SOME ASPECTS OF MONOTONE ITERATIVE METHODS FOR CLASSICAL AND FRACTIONAL DIFFERENTIAL EQUATIONS

Ph.D. Thesis

By
Rupsha Roy

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SOME ASPECTS OF MONOTONE ITERATIVE METHODS FOR CLASSICAL AND FRACTIONAL DIFFERENTIAL EQUATIONS

A THESIS

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Ruspha Roy

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **SOME ASPECTS OF MONOTONE ITERATIVE METHODS FOR CLASSICAL AND FRACTIONAL DIFFERENTIAL EQUATIONS** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DISCIPLINE OF MATHEMATICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from January 2013 to August 2018 under the supervision of Dr. V. Antony Vijesh, Associate Professor, IIT Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Signature of the student with date
(**RUPSHA ROY**)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Signature of Thesis Supervisor with date
(**DR. V. ANTONY VIJESH**)

RUPSHA ROY has successfully given her Ph.D. Oral Examination held on 26^{th} July 2019.

Sign. of Chairperson(OEB) Date:	Sign. of External Examiner Date:	Sign. of Thesis Supervisor Date:
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Sign. of Head of Discipline Date:		

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DEDICATION

I dedicate this thesis to my mother Mina Roy,
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LIST OF PUBLICATIONS

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3. R. Roy, V. A. Vijesh and G. Chandhini, *Iterative methods for fractional order Volterra population model*, J. Integral Equations Appl., to appear.
4. L. A. Sunny, R. Roy and V. A. Vijesh, *An alternative technique for solving a coupled PDE system in interphase heat transfer*, Appl. Anal., to appear.
5. R. Roy, *Quasilinearization method for two-point boundary value problem of fractional order* $1 < q < 2$, communicated.

ABSTRACT

The present thesis, in six chapters, proposes existence and uniqueness theorems for classical and fractional order differential equations with initial conditions and/or boundary conditions, via monotone iterative methods. The proposed iterative schemes are easy to apply and computationally inexpensive. Based on the iterative schemes, numerical methods are developed to solve the problems numerically. To show the efficiency of the proposed schemes, they are compared with existing methods in the literature, wherever necessary.

Chapter 1 presents a short literature survey of monotone iterative methods as well as fractional differential equations. This chapter also provides basic definitions, properties and formulas which are useful in later chapters.

In **Chapter 2**, an existence and uniqueness theorem for solving a nonlinear fractional order initial value problem of Caputo type of order $q \in (0, 1]$ is proposed using the method of modified quasilinearization. The main theorem has been illustrated numerically using appropriate examples which show that the proposed quasilinearization method is robust and easy to apply.

Chapter 3 proposed an existence and uniqueness theorem for fractional order Volterra population model via an efficient monotone iterative scheme. By coupling spectral method with the proposed iterative scheme, the fractional order integro differential equation is solved numerically. The numerical experiments support the fact that the proposed iterative scheme is efficient than the existing iterative scheme in the literature.

In **Chapter 4** a proof, via monotone quasilinearization method, for the existence and uniqueness of the solution for a two-point nonlinear boundary value problem of fractional order $1 < q < 2$ is proposed. Using the lower and upper solution, two sequences are constructed that converge uniformly, monotonically and quadratically to the unique solution of the problem. An interesting numerical study is also provided to support the proposed theory.

Chapter 5 provides an existence and uniqueness solution of a class of parabolic partial integro-differential equations via a monotone iterative scheme. A bivariate spectral collocation method is also proposed to solve these problems numerically. Finally, the robustness and efficiency of the numerical scheme is illustrated using partial integro differential equation that arises in population dynamics.

Chapter 6 provides a short note on monotone iterative methods available for handling a coupled system of partial differential equation that arises in catalytic converter. Chapter primarily focuses on providing a theoretical justification for the better performance of some monotone methods over other existing monotone methods in the literature.

KEYWORDS: Caputo fractional derivative, Catalytic converter, Finite difference method, Fixed point theorem, Fractional order Riccati equation, Monotone iterative method, Modified Quasilinearization, Ordered Banach space, Parabolic integro differential equation, Quasilinearization, Spectral collocation method, Volterra population model.

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NOTATION

List of symbols

$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of positive real numbers
$\mathbb{R}^-$	the set of negative real numbers
$\mathbb{R}_0^+$	the set of non negative real numbers
D^n	differential operator of order n
${}^cD^n$	Caputo fractional differential operator of order n
J^n	Riemann-Liouville fractional integral operator of order n
$A^k[a,b]$	set of functions with an absolutely continuous $(k-1)^{th}$ derivative
$L_p[a,b]$	$\{f : [a,b] \to \mathbb{R};\ f$ is measurable on $[a,b]$ and $\int_a^b \lvert f(x)\rvert^p \mathrm{d}x < \infty\}$
$C^k[a,b]$	$\{f : [a,b] \to \mathbb{R};\ f$ has continuous k^{th} derivative$\}$
$C^{k_1,k_2}[a,b]$	$\{f : [a,b] \times [c,d] \to \mathbb{R};\ f$ has continuous k_1^{th} and k_2^{th} derivative in 1^{st} and 2^{nd} variable respectively$\}$
$C_p[t_0,T]$	$\{f \in C(t_0,T];\ (t-t_0)^p f \in C[t_0,T]\}$
$E_n(z)$	Mittag-Leffler function
$E_{n_1,n_2}(z)$	two parameter Mittag-Leffler function

CHAPTER 1

INTRODUCTION

1.1. A Brief Literature Review

Various real life problems arising from business, engineering, science and technology naturally lead to differential equations. Most of the time the differential equations arise from real life models are nonlinear in behavior and its closed form solutions are rarely available. Studying the qualitative and quantitative properties of solutions of nonlinear differential equations is not for only theoretical interest, but also has practical relevance. Though there are many well established classical approaches to study the qualitative and quantitative behavior of nonlinear differential equations, studying these properties through an iterative procedure are very relevant due to the importance in developing numerical methods via these iterative schemes. Developing an efficient iterative procedure, studying its convergence analysis and developing numerical schemes based on this iterative procedure to solve the nonlinear differential equations is still an active area of research. Recently, various experiments [**36**] support the fact that fractional order model predicts some of the physical phenomena more accurately than the corresponding classical model. Unfortunately, the methods to obtain the closed form solution are very limited for fractional order differential equations. Hence studying the qualitative and quantitative behavior of fractional differential equations and developing numerical methods to solve the fractional differential equations have more physical relevance. In this section, a short literature review is presented for monotone iterative methods and fractional differential equations. The historical development of monotone iterative methods presented in this section is based on the books [**37, 88, 114, 131**], whereas most of the historical informations on fractional calculus and fractional differential equations are based on the references [**43, 94, 116**].

1.1.1. Monotone Iterative Methods

The monotone iterative methods are one of the classical techniques available in the literature to prove existence and uniqueness of solutions of nonlinear differential equations. As the name suggests, in this approach, the solution of a nonlinear differential equation is approximated by a monotone sequence of functions. Usually these functions are constructed as a solution of sequence of linear differential equations. This approach not only provides proof for an existence-uniqueness theorem and an iterative procedure to approximate the solution, but also supplement with lower and upper bounds for the solution.

The earlier use of the concept of monotone iteration can be traced back to 1893, which is developed by E. Picard [**37**] for proving the existence of solution for the following two-point boundary value problem.

$$y'' + f(t, y) = 0, \quad y(a) = 0, \ y(b) = 0. \tag{1.1}$$

By assuming $f(t, \cdot)$ as an increasing function and along with few additional assumptions, Picard was able to show that the successive iterative scheme produces a monotonic increasing sequence $\{u_n\}$ which converges to the nontrivial solution of Eqn.(1.1). Later independently in 1915, O. Perron used the idea of comparing the solutions of differential inequalities during the study of the following first order Cauchy problem,

$$y' + f(t, y) = 0, \quad y(a) = y_0. \tag{1.2}$$

This idea was further extended by M. Muller for systems in 1926.

It is worth mentioning that the monotone property of $f(t, y)$ in Eqn.(1.1) was relaxed by B. N. Babkin in 1954 by assuming $f(t, y) + Ky$ is an increasing function in y for some $K > 0$. With suitable assumption on the initial guess u_0 and v_0, he showed that the sequences $\{u_n\}$ and $\{v_n\}$ defined recursively by

$$-u_n'' + Ku_n = f(t, u_{n-1}) + Ku_{n-1}, \quad u_n(a) = 0, \ u_n(b) = 0,$$

$$-v_n'' + Kv_n = f(t, v_{n-1}) + Kv_{n-1}, \quad v_n(a) = 0, \ v_n(b) = 0,$$

converge to the unique solution of Eqn.(1.1) monotonically.

A study by G. Scorza Dragoni in 1931 on the two point boundary value problem, a generalization of Eqn.(1.1), was the first work that has considered the role of lower and upper solutions explicitly. More specifically, he obtained conditions for the existence of solution to the following boundary value problem.

$$y'' = f(t, y, y'); \quad y(a) = A, \ y(b) = B. \tag{1.3}$$

By assuming the existence of two functions $u, v \in C^2[a, b]$ such that $u \leq v$ and satisfying

$$u''(t) + f(t, u, x) \geq 0 \quad \text{if } t \in [a, b], \quad x \leq u'(t), \ u(a) \leq A, \ u(b) \leq B;$$

$$v''(t) + f(t, v, x) \leq 0 \quad \text{if } t \in [a, b], \quad x \geq v'(t), \ v(a) \geq A, \ v(b) \geq B,$$

he has shown that Eqn.(1.3) has a solution y such that $u \leq y \leq v$. This significance of u and v led to the study of the existence as well as the construction of lower and upper solutions by K. Ako, R. E. Gaines [**53**] and others.

By merging the idea of lower and upper solutions and successive approximation, K. Schmitt [**125**] proved the existence of a solution to the following boundary value problem

$$\begin{aligned} y'' &= f(t, y, y') \\ a_1 y(a) - a_2 y'(a) &= A_0, \quad a_1 + a_2 > 0 \\ b_1 y(b) + b_2 y'(b) &= B_0, \quad b_1 + b_2 > 0 \end{aligned} \tag{1.4}$$

where $a_i, \ b_i \geq 0$ for $i = 1, 2$ and $a_1 + b_1 > 0$. In his study, Schmidt showed that by considering lower solution as an initial guess, successive iteration produces an increasing sequence $\{u_n\}$ which converges to the solution of Eqn.(1.4). Similarly, by considering upper solution as an initial guess successive iteration produces a decreasing sequence $\{v_n\}$ which converges to the solution of Eqn.(1.4). Moreover, u_n and v_n satisfy

$$u_1 \leq u_2 \leq \cdots \leq u_n \leq v_n \leq \cdots \leq v_2 \leq v_1.$$

It is observed that the sequences constructed by the successive approximation and its variations produce only linear order of convergence. To accelerate the iterative scheme, R.

E. Kalaba [**71**] constructed the sequence of linear problems using the idea of quasilinearization studied by R. E. Bellman [**21**] for handling dynamic programming problems. In [**71**], Kalaba demonstrated the quasilinerization iterative procedure by successfully applying to initial value problems, two point boundary value problems and partial differential equations. Under suitable assumption he is able to show that the quasilinearization iterative scheme converges monotonically and quadratically to the solution of the corresponding differential equation. For example, for the two point boundary value problem

$$y'' = f(t,y), \quad y(0) = 0,\ y(b) = 0, \tag{1.5}$$

he proposed the following iterative scheme

$$y''_{n+1} = f(t,y_n) + f_y(t,y_n)(y_{n+1} - y_n), \quad y_{n+1}(0) = 0,\ y_{n+1}(b) = 0.$$

With suitable assumption on f, he has proved the monotone behavior of $\{y_n\}$ and its quadratic convergence.

In 1964, a different process to linearize the following two-point nonlinear problem was proposed by G. V. Gendzhoyan.

$$y'' + f(t,y,y') = 0, \quad y(a) = 0,\ y(b) = 0. \tag{1.6}$$

Using the lower solution u_0 and upper solution $v_0 \geq u_0$ as the initial guesses the monotone behavior and the convergence of the following iterative schemes to the solution of Eqn.(1.6) was discussed by Gendzhoyan with suitable error bound.

$$-u''_n + l(t)u'_n + k(t)u_n = f(t,u_{n-1},u'_{n-1}) + l(t)u'_{n-1} + k(t)u_{n-1},$$

$$u_n(a) = 0,\ u_n(b) = 0,$$

$$-v''_n + l(t)v'_n + k(t)v_n = f(t,v_{n-1},v'_{n-1}) + l(t)v'_{n-1} + k(t)v_{n-1},$$

$$v_n(a) = 0,\ v_n(b) = 0.$$

Later the monotone quasilinearization method was extended to problems arising in dynamic systems on time scale [**88**], integro differential equations [**8**], functional differential equations [**68**], impulsive differential equations [**51**], stochastic differential equations [**88**], differential algebraic equations [**137**] and differential equations in abstract spaces

[**83**] by many authors. One of the quasilinearization method for proving the existence and uniqueness solution of the following integro differential equation is given below:

$$y'(t) = f(t, y) + \int_{t_0}^{t} K(t, s, y(s))\mathrm{d}s,\ y(t_0) = y_0, \tag{1.7}$$

with $t \in I = [t_0, t_0 + T],\ t_0 \geq 0,\ T > 0$.

Theorem 1.1.1. [**88**] *Assume that*

(i) $f \in C^2[I\times\mathbb{R}, \mathbb{R}], K \in C^2[I\times I\times\mathbb{R}, \mathbb{R}]$ *and* $K(t, s, y)$ *is monotonically nondecreasing in* y *for each fixed* $(t, s) \in I \times I$,

(ii) $u_0, v_0 \in C^1[I, \mathbb{R}]$ *such that* $u_0 \leq v_0$ *and*

$$\begin{aligned} u_0' &\leq f(t, u_0) + \int_{t_0}^{t} K(t, s, u_0(s))\mathrm{d}s, \\ v_0' &\geq f(t, v_0) + \int_{t_0}^{t} K(t, s, v_0(s))\mathrm{d}s \end{aligned}$$

where $u_0(t_0) \leq y_0 \leq v_0(t_0)$,

(iii) $f_{yy}(t, y) \geq 0$ *for each* $t \in I$ *and* $K_{yy}(t, s, y) \geq 0$ *for each* $(t, s) \in I \times I$.

Then there exist monotone sequences $\{u_n\}$ *and* $\{v_n\}$ *which converge uniformly and quadratically to the unique solution of Eqn.*(1.7).

By relaxing the convexity condition, in 1994 V. Lakshmikantham and S. Malek [**86**] have proposed a quadratically convergent quasilinearization scheme for first order initial value problem and the sequence is also monotonic in nature. Later, F. A. McRae [**93**] extended the technique discussed in [**86**] to stochastic initial value problems. More specifically, he considered the following stochastic initial value problem.

$$y'(t, \omega) = f(t, y, \omega) \text{ a.e. on } I, \tag{1.8a}$$

$$y(0, \omega) = y_0(\omega) \tag{1.8b}$$

where $f : I \times \mathbb{R} \times \Omega \to \mathbb{R}$, Ω is a probability measure space $(\Omega, \mathscr{Y}, P)$ and $y_0 : \Omega \to \mathbb{R}$ is a given measurable function. The statement of the main result of F. A. McRae is given in the following.

Theorem 1.1.2. [**93**] *Assume that*

(i) $f(t, y, \cdot)$ *is measurable in probability for all* (t, y), $f(\cdot, y, \cdot)$ *is product measurable for every* y *and* $f(t, \cdot, \omega)$ *is continuous for all* (t, ω),

(ii) $|f(t, y, \omega)| \leq K(t, \omega)$ *on* $I \times \mathbb{R} \times \Omega$ *where* $K : I \times \Omega \to \mathbb{R}_+$ *is measurable in* t *and* $\int_I K(t, \omega)\mathrm{d}t < \infty$ *on* Ω,

(iii) u_0, v_0 *are lower and upper sample solutions of Eqn.*(1.8) *such that* $u_0 \leq v_0$ *on* $I \times \Omega$,

(iv) $f_y(t, y, \omega), f_{yy}(t, y, \omega)$ *exist, are continuous in* y, *product measurable in* (t, ω) *and satisfy* $f_{yy}(t, y, \omega) + 2M(t, \omega) \geq 0$ *where* $M(t, \omega) \geq 0$ *is product measurable in* (t, ω) *and* $\int_I M(t, \omega)\mathrm{d}t < \infty$ *on* Ω.

Then there exists a monotone nondecreasing sequence $\{u_n\}$ *which converges uniformly and monotonically for P-almost all* $\omega \in \Omega$ *to the sample solution* $y(t, \omega)$ *of Eqn.*(1.8) *such that* $u_0(t, \omega) \leq u_1(t, \omega) \leq \cdots u_n(t, \omega) \leq y(t, \omega) \leq v_0(t, \omega)$.

Later, V. Lakshmikantham and N. Shahzad [**87**] generalized the quasilinearization method for an initial value problem when the function involved in the initial value problem admits a decomposition as sum of convex and concave function. One of the generalized quasilinearization method for the following initial value problem due to V. Lakshmikantham and N. Shahzad is given below.

$$y'(t) = F(t, y),\ y(0) = y_0,\ t \in I = [0, T] \tag{1.9}$$

where $F \in C[I \times \mathbb{R}, \mathbb{R}]$ admits a decomposition $F = f + g$.

Theorem 1.1.3. [**87**] *Assume that*

(i) $u_0, v_0 \in C^1[I, \mathbb{R}]$ *such that* $u_0' \leq F(t, u_0)$, $v_0' \geq F(t, v_0)$ *and* $u_0 \leq v_0$ *on* I,

(ii) $F \in C[\Omega, \mathbb{R}]$, f_y, g_y, f_{yy}, g_{yy} *exist and are continuous satisfying* $f_{yy} + \psi_{yy} \leq 0$ *and* $g_{yy} + \phi_{yy} \geq 0$ *on* Ω *where* $\phi, \psi \in C[\Omega, \mathbb{R}]$, $\phi_y, \psi_y, \phi_{yy}, \psi_{yy}$ *exist, are continuous and* $\psi_{yy} \leq 0, \phi_{yy} \geq 0$ *on* Ω *where* $\Omega = \{(t, y) :\ u_0 \leq y \leq v_0,\ t \in I\}$.

Then there exist monotone sequences $\{u_n\}$ *and* $\{v_n\}$ *which converge uniformly to the unique solution of Eqn.*(1.9) *and the convergence is quadratic.*

For more details on generalized quasilinearization method one can refer the work of V. Lakshmikantham and A. S. Vatsala [**88**]. The work of A. Cabada et.al. [**24**] proposed a technique to accelerate the order of convergence of monotone iterative technique for the first order boundary value problem

$$y'(t) = f(t, y), \ ay(0) - by(t_0) = c, \ t \in I = [0, T], \ T > 0, \tag{1.10}$$

where $f \in C[I \times \mathbb{R}, \mathbb{R}]$, $t_0 \in (0, T]$ and $a, b \geq 0$ with $a + b > 0$.

Theorem 1.1.4. [**24**] *Assume that there exist*

(i) $u_0, v_0 \in C^1(I)$ *such that* $u_0 \leq v_0$ *and satisfy*

$$u_0'(t) \leq f(t, u_0), \ au_0(0) - bu_0(t_0) \leq c, \ t \in I,$$

$$v_0'(t) \geq f(t, v_0), \ av_0(0) - bv_0(t_0) \geq c, \ t \in I,$$

(ii) $k \geq 1$ *such that* $\frac{\partial^k f}{\partial y^k}$ *is continuous in* $\Omega = \{(t, y) \in I \times \mathbb{R};\ u_0 \leq y \leq v_0\}$ *and for each* $\xi \in [u_0, v_0]$, $a - b^{\sigma(t_0)} > \delta > 0$, *where* $\sigma(t) = \int_0^t \frac{\partial f}{\partial u}(s, \xi(s))\mathrm{d}s$ *and* $\sigma(T) < \delta < 0$.

Then there exist two monotone sequences $\{u_n\}$ *and* $\{v_n\}$ *which converge uniformly to the extremal solutions* ψ *and* ϕ *of Eqn.*(1.10) *in* $[u_0, v_0]$. *The convergence is of order* k.

When considering the work done on qualitative and quantitative study on nonlinear partial differential equations, it is worth mentioning that vast literature is available using monotone iterative methods. The study of the monotone iterative methods for partial differential equation was initiated around 1950s by M. Nagumo [**114**]. He also extended the idea of lower and upper solutions to quasi subsolution and quasi supersolution to handle nonlinear partial differential equation of the form

$$\Delta u + f(t, u, \nabla u) = 0 \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega.$$

Nagumo's result is based on the assumption that f is a Hölder continuous function satisfying $|f(t, u, v)| \leq B\|v\|^2 + C$ and

$$16MB < 1; \quad M = \max\{\|u_0\|_\infty, \|v_0\|_\infty\}, \tag{1.11}$$

where u_0, v_0 are lower and upper solutions. Later in 1969, F. Tomi improved Nagumo's result by relaxing the condition (1.11). Monotone iterative methods for partial differential equations received more attentions around 1960s after the work of K. Ako [**13**], R. Courant and D. Hilbert [**38**], S. I. Hudjaev [**66**], H. B. Keller and D. S. Cohen [**73**] on nonlinear elliptic differential equations. It is worth mentioning that the work of H. Amann [**14**] and D. H. Sattinger [**124**] have more systematic way of constructing the monotone sequences using lower and upper solutions for the nonlinear partial differential equations. The idea of Sattinger [**124**] for nonlinear parabolic partial differential equations was extended by Pao [**112**] for parabolic partial differential equations with nonlinear boundary conditions. Similar to the study of quasilinearization method for nonlinear ordinary differential equations, quasilinearization method as well as its generalization for nonlinear partial differential equations was also studied in detail by many authors including A. Buică, S. Carl, S. Koksal, K. Heikkilä, V. Lakshmikantham, D. O′Regan, C. V. Pao and A. S. Vatsala.

For instance, one of the existence and uniqueness theorems for the solution of nonlinear parabolic partial integro differential equation via generalized quasilinearization method due to A. S. Vatsala and L. Wang [**140**] is presented below. To provide the result consider the following integro differential equation.

$$
\begin{aligned}
&Lu = f(t,x,u(t,x)) + \int_0^t g(t,x,s,u(s,x))\mathrm{d}s, \quad t \in I = [0,T],\ x \in \Omega \\
(1.12) \qquad &u(0,x) = u_0(x), \quad x \in \Omega \\
&u(t,x)|_{x\in\partial\Omega} = h(t,x)
\end{aligned}
$$

where Ω is a bounded open domain in $\mathbb{R}^n$, $L = \frac{\partial}{\partial t} - A$ is a partial differential operator with

$$A = \sum_{i,j=1}^{N} a_{i,j}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(t,x)\frac{\partial}{\partial x_i} + c(t,x),$$

$a_{i,j}, b_i, c \in C^{\alpha,\alpha/2}(\bar{Q}_T)$, $\sigma_0|\xi|^2 \le \sum_{i,j=1}^{N} a_{i,j}\xi_i\xi_j \le \sigma_1|\xi|^2, \sigma_0, \sigma_1 > 0$ for $(t,x) \in \bar{Q}_T$, $\xi \in \mathbb{R}^n$, $u_0 \in C^{2+\alpha}(\bar{\Omega}), h \in C^{2+\alpha,1+\alpha/2}((0,T)\times\partial\Omega)$ and $u_0(x) = h(0,x)$, $h_t = Au_0 + f(0,x,u_0)$ for $t = 0$ and $x \in \partial\Omega$.

Theorem 1.1.5. [**140**] *Assume that*

(i) $g(t,x,s,u)$ *is monotonically nondecreasing in* u *for each fixed* $(t,x,s) \in I \times \Omega \times I$, $f_{uu} \geq 0$,

(ii) $v_0(t,x)$ *and* $w_0(t,x)$ *satisfy*

$$
\begin{aligned}
Lv_0 &\leq f(t,x,v_0(t,x)) + \int_0^t g(t,x,s,v_0(s,x))\mathrm{d}s, \\
Lw_0 &\geq f(t,x,w_0(t,x)) + \int_0^t g(t,x,s,w_0(s,x))\mathrm{d}s,
\end{aligned}
$$

and

$$v_0(0,x) \leq w_0(0,x),$$

$$v_0(t,x)|_{x\in\partial\Omega} \leq w_0(t,x)|_{x\in\partial\Omega},$$

(iii) $f_u, g_u \in C^{\alpha,\alpha/2}[I \times \Omega]$, *and* f_{uu}, g_{uu} *exist and are continuous such that* $f_{uu} \geq 0$. *There exist functions* $\phi(t,x,s,u)$ *and* $G(t,x,s,u) = g(t,x,s,u) + \phi(t,x,s,u)$ *such that* $G_{uu} \geq 0, \phi_{uu} \geq 0$,

(iv) $G_u(t,x,s,u_1) - \phi_u(t,x,s,u_2) \geq 0$ *for* $v_0 \leq u_1 \leq u_2 \leq w_0$.

Then there exist monotone sequences $\{v_n\}$ *and* $\{w_n\}$ *that converge monotonically and quadratically to the unique solution of Eqn.*(1.12).

One can formulate the solution of a differential and an integro differential equation equivalent to a corresponding fixed point problem of a nonlinear operator in abstract space. Consequently, one can obtain monotone iterative results for differential equations and integro differential equations via fixed point theorems in partially ordered abstract space. Extensive literature (for eg. [**48, 58, 59, 152**]) is available for fixed point theorems and its applications. It is interesting to observe that solutions of fixed point problems via successive iterative scheme can have only first order convergence. To accelerate the iterative procedure, iterative schemes via quasilinearization for solving fixed point problems in abstract space are also available in the literature. Usually these iterative schemes have second order convergence. Solutions of fixed point problems via quasilinearization scheme in abstract space and its applications to differential equations are studied by A.

Buică and R. Precup [**23**], M. A. El-Gebeily et.al. [**54**], V. Lakshmikantham et. al. [**83**] and V. A. Vijesh and K. H. Kumar [**143**], to mention a few.

1.1.2. Fractional Differential Equations

The derivative of non integer order is a consequence of the discussion between Leibnitz and l′Hospital during 1960s. Since that time, the concept of derivative of arbitrary order attracted the attention of many famous mathematicians including Euler, Fourier, Lacroix, Laplace, Laurent, Liouville, Riemann and Weyl. The definition of fractional derivative in terms of an integral was expressed by Laplace and Fourier in 1812 and 1822, respectively. It is worth mentioning that the book entitled "Traité du Calcul Différentiel et du Calcul Intégral" by S. F. Lacroix had a discussion on fractional calculus and showed that $\frac{\mathrm{d}^{1/2}v}{\mathrm{d}v^{1/2}} = \frac{2\sqrt{v}}{\sqrt{\pi}}$. In contrast to the classical derivative, many definitions are proposed for the fractional derivative of a function in the literature. Among them, the most frequently considered definitions include *Caputo-derivative, Grunwall-Letnikov derivative* and *Riemann-Liouville derivative.*

Problems involving fractional derivatives occur naturally in real life. For example, the solution of the well known integral equation of Abel that arises from the tautochronous problem can be expressed as Riemann-Liouville derivative of order 1/2 of a known function. Around mid of 19^{th} century, scientists and engineers suggested that replacing the classical derivative by the fractional derivative may improve the classical mathematical models to predict the physical phenomena more accurately [**116**]. For example, S. Blair proposed a model based on fractional derivative to obtain the relation between stress and strain for viscoelastic material having the property of solids and fluids. Bagley-Torvik concluded that fractional calculus models of viscoelastic materials are consistent with the physical principles governing the behavior of such materials. Around the same time, A. N. Gerasimov [**116**] generalized the basic law of deformation and studied the movement of viscous fluid between two moving surfaces. He successfully solved a partial differential equation involving fractional derivative that arose from his study.

It is observed in the literature that the data obtained from various scientific experiments show better accuracy with the solutions of mathematical models involving fractional order models than their classical counterpart (for eg. [**36, 70**]). This relevance has attracted many researchers to obtain solutions for many important mathematical equations involving fractional derivatives. However, in contrast to the classical derivatives, fractional derivatives of elementary functions need not be an elementary function. Hence, solving the equations involving fractional derivatives is more complicated than the equations involving classical derivatives. Thus the study of existence and uniqueness for the equations involving fractional derivatives plays an important role. In this direction, the understanding of the convergence of iterative methods to the solution of fractional differential equations is very crucial. Further, in numerical analysis, iterative methods play an important role in obtaining approximate solutions, especially for nonlinear problems.

In a few situations, the solutions of linear fractional differential equations can be obtained through methods such as Laplace transform and then expressed in terms of special functions like Fox H function, Mittag-Leffler function etc. Unlike linear fractional problems, applying transform techniques or expressing the solution in terms of special functions may not be possible for nonlinear fractional differential equations. Then iterative methods become relevant as an alternative approach to solve nonlinear fractional models. In this direction, the first existence and uniqueness result for a nonlinear fractional order initial value problem involving Riemann-Liouville derivative was obtained by D. Delbosco and L. Rodino [**40**] by suitably using classical Schauder's fixed point theorem and Banach contraction principle. One of the main theorem of their study is given below.

Theorem 1.1.6. [**40**] *Let* $0 \leq \sigma < q < 1$ *and* $t^\sigma f(t,y)$ *be a continuous function on* $[0,1] \times \mathbb{R}$. *Assume that*

$$|f(t,u) - f(t,v)| \leq \frac{L}{t^\sigma}|u-v|$$

for some positive constant L *independent of* $u, v \in \mathbb{R},\ t \in (0,1]$. *Then the equation*

$$D^q u = f(x,u)$$

has a unique solution.

Later, various methods developed for classical differential equations were suitably modified and applied to prove existence and uniqueness theorems for fractional order initial value problems, fractional order boundary value problems and fractional order integro differential equations. For instance, the monotone iterative methods established for classical problems were studied for fractional order differential equations by Lakshmikantham and his group. One of the existence-uniqueness theorem via monotone iterative method for the nonlinear fractional order initial value problem $D^q(y(t)-y(0)) = f(t,y)$, $y(0) = y_0$ involving Riemann-Liouville derivative is given below.

Theorem 1.1.7. [**89**] *Assume that* $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$ *and*

(i) $u_0, v_0 : [0,T] \to \mathbb{R}$ *be locally Hölder continuous and satisfy*

$$\begin{aligned} D^q(u_0(t) - u_0(0)) &\leq f(t, u_0) \\ D^q(v_0(t) - v_0(0)) &\geq f(t, v_0) \end{aligned}$$

such that $u_0(t) \leq v_0(t),\ 0 \leq t \leq T,$

(ii) $f(t,x) - f(t,y) \geq -M(x-y)$ *wherever* $u_0 \leq y \leq x \leq v_0$ *and* $0 \leq M \leq \frac{1}{T^q\Gamma(1-q)}$.

Then there exist monotone sequences $\{u_n\}$ *and* $\{v_n\}$ *such that* $u_n \to u^*,\ v_n \to v^*$ *as* $n \to \infty$ *uniformly and monotonically on* $[0,T]$ *and* (u^*, v^*) *are extremal solutions of the above initial value problem on* $[0,T]$.

The following theorem due to G. Wang et. al. [**144**] is an extension of monotone iteration method to system of nonlinear fractional order initial value problem involving Riemann-Liouville fractional order derivative.

$$\begin{aligned} &D^q x(t) = f(t,x,y), \quad t \in (0,T], \\ &D^q y(t) = g(t,y,x), \quad t \in (0,T], \qquad (1.13) \\ &t^{1-q}x(t)|_{t=0} = x_0, \quad t^{1-q}y(t)|_{t=0} = y_0, \end{aligned}$$

where $0 < t < \infty, f, g \in C([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), x_0, y_0 \in \mathbb{R}$ satisfying $x_0 \leq y_0$ and D^q is the Riemann-Liouville derivative of order $0 < q \leq 1$.

Theorem 1.1.8. [**144**] *Assume that*

(i) *There exist* $u_0, v_0 \in C_{1-q}[0,T]$ *and* $u_0 \leq v_0$ *such that*

$$D^q u_0 \leq f(t, u_0, v_0), \quad t \in (0,T],$$

$$D^q v_0 \geq g(t, v_0, u_0), \quad t \in (0,T],$$

$$t^{1-q} u_0|_{t=0} \leq x_0, \quad t^{1-q} v_0|_{t=0} \geq y_0,$$

(ii) *There exist constants* $M \in \mathbb{R}$ *and* $N \geq 0$ *such that*

$$\begin{aligned} f(t,x,y) - f(t,\bar{x},\bar{y}) &\geq -M(x-\bar{x}) - N(y-\bar{y}) \\ g(t,x,y) - g(t,\bar{x},\bar{y}) &\geq -M(x-\bar{x}) - N(y-\bar{y}) \end{aligned}$$

where $u_0 \leq \bar{x} \leq x \leq v_0$, $u_0 \leq y \leq \bar{y} \leq v_0$ *and*

$$g(t,y,x) - f(t,x,y) \geq M(x-y) + N(y-x)$$

with $u_0 \leq x \leq y \leq v_0$.

Then, there is $(u^*, v^*) \in [u_0, v_0] \times [u_0, v_0]$ *an extremal solution of the nonlinear Eqn.*(1.13). *Moreover, there exist monotone iterative sequences* $\{u_n\}, \{v_n\} \subset [u_0, v_0]$ *such that* $u_n \to u^*, v_n \to v^*$ *as* $n \to \infty$ *uniformly at* $t \in (0,T]$ *and* $u_0 \leq u_1 \leq \cdots \leq u_n \leq u^* \leq v^* \leq v_n \leq \cdots \leq v_1 \leq v_0$.

Similar to the development of quasilinerization monotone iterative methods for classical differential equations, there are many efforts in developing quasilinearization monotone iterative schemes for various types of fractional order differential equations. One such result for nonlinear fractional order initial value problem of the form ${}^cD^q y(t) = f(t,y) + g(t,y)$, $y(t_0) = y_0$ involving Caputo fractional derivative due to J. V. Devi et. al. [**42**] is given as follows:

Theorem 1.1.9. *Assume that*

(i) $f, g \in C([t_0, T] \times \mathbb{R}, \mathbb{R}), u_0, v_0 \in C_q([t_0, T], \mathbb{R})$ *and*

$$\begin{aligned} {}^cD^q u_0 &\leq f(t, u_0) + g(t, u_0),\ u_0(t_0) \leq y_0 \\ {}^cD^q v_0 &\geq f(t, v_0) + g(t, v_0),\ v_0(t_0) \geq y_0 \end{aligned}$$

$u_0(t) \leq v_0(t)$ *on* $I = [t_0, T]$ *and* $u_0(t_0) \leq y_0 \leq v_0(t_0)$,

(ii) $f_y(t, y),\ g_y(t, y)$ *exist, are respectively decreasing and increasing in* x *for each* t,

$$\begin{aligned} f(t, y) &\geq f(t, x) + f_y(t, x)(y - x),\ y \geq x, \\ g(t, y) &\geq g(t, x) + g_y(t, y)(y - x),\ y \geq x \end{aligned}$$

and

$$\begin{aligned} |f_y(t, y) - f_y(t, x)| &\leq L_1|y - x|, \\ |g_y(t, y) - g_y(t, x)| &\leq L_2|y - x|. \end{aligned}$$

Then there exist monotone sequences $\{u_n\}, \{v_n\}$ *such that* $u_n \to u^*, v_n \to v^*$ *uniformly and monotonically to the unique solution* $u^* = v^* = y$ *of the initial value problem on* I *and the convergence is quadratic.*

Recently various classical techniques are suitably enhanced to study various types of fractional order differential equations. Further theoretical developments like existence-uniqueness results for fractional order versions of initial value problems, boundary value problems, integro differential equations, stochastic differential equations, Darboux problems, $q-$derivative problems, control problems, fuzzy differential equations, functional differential equations and fractional order differential equations in the abstract setting are available in the literature: S. Abbas [**2, 3**], R. P. Agarwal [**9, 10, 150**], D. Bahugana [**18, 72, 81**], M. Benchohra [**1, 22**], S. N. Bora [**28, 29**], A. Cabada [**25, 26**], J. V. Devi [**42**], K. Diethelm [**43, 45**], V. Gejji [**17, 35, 55**], R. K. George [**141, 142**], T. Jankowski [**69**], I. Koca [**80**], V. Lakshmikantham [**84, 85, 90**], J. J. Nieto [**102, 103**], M. Al-Refai [**119**], J. J. Trujillo [**20, 104**], A. S. Vatsala [**89, 117**], G. Wang [**144, 145**].

Similarly, there are various numerical methods that are suitably modified to solve fractional order differential equations. Methods like Adomian decomposition, differential transform, finite difference method, finite element method, homotopy perturbation method, Pade approximation method, predictor corrector method, radial basis function method, spectral method, wavelet method are available in the literature. Few references are mentioned here. Om P. Agarwal [**5, 6, 7**], G. Chandhini [**30**], Li Changpin [**32, 33, 34**], K. Diethelm [**43, 44**], E. H. Doha [**46, 47**], V. S. Erturk [**52**], N. J. Ford [**49, 50**], V. Gejji [**56, 57**], D. Hengfei [**61**], M. H. Heydari [**62, 63**], A. A. Kilbas [**76, 77, 78**], S. Momani [**95, 96, 97**], M. Stynes [**129**], Li Zhu [**155, 156**].

Throughout the proposed thesis various types of monotone iterative methods are studied to prove existence and uniqueness theorem for classical and fractional differential equations and integro differential equations. Based on the proposed monotone iterative methods, numerical techniques have been developed to solve the equations numerically. The proposed iterative scheme for fractional order differential equation is computationally less expensive than certain monotone iterative method available in the recent literature. Based on the proposed monotone iterative method for fractional order integro differential equation, an efficient numerical technique has been developed using spectral method to solve the fractional order Volterra population model. The proposed iterative algorithm is not only efficient in predicting the solution, but also less sensitive to various parameters in the mathematical model, compared to the other iterative methods available in the literature. An alternative approach for obtaining the monotone iteration for fractional order integro differential equation has also been discussed in this thesis. The ideas discussed in this thesis have been extended for handling a class of nonlinear parabolic integro differential equation with initial and boundary conditions. An interesting theoretical justification is also provided to select an efficient iterative method from the existing literature to solve the catalytic converter model.

