CHARACTERIZATION AND EXPLICIT CONSTRUCTION OF PAIRWISE ORTHOGONAL PARSEVAL FRAMES ON LCA GROUPS

Ph.D. Thesis

By

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DEPARTMENT OF MATHEMATICS
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CHARACTERIZATION AND EXPLICIT CONSTRUCTION OF PAIRWISE ORTHOGONAL PARSEVAL FRAMES ON LCA GROUPS

A THESIS

Submitted in partial fulfillment of the requirements for the award of the degree

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by

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I hereby certify that the work which is being presented in the thesis entitled **CHARACTERIZATION AND EXPLICIT CONSTRUCTION OF PAIRWISE ORTHOGONAL PARSEVAL FRAMES ON LCA GROUPS** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DEPARTMENT OF MATHEMATICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from August 2020 to March 2026 under the supervision of **Dr. Niraj Kumar Shukla**, Professor, Department of Mathematics, Indian Institute of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

16/10/2025
Signature of the student with date
(Navneet)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

16/10/2025
Signature of Thesis Supervisor with date
(Dr. Niraj Kumar Shukla)

Navneet has successfully given her Ph.D. Oral Examination held on **March 10, 2026**

10/03/2026
Signature of Thesis Supervisor with date
(Dr. Niraj Kumar Shukla)

DEDICATION

To My Grandparents and Parents

LIST OF PUBLICATIONS

List of Published/Communicated Research Papers from the Thesis

1. **Redhu N.**, Gumber A., Shukla N. K. (2025), *Constructions of pairwise orthogonal Parseval frames generated by filter on LCA groups*, **Applied and computational Harmonic analysis**, **74**, paper No.101708, 27pp.
   DOI: 10.1016/j.acha.2024.101708
2. Christensen O., **Redhu N.**, Shukla N. K., (2026) *Cyclic frames in finite-dimensional Hilbert spaces*, **Linear Algebra and its Applications**, **728**, 63-81.
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3. **Redhu N.**, Gumber A., Shukla N. K. (2025), *The geometry of sampling on unions of co-compact subgroups*, **Under review**.
4. **Redhu N.**, Gumber A., Führ H., Shukla N. K., *Characterization and Explicit Construction of pairwise Orthogonal Parseval Frames in LCA Groups*, *Submitting soon*.

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ABSTRACT

KEYWORDS: B-splines; Co-compact Gabor system; Cyclic frames; Dual Gramian; Dynamical frames; Frame; Frame operator; Generalized translation invariant (GTI) systems; local integrablity condition (LIC); Locally compact abelian (LCA) group; Mercedes-Benz frame; Orthogonal frames; Orthogonal Samples; Parseval frames; Sampling transforms; Shearlet; tight samples; Translation invariant systems; Unconditional convergence property (UCP); Wave-packet system; Wavelet system

The theory of generalized translation invariant (GTI) systems provides a powerful and unified framework for analyzing a wide range of structured systems in harmonic analysis. Notable examples such as wavelet and Gabor systems arise naturally as special cases within the GTI framework. A central concept in frame theory is the orthogonality or strong disjointness between two frames in a Hilbert space, introduced by Balan, Han, and Larson, which characterizes pairs of frames that do not interfere with each other. These orthogonal frames have significant practical applications, including constructing new frames from existing ones, developing dual frames, secure communication protocols, and synthesizing complex super-frames. The broad applicability and flexibility of GTI systems make them indispensable in both theoretical studies and practical fields, such as signal processing and data transmission.

This thesis develops novel construction techniques for generating pairwise orthogonal Parseval frames within locally compact abelian (LCA) groups. It establishes sufficient conditions for constructing such pairwise orthogonal Parseval GTI frames in the space $L^2(G)$, where G is a second countable LCA group. The constructions leverage the local integrability condition (LIC). Initially, GTI frames are built through filter based methods, ensuring the LIC holds. Subsequently, these GTI systems are shown to be Parseval frames and pairwise orthogonal. As a key result, explicit constructions of pairwise orthogonal Parseval frames are provided in both $L^2(\mathbb{R})$ and $L^2(G)$, using B-splines as generating

functions. Furthermore, a new approach is introduced to construct N pairwise orthogonal Parseval frames with GTI structure starting from a single Parseval frame.

