -1$ the function $\rho \mapsto \frac{F_L'(\eta,\rho)}{F_L(\eta,\rho)}$ is decreasing on the interval $(x_{L,\eta,k}, x_{L,\eta,k+1})$, where $k \in \{1,2,\dots\}$. Moreover, the expression $\frac{F_L'(\eta,\rho)}{F_L(\eta,\rho)}$ tends to $-\infty$ as $\rho \nearrow x_{L,\eta,k+1}$ and tends to ∞ as $\rho \searrow x_{L,\eta,k}$.

Since for $L > -1/2$ and $\eta \in \mathbb{R}$ the zeros of $\rho \mapsto F_L'(\eta,\rho)$ are real and simple, it follows that $\rho \mapsto \frac{F_L'(\eta,\rho)}{F_L(\eta,\rho)}$ intersects once and only once the horizontal axis, and the abscissa of the intersection point is actually the kth positive zero of $\rho \mapsto F_L'(\eta,\rho)$. A similar argument applies to the negative zeros. □

CHAPTER 4

Conclusion and Future Works

4.1 Conclusion

This thesis has provided a detailed investigation of the Bessel and Coulomb wave functions, focusing on their analytical properties, zero distributions, and interrelations. Key contributions include: **Bessel Functions:**

- Derived fundamental properties, including recurrence relations, Wronskians, and infinite product representations.
- Established rigorous proofs for the reality, simplicity, and interlacing of zeros, extending classical results such as Hurwitz's theorem.
- Explored the behavior of derivatives of Bessel functions, confirming the realness and distribution of their zeros under various conditions.

Coulomb Wave Functions:

- Demonstrated that the regular Coulomb wave function $F_L(\eta, \rho)$ generalizes the Bessel function, with analogous properties in zero distributions.

- Applied determinantal criteria (Grommer and Chebotarev theorems) and Hankel matrices to analyze the nature of zeros, unifying results from Bessel theory.

- Derived Mittag-Leffler expansions and Turán-type inequalities, providing deeper insights into the functional behavior of Coulomb wave solutions.

The study bridges classical special function theory with modern analytical techniques, reinforcing the importance of these functions in mathematical physics, quantum mechanics, and applied mathematics.

4.2 Future Work

Building upon the results presented in this thesis, several directions for future research emerge. I intend to further investigate the deeper mathematical properties of Bessel functions and regular Coulomb wave functions, particularly their behavior under various transformations and their connections to other special functions and orthogonal polynomials. Special attention will be given to the generalization of these functions to complex parameters and arguments, as well as their applications in advanced physical problems such as quantum scattering and wave propagation.

Additionally, I aim to develop more efficient computational algorithms for evaluating these functions and their zeros, especially for large arguments or non-integer orders. Understanding the asymptotic expansions of the zeros, similar to McMahon-type expansions for Bessel functions, could also be a fruitful area of exploration. Another direction of interest is the study of inequalities, such as Turán-type and logarithmic convexity, for both Bessel and Coulomb wave functions.

Several alternative proofs of Hurwitz's theorem for the Bessel function of the first kind exist, one classical approach involves Lommel polynomials. In particular, the work of Runckel [20] presents a proof using meromorphic continued fractions. A promising direction for future research is to investigate whether this continued fraction approach can be extended to establish a Hurwitz-type theorem for the regular Coulomb wave functions.

By pursuing these avenues, I hope to contribute to both the theoretical understanding and practical applications of Bessel and Coulomb wave functions in mathematical physics and engineering.

Bibliography

[1] Á. Baricz, *Turán Type Inequalities for Regular Coulomb Wave Functions*, J. Math. Anal. Appl. , vol. 430, no. 1, pp. 166–180, 2015.

[2] Á. Baricz and F. Štampach, *The Hurwitz-Type Theorem for the Regular Coulomb Wave Function via Hankel Determinants*, Linear Algebra Appl., vol. 548, pp. 259–272, 2018.

[3] Á. Baricz and S. Singh, *Zeros of Some Special Entire Functions*, Proc. Amer. Math. Soc., vol. 146, no. 5, pp. 2207–2216, 2018.

[4] Á. Baricz, C. Kokologiannaki, and Tibor Pogány, *Zeros of Bessel function derivatives*, Proc. Amer. Math. Soc. Vol. 146, No. 1, pp. 209–222, 2018.

[5] B. Ya. Levin, *Lectures on Entire Functions*, vol. 150, Amer. Math. Soc., 1996.

[6] E. A. Skelton, "A new identity for the infinite product of zeros of Bessel functions of the first kind or their derivatives," *Proc. Amer. Math. Soc.* , vol. 267, no. 1, pp. 338–344, 2002.

[7] F. Štampach and P. Št'ovíček, *Orthogonal Polynomials Associated with Coulomb Wave Functions*, J. Math. Anal. Appl., vol. 419, no. 1, pp. 231–254, 2014.

[8] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, vol. 3, The University Press, 1922.

[9] H. Skovgaard, *On inequalities of the Turán type,* Math. Scand., pp. 65–73, 1954.

[10] J. Grommer, *Ganze transzendente Funktionen mit lauter reellen Nullstellen*, published by Walter de Gruyter, 1914.

[11] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*, vol. 55, Courier Corp., 1965.

[12] M. E. H. Ismail and M. E. Muldoon, *Bounds for the Small Real and Purely Imaginary Zeros of Bessel and Related Functions*, Methods Appl. Anal., vol. 2, no. 1, pp. 1–21, 1995.

[13] M. K. Kerimov,*Studies on the zeros of Bessel functions and methods for their computation*, Comput. Math. Math. Phys., Vol. 54, pp. 1337–1388, 2014.

[14] N. Obreschkoff, *Über die Nullstellen der Besselschen Funktionen*, Jahresber. Dtsch. Math.-Ver., vol. 38, pp. 156–160, 1929.

[15] N. Tschebotareff, *Über die Realität von Nullstellen ganzer transzendenter Funktionen*, Math. Ann., vol. 99, no. 1, pp. 660–686, 1928.

[16] R. P. Boas Jr., *Entire Functions*, Academic Press Inc., New York, 1954.

[17] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Courier Corp., 2011.

[18] William W. Bell, *Special Functions for Scientists and Engineers*, Courier Corp., 2004.

[19] Y. Miyazaki, Y. Kikuchi, D. Cai, and Y. Ikebe, *Error Analysis for the Computation of Zeros of Regular Coulomb Wave Function and Its First Derivative*, Math. Comp. vol. 70, no. 235, pp. 1195–1204, 2001.

[20] H.-J. Runckel, Distribution of zeros of certain meromorphic continued fractions, *Mathematische Annalen*, **203** (1973), 191–200.