1.2. Basic Results

In this section, a few basic definitions as well as properties of fractional integral and derivatives that are used in the upcoming chapters are presented. For more details, one can refer [**43, 94, 77, 116**]. Euler's gamma function plays an important role in fractional calculus. Based on the definition of gamma function one can define Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative and Caputo fractional derivative as follows.

Definition 1.2.1. *The Euler's gamma function is denoted by* Γ *and defined as*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t.$$

Definition 1.2.2. *Let* $q \in \mathbb{R}_+$. *For* $f \in L_1[a,b]$, *the Riemann-Liouville fractional integral of order* q *is defined by*

$$J^q f(x) = \frac{1}{\Gamma q} \int_a^x (x-t)^{q-1} f(t) \mathrm{dt}$$

and denoted by $J^q f(x)$. *For* $q = 0$, *we set* $J^0 = I$, *the identity operator.*

Definition 1.2.3. *Let* $q \in \mathbb{R}_+$ *and* $m = \lceil q \rceil$. *Define an operator* D^q *as*

$$D^q f(x) = D^m J^{m-q} f(x) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_a^x (x-t)^{m-q-1} f(t) \mathrm{d}t.$$

$D^q f$ *is known as Riemann-Liouville fractional derivative of the function* f *of order* q. *For* $q = 0$, *we set* $D^0 = I$, *the identity operator.*

Definition 1.2.4. *Let* $q \geq 0$ *and* $m = \lceil q \rceil$. *Define an operator* ${}^cD^q$ *as*

$${}^cD^q f(x) = J^{m-q} D^m f(x).$$

${}^cD^q f$ *is known as Caputo fractional derivative of the function* f *of order* q.

The following theorems provide the relation between Riemann-Liouville fractional integral and derivative and similarly between Riemann-Liouville fractional integral and Caputo fractional derivative.

Theorem 1.2.1. [**43**, p. 30] *Let $q \geq 0$. Then, for every $f \in L_1[a,b]$, $D^q\, J^q f = f$ almost everywhere.*

Theorem 1.2.2. [**43**, p. 39] *Let $q \geq 0$ and $m = \lfloor q \rfloor + 1$. Assume that f is such that $J^{m-q} f \in A^m[a,b]$. Then*

$$J^q\, D^q f(x) = f(x) - \sum_{k=0}^{m-1} \frac{(x-a)^{q-k-1}}{\Gamma(q-k)} \lim_{z \to a_+} D^{m-k-1} J^{m-q} f(z).$$

Specifically, for $0 < q < 1$ we have

$$J^q\, D^q f(x) = f(x) - \frac{(x-a)^{q-1}}{\Gamma(q)} \lim_{z \to a_+} J^{1-q} f(z).$$

Theorem 1.2.3. [**43**, p. 53] *If f is continuous and $q \geq 0$, then ${}^cD^q J^q f = f$.*

Theorem 1.2.4. [**43**, p. 54] *Assume that $q \geq 0, m = \lceil q \rceil$ and $f \in A^m[a,b]$. Then*

$$J^q\, {}^cD^q f(x) = f(x) - \sum_{k=0}^{m-1} \frac{D^k f(a)}{k!} (x-a)^k.$$

Remark 1.2.1. *From Theorem 1.2.1 and Theorem 1.2.3, one can easily assure that Riemann-Liouville fractional integral is right inverse of both Riemann-Liouville fractional derivative as well as Caputo fractional derivative. From Theorem 1.2.2 and Theorem 1.2.4, it is easy to conclude that Caputo fractional derivative generalizes the property of classical derivative, whereas Riemann-Liouville fractional derivative fails.*

1.2.1. Adams Type Predictor-Corrector Method

Consider the initial value problem

$$ {}^cD^q y(t) = f(t,y),\ y(0) = y_0,\ t \in [0,T]. \tag{1.14}$$

The algorithm for predictor-corrector method for the initial value problem is given below. For more details one can refer [**43**] and the references therein. Let y^p_{k+1} denotes the predicted value of y at t_{k+1}, y_{k+1} denotes the corrected value of y at t_{k+1} and h denotes the step size.

$$
\begin{aligned}
y_{k+1}^{p} &= \sum_{j=0}^{m-1} \frac{t_{k+1}^{j}}{j!} y_0^j + \frac{1}{\Gamma q} \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j) \\
y_{k+1} &= \sum_{j=0}^{m-1} \frac{t_{k+1}^{j}}{j!} y_0^j + \frac{1}{\Gamma q} \left(\sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + a_{k+1,k+1} f(t_{k+1}, y_{k+1}^{p}) \right)
\end{aligned}
$$

where $b_{j,k+1} = \frac{h^q}{q}((k+1-j)^q - (k-j)^q)$, $t_j = jh$, $j = 0, 1, \cdots, k+1$ and

$$
a_{j,k+1} = \begin{cases} \frac{h^q}{q(q+1)}(k^{q+1} - (k-q)(k+1)^q) & \text{if} \quad j = 0, \\ \frac{h^q}{q(q+1)}((k-j+2)^{q+1} - 2(k-j+1)^{q+1} + (k-j)^{q+1}) & \text{if} \quad 1 \le j \le k, \\ \frac{h^q}{q(q+1)} & \text{if} \quad j = k+1. \end{cases}
$$

1.2.2. Finite Difference Approximation of Derivatives

The following formulas are used to approximate the derivatives of a function $y(x,t)$ where h and k denote the step size in x and t direction respectively.

- Forward difference approximation of $y_x(x,t)$

$$y_x(x,t) \approx \frac{y(x+h,t) - y(x,t)}{h}$$

- Forward difference approximation of $y_t(x,t)$

$$y_t(x,t) \approx \frac{y(x,t+k) - y(x,t)}{k}$$

- Backward difference approximation of $y_x(x,t)$

$$y_x(x,t) \approx \frac{y(x,t) - y(x-h,t)}{h}$$

- Backward difference approximation of $y_t(x,t)$

$$y_t(x,t) \approx \frac{y(x,t) - y(x,t-k)}{k}$$

1.3. Outline of the Thesis

Chapter 2 provides an interesting existence and uniqueness theorem for the fractional order differential equation via monotone modified quasilinearization method. In particular, the following nonlinear initial value problem

$$ {}^cD^q x(t) = f(t, x(t)) + g(t, x(t)) + h(t, x(t)), \qquad x(t_0) = x_0 \tag{1.15} $$

where $f, g, h \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$ and ${}^cD^q$ is the Caputo fractional derivative of order $q,\ 0 < q \leq 1$ is considered. The nonlinear problem is linearized through modified quasilinearization method. Using the lower and upper solutions, two sequences are constructed that converge uniformly and monotonically to the unique solution of the initial value problem (1.15). One of the main theorems of this chapter is given below.

Theorem 1.3.1. *Let $\alpha \in C([t_0, t_0 + T], \mathbb{R})$, $\beta \in C([\tau_0, \tau_0 + T], \mathbb{R})$, $f, g \in C^{0,1}([t_0, \tau_0 + T] \times \mathbb{R}, \mathbb{R})$ and $h \in C([t_0, \tau_0 + T] \times \mathbb{R}, \mathbb{R})$ where $\tau_0 \geq 0$. Suppose*

(i) ${}^cD^q \alpha(t) \leq f(t, \alpha(t)) + g(t, \alpha(t)) + h(t, \alpha(t)), \quad t_0 \leq t \leq t_0 + T$
${}^cD^q \beta(t) \geq f(t, \beta(t)) + g(t, \beta(t)) + h(t, \beta(t)), \quad \tau_0 \leq t \leq \tau_0 + T$
with $\alpha(t_0) \leq x(s_0) \leq \beta(\tau_0)$ and $t_0 \leq s_0 \leq \tau_0$ where $\alpha(t) \leq \beta(t+\eta_1)$, $t_0 \leq t \leq t_0+T$ and $\eta_1 = \tau_0 - t_0$.

(ii) *$\exists$ two functions $\phi, \psi \in C^{0,1}([t_0, \tau_0 + T] \times \mathbb{R}, \mathbb{R})$ such that ϕ_x and $\phi_x + f_x$ are nondecreasing and ψ_x and $\psi_x + g_x$ are nonincreasing in x for each t.*

(iii) *For each x, $f(t, x)$, $g(t, x)$ and $h(t, x)$ are nondecreasing in t.*

(iv) *For some constant $K > 0$ and each t, $|h(t, x_1) - h(t, x_2)| \leq K|x_1 - x_2|$, $\forall x_1, x_2 \in \mathbb{R}$.*

Then there exist monotone sequences $\{\tilde{\alpha}_n\}$ and $\{\tilde{\beta}_n\}$ which converge uniformly and monotonically to the unique solution of (1.15) with $x(s_0) = x_0$ on $[s_0, s_0+T]$ and the convergence is linear.

Chapter 3 deals with a fractional order integro differential equation of the form

$$ {}^cD^q x(t) = f(t, x, \tilde{x}), \qquad x(0) = x_0 \tag{1.16} $$

where $\tilde{x}(t) = \int_0^t x(s)\mathrm{d}s$, $f \in C([0,T]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and ${}^cD^q$ is the Caputo fractional derivative of order $q \in (0,1]$. An existence and uniqueness result for the initial value problem (1.16) is obtained through a monotone iterative scheme. As an application of the main theorem, existence and uniqueness result is proved for the fractional order Volterra population model

$$(1.17) \qquad {}^cD^q x(t) = ax(t) - bx^2(t) - cx(t)\int_0^t x(s)\mathrm{d}s; \quad x(0) = x_0$$

where $a > 0$ is the birth rate coefficient, $b > 0$ is the intraspecies competition, $c > 0$ is the toxicity coefficient, x_0 is the initial population and $x(t)$ is the population at time t. One of the main theorems of this chapter is given below.

Define $m_1 = \min\limits_{t\in[0,T]}\{\alpha_0, \beta_0\}$, $m_2 = \min\limits_{t\in[0,T]}\{\tilde{\alpha}_0, \tilde{\beta}_0\}$, $M_1 = \max\limits_{t\in[0,T]}\{\alpha_0, \beta_0\}$, $M_2 = \max\limits_{t\in[0,T]}\{\tilde{\alpha}_0, \tilde{\beta}_0\}$ and f_2 and f_3 denote the first order partial derivative of f with respect to the second and third variables respectively. Denote the interval $[0,\infty)$ by $\mathbb{R}_0^+$.

Theorem 1.3.2. *Let $\alpha_0, \beta_0 \in C^1([0,T],\mathbb{R})$ be a coupled lower and upper solutions of* (1.16) *with f, $f_2 \in C([0,T]\times[m_1,M_1]\times[m_2,M_2],\mathbb{R})$ and $-f_3 \in C([0,T]\times[m_1,M_1]\times[m_2,M_2],\mathbb{R}_0^+)$. Then there exist monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that converge uniformly and monotonically to the unique solution of* (1.16) *in* $[\alpha_0, \beta_0]$.

In **Chapter 4**, an existence and uniqueness theorem for the following fractional order two-point nonlinear boundary value problem

$$(1.18a) \qquad -{}^cD^q x(t) = f(t, x(t)); \ t \in (0,1)$$

$$(1.18b) \qquad x(0) - \alpha_0 x'(0) = \gamma_0 \text{ and } x(1) + \alpha_1 x'(1) = \gamma_1,$$

where $f \in C([0,1]\times\mathbb{R},\mathbb{R})$, $\alpha_0 \geq \frac{1}{q-1}$, $\alpha_1 \geq 0$ and ${}^cD^q$ is the Caputo fractional derivative of order $1 < q < 2$ is discussed via a monotone quasilinearization method. The quadratic convergence of the quasilinearization scheme is also proved. One of the main theorems of this chapter is given below.

Define $m_1 = \min\limits_{t\in[0,1]}\{v^0, u^0\}$ and $m_2 = \max\limits_{t\in[0,1]}\{v^0, u^0\}$ and f_x denotes first order partial derivative of f with respect to the second variable and $\mathbb{R}^-$ denotes the interval $(-\infty, 0)$.

Theorem 1.3.3. *Let v^0, $u^0 \in C^2[0,1]$ represent, respectively the lower and upper solutions of* (1.18)*, $f \in C([0,1]\times[m_1,m_2],\mathbb{R})$, $f_x \in C([0,1]\times[m_1,m_2],\mathbb{R}^-)$. Further assume that*

(i) $|f_x(x,y_1) - f_x(x,y_2)| \le M_2|y_1 - y_2|, \quad M_2 > 0,$

(ii) *for each t, $f_x(t,x)$ is nondecreasing in x.*

Then there exist two sequences that converge uniformly and monotonically to the unique solution of the problem (1.18) *in $[v^0, u^0]$ and the order of convergence is quadratic.*

In **Chapter 5**, an existence and uniqueness theorems for a partial integro differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u,\tilde{u}) \quad \text{in } Q, \quad u|_{\partial_p Q} = \phi, \tag{1.19}$$

is provided using a monotone iterative technique, where $Q = (0,1)\times(0,T)$. Further $\partial_p Q = \partial Q \backslash ((0,1)\times\{T\})$ denotes the parabolic boundary of Q and $\tilde{u} = \int_0^t \kappa(t-s)u(x,s)\mathrm{d}s$. Also $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is assumed to be a continuous function and ϕ is the restriction of Φ on $\partial_p Q$ where $\Phi \in C^{2,1}(\overline{Q})$. The proof is based on a fixed point theorem in partially ordered Banach space. This chapter also extends the bivariate spectral collocation method for partial differential equation to solve the above partial integro differential equation. It also provides an alternative proof for the results in Chapter 3.

Define $m = \min\limits_{(x,t)\in\overline{Q}}\{v_0, w_0\}$, $M = \max\limits_{(x,t)\in\overline{Q}}\{v_0, w_0\}$, $\tilde{m} = \min\limits_{(x,t)\in\overline{Q}}\{\tilde{v}_0, \tilde{w}_0\}$ and $\tilde{M} = \max\limits_{(x,t)\in\overline{Q}}\{\tilde{v}_0, \tilde{w}_0\}$. Denote the partial derivative of f with respect to the second and third variable by f_1 and f_2 respectively. Consider the following assumptions:

(i) Let v_0 and w_0 in $C^{2,1}(\overline{Q})$ be an ordered coupled lower and upper solution of (1.19).

(ii) For some $\delta > 0$, $f, f_1, f_2 : C[m-\delta, M+\delta]\times[\tilde{m}-\delta, \tilde{M}+\delta] \to \mathbb{R}$ are continuous functions and for all $s_1 \in [m,M]$, $s_2 \in [\tilde{m},\tilde{M}]$

$$f_1(s_1,s_2) + \lambda \ge 0 \text{ and } f_2(s_1,s_2) \le 0.$$

Under these hypothesis, the main theorem can be stated as,

Theorem 1.3.4. *The parabolic partial integro differential equation (1.19) has a unique solution in* $[v_0, w_0]$. *Moreover, there exist two sequences that converge to the unique solution monotonically.*

In **Chapter 6** an interesting short note on monotone iterative method for the following coupled system of partial differential equations is provided.

$$
\begin{cases}
\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + cu = cv, & t > 0, 0 < x \le l \\
\frac{\partial v}{\partial t} + bv = bu + \lambda \exp(v), & t > 0, 0 < x \le l \\
u(0,t) = \eta(t), & t \ge 0 \\
u(x,0) = u_0(x),\ v(x,0) = v_0(x), & 0 \le x \le l.
\end{cases}
\tag{1.20}
$$

The above equation arises in catalytic converter model. Recently, Linia et.al. [**132**] proposed an alternative monotone iterative procedure to Pao et. al. [**115**]. Though the numerical simulations assure that the iterative scheme in [**132**] produces faster convergence than the method in [**115**], no theoretical justification is provided in [**132**]. In this chapter a theoretical justification is provided that ensures that the iterative scheme in [**132**] always requires less number of iterations than that in [**115**]. One of the main theorems of this chapter is given below.

Theorem 1.3.5. *Let* (u^*, v^*) *be a solution of* (1.20). *If* $(\overline{u}^{(0)}, \overline{v}^{(0)})$ *and* $(\underline{u}^{(0)}, \underline{v}^{(0)})$ *are ordered upper and lower solutions for the Eqn.*(1.20), *then for all* $n \in \mathbb{N}$,

$$(\underline{u}^{(n)}, \underline{v}^{(n)}) \le (\underline{\alpha}^{(n)}, \underline{\beta}^{(n)}) \le (u^*, v^*) \le (\overline{\alpha}^{(n)}, \overline{\beta}^{(n)}) \le (\overline{u}^{(n)}, \overline{v}^{(n)}).$$

CHAPTER 2

A MODIFIED QUASILINEARIZATION METHOD FOR FRACTIONAL ORDER INITIAL VALUE PROBLEM

2.1. Introduction

This chapter discusses the existence and uniqueness of a solution for the following problem.

$$ {}^{c}D^{q}x(t) = F(t, x(t)), \quad x(t_0) = x_0 \tag{2.1} $$

where $F \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$ and ${}^{c}D^{q}$ is the Caputo fractional derivative of order q, $0 < q \leq 1$.

The applications of fractional calculus in various branches of modern science and technology brought considerable attention to the study of fractional differential equations. Recently, quasilinearization method [**41, 42, 148, 149**], has been successfully applied by researchers to prove the existence of solutions for various types of fractional differential equations. The advantage of this method is not only in proving the existence of solution, but also in providing an iterative scheme to obtain an approximate solution. However this method [**41, 42, 148, 149**] rests on the hypothesis that F has a monotone derivative. In several cases neither F is differentiable nor has a monotone derivative. Recently, Devi et al. [**42**] proposed a quasilinearization method where a non-monotone function F has been decomposed into two monotone functions and obtained quadratic convergence. A more generalized form of quasilinearization has been considered in [**41**], where the monotone property has been further relaxed and still quadratic convergence has been obtained. In [**149**] the authors extended the quasilinearization method to functions that are neither differentiable nor have a monotone derivative. Besides, the construction of monotone sequences in the work presented in [**41, 42, 148, 149**] requires to evaluate the partial derivative of F at each iteration. It is interesting to note that in [**41, 42**] the partial derivative should be evaluated at least for two points at each iteration to ensure the quadratic convergence. Even, the theorems proved in [**41, 42**] fail to guarantee the existence of the solution as well as the convergence of the quasilinearization when F is not differentiable. However, in the proposed modified quasilinearization method, the monotone sequences are constructed by evaluating the partial derivative of F only once at specified points. Moreover the proof for the convergence of the monotone sequence with systematic error analysis uses only mild conditions on F unlike in [**42, 148, 149**]. Consequently, the present method is more robust with less computational complexity compared to existing quasilinearization approaches.

The organization of the chapter is as follows. In Section 2, we provide the basic materials relevant to the main theorem. In Section 3, we prove the linear convergence of the modified quasilinearization method with systematic error analysis. The modified approach is illustrated in Section 4 by applying to various examples including fractional order Riccati equation. Our numerical results are also compared with other numerical results obtained using Haar wavelet method [**91, 121**], and the modified homotopy perturbation method [**105**]. We conclude the discussion in Section 5, by stating the merits of the proposed modified quasilinearization method.

2.2. Preliminaries

In this section, we provide some basic definitions and results relevant to the main theorem. First we define the lower and upper solution of the problem (2.1).

Definition 2.2.1. *A function $v \in C([t_0, T], \mathbb{R})$ is called a lower solution of (2.1) if for all $t \in [t_0, T]$*

$$^{c}D^{q}v(t) \leq F(t, v(t)), \quad v(t_0) \leq x(t_0).$$

If the inequalities are reversed then the corresponding solution is called an upper solution.

Next we introduce the one parameter and two parameter Mittag-Leffler functions which play a crucial role in the solution of the following non-homogeneous linear fractional differential equation,

$$^{c}D^{q}x(t) = \lambda x(t) + f(t), \quad x(t_0) = x_0 \tag{2.2}$$

where λ is a real number and $f \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$. Using the Laplace Transform technique, the solution for nonhomogeneous initial value problem (2.2) is obtained as follows

$$x(t) = x_0 E_q(\lambda(t-t_0)^q) + \int_{t_0}^{t} (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) f(s) \mathrm{d}s, \quad t \in [t_0, T] \tag{2.3}$$

where $E_q(t) = \displaystyle\sum_{k=0}^{\infty} \frac{t^k}{\Gamma(qk+1)}$ and $E_{q,q}(t) = \displaystyle\sum_{k=0}^{\infty} \frac{t^k}{\Gamma(qk+q)}$ are the Mittag-Leffler functions of one parameter and two parameter respectively. If $f(t) = 0$ then the solution of the corresponding homogeneous initial value problem is given by

$$x(t) = x_0 E_q(\lambda(t-t_0)^q), \quad t \in [t_0, T]. \tag{2.4}$$

In the following we state Gronwall-type inequality for fractional differential equation without proof. The proof can be found in [**43**].

Remark 2.2.1. *Let $^{c}D^{q}u(t) \le Lu(t)$, $u(t_0) = u_0$ where $u(t) \in C([t_0, T], \mathbb{R}_+)$ and L is a positive constant. Then we have the estimate*

$$u(t) \le u_0 E_q(L(t-t_0)^q) \text{ on } [t_0, T].$$

Note that if $u_0 = 0$, then $u(t) = 0$ identically on $[t_0, T]$.

This section is concluded by stating the following comparison theorem, an important tool in proving the main theorem.

Theorem 2.2.1. *(Theorem 3.1 in [**148**]) Let $v(t), w(t) \in C([t_0, T], \mathbb{R})$. Suppose $F \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$ and*

(i) $^{c}D^{q}v(t) \le F(t, v(t))$,
(ii) $^{c}D^{q}w(t) \ge F(t, w(t))$,
(iii) $F(t, x) - F(t, y) \le L(x - y), \quad x \ge y$ *and* $L > 0$.

Then $v(t_0) \le w(t_0)$ *implies*

$$v(t) \le w(t), \quad t_0 \le t \le T. \tag{2.5}$$

Corollary 2.2.1. *The function* $F(t,u) = \sigma(t)u(t)$ *where* $\sigma(t) \le L$ *is admissible in Theorem 2.2.1 to yield* $u(t) \le 0$, *on* $t_0 \le t \le T$.

Note that the dual of the Corollary 2.2.1 is also valid.

2.3. Convergence Analysis

This section gives the proof for the main theorem which is derived by constructing the monotone sequences using the modified quasilinearization idea. Then the linear convergence of the sequence with systematic error analysis has also been stated with the proof. We will use the notation $[\alpha, \beta]$ to denote the sector $\{x : \alpha(t) \le x(t) \le \beta(t), \forall t\}$. Throughout this section we assume that F has a decomposition of the form $F(t,x) = f(t,x) + g(t,x) + h(t,x)$ where f, g and $h \in C([t_0, \tau_0 + T] \times \mathbb{R}, \mathbb{R})$, $t_0 \ge 0$ and $T > 0$.

Theorem 2.3.1. *Let* $\alpha \in C([t_0, t_0 + T], \mathbb{R})$, $\beta \in C([\tau_0, \tau_0 + T], \mathbb{R})$, $f, g \in C^{0,1}([t_0, \tau_0 + T] \times \mathbb{R}, \mathbb{R})$ *and* $h \in C([t_0, \tau_0 + T] \times \mathbb{R}, \mathbb{R})$ *where* $\tau_0 \ge 0$. *Suppose*

(i) ${}^cD^q\alpha(t) \le f(t,\alpha(t)) + g(t,\alpha(t)) + h(t,\alpha(t)), \quad t_0 \le t \le t_0 + T,$
${}^cD^q\beta(t) \ge f(t,\beta(t)) + g(t,\beta(t)) + h(t,\beta(t)), \quad \tau_0 \le t \le \tau_0 + T$
with $\alpha(t_0) \le x(s_0) \le \beta(\tau_0)$ *and* $t_0 \le s_0 \le \tau_0$ *where* $\alpha(t) \le \beta(t + \eta_1)$, $t_0 \le t \le t_0 + T$ *and* $\eta_1 = \tau_0 - t_0$,

(ii) $\exists$ *two functions* $\phi, \psi \in C^{0,1}([t_0, \tau_0 + T] \times \mathbb{R}, \mathbb{R})$ *such that* ϕ_x *and* $\phi_x + f_x$ *are nondecreasing and* ψ_x *and* $\psi_x + g_x$ *are nonincreasing in* x *for each* t,

(iii) *for each* x, $f(t,x)$, $g(t,x)$ *and* $h(t,x)$ *are nondecreasing in* t,

(iv) *for some constant* $K > 0$ *and each* t, $|h(t,x_1) - h(t,x_2)| \le K|x_1 - x_2|$, $\forall x_1, x_2 \in \mathbb{R}$.

Then there exist monotone sequences $\{\tilde{\alpha}_n\}$ *and* $\{\tilde{\beta}_n\}$ *which converge uniformly and monotonically to the unique solution of (2.1) with* $x(s_0) = x_0$ *on* $[s_0, s_0 + T]$ *and the convergence is linear.*

Proof: Set $\tilde{\beta}_0(t) = \beta(t+\eta_1)$ and $\tilde{\alpha}_0(t) = \alpha(t), t_0 \le t \le t_0 + T$ where $\eta_1 = \tau_0 - t_0$. Then we have $\tilde{\beta}_0(t_0) = \beta(t_0+\eta_1) = \beta(\tau_0) \ge \alpha(t_0) = \tilde{\alpha}_0(t_0)$. Note that

$$\begin{aligned}
{}^cD^q\tilde{\beta}_0(t) &= {}^cD^q\beta(t+\eta_1) \\
&\ge f(t+\eta_1, \beta(t+\eta_1)) + g(t+\eta_1, \beta(t+\eta_1)) + h(t+\eta_1, \beta(t+\eta_1)) \\
&= f(t+\eta_1, \tilde{\beta}_0(t)) + g(t+\eta_1, \tilde{\beta}_0(t)) + h(t+\eta_1, \tilde{\beta}_0(t)) \\
{}^cD^q\tilde{\beta}_0(t) &\ge f(t, \tilde{\beta}_0(t)) + g(t, \tilde{\beta}_0(t)) + h(t, \tilde{\beta}_0(t)).
\end{aligned}$$

Thus $\tilde{\beta}_0(t)$ is an upper solution. In a similar manner we can show that $\tilde{\alpha}_0(t)$ is a lower solution. Set $\eta_2 = s_0 - t_0$. The monotonic sequences of solution can be constructed by solving the following modified quasilinear fractional differential equations.

$$(2.6) \qquad {}^cD^q\tilde{\alpha}_{n+1} = P_1(t, \tilde{\alpha}_{n+1}, \tilde{\alpha}_n, \tilde{\alpha}_0, \tilde{\beta}_0), \quad \tilde{\alpha}_{n+1}(t_0) = x_0$$

where

$$\begin{aligned}
P_1(t, \tilde{\alpha}_{n+1}, \tilde{\alpha}_n, \tilde{\alpha}_0, \tilde{\beta}_0) = f(t+\eta_2, \tilde{\alpha}_n) + g(t+\eta_2, \tilde{\alpha}_n) + h(t+\eta_2, \tilde{\alpha}_n) + [P_x(t+\eta_2, \tilde{\alpha}_0) \\
-\phi_x(t+\eta_2, \tilde{\beta}_0) + G_x(t+\eta_2, \tilde{\beta}_0) - \psi_x(t+\eta_2, \tilde{\alpha}_0) - K](\tilde{\alpha}_{n+1} - \tilde{\alpha}_n)
\end{aligned}$$

where $P(t,x) = f(t,x) + \phi(t,x)$ and $G(t,x) = g(t,x) + \psi(t,x)$.

$$(2.7) \qquad {}^cD^q\tilde{\beta}_{n+1}(t) = G_1(t, \tilde{\beta}_{n+1}, \tilde{\beta}_n, \tilde{\alpha}_0, \tilde{\beta}_0), \quad \tilde{\beta}_{n+1}(t_0) = x_0$$

where

$$\begin{aligned}
G_1(t, \tilde{\beta}_{n+1}, \tilde{\beta}_n, \tilde{\alpha}_0, \tilde{\beta}_0) = f(t+\eta_2, \tilde{\beta}_n) + g(t+\eta_2, \tilde{\beta}_n) + h(t+\eta_2, \tilde{\beta}_n) + [P_x(t+\eta_2, \tilde{\alpha}_0) \\
-\phi_x(t+\eta_2, \tilde{\beta}_0) + G_x(t+\eta_2, \tilde{\beta}_0) - \psi_x(t+\eta_2, \tilde{\alpha}_0) - K](\tilde{\beta}_{n+1} - \tilde{\beta}_n).
\end{aligned}$$

Note that, for the problems (2.6) and (2.7) unique solution exists since the right hand side of these equations satisfy Lipschitz condition.
Next using induction we prove that, for all $n \in \mathbb{N}$,

$$(2.8) \qquad \tilde{\alpha}_0 \le \tilde{\alpha}_1 \le \cdots \le \tilde{\alpha}_n \le \tilde{\beta}_n \le \cdots \le \tilde{\beta}_1 \le \tilde{\beta}_0 \quad \text{on} \quad [t_0, t_0+T].$$

For $n = 1$, we have to show

$$(2.9) \qquad \tilde{\alpha}_0 \le \tilde{\alpha}_1 \le \tilde{\beta}_1 \le \tilde{\beta}_0 \quad \text{on} \quad [t_0, t_0+T].$$

Let $p(t) = \tilde{\alpha}_1 - \tilde{\alpha}_0$. Note that $p(t_0) \geq 0$ and

$$\begin{aligned}
{}^cD^q p(t) &= {}^cD^q \tilde{\alpha}_1 - {}^cD^q \tilde{\alpha}_0 \\
&\geq f(t+\eta_2, \tilde{\alpha}_0) + g(t+\eta_2, \tilde{\alpha}_0) + h(t+\eta_2, \tilde{\alpha}_0) + [P_x(t+\eta_2, \tilde{\alpha}_0) \\
&\quad -\phi_x(t+\eta_2, \tilde{\beta}_0) + G_x(t+\eta_2, \tilde{\beta}_0) - \psi_x(t+\eta_2, \tilde{\alpha}_0) - K] \\
&\quad (\tilde{\alpha}_1 - \tilde{\alpha}_0) - f(t, \tilde{\alpha}_0) - g(t, \tilde{\alpha}_0) - h(t, \tilde{\alpha}_0) \\
{}^cD^q p(t) &\geq [P_x(t+\eta_2, \tilde{\alpha}_0) - \phi_x(t+\eta_2, \tilde{\beta}_0) + G_x(t+\eta_2, \tilde{\beta}_0) \\
&\quad -\psi_x(t+\eta_2, \tilde{\alpha}_0) - K]p(t).
\end{aligned}$$

By Corollary 2.2.1, $p(t) \geq 0$. Thus $\tilde{\alpha}_0 \leq \tilde{\alpha}_1$ on $[t_0, t_0+T]$. Similarly, it is straightforward to show that $\tilde{\beta}_0 \geq \tilde{\beta}_1$ on $[t_0, t_0+T]$. Let $p(t) = \tilde{\beta}_1 - \tilde{\alpha}_1$.

$$\begin{aligned}
{}^cD^q p(t) &= {}^cD^q \tilde{\beta}_1 - {}^cD^q \tilde{\alpha}_1 \\
&= [P_x(t+\eta_2, \tilde{\alpha}_0) - \phi_x(t+\eta_2, \tilde{\beta}_0) + G_x(t+\eta_2, \tilde{\beta}_0) - \psi_x(t+\eta_2, \tilde{\alpha}_0) \\
&\quad -K](\tilde{\beta}_1 - \tilde{\beta}_0 - \tilde{\alpha}_1 + \tilde{\alpha}_0) + [f(t+\eta_2, \tilde{\beta}_0) - f(t+\eta_2, \tilde{\alpha}_0)] \\
&\quad +[g(t+\eta_2, \tilde{\beta}_0) - g(t+\eta_2, \tilde{\alpha}_0)] + [h(t+\eta_2, \tilde{\beta}_0) - h(t+\eta_2, \tilde{\alpha}_0)] \\
&\geq [P_x(t+\eta_2, \tilde{\alpha}_0) - \phi_x(t+\eta_2, \tilde{\beta}_0) + G_x(t+\eta_2, \tilde{\beta}_0) - \psi_x(t+\eta_2, \tilde{\alpha}_0) \\
&\quad -K](\tilde{\beta}_1 - \tilde{\beta}_0 - \tilde{\alpha}_1 + \tilde{\alpha}_0) + \phi(t+\eta_2, \tilde{\alpha}_0) - \phi(t+\eta_2, \tilde{\beta}_0) \\
&\quad +P_x(t+\eta_2, \tilde{\alpha}_0)(\tilde{\beta}_0 - \tilde{\alpha}_0) + \psi(t+\eta_2, \tilde{\alpha}_0) - \psi(t+\eta_2, \tilde{\beta}_0) \\
&\quad +G_x(t+\eta_2, \tilde{\beta}_0)(\tilde{\beta}_0 - \tilde{\alpha}_0) - K(\tilde{\beta}_0 - \tilde{\alpha}_0) \\
&\geq [P_x(t+\eta_2, \tilde{\alpha}_0) - \phi_x(t+\eta_2, \tilde{\beta}_0) + G_x(t+\eta_2, \tilde{\beta}_0) - \psi_x(t+\eta_2, \tilde{\alpha}_0) \\
&\quad -K](\tilde{\beta}_1 - \tilde{\beta}_0 - \tilde{\alpha}_1 + \tilde{\alpha}_0) + \phi_x(t+\eta_2, \tilde{\beta}_0)(\tilde{\alpha}_0 - \tilde{\beta}_0) \\
&\quad +P_x(t+\eta_2, \tilde{\alpha}_0)(\tilde{\beta}_0 - \tilde{\alpha}_0) + \psi_x(t+\eta_2, \tilde{\alpha}_0)(\tilde{\alpha}_0 - \tilde{\beta}_0) \\
&\quad +G_x(t+\eta_2, \tilde{\beta}_0)(\tilde{\beta}_0 - \tilde{\alpha}_0) - K(\tilde{\beta}_0 - \tilde{\alpha}_0) \\
{}^cD^q p(t) &\geq [P_x(t+\eta_2, \tilde{\alpha}_0) - \phi_x(t+\eta_2, \tilde{\beta}_0) + G_x(t+\eta_2, \tilde{\beta}_0) - \psi_x(t+\eta_2, \tilde{\alpha}_0) - K]p(t).
\end{aligned}$$

Then using the condition $p(t_0) = 0$ and Corollary 2.2.1, we obtain $p(t) \geq 0$. Hence $\tilde{\beta}_1 \geq \tilde{\alpha}_1$ on $[t_0, t_0+T]$. Consequently, (2.9) is proved. Assume (2.8) is true for k;

$$\text{(2.10)} \qquad \text{i.e.,} \quad \tilde{\alpha}_0 \le \tilde{\alpha}_1 \le \cdots \le \tilde{\alpha}_k \le \tilde{\beta}_k \le \cdots \le \tilde{\beta}_1 \le \tilde{\beta}_0 \quad \text{on} \quad [t_0, t_0+T].$$

Then it is enough to show that

$$\text{(2.11)} \qquad \tilde{\alpha}_k \le \tilde{\alpha}_{k+1} \le \tilde{\beta}_{k+1} \le \tilde{\beta}_k \quad \text{on} \quad [t_0, t_0+T].$$

Let $p(t) = \tilde{\alpha}_{k+1} - \tilde{\alpha}_k$. Note that $p(t_0) = 0$ and

$$\begin{aligned}
{}^cD^q p(t) &= {}^cD^q \tilde{\alpha}_{k+1} - {}^cD^q \tilde{\alpha}_k \\
&= [P_x(t+\eta_2, \tilde{\alpha}_0) - \phi_x(t+\eta_2, \tilde{\beta}_0) + G_x(t+\eta_2, \tilde{\beta}_0) - \psi_x(t+\eta_2, \tilde{\alpha}_0) \\
&\quad -K](\tilde{\alpha}_{k+1} - 2\tilde{\alpha}_k + \tilde{\alpha}_{k-1}) + [f(t+\eta_2, \tilde{\alpha}_k) - f(t+\eta_2, \tilde{\alpha}_{k-1})] \\
&\quad +[g(t+\eta_2, \tilde{\alpha}_k) - g(t+\eta_2, \tilde{\alpha}_{k-1})] + [h(t+\eta_2, \tilde{\alpha}_k) - h(t+\eta_2, \tilde{\alpha}_{k-1})] \\
&\ge [P_x(t+\eta_2, \tilde{\alpha}_0) - \phi_x(t+\eta_2, \tilde{\beta}_0) + G_x(t+\eta_2, \tilde{\beta}_0) - \psi_x(t+\eta_2, \tilde{\alpha}_0) \\
&\quad -K](\tilde{\alpha}_{k+1} - 2\tilde{\alpha}_k + \tilde{\alpha}_{k-1}) + \phi(t+\eta_2, \tilde{\alpha}_{k-1}) - \phi(t+\eta_2, \tilde{\alpha}_k) \\
&\quad +P_x(t+\eta_2, \tilde{\alpha}_{k-1})(\tilde{\alpha}_k - \tilde{\alpha}_{k-1}) + \psi(t+\eta_2, \tilde{\alpha}_{k-1}) - \psi(t+\eta_2, \tilde{\alpha}_k) \\
&\quad +G_x(t+\eta_2, \tilde{\alpha}_k)(\tilde{\alpha}_k - \tilde{\alpha}_{k-1}) - K(\tilde{\alpha}_k - \tilde{\alpha}_{k-1}) \\
&\ge [P_x(t+\eta_2, \tilde{\alpha}_0) - \phi_x(t+\eta_2, \tilde{\beta}_0) + G_x(t+\eta_2, \tilde{\beta}_0) - \psi_x(t+\eta_2, \tilde{\alpha}_0) \\
&\quad -K](\tilde{\alpha}_{k+1} - \tilde{\alpha}_k) + [P_x(t+\eta_2, \tilde{\alpha}_{k-1}) - P_x(t+\eta_2, \tilde{\alpha}_0)](\tilde{\alpha}_k - \tilde{\alpha}_{k-1}) \\
&\quad +[G_x(t+\eta_2, \tilde{\alpha}_k) - G_x(t+\eta_2, \tilde{\beta}_0)](\tilde{\alpha}_k - \tilde{\alpha}_{k-1}) + \phi_x(t+\eta_2, \tilde{\beta}_0) \\
&\quad (\tilde{\alpha}_k - \tilde{\alpha}_{k-1}) + \psi_x(t+\eta_2, \tilde{\alpha}_0)(\tilde{\alpha}_k - \tilde{\alpha}_{k-1}) - \phi_x(t+\eta_2, \tilde{\beta}_0) \\
&\quad (\tilde{\alpha}_k - \tilde{\alpha}_{k-1}) - \psi_x(t+\eta_2, \tilde{\alpha}_0)(\tilde{\alpha}_k - \tilde{\alpha}_{k-1}) \\
{}^cD^q p(t) &\ge [P_x(t+\eta_2, \tilde{\alpha}_0) - \phi_x(t+\eta_2, \tilde{\beta}_0) + G_x(t+\eta_2, \tilde{\beta}_0) - \psi_x(t+\eta_2, \tilde{\alpha}_0) \\
&\quad -K]p(t).
\end{aligned}$$