Next, the thesis investigates the applications of orthogonal frames in sampling theory, with a particular focus on orthogonal sampling transforms associated with two unions of co-compact subgroups (not necessarily lattices) defined over distinct frequency bands. The analysis is then extended to the case of a general union of co-compact subgroups. It is shown that two unions of sampling sets are orthogonal if and only if each corresponding pair of individual sampling sets is orthogonal. Furthermore, a union of sampling sets is shown to be tight if and only if each individual sampling set is tight.

Building on this foundation, the thesis offers characterizations of GTI Parseval Bessel and frame systems that are pairwise orthogonal, highlighting conditions under which these systems form Parseval frames. These characterizations pivot around the *unconditional convergence property (UCP)*, a weaker and more flexible assumption than the LIC. The results extend to various structured systems, including wavelets and shearlets in $L^2(\mathbb{R}^d)$. Utilizing these insights, explicit filter based constructions of pairs of GTI systems satisfying the ∞-UCP are developed, forming pairwise orthogonal Parseval frames. This advancement relaxes some restrictive assumptions in the previous method.

Finally, the thesis shifts focus to frames in finite-dimensional Hilbert spaces, which are critical in practical applications such as coding theory, signal processing, and data transmission. The study emphasizes frames with a dynamical structure, with cyclic frames appearing as a significant subclass. Dynamical frames, thoroughly analyzed by Aldroubi et al. and others, are notable for their robustness in erasure problems and error correction. By employing an alternative approach, the thesis presents new insights into both general dynamical frames and cyclic frames. In particular, it provides a complete characterization of dynamical frames, which naturally yields a characterization of cyclic frames. These results deepen the understanding of frame structures vital for reliable data encoding and recovery.

CHAPTER 1

INTRODUCTION

Frames are special sets of vectors in a Hilbert space that are used to represent signals flexibly and robustly. For any signal f, we often seek a way to write it as a sum of building blocks:

$$f = \sum_{i \in I} c_i f_i,$$

where each f_i is a building block and c_i's are coefficients. With frames, unlike with orthonormal bases where coefficients are fixed, we have many choices for c_i which helps control noise and gives more freedom in representing signals. This flexibility was first explored by Duffin and Schaeffer in non-harmonic Fourier analysis in 1952 (see, [45]). It made frames useful for signal and image processing, data compression, and have many related applications in science and engineering (see, [12, 13, 47, 52, 58, 96, 108, 114] and references therein).

In this scenario, many mathematicians and researchers have contributed to studying and understanding various interesting properties and results of frame theory in different contexts, to mention a few (see, [27, 33, 48, 46, 51] and references therein).

Over the past two decades, the study of frames in the setting of locally compact abelian (LCA) groups has emerged as a vibrant area of research, both in theory and in applications. The LCA group framework offers several advantages:

- It unifies the continuous theory (integral representations) and the discrete theory (series expansions), enabling frame analysis from a broad, abstract perspective.
- It provides a single framework for studying the fundamental groups $\mathbb{R}, \mathbb{Z}, \mathbb{T}, \mathbb{Z}_m$ and their higher-dimensional counterparts as well as groups of the form $\mathbb{R}^p \times \mathbb{T}^q \times \mathbb{Z}^r \times \mathbb{Z}_N^s$.
- It becomes highly useful in signal and image processing, where products of LCA groups naturally arise; for instance, multichannel video signals involve the product group $\mathbb{Z}^d \times \mathbb{Z}_m$.

In light of these advantages, several researchers have made noteworthy contributions to the development of the theoretical framework necessary for analyzing frame properties on LCA groups (see, [8, 20, 27, 33, 35, 29, 46, 57, 60, 61, 62, 79, 80, 83, 88] and references therein).