Again by Corollary 2.2.1, $p(t) \ge 0 \Rightarrow \tilde{\alpha}_{k+1} \ge \tilde{\alpha}_k$ on $[t_0, t_0+T]$. In a similar way we can prove that $\tilde{\beta}_k \ge \tilde{\beta}_{k+1}$ on $[t_0, t_0+T]$. Let $p(t) = \tilde{\beta}_{k+1} - \tilde{\alpha}_{k+1}$.

$$
\begin{aligned}
{}^cD^q p(t) &= {}^cD^q \tilde{\beta}_{k+1} - {}^cD^q \tilde{\alpha}_{k+1} \\
&= [P_x(t+\eta_2, \tilde{\alpha}_0) - \phi_x(t+\eta_2, \tilde{\beta}_0) + G_x(t+\eta_2, \tilde{\beta}_0) - \psi_x(t+\eta_2, \tilde{\alpha}_0) \\
&\quad -K](\tilde{\beta}_{k+1} - \tilde{\beta}_k - \tilde{\alpha}_{k+1} + \tilde{\alpha}_k) + [f(t+\eta_2, \tilde{\beta}_k) - f(t+\eta_2, \tilde{\alpha}_k)] \\
&\quad +[g(t+\eta_2, \tilde{\beta}_k) - g(t+\eta_2, \tilde{\alpha}_k)] + [h(t+\eta_2, \tilde{\beta}_k) - h(t+\eta_2, \tilde{\alpha}_k)] \\
&\geq [P_x(t+\eta_2, \tilde{\alpha}_0) - \phi_x(t+\eta_2, \tilde{\beta}_0) + G_x(t+\eta_2, \tilde{\beta}_0) - \psi_x(t+\eta_2, \tilde{\alpha}_0) \\
&\quad -K](\tilde{\beta}_{k+1} - \tilde{\beta}_k - \tilde{\alpha}_{k+1} + \tilde{\alpha}_k) + \phi(t+\eta_2, \tilde{\alpha}_k) - \phi(t+\eta_2, \tilde{\beta}_k) \\
&\quad +P_x(t+\eta_2, \tilde{\alpha}_k)(\tilde{\beta}_k - \tilde{\alpha}_k) + \psi(t+\eta_2, \tilde{\alpha}_k) - \psi(t+\eta_2, \tilde{\beta}_k) \\
&\quad +G_x(t+\eta_2, \tilde{\beta}_k)(\tilde{\beta}_k - \tilde{\alpha}_k) - K(\tilde{\beta}_k - \tilde{\alpha}_k) \\
&\geq [P_x(t+\eta_2, \tilde{\alpha}_0) - \phi_x(t+\eta_2, \tilde{\beta}_0) + G_x(t+\eta_2, \tilde{\beta}_0) - \psi_x(t+\eta_2, \tilde{\alpha}_0) \\
&\quad -K](\tilde{\beta}_{k+1} - \tilde{\alpha}_{k+1}) + [P_x(t+\eta_2, \tilde{\alpha}_k) - P_x(t+\eta_2, \tilde{\alpha}_0)](\tilde{\beta}_k - \tilde{\alpha}_k) \\
&\quad +[G_x(t+\eta_2, \tilde{\beta}_k) - G_x(t+\eta_2, \tilde{\beta}_0)](\tilde{\beta}_k - \tilde{\alpha}_k) + \phi_x(t+\eta_2, \tilde{\beta}_0) \\
&\quad (\tilde{\beta}_k - \tilde{\alpha}_k) + \psi_x(t+\eta_2, \tilde{\alpha}_0)(\tilde{\beta}_k - \tilde{\alpha}_k) - \phi_x(t+\eta_2, \tilde{\beta}_0)(\tilde{\beta}_k - \tilde{\alpha}_k) \\
&\quad -\psi_x(t+\eta_2, \tilde{\alpha}_0)(\tilde{\beta}_k - \tilde{\alpha}_k) \\
{}^cD^q p(t) &\geq [P_x(t+\eta_2, \tilde{\alpha}_0) - \phi_x(t+\eta_2, \tilde{\beta}_0) + G_x(t+\eta_2, \tilde{\beta}_0) - \psi_x(t+\eta_2, \tilde{\alpha}_0) \\
&\quad -K]p(t).
\end{aligned}
$$

Then using the condition $p(t_0) = 0$ and Corollary 2.2.1, we obtain $p(t) \geq 0$. Hence $\tilde{\beta}_{k+1} \geq \tilde{\alpha}_{k+1}$ on $[t_0, t_0+T]$. Consequently, (2.11) is proved.

It is clear that the sequences $\{\tilde{\alpha}_n\}$ and $\{\tilde{\beta}_n\}$ are uniformly bounded and monotone on $[t_0, t_0+T]$. We can also show that the sequences $\{\tilde{\alpha}_n(t)\}$ and $\{\tilde{\beta}_n(t)\}$ are equicontinuous on $[t_0, t_0+T]$. Hence by Ascoli-Arzela's Theorem and using the property of monotonicity, $\{\tilde{\alpha}_n\}$ and $\{\tilde{\beta}_n\}$ converge uniformly to ρ_1 and ρ_2, respectively. Using the equivalent integral representation of (2.6) and (2.7) one can easily prove that ρ_1 and ρ_2 are solutions of the initial value problem

$$
{}^cD^q \tilde{x}(t) = F(t+\eta_2, \tilde{x}(t)), \quad \tilde{x}(t_0) = x_0. \tag{2.12}
$$

From the hypotheses it is clear that f, g and h are Lipschitz and $f_x(t,x)$ and $g_x(t,x)$ are bounded on the sector $[\tilde{\alpha}_0, \tilde{\beta}_0]$. Hence it can be concluded that $\rho_1 = \rho_2 = \tilde{x}$. Thus (2.12) has a unique solution. Using the change of variable $s = t + \eta_2$, (2.12) is equivalent to

$$ {}^cD^q x(s) = F(s, x(s)), \quad x(s_0) = x_0. \tag{2.13} $$

To prove the linear convergence of the modified quasilinearization, define

$k_1 = \sup\limits_{\substack{\alpha\in[\tilde{\alpha}_0,\tilde{\beta}_0] \\ t\in[t_0,t_0+T]}} |f_x(t+\eta_2, \alpha)|$, $k_1' = \sup_{t\in[t_0,t_0+T]}(\phi(t+\eta_2,\tilde{\beta}_0) - \phi(t+\eta_2,\tilde{\alpha}_0))$,

$k_2 = \sup\limits_{\substack{\alpha\in[\tilde{\alpha}_0,\tilde{\beta}_0] \\ t\in[t_0,t_0+T]}} |g_x(t+\eta_2, \alpha)|$, $k_2' = \sup_{t\in[t_0,t_0+T]}(\psi(t+\eta_2,\tilde{\alpha}_0) - \psi(t+\eta_2,\tilde{\beta}_0))$,

$p_n = \tilde{x} - \tilde{\alpha}_n$, $r_n = \tilde{\beta}_n - \tilde{x}$, $|p_n|_0 = \sup\limits_{t\in[t_0,t_0+T]} |p_n(t)|$, $|r_n|_0 = \sup\limits_{t\in[t_0,t_0+T]} |r_n(t)|$ for all $n \in \mathbb{N}$.

Then

$$
\begin{aligned}
{}^cD^q p_{n+1}(t) &= {}^cD^q \tilde{x}(t) - {}^cD^q \tilde{\alpha}_{n+1}(t) \\
&= [P_x(t+\eta_2,\tilde{\alpha}_0) - \phi_x(t+\eta_2,\tilde{\beta}_0) + G_x(t+\eta_2,\tilde{\beta}_0) - \psi_x(t+\eta_2,\tilde{\alpha}_0) \\
&\quad -K](\tilde{\alpha}_n - \tilde{\alpha}_{n+1}) + [f(t+\eta_2,\tilde{x}) - f(t+\eta_2,\tilde{\alpha}_n)] \\
&\quad +[g(t+\eta_2,\tilde{x}) - g(t+\eta_2,\tilde{\alpha}_n)] + [h(t+\eta_2,\tilde{x}) - h(t+\eta_2,\tilde{\alpha}_n)] \\
&= [P_x(t+\eta_2,\tilde{\alpha}_0) - \phi_x(t+\eta_2,\tilde{\beta}_0) + G_x(t+\eta_2,\tilde{\beta}_0) - \psi_x(t+\eta_2,\tilde{\alpha}_0) \\
&\quad -K](p_{n+1} - p_n) + \int_0^1 f_x(t+\eta_2, \theta\tilde{x} + (1-\theta)\tilde{\alpha}_n)(\tilde{x} - \tilde{\alpha}_n)\mathrm{d}\theta \\
&\quad + \int_0^1 g_x(t+\eta_2, \theta\tilde{x} + (1-\theta)\tilde{\alpha}_n)(\tilde{x} - \tilde{\alpha}_n)\mathrm{d}\theta + K(\tilde{x} - \tilde{\alpha}_n) \\
&= [P_x(t+\eta_2,\tilde{\alpha}_0) - \phi_x(t+\eta_2,\tilde{\beta}_0) + G_x(t+\eta_2,\tilde{\beta}_0) - \psi_x(t+\eta_2,\tilde{\alpha}_0) \\
&\quad -K]p_{n+1} + \Bigg[\int_0^1 f_x(t+\eta_2, \theta\tilde{x} + (1-\theta)\tilde{\alpha}_n)\mathrm{d}\theta \\
&\quad + \int_0^1 g_x(t+\eta_2, \theta\tilde{x} + (1-\theta)\tilde{\alpha}_n)\mathrm{d}\theta - P_x(t+\eta_2,\tilde{\alpha}_0) \\
&\quad +\phi_x(t+\eta_2,\tilde{\beta}_0) - G_x(t+\eta_2,\tilde{\beta}_0) + \psi_x(t+\eta_2,\tilde{\alpha}_0) + 2K\Bigg]p_n(t) \\
{}^cD^q p_{n+1}(t) &\leq M p_n(t) + M' p_{n+1}(t)
\end{aligned}
$$

where

$$
\begin{aligned}
M' &= \sup_{t\in[t_0,t_0+T]} \Big| P_x(t+\eta_2,\tilde{\alpha}_0) - \phi_x(t+\eta_2,\tilde{\beta}_0) + G_x(t+\eta_2,\tilde{\beta}_0) \\
&\qquad -\psi_x(t+\eta_2,\tilde{\alpha}_0) - K\Big| \\
M &= (2(k_1+k_2)+k_1'+k_2'+2K)
\end{aligned}
$$

$$
{}^cD^q p_{n+1}(t) \leq M|p_n|_0 + M'p_{n+1}(t). \tag{2.14}
$$

From the inequality (2.14), we get

$$
\begin{aligned}
p_{n+1}(t) &\leq M|p_n|_0 \int_{t_0}^{t} (t-s)^{q-1} E_{q,q}(M'(t-s)^q)\mathrm{d}s \\
&\leq \frac{M|p_n|_0 (t-t_0)^q}{q} E_{q,q}(M'(t-t_0)^q) \\
&\leq \frac{M|p_n|_0 T^q}{q} E_{q,q}(M'T^q) \\
p_{n+1}(t) &\leq |p_n|_0 N
\end{aligned}
$$

where $N = \dfrac{MT^q}{q} E_{q,q}(M'T^q)$. Hence we have,

$$
|p_{n+1}|_0 \leq N|p_n|_0. \tag{2.15}
$$

A similar calculation shows that

$$
|r_{n+1}|_0 \leq N|r_n|_0. \tag{2.16}
$$

Hence the Theorem.

Remark 2.3.1. *If $t_0 = \tau_0$ in Theorem 2.3.1, then the hypothesis (iii) can be omitted.*

Remark 2.3.2. *An interesting observation from the proof of Theorem 2.3.1 is that the condition (ii) can be replaced as follows. Suppose $f, g \in C^{0,2}([t_0, \tau_0 + T] \times \mathbb{R}, \mathbb{R})$ and $\exists$ two functions $\phi, \psi \in C^{0,2}([t_0, \tau_0 + T] \times \mathbb{R}, \mathbb{R})$ such that $\phi_{xx}(t,x) \geq 0, \phi_{xx}(t,x) + f_{xx}(t,x) \geq 0, \psi_{xx}(t,x) \leq 0$ and $\psi_{xx}(t,x) + g_{xx}(t,x) \leq 0$.*

Corollary 2.3.1. *Let $\alpha, \beta \in C([t_0, t_0 + T], \mathbb{R})$ and $f \in C^{0,1}([t_0, t_0 + T] \times \mathbb{R}, \mathbb{R})$. Suppose*

(i) ${}^cD^q\alpha(t) \le f(t,\alpha(t)), \quad t_0 \le t \le t_0+T,$

${}^cD^q\beta(t) \ge f(t,\beta(t)), \quad t_0 \le t \le t_0+T$

with $\alpha(t_0) \le x_0 \le \beta(t_0)$ *where* $\alpha(t) \le \beta(t)$, $t_0 \le t \le t_0+T$,

(ii) $\exists$ *a function* $\phi \in C^{0,1}([t_0,t_0+T]\times\mathbb{R},\mathbb{R})$ *such that* ϕ_x *and* ϕ_x+f_x *are nondecreasing in* x *for each* t.

Then there exist monotone sequences $\{\alpha_n\}$ *and* $\{\beta_n\}$ *which converge uniformly and monotonically to the unique solution of* ${}^cD^q x(t) = f(t,x(t))$ *with* $x(t_0)=x_0$ *on* $[t_0,t_0+T]$ *and the convergence is linear.*

Proof: For the choice $g \equiv \psi \equiv 0$, $h \equiv 0$, $\tilde{\alpha_0} = \alpha_0 = \alpha$ and $\tilde{\beta_0} = \beta_0 = \beta$ all the hypotheses of Theorem 2.3.1 are satisfied. Hence the initial value problem

$$(2.17) \qquad {}^cD^q x(t) = f(t,x(t)), \qquad x(t_0)=x_0$$

has a unique solution in $[\alpha,\beta]$. Moreover the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ defined by the solutions of the following linear fractional differential equation converge uniformly and monotonically to the unique solution of the initial value problem 2.17.

$${}^cD^q\alpha_{n+1}(t) = f(t,\alpha_n) + [P_x(t,\alpha_0) - \phi_x(t,\beta_0)](\alpha_{n+1}-\alpha_n), \quad \alpha_{n+1}(t_0)=x_0,$$

$${}^cD^q\beta_{n+1}(t) = f(t,\beta_n) + [P_x(t,\alpha_0) - \phi_x(t,\beta_0)](\beta_{n+1}-\beta_n), \quad \beta_{n+1}(t_0)=x_0$$

where $P(t,x) = f(t,x)+\phi(t,x)$.

Corollary 2.3.2. *Let* $\alpha,\beta \in C([t_0,t_0+T],\mathbb{R})$ *and* $g \in C^{0,1}([t_0,t_0+T]\times\mathbb{R},\mathbb{R})$. *Suppose*

(i) ${}^cD^q\alpha(t) \le g(t,\alpha(t)), \quad t_0 \le t \le t_0+T,$

${}^cD^q\beta(t) \ge g(t,\beta(t)), \quad t_0 \le t \le t_0+T$

with $\alpha(t_0) \le x_0 \le \beta(t_0)$ *where* $\alpha(t) \le \beta(t)$, $t_0 \le t \le t_0+T$,

(ii) $\exists$ *a function* $\psi \in C^{0,1}([t_0,t_0+T]\times\mathbb{R},\mathbb{R})$ *such that* ψ_x *and* ψ_x+g_x *are nonincreasing in* x *for each* t.

Then there exist monotone sequences $\{\alpha_n\}$ *and* $\{\beta_n\}$ *which converge uniformly and monotonically to the unique solution of* ${}^cD^q x(t) = g(t,x(t))$ *with* $x(t_0)=x_0$ *on* $[t_0,t_0+T]$ *and the convergence is linear.*

Proof: For the choice $f \equiv \phi \equiv 0$, $h \equiv 0$, $\tilde{\alpha_0} = \alpha_0 = \alpha$ and $\tilde{\beta_0} = \beta_0 = \beta$ all the hypotheses of Theorem 2.3.1 are satisfied. Hence the initial value problem

$$ {}^cD^q x(t) = g(t, x(t)), \qquad x(t_0) = x_0 \tag{2.18}$$

has a unique solution in $[\alpha, \beta]$. Moreover the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ defined by the solutions of the following linear fractional differential equation converge uniformly and monotonically to the unique solution of the initial value problem 2.18.

$$ {}^cD^q \alpha_{n+1}(t) = g(t, \alpha_n) + [G_x(t, \beta_0) - \psi_x(t, \alpha_0)](\alpha_{n+1} - \alpha_n), \quad \alpha_{n+1}(t_0) = x_0,$$

$$ {}^cD^q \beta_{n+1}(t) = g(t, \beta_n) + [G_x(t, \beta_0) - \psi_x(t, \alpha_0)](\beta_{n+1} - \beta_n), \quad \beta_{n+1}(t_0) = x_0$$

where $G(t, x) = g(t, x) + \psi(t, x)$.

Corollary 2.3.3. *Let $\alpha, \beta \in C([t_0, t_0 + T], \mathbb{R})$ and $h \in C([t_0, t_0 + T] \times \mathbb{R}, \mathbb{R})$. Suppose*

(i) ${}^cD^q \alpha(t) \leq h(t, \alpha(t)), \quad t_0 \leq t \leq t_0 + T,$
${}^cD^q \beta(t) \geq h(t, \beta(t)), \quad t_0 \leq t \leq t_0 + T$
with $\alpha(t_0) \leq x_0 \leq \beta(t_0)$ where $\alpha(t) \leq \beta(t)$, $t_0 \leq t \leq t_0 + T$,

(ii) *for some constant $K > 0$ and each t, $|h(t, x_1) - h(t, x_2)| \leq K|x_1 - x_2|$, $\forall x_1, x_2 \in \mathbb{R}$.*

Then there exist monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ which converge uniformly and monotonically to the unique solution of ${}^cD^q x(t) = h(t, x(t))$ with $x(t_0) = x_0$ on $[t_0, t_0 + T]$ and the convergence is linear.

Proof: For the choice $f \equiv g \equiv \phi \equiv \psi \equiv 0$, $\tilde{\alpha_0} = \alpha_0 = \alpha$ and $\tilde{\beta_0} = \beta_0 = \beta$ all the hypotheses of Theorem 2.3.1 are satisfied. Hence the initial value problem

$$ {}^cD^q x(t) = h(t, x(t)), \qquad x(t_0) = x_0 \tag{2.19}$$

has a unique solution in $[\alpha, \beta]$. Moreover the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ defined by the solutions of the following linear fractional differential equation converge uniformly and monotonically to the unique solution of the initial value problem 2.19.

$$\begin{aligned} {}^cD^q \alpha_{n+1}(t) &= h(t, \alpha_n) - K(\alpha_{n+1} - \alpha_n), \quad \alpha_{n+1}(t_0) = x_0, \\ {}^cD^q \beta_{n+1}(t) &= h(t, \beta_n) - K(\beta_{n+1} - \beta_n), \quad \beta_{n+1}(t_0) = x_0. \end{aligned}$$

Corollary 2.3.4. *Let $\alpha, \beta \in C([t_0, t_0+T], \mathbb{R})$ and $f, g \in C^{0,1}([t_0, t_0+T] \times \mathbb{R}, \mathbb{R})$. Suppose*

(i) ${}^cD^q\alpha(t) \leq f(t, \alpha(t)) + g(t, \alpha(t)), \quad t_0 \leq t \leq t_0 + T,$
${}^cD^q\beta(t) \geq f(t, \beta(t)) + g(t, \beta(t)), \quad \tau_0 \leq t \leq \tau_0 + T$
with $\alpha(t_0) \leq x_0 \leq \beta(t_0)$ where $\alpha(t) \leq \beta(t)$, $t_0 \leq t \leq t_0 + T$,

(ii) *$\exists$ two functions $\phi, \psi \in C^{0,1}([t_0, t_0 + T] \times \mathbb{R}, \mathbb{R})$ such that ϕ_x and $\phi_x + f_x$ are nondecreasing and ψ_x and $\psi_x + g_x$ are nonincreasing in x for each t.*

Then there exist monotone sequences $\{\tilde{\alpha}_n\}$ and $\{\tilde{\beta}_n\}$ which converge uniformly and monotonically to the unique solution of ${}^cD^qx(t) = f(t, x(t)) + g(t, x(t))$ with $x(t_0) = x_0$ on $[t_0, t_0 + T]$ and the convergence is linear.

Proof: For the choice $h \equiv 0$, $\tilde{\alpha_0} = \alpha_0 = \alpha$ and $\tilde{\beta_0} = \beta_0 = \beta$ all the hypotheses of the Theorem 2.3.1 are satisfied. Hence the initial value problem

$$
{}^cD^qx(t) = f(t, x(t)) + g(t, x(t)), \qquad x(t_0) = x_0 \tag{2.20}
$$

has a unique solution in $[\alpha, \beta]$. Moreover the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ defined by the solutions of the following linear fractional differential equation converge uniformly and monotonically to the unique solution of the initial value problem (2.20).

$$
\begin{aligned}
{}^cD^q\alpha_{n+1}(t) &= f(t, \alpha_n) + g(t, \alpha_n) + [P_x(t, \alpha_0) - \phi_x(t, \beta_0) + G_x(t, \beta_0) \\
&\quad -\psi_x(t, \alpha_0)](\alpha_{n+1} - \alpha_n), \qquad \alpha_{n+1}(t_0) = x_0, \\
{}^cD^q\beta_{n+1}(t) &= f(t, \beta_n) + g(t, \beta_n) + [P_x(t, \alpha_0) - \phi_x(t, \beta_0) + G_x(t, \beta_0) \\
&\quad -\psi_x(t, \alpha_0)](\beta_{n+1} - \beta_n), \qquad \beta_{n+1}(t_0) = x_0.
\end{aligned}
$$

Remark 2.3.3. *The other possible corollaries of combination f, h and g, h are similar to Corollary 2.3.4 and hence not discussed here.*

We will conclude this section by stating a generalized version of quasilinearization which will unify the existing method presented in [**41, 42, 148**]. This Theorem will be stated in the setting of the function space

$$
C_p[t_0, T] = \{u \in C((t_0, T], \mathbb{R}) : (t - t_0)^p u \in C([t_0, T], \mathbb{R})\}, \quad p = 1 - q.
$$

Theorem 2.3.2. *Let* $\alpha \in C_p([t_0, t_0+T])$, $\beta \in C_p([\tau_0, \tau_0+T])$, $f, g \in C^{0,1}([t_0, \tau_0+T]\times\mathbb{R}, \mathbb{R})$ *where* $\tau_0 \geq 0$. *Suppose*

(i) ${}^cD^q\alpha(t) \leq f(t, \alpha(t)) + g(t, \alpha(t)), \quad t_0 \leq t \leq t_0 + T,$
${}^cD^q\beta(t) \geq f(t, \beta(t)) + g(t, \beta(t)), \quad \tau_0 \leq t \leq \tau_0 + T$
with $\alpha(t_0) \leq x(s_0) \leq \beta(\tau_0)$ *and* $t_0 \leq s_0 \leq \tau_0$ *where* $\alpha(t) \leq \beta(t+\eta_1)$, $t_0 \leq t \leq t_0+T$ *and* $\eta_1 = \tau_0 - t_0$,

(ii) $\exists$ *two functions* $\phi, \psi \in C^{0,1}([t_0, \tau_0 + T] \times \mathbb{R}, \mathbb{R})$ *such that* ϕ_x *and* $\phi_x + f_x$ *are nondecreasing and* ψ_x *and* $\psi_x + g_x$ *are nonincreasing in* x *for each* t,

(iii) *for each* x, $f(t, x)$ *and* $g(t, x)$ *are nondecreasing in* t,

(iv) *for some positive constants* L_1, L_2, L_3, L_4

$$\begin{aligned} |f(t,x) - f(t,y)| &\leq L_1|x-y|, \quad |g(t,x) - g(t,y)| \leq L_2|x-y|, \\ |\phi(t,x) - \phi(t,y)| &\leq L_3|x-y|, \quad |\psi(t,x) - \psi(t,y)| \leq L_4|x-y|. \end{aligned}$$

Then there exist monotone sequences $\{\tilde{\alpha}_n\}$ *and* $\{\tilde{\beta}_n\}$ *which converge uniformly and monotonically to the unique solution of (2.1) with* $x(s_0) = x_0$ *on* $[s_0, s_0+T]$ *and the convergence is quadratic.*

Proof. It is easy to show that the sequences $\{\tilde{\alpha}_n\}$ and $\{\tilde{\beta}_n\}$ defined by the solutions of the following linear fractional differential equations converge uniformly, quadratically and monotonically to the unique solution of the initial value problem 2.20.

$$\begin{aligned} {}^cD^q\tilde{\alpha}_{n+1}(t) &= f(t+\eta_2, \tilde{\alpha}_n) + g(t+\eta_2, \tilde{\alpha}_n) + [P_x(t+\eta_2, \tilde{\alpha}_n) - \phi_x(t+\eta_2, \tilde{\beta}_n) \\ &\quad + G_x(t+\eta_2, \tilde{\beta}_n) - \psi_x(t+\eta_2, \tilde{\alpha}_n)](\tilde{\alpha}_{n+1} - \tilde{\alpha}_n),\ \tilde{\alpha}_{n+1}(t_0) = x_0, \\ {}^cD^q\tilde{\beta}_{n+1}(t) &= f(t+\eta_2, \tilde{\beta}_n) + g(t+\eta_2, \tilde{\beta}_n) + [P_x(t+\eta_2, \tilde{\alpha}_n) - \phi_x(t+\eta_2, \tilde{\beta}_n) \\ &\quad + G_x(t+\eta_2, \tilde{\beta}_n) - \psi_x(t+\eta_2, \tilde{\alpha}_n)](\tilde{\beta}_{n+1} - \tilde{\beta}_n),\ \tilde{\beta}_{n+1}(t_0) = x_0 \end{aligned}$$

where $P(t,x) = f(t,x) + \phi(t,x)$ and $G(t,x) = g(t,x) + \psi(t,x)$. □

2.4. Numerical Examples

In this section, the modified quasilinearization method is illustrated by successfully applying to different examples including fractional order Riccati equation. For each example we verified the existence and uniqueness of the solution and convergence of the proposed method using Theorem 2.3.1. To solve the examples numerically, at each iteration, the corresponding linear initial value problem has been solved using Adams type predictor-corrector method [**43**]. This section also provides examples where the proposed modified quasilinearization performs even better than the generalized quasilinearization method as given in Theorem 2.3.2 and the theorems in [**41**].

Example 2.4.1. *Consider the fractional order Riccati equation of order q, $0 < q \leq 1$.*

$$ {}^cD^q x(t) = 2x(t) - x^2(t) + 1, \quad x(0) = 0, \quad 0 \leq t \leq 1. \tag{2.21} $$

Numerical solution of (2.21), has been discussed using Haar wavelet method [**91, 121**] and modified homotopy perturbation method [**105**] for various choices of q. First a comparison study for the case $q = 1$ with the results in [**91, 105, 121**] is provided (Table 2.1). It can be seen that for the choice of $t_0 = \tau_0 = 0$, $T = 1$, $L = 2$, $g(t, x) = 2x - x^2 + 1$, $f \equiv h \equiv \phi \equiv \psi = 0$, $\alpha(t) \equiv 0$ and $\beta(t) \equiv 3$ all the hypotheses of Theorem 2.3.1 are satisfied. Hence the initial value problem (2.21) has a unique solution in $[\alpha, \beta]$. Moreover, the modified quasilinearization defined by (2.6) and (2.7) converges uniformly to the solution.

It is interesting to note that g_x is not a nondecreasing function, which is a crucial condition in [**148**]. Hence the quasilinearization technique discussed in [**148**] cannot ensure the convergence of the iterative procedure as well as the existence and uniqueness of solution of (2.21) but the modified quasilinearization technique ensures both (Figure 2.1).

t	Exact	Present	Ref.[**91**]	Ref.[**105**]	Ref.[**121**]
0.1	0.1103	0.1103	0.1103	0.1103	0.1103
0.3	0.3951	0.3951	0.3951	0.3951	0.3951
0.5	0.7560	0.7560	0.7560	0.7576	0.7560
0.7	1.1529	1.1529	1.1530	1.1635	1.1529
0.9	1.5269	1.5269	1.5269	1.5550	1.5269
1.0	1.6895	1.6895	1.6895	1.7238	1.6895

TABLE 2.1. Comparison of present method with the methods discussed in [**91, 105, 121**] for Example 2.4.1.

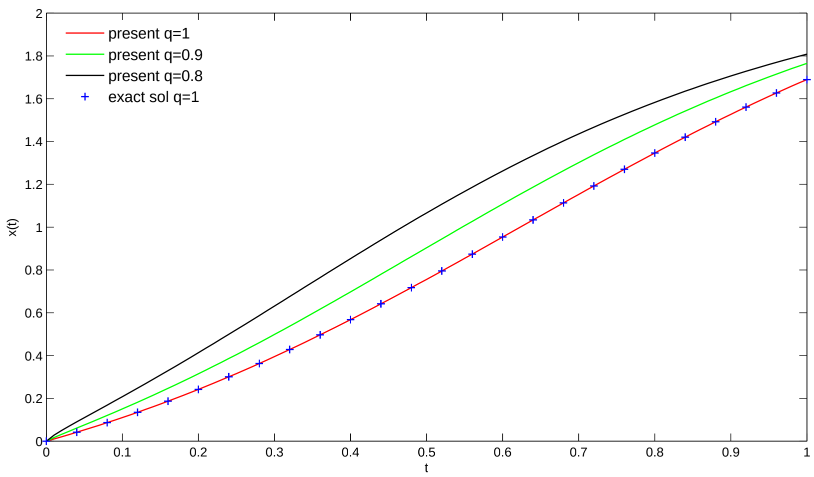

FIGURE 2.1. Solution of Example 2.4.1 for various values of q.

Example 2.4.2. *Consider the fractional differential equation of order q, $0 < q \leq 1$*

$$ {}^cD^q x(t) = 1 - x^2 + |x|, \quad x(0) = 0, \quad 0 \leq t \leq 1. \tag{2.22}$$

For the choice of $t_0 = \tau_0 = 0$, $T = 1$, $K = 1$, $h(t,x) = |x|$, $f \equiv \phi \equiv \psi = 0$, $g = 1 - x^2$, $\alpha(t) \equiv 0$ and $\beta(t) = 3$ all the hypotheses of Theorem 2.3.1 are satisfied (Table 2.2). Hence the initial value problem (2.22) has a unique solution in $[\alpha, \beta]$. Moreover the modified quasilinearization defined by (2.6) and (2.7) converges uniformly to the solution. It is easy to see that h is not a differentiable function but monotonicity of h_x is a crucial condition in [**41, 42, 148**]. Hence, even for this example the quasilinearization technique discussed in [**41, 42, 148**] cannot ensure the convergence of the iterative procedure as well as the existence and uniqueness of the solution of the (2.22) but the modified quasilinearization technique ensures both (Figure 2.2). It is also interesting to note that though the result in [**149**] ensures both linear convergence and uniqueness of the solution, computational complexity is much more than the proposed modified quasilinearization method.

t	$q = 1$	$q = 0.9$	$q = 0.8$
0.1	0.1048	0.1398	0.1864
0.3	0.3381	0.4067	0.4854
0.5	0.5868	0.6639	0.7407
0.7	0.8271	0.8907	0.9439
0.9	1.0384	1.0747	1.0971
1	1.1291	1.1500	1.1576

TABLE 2.2. Numerical solution of Example 2.4.2 for various values of q.

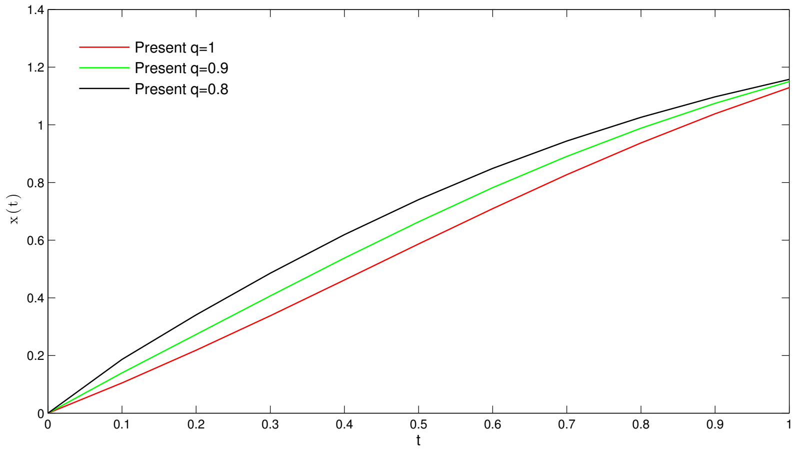

FIGURE 2.2. Solution of Example 2.4.2 for various values of q.

2.5. Conclusion

In this chapter, a modification to the quasilinearization method discussed in [**41, 42, 148, 149**] has been proposed. The error analysis for the proposed modification is done under mild conditions. To illustrate the proposed method, numerical examples are provided including fractional order Riccati equation. Since the modified quasilinearization method avoids the evaluation of the derivative at each iteration, computational cost has been greatly reduced. More classes of fractional order initial value problems could be solved using modified quasilinearization as the conditions on $F(t,x)$ has been relaxed. Thus the results obtained using the modified approach along with Adams method are better than those in some of the recent literature [**41, 42, 148, 149**] and are in good agreement with the exact solutions as discussed in Section 4.

CHAPTER 3

FRACTIONAL ORDER VOLTERRA POPULATION MODEL

3.1. Introduction

Research in recent years has shown that fractional calculus helps in modeling various physical phenomena more accurately than its classical counterparts. Hence, developing methods for solving fractional models has become one of the most emerging research fields. One such model of interest is the fractional order population model

$$ {}^{c}D^{q}x(t) = ax(t) - bx^{2}(t) - cx(t)\int_{0}^{t} x(s)\mathrm{d}s, \quad x(0) = x_{0} \tag{3.1} $$

where ${}^{c}D^{q}$ is the Caputo fractional derivative of order $q \in (0, 1]$, $a > 0$ is the birth rate coefficient, $b > 0$ is the intra species competition, $c > 0$ is the toxicity coefficient, x_0 is the initial population and $x(t)$ is the population at time t. For the choice $q = 1$, (3.1) represents the classical population growth model discussed in [**136**].

Various semi analytical methods and classical numerical techniques are suitably modified in the literature to solve (3.1). For example, methods based on Bessel collocation [**151**], differential transform [**52**], Euler wavelet [**146**], fractional polynomial [**79, 109**], homotopy analysis [**60**], Legendre wavelet [**63**], Pade approximation [**97**], pseudo spectral method [**92**] are applied to (3.1). All these methods assume that (3.1) has a unique solution. In theoretical aspects, very few results are available in the literature [**69, 117, 145**] for fractional order integro differential equation. All the results are based on various types of successive iterative schemes. The results discussed in [**69**] are not applicable for (3.1) with larger time domain due to the stringent assumption on the time domain for $0 < q \leq \frac{1}{2}$.

After transforming the integro differential equation into an integral equation, the convergence of the successive iterative scheme is discussed in [**145**]. Any numerical method based on this iterative scheme has an additional job of inverting the transformation. One of the simplest successive iterative schemes for (3.1) is available in [**117**]. The main drawback of this iteration is that it is very sensitive to various parameters, including the order of the fractional derivative, and may fail to converge. It is also interesting to note that though various numerical methods [**79, 97, 109, 146, 151**] assumed the convergence of quasilinearization, theoretical results for the quadratic convergence is not available in the literature for fractional order integro differential equations. Hence the major contributions of this chapter are the following.

1. An efficient iterative scheme is proposed for (3.1) which is independent of any transformation as well as having no restrictions on the length of the time interval.
2. The proposed iterative scheme is easy to apply and when it is combined with spectral method it shows greater flexibility with respect to the parameters in (3.1).
3. A set of sufficient conditions is provided to select the initial guess which ensures the quadratic convergence of the quasilinearization scheme.

The organization of the chapter is as follows. The preliminary results relevant to the main theorem are given in Section 2. Section 3 provides existence and uniqueness theorems for (3.1) by a monotone iterative method and a local convergence theorem for quasilinearization method, which also ensures the quadratic convergence of the quasilinearization scheme. In Section 4, the proposed result is demonstrated successfully by combining the proposed iterative scheme and spectral method for Volterra population model.

3.2. Preliminaries

In this section, some basic definitions and preliminary theorems relevant to the main theorem are presented. First, lower and upper solutions of the following problem are provided.

$$(3.2) \qquad {}^cD^q x(t) = f(t, x, \tilde{x}), \qquad x(0) = x_0$$

where $\tilde{x}(t) = \displaystyle\int_0^t x(s)\mathrm{d}s$, $f \in C([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Definition 3.2.1. *A function $\alpha_0 \in C^1([0,T], \mathbb{R})$ is called a lower solution of (3.2) if for all $t \in [0,T]$,*

$$^cD^q \alpha_0(t) \leq f(t, \alpha_0, \tilde{\alpha}_0), \qquad \alpha_0(0) \leq x_0.$$

If the inequalities are reversed, then the corresponding solution is called an upper solution of (3.2).

Definition 3.2.2. *Functions $\alpha_0, \beta_0 \in C^1([0,T], \mathbb{R})$ are called coupled lower and upper solutions of (3.2) if for all $t \in [0,T]$,*

$$\begin{aligned} ^cD^q \alpha_0(t) &\leq f(t, \alpha_0, \tilde{\beta}_0), \\ ^cD^q \beta_0(t) &\geq f(t, \beta_0, \tilde{\alpha}_0) \end{aligned}$$

and $\alpha_0(0) \leq x_0 \leq \beta_0(0)$.

Throughout this chapter f_2 and f_3 denote the first order partial derivative of f with respect to the second and third variables respectively. The interval $[0, \infty)$ is denoted by $\mathbb{R}_0^+$.

Lemma 3.2.1. *(Corollary 2.11 of [**117**]) Let $x \in C^1([a,b], \mathbb{R})$ be such that*

$$\begin{aligned} ^cD^q x(t) &\leq Lx(t) + M \int_a^t x(s)\mathrm{d}s \\ x(a) &\leq 0 \end{aligned}$$

for $L > 0$, $M \geq 0$. Then $x(t) \leq 0$ for $a \leq t \leq b$. Similarly, if $x \in C^1([a,b], \mathbb{R})$ is such that

$$\begin{aligned} ^cD^q x(t) &\geq -Lx(t) - M \int_a^t x(s)\mathrm{d}s \\ x(a) &\geq 0 \end{aligned}$$

for $L > 0$, $M \geq 0$. Then $x(t) \geq 0$ for $a \leq t \leq b$. Moreover, if ${}^cD^q x(t) \leq 0$ on $[a,b]$ with $x(a) \leq 0$, then $x(t) \leq 0$.

Lemma 3.2.2. *The fractional order integro-differential equation*

$$
{}^cD^q x(t) = \lambda_1(t)x(t) + \lambda_2(t)\tilde{x}(t) + f(t), \quad x(0) = x_0 \tag{3.3}
$$

has a unique solution, where λ_1, λ_2 and f are continuous on $[0,T]$.

Proof. The integro-differential equation (3.3) is equivalent to

$$
\begin{aligned}
x(t) &= x_0 + \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}\lambda_1(s)x(s)\mathrm{d}s + \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}\lambda_2(s)\tilde{x}(s)\mathrm{d}s \\
&\quad + \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1} f(s)\mathrm{d}s.
\end{aligned}
$$

Let $M_1 = \sup_{s\in[0,T]} |\lambda_1(s)|$, $M_2 = \sup_{s\in[0,T]} |\lambda_2(s)|$ and $\|x\| = \max_{t\in[0,T]} |x(t)|$. Define a function $\mathscr{F} : C[0,T] \to C[0,T]$ by

$$
\begin{aligned}
\mathscr{F}x(t) &= x_0 + \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}\lambda_1(s)x(s)\mathrm{d}s + \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}\lambda_2(s)\tilde{x}(s)\mathrm{d}s \\
&\quad + \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1} f(s)\mathrm{d}s.
\end{aligned}
$$

Now

$$
\begin{aligned}
|\mathscr{F}x(t) - \mathscr{F}y(t)| &\le \frac{1}{\Gamma q}\left|\int_0^t (t-s)^{q-1}\lambda_1(s)(x(s)-y(s))\mathrm{d}s\right| \\
&\quad + \frac{1}{\Gamma q}\left|\int_0^t (t-s)^{q-1}\lambda_2(s)(\tilde{x}(s)-\tilde{y}(s))\mathrm{d}s\right| \\
&\le \frac{t^q}{\Gamma(q+1)}\left(M_1 + \frac{M_2 T}{q+1}\right)\|x-y\| \\
|\mathscr{F}x(t) - \mathscr{F}y(t)| &\le \frac{Lt^q}{\Gamma(q+1)}\|x-y\| \quad \text{where } L = M_1 + \frac{M_2 T}{q+1}.
\end{aligned}
$$

Assume that $|\mathscr{F}^k x(t) - \mathscr{F}^k y(t)| \le \dfrac{(Lt^q)^k}{\Gamma(kq+1)}\|x-y\|$ for $k = 1, 2, \cdots, n-1$. Then,

$$\begin{aligned}
|\mathscr{F}^n x(t)-\mathscr{F}^n y(t)| &\leq \frac{1}{\Gamma q}\left|\int_0^t (t-s)^{q-1}\lambda_1(s)(\mathscr{F}^{n-1}x(s)-\mathscr{F}^{n-1}y(s))\mathrm{d}s\right| \\
&\quad +\frac{1}{\Gamma q}\left|\int_0^t (t-s)^{q-1}\lambda_2(s)\left(\int_0^t (\mathscr{F}^{n-1}x(z)-\mathscr{F}^{n-1}y(z))\mathrm{d}z\right)\mathrm{d}s\right| \\
&\leq \frac{M_1 L^{n-1}\|x-y\|}{\Gamma q\Gamma((n-1)q+1)}\int_0^t (t-s)^{q-1}s^{(n-1)q}\mathrm{d}s \\
&\quad +\frac{M_2 L^{n-1}\|x-y\|}{\Gamma q\Gamma((n-1)q+1)}\int_0^t (t-s)^{q-1}\frac{s^{(n-1)q+1}}{(n-1)q+1}\mathrm{d}s \\
&\leq \frac{L^{n-1}t^{nq}}{\Gamma(nq+1)}\left(M_1+\frac{M_2 T}{nq+1}\right)\|x-y\| \\
\|\mathscr{F}^n x-\mathscr{F}^n y\| &\leq \frac{(LT^q)^n}{\Gamma(nq+1)}\|x-y\|.
\end{aligned}$$

Choose n sufficiently large such that $\dfrac{(LT^q)^n}{\Gamma(nq+1)}=\rho<1$. Consequently, $\mathscr{F}^n$ is a contraction map. Then by contraction principle, $\mathscr{F}$ has a unique fixed point. Equivalently, the integral equation has a unique solution and hence (3.3) has a unique solution. □

3.3. Iterative Schemes

This section provides two interesting existence and uniqueness theorems for (3.2) by monotone iteration methods. Sufficient conditions are provided for the convergence of the proposed monotone iterations. This section also proves a local convergence theorem for quasilinearization method for (3.2) which ensures the quadratic convergence of quasilinearization scheme. Throughout this section, $[\alpha_0,\beta_0]$ denotes the sector $\{x:\alpha_0(t)\leq x(t)\leq \beta_0(t), \forall\ t\in[0,T]\}$. Define $m_1=\min\limits_{t\in[0,T]}\{\alpha_0,\beta_0\}$, $m_2=\min\limits_{t\in[0,T]}\{\tilde{\alpha}_0,\tilde{\beta}_0\}$, $M_1=\max\limits_{t\in[0,T]}\{\alpha_0,\beta_0\}$ and $M_2=\max\limits_{t\in[0,T]}\{\tilde{\alpha}_0,\tilde{\beta}_0\}$.

Theorem 3.3.1. *Let $\alpha_0,\beta_0\in C^1([0,T],\mathbb{R})$ be the lower and upper solutions of (3.2) respectively with f, $f_2\in C([0,T]\times[m_1,M_1]\times[m_2,M_2],\mathbb{R})$ and $f_3\in C([0,T]\times[m_1,M_1]\times[m_2,M_2],\mathbb{R}_0^+)$. Then there exist monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that converge uniformly and monotonically to the unique solution of (3.2) in $[\alpha_0,\beta_0]$.*

Proof. Consider the following successive iterative schemes

$$\begin{aligned} {}^cD^q\alpha_{n+1}(t) + \lambda\alpha_{n+1}(t) &= f(t, \alpha_n, \tilde{\alpha}_n) + \lambda\alpha_n, \quad \alpha_{n+1}(0) = x_0, && (3.4) \\ {}^cD^q\beta_{n+1}(t) + \lambda\beta_{n+1}(t) &= f(t, \beta_n, \tilde{\beta}_n) + \lambda\beta_n, \quad \beta_{n+1}(0) = x_0 && (3.5) \end{aligned}$$

where $\lambda > 0$ such that $\lambda + f_2 \geq 0$ on $[0, T] \times [m_1, M_1] \times [m_2, M_2]$. From Lemma 3.2.2, it is clear that the iterative schemes (3.4) and (3.5) are well defined. That is, at each step the linear differential equation has a unique solution. Using induction on n, it can be proved that for all $n \in \mathbb{N}$,

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0 \quad \text{on} \quad [0, T]. \tag{3.6}$$

Let $p(t) = \alpha_1 - \alpha_0$. Note that $p(0) \geq 0$ and

$$\begin{aligned} {}^cD^qp(t) + \lambda p(t) &= {}^cD^q\alpha_1 - {}^cD^q\alpha_0 + \lambda(\alpha_1 - \alpha_0) \\ &\geq f(t, \alpha_0, \tilde{\alpha}_0) - f(t, \alpha_0, \tilde{\alpha}_0) \\ {}^cD^qp(t) + \lambda p(t) &\geq 0. \end{aligned}$$

By Lemma 3.2.1, $p(t) \geq 0$. Thus $\alpha_1 \geq \alpha_0$ on $[0, T]$. Similarly $\beta_0 \geq \beta_1$ on $[0, T]$. Let $p(t) = \beta_1 - \alpha_1$. Note that $p(0) = 0$ and

$$\begin{aligned} {}^cD^qp(t) + \lambda p(t) &= {}^cD^q\beta_1 - {}^cD^q\alpha_1 + \lambda(\beta_1 - \alpha_1) \\ &= f(t, \beta_0, \tilde{\beta}_0) + \lambda\beta_0 - f(t, \alpha_0, \tilde{\alpha}_0) - \lambda\alpha_0 \\ &= [f_2(t, \delta_1, \delta_2) + \lambda](\beta_0 - \alpha_0) + f_3(t, \delta_1, \delta_2)(\tilde{\beta}_0 - \tilde{\alpha}_0) \\ {}^cD^qp(t) + \lambda p(t) &\geq 0 \end{aligned}$$

where $\alpha_0 \leq \delta_1 \leq \beta_0$ and $\tilde{\alpha}_0 \leq \delta_2 \leq \tilde{\beta}_0$. By Lemma 3.2.1, $p(t) \geq 0$. Thus $\beta_1 \geq \alpha_1$ on $[0, T]$. Consequently, $\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0$ on $[0, T]$. Assume that (3.6) is true for $n = k$. That is,

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_{k-1} \leq \alpha_k \leq \beta_k \leq \beta_{k-1} \leq \cdots \leq \beta_1 \leq \beta_0 \quad \text{on } [0, T]. \tag{3.7}$$

To complete the induction argument, it is enough to prove that

$$\alpha_k \leq \alpha_{k+1} \leq \beta_{k+1} \leq \beta_k \quad \text{on} \quad [0, T]. \tag{3.8}$$

Let $p(t) = \alpha_{k+1} - \alpha_k$. Note that $p(0) = 0$ and

$$\begin{aligned}
{}^cD^q p(t) + \lambda p(t) &= {}^cD^q \alpha_{k+1} - {}^cD^q \alpha_k + \lambda(\alpha_{k+1} - \alpha_k) \\
&= f(t, \alpha_k, \tilde{\alpha}_k) + \lambda \alpha_k - f(t, \alpha_{k-1}, \tilde{\alpha}_{k-1}) - \lambda \alpha_{k-1} \\
&= [f_2(t, \delta_1, \delta_2) + \lambda](\alpha_k - \alpha_{k-1}) + f_3(t, \delta_1, \delta_2)(\tilde{\alpha}_k - \tilde{\alpha}_{k-1}) \\
{}^cD^q p(t) + \lambda p(t) &\geq 0
\end{aligned}$$

where $\alpha_{k-1} \leq \delta_1 \leq \alpha_k$ and $\tilde{\alpha}_{k-1} \leq \delta_2 \leq \tilde{\alpha}_k$. By Lemma 3.2.1, $p(t) \geq 0$. Thus $\alpha_{k+1} \geq \alpha_k$ on $[0, T]$. In a similar manner it can be proved that $\beta_k \geq \beta_{k+1}$ on $[0, T]$. Let $p(t) = \beta_{k+1} - \alpha_{k+1}$. Note that $p(0) = 0$ and

$$\begin{aligned}
{}^cD^q p(t) + \lambda p(t) &= {}^cD^q \beta_{k+1} - {}^cD^q \alpha_{k+1} + \lambda(\beta_{k+1} - \alpha_{k+1}) \\
&= f(t, \beta_k, \tilde{\beta}_k) + \lambda \beta_k - f(t, \alpha_k, \tilde{\alpha}_k) - \lambda \alpha_k \\
&= [f_2(t, \delta_1, \delta_2) + \lambda](\beta_k - \alpha_k) + f_3(t, \delta_1, \delta_2)(\tilde{\beta}_k - \tilde{\alpha}_k) \\
{}^cD^q p(t) &\geq 0
\end{aligned}$$

where $\alpha_k \leq \delta_1 \leq \beta_k$ and $\tilde{\alpha}_k \leq \delta_2 \leq \tilde{\beta}_k$. By Lemma 3.2.1, $p(t) \geq 0$. Thus $\beta_{k+1} \geq \alpha_{k+1}$ on $[0, T]$. Consequently (3.8) is proved. Hence the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are monotone and uniformly bounded on $[0, T]$. Using the similar argument as in [**117**], one can show that $\{\alpha_n\}$ and $\{\beta_n\}$ are equicontinuous on $[0, T]$. Hence there exist ρ_1, $\rho_2 \in C^1([0, T])$ such that $\{\alpha_n\}$ and $\{\beta_n\}$ converge uniformly and monotonically to ρ_1 and ρ_2 respectively. Clearly $\rho_1 \leq \rho_2$. For uniqueness it is enough to show that $\rho_2 \leq \rho_1$. Define $p(t) = \rho_2 - \rho_1$ on $[0, T]$. Then $p(0) = 0$ and it is easy to show that ${}^cD^q p(t) \leq C_1 p(t) + C_2 \tilde{p}(t)$ for some positive constants C_1 and C_2. Consequently $p(t) \leq 0$ by Lemma 3.2.1. Hence (3.2) has a unique solution in $[\alpha_0, \beta_0]$. □

Remark 3.3.1. *Since the condition* $f_3 \geq 0$ *is not true for* (3.1)*, Theorem 3.3.1 is not applicable for* (3.1)*. To handle* (3.1)*, the following monotone iterative method is proposed which is based on coupled lower and upper solutions.*

Theorem 3.3.2. *Let* $\alpha_0, \beta_0 \in C^1([0, T], \mathbb{R})$ *be a coupled lower and upper solutions of* (3.2) *with* f, $f_2 \in C([0, T] \times [m_1, M_1] \times [m_2, M_2], \mathbb{R})$ *and* $-f_3 \in C([0, T] \times [m_1, M_1] \times$

$[m_2, M_2], \mathbb{R}_0^+)$. *Then there exist monotone sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *that converge uniformly and monotonically to the unique solution of* (3.2) *in* $[\alpha_0, \beta_0]$.

Proof. Consider the following successive iterative schemes

$$
\begin{aligned}
{}^cD^q\alpha_{n+1}(t) + \lambda\alpha_{n+1}(t) &= f(t, \alpha_n, \tilde{\beta}_n) + \lambda\alpha_n, \quad \alpha_{n+1}(0) = x_0, && (3.9)\\
{}^cD^q\beta_{n+1}(t) + \lambda\beta_{n+1}(t) &= f(t, \beta_n, \tilde{\alpha}_n) + \lambda\beta_n, \quad \beta_{n+1}(0) = x_0 && (3.10)
\end{aligned}
$$

where $\lambda > 0$ such that $\lambda + f_2 \geq 0$ on $[0, T] \times [m_1, M_1] \times [m_2, M_2]$. From Lemma 3.2.2, it is clear that the iterative schemes (3.9) and (3.10) are well defined. That is, at each step, the linear integro differential equation has a unique solution. For all $n \in \mathbb{N}$, one can obtain

$$
\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0 \quad \text{on} \quad [0, T] \tag{3.11}
$$

by using induction on n. Let $p(t) = \alpha_1 - \alpha_0$. Note that $p(0) \geq 0$ and

$$
\begin{aligned}
{}^cD^q p(t) + \lambda p(t) &= {}^cD^q\alpha_1 - {}^cD^q\alpha_0 + \lambda(\alpha_1 - \alpha_0)\\
&\geq f(t, \alpha_0, \tilde{\beta}_0) - f(t, \alpha_0, \tilde{\beta}_0)\\
{}^cD^q p(t) + \lambda p(t) &\geq 0.
\end{aligned}
$$

By Lemma 3.2.1, $p(t) \geq 0$. Thus $\alpha_1 \geq \alpha_0$ on $[0, T]$. Similarly, $\beta_0 \geq \beta_1$ on $[0, T]$. Let $p(t) = \beta_1 - \alpha_1$. Note that $p(0) = 0$ and

$$
\begin{aligned}
{}^cD^q p(t) + \lambda p(t) &= {}^cD^q\beta_1 - {}^cD^q\alpha_1 + \lambda(\beta_1 - \alpha_1)\\
&= f(t, \beta_0, \tilde{\alpha}_0) + \lambda\beta_0 - f(t, \alpha_0, \tilde{\beta}_0) - \lambda\alpha_0\\
&= [f_2(t, \delta_1, \delta_2) + \lambda](\beta_0 - \alpha_0) + f_3(t, \delta_1, \delta_2)(\tilde{\alpha}_0 - \tilde{\beta}_0)\\
{}^cD^q p(t) + \lambda p(t) &\geq 0
\end{aligned}
$$

where $\alpha_0 \leq \delta_1 \leq \beta_0$ and $\tilde{\alpha}_0 \leq \delta_2 \leq \tilde{\beta}_0$. By Lemma 3.2.1, $p(t) \geq 0$. Thus $\beta_1 \geq \alpha_1$ on $[0, T]$. Consequently, $\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0$ and $[0, T]$. Assume that (3.11) is true for $n = k$. That is,

$$
\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_{k-1} \leq \alpha_k \leq \beta_k \leq \beta_{k-1} \leq \cdots \leq \beta_1 \leq \beta_0 \quad \text{on } [0, T]. \tag{3.12}
$$

To complete the induction argument it is enough to prove that

$$\alpha_k \leq \alpha_{k+1} \leq \beta_{k+1} \leq \beta_k \quad \text{on} \quad [0,T]. \tag{3.13}$$

Let $p(t) = \alpha_{k+1} - \alpha_k$. Note that $p(0) = 0$ and

$$\begin{aligned}
{}^cD^q p(t) + \lambda p(t) &= {}^cD^q \alpha_{k+1} - {}^cD^q \alpha_k + \lambda(\alpha_{k+1} - \alpha_k) \\
&= f(t, \alpha_k, \tilde{\beta}_k) + \lambda\alpha_k - f(t, \alpha_{k-1}, \tilde{\beta}_{k-1}) - \lambda\alpha_{k-1} \\
&= [f_2(t, \delta_1, \delta_2) + \lambda](\alpha_k - \alpha_{k-1}) + f_3(t, \delta_1, \delta_2)(\tilde{\beta}_k - \tilde{\beta}_{k-1}) \\
{}^cD^q p(t) + \lambda p(t) &\geq 0
\end{aligned}$$

where $\alpha_{k-1} \leq \delta_1 \leq \alpha_k$ and $\tilde{\beta}_k \leq \delta_2 \leq \tilde{\beta}_{k-1}$. By Lemma 3.2.1, $p(t) \geq 0$. Thus $\alpha_{k+1} \geq \alpha_k$ on $[0,T]$. In a similar manner it can be proved that $\beta_k \geq \beta_{k+1}$ on $[0,T]$. Let $p(t) = \beta_{k+1} - \alpha_{k+1}$. Note that $p(0) = 0$ and

$$\begin{aligned}
{}^cD^q p(t) + \lambda p(t) &= {}^cD^q \beta_{k+1} - {}^cD^q \alpha_{k+1} + \lambda p(t) \\
&= f(t, \beta_k, \tilde{\alpha}_k) + \lambda\beta_k - f(t, \alpha_k, \tilde{\beta}_k) - \lambda\alpha_k \\
&= [f_2(t, \delta_1, \delta_2) + \lambda](\beta_k - \alpha_k) + f_3(t, \delta_1, \delta_2)(\tilde{\alpha}_k - \tilde{\beta}_k) \\
{}^cD^q p(t) + \lambda p(t) &\geq 0
\end{aligned}$$

where $\alpha_k \leq \delta_1 \leq \beta_k$ and $\tilde{\alpha}_k \leq \delta_2 \leq \tilde{\beta}_k$. By Lemma 3.2.1, $p(t) \geq 0$. Thus $\beta_{k+1} \geq \alpha_{k+1}$ on $[0,T]$. Consequently (3.13) is proved. Hence the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are monotone and uniformly bounded on $[0,T]$. Using the similar argument as in [**117**], one can show that $\{\alpha_n\}$ and $\{\beta_n\}$ are equicontinuous on $[0,T]$. Hence there exist ρ_1, $\rho_2 \in C^1([0,T])$ such that $\{\alpha_n\}$ and $\{\beta_n\}$ converge uniformly and monotonically to ρ_1 and ρ_2 respectively. Clearly, $\rho_1 \leq \rho_2$. To show that $\rho_2 \leq \rho_1$, define $p(t) = \rho_2 - \rho_1$ on $[0,T]$. It is easy to show that ${}^cD^q p(t) \leq C_1 p(t) + C_2 \tilde{p}(t)$ for some positive constants C_1 and C_2. Also we have $p(0) = 0$. These conditions imply that $p(t) \leq 0$, due to Lemma 3.2.1. Hence $\rho_1 = \rho_2 = x$ is the unique solution of (3.2) in $[\alpha_0, \beta_0]$. □

Remark 3.3.2. *Note that $\alpha_0 \equiv 0$ and $\beta_0 \equiv 1$ is a coupled lower and upper solution for* (3.1) *when $b \geq a > 0$. Similarly, $\alpha_0 \equiv 0$ and $\beta_0 \equiv \dfrac{a}{b}$ is a coupled lower and upper* *solution for* (3.1) *when* $a > b > 0$. *Consequently* (3.1) *satisfies all the hypotheses of Theorem 3.3.2. Hence the fractional order population model has a unique solution in* $[\alpha_0, \beta_0]$. *Though Theorem 3.3.2 ensures the existence, uniqueness as well as an iterative scheme to approximate the unique solution of* (3.1), *the order of convergence of the iterative schemes is linear.*

One of the ways to accelerate the iterative procedure for (3.1) is to replace the successive scheme by quasilinearization scheme. The convergence analysis of quasilinearization highly depends on the initial guess.

In this context, the following local convergence theorem for quasilinearization scheme is proved. The theorem provides a sufficient condition on the initial guess that ensures the quadratic convergence of the quasilinearization scheme. The other advantage of the following theorem is that at each step one has to solve exactly one linear equation in contrast to the iterative scheme in Theorem 3.3.2.

The following notations and assumptions are used in the following Theorem 3.3.3. Let $C[0,T]$ be the collection of all continuous functions on $[0,T]$ endowed with the norm $\|x\|_\rho = \sup_{t\in[0,T]} \dfrac{|x(t)|}{E_q(\rho t^q)}$, $\rho > 0$ [**30**]. Let x^* be a solution of (3.2). Let $B_\rho(x^*, r)$ denotes the closed ball with center x^* and radius $r > 0$ i.e., $B_\rho(x^*, r) = \{x \in C[a,b] : \|x - x^*\|_\rho \leq r\}$. Let f, f_2, $f_3 \in C([0,T]\times\mathbb{R}\times\mathbb{R}, \mathbb{R})$ be the set of all continuous functions defined on $[0,T]\times\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$. Let C_1 and C_2 be constants such that $\sup_A |f_2(t,x_1,x_2)| \leq C_1$ and $\sup_A |f_3(t,x_1,x_2)| \leq C_2$, where $A = \{(t,x_1,x_2) : t \in [0,T], x_1 \in B_\rho(x^*, r), x_2 \in B_\rho(\tilde{x}^*, rTE_q(\rho T^q))\}$.

Theorem 3.3.3. *If there exists* $\rho > 0$ *such that* $0 \leq \dfrac{2C_1\rho + 2C_2T^{1-q}}{\rho^2 - C_1\rho - C_2T^{1-q}} < 1$ *and* $\alpha_0 \in B_\rho(x^*, r)$, *then* $\forall\, n \in \mathbb{N}$ *the quasilinearization scheme*

(3.14)

$$^cD^q\alpha_{n+1}(t) = f(t,\alpha_n,\tilde{\alpha}_n) + f_2(t,\alpha_n,\tilde{\alpha}_n)(\alpha_{n+1}-\alpha_n) + f_3(t,\alpha_n,\tilde{\alpha}_n)(\tilde{\alpha}_{n+1}-\tilde{\alpha}_n),\ \alpha_{n+1}(0) = x_0$$

is well defined, $\alpha_n \in B_\rho(x^*, r)$ *and it converges to the unique solution* x^* *of* (3.2) *in* $B_\rho(x^*, r)$. *Moreover, if* f_2 *and* f_3 *are Lipschitz in their respective domain, then the quasilinearization scheme converges quadratically to the unique solution of* (3.2).

Proof. From Lemma 3.2.2, it is clear that the iterative scheme (3.14) is well defined for all $n \in \mathbb{N}$. Define $p_{n+1} = \alpha_{n+1} - x^*$.

$$
\begin{aligned}
{}^cD^q p_{n+1}(t) &= f(t,\alpha_n,\tilde{\alpha}_n) - f(t,x^*,\tilde{x}^*) + f_2(t,\alpha_n,\tilde{\alpha}_n)(\alpha_{n+1}-\alpha_n) \\
&\quad + f_3(t,\alpha_n,\tilde{\alpha}_n)(\tilde{\alpha}_{n+1}-\tilde{\alpha}_n) \\
p_{n+1}(t) &= \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}\int_0^1 f_2(s,\theta\alpha_n+(1-\theta)x^*,\theta\tilde{\alpha}_n+(1-\theta)\tilde{x}^*)p_n(s)\mathrm{d}\theta\mathrm{d}s \\
&\quad + \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}\int_0^1 f_3(s,\theta\alpha_n+(1-\theta)x^*,\theta\tilde{\alpha}_n+(1-\theta)\tilde{x}^*)\tilde{p}_n(s)\mathrm{d}\theta\mathrm{d}s \\
&\quad + \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1} f_2(s,\alpha_n,\tilde{\alpha}_n)(p_{n+1}(s)-p_n(s))\mathrm{d}s \\
&\quad + \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1} f_3(s,\alpha_n,\tilde{\alpha}_n)(\tilde{p}_{n+1}(s)-\tilde{p}_n(s))\mathrm{d}s \\
|p_{n+1}(t)| &\leq \frac{C_1}{\Gamma q}\int_0^t (t-s)^{q-1}\frac{|p_n(s)|}{E_q(\rho s^q)}E_q(\rho s^q)\mathrm{d}s \\
&\quad + \frac{C_2}{\Gamma q}\int_0^t (t-s)^{q-1}\int_0^s \frac{|p_n(\tau)|}{E_q(\rho\tau^q)}E_q(\rho\tau^q)(s-\tau)^{q-1}(s-\tau)^{1-q}\mathrm{d}\tau\mathrm{d}s \\
&\quad + \frac{C_1}{\Gamma q}\int_0^t (t-s)^{q-1}\frac{|p_{n+1}(s)-p_n(s)|}{E_q(\rho s^q)}E_q(\rho s^q)\mathrm{d}s \\
&\quad + \frac{C_2}{\Gamma q}\int_0^t (t-s)^{q-1}\int_0^s \frac{|p_{n+1}(\tau)-p_n(\tau)|}{E_q(\rho\tau^q)}E_q(\rho\tau^q)(s-\tau)^{q-1}(s-\tau)^{1-q}\mathrm{d}\tau\mathrm{d}s \\
|p_{n+1}(t)| &\leq \frac{C_1\|p_n\|_\rho E_q(\rho t^q)}{\rho} + \frac{C_2\|p_n\|_\rho \Gamma q T^{1-q}E_q(\rho t^q)}{\rho^2} \\
&\quad + \frac{C_1(\|p_{n+1}\|_\rho+\|p_n\|_\rho)E_q(\rho t^q)}{\rho} + \frac{C_2(\|p_{n+1}\|_\rho+\|p_n\|_\rho)\Gamma q T^{1-q}E_q(\rho t^q)}{\rho^2} \\
\|p_{n+1}\|_\rho &\leq \frac{2C_1\|p_n\|_\rho}{\rho} + \frac{2C_2\|p_n\|_\rho\Gamma q T^{1-q}}{\rho^2} + \frac{C_1\|p_{n+1}\|_\rho}{\rho} + \frac{C_2\|p_{n+1}\|_\rho\Gamma q T^{1-q}}{\rho^2} \\
\|p_{n+1}\|_\rho &\leq \left(\frac{2C_1\rho+2C_2\Gamma q T^{1-q}}{\rho^2-C_1\rho-C_2\Gamma q T^{1-q}}\right)\|p_n\|_\rho \leq \left(\frac{2C_1\rho+2C_2\Gamma q T^{1-q}}{\rho^2-C_1\rho-C_2\Gamma q T^{1-q}}\right)^{n+1}\|p_0\|_\rho.
\end{aligned}
$$

Thus $\alpha_{n+1} \in B_\rho(x^*, r)$. Consequently, α_n converges to x^*. Now it is required to show that x^* is the unique solution of (3.2) in $B_\rho(x^*, r)$. Let x_1 and x_2 be two solutions of (3.2) in $B_\rho(x^*, r)$ and $p(t) = x_1 - x_2$. Then by (3.2) and using mean value theorem,

$$\begin{aligned}
p(t) &= \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}(f(s,x_1,\tilde{x}_1)-f(s,x_2,\tilde{x}_2))\mathrm{d}s \\
&= \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}\int_0^1 f_2(s,\theta x_1+(1-\theta)x_2,\theta\tilde{x}_1+(1-\theta)\tilde{x}_2)p(s)\mathrm{d}\theta\mathrm{d}s \\
&\quad +\frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}\int_0^1 f_3(s,\theta x_1+(1-\theta)x_2,\theta\tilde{x}_1+(1-\theta)\tilde{x}_2)\tilde{p}(s)\mathrm{d}\theta\mathrm{d}s \\
|p(t)| &\le \frac{C_1}{\Gamma q}\int_0^t (t-s)^{q-1}\frac{|p(s)|}{E_q(\rho s^q)}E_q(\rho s^q)\mathrm{d}s \\
&\quad +\frac{C_2}{\Gamma q}\int_0^t (t-s)^{q-1}\int_0^s \frac{|p(\tau)|}{E_q(\rho\tau^q)}E_q(\rho\tau^q)(s-\tau)^{q-1}(s-\tau)^{1-q}\mathrm{d}\tau\mathrm{d}s \\
&\le \frac{C_1\|p\|_\rho E_q(\rho t^q)}{\rho}+\frac{C_2\|p\|_\rho T^{1-q}}{\rho}\int_0^t (t-s)^{q-1}E_q(\rho s^q)\mathrm{d}s \\
&\le \frac{C_1\|p\|_\rho E_q(\rho t^q)}{\rho}+\frac{C_2\|p\|_\rho \Gamma q T^{1-q}E_q(\rho t^q)}{\rho^2} \\
\|p\|_\rho &\le \frac{C_1\rho+C_2\Gamma q T^{1-q}}{\rho^2}\|p\|_\rho.
\end{aligned}$$

Thus $\|p\|_\rho \le 0$, which implies $p(t)=0$. Consequently $x_1=x_2=x^*$ is the unique solution of (3.2). Now we have

$$\begin{aligned}
{}^cD^q p_{n+1}(t) &= f(t,\alpha_n,\tilde{\alpha}_n)-f(t,x^*,\tilde{x}^*)+f_2(t,\alpha_n,\tilde{\alpha}_n)(\alpha_{n+1}-\alpha_n) \\
&\quad +f_3(t,\alpha_n,\tilde{\alpha}_n)(\tilde{\alpha}_{n+1}-\tilde{\alpha}_n) \\
&= \int_0^1 f_2(t,\theta\alpha_n+(1-\theta)x^*,\theta\tilde{\alpha}_n+(1-\theta)\tilde{x}^*)p_n(t)\mathrm{d}\theta \\
&\quad +\int_0^1 f_3(t,\theta\alpha_n+(1-\theta)x^*,\theta\tilde{\alpha}_n+(1-\theta)\tilde{x}^*)\tilde{p}_n(t)\mathrm{d}\theta \\
&\quad +f_2(t,\alpha_n,\tilde{\alpha}_n)(p_{n+1}-p_n)+f_3(t,\alpha_n,\tilde{\alpha}_n)(\tilde{p}_{n+1}-\tilde{p}_n) \\
{}^cD^q p_{n+1}(t) &= f_2(t,\alpha_n,\tilde{\alpha}_n)p_{n+1}+f_3(t,\alpha_n,\tilde{\alpha}_n)\tilde{p}_{n+1} \\
&\quad +\int_0^1 (f_2(t,\theta\alpha_n+(1-\theta)x^*,\theta\tilde{\alpha}_n+(1-\theta)\tilde{x}^*)-f_2(t,\alpha_n,\tilde{\alpha}_n))p_n(t)\mathrm{d}\theta \\
&\quad +\int_0^1 (f_3(t,\theta\alpha_n+(1-\theta)x^*,\theta\tilde{\alpha}_n+(1-\theta)\tilde{x}^*)-f_3(t,\alpha_n,\tilde{\alpha}_n))\tilde{p}_n(t)\mathrm{d}\theta
\end{aligned}$$

$$
\begin{aligned}
p_{n+1}(t) &= \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1} f_2(s,\alpha_n,\tilde{\alpha}_n)p_{n+1}(s)\mathrm{d}s + \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1} f_3(s,\alpha_n,\tilde{\alpha}_n)\tilde{p}_{n+1}(s)\mathrm{d}s \\
&+ \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}\int_0^1 (f_2(s,\theta\alpha_n+(1-\theta)x^*,\theta\tilde{\alpha}_n+(1-\theta)\tilde{x}^*) - f_2(s,\alpha_n,\tilde{\alpha}_n))p_n\mathrm{d}\theta\mathrm{d}s \\
&+ \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}\int_0^1 (f_3(s,\theta\alpha_n+(1-\theta)x^*,\theta\tilde{\alpha}_n+(1-\theta)\tilde{x}^*) - f_3(s,\alpha_n,\tilde{\alpha}_n))\tilde{p}_n\mathrm{d}\theta\mathrm{d}s \\
|p_{n+1}(t)| &\le \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}|f_2||p_{n+1}|\mathrm{d}s + \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}|f_3||\tilde{p}_{n+1}|\mathrm{d}s \\
&+ \frac{L_1}{\Gamma q}\int_0^t (t-s)^{q-1}\int_0^1 (1-\theta)(|p_n|+|\tilde{p}_n|)|p_n|\mathrm{d}\theta\mathrm{d}s \\
&+ \frac{L_2}{\Gamma q}\int_0^t (t-s)^{q-1}\int_0^1 (1-\theta)(|p_n|+|\tilde{p}_n|)|\tilde{p}_n|\mathrm{d}\theta\mathrm{d}s, \text{ for some } L_1, L_2 > 0 \\
&\le \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}|f_2|\frac{|p_{n+1}|}{E_q(\rho s^q)}E_q(\rho s^q)\mathrm{d}s \\
&+ \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}|f_3|\int_0^s \frac{|p_{n+1}|}{E_q(\rho\tau^q)}E_q(\rho\tau^q)(s-\tau)^{q-1}(s-\tau)^{1-q}\mathrm{d}\tau\mathrm{d}s \\
&+ \frac{L_1}{2\Gamma q}\int_0^t (t-s)^{q-1}(|p_n|+|\tilde{p}_n|)|p_n|\mathrm{d}s + \frac{L_2}{2\Gamma q}\int_0^t (t-s)^{q-1}(|p_n|+|\tilde{p}_n|)|\tilde{p}_n|\mathrm{d}s \\
&\le \frac{C_1\|p_{n+1}\|_\rho E_q(\rho t^q)}{\rho} + \frac{C_2 T^{1-q}\|p_{n+1}\|_\rho}{\rho}\int_0^t (t-s)^{q-1}E_q(\rho s^q)\mathrm{d}s \\
&+ \frac{L_1}{2\Gamma q}\int_0^t (t-s)^{q-1}\frac{|p_n|^2}{(E_q(\rho s^q))^2}(E_q(\rho s^q))^2\mathrm{d}s \\
&+ \frac{(L_1+L_2)}{2\Gamma q}\int_0^t (t-s)^{q-1}|p_n|\int_0^s \frac{|p_n|}{E_q(\rho\tau^q)}E_q(\rho\tau^q)(s-\tau)^{q-1}(s-\tau)^{1-q}\mathrm{d}\tau\mathrm{d}s \\
&+ \frac{L_2}{2\Gamma q}\int_0^t (t-s)^{q-1}\left(\int_0^s \frac{|p_n|}{E_q(\rho\tau^q)}E_q(\rho\tau^q)(s-\tau)^{q-1}(s-\tau)^{1-q}\mathrm{d}\tau\right)^2\mathrm{d}s \\
|p_{n+1}(t)| &\le \frac{C_1 E_q(\rho t^q)\|p_{n+1}\|_\rho}{\rho} + \frac{C_2\Gamma q T^{1-q}E_q(\rho t^q)\|p_{n+1}\|_\rho}{\rho^2} \\
&+ \frac{(L_1+L_2)T^{1-q}\|p_n\|_\rho}{2\rho}\int_0^t (t-s)^{q-1}\frac{|p_n|}{E_q(\rho s^q)}(E_q(\rho s^q))^2\mathrm{d}s \\
&+ \frac{L_1 E_q(\rho T^q)E_q(\rho t^q)\|p_n\|_\rho^2}{2\rho} + \frac{L_2 T^{2-2q}\Gamma q\|p_n\|_\rho^2}{2\rho^2}\int_0^t (t-s)^{q-1}(E_q(\rho s^q))^2\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
|p_{n+1}(t)| \;\leq\; & \frac{C_1 E_q(\rho t^q)\|p_{n+1}\|_\rho}{\rho} + \frac{C_2 \Gamma q T^{1-q} E_q(\rho t^q)\|p_{n+1}\|_\rho}{\rho^2} \\
& + \frac{(L_1+L_2)T^{1-q}\Gamma q E_q(\rho T^q) E_q(\rho t^q)\|p_n\|_\rho^2}{2\rho^2} \\
& + \frac{L_1 E_q(\rho T^q) E_q(\rho t^q)\|p_n\|_\rho^2}{2\rho} + \frac{L_2 T^{2-2q}(\Gamma q)^2 E_q(\rho T^q) E_q(\rho t^q)\|p_n\|_\rho^2}{2\rho^3} \\
\|p_{n+1}\|_\rho \;\leq\; & \left(\frac{C_1\rho + C_2 \Gamma q T^{1-q}}{\rho^2}\right)\|p_{n+1}\|_\rho \\
& + \left(\frac{L_1\rho^2 + (L_1+L_2)\Gamma q T^{1-q}\rho + L_2(\Gamma q)^2 T^{2-2q}}{2\rho^3}\right) E_q(\rho T^q)\|p_n\|_\rho^2.
\end{aligned}
$$

Hence, $\|p_{n+1}\|_\rho \leq N\|p_n\|_\rho^2$, for some N. □

Remark 3.3.3. *It can be observed that the set A depends on the radius r. Consequently, the constants C_1 and C_2 depend on r. Further, the condition on ρ given in the statement of Theorem 3.3.3 suggests that the choice of ρ is also influenced by r. An illustration of Theorem 3.3.3 is shown through following example.*

Consider the problem:

$$
{}^cD^{0.5}x(t) = \frac{1}{70}(x(t) - x^2(t) - x(t)\int_0^t x(s)\mathrm{d}s) + g(t),\ t \in [0, 0.5], \quad x(0) = 0 \tag{3.15}
$$

where $g(t) = \dfrac{2t^{1.5}}{\Gamma(2.5)} - \dfrac{t^2}{70} + \dfrac{t^4}{70} + \dfrac{t^5}{210}$. It is easy to verify that the exact solution is t^2. For the choice of $\rho = 0.9$, $r = 1$, $C_1 = 0.2017$, $C_2 = 0.0521$, $L_1 = \frac{1}{35}$ and $L_2 = \frac{1}{70}$ all the hypotheses of the Theorem 3.3.3 are satisfied. Hence if the initial guess is chosen in $B_\rho(t^2, 1)$ then the quasilinearization scheme converges quadratically and uniformly in $B_\rho(t^2, 1)$.

3.4. Numerical Illustration

To make the presentation self contained, the implementation of spectral method is outlined in this section. For more details on spectral method, one can refer [**27, 99**]. The numerical implementation is demonstrated for the Volterra population model by coupling the iterative scheme (3.9) and (3.10) with spectral method. To validate the proposed theory, the following normalization of (3.1) is considered.

$$k\, {}^cD^q x(t) = x(t) - x^2(t) - x(t)\int_0^t x(s)\mathrm{d}s, \quad x(0) = x_0 \tag{3.16}$$

where $t \in [0, T]$. Since the present scheme uses Chebyshev-Gauss-Lobatto points, the time domain $t \in [0, T]$ is transformed to the computational domain $[-1, 1]$ by the linear transformation $t = \dfrac{T(\tau + 1)}{2}$. Hence (3.16) becomes,

$$\frac{2k}{T}\, {}^cD^q x(\tau) = x(\tau) - x^2(\tau) - \frac{T}{2}x(\tau)\int_{-1}^{\tau} x(s)\mathrm{d}s, \quad x(-1) = x_0. \tag{3.17}$$

To proceed further, assume that α_0 and β_0 are the coupled lower and upper solutions of (3.16). Applying the proposed iterative scheme (3.9) and (3.10) for (3.17) leads to

$$\frac{2k}{T}\, {}^cD^q \alpha_{n+1}(\tau) + \lambda\alpha_{n+1}(\tau) = \lambda\alpha_n(\tau) + \alpha_n(\tau) - \alpha_n^2(\tau) - \frac{T}{2}\alpha_n(\tau)\int_{-1}^{\tau} \beta_n(s)\mathrm{d}s \tag{3.18}$$

and

$$\frac{2k}{T}\, {}^cD^q \beta_{n+1}(\tau) + \lambda\beta_{n+1}(\tau) = \lambda\beta_n(\tau) + \beta_n(\tau) - \beta_n^2(\tau) - \frac{T}{2}\beta_n(\tau)\int_{-1}^{\tau} \alpha_n(s)\mathrm{d}s \tag{3.19}$$

with the initial conditions $\alpha_{n+1}(-1) = x_0 = \beta_{n+1}(-1)$. Assume that the solution of (3.18) can be approximated by a Lagrange interpolation polynomial of the form

$$\alpha_n(\tau) = \sum_{j=0}^{N} \alpha_n(\tau_j)L_j(\tau), \text{ for any } \tau \in [-1, 1] \tag{3.20}$$

where $\tau_j = \cos\left(\dfrac{\pi j}{N}\right)$; $j = 0, 1, \cdots, N$ are Chebyshev-Gauss-Lobatto grid points and the functions $L_j(\tau)$ are the characteristic Lagrange polynomials given by

$$L_j(\tau) = \prod_{k=0, k\neq j}^{N} \frac{\tau - \tau_k}{\tau_j - \tau_k}.$$

It can be seen that each Lagrange polynomial satisfies cardinality property

$$L_j(\tau_k) = \delta_{jk} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

for $j,\ k = 1, 2, \cdots, N$. The values of the time derivative at the Chebyshev-Gauss-Lobatto points τ_j are computed as

$$^{c}D^{q}\ \alpha_n(\tau)|_{\tau=\tau_j} = \sum_{k=0}^{N} \alpha_n(\tau_k)^{c}D^{q}L_k(\tau_j) = \sum_{k=0}^{N} d_{j,k}\alpha_n(\tau_k)$$

where $[d_{j,k}] = [^{c}D^{q}L_k(\tau_j)]$ is the Chebyshev Caputo differentiation matrix of order q and size $(N+1)\times(N+1)$. The values of the time integral at the Chebyshev-Gauss-Lobatto points τ_j are computed as

$$\int_{-1}^{\tau} \alpha_n(s)\mathrm{d}s\bigg|_{\tau=\tau_j} = \sum_{k=0}^{N} \alpha_n(\tau_k)\int_{-1}^{\tau_j} L_k(\tau)\mathrm{d}\tau = \sum_{k=0}^{N} f_{j,k}\alpha_n(\tau_k)$$

where $[f_{j,k}] = \left[\int_{-1}^{\tau_j} L_k(\tau)\mathrm{d}\tau\right]$ is the Chebyshev integration matrix of size $(N+1)\times(N+1)$. Using the initial condition, (3.18) can be written as

$$\frac{2k}{T}\sum_{k=0}^{N} d_{j,k}\alpha_{n+1}(\tau_k) + \lambda\alpha_{n+1}(\tau_j) = R_j; \quad j = 0, 1, \cdots, N-1 \tag{3.21}$$

where $R_j = \lambda\alpha_n(\tau_j) + \alpha_n(\tau_j) - \alpha_n^2(\tau_j) - \frac{T}{2}\alpha_n(\tau_j)\sum_{k=0}^{N} f_{j,k}\beta_n(\tau_k) - \frac{2k}{T}d_{j,N}x_0$. Now (3.21) can be written in the matrix form as follows:

$$\begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,N-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1,0} & a_{N-1,1} & \cdots & a_{N-1,N-1} \end{bmatrix} \begin{bmatrix} \alpha_{n+1}(\tau_0) \\ \alpha_{n+1}(\tau_1) \\ \vdots \\ \alpha_{n+1}(\tau_{N-1}) \end{bmatrix} = \begin{bmatrix} R_0 \\ R_1 \\ \vdots \\ R_{N-1} \end{bmatrix} \tag{3.22}$$

where $a_{i,i} = \frac{2k}{T}d_{i,i} + \lambda$ and $a_{i,j} = \frac{2k}{T}d_{i,j};\ i \neq j$. A similar procedure applied to (3.19) leads to

$$\begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,N-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1,0} & a_{N-1,1} & \cdots & a_{N-1,N-1} \end{bmatrix} \begin{bmatrix} \beta_{n+1}(\tau_0) \\ \beta_{n+1}(\tau_1) \\ \vdots \\ \beta_{n+1}(\tau_{N-1}) \end{bmatrix} = \begin{bmatrix} R'_0 \\ R'_1 \\ \vdots \\ R'_{N-1} \end{bmatrix} \tag{3.23}$$

where $R'_j = \lambda\beta_n(\tau_j) + \beta_n(\tau_j) - \beta_n^2(\tau_j) - \frac{T}{2}\beta_n(\tau_j)\sum_{k=0}^{N} f_{j,k}\alpha_n(\tau_k) - \frac{2k}{T}d_{j,N}x_0$. Using $\alpha_0,\ \beta_0$ one can obtain the rest of $\alpha'_n s,\ \beta'_n s$.

Figure 3.1 and Figure 3.2 represent the numerical solution of (3.16) for various choices of k and q. Table 3.1 presents the comparison of the spectral method based on the proposed iterative scheme and the iterative scheme in [**117**]. From Table 3.1 it can be easily observed that the spectral method based on the iterative scheme in [**117**] is very sensitive to various parameters such as length of the time domain $[0, T]$ and order of the derivative. On the other hand, the proposed iterative scheme easily handles all the situation in which the iterative scheme in [**117**] fails. Hence the proposed iterative scheme is more efficient than the iterative scheme studied in [**117**]. In the Table 3.1 "-" denotes no convergence. All the numerical simulations are performed using Matlab R2010b.

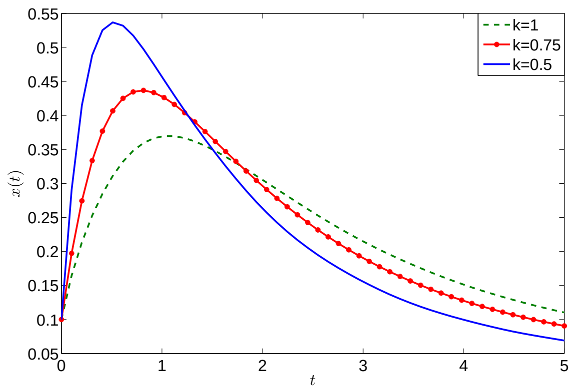

FIGURE 3.1. Approximate solution $x(t)$ for various k when $q = 0.5$, $x_0 = 0.1$ and $\lambda = 6$.

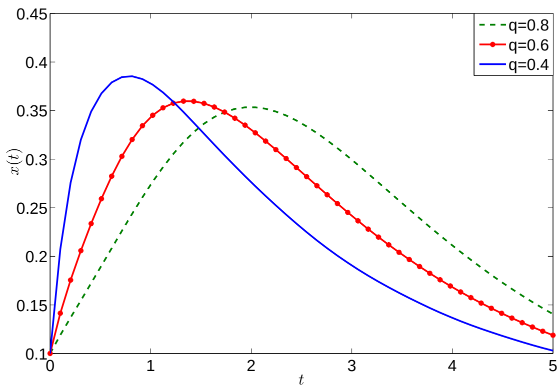

FIGURE 3.2. Approximate solution $x(t)$ for various q when $k = 1$, $x_0 = 0.1$ and $\lambda = 6$.

3.5. Conclusion

In this chapter, an existence and uniqueness result is obtained for a fractional order Volterra population model. The proposed analysis supplements the monotone property as well as the convergence of the iterative scheme for (3.2). The quadratic convergence of the quasilinearization scheme to the unique solution of the problem is also discussed. Finally, to show the efficiency of the proposed successive iterative scheme, a spectral method is

T	k	q	Method[**117**]	Proposed
.4	1	0.75	12	13
		0.5	13	15
		0.25	15	17
	0.75	0.75	13	15
		0.5	15	18
		0.25	18	21
	0.5	0.75	16	18
		0.5	20	22
		0.25	27	27

T	k	q	Method[**117**]	Proposed
1	1	0.75	19	22
		0.5	24	29
		0.25	37	38
	0.75	0.75	25	26
		0.5	-	35
		0.25	-	47
	0.5	0.75	-	32
		0.5	-	44
		0.25	-	62

T	k	q	Method[**117**]	Proposed
2	1	0.75	-	39
		0.5	-	56
		0.25	-	81
	0.75	0.75	-	46
		0.5	-	66
		0.25	-	98
	0.5	0.75	-	61
		0.5	-	79
		0.25	-	118

T	k	q	Method[**117**]	Proposed
5	1	0.75	-	117
		0.5	-	156
		0.25	-	240
	0.75	0.75	-	144
		0.5	-	182
		0.25	-	266
	0.5	0.75	-	200
		0.5	-	245
		0.25	-	315

TABLE 3.1. Comparison table for no. of iterations where $x_0 = 0.1$, $\lambda = 1 + T$ and $N = 8$.

coupled with the proposed iterative scheme and compared favorably with the iterative scheme in [**117**].

CHAPTER 4

FRACTIONAL ORDER TWO-POINT BOUNDARY VALUE PROBLEM

4.1. Introduction

Developments in last few decades, shows that fractional differential equations provide better and accurate models for various applications in fluid mechanics, visco-elasticity, physics, biology and economics. Several existence and uniqueness results are available in the literature that uses classical fixed point theorems and monotone iterative techniques for different types of fractional order differential equations. Apart from this, both theoretical as well as numerical results have also been obtained for two point boundary value problems of fractional order $1 < q < 2$ with various boundary conditions. For example, one can refer [**11, 19, 22, 65, 119**] for Dirichlet boundary conditions, [**4**] for Neumann boundary conditions and [**12, 128, 130, 153, 154**] for mixed boundary conditions. It is worth mentioning that attempts are also made to handle fractional order boundary value problems of higher order cases [**134**]. In [**118**], Al-Refai established an existence and uniqueness result for the following two point boundary value problem in the more general setting:

$$ {}^{c}D^{q}u(t) + g(t)u' + h(t)u = -\lambda k(t,u),\ t \in (0,1),\ 1 < q < 2 \tag{4.1a} $$

$$ u(0) - \alpha u'(0) = 0, \quad u(1) + \beta u'(1) = 0,\ \alpha, \beta \geq 0 \tag{4.1b} $$

where $k \in C^1([0,1] \times \mathbb{R}),\ g, h \in C[0,1]$ and ${}^{c}D^{q}$ is the Caputo fractional derivative of order q.

It is interesting to note that most of the techniques available in the recent literature [**118, 138**] are based on successive approximation. More specifically, by combining successive iteration with monotone method, existence and uniqueness results are obtained in [**118, 138**]. Consequently, the order of convergence thus obtained is linear. Hence, the main aim of this work is to prove an existence and uniqueness result for the following class of problem using an accelerated iterative procedure:

$$-{}^{c}D^{q}x(t) = f(t, x(t)),\ t \in (0,1) \tag{4.2a}$$

$$x(0) - \alpha_0 x'(0) = \gamma_0 \text{ and } x(1) + \alpha_1 x'(1) = \gamma_1 \tag{4.2b}$$

where $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $\alpha_0 \geq \frac{1}{q-1}$, $\alpha_1 \geq 0$ and ${}^{c}D^{q}$ is the Caputo fractional derivative of order $1 < q < 2$.

Present work focuses mainly on proving existence, uniqueness and quadratic convergence through monotone quasilinearization approach. This accelerated convergence is validated by combining proposed iteration with a finite difference discretization method [**129**]. Further the results are compared with the numerical scheme that combines successive iteration and above finite difference method. The results thus show that the proposed iteration outperforms successive approximation based scheme.

The organization of the chapter is as follows. Section 2 provides the definition of lower and upper solutions and few important results required for the main theorem. In Section 3, the existence and uniqueness of (4.2) is proved using quasilinearization iterative scheme. Section 4 provides numerical examples to show the efficiency of the proposed results.

4.2. Preliminaries

In this section, some basic definitions and results relevant to the main theorem are presented. First, lower and upper solutions of (4.2) are provided.

Definition 4.2.1. *A function $v(t) \in C^2[0,1]$ is called a lower solution of* (4.2) *if*

$$-{}^{c}D^{q}v(t) \leq f(t, v(t)),\ t \in (0,1) \tag{4.3a}$$

(4.3b) $$v(0) - \alpha_0 v'(0) \leq \gamma_0 \text{ and } v(1) + \alpha_1 v'(1) \leq \gamma_1.$$

It is called an upper solution of (4.2) *if the inequalities are reversed.*

Lemma 4.2.1. *[[**118**], Lemma 3.3] Let $x(t) \in C^2[0,1]$ and $b(t),\ c(t) \in C[0,1]$ with $c(t) > 0$ for all $t \in (0,1)$. Assume that $x(t)$ satisfies the inequalities*

(4.4a) $$-{}^cD^q x(t) + bx'(t) + cx(t) \geq 0,\ \ t \in (0,1)$$

(4.4b) $$x(0) - \alpha_0 x'(0) \geq 0 \text{ and } x(1) + \alpha_1 x'(1) \geq 0$$

where $\alpha_0 \geq \dfrac{1}{q-1}$ and $\alpha_1 \geq 0$. Then $x(t) \geq 0$ for all $t \in [0,1]$.

Lemma 4.2.2. *A function $x(t) \in C^2[0,1]$ is a solution of* (4.2) *if and only if it is a solution of the integral equation*

(4.5)
$$\begin{aligned} x(t) = \gamma_0 + \frac{\alpha_0 + t}{1 + \alpha_0 + \alpha_1} \Bigg(& \gamma_1 - \gamma_0 + \frac{1}{\Gamma q} \int_0^1 (1-s)^{q-1} f(s,x) \mathrm{d}s \\ + \frac{\alpha_1 (q-1)}{\Gamma q} \int_0^1 (1-s)^{q-2} f(s,x) \mathrm{d}s \Bigg) & - \frac{1}{\Gamma q} \int_0^t (t-s)^{q-1} f(s,x) \mathrm{d}s. \end{aligned}$$

Proof. The proof is same as the proof in [**118**]. □

4.3. Convergence Analysis

This section presents an existence and uniqueness result for the solution of (4.2) by constructing two monotone sequences using quasilinearization scheme. Henceforth following notations are considered throughout: $[v_0, u_0]$ denotes the sector $\{x : v_0 \leq x \leq u_0\}$, f_2 denotes first order partial derivative of f with respect to the second variable and $\mathbb{R}^-$ denotes the interval $(-\infty, 0)$. Also, define $m_1 = \min\limits_{t\in[0,1]} \{v_0, u_0\}$, $m_2 = \max\limits_{t\in[0,1]} \{v_0, u_0\}$ and $\|x\| = \sup\limits_{t\in[0,1]} |x(t)|$.

Theorem 4.3.1. *Let v_0, $u_0 \in C^2([0,1], \mathbb{R})$ represent, respectively, the lower and upper solutions of* (4.2)*, $f \in C([0,1] \times [m_1, m_2], \mathbb{R})$, $f_2 \in C([0,1] \times [m_1, m_2], \mathbb{R}^-)$. Further assume that*

(i) $|f_2(x, y_1) - f_2(x, y_2)| \leq M_2 |y_1 - y_2|, \quad M_2 > 0,$

(ii) *for each t, $f_2(t,x)$ is nondecreasing in x.*

Then the iterative schemes

$$-{}^cD^q v_{n+1}(t) = f(t, v_n(t)) + f_2(t, v_n(t))(v_{n+1}(t) - v_n(t)),\ \ t \in (0,1) \tag{4.6a}$$

$$v_{n+1}(0) - \alpha_0 {v_{n+1}}'(0) = \gamma_0 \text{ and } v_{n+1}(1) + \alpha_1 {v_{n+1}}'(1) = \gamma_1 \tag{4.6b}$$

and

$$-{}^cD^q u_{n+1}(t) = f(t, u_n(t)) + f_2(t, v_n(t))(u_{n+1}(t) - u_n(t)),\ \ t \in (0,1) \tag{4.7a}$$

$$u_{n+1}(0) - \alpha_0 {u_{n+1}}'(0) = \gamma_0 \text{ and } u_{n+1}(1) + \alpha_1 {u_{n+1}}'(1) = \gamma_1 \tag{4.7b}$$

converge uniformly and monotonically to the unique solution of (4.2) *in* $[v_0, u_0]$ *and the order of convergence is quadratic.*

Proof. It is clear that the iterative schemes (4.6) and (4.7) are well defined and have a unique solution at each step ([**118**]). Using induction on n, it can be proved that for all $n \in \mathbb{N}$ and $t \in [0,1]$,

$$v_0 \le v_1 \le \cdots \le v_n \le u_n \le \cdots \le u_1 \le u_0 \quad \text{on } [0,1]. \tag{4.8}$$

Let $p(t) = v_1 - v_0$. Then $p(0) - \alpha_0 p'(0) \ge 0$, $p(1) + \alpha_1 p'(1) \ge 0$ and

$$\begin{aligned} -{}^cD^q p(t) &= -{}^cD^q v_1 + {}^cD^q v_0 \\ &\ge f(t, v_0) + f_2(t, v_0)(v_1 - v_0) - f(t, v_0) \\ -{}^cD^q p(t) - f_2(t, v_0)p(t) &\ge 0. \end{aligned}$$

By Lemma 4.2.1, $p(t) \ge 0$. Thus $v_0 \le v_1$. Similarly, one can show that $u_1 \le u_0$. Let $p(t) = u_1 - v_1$. Then $p(0) - \alpha_0 p'(0) = 0$, $p(1) + \alpha_1 p'(1) = 0$ and

$$\begin{aligned} -{}^cD^q p(t) &= -{}^cD^q u_1 + {}^cD^q v_1 \\ &\ge f(t, u_0) + f_2(t, v_0)(u_1 - u_0) - f(t, v_0) \\ &\quad -f_2(t, v_0)(v_1 - v_0) \\ -{}^cD^q p(t) - f_2(t, v_0)p(t) &\ge f_2(t, v_0)(u_0 - v_0) - f_2(t, v_0)(u_0 - v_0) \\ -{}^cD^q p(t) - f_2(t, v_0)p(t) &\ge 0. \end{aligned}$$

By Lemma 4.2.1, $p(t) \geq 0$. Thus $v_1 \leq u_1$. Consequently,

$$v_0 \leq v_1 \leq u_1 \leq u_0 \quad \text{on } [0,1]. \tag{4.9}$$

Assume that (4.8) is true for $n = k$. That is,

$$v_0 \leq v_1 \leq \cdots \leq v_{k-1} \leq v_k \leq u_k \leq u_{k-1} \leq \cdots \leq u_1 \leq u_0 \quad \text{on } [0,1]. \tag{4.10}$$

To complete the induction argument it is enough to prove that

$$v_k \leq v_{k+1} \leq u_{k+1} \leq u_k \quad \text{on } [0,1]. \tag{4.11}$$

Let $p(t) = v_{k+1} - v_k$. Then $p(0) - \alpha_0 p'(0) = 0$, $p(1) + \alpha_1 p'(1) = 0$ and

$$\begin{aligned}
-{}^cD^q p(t) &= -{}^cD^q v_{k+1} + {}^cD^q v_k \\
&\geq f(t, v_k) + f_2(t, v_k)(v_{k+1} - v_k) - f(t, v_{k-1}) \\
&\quad -f_2(t, v_{k-1})(v_k - v_{k-1}) \\
-{}^cD^q p(t) - f_2(t, v_k)p(t) &\geq f_2(t, v_{k-1})(v_k - v_{k-1}) - f_2(t, v_{k-1})(v_k - v_{k-1}) \\
-{}^cD^q p(t) - f_2(t, v_k)p(t) &\geq 0.
\end{aligned}$$

By Lemma 4.2.1, $p(t) \geq 0$. Thus $v_k \leq v_{k+1}$. Similarly, one can show that $u_{k+1} \leq v_k$. Let $p(t) = u_{k+1} - v_{k+1}$. Then $p(0) - \alpha_0 p'(0) = 0$, $p(1) + \alpha_1 p'(1) = 0$ and

$$\begin{aligned}
-{}^cD^q p(t) &= -{}^cD^q u_{k+1} + {}^cD^q v_{k+1} \\
&\geq f(t, v_k) + f_2(t, v_k)(u_{k+1} - v_k) - f(t, v_k) \\
&\quad -f_2(t, v_k)(v_{k+1} - v_k) \\
-{}^cD^q p(t) - f_2(t, v_k)p(t) &\geq f_2(t, v_k)(u_k - v_k) - f_2(t, v_k)(u_k - v_k) \\
-{}^cD^q p(t) - f_2(t, v_k)p(t) &\geq 0.
\end{aligned}$$

By Lemma 4.2.1, $p(t) \geq 0$. Thus $v_{k+1} \leq u_{k+1}$. Consequently, (4.11) is proved. To complete the proof, it is enough to show that $\{v_{n+1}\}$ and $\{u_{n+1}\}$ are equicontinuous. Define $H(t, v_{n+1}) = f(t, v_n(t)) + f_2(t, v_n(t))(v_{n+1}(t) - v_n(t))$, $\|f_2\| \leq M_1$, $M = \sup\limits_{\substack{\nu \in [v_0, u_0] \\ t \in [0,1]}} |f(t, \nu) +$

$2u_0M_1|$ and $A = \dfrac{|\gamma_1-\gamma_0|}{1+\alpha_0+\alpha_1} + \dfrac{M(1+q\alpha_1)}{\Gamma(q+1)(1+\alpha_0+\alpha_1)} + \dfrac{2M}{\Gamma(q+1)}$. For any $t_1 < t_2$,

$$\begin{aligned}
|v_{n+1}(t_1) - v_{n+1}(t_2)| &\leq \frac{|\gamma_1-\gamma_0||t_1-t_2|}{1+\alpha_0+\alpha_1} + \frac{1}{\Gamma q}\bigg|\int_0^1 (1-s)^{q-1}H(s,v_{n+1})\mathrm{d}s \\
&\quad +\alpha_1(q-1)\int_0^1 (1-s)^{q-2}H(s,v_{n+1})\mathrm{d}s\bigg|\frac{|t_1-t_2|}{1+\alpha_0+\alpha_1} \\
&\quad +\frac{1}{\Gamma q}\bigg|\int_0^{t_1}(t_1-s)^{q-1}H(s,v_{n+1})\mathrm{d}s - \int_0^{t_2}(t_2-s)^{q-1}H(s,v_{n+1})\mathrm{d}s\bigg| \\
&\leq \frac{|\gamma_1-\gamma_0||t_1-t_2|}{1+\alpha_0+\alpha_1} + \frac{M}{\Gamma q}\left(\frac{1}{q}+\alpha_1\right)\frac{|t_1-t_2|}{1+\alpha_0+\alpha_1} \\
&\quad +\frac{M}{\Gamma(q+1)}\left(2(t_2-t_1)^q + t_1^q - t_2^q\right) \\
&\leq \left(|\gamma_1-\gamma_0| + \frac{M(1+q\alpha_1)}{\Gamma(q+1)}\right)\frac{|t_1-t_2|}{1+\alpha_0+\alpha_1} + \frac{2M(t_2-t_1)^q}{\Gamma(q+1)} \\
&\leq A|t_1-t_2|.
\end{aligned}$$

Thus $\{v_n\}$ is equicontinuous. Similarly, $\{u_n\}$ can also proved to be equicontinuous. Thus it is clear that the sequences $\{v_n\}$ and $\{u_n\}$ are uniformly bounded and equicontinuous on $[0,1]$. Hence by Ascoli-Arzela's Theorem, there exist subsequences that converge uniformly on $[0,1]$. In view of (4.8), it follows that the sequences $\{v_n\}$ and $\{u_n\}$ converge uniformly and monotonically to ρ_1 and ρ_2 respectively. It is clear that $\rho_1 \leq \rho_2$ on $[0,1]$. Define $p(t) = \rho_1 - \rho_2$ on $[0,1]$. Clearly $p(0) - \alpha_0 p'(0) = 0$ and $p(1) + \alpha_1 p'(1) = 0$. It is easy to show that $-{}^cD^q p(t) - f_2(t,u_0)p(t) \geq 0$. Consequently, $p(t) \geq 0$. Hence, (4.2) has a unique solution. To prove the quadratic convergence of the quasilinearization scheme, define $p_{n+1} = x - v_{n+1}$ and $r_{n+1} = u_{n+1} - x$. Then $p_{n+1}(0) - \alpha_0 {p_{n+1}}'(0) = 0$, $p_{n+1}(1) + \alpha_1 {p_{n+1}}'(1) = 0$ and

$$\begin{aligned}
-{}^cD^q p_{n+1}(t) &= -{}^cD^q x(t) + {}^cD^q v_{n+1}(t) \\
&= f(t,x) - f(t,v_n) - f_2(t,v_n)(v_{n+1}-v_n) \\
&= f_2(t,\delta)p_n - f_2(t,v_n)(p_n - p_{n+1})
\end{aligned}$$

$$-{}^cD^q p_{n+1}(t) = f_2(t,v_n)p_{n+1} + (f_2(t,\delta) - f_2(t,v_n))p_n \tag{4.12}$$

where $v_n \leq \delta \leq x$. By Lemma 4.2.2,

$$
\begin{aligned}
p_{n+1}(t) &= \frac{\alpha_0+t}{1+\alpha_0+\alpha_1}\left(\frac{1}{\Gamma q}\int_0^1 (1-s)^{q-1}(f_2(s,v_n)p_{n+1}+(f_2(s,\delta)-f_2(s,v_n))p_n)\mathrm{d}s\right.\\
&\quad\left.+\frac{\alpha_1(q-1)}{\Gamma q}\int_0^1 (1-s)^{q-2}(f_2(s,v_n)p_{n+1}+(f_2(s,\delta)-f_2(s,v_n))p_n)\mathrm{d}s\right)\\
&\quad-\frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}(f_2(s,v_n)p_{n+1}+(f_2(s,\delta)-f_2(s,v_n))p_n)\mathrm{d}s\\
p_{n+1}(t) &\leq \frac{(\alpha_0+t)M_2}{1+\alpha_0+\alpha_1}\left(\frac{1}{\Gamma q}\int_0^1 (1-s)^{q-1}p_n^2\mathrm{d}s+\frac{\alpha_1(q-1)}{\Gamma q}\int_0^1 (1-s)^{q-2}p_n^2\mathrm{d}s\right)\\
&\quad-\frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}f_2(s,v_n)p_{n+1}\mathrm{d}s, \quad [\ f_2\leq 0 \text{ and } p_{n+1}\geq 0]\\
\|p_{n+1}\| &\leq \frac{M_2\|p_n\|^2}{\Gamma(q+1)}+\frac{\alpha_1 M_2\|p_n\|^2}{\Gamma q}+\frac{M_1\|p_{n+1}\|}{\Gamma(q+1)}\\
\|p_{n+1}\| &\leq \frac{M_2(1+q\alpha_1)}{\Gamma(q+1)-M_1}\|p_n\|^2 = N\|p_n\|^2
\end{aligned}
$$

where $N = \dfrac{M_2(1+q\alpha_1)}{\Gamma(q+1)-M_1}$. Similarly, $\|r_{n+1}\| \leq N_1\|r_n\|^2$ for some N_1. □

4.4. Numerical Illustration

In this section, the relevance of the proposed iterative scheme is illustrated using numerical examples. To solve the problem numerically using the proposed iterative scheme or successive iterative scheme, at each step, one has to solve a linear two point boundary value problem of fractional order $1 < q < 2$. For all examples, these linear problems are solved using a finite difference method [**129**]. For all the numerical simulations, the stopping criterion is $\|v_{n+1} - v_n\| \leq 10^{-8}$. Throughout this section, N denotes the number of grid points.

Example 4.4.1. *Consider the boundary value problem for* $1 < q < 2$

(4.13a) $$-{}^cD^q x(t) = 10x^2(t) - x(t) + f(t),\ \ t \in (0,1)$$

(4.13b) $$x(0) - 12x'(0) = 0 \ \textit{and}\ x(1) + 2x'(1) = 0,$$

where $f(t) = -10t^{24} + 100t^{14} - t^{12} - 100t^4 + 5t^2 - \frac{10t^{2-q}}{\Gamma(3-q)} + \frac{\Gamma(13)t^{12-q}}{\Gamma(13-q)}$.

For the choice of $v_0 = -4,\ u_0 = 0,\ q \in [1.1, 2)$ all the hypotheses of the Theorem 4.3.1 are satisfied. Hence the boundary value problem 4.13 has a unique solution in $[-4, 0]$. Figure 4.1 gives the numerical solution of Example 4.4.1 for various values of q, whereas Figure 4.2 displays the monotone property of the sequences $\{v_n\}$ and $\{u_n\}$. Table 4.1 presents the comparison of number of iterations based on the iterative scheme discussed in [**118**] and the proposed iterative scheme. The iteration in [**118**] depends on a constant $c > 0$, which is chosen as $c = 82$.

Example 4.4.2. *Consider the boundary value problem for* $1 < q < 2$

(4.14a) $$-{}^cD^q x(t) = 10e^{-x(t)} - x(t),\ t \in (0, 1)$$

(4.14b) $$x(0) - 10x'(0) = 1 \text{ and } x(1) + 3x'(1) = 4.$$

For the choice of $v_0 = 1,\ u_0 = 4,\ q \in [1.1, 2)$ all the hypotheses of the Theorem 4.3.1 are satisfied. Hence the boundary value problem 4.14 has a unique solution in $[1, 4]$. Similar to Example 4.4.1, Figure 4.3 shows the numerical solution of Example 4.4.2 for various values of q, whereas Figure 4.4 displays the monotone property of the sequences $\{v_n\}$ and $\{u_n\}$. Table 4.2 presents the comparison of number of iterations based on the iterative scheme discussed in [**118**] and the proposed iterative scheme. The constant 'c' required for the iteration is chosen as $c = 5$.

4.5. Conclusion

In this chapter, a two point boundary value problem of fractional order $1 < q < 2$ is considered. Using quasilinearization technique, two well defined sequences are constructed that converge uniformly, monotonically and quadratically to the unique solution of the problem. Based on the proposed accelerated iterative procedure a finite difference based numerical method is also proposed to solve nonlinear two point boundary value problem.

q	Method in [**118**]	Proposed
1.1	76	9
1.2	74	8
1.3	71	8
1.4	66	8
1.5	62	8
1.6	58	8
1.7	54	8
1.8	51	8
1.9	48	8

TABLE 4.1. Comparison of number of iterations for Example 4.4.1 where $N = 1000$.

q	Method in [**118**]	Proposed
1.1	28	7
1.2	28	7
1.3	28	7
1.4	28	7
1.5	28	7
1.6	28	7
1.7	28	7
1.8	28	7
1.9	28	7

TABLE 4.2. Comparison of number of iterations for Example 4.4.2 where $N = 1000$.

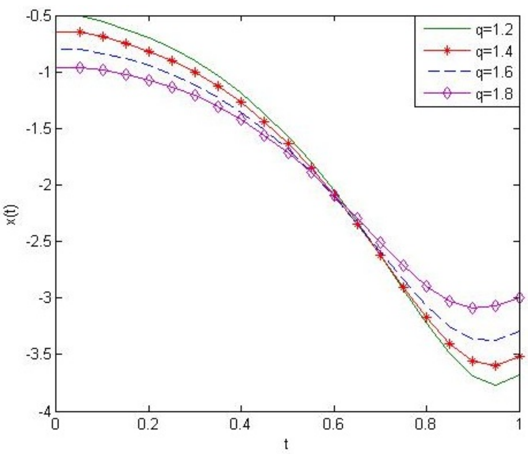

FIGURE 4.1. Approximate solution of Example 4.4.1 for various values of q.

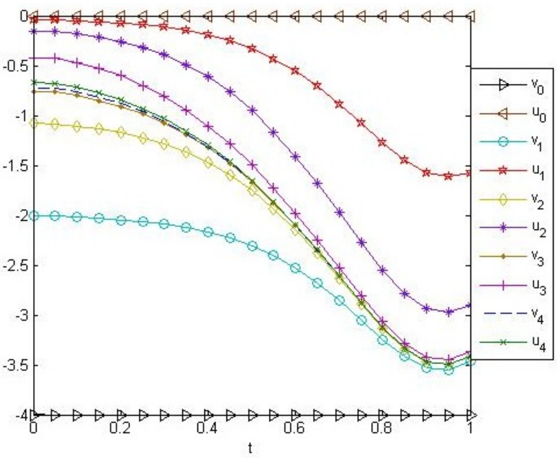

FIGURE 4.2. A plot of v_n and u_n, $n = 0, 1, 2, 3, 4$ for Example 4.4.1 when $q = 1.5$.

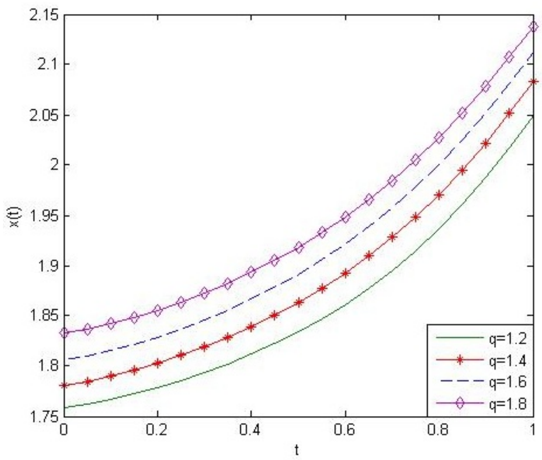

FIGURE 4.3. Approximate solution of Example 4.4.2 for various values of q.

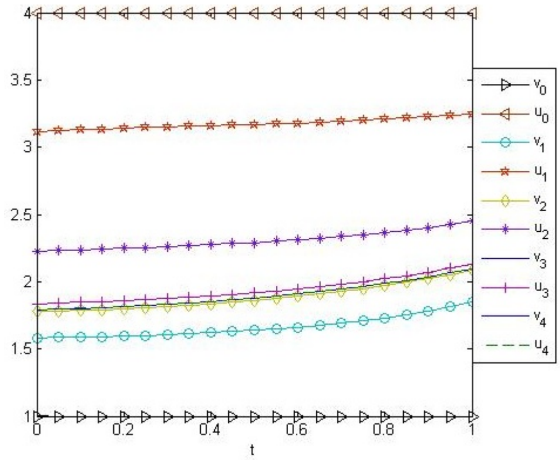

FIGURE 4.4. A plot of v_n and u_n, $n = 0, 1, 2, 3, 4$ for Example 4.4.2 when $q = 1.5$.

CHAPTER 5

NONLINEAR INTEGRO PARTIAL DIFFERENTIAL EQUATION

5.1. Introduction

In literature, differential equations arising in various mathematical models have been improved by adding certain non-local integral terms either in the governing equation or in the boundary condition for more accurate results (for example [**82, 100, 101, 106, 113, 114, 127, 136**]). The following Volterra partial integro-differential equation with positive memory may be considered as a modification of the well known Fisher equation arising in the population model.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au - bu^2 - cu \int_0^t \kappa(t-s)u(x,s)\mathrm{d}s + g(x,t) \quad \text{in } Q, \quad u|_{\partial_p Q} = \phi, \tag{5.1}$$

where a, b and c are non negative constants, $Q = (0,1) \times (0,T)$ and $\partial_p Q = \partial Q \backslash ((0,1) \times \{T\})$ denotes the parabolic boundary of Q. Here ϕ is the restriction of some smooth function $\Phi \in C^{2,1}(\overline{Q})$ on $\partial_p Q$ and κ is a positive continuous function on $\mathbb{R}$. For the choice of $b = 0$, $g(x,t) \equiv 0$ and $\kappa \equiv 1$, (5.1) arises in the analysis of space-time dependent nuclear reactor dynamics if the effect of a linear temperature feedback is taken into consideration [**106, 113, 114, 122, 123, 135**]. For the choice of $a = b = c = 1$ and $\kappa(t) = \frac{t}{T^2}\exp(-\frac{t}{T})$, (5.1) represents the mathematical population model for the evolution of a community of species that is allowed to diffuse spatially [**64**]. (5.1) can also be considered as a generalization of the following ordinary integro-differential equation,

$$x'(t) = ax(t) - bx^2(t) - cx(t)\int_0^t x(s)\mathrm{d}s, \quad x(0) = x_0 \geq 0 \tag{5.2}$$

arising in the population model in a closed system. There are ample number of numerical methods, including different type of spectral methods, available in the literature for the ordinary integro-differential equation (5.2) (see [**39, 74, 107, 108, 110, 111**] and the reference therein). It is worth mentioning that similar to the classical model (5.2), generous numerical techniques [**63, 92, 97, 109, 120**] are available to handle corresponding fractional order model

$$(5.3) \qquad {}^{c}D^{q}x(t) = ax(t) - bx^{2}(t) - cx(t)\int_{0}^{t} x(s)\mathrm{d}s, \quad x(0) = x_0 \geq 0.$$

However, the numerical methods for solving the partial integro-differential equation (5.1) are very limited.

The present work proposes an efficient numerical method for a class of partial integro-differential equation (5.1) by combining bivariate spectral method with a monotone iterative scheme. Past few decades have seen tremendous development of various numerical schemes which could replace traditional methods such as finite difference and finite element schemes. One such class of schemes, which has seen extensive development and applications, is spectral methods. The convergence rate of spectral methods depends only on the smoothness of the solution and hence produces highly accurate solutions with a small number of grid points. One popular choice of basis functions for spectral collocation methods is Lagrange polynomials, which does not require periodic boundary conditions. Another advantage of using Chebyshev spectral collocation method is that these polynomials are well defined throughout the domain due to which method yield good accuracy even on non-collocation points. Also, to avoid the error intrinsic in higher order polynomial approximation on equidistant nodes, Chebyshev-Gauss-Lobotto points are in general, considered [**27, 139**].

Hence in the proposed work, the authors have extended the bivariate Chebyshev spectral collocation method by Motsa et al [**98, 99**] to the initial boundary value problems governed by the partial integro-differential equation (5.1). Both time and space operators have been approximated using Chebyshev spectral collocation method with Lagrange interpolation polynomial, which differentiate present scheme from other Chebyshev spectral collocation methods [**67, 75**]. The authors not only extend the method numerically, but also prove the existence and uniqueness of the solution of (5.3) as well as the following partial integro differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u, \tilde{u}) \quad \text{in } Q, \quad u|_{\partial_p Q} = \phi, \tag{5.4}$$

where $Q = (0, 1) \times (0, T)$, $\Omega = (0, 1)$ and $\partial_p Q = \partial Q \backslash ((0, 1) \times \{T\})$ denotes the parabolic boundary of Q and $\tilde{u}$ denotes $\int_0^t \kappa(t - s)u(x, s)\mathrm{d}s$. Here $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and ϕ is the restriction of Φ on $\partial_p Q$ where $\Phi \in C^{2,1}(\overline{Q})$.

Although the results on the existence and uniqueness as well as the convergence of the monotone iterative scheme for the problems (5.3) and (5.4) are available in the literature [**113, 114, 120**], present study proposes a novel proof completely different from the existing ones. The proposed proof is based on the concepts on operator theory in partially ordered Banach space as done by Lakshmikantham et al in [**83**]. This idea is extended to prove the existence and uniqueness of (5.3) and (5.4) as well as the convergence analysis of the associated monotone iterative scheme. It is worth mentioning that the results in [**83**] failed to handle the (5.3) and (5.4) as the associated operator fails to satisfy certain positivity condition.

The organization of the chapter is as follows. Section 2 presents the basic definition, notations and results required to prove the main result. This section also demonstrates the operator theory methods by proving the existence and uniqueness result for the fractional order integro differential equation (5.3). Our main results, the existence and uniqueness of the solution of the partial integro differential equation (5.4), the convergence of the iterative scheme as well as its monotone property, are proved in Section 3. The derivation of the bivariate Chebyshev spectral collocation using proposed monotone iteration is detailed in Section 4. In Section 5, the developed scheme is illustrated by solving the partial integro differential equation arising from population dynamics. Few concluding remarks are given in Section 6.

5.2. Preliminaries

This section supplies some basic definitions, notations and results relevant to the main theorem. Throughout this chapter, we assume that $E = (E, \leq, \|\cdot\|)$ is an ordered Banach space with order cone E_+, $T : E \times E \to E$ is a continuous compact operator. Let $F : E \to E$ be a nonlinear operator defined by $F(u) = T(u, L(u))$, where $L : E \to E$ is a positive continuous linear operator. For $i = 1, 2$, T_u^i denotes the partial Frechet derivative of T with respect to the i^{th} variable. First we present a basic result which will ensure the existence of solution to the operator equation $u = F(u)$ via monotone iteration. To understand further one requires the following definition.

Definition 5.2.1. *A pair of function $(v_0, w_0) \in E \times E$ is said to be an ordered coupled lower and upper solution of the operator equation $u = F(u)$ if $v_0 \leq w_0$ and*

$$\begin{aligned} v_0 &\leq T(v_0, L(w_0)), \\ w_0 &\geq T(w_0, L(v_0)). \end{aligned}$$

Throughout the chapter, $[v_0, w_0]$ denotes the sector $\{u \in E : v_0 \leq u \leq w_0\}$. The following lemma is an important tool to prove the existence and uniqueness of (5.3) and (5.4).

Lemma 5.2.1. *Let E be an ordered Banach space with a normal order cone E_+. Assume that $T : E \times E \to E$ satisfies the following hypotheses.*

(i) *$(v_0, w_0) \in E \times E$ be an ordered coupled lower and upper solution for the operator equation $u = T(u, L(u))$,*
(ii) *The Frechet derivative $T_u' = (T^1, T^2)_u$ exists for every $u \in [v_0, w_0] \times [L(v_0), L(w_0)]$,*
(iii) *$T^1_{(u,\cdot)} : E \to E$ is a positive operator for every $u \in [v_0, w_0]$,*
(iv) *$T^2_{(\cdot,u)} : E \to E$ is a negative operator for every $u \in [L(v_0), L(w_0)]$.*

Then for $n \in \mathbb{N}$, relations

$$v_{n+1} = T(v_n, L(w_n)) \tag{5.5}$$

$$w_{n+1} = T(w_n, L(v_n)) \tag{5.6}$$

define a non decreasing sequence $\{v_n\}$ and a non increasing sequence $\{w_n\}$ which converges to the solutions of the operator equation $v = T(v, L(w))$ and $w = T(w, L(v))$ respectively.

Proof. From the construction (5.5), $v_1 = T(v_0, L(w_0))$. Using hypothesis (i), it is easy to verify that $v_0 \leq v_1$. Similarly, $w_1 \leq w_0$. Define $\Theta_n = \theta(w_n, L(v_n)) + (1-\theta)(v_n, L(w_n))$, $0 \leq \theta \leq 1$ and $p_{n+1} = w_n - v_n$, $q_n = v_{n+1} - v_n$ and $r_n = w_n - w_{n+1}$ for $n = 0, 1, 2, \cdots$. Then

$$\begin{aligned}
p_1 &= T(w_0, L(v_0)) - T(v_0, L(w_0)) \\
&= \int_0^1 T'_{\Theta_0}((w_0, L(v_0)) - (v_0, L(w_0)))\mathrm{d}\theta \\
&= \int_0^1 T^1_{\theta w_0 + (1-\theta)v_0}(w_0 - v_0)\mathrm{d}\theta + \int_0^1 T^2_{\theta L(v_0)+(1-\theta)L(w_0)}(L(v_0) - L(w_0))\mathrm{d}\theta \\
p_1 &\geq 0.
\end{aligned}$$

Similarly, one can show that for all $n \in \mathbb{N}$, p_n, q_n and r_n are non negative. Using the compactness property as well as property of the normal cone one can easily conclude that $\{v_n\}$ and $\{w_n\}$ are convergent sequences. Let v and w are the limits of the sequences $\{v_n\}$ and $\{w_n\}$ respectively. Thus v and w satisfy the operator equation $v = T(v, L(w))$ and $w = T(w, L(v))$. □

Remark 5.2.1. *In addition to $(i) - (iv)$, if one assumes that $T(u, L(u))$ is a contraction map then one can easily conclude that $v = w$. Consequently, the operator equation $u = T(u, L(u))$ has a unique solution in $[v_0, w_0]$. Thus the operator equation $u = F(u)$ has a unique solution in $[v_0, w_0]$.*

5.3. Existence and Uniqueness

In this section, as an application of Lemma 5.2.1 an existence and uniqueness result for a fractional order integro differential equation via monotone iterative scheme is obtained. Consider the following initial value problem

(5.7) $$ {}^cD^q x(t) = f(x(t), \tilde{x}(t)), \quad x(0) = x_0 $$

where $\tilde{x}(t) = \int_0^t x(s)\mathrm{d}s$, ${}^cD^q$ is the Caputo fractional derivative of order $q, 0 < q \leq 1$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Denote partial derivative of f with respect to the first and second variable by f_1 and f_2 respectively.

Remark 5.3.1. *For a real number λ and $\mathfrak{g} \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$, it is easy to verify that the fractional order differential equation*

$$ {}^cD^q x(t) = \lambda x(t) + \mathfrak{g}(t), \quad x(t_0) = x_0 \tag{5.8}$$

is equivalent to the integral equation

$$x(t) = x_0 E_q(\lambda(t - t_0)^q) + \int_{t_0}^{t} (t - s)^{q-1} E_{q,q}(\lambda(t - s)^q)\mathfrak{g}(s)\mathrm{d}s, \quad t \in [t_0, T] \tag{5.9}$$

where $E_q(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(qk + 1)}$ *and* $E_{q,q}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(qk + q)}$ *are the Mittag-Leffler functions of one parameter and two parameters, respectively.*

Define $m = \min_{t\in[0,T]} \{v_0, w_0\}$, $M = \max_{t\in[0,T]} \{v_0, w_0\}$, $\tilde{m} = \min_{t\in[0,T]} \{\tilde{v}_0, \tilde{w}_0\}$ and $\tilde{M} = \max_{t\in[0,T]} \{\tilde{v}_0, \tilde{w}_0\}$. Throughout this subsection the following assumptions are considered.

(i) Let $v_0, w_0 \in C[0, T]$ satisfy $v_0 \leq w_0$ and

$$\begin{aligned} {}^cD^q v_0 &\leq f(v_0, \tilde{w}_0), \quad v_0(0) \leq x_0, \\ {}^cD^q w_0 &\geq f(w_0, \tilde{v}_0), \quad w_0(0) \geq x_0. \end{aligned}$$

(ii) For some $\delta > 0$, $f, f_1, f_2 : C[m - \delta, M + \delta] \times [\tilde{m} - \delta, \tilde{M} + \delta] \to \mathbb{R}$ is continuous and for all $s_1 \in [m, M]$, $s_2 \in [\tilde{m}, \tilde{M}]$

$$f_1(s_1, s_2) + \lambda \geq 0 \text{ and } f_2(s_1, s_2) \leq 0$$

Theorem 5.3.1. *Let the hypotheses (i) and (ii) be satisfied then the initial value problem (5.7) has a unique solution in $[v_0, w_0]$. Moreover, there exist monotone sequences $\{v_n\}$ and $\{w_n\}$ which converge uniformly and monotonically to the unique solution of (5.7).*

Proof. The initial value problem (5.7) can be rewritten as

$$ {}^cD^q x(t) + \lambda x(t) = f(x(t), \tilde{x}(t)) + \lambda x(t); \quad x(0) = x_0. \tag{5.10}$$

The above initial value problem (5.10) is equivalent to the integral equation

$$x(t) = x_0 E_q(-\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(-\lambda(t-s)^q)(f(x(s),\tilde{x}(s)) + \lambda x(s))\mathrm{d}s. \tag{5.11}$$

Define operators $F : C[0,T] \to C[0,T]$ by

$$Fx(t) = x_0 E_q(-\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(-\lambda(t-s)^q)(f(x(s),\tilde{x}(s)) + \lambda x(s))\mathrm{d}s. \tag{5.12}$$

It is easy to verify that the operator is well defined and the solution of the initial value problem (5.7) is nothing, but the solution of the operator equation $Fx = x$. For each $(x,y) \in C[0,T] \times C[0,T]$ define $T : C[0,T] \times C[0,T] \to C[0,T]$ by

$$T(x(t),y(t)) = x_0 E_q(-\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(-\lambda(t-s)^q)(f(x(s),y(s)) + \lambda x(s))\mathrm{d}s. \tag{5.13}$$

Thus the operator equation $F(x) = x$ can be reformulated as $T(u, L(u)) = u$. For $u, w \in [v_0, w_0]$, define operators $T^i_{(u,L(w))} : C[0,T] \to C[0,T]$, $i = 1,2$ by

$$\begin{aligned}
T^1_{(u,L(w))}h(t) &= \int_0^t (t-s)^{q-1} E_{q,q}(-\lambda(t-s)^q)(f_1(u,L(w)) + \lambda)h(s)\mathrm{d}s && (5.14)\\
T^2_{(u,L(w))}h(t) &= \int_0^t (t-s)^{q-1} E_{q,q}(-\lambda(t-s)^q) f_2(u,L(w))L(h(s))\mathrm{d}s. && (5.15)
\end{aligned}$$

It is easy to verify that $T^1_{(u,L(w))}$ and $T^2_{(u,L(w))}$ are the partial Frechet derivative of $T(u, L(w))$ with respect to the first and second variable respectively. Combining with this choice of λ, it can be concluded that for any u, w in $[v_0, w_0]$ the operators $T^1_{(u,L(w))}$ and $T^2_{(u,L(w))}$ are positive and negative operators respectively. Define a norm on $C[0,T] \times C[0,T]$ by $\|h\|_\rho = \|(h_1,h_2)\|_\rho = \max\left\{\sup_{t\in[0,T]}\left|\frac{h_1(t)}{E_q(\rho t^q)}\right|, \sup_{t\in[0,T]}\left|\frac{h_2(t)}{E_q(\rho t^q)}\right|\right\}$. Define $N = \max\{\sup_{(s_1,s_2)\in\Gamma}|f_1(s_1,s_2) + \lambda|, \sup_{(s_1,s_2)\in\Gamma}|f_2(s_1,s_2)|\}$ and $\Gamma = [m-\delta, M+\delta] \times [\tilde{m}-\delta, \tilde{M}+\delta]$. For any $u, w \in [v_0, w_0]$

and $h \in C[0,T] \times C[0,T]$ with $\|h\|_\rho = 1$.

$$\begin{aligned}
\left|T'_{(u,L(w))}h(t)\right| &\leq \left|\int_0^t (t-s)^{q-1}E_{q,q}(-\lambda(t-s)^q)(f_1(u,\tilde{u})+\lambda)h_1(s)\mathrm{d}s\right| \\
&\quad + \left|\int_0^t (t-s)^{q-1}E_{q,q}(-\lambda(t-s)^q)f_2(u,\tilde{u})\tilde{h_2}(s)\mathrm{d}s\right| \\
&\leq \frac{N\Gamma(q)}{\Gamma(q)}\int_0^t (t-s)^{q-1}|h_1(s)|\mathrm{d}s + \frac{N\Gamma(q)}{\Gamma(q)}\int_0^t (t-s)^{q-1}|\tilde{h}_2(s)|\mathrm{d}s \\
&\leq \frac{N\Gamma(q)}{\Gamma(q)}\int_0^t (t-s)^{q-1}E_q(\rho s^q)\left|\frac{h_1(s)}{E_q(\rho s^q)}\right|\mathrm{d}s \\
&\quad + \frac{N\Gamma(q)}{\Gamma(q)}\int_0^t (t-s)^{q-1}\int_0^s \frac{|h_2(\tau)|}{E_q(\rho\tau^q)}E_q(\rho\tau^q)\mathrm{d}\tau\mathrm{d}s \\
&\leq \left(\frac{N\Gamma(q)}{\rho} + \frac{N(\Gamma(q))^2T^{1-q}}{\rho^2}\right)\|h\|_\rho E_q(\rho t^q) \\
\left|\frac{T'_{u,L(w))}h(t)}{E_q(\rho t^q)}\right| &\leq \frac{N\Gamma(q)}{\rho} + \frac{N(\Gamma(q))^2T^{1-q}}{\rho^2}.
\end{aligned}$$

Thus $\|T'_{(u,L(w))}\| \leq \frac{N\Gamma(q)}{\rho} + \frac{N(\Gamma(q))^2T^{1-q}}{\rho^2} = \theta < 1$ for large ρ. Thus $T(u,L(u))$ satisfies the Lipschitz condition with Lipschitz constant θ. Thus all the hypotheses of Lemma 5.2.1 are satisfied. Consequently, the operator equation $T(u,L(u)) = u$ has a unique solution in $[v_0, w_0]$. Hence the initial value problem (5.7) has a unique solution in $[v_0, w_0]$. Moreover the iterative schemes (5.5) and (5.6) converge monotonically and uniformly to the unique solution of the initial value problem (5.7). □

Remark 5.3.2. *For the initial value problem* (5.7)*, the above mentioned abstract iterative scheme* (5.5) *and* (5.6) *is equivalent to the iterative scheme*

$$\begin{aligned}
{}^cD^q v_{n+1} + \lambda v_{n+1} &= \lambda v_n + f(v_n, \tilde{w_n}), \quad v_{n+1}(0) = x_0, \\
{}^cD^q w_{n+1} + \lambda w_{n+1} &= \lambda w_n + f(w_n, \tilde{v_n}), \quad w_{n+1}(0) = x_0.
\end{aligned}$$

Corollary 5.3.1. *The fractional order population model* (5.3) *has a unique solution in* $[0,1]$ *if* $b \geq a > 0$*. If* $a > b > 0$ *then the fractional order population model* (5.3) *has a unique solution in* $[0, \frac{a}{b}]$*.*

Proof. For the choice of $f(x,\tilde{x}) = ax - bx^2 - cx\tilde{x}$, if $b \geq a > 0$, then $(v_0, w_0) = (0,1)$ is a coupled lower and upper solution and all the hypothesis of Theorem 5.3.1 are satisfied.

Hence the (5.3) has a unique solution in the sector [0,1]. Similarly, if $a > b > 0$, then $(v_0, w_0) = (0, \frac{a}{b})$ is coupled lower and upper solution and all the hypothesis of Theorem 5.3.1 are satisfied. Hence the (5.3) has a unique solution in the sector $[0, \frac{a}{b}]$. $\square$

We extend the technique to prove the existence and uniqueness result via monotone iteration for the following partial integro differential equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u, \tilde{u}) \quad \text{in } Q, \quad u|_{\partial_p Q} = \phi \tag{5.16}$$

where $Q = (0,1) \times (0,T)$, $\Omega = (0,1)$ and $\partial_p Q = \partial Q \backslash ((0,1) \times \{T\})$ denotes the parabolic boundary of Q and $\tilde{u}$ denotes $\int_0^t \kappa(t-s)u(x,s)\mathrm{d}s$. Here $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and ϕ is the restriction of Φ on $\partial_p Q$ where $\Phi \in C^{2,1}(\overline{Q})$. For $u \in C([0,1] \times [0,T])$ define $L(u) = \tilde{u}$. The following definition suitably modify the coupled lower and upper solution for the problem (5.16).

Definition 5.3.1. *A pair of functions $v, w \in C^{2,1}(\overline{Q})$ is called an ordered coupled lower and upper solutions of (5.4) if $v \leq w$ and*

$$\begin{aligned} \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} &\leq f(v, \tilde{w}) \quad \text{in } Q, \quad v|_{\partial_p Q} \leq \phi, \\ \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} &\geq f(w, \tilde{v}) \quad \text{in } Q, \quad w|_{\partial_p Q} \geq \phi. \end{aligned}$$

Define $m = \min\limits_{(x,t)\in\overline{Q}} \{v_0, w_0\}$, $M = \max\limits_{(x,t)\in\overline{Q}} \{v_0, w_0\}$, $\tilde{m} = \min\limits_{(x,t)\in\overline{Q}} \{\tilde{v}_0, \tilde{w}_0\}$ and $\tilde{M} = \max\limits_{(x,t)\in\overline{Q}} \{\tilde{v}_0, \tilde{w}_0\}$. Throughout this section we assumed the following:

(i) Let v_0 and w_0 in $C^{2,1}(\overline{Q})$ be an ordered coupled lower and upper solution of (5.16).

(ii) For some $\delta > 0$, $f, f_1, f_2 : C[m-\delta, M+\delta] \times [\tilde{m}-\delta, \tilde{M}+\delta] \to \mathbb{R}$ are continuous and for all $s_1 \in [m, M]$, $s_2 \in [\tilde{m}, \tilde{M}]$

$$f_1(s_1, s_2) + \lambda \geq 0 \text{ and } f_2(s_1, s_2) \leq 0.$$

Theorem 5.3.2. *The parabolic partial integro differential equation (5.16) has a unique solution in* $[v_0, w_0]$*. Moreover the sequences* $\{v_n\}$ *and* $\{w_n\}$ *generated by*

$$\frac{\partial v_{n+1}}{\partial t} - \frac{\partial^2 v_{n+1}}{\partial x^2} + \lambda v_{n+1} = \lambda v_n + f(v_n, \tilde{w}_n), \quad v_{n+1}|_{\partial_p Q} = \phi \tag{5.17}$$

$$\frac{\partial w_{n+1}}{\partial t} - \frac{\partial^2 w_{n+1}}{\partial x^2} + \lambda w_{n+1} = \lambda w_n + f(w_n, \tilde{v}_n), \quad w_{n+1}|_{\partial_p Q} = \phi \tag{5.18}$$

are well-defined and converge to the unique solution monotonically.

Proof: It is enough to prove this theorem for (5.16) with homogeneous initial condition

$$\frac{\partial u}{\partial t} = \triangle u + f(u, \tilde{u}) \quad \text{in } Q, \quad u|_{\partial_p Q} = 0. \tag{5.19}$$

One can convert this problem as a fixed point problem in Banach space $C(\overline{Q})$. The Eqn. (5.19) can be written as

$$\frac{\partial u}{\partial t} - \triangle u + \lambda u = f(u, \tilde{u}) + \lambda u \quad \text{in } Q, \quad u|_{\partial_p Q} = 0. \tag{5.20}$$

Define an operator $F : [v_0, w_0] \subset C(\overline{Q}) \to C(\overline{Q})$ by $F(u) = v$ where v is the solution of the linear partial differential equation

$$\frac{\partial v}{\partial t} - \triangle v + \lambda v = f(v, \tilde{v}) + \lambda v \quad \text{in } Q, \quad v|_{\partial_p Q} = 0. \tag{5.21}$$

From Theorem 9.2.5 of [**147**], the operator F is well defined. Clearly the solution of $Fx = x$ is the solution of (5.19). Using this fact on compact embedding i.e. for $p > \frac{3}{2}$, $W_p^{2,1}(Q) \hookrightarrow\hookrightarrow C(\overline{Q})$ one can easily conclude that F is a compact operator. Define an operator $T : [v_0, w_0] \times [L(v_0), L(w_0)] \subset C(\overline{Q}) \times C(\overline{Q}) \to C(\overline{Q})$ for each $u, w \in [v_0, w_0]$ by $T(u, w) = v$ where v is the solution of the partial differential equation

$$\frac{\partial v}{\partial t} - \triangle v + \lambda v = f(u, w) + \lambda u \quad \text{in } Q, \quad v|_{\partial_p Q} = 0. \tag{5.22}$$

Thus the operator equation $F(x) = x$ can be reformulated as $T(u, L(u)) = u$. For each $(u, w) \in [v_0, w_0]$ and $h \in C[0, T]$ define operators $T^i_{(u,L(w))} : C(\overline{Q}) \to C(\overline{Q})$, $i = 1, 2$ by $T^1_{(u,L(w))}h(t) = z_1(h)$ and $T^2_{(u,L(w))}h(t) = z_2(h)$ that are solutions of

$$\frac{\partial z_1}{\partial t} - \triangle z_1 + \lambda z_1 = (f_1(u, \tilde{w}) + \lambda)h \quad \text{in } Q, \quad z|_{\partial_p Q} = 0 \tag{5.23}$$

$$\frac{\partial z_2}{\partial t} - \triangle z_2 + \lambda z_2 = f_2(u, \tilde{w})\tilde{h} \quad \text{in } Q, \quad z|_{\partial_p Q} = 0 \tag{5.24}$$

respectively. Note that $\|z_1(h)\|_{C(\overline{Q})} \leq c\|z_1(h)\|_{W_p^{2,1}(Q)} \leq \tilde{c}\|h\|_{C(\overline{Q})}$. Thus z_1 is a continuous linear transformation from $C(\overline{Q})$ into itself. Similarly, z_2 is a continuous linear transformation from $C(\overline{Q})$ into itself. Define for $h \in C(\overline{Q})$, $g = g(h) = T(u+h, L(w)) - T(u, L(w)) - z_1(h)$. Then g is a solution of the following partial differential equation

$$\frac{\partial g}{\partial t} - \triangle g + \lambda g = f(u+h, \tilde{w}) - f(u, \tilde{w}) - f_1(u, \tilde{w})h \quad \text{in } Q, \quad g|_{\partial_p Q} = 0. \tag{5.25}$$

Thus

$$\|g(h)\|_{C(\overline{Q})} \leq c\|g(h)\|_{W_p^{2,1}(Q)} \leq \tilde{c}\|f(u+h, \tilde{w}) - f(u, \tilde{w}) - f_1(u, \tilde{w})h\|_{C(\overline{Q})}.$$

Using the assumption (ii), one can write $f(u+h, \tilde{w}) = f(u, \tilde{w}) + f_1(u, \tilde{w})h + r(h)h$ where r satisfies $\lim_{\|h\|_{C(\overline{Q})}\to 0} \|r(h)\|_{C(\overline{Q})} = 0$. Thus $\lim_{\|h\|_{C(\overline{Q})}\to 0} \frac{\|g(h)\|_{C(\overline{Q})}}{\|h\|_{C(\overline{Q})}} = 0$. Hence the operator z_1 is Partial Frechet derivative of $T(u, L(w))$ with respect to the first variable at $(u, L(w))$. Similarly, one can show that z_2 is Partial Frechet derivative of $T(u, L(w))$ with respect to the second variable at $(u, L(w))$. It is easy to verify that $T^1_{(u,L(w))}$ and $T^2_{(u,L(w))}$ are positive and negative operators respectively for $u, w \in [v_0, w_0]$. Let $Q_t = \Omega \times (0, t)$, $t \in [0, T]$. Thus $z_1(h) = T^1_{(u,L(w))}h(t)$ is a solution for the following differential equation in $Q_t \subset Q$.

$$\frac{\partial z_1(h)}{\partial t} - \triangle z_1(h) + \lambda z_1(h) = (f_1(u, \tilde{w}) + \lambda)h \quad \text{in } Q_t, \quad z|_{\partial_p Q_t} = 0. \tag{5.26}$$

Thus $z_1(h)$ satisfies $\|z_1(h)\|_{W_p^{2,1}(Q_t)} \leq c\|(f_1(u, \tilde{w}) + \lambda)h\|_{L^p(Q_t)}$. For rest of the discussion assume that $c > 0$ is a generic constant. Using (ii), $|f_1(u, \tilde{u}) + \lambda| \leq c$ and $|f_2(u, \tilde{u})| \leq c$ for all $(x, t) \in \overline{Q}$ and $u \in [v_0, w_0]$.

$$\begin{aligned}
\|z_1\|^p_{C(\overline{Q_t})} &\leq c\|z_1\|^p_{W_p^{2,1}(Q_t)} \leq c\int_0^t \int_\Omega |h(x,\tau)|^p \mathrm{d}x\mathrm{d}\tau \\
&\leq c\int_0^t \int_\Omega |h(x,\tau)|^p \mathrm{e}^{-\alpha p\tau}\mathrm{e}^{\alpha p\tau} \mathrm{d}x\mathrm{d}\tau \\
\|z_1\|^p_{C(\overline{Q_t})} &\leq c|\Omega|\ \|h\|_{\alpha,\overline{Q_t}} \left(\frac{1}{\alpha p}\right) \mathrm{e}^{\alpha pt}.
\end{aligned}$$

Consequently, $\|z_1\|_{\alpha,\overline{Q}} \leq \|h\|_{\alpha,\overline{Q}} \frac{1}{\alpha p}$. Thus $\|T^1_{(u,L(w))}\| \leq \frac{1}{\alpha p}$ for $u, w \in [v_0, w_0]$. Similarly, $z_2(h) = T^2_{(u,L(w))}h(t)$ is a solution for the following differential equation in $Q_t \subset Q$.

$$\frac{\partial z_2(h)}{\partial t} - \triangle z_2(h) + \lambda z_2(h) = f_2(u, \tilde{w})\tilde{h} \quad \text{in } Q_t, \quad z|_{\partial_p Q_t} = 0. \tag{5.27}$$

Thus $z_2(h)$ satisfies $\|z_2(h)\|_{W_p^{2,1}(Q_t)} \le c\|f_2(u,\tilde{w})\tilde{h}\|_{L^p(Q_t)}$. Let $c > 0$ be a generic constant.

$$\begin{aligned}
\|z_2\|^p_{C(\overline{Q_t})} &\le c\|z_1\|^p_{W_p^{2,1}(Q_t)} \le c\int_0^t\int_\Omega |\tilde{h}(x,\tau)|^p \mathrm{d}x\mathrm{d}\tau \\
&\le c\int_0^t\int_\Omega \left\{\int_0^\tau |h(x,s)|\mathrm{d}s\right\}^p \mathrm{d}x\mathrm{d}\tau \\
\|z_2\|^p_{C(\overline{Q_t})} &\le c|\Omega|\ \|h\|_{\alpha,\overline{Q_t}}\left(\frac{1}{\alpha^{1+p}p}\right)\mathrm{e}^{\alpha pt}.
\end{aligned}$$

Consequently, $\|z_2\|_{\alpha,\overline{Q}} \le \|h\|_{\alpha,\overline{Q}}\frac{1}{\alpha^{1+p}p}$. Thus $\|T^1_{(u,L(w))}\| \le \frac{1}{\alpha^{1+p}p}$ for $u, w \in [v_0, w_0]$. Choose α large enough so that $\max\{\frac{1}{\alpha p}, \frac{1}{\alpha^{1+p}p}\} < \frac{1}{4}$. Hence $T(u, L(w))$ is a contraction map. Thus, T satisfies all the hypotheses of Lemma 5.2.1. Hence the operator equation $T(u, L(u)) = u$ has a unique solution in $[v_0, w_0]$. Consequently, the operator equation $F(u) = u$ has a unique solution in $[v_0, w_0]$. Moreover the following iterative procedure

$$v_{n+1} = T(v_n, \tilde{w}_n) \tag{5.28}$$

$$w_{n+1} = T(w_n, \tilde{v}_n) \tag{5.29}$$

converges to the unique solution of $F(u) = u$. Equivalently, (5.19) has a unique solution in $[v_0, w_0]$ and the iterative scheme (5.17) and (5.18) converge monotonically to the unique solution of the parabolic integro differential equation (5.19).

Corollary 5.3.2. *The partial integro differential equation* (5.1) *arising from the population dynamics has a unique solution in* $[0, 1]$ *if* $b \ge a > 0$*. If* $a > b > 0$ *then the partial integro differential equationl* (5.3) *has a unique solution in* $[0, \frac{a}{b}]$.

Proof. For the choice of $f(x,\tilde{x}) = ax - bx^2 - cx\tilde{x}$, if $b \ge a > 0$, then $(v_0, w_0) = (0, 1)$ is a coupled lower and upper solution and all the hypothesis of Theorem 5.3.2 are satisfied. Hence the Equation (5.1) has a unique solution in the sector [0,1]. Similarly, if $a > b > 0$, then $(v_0, w_0) = (0, \frac{a}{b})$ is a coupled lower and upper solution and all the hypothesis of Theorem 5.3.2 are satisfied. Hence the Equation (5.1) has a unique solution in the sector $[0, \frac{a}{b}]$. □

Remark 5.3.3. *Recently the authors in* [**15, 16, 67, 126**] *studied the population model represented by the following partial integro differential equation*

$$(5.30)\qquad \frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial \eta^2}+au-bu^2-cu\int_0^1\kappa(\eta-y)u(y,\tau)\mathrm{d}y+g(\eta,\tau)\quad \textit{in } Q,\quad u|_{\partial_p Q}=\phi.$$

This equation is similar to (5.4) *but it is Fredholdm type partial integro differential equation and the integration is over the spatial domain. Existence and uniqueness result for* (5.30) *is discussed in* [**114**]*[P.84]. It is worth mentioning that similar existence theorem can be proved using the operator theory technique in partially ordered Banach space. By setting* $h(u,\overline{u})=au-bu^2-cu\overline{u}+g(\eta,\tau)$, $\overline{u}=\int_0^1\kappa(\eta-y)u(y,\tau)\mathrm{d}y$ *one can prove the following theorem.*

Theorem 5.3.3. *If* $v_0,w_0\in C^{2,1}(\overline{Q})$ *are ordered coupled lower and upper solution for the Fredholm partial integro differential equation* (5.30)*, then (5.30) has a unique solution in* $[\alpha_0,\beta_0]$*. Moreover the iterative scheme*

$$\begin{aligned}\frac{\partial v_{n+1}}{\partial \tau}-\frac{\partial^2 v_{n+1}}{\partial \eta^2}+\lambda v_{n+1} &= \lambda v_n+h(v_n,\overline{w}_n),\quad v_{n+1}|_{\partial_p Q}=\phi\\ \frac{\partial w_{n+1}}{\partial \tau}-\frac{\partial^2 w_{n+1}}{\partial \eta^2}+\lambda w_{n+1} &= \lambda w_n+h(w_n,\overline{v}_n),\quad w_{n+1}|_{\partial_p Q}=\phi\end{aligned}$$

converges monotonically to the unique solution in $[v_0,w_0]$.

Proof: The proof is, again, based on Lemma 5.2.1 and similar to that of Theorem 5.3.2.

Remark 5.3.4. *Though the Theorem 5.3.2 is discussed for the one dimensional Volterra partial integro differential equation, it can be easily extended for* n *dimensional problem provided the spatial domain* $\Omega\subset\mathbb{R}^n$ *has a smooth boundary* $\partial\Omega\subset C^2$*. To ensure the compact embedding, choose* p *such that* $p>\frac{n+2}{2}$.

5.4. Bivariate Interpolated Spectral Method

The formulation of the bivariate interpolated spectral iterative method (BISIM) to find the solution of nonlinear Volterra type partial integro-differential equations is detailed in this section. More specifically, consider

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \eta^2} + f\left(u, \int_0^{\tau} u(\eta, s)\mathrm{d}s\right) + g(\eta, \tau) \tag{5.31}$$

where $\tau \in [0, T]$, $\eta \in [a_1, b_1]$. Here, τ and η represent, respectively, the time and space variables in the given physical domain $[a_1, b_1] \times [0, T]$. Since the present scheme uses Chebyshev-Gauss-Lobatto points for both time and space discretization, the physical domain $[a_1, b_1] \times [0, T]$ has been transformed to the computational domain $[-1, 1] \times [-1, 1]$ by the linear transformations $\tau = \dfrac{T(t+1)}{2}$ and $\eta = \dfrac{(b_1 - a_1)x}{2} + \dfrac{b_1 + a_1}{2}$. Here $a_1 = 0$ and $b_1 = 1$. Using these relations (5.31) becomes,

$$\frac{2}{T}\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} + f(u, \tilde{u}) + g\left(\frac{x+1}{2}, \frac{T(t+1)}{2}\right) \tag{5.32}$$

where $\tilde{u} = \frac{T}{2}\int_{-1}^{t} u(x, s)\mathrm{d}s$. To proceed further, assume that the solution of (5.32) can be approximated by a bivariate Lagrange interpolation polynomial of the form

$$u(x, t) = \sum_{i=0}^{N_x}\sum_{j=0}^{N_t} u(x_i, t_j)L_i(x)L_j(t), \text{ for any } (x, t) \in [-1, 1] \times [-1, 1] \tag{5.33}$$

where $x_i = \cos\left(\dfrac{\pi i}{N_x}\right)$ and $t_j = \cos\left(\dfrac{\pi j}{N_t}\right)$; $\begin{cases} i = 0, 1, \dots, N_x \\ j = 0, 1, \dots, N_t \end{cases}$ are Chebyshev-Gauss-Lobatto grid points and the functions $L_i(x)$ are the characteristic Lagrange cardinal polynomials given by

$$L_i(x) = \prod_{k=0, k\neq i}^{N_x} \frac{x - x_k}{x_i - x_k}.$$

It can be seen that each Lagrange polynomial satisfies cardinality property. i.e.,

$$L_i(x_k) = \delta_{ik}, \ i, k = 1, 2, \dots, N_x.$$

Similarly for $L_j(t)$. The values of the time derivative at the Chebyshev-Gauss-Lobatto points (x_i, t_j) are computed as

$$\begin{aligned} \left.\frac{\partial u}{\partial t}\right|_{x=x_i, t=t_j} &= \sum_{p=0}^{N_x}\sum_{k=0}^{N_t} u(x_p, t_k)L_p(x_i)\frac{\mathrm{d}L_k(t_j)}{\mathrm{d}t} \\ &= \sum_{k=0}^{N_t} \mathrm{d}_{jk} u(x_i, t_k) \end{aligned} \tag{5.34}$$

where $d_{jk} = \dfrac{\mathrm{d}L_k(t_j)}{\mathrm{d}t}$ is the standard first derivative Chebyshev differentiation matrix of size $(N_t+1)\times(N_t+1)$ [**139**]. The values of the space derivative at the Chebyshev-Gauss-Lobatto points (x_i, t_j) are computed as

$$\begin{aligned}
\left.\frac{\partial u}{\partial x}\right|_{x=x_i,t=t_j} &= \sum_{p=0}^{N_x}\sum_{k=0}^{N_t} u(x_p,t_k)\frac{\mathrm{d}L_p(x_i)}{\mathrm{d}x}L_k(t_j) \\
&= \sum_{p=0}^{N_x} D_{ip}u(x_p,t_j)
\end{aligned}$$

where $D_{ip} = \dfrac{\mathrm{d}L_p(x_i)}{dx}$ is the standard first derivative Chebyshev differentiation matrix of size $(N_x+1)\times(N_x+1)$. Similarly for n^{th} order derivative we have

$$\left.\frac{\partial^n u}{\partial x^n}\right|_{x=x_i,t=t_j} = \sum_{p=0}^{N_x} D_{ip}^n u(x_p,t_j).$$

The values of the time integral at the Chebyshev-Gauss-Lobatto points (x_i, t_j) are computed as

$$\begin{aligned}
\left.\int_{-1}^{t} u(x,s)\mathrm{d}s\right|_{x=x_i,t=t_j} &= \sum_{p=0}^{N_x}\sum_{k=0}^{N_t} u(x_p,t_k)L_p(x_i)\int_{-1}^{t_j} L_k(t)\mathrm{d}t \\
&= \sum_{k=0}^{N_t} u(x_i,t_k)r_{jk} \\
&= \sum_{k=0}^{N_t} r_{jk}u(x_i,t_k) \qquad (5.35)
\end{aligned}$$

where $r_{jk} = \displaystyle\int_{-1}^{t_j} L_k(t)\mathrm{d}t$ is the Chebyshev integration matrix of size $(N_t+1)\times(N_t+1)$. The iterative scheme for the equation (5.32) is

$$\frac{2}{T}\frac{\partial v_{n+1}}{\partial t} - 4\frac{\partial^2 v_{n+1}}{\partial x^2} + \lambda v_{n+1} = \overline{R} \qquad (5.36)$$

and

$$\frac{2}{T}\frac{\partial w_{n+1}}{\partial t} - 4\frac{\partial^2 w_{n+1}}{\partial x^2} + \lambda w_{n+1} = \tilde{R} \qquad (5.37)$$

where $\overline{R} = \lambda v_n + f(v_n, \tilde{w}_n) + g\left(\frac{x+1}{2}, \frac{T(t+1)}{2}\right)$ and $\tilde{R} = \lambda w_n + f(w_n, \tilde{v}_n) + g\left(\frac{x+1}{2}, \frac{T(t+1)}{2}\right)$. After approximating the above linearized equation (5.36) for each $(x_i, t_j) \in (-1,1)\times$

$(-1, 1]$, using (5.34) - (5.35), and on applying initial and boundary conditions the resulting system can be written in matrix form as,

$$4\mathcal{D}^2\bar{V}_j^{n+1} - \lambda I\bar{V}_j^{n+1} - \frac{2}{T}\sum_{k=0}^{N_t-1} d_{j,k}\bar{V}_k^{n+1} = \overline{R}_j^{'}; \quad j = 0, \cdots, N_t - 1 \tag{5.38}$$

where $\mathcal{D}$ is a square matrix of size $N_x - 1$ obtained by just removing first and last rows and columns of the differentiation matrix D (D is the standard n^{th} derivative Chebyshev differentiation matrix of size $(N_x + 1) \times (N_x + 1)$), $\bar{V}_j$ is a column vector of size $N_x - 1$ obtained by removing the first and last elements of the column vector $V_j = [v(x_0, t_j), v(x_1, t_j), \ldots, v(x_{N_x}, t_j)]^T$ and I is an identity matrix of size $(N_x - 1) \times (N_x - 1)$. Combining all the matrix equations in (5.38) for each $t_j,\ j = 0, 1, \ldots, N_t - 1$, we obtain the following system.

$$\begin{bmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,N_t-1} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,N_t-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N_t-1,0} & A_{N_t-1,1} & \cdots & A_{N_t-1,N_t-1} \end{bmatrix} \begin{bmatrix} \bar{V}_0^{n+1} \\ \bar{V}_1^{n+1} \\ \vdots \\ \bar{V}_{N_t-1}^{n+1} \end{bmatrix} = \begin{bmatrix} \overline{R}_0^{'} \\ \overline{R}_1^{'} \\ \vdots \\ \overline{R}_{N_t-1}^{'} \end{bmatrix} \tag{5.39}$$

where $\overline{R}_j^{'} = -\overline{R}_j + \frac{2}{T}d_{j,N_t}\bar{V}_{N_t} - 4\left[\bar{D}^2(:,0)v(x_0,t_j) + \bar{D}^2(:,N_x)v(x_{N_x},t_j)\right]$, $A_{j,k} = -\frac{2}{T}d_{j,k}I$, $A_{j,j} = 4\mathcal{D}^2 - \lambda I - \frac{2}{T}d_{j,j}I$, $\bar{D}$ is a matrix of size $(N_x - 1) \times (N_x + 1)$ obtained by removing the first and last rows of the differentiation matrix D and the vector $\overline{R}_j$ corresponds to the discretized form of $\overline{R}$ in (5.36). Similarly, one can obtain the matrix system for (5.37). Using v_0 and w_0 one can obtain the rest of $v_n's$ and $w_n's$.

5.5. Numerical Experiment

This section illustrates the proposed BISIM for the initial boundary value problems (IBVP) governed by a class of partial integro-differential equations using various examples. The existence and uniqueness of the solution and convergence of the proposed scheme has been verified using the proposed Theorem. At each iteration, the corresponding linear IBVP has been solved numerically using the proposed bivariate Chebyshev spectral method.

Example 5.5.1. *Consider the following differential equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^2 - u \int_0^t u(x,s)\mathrm{d}s \tag{5.40}$$

with initial and boundary conditions: $u(x,0) = \frac{1}{1+e^x}$, $u(0,t) = \frac{1}{1+e^t}$ *and* $u(1,t) = \frac{1}{1+e^{1+t}}$.

Clearly $v_0 = 0$ and $w_0 = 1$ are coupled lower and upper solution of (5.40) respectively. For the choice $\lambda = 1 + T$ all the assumptions of Theorem 5.3.2 are satisfied. Figure 5.1 shows the monotone behavior of lower and upper sequences for a fixed $x = 0.5$ in the interval $t \in [0, 20]$ and Figure 5.2 provides the solution of (5.40) for $x = 0.3, 0.5, 0.7$ in the interval $t \in [0, 20]$.

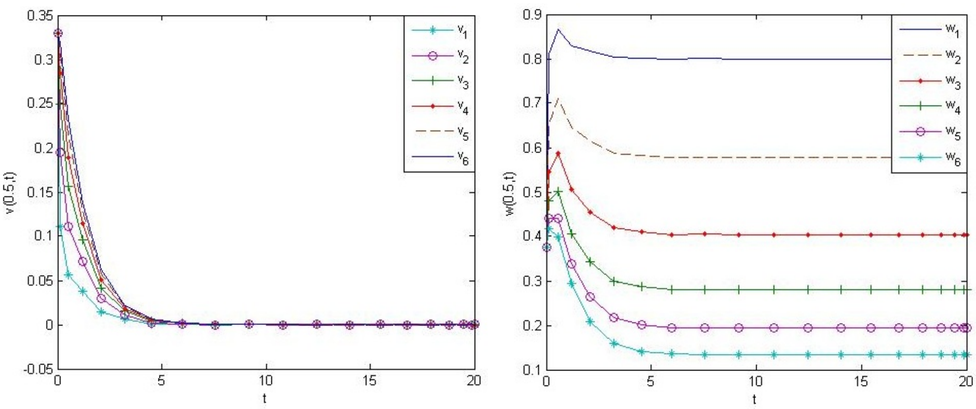

FIGURE 5.1. Monotone behavior of v_n and w_n for Example 5.5.1.

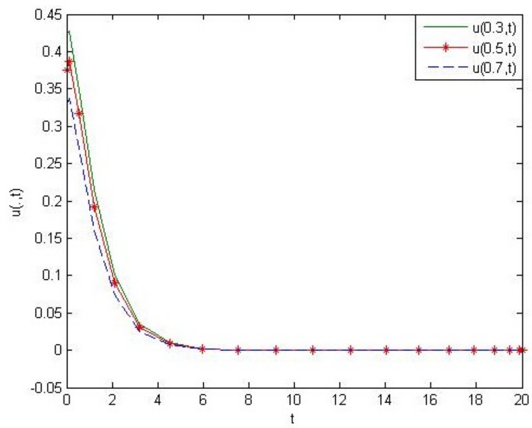

FIGURE 5.2. Approximate solution of Example 5.5.1.

Example 5.5.2. *Consider the following differential equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^2 - u\int_0^t u(x,s)\mathrm{d}s + \sin x + \sin^2 x(t^2 + t^3/2) \tag{5.41}$$

with initial and boundary conditions: $u(x,0) = 0, u(0,t) = 0$ *and* $u(1,t) = t\sin 1$.

The exact solution of the problem is $u(x,t) = t\sin x$. It is easy to verify that $v_0 = 0$ and $w_0 = 9$ are coupled lower and upper solution of (5.41) respectively when $T = 5$. For the choice of $\lambda = 62$ all the conditions of Theorem 5.3.2 satisfied. Table 5.1 provides the absolute error at various non-collocation points for the grid size 6×6.

$x \downarrow t \rightarrow$	1	2	3	4	5
0.25	$2.6894e^{-10}$	$3.1904e^{-09}$	$2.4194e^{-09}$	$3.3821e^{-09}$	$3.7453e^{-09}$
0.5	$1.1037e^{-09}$	$1.4585e^{-09}$	$8.7211e^{-10}$	$5.8498e^{-10}$	$1.1050e^{-09}$
0.75	$7.0496e^{-10}$	$4.1389e^{-09}$	$4.5190e^{-09}$	$7.0616e^{-09}$	$9.3318e^{-09}$

TABLE 5.1. Absolute error at various non collocation points of Example 5.5.2.

5.6. Conclusion

In this chapter, a bivariate spectral collocation method has been proposed for solving partial integro-differential equation of type (5.4). The nonlinear partial integro-differential equation has been linearized by monotone iterative scheme and then discretized using BISIM, where both spatial and time derivatives as well as integrals have been approximated using spectral Chebyshev collocation method. An independent existence and uniqueness of the solution as well as the convergence of monotone iterative scheme for fractional order Volterra population model and a partial integro differential equation arise commonly in population dynamics have also been proved. Though the existence and uniqueness as well as the convergence of the proposed iterative scheme are discussed only for one dimensional problems, it is easy to extend these results to corresponding higher dimensional problems, whereas numerical implementation can be a challenging task.

CHAPTER 6

CATALYTIC CONVERTER MODEL

6.1. Introduction

The following coupled partial differential equation arises from the mathematical modeling of catalytic converter model

$$
\begin{cases}
\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + cu = cv, & t > 0,\ 0 < x \le l \\
\frac{\partial v}{\partial t} + bv = bu + \lambda \exp(v), & t > 0,\ 0 < x \le l \\
u(0,t) = \eta(t), & t > 0 \\
u(x,0) = u_0(x),\ v(x,0) = v_0(x), & 0 < x \le l
\end{cases}
\tag{6.1}
$$

where a, b, c, l are positive constants, $u_0(x), v_0(x)$ and $\eta(t)$ are non negative continuous functions with $u_0(0) = \eta(0)$. Existence and uniqueness theorem as well as finite difference method based on monotone iteration are studied in [**31, 115, 132, 133**] for Eqn. (6.1). Chang et.al. [**31**] first studied an existence and uniqueness result as well as the blowup property for Eqn. (6.1) based on successive monotone iteration. Later in [**115**], Pao et.al. developed a finite difference method based on the iterative scheme in [**31**] to solve the Eqn. (6.1) numerically and studied the blowup property. Recently, Linia et. al [**132, 133**] proposed two alternative monotone iterative methods to prove the existence and uniqueness result for Eqn. (6.1) as well finite difference method to solve the Eqn. (6.1) numerically. More specifically, the iterative schemes in [**132, 133**] are based on quasilinearization and modified quasilinearization respectively. Though the iterative schems in [**31, 115, 132, 133**] have same linear order of convergence but the numerical experiments in [**115, 132, 133**] show that both iterative schemes in [**132, 133**] converge always faster than the iterative scheme in [**115**]. It is worth mentioning that there is no theoretical justification provided in [**132, 133**] for the faster convergence of the iterative schemes in [**132, 133**] over the iterative schemes in [**31, 115**]. Also in [**132, 133**], there is no discussion on the prediction of blowup property by the iterative schemes.

This short note provides a theoretical justification to show that the iterative schemes in [**132, 133**] converge faster than the iterative scheme in [**31, 115**] under the assumptions in [**31, 115**]. This short note also guarantees the prediction of blowup property of the iterative scheme studied in [**132, 133**].

This chapter is organized as follows. To make the presentation self contained, Section 2 provides the iterative schemes of [**31, 115, 132, 133**], basic definitions, results and notations that are used in the succeeding sections. In Section 3, the relation between the iterative schemes discussed in [**132**] and [**31, 115**] and the relation between the iterative schemes discussed in [**133**] and [**31, 115**] are obtained.

6.2. Preliminaries

Denote $Q = (0, l] \times (0, T]$ and $C^1(\overline{Q})$ be the set of all continuously differentiable real valued functions on $\overline{Q}$, where T is an arbitrary positive constant. Throughout this discussion L and H denote the differential operators $Lu = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + cu$ and $Hv = \frac{\partial v}{\partial t} + bv$. The lower and upper solutions for Eqn. (6.1) is provided below.

Definition 6.2.1. [**31**] *A function $(\overline{u}, \overline{v}) \in C^1(\overline{Q}) \times C^1(\overline{Q})$ is called an upper solution of* (6.1) *if it satisfies*

$$\begin{cases} \frac{\partial \overline{u}}{\partial t} + a\frac{\partial \overline{u}}{\partial x} + c\overline{u} \geq c\overline{v}, & (t,x) \in Q \\ \frac{\partial \overline{v}}{\partial t} + b\overline{v} \geq b\overline{u} + \lambda \exp(\overline{v}), & (t,x) \in Q \\ \overline{u}(0,t) \geq \eta, & 0 \leq t \leq T \\ \overline{u}(x,0) \geq u_0(x), \overline{v}(x,0) \geq v_0(x), & 0 \leq x \leq l. \end{cases} \tag{6.2}$$

Similarly, $(\underline{u}, \underline{v}) \in C^1(\overline{Q}) \times C^1(\overline{Q})$ is called a lower solution if it satisfies (6.2) *with the reversed inequalities.*

For a given pair of ordered lower and upper solutions, the sector S is defined as $S = \{(u,v) \in C^1(\overline{Q}) \times C^1(\overline{Q}) : (\widehat{u}, \widehat{v}) \leq (u,v) \leq (\widetilde{u}, \widetilde{v})\}$. The following Lemmas are useful tools to obtain the main results.

Lemma 6.2.1. [**31**] *If $w \in C^1(\overline{Q})$ satisfies the inequalities*

$$\begin{cases} \frac{\partial w}{\partial t} + a\frac{\partial w}{\partial x} + bw \geq 0, & 0 < t \leq T,\ 0 < x \leq l \\ w(0,t) \geq 0, & 0 \leq t \leq T \\ w(x,0) \geq 0, & 0 \leq x \leq l \end{cases}$$

where $a \geq 0$ and $b > 0$ are constants, then $w \geq 0$ on $\overline{Q}$.

Lemma 6.2.2. [**132**] *Let $v \in C(\overline{Q})$ be continuously differentiable with respect to t such that*

$$\frac{\partial v}{\partial t} - f(x,t)v \geq 0,$$

where $f(x,t)$ is a continuous function defined on $\overline{Q}$ with $v(x,0) \geq 0$ for $0 < x \leq l$. Then $v(x,t) \geq 0$ on $\overline{Q}$.

Lemma 6.2.3. *Let $a_1, a_{i,j} > 0$ for all $(i,j) \in \bar{\Lambda}, b_1, b_2, c_1$ and $c_2 \geq 0$. If $w_{i,j}$ and $z_{i,j}$ satisfy*

$$a_1 w_{i,j} - b_1 w_{i-1,j} - c_1 w_{i,j-1} \geq 0; \quad (i,j) \in \Lambda \tag{6.3}$$

$$a_{i,j} z_{i,j} - b_2 w_{i,j} - c_2 z_{i,j-1} \geq 0; \quad (i,j) \in \Lambda \tag{6.4}$$

$$\textit{with } w_{0,j} \geq 0, w_{i,0} \geq 0, z_{i,0} \geq 0 \tag{6.5}$$

then $w_{i,j} \geq 0$ and $z_{i,j} \geq 0$ for all $(i,j) \in \bar{\Lambda}$.

Note that in [**31**], for an initial guess $(u^{(0)}, v^{(0)})$, the sequence $\{(u^{(n)}, v^{(n)})\}$ has been constructed using the iterative process

$$\begin{cases} Lu^{(n)} = cv^{(n-1)}, & (x,t) \in Q \\ Hv^{(n)} = bu^{(n-1)} + \lambda \exp(v^{(n-1)}), & (x,t) \in Q \\ u^{(n)}(0,t) = \eta(t), & 0 \leq t \leq T \\ u^{(n)}(x,0) = u_0(x),\ v^{(n)}(x,0) = v_0(x), & 0 \leq x \leq l. \end{cases} \tag{6.6}$$

Throughout this chapter, denote the maximal sequence of Chang et.al. [**31**] by $\{(\overline{u}^{(n)}, \overline{v}^{(n)})\}$ with $(u^{(0)}, v^{(0)}) = (\overline{u}^{(0)}, \overline{v}^{(0)})$ and the minimal sequence by $\{(\underline{u}^{(n)}, \underline{v}^{(n)})\}$ with $(u^{(0)}, v^{(0)}) = (\underline{u}^{(0)}, \underline{v}^{(0)})$. The maximal sequence of Linia et.al. [**132**] is denoted by $\{(\overline{\alpha}^{(n)}, \overline{\beta}^{(n)})\}$ and minimal sequence by $\{(\underline{\alpha}^{(n)}, \underline{\beta}^{(n)})\}$. The maximal sequence and minimal sequence of Linia et.al.[**132**] are generated by the following iterative scheme for the initial guess $(u^{(0)}, v^{(0)}) = (\overline{\alpha}^{(0)}, \overline{\beta}^{(0)})$ and $(u^{(0)}, v^{(0)}) = (\underline{\alpha}^{(0)}, \underline{\beta}^{(0)})$ respectively.

$$
\begin{cases}
Lu^{(n)} = cv^{(n-1)}, & (x,t) \in Q \\
Hv^{(n)} = bu^{(n)} + \lambda \exp(v^{(n-1)}) + \lambda \exp(v^{(n-1)})(v^{(n)} - v^{(n-1)}), & (x,t) \in Q \\
u^{(n)}(0,t) = \eta(t), & 0 \leq t \leq T \\
u^{(n)}(x,0) = u_0(x),\ v^{(n)}(x,0) = v_0(x), & 0 \leq x \leq l.
\end{cases}
\tag{6.7}
$$

The maximal sequence of Linia et.al. [**133**] is denoted by $\{(\overline{\omega}^{(n)}, \overline{\chi}^{(n)})\}$ and minimal sequence by $\{(\underline{\omega}^{(n)}, \underline{\chi}^{(n)})\}$. The maximal sequence and minimal sequence of Linia et.al. [**133**] are generated by the following iterative scheme for the initial guess $(u^{(0)}, v^{(0)}) = (\overline{\omega}^{(0)}, \overline{\chi}^{(0)})$ and $(u^{(0)}, v^{(0)}) = (\underline{\omega}^{(0)}, \underline{\chi}^{(0)})$ respectively

$$
\begin{cases}
Lu^{(n)} = cv^{(n)}, & (x,t) \in Q \\
Hv^{(n)} = bu^{(n)} + \lambda \exp(v^{(n-1)}) + \lambda \exp(v^{0)})(v^{(n)} - v^{(n-1)}), & (x,t) \in Q \\
u^{(n)}(0,t) = \eta(t), & 0 \leq t \leq T \\
u^{(n)}(x,0) = u_0(x),\ v^{(n)}(x,0) = v_0(x), & 0 \leq x \leq l.
\end{cases}
\tag{6.8}
$$

To solve the Eqn. (6.1) numerically, all the works in [**115, 132, 133**] used the backward finite difference method to discretize the problem. This dicretization leads to the following nonlinear system

$$
\begin{cases}
(1 + kc + \frac{ka}{h})u_{i,j} = u_{i,j-1} + \frac{ka}{h}u_{i-1,j} + kcv_{i,j}, \\
(1 + kb)v_{i,j} = v_{i,j-1} + kbu_{i,j} + k\lambda \exp(v_{i,j}), \\
u_{0,j} = \eta_j,\ u_{i,0} = \psi_i,\ v_{i,0} = \phi_i,\ i = 1, 2, \cdots, M,\ j = 1, 2, \cdots, N
\end{cases}
\tag{6.9}
$$

where $\eta_j = \eta(t_j), \psi_i = u_0(x_i)$ and $\phi_i = v_0(x_i)$. Based on the iteration in [**31**], Pao et. al. [**115**] obtained the following iterative scheme to solve the nonlinear system (6.9).

$$
\begin{cases}
(1 + kc + \frac{ka}{h})u_{i,j}^{(n)} = u_{i,j-1}^{(n)} + \frac{ka}{h}u_{i-1,j}^{(n)} + kcv_{i,j}^{(n-1)}, \\
(1 + kb)v_{i,j}^{(n)} = v_{i,j-1}^{(n)} + kbu_{i,j}^{(n)} + k\lambda \exp(v_{i,j}^{(n-1)}), \\
u_{0,j}^{(n)} = \eta_j,\ u_{i,0}^{(n)} = \psi_i,\ v_{i,0}^{(n)} = \phi_i,\ i = 1, 2, \cdots, M,\ j = 1, 2, \cdots, N.
\end{cases}
\tag{6.10}
$$

Denote the maximal sequence of the discretized version of Pao et. al. [**115**] by $\{(\overline{u}_{i,j}^{(n)}, \overline{v}_{i,j}^{(n)})\}$ with $(u_{i,j}^{(0)}, v_{i,j}^{(0)}) = (\overline{u}_{i,j}^{(0)}, \overline{v}_{i,j}^{(0)})$. Denote the minimal sequence of the discretized version of Pao et. al. [**115**] by $\{(\underline{u}_{i,j}^{(n)}, \underline{v}_{i,j}^{(n)})\}$ with $(u_{i,j}^{(0)}, v_{i,j}^{(0)}) = (\underline{u}_{i,j}^{(0)}, \underline{v}_{i,j}^{(0)})$. The maximal sequence of discretized version of Linia et. al. [**132**] is denoted by $\{(\overline{\alpha}_{i,j}^{(n)}, \overline{\beta}_{i,j}^{(n)})\}$ and minimal sequence by $\{(\underline{\alpha}_{i,j}^{(n)}, \underline{\beta}_{i,j}^{(n)})\}$. The maximal sequence and minimal sequence of Linia et. al. [**132**] corresponding to the discretized problem are generated by the following iterative scheme for the initial guess $(u_{i,j}^{(0)}, v_{i,j}^{(0)}) = (\overline{\alpha}_{i,j}^{(0)}, \overline{\beta}_{i,j}^{(0)})$ and $(u_{i,j}^{(0)}, v_{i,j}^{(0)}) = (\underline{\alpha}_{i,j}^{(0)}, \underline{\beta}_{i,j}^{(0)})$ respectively.

(6.11)
$$\begin{cases} (1 + kc + \frac{ka}{h})u_{i,j}^{(n)} = u_{i,j-1}^{(n)} + \frac{ka}{h}u_{i-1,j}^{(n)} + kcv_{i,j}^{(n-1)}, \\ (1 + kb)v_{i,j}^{(n)} = v_{i,j-1}^{(n)} + kbu_{i,j}^{(n)} + k\lambda \exp(v_{i,j}^{(n-1)}) + k\lambda \exp(v_{i,j}^{(n-1)})(v_{i,j}^{(n)} - v_{i,j}^{(n-1)}), \\ u_{0,j}^{(n)} = \eta_j,\ u_{i,0}^{(n)} = \psi_i,\ v_{i,0}^{(n)} = \phi_i,\ i = 1, 2, \cdots, M,\ j = 1, 2, \cdots, N. \end{cases}$$

The maximal sequence of discretized version of Linia et.al.[**133**] is denoted by $\{(\overline{\omega}_{i,j}^{(n)}, \overline{\chi}_{i,j}^{(n)})\}$ and minimal sequence by $\{(\underline{\omega}_{i,j}^{(n)}, \underline{\chi}_{i,j}^{(n)})\}$. The maximal sequence and minimal sequence of Linia et. al. [**133**] corresponding to the discretized problem are generated by the following iterative scheme for the initial guess $(u_{i,j}^{(0)}, v_{i,j}^{(0)}) = (\overline{\omega}_{i,j}^{(0)}, \overline{\chi}_{i,j}^{(0)})$ and $(u_{i,j}^{(0)}, v_{i,j}^{(0)}) = (\underline{\omega}_{i,j}^{(0)}, \underline{\chi}_{i,j}^{(0)})$ respectively.

$$\begin{cases} (1 + kc + \frac{ka}{h})u_{i,j}^{(n)} = u_{i,j-1}^{(n)} + \frac{ka}{h}u_{i-1,j}^{(n)} + kcv_{i,j}^{(n)}, \\ (1 + kb)v_{i,j}^{(n)} = v_{i,j-1}^{(n)} + kbu_{i,j}^{(n)} + k\lambda \exp(v_{i,j}^{(n-1)}) + k\lambda \exp(v_{i,j}^{(0)})(v_{i,j}^{(n)} - v_{i,j}^{(n-1)}), \\ u_{0,j}^{(n)} = \eta_j,\ u_{i,0}^{(n)} = \psi_i,\ v_{i,0}^{(n)} = \phi_i,\ i = 1, 2, \cdots, M,\ j = 1, 2, \cdots, N. \end{cases} \tag{6.12}$$

6.3. Relation between the monotone iterations

This section provides theoretical justification for the faster convergence of the iterative schemess (6.7) and (6.8) over the iterative scheme (6.6) for the problem (6.1). Similarly, this section also proves that the iterative schemes (6.11) and (6.12) always converge faster than the iterative scheme (6.10) for the nonlinear system (6.9). The following theorem provides the relation between the iterative schemes (6.6) and (6.7).

Theorem 6.3.1. *Let (u^*, v^*) be a solution of* (6.1). *If $(\overline{u}^0, \overline{v}^0)$ and $(\underline{u}^0, \underline{v}^0)$ are ordered upper and lower solution for Eqn.* (6.1) *then for all $n \in \mathbb{N}$,*

$$(\underline{u}^{(n)}, \underline{v}^{(n)}) \leq (\underline{\alpha}^{(n)}, \underline{\beta}^{(n)}) \leq (u^*, v^*) \leq (\overline{\alpha}^{(n)}, \overline{\beta}^{(n)}) \leq (\overline{u}^{(n)}, \overline{v}^{(n)}). \tag{6.13}$$

Proof. It is enough to show that for $n \in \mathbb{N}$,

$$(\underline{u}^{(n)}, \underline{v}^{(n)}) \leq (\underline{\alpha}^{(n)}, \underline{\beta}^{(n)}) \leq (\overline{\alpha}^{(n)}, \overline{\beta}^{(n)}) \leq (\overline{u}^{(n)}, \overline{v}^{(n)}). \tag{6.14}$$

Using mathematical induction the inequality (6.14) is proved. From the definitions of $(\underline{\alpha}^{(n)}, \underline{\beta}^{(n)})$ and $(\underline{u}^{(n)}, \underline{v}^{(n)})$ one can get

$$\begin{cases} L(\underline{\alpha}^{(n)} - \underline{u}^{(n)}) = c(\underline{\beta}^{(n-1)} - \underline{v}^{(n-1)}), & (x,t) \in Q \\ H(\underline{\beta}^{(n)} - \underline{v}^{(n)}) = b(\underline{\alpha}^{(n)} - \underline{u}^{(n-1)}) + \lambda(\exp(\underline{\beta}^{(n-1)}) - \exp(\underline{v}^{(n-1)})) & \\ \qquad\qquad +\lambda \exp(\underline{\beta}^{(n-1)})(\underline{\beta}^{(n)} - \underline{\beta}^{(n-1)}), & (x,t) \in Q \\ \underline{\alpha}^{(n)}(0,t) - \underline{u}^{(n)}(0,t) = 0, & 0 \leq t \leq T \\ \underline{\alpha}^{(n)}(x,0) - \underline{u}^{(n)}(x,0) = 0, \ \underline{\beta}^{(n)}(x,0) - \underline{v}^{(n)}(x,0) = 0, & 0 \leq x \leq l. \end{cases} \tag{6.15}$$

Note that both the minimal sequences $(\underline{u}^{(n)}, \underline{v}^{(n)})$ and $(\underline{\alpha}^{(n)}, \underline{\beta}^{(n)})$ corresponding to the iterative schemes (6.6) and (6.7) respectively have the same initial guess i.e. $(\underline{u}^{(0)}, \underline{v}^{(0)}) = (\underline{\alpha}^{(0)}, \underline{\beta}^{(0)})$. Thus for the choice $n = 1$ in (6.15), one can get

$$\begin{cases} L(\underline{\alpha}^{(1)} - \underline{u}^{(1)}) = c(\underline{\beta}^{(0)} - \underline{v}^{(0)}) = 0, & (x,t) \in Q \\ H(\underline{\beta}^{(1)} - \underline{v}^{(1)}) = b(\underline{\alpha}^{(1)} - \underline{u}^{(0)}) + \lambda(\exp(\underline{\beta}^{(0)}) - \exp(\underline{v}^{(0)})) & \\ \qquad\qquad +\lambda \exp(\underline{\beta}^{(0)})(\underline{\beta}^{(1)} - \underline{\beta}^{(0)}) \geq 0, & (x,t) \in Q \\ \underline{\alpha}^{(1)}(t,0) - \underline{u}^{(1)}(t,t) = 0, & 0 \leq t \leq T \\ \underline{\alpha}^{(1)}(x,0) - \underline{u}^{(1)}(x,0) = 0, \ \underline{\beta}^{(1)}(x,0) - \underline{v}^{(1)}(x,0) = 0, & 0 \leq x \leq l. \end{cases} \tag{6.16}$$

Using Lemma (6.2.1) and (6.2.2), one can have $(\underline{\alpha}^{(1)}, \underline{\beta}^{(1)}) \geq (\underline{u}^{(1)}, \underline{v}^{(1)})$. Similarly, one can prove $(\overline{\alpha}^{(1)}, \overline{\beta}^{(1)}) \leq (\overline{u}^{(1)}, \overline{v}^{(1)})$. Thus the inequality (6.14) holds true for $n = 1$. Assume that inequality (6.14) is true for $n = 1, 2, \cdots, m$. For the choice $n = m+1$ in Eqn. (6.15) leads to

$$\begin{aligned} L(\underline{\alpha}^{(m+1)} - \underline{u}^{(m+1)}) &= c(\underline{\beta}^{(m)} - \underline{v}^{(m)}) \geq 0, \qquad\qquad (x,t) \in Q \\ H(\underline{\beta}^{(m+1)} - \underline{v}^{(m+1)}) &= b(\underline{\alpha}^{(m+1)} - \underline{u}^{(m)}) + \lambda(\exp(\underline{\beta}^{(m)}) - \exp(\underline{v}^{(m)})) \\ &\quad +\lambda \exp(\underline{\beta}^{(m)})(\underline{\beta}^{(m+1)} - \underline{\beta}^{(m)}) \geq 0, (x,t) \in Q \end{aligned}$$

Moreover, for all $t \in [0,T]$, $\underline{\alpha}^{(m+1)}(0,t) - \underline{u}^{(m+1)}(0,t) = 0$ and for $x \in [0,l]$, $\underline{\alpha}^{(m+1)}(x,0) - \underline{u}^{(m+1)}(x,0) = 0$ and $\underline{\beta}^{(m+1)}(x,0) - \underline{v}^{(m+1)}(x,0) = 0$. Once again using Lemma (6.2.1) and (6.2.2), one get $(\underline{\alpha}^{(m+1)}, \underline{\beta}^{(m+1)}) \geq (\underline{u}^{(m+1)}, \underline{v}^{(m+1)})$. Similarly, one can show that $(\overline{\alpha}^{(m+1)}, \overline{\beta}^{(m+1)}) \leq (\overline{u}^{(m+1)}, \overline{v}^{(m+1)})$. Hence (6.14) is true for $n = m+1$. Consequently, (6.14) holds true for all $n \in \mathbb{N}$. $\square$

The following theorem provides the relation between the iterative schemes (6.10) and (6.11) for the nonlinear system (6.9).

Theorem 6.3.2. *Let $(u^*_{i,j}, v^*_{i,j})$ be a solution of (6.9). If $(\overline{u}^{(0)}_{i,j}, \overline{v}^{(0)}_{i,j}$ and $(\underline{u}^{(0)}_{i,j}, \underline{v}^{(0)}_{i,j})$ are ordered upper and lower solution for Eqn. (6.9) then for all $n \in \mathbb{N}$,*

$$(6.17)\qquad (\underline{u}^{(n)}_{i,j}, \underline{v}^{(n)}_{i,j}) \leq (\underline{\alpha}^{(n)}_{i,j}, \underline{\beta}^{(n)}_{i,j}) \leq (u^*_{i,j}, v^*_{i,j}) \leq (\overline{\alpha}^{(n)}_{i,j}, \overline{\beta}^{(n)}_{i,j}) \leq (\overline{u}^{(n)}_{i,j}, \overline{v}^{(n)}_{i,j}).$$

Proof. It is enough to show that for $n \in \mathbb{N}$,

$$(6.18)\qquad (\underline{u}^{(n)}_{i,j}, \underline{v}^{(n)}_{i,j}) \leq (\underline{\alpha}^{(n)}_{i,j}, \underline{\beta}^{(n)}_{i,j}) \leq (\overline{\alpha}^{(n)}_{i,j}, \overline{\beta}^{(n)}_{i,j}) \leq (\overline{u}^{(n)}_{i,j}, \overline{v}^{(n)}_{i,j}).$$

Using mathematical induction the inequality (6.18) is proved. From the definitions of $(\underline{\alpha}^{(n)}_{i,j}, \underline{\beta}^{(n)}_{i,j})$ and $(\underline{u}^{(n)}_{i,j}, \underline{v}^{(n)}_{i,j})$ one can get,

$$(6.19)\qquad \begin{cases} (1 + kc + \frac{ka}{h})(\underline{\alpha}^{(n)}_{i,j} - \underline{u}^{(n)}_{i,j}) = (\underline{\alpha}^{(n)}_{i,j-1} - \underline{u}^{(n)}_{i,j-1}) + kc(\underline{\beta}^{(n-1)}_{i,j} - \underline{v}^{(n-1)}_{i,j}) \\ \qquad\qquad + \frac{ka}{h}(\underline{\alpha}^{(n)}_{i-1,j} - \underline{u}^{(n)}_{i-1,j}), \\ (1 + kb)(\underline{\beta}^{(n)}_{i,j} - \underline{v}^{(n)}_{i,j}) = kb(\underline{\alpha}^{(n)}_{i,j} - \underline{u}^{(n)}_{i,j}) + k\lambda(\exp(\underline{\beta}^{(n-1)}_{i,j}) - \exp(\underline{v}^{(n-1)}_{i,j})) \\ \qquad\qquad + (\underline{\beta}^{(n)}_{i,j-1} - \underline{v}^{(n)}_{i,j-1}) + k\lambda\exp(\underline{\beta}^{(n-1)}_{i,j})(\underline{\beta}^{(n)}_{i,j} - \underline{\beta}^{(n-1)}_{i,j}), \\ \underline{\alpha}^{(n)}_{0,j} - \underline{u}^{(n)}_{0,j} = 0, \qquad j = 1, 2, \cdots, N \\ \underline{\alpha}^{(n)}_{i,0} - \underline{u}^{(n)}_{i,0} = 0,\ \underline{\beta}^{(n)}_{i,0} - \underline{v}^{(n)}_{i,0} = 0, \qquad i = 1, 2, \cdots, M. \end{cases}$$

Note that both the minimal sequences $(\underline{u}^{(n)}_{i,j}, \underline{v}^{(n)}_{i,j})$ and $(\underline{\alpha}^{(n)}_{i,j}, \underline{\beta}^{(n)}_{i,j})$ corresponding to the iterative schemes (6.10) and (6.11) respectively have the same initial guess i.e. $(\underline{u}^{(0)}_{i,j}, \underline{v}^{(0)}_{i,j}) =$

$(\underline{\alpha}_{i,j}^{(0)}, \underline{\beta}_{i,j}^{(0)})$. Thus for the choice $n=1$ in (6.19), one get

$$
(6.20)\quad \begin{cases}
(1+kc+\frac{ka}{h})(\underline{\alpha}_{i,j}^{(1)}-\underline{u}_{i,j}^{(1)}) = (\underline{\alpha}_{i,j-1}^{(1)}-\underline{u}_{i,j-1}^{(1)}) + kc(\underline{\beta}_{i,j}^{(0)}-\underline{v}_{i,j}^{(0)}) \\
\qquad\qquad +\frac{ka}{h}(\underline{\alpha}_{i-1,j}^{(1)}-\underline{u}_{i-1,j}^{(1)}), \\
(1+kb)(\underline{\beta}_{i,j}^{(1)}-\underline{v}_{i,j}^{(1)}) = kb(\underline{\alpha}_{i,j}^{(1)}-\underline{u}_{i,j}^{(1)}) + k\lambda(\exp(\underline{\beta}_{i,j}^{(0)})-\exp(\underline{v}_{i,j}^{(0)})) \\
\qquad\qquad +(\underline{\beta}_{i,j-1}^{(1)}-\underline{v}_{i,j-1}^{(1)}) + k\lambda\exp(\underline{\beta}_{i,j}^{(0)})(\underline{\beta}_{i,j}^{(1)}-\underline{\beta}_{i,j}^{(0)}), \\
\underline{\alpha}_{0,j}^{(1)}-\underline{u}_{0,j}^{(1)}=0, \qquad j=1,2,\cdots,N \\
\underline{\alpha}_{i,0}^{(1)}-\underline{u}_{i,0}^{(1)}=0,\ \underline{\beta}_{i,0}^{(1)}-\underline{v}_{i,0}^{(1)}=0, \qquad i=1,2,\cdots,M.
\end{cases}
$$

The first two equation of (6.20) can be written as
$(1+kc+\frac{ka}{h})(\underline{\alpha}_{i,j}^{(1)}-\underline{u}_{i,j}^{(1)}) - (\underline{\alpha}_{i,j-1}^{(1)}-\underline{u}_{i,j-1}^{(1)}) - \frac{ka}{h}(\underline{\alpha}_{i-1,j}^{(1)}-\underline{u}_{i-1,j}^{(1)}) = kc(\underline{\beta}_{i,j}^{(0)}-\underline{v}_{i,j}^{(0)}) = 0$ and $(1+kb)(\underline{\beta}_{i,j}^{(1)}-\underline{v}_{i,j}^{(1)}) - kb(\underline{\alpha}_{i,j}^{(1)}-\underline{u}_{i,j}^{(1)}) - (\underline{\beta}_{i,j-1}^{(1)}-\underline{v}_{i,j-1}^{(1)}) = k\lambda\exp(\underline{\beta}_{i,j}^{(0)})(\underline{\beta}_{i,j}^{(1)}-\underline{\beta}_{i,j}^{(0)}) \geq 0$. Using Lemma (6.2.3) together with the initial and boundary conditions of (6.20), one can get $(\underline{\alpha}_{i,j}^{(1)}, \underline{\beta}_{i,j}^{(1)}) \geq (\underline{u}_{i,j}^{(1)}, \underline{v}_{i,j}^{(1)})$ for all i and j. Similarly, One can show that $(\overline{\alpha}_{i,j}^{(1)}, \overline{\beta}_{i,j}^{(1)}) \leq (\overline{u}_{i,j}^{(1)}, \overline{v}_{i,j}^{(1)})$. Thus the inequality (6.18) holds true for $n=1$. Assume that inequality (6.18) is true for $n=1,2,\cdots,m$. For the choice $n=m+1$ in Eqn. (6.19) leads to

$$
(6.21)\quad \begin{cases}
(1+kc+\frac{ka}{h})(\underline{\alpha}_{i,j}^{(m+1)}-\underline{u}_{i,j}^{(m+1)}) = (\underline{\alpha}_{i,j-1}^{(m+1)}-\underline{u}_{i,j-1}^{(m+1)}) + kc(\underline{\beta}_{i,j}^{(m)}-\underline{v}_{i,j}^{(m)}) \\
\qquad\qquad +\frac{ka}{h}(\underline{\alpha}_{i-1,j}^{(m+1)}-\underline{u}_{i-1,j}^{(m+1)}), \\
(1+kb)(\underline{\beta}_{i,j}^{(m+1)}-\underline{v}_{i,j}^{(m+1)}) = kb(\underline{\alpha}_{i,j}^{(m+1)}-\underline{u}_{i,j}^{(m+1)}) + k\lambda(\exp(\underline{\beta}_{i,j}^{(m)})-\exp(\underline{v}_{i,j}^{(m)})) \\
\qquad\qquad +(\underline{\beta}_{i,j-1}^{(m+1)}-\underline{v}_{i,j-1}^{(m+1)}) + k\lambda\exp(\underline{\beta}_{i,j}^{(m)})(\underline{\beta}_{i,j}^{(m+1)}-\underline{\beta}_{i,j}^{(m)}), \\
\underline{\alpha}_{0,j}^{(m+1)}-\underline{u}_{0,j}^{(m+1)}=0, \qquad j=1,2,\cdots,N \\
\underline{\alpha}_{i,0}^{(m+1)}-\underline{u}_{i,0}^{(m+1)}=0,\ \underline{\beta}_{i,0}^{(m+1)}-\underline{v}_{i,0}^{(m+1)}=0, \qquad i=1,2,\cdots,M.
\end{cases}
$$

The first two equations in (6.21) can be written as
$(1+kc+\frac{ka}{h})(\underline{\alpha}_{i,j}^{(m+1)}-\underline{u}_{i,j}^{(m+1)}) - (\underline{\alpha}_{i,j-1}^{(m+1)}-\underline{u}_{i,j-1}^{(m+1)}) - \frac{ka}{h}(\underline{\alpha}_{i-1,j}^{(m+1)}-\underline{u}_{i-1,j}^{(m+1)}) \geq 0$
and $(1+kb)(\underline{\beta}_{i,j}^{(m+1)}-\underline{v}_{i,j}^{(m+1)}) - kb(\underline{\alpha}_{i,j}^{(m+1)}-\underline{u}_{i,j}^{(m+1)}) - (\underline{\beta}_{i,j-1}^{(m+1)}-\underline{v}_{i,j-1}^{(m+1)}) \geq 0$.
Using Lemma (6.2.3) together with the initial and boundary conditions of (6.21), one can get $(\underline{\alpha}_{i,j}^{(m+1)}, \underline{\beta}_{i,j}^{(m+1)}) \geq (\underline{u}_{i,j}^{(m+1)}, \underline{u}_{i,j}^{(m+1)})$. Similarly, one can show that $(\overline{\alpha}_{i,j}^{(m+1)}, \overline{\beta}_{i,j}^{(m+1)}) \leq (\overline{u}_{i,j}^{(m+1)}, \overline{u}_{i,j}^{(m+1)})$. Hence (6.18) is true for $n=m+1$. Consequently, (6.18) holds true for all $n \in \mathbb{N}$. $\square$

The following Theorems provide the relation between the iterative schemes discussed in [**31, 115**] and [**133**]. The proof is similar to the proof of Theorem 6.3.1 and Theorem 6.3.2 respectively. Hence the proof is omitted here.

Theorem 6.3.3. *Let* (u^*, v^*) *be a solution of* (6.1)*. If* $(\overline{u}^{(0)}, \overline{v}^{(0)})$ *and* $(\underline{u}^{(0)}, \underline{v}^{(0)})$ *are ordered upper and lower solution for Eqn.* (6.1) *then for all* $n \in \mathbb{N}$*,*

$$(\underline{u}^{(n)}, \underline{v}^{(n)}) \leq (\underline{\omega}^{(n)}, \underline{\chi}^{(n)}) \leq (u^*, v^*) \leq (\overline{\omega}^{(n)}, \overline{\chi}^{(n)}) \leq (\overline{u}^{(n)}, \overline{v}^{(n)}). \tag{6.22}$$

Theorem 6.3.4. *Let* $(u^*_{i,j}, v^*_{i,j})$ *be a solution of* (6.9)*. If* $(\overline{u}^{(0)}_{i,j}, \overline{v}^{(0)}_{i,j})$ *and* $(\underline{u}^{(0)}_{i,j}, \underline{v}^{(0)}_{i,j})$ *are ordered upper and lower solution for Eqn.* (6.9) *then for all* $n \in \mathbb{N}$*,*

$$(\underline{u}^{(n)}_{i,j}, \underline{v}^{(n)}_{i,j}) \leq (\underline{\omega}^{(n)}_{i,j}, \underline{\chi}^{(n)}_{i,j}) \leq (u^*_{i,j}, v^*_{i,j}) \leq (\overline{\omega}^{(n)}_{i,j}, \overline{\chi}^{(n)}_{i,j}) \leq (\overline{u}^{(n)}_{i,j}, \overline{v}^{(n)}_{i,j}). \tag{6.23}$$

Remark 6.3.1. *Inequalities 6.13,6.17,6.22 and 6.23 not only ensure that the iterative schemes in* [**132, 133**] *converges faster than the iterative scheme in* [**31, 115**] *but also guarantee that all the blowup properties discussed for the iterative scheme in* [**31, 115**] *also hold true for the iterative schemes in* [**132, 133**].

6.4. Conclusion

In this chapter, theoretical justification is provided to show that the iterative schemes in [**132, 133**] always requires less number of iterations than the iterative scheme in [**31, 115**]. Moreover, it also obtains the blowup results related to the iterative schemes in [**132, 133**].

BIBLIOGRAPHY

[1] S. Abbas, M. Benchohra and G. M. N′Guerekata, *Topics in fractional differential equations*, Springer, New York, Vol. 27, 2012.

[2] S. Abbas, M. Benchohra and J. J. Nieto, *Global uniqueness results for fractional order partial hyperbolic functional differential equations*, Adv. Difference Equ., Vol. 2011, Art. ID 379876, 25pages.

[3] S. Abbas, M. Benchohra and A. Petrusel, *Ulam stability for Hilfer type fractional differential inclusions via the weakly Picard operators theory*, Fract. Calc. Appl. Anal., 20(2017), 384–398.

[4] A. B. Abdulla, M. Al-Refai and A. Al-Rawashdeh, *On the existence and uniqueness of solutions for a class of nonlinear fractional boundary value problems*, J. King Saud Univ. Sci., 28(2016), 103–110.

[5] Om P. Agarwal, *A general finite element formulation for fractional variational problems*, J. Math. Anal. Appl., 337(2008), 1–12.

[6] Om P. Agarwal, *A general solution for the fourth-order fractional diffusion-wave equation*, Fract. Calc. Anal. Appl., 3(2000), 1–12.

[7] Om P. Agarwal, *Solution for a fractional diffusion-wave equation defined in a bounded domain*, Nonlinear Dynam., 29(2002), 145–155.

[8] R. P. Agarwal, B. Ahmad and A Alsaedi, *Method of quasilinearization for a nonlocal singular boundary value problem in weighted spaces*, Bound. Value Probl., 2013, 2013:261, 17pages.

[9] R. P. Agarwal, D. Baleanu, J. J. Nieto, D. F. M. Torres and Y. Zhou, *A survey on fuzzy fractional differential and optimal control nonlocal evolution equations*, J. Comput. Appl. Math., 339(2018), 3–29.

[10] R. P. Agarwal, M. Benchohra, S. Hamani and S. Pinelas, *Upper and lower solutions method for impulsive differential equations involving the Caputo fractional derivative*, Mem. Differ. Equ. Math. Phys., 53(2011), 1–12.

[11] B. Ahmad, *Sharp estimates for the unique solution of two-point fractional-order boundary value problems*, Appl. Math. Lett., 65(2017), 77–82.

[12] B. Ahmad and J. J. Nieto, *Existence results for nonlinear boundary value problems of fractional integro differential equations with integral boundary conditions*, Bound. Value Probl., 2009, Art. ID 708576, 11 pages.

[13] K. Ako, *Subfunctions for ordinary differential equations. V.*, Funkcial. Ekvac., 12(1969), 239–249.

[14] H. Amann, *On the existence of positive solutions of nonlinear elliptic boundary value problems*, Indiana Univ. Math. J., 21(1971), 125–146.

[15] N. Apreutesei, N. Bessonov, V. Volpert and V. Vougalter, *Spatial structures and generalized travelling waves for an integro-differential equation*, Discrete Contin. Dyn. Syst. Ser. B, 13(2010), 537–557.

[16] M. Aslefallah and E. Shivanian, *A nonlinear partial integro-differential equation in population dynamic via radial basis functions and theta-method*, J. Math. Comp. Sci., 13(2014), 14–25.

[17] A. Babakhani and V. D. Gejji, *Existence of positive solutions for multi-term non-autonomous fractional differential equations with polynomial coefficients*, Electron. J. Differential Equations, 2006(2006), 1–12.

[18] D. Bahuguna, R. Sakthivel and A. Chadha, *Asymptotic stability of fractional impulsive neutral stochastic partial integro-differential equations with infinite delay*, Stoch. Anal. Appl., 35(2017), 63–88.

[19] Z. Bai and H. Lu, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl., 311(2005), 495–505.

[20] K. Balachandran, S. Divya, L. R. Germa and J. J. Trujillo, *Relative controllability of nonlinear neutral fractional integro-differential systems with distributed delays in control*, Math. Methods Appl. Sci., 39(2016), 214–224.

[21] R. E. Bellman, *Functional Equations in the theory of dynamic programming-II, Nonlinear differential equations*, Proc. Nat. Acad. Sci. U.S.A., 41(1955), 482–486.

[22] M. Benchohra, S. Hamani and S. K. Ntouyas, *Boundary value problems for differential equations with fractional order and nonlocal conditions*, Nonlinear Anal., 71(2009), 2391–2396.

[23] A. Buică and R. Precup, *Abstract generalized quasilinearization method for coincidences*, Nonlinear Stud., 9(2002), 371–386.

[24] A. Cabada, J. J. Nieto and S. Heikkilä, *Rapid convergence of approximate solutions for first order nonlinear boundary value problems*, Int. J. Math. Math. Sci., 21(1998), 499–506.

[25] A. Cabada and T. Kisela, *Existence of positive periodic solutions of some nonlinear fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul., 50(2017), 51–67.

[26] A. Cabada and T. Kisela, *The existence of a positive solution for nonlinear fractional differential equations with integral boundary value conditions.*, Math. Methods Appl. Sci., 40(2017), 1880-1891.

[27] C. Canuto, A. Quarteroni, M. Y. Hussaini and T. A. Zang, *Spectral methods: fundamentals in single domains*, Springer, New York, 2006.

[28] A. Chadha and S. N. Bora, *Approximate controllability of impulsive neutral stochastic differential equations driven by poisson jumps*, J. Dyn. Control Syst., 24(2018), 101–128.

[29] A. Chadha and S. N. Bora, *Existence and exponential stability for neutral stochastic fractional differential equations with impulses driven by Poisson jumps.*, Stochastics, 90(2018), 663–681.

[30] G. Chandhini, K. S. Prashanthi and V. A. Vijesh, *A radial basis function method for fractional Darboux problems*, Eng. Anal. Bound. Elem., 86(2018), 1–18.

[31] Y. H. Chang, G. C. Jau and C. V. Pao, *Blowup and global existence of solutions for a catalytic converter in interphase heat transfer*, Nonlinear Anal. Real World Appl., 9(2008), 822–829.

[32] Li Changpin and Z. Fanhai , *The finite difference methods for fractional ordinary differential equations*, Numer. Funct. Anal. Optim., 34(2013), 149–179.

[33] Li Changpin, Li Qian and An Chen, *Finite difference methods with non-uniform meshes for nonlinear fractional differential equations*, J. Comput. Phys., 316(2016), 614–631.

[34] Li Changpin and W. Yihong, *Numerical algorithm based on Adomian decomposition for fractional differential equations*, Comput. Math. Appl., 57(2009), 1672–1681.

[35] S. Choudhary and V. D. Gejji, *Existence uniqueness theorems for multi-term fractional delay differential equations*, Fract. Calc. Appl. Anal., 18(2015), 1113–1127.

[36] F. Ciuchi, A. Mazzulla, N. Scaramuzza, E. K. Lenzi and L. R. Evangelista, *Fractional diffusion equation and the electrical impedance: experimental evidence in liquid-crystalline cells*, J. Phys. Chem. C, 116(2012), 8773–8777.

[37] C. D. Coster and P. Habets, *Two point boundary value problems: lower and upper solutions*, Elsevier, Netherlands, 2006.

[38] R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. II: Partial Differential Equations, Interscience Publishers, New York, 1962.

[39] M. Dehghan and M. Shahini, *Rational pseudospectral approximation to the solution of a nonlinear integro-differential equation arising in modeling of the population growth*, Appl. Math. Model., 39(2015), 5521–5530.

[40] D. Delbosco and L. Rodino, *Existence and uniqueness for a nonlinear fractional differential equation*, J. Math. Anal. Appl., 204(1996), 609–625.

[41] Z. Denton, P. W. Ng and A. S. Vatsala, *Quasilinearization method via lower and upper solutions for Riemann-Liouville fractional differential equation*, Nonlinear Dyn. Syst. Theory, 11(2011), 239–251.

[42] J. V. Devi, F. A. McRae and Z. Drici, *Generalized quasilinearization for fractional differential equations*, Comput. Math. Appl., 59(2010), 1057–1062.

[43] K. Diethelm, *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*, Springer-Verlag, Berlin, 2010.

[44] K. Diethelm, N. J. Ford and A. D Freed, *A predictor-corrector approach for the numerical solution of fractional differential equations. Fractional order calculus and its applications*, Nonlinear Dynam., 29(2002), 3–22.

[45] K. Diethelm, *An extension of the well-posedness concept for fractional differential equations of Caputo's type*, Appl. Anal., 93(2014), 2126–2135.

[46] E. H. Doha, A. H. Bhrawy and S. S. Ezz-Eldien, *A chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order*, Comput. Math. Appl., 62(2011), 2364–2373.

[47] E. H. Doha, A. H. Bhrawy and S. S. Ezz-Eldien, *Efficient Chebyshev spectral methods for solving multi-term fractional orders differential equations*, Appl. Math. Model., 35(2011), 5662–5672.

[48] P. Drábek and J. Milota, *Methods of nonlinear analysis: Applications to differential equations*, 2nd Edition, Springer, New York, 2010.

[49] N. J. Ford and J. A. Connolly, *Comparison of numerical methods for fractional differential equations*, Commun. Pure Appl. Anal., 5(2006), 289–307.

[50] N. J. Ford and A. C. Simpson, *The numerical solution of fractional differential equations: Speed versus accuracy*, Numer. Algorithms, 26(2001), 333–346.

[51] P. W. Eloe and S. G. Hristova, *Method of the quasilinearization for nonlinear impulsive differential equations with linear boundary conditions*, Electron. J. Qual. Theory Differ. Equ., 10(2002), 1–14.

[52] V. S. Erturk, A. Yildirim, S. Momanic and Y. Khan, *The differential transform method and Pade approximants for a fractional population growth model*, Internat. J. Numer. Methods Heat Fluid Flow, 22(2012), 791–802.

[53] R. E. Gaines, *A priori bounds and upper and lower solutions for nonlinear second order boundary value problems*, J. Differential Equations, 12(1972), 291–312.

[54] M. A. El-Gebeily, K. Al Shammari and D. O′Regan, *Existence and quasilinearization in Banach spaces*, J. Math. Anal. Appl., 358(2009), 345–354.

[55] V. D. Gejji, *Positive solutions of a system of non-autonomous fractional differential equations*, J. Math. Anal. Appl., 302(2005), 56–64.

[56] V. D. Gejji and S. Bhalekar, *Solving multi-term linear and non-linear diffusion–wave equations of fractional order by Adomian decomposition method*, Appl. Math. Comput., 202(2008), 113–120.

[57] V. D. Gejji and H. Jafari, *Solving a multi-order fractional differential equation using adomian decomposition*, Appl. Math. Comput., 189(2007), 541–548.

[58] A. Granas and J. Dugundji, *Fixed point theory*, Spring-Verlag, New York, 2003.

[59] D. Guo, Y. J. Cho and J. Zhu, *Partial ordering methods in nonlinear problems*, Nova Science Publishers, New York, 2004.

[60] X. Hang, *Analytical approximations for a population growth model with fractional order*, Commun. Nonlinear Sci. Numer. Simul., 14(2009), 1978–1983.

[61] D. Hengfei and Li Changpin, *High–order compact difference schemes for the modified anomalous subdiffusion equation*, Numer. Methods Partial Differential Equations, 32(2016), 213–242.

[62] M. H. Heydari, M. R. Hooshmandasl, C. Cattani and G. Hariharan, *An optimization wavelet method for multi variable-order fractional differential equations*, Fund. Inform., 151(2017), 255–273.

[63] M. H. Heydari, M. R. Hooshmandasl, C. Cattani and Li Ming, *Legendre wavelets method for solving fractional population growth model in a closed system*, Math. Probl. Eng., 2013, Art. ID 161030, 8pages.

[64] P. J. V. D. Houwen and B. P. Sommeijer, *Euler-chebyshev methods for integro-differential equations*, Appl. Numer. Math., 24(1997), 203–218.

[65] C. Hu, B. Liu and S. Xie, *Monotone iterative solutions for nonlinear boundary value problems of fractional differential equation*, Abstr. Appl. Anal., Vol. 2013, Art. ID 493164, 8 pages.

[66] S. I. Hudjaev, *Test for solvability of the Dirichlet problem for elliptic equations*, Soviet Math. Dokl., 4(1963), 40–43.

[67] F. F. Izadi and M. Dehghan, *An efficient pseudo-spectral Legendre-Galerkin method for solving a nonlinear partial integro-differential equation arising in population dynamics*, Math. Methods Appl. Sci., 36(2012), 1485–1511.

[68] T. Jankowski, *Functional differential equations of second order*, Bull. Belg. Math. Soc. Simon Stevin, 10(2003), 291–298.

[69] T. Jankowski, *Fractional equations of Volterra type involving a Riemann-Liouville derivative*, Appl. Math. Lett., 26(2013), 344–350.

[70] V. Joshi, R. B. Pachori and V. A. Vijesh, *Classification of ictal and seizure-free EEG signals using fractional linear prediction*, Biomed. Signal Process. Control, 9(2014), 1–5.

[71] R. E. Kalaba, *On nonlinear differential equations, the maximum operation and monotone convergences*, J. Math. Mech., 8(1959), 519–574.

[72] J. Kamal and D. Bahuguna, *Monotone iterative technique for nonlocal fractional differential equations with finite delay in a Banach space*, Electron. J. Qual. Theory Differ. Equ., 9(2015), 1–16.

[73] H. B. Keller and D. S. Cohen, *Some positone problems suggested by nonlinear heat generation*, J. Math. Mech., 16(1967), 1361–1376.

[74] N. A. Khan, A. Ara and M. Jamil, *Approximations of the nonlinear Volterra's population model by an efficient numerical method*, Math. Methods Appl. Sci., 34(2011), 1733–1738.

[75] A. H. Khater, R. S. Temsah and M. M. Hassan, *A Chebyshev spectral collocation method for solving Burgers'-type equations*, J. Comput. Appl. Math., 222(2008), 333–350.

[76] A. A. Kilbas, *Partial fractional differential equations and some of their applications*, Analysis (Munich), 30(2010), 35–66.

[77] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Elsevier, New York, 2006.

[78] A. A. Kilbas and J. J. Trujillo, *Differential equations of fractional order: Methods, results and problems. II*, Appl. Anal., 78(2001), 153–192.

[79] K. Krishnarajulu, K. Krithivasan and R. B. Sevugan, *Fractional polynomial method for solving fractional order population growth model*, Commun. Korean Math. Soc., 31(2016), 869–878.

[80] I. Koca, *A method for solving differential equations of q-fractional order*, Appl. Math. Comput., 266(2015), 1–5.

[81] P. Kumar, D. N. Pandey and D. Bahuguna, *Approximations of solutions to a retarded type fractional differential equation with a deviated argument*, J. Integral Equations Appl., 26(2014), 215–242.

[82] K. H. Kumar and V. A. Vijesh, *Wavelet based iterative methods for a class of 2D-partial integro differential equations*, Comput. Math. Appl., 75(2018), 187– 198.

[83] V. Lakshmikantham, S. Carl and S. Heikkilä, *Fixed point theorems in ordered Banach spaces via quasilinearization*, Nonlinear Anal., 71(2009), 3448–3458.

[84] V. Lakshmikantham and S. Leela, *Nagumo-type uniqueness result for fractional differential equations*, Nonlinear Anal., 71(2009), 2886–2889.

[85] V. Lakshmikantham, S. Leela and M. Sambandham, *Lyapunov theory for fractional differential equations*, Commun. Appl. Anal., 12(2008), 365–376.

[86] V. Lakshmikantham and S. Malek, *Generalized quasilinearization*, Nonlinear World, 1(1994), 59–63.

[87] V. Lakshmikantham and N. Shahzad, *Further generalization of generalized quasilinearization method*, J. Appl. Math. Stoch. Anal., 7(1994), 545–552.

[88] V. Lakshmikantham and A. S. Vatsala, *Generalized quasilinearization for nonlinear problems*, Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 1998.

[89] V. Lakshmikantham and A. S. Vatsala, *General uniqueness and monotone iterative technique for fractional differential equations*, Appl. Math. Lett., 21(2008), 828-–834.

[90] V. Lakshmikantham and A. S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal., 69(2008), 2677–2682.

[91] Y. Li and L. Hu, *Solving fractional Riccati differential equations using Haar wavelet*, 2010 Third International Conference on Information and Computing, 1(2010), 314–317.

[92] M. Maleki and M. T. Kajani, *Numerical approximations for Volterra's population growth model with fractional order via a multi-domain pseudospectral method*, Appl. Math. Model., 39(2015), 4300–4308.

[93] F. A. McRae, *Generalized quasilinearization of stochastic initial value problem*, Stoch. Anal. Appl., 13(1995), 205–210.

[94] K. S. Miller and B. Ross, *An introduction to fractional calculus and fractional differential equations*, John Wiley & Sons, Inc., New-York, 1993.

[95] S. Momani and Z. Odibat, *Comparison between the homotopy perturbation method and the variational iteration method for linear fractional partial differential equations*, Comput. Math. Appl., 54(2007), 910–919.

[96] S. Momani and Z. Odibat, *Homotopy perturbation method for nonlinear partial differential equations of fractional order*, Phys. Lett. A, 365(2007), 345–350.

[97] S. Momani and R. Qaralleh, *Numerical approximations and Pade approximants for a fractional population growth model*, Appl. Math. Model., 31(2007), 1907–1914.

[98] S. S. Motsa, *On the bivariate spectral homotopy analysis method approach for solving nonlinear evolution partial differential equations*, Abstr. Appl. Anal., 2014, Art. ID 350529, 8 pages.

[99] S. S. Motsa, V. M. Magagula and P. Sibanda, *A bivariate Chebyshev spectral collocation quasilinearization method for nonlinear evolution parabolic equations*, The Scientific World Journal, Vol. 2014, Art. ID 581987, 13 pages.

[100] A. S. V. Murthy and J. G. Verwer, *Solving parabolic integro-differential equations by an explicit integration method*, J. Comput. Appl. Math., 39(1992), 121–132.

[101] R. Narasimha, *Non-linear vibration of an elastic string*, J. Sound Vibration, 8(1968), 134–146.

[102] J. J. Nieto and B. Samet, *Solvability of an implicit fractional integral equation via a measure of noncompactness argument*, Acta Math. Sci., 37(2017), 195–204.

[103] J. J. Nieto, G. Stamov and I. Stamova, *A fractional-order impulsive delay model of price fluctuations in commodity markets: almost periodic solutions*, Eur. Phys. J. Spec. Top., 226(2017), 3811–3825.

[104] R. J. Nirmala, K. Balachandran and J. J. Trujillo, *Null controllability of fractional dynamical systems with constrained control*, Fract. Calc. Appl. Anal., 20(2017), 553–565.

[105] Z. Odibat and S. Momani, *Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order*, Chaos Solitons Fractals, 36(2008), 167–174.

[106] B. G. Pachpatte, *On a nonlinear diffusion system arising in reactor dynamics*, J. Math. Anal. Appl., 94(1983), 501–508.

[107] K. Parand, S. Abbasbandy, S. Kazem and J. A. Rad, *A novel application of radial basis functions for solving a model of first-order integro-ordinary differential equation*, Commun. Nonlinear Sci. Numer. Simul., 16(2011), 4250–4258.

[108] K. Parand, Z. Delafkar, N. Pakniat, A. Pirkhedri and M. K. Haji, *Collocation method using sinc and rational Legendre functions for solving Volterra's population model*, Commun. Nonlinear Sci. Numer. Simul., 16(2011), 1811-1819.

[109] K. Parand and M. Delkhosh, *Solving Volterra's population growth model of arbitrary order using the generalized fractional order of the Chebyshev functions*, Ric. Mat., 65(2016), 307–328.

[110] K. Parand and M. Razzaghi, *Rational Chebyshev tau method for solving Volterra's population model*, Appl. Math. Comput., 149(2004), 893–900.

[111] K. Parand, A. R. Rezaei and A. Taghavi, *Numerical approximations for population growth model by rational Chebyshev and Hermite functions collocation approach: a comparison*, Math. Methods Appl. Sci., 33(2010), 2076–2086.

[112] C. V. Pao, *Positive solutions of a nonlinear boundary-value problem of parabolic type*, J. Differential Equations, 22(1976), 145–163.

[113] C. V. Pao, *Bifurcation analysis on a nonlinear diffusion system in reactor dynamics*, Appl. Anal., 9(1979), 107–119.

[114] C. V. Pao, *Nonlinear parabolic and elliptic equations*, Plenum Press, New York, 1992.

[115] C. V. Pao, Y. H. Chang and G. C. Jau, *Numerical methods for a coupled system of differential equations arising from a thermal ignition problem*, Numer. Methods Partial Differential Equations, 29(2013), 251–279.

[116] I. Podlubny, *Fractional differential equations: Mathematics in science and engineering*, Academic Press, London, Vol. 198, 1999.

[117] J. D. Ramirez and A. S. Vatsala, *Generalized monotone iterative techniques for Caputo fractional integro-differential equations with initial condition*, Neural Parallel Sci. Comput., 23(2015), 219–237.

[118] M. Al-Refai, *Basic results on nonlinear eigenvalue problems of fractional order*, Electron. J. Differential Equations., 2012(2012), 1–12.

[119] M. Al-Refai and M. A. Hajji, *Monotone iterative sequences for nonlinear boundary value problems of fractional order*, Nonlinear Anal., 74(2011), 3531–3539.

[120] R. Roy, V. A. Vijesh and G. Chandhini, *Iterative methods for a fractional order Volterra population model*, J. Integral Equations Appli., https://projecteuclid.org/euclid.jiea/1522980020.

[121] U. Saeed and M. Rehman, *Haar wavelet-quasilinearization technique for fractional nonlinear differential equations*, Appl. Math. Comput., 220(2013), 630-648.

[122] K. Sakthivel, K. Balachandran and B. R. Nagaraj, *On a class of non-linear parabolic control systems with memory effects*, Internat. J. Control, 81(2008), 764–777.

[123] K. Sakthivel, K. Balachandran, J. Y. Park and G. Devipriya, *Null controllability of a nonlinear diffusion system in reactor dynamics*, Kybernetika, 46(2010), 890–906.

[124] D. H. Sattinger, *Monotone methods in nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Anal., 29(1972), 979–1000.

[125] K. Schmitt, *A nonlinear boundary value problem*, J. Differential Equations, 7(1970), 527–537.

[126] E. Shivanian, *Analysis of meshless local radial point interpolation (MLRPI) on a nonlinear partial integro-differential equations arising in population dynamics.*, Eng. Anal. Bound. Elem., 37(2013), 1693–1702.

[127] R. D. Small, *Population growth in a closed system*, SIAM Rev., 25(1983), 93–95.

[128] M. Stynes, *A Caputo two-point boundary value problems: existence, uniqueness and regularity of a solution*, Model. Anal. Inf. Syit., 23(2016), 370–376.

[129] M. Stynes and J. L. Gracia, *A finite difference method for a two-point boundary value problem with a Caputo fractional derivative*, IMA J. Numer. Anal., 35(2015), 698–721.

[130] M. Stynes and J. L. Gracia, *Boundary layers in a two-point boundary value problem with a Caputo fractional derivative*, Comput. Methods Appl. Math., 15(2015), 79–95.

[131] L. A. Sunny, *Monotone iterative methods for nonlinear partial differential equations*, Thesis(Ph.D), I. I. T. Indore, 2017.

[132] L. A. Sunny, R. Roy and V. A. Vijesh, *An accelerated technique for solving a system of differential equations for a catalytic converter in interphase heat transfer*, J. Math. Anal. Appl., 445(2017), 318–336.

[133] L. A. Sunny, R. Roy and V. A. Vijesh, *An alternative technique for solving a coupled PDE system in interphase heat transfer*, Appl. Anal., https://doi.org/10.1080/00036811.2018.1478077.

[134] M. Syam and M. Al-Refai, *Positive solutions and monotone iterative sequences for a class of higher order boundary value problems of fractional order*, J. Fract. Calc. Appl., 4(2013), 1–13.

[135] Q. Tao and Z. A. Yao, *The time optimal control of parabolic integro differential equation in reactor dynamics*, Taiwanese J. Math., 18(2014), 409–417.

[136] K. G. TeBeest, *Classroom note: numerical and analytical solutions of Volterra's population model*, SIAM Rev., 39(1997), 484–493.

[137] A. D. Tlalka and T. Jankowski, *Quadratic convergence of monotone iterations of differential-algebraic equations*, Math. Slovaca, 52(2002), 315–330.

[138] F. Toumi, *Monotone iterative technique for systems of nonlinear Caputo fractional differential equations*, Differential and Difference Equations with Applications, ICDDEA (2015), 99–107.

[139] L. N. Trefethen, *Spectral methods in MATLAB*, SIAM, 2000.

[140] A. S. Vatsala and L. Wang, *The generalized quasilinearization method for parabolic integro-differential equations*, Quart. Appl. Math., 59(2001), 459–470.

[141] G. Venkatesan and R. K. George, *Functional approach to observability and controllability of linear fractional dynamical systems*, J. Dyn. Syst. Geom. Theor., 15(2017), 111–129.

[142] G. Venkatesan, M. Malik and R. K. George, *Trajectory controllability of fractional dynamical systems*, J. Control Decis., 4(2017), 114–130.

[143] V. A. Vijesh and K. H. Kumar, *Wavelet based quasilinearization method for semi-linear parabolic initial boundary value problems*, Appl. Math. Comput., 266(2015), 1163–1176.

[144] G. Wang, R. P. Agarwal and A. Cabada, *Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations*, Appl. Math. Lett., 25(2012), 1019–1024.

[145] G. Wang, S. Liu and L. Zhang, *Neutral fractional integro-differential equation with nonlinear term depending on lower order derivative*, J. Comput. Appl. Math., 260(2014), 167–172.

[146] Y. Wang and Li Zhu, *Solving nonlinear Volterra integro-differential equations of fractional order by using Euler wavelet method*, Adv. Difference Equ., 27(2017), 16 pages.

[147] Z. Wu, J. Yin and C. Wang, *Elliptic and parabolic equations*, World Scientific, Singapore, 2006.

[148] A. Yakar, *Initial time difference quasilinearization for Caputo fractional differential equations*, Adv. Difference Equ., 2012, 2012:99, 9 pages.

[149] C. Yakar and A. Yakar, *A refinement of quasilinearization method for Caputo's sense fractional-order differential equations*, Abstr. Appl. Anal., 2010, Art. ID 704367, 10 pages.

[150] He Yang, R. P. Agarwal, H. K. Nashine and Y. Liang, *Fixed point theorems in partially ordered Banach spaces with applications to nonlinear fractional evolution equations*, J. Fixed Point Theory Appl., 19(2017), 1661–1678.

[151] S. Yüzbasi, *A numerical approximation for Volterra's population growth model with fractional order*, Appl. Math. Model., 37(2013), 3216–3227.

[152] E. Zeidler, *Nonlinear functional analysis and its applications I: Fixed point theorems*, Springer-Verlag, New York, 1986.

[153] S. Zhang, *Positive solutions for boundary value problems of nonlinear fractional differential equations*, Electron. J. Differential Equations, 2006(2006), 1–12.

[154] W. X. Zhou and Y. D. Chu, *Existence of solutions for fractional differential equations with multi-point boundary conditions*, Commun. Nonlinear Sci. Numer. Simul., 17(2012), 1142–1148.

[155] Li Zhou and Q. Fan, *Solving fractional nonlinear Fredholm integro-differential equations by the second kind Chebyshev wavelet*, Commun. Nonlinear Sci. Numer. Simul., 17(2012), 2333–2341.

[156] Li Zhou and Y. Wang, *Solving fractional partial differential equations by using the second Chebyshev wavelet operational matrix method*, Nonlinear Dynam., 89(2017), 1915–1925.