Among these properties, the orthogonality or strongly disjointness of frame pairs in Hilbert spaces is very useful. This concept was first introduced and studied by Han and Larson [71], and Balan [12] in the context of multiplexing. Two frames are said to be pairwise orthogonal if their analysis operators have orthogonal ranges. This property has significant practical implications. For instance:

- *Multiple access communication*: Pairwise orthogonal Parseval frames enable multiplexing signals so that multiple users can share a common communication channel. Signals can be recovered perfectly from summed coefficients, as seen in applications like radio and television broadcasting, as well as computer networks [13, 14].
- *Duality*: In [24], it has been demonstrated that noncanonical dual frames result in a smaller reconstruction error compared to canonical dual frames. This finding implies that in certain practical scenarios, exploring multiple duals with respect to the given frame is desirable. Orthogonality allows the easy construction of such noncanonical dual frames, expanding flexibility in signal representation.
- *Superframes*: Superframes arise naturally when two pairwise orthogonal frames from separate spaces combine to form a frame in the direct sum space, allowing coordinated processing across distinct signal components [13].
- *Perfect reconstruction in sampling theory*: In some sampling problems, frames are used to represent signals in subspaces even though the frame elements are not necessarily in those subspaces. Thanks to orthogonality, it remains possible to perfectly reconstruct signals in these cases, which are useful for practical applications in signal processing and communications [1, 115, 116].

Motivated by the aforementioned applications, many researchers have developed construction methods for such frames with various structural systems, such as wavelet and shift-invariant systems (see, [7, 22, 60, 61, 83, 85, 92, 99, 107, 109, 111, 115] and references therein).

In this thesis, the main focus is on the study of pairwise orthogonal frames in locally compact abelian (LCA) groups and their applications in sampling theory. Specifically, we investigate the orthogonality of frame pairs for generalized translation invariant (GTI) systems in LCA groups. We provide characterizations and construction techniques for pairwise orthogonal Parseval frames with GTI structures. We also explicitly construct such frames generated by B-splines [36, 98]. Furthermore, we develop the theory for certain special cases of GTI systems and LCA groups. For example, we present characterizations and construction methods for pairwise orthogonal wavelet frames in $L^2(\mathbb{R})$. Additionally, we explore conditions under which unions of samples form tight and orthogonal systems. Finally, we study the properties of dynamical and cyclic frames in the finite-dimensional Hilbert spaces.

1.1. Motivation and objective

Frames generated by unitary group representation provide a unified framework for analyzing a broad class of function systems, such as wavelet, Gabor, shearlet, translation invariant (TI), shift invariant (SI), generalized shift invariant (GSI), and wave-packet systems. In this context, many researchers have studied the structural and analytical properties of frame theory (see, e.g., [19, 27, 33, 34, 60, 61, 79] and references therein).

Motivated by these developments, this thesis focuses on the characterization and explicit construction of orthogonal frames with generalized translation invariant (GTI) structures, which is a class of systems introduced recently by Jakobsen and Lemvig in [79]. A GTI system is defined as one generated by translating a collection of functions over a countable family of closed, co-compact subgroups Γ_j of a second-countable LCA group G, within the separable Hilbert space $L^2(G)$, where j belongs to a countable index set $\mathcal{J}$.

In this connection, we note that our construction technique for pairwise orthogonal frames with GTI structures is inspired by two recent characterizations: the characterization of pairwise orthogonal frames with GTI structures by Gumber and Shukla [60], and the characterization of GTI Parseval frames by Jakobsen and Lemvig [79], together with the unitary extension principle (UEP) introduced by Christensen et al. [36]. The GTI framework serves as a bridge between the well-established discrete frame theory of GSI systems and its continuous counterpart. Consequently, the construction of pairwise orthogonal GTI frame systems provides a unified approach for deriving analogous results across several function systems, including the GSI systems studied by Kutyniok and Labate [88] and the TI systems considered by Bownik and Ross [27].

In this direction, Kutyniok and Labate [88] developed a unified framework for several classical function systems, such as Gabor and GSI systems on $\mathbb{R}^d$, by formulating GSI systems in the general setting of LCA groups. Their approach extends earlier works of Hernández, Labate, and Weiss [73], as well as Ron and Shen [103], on GSI systems in $L^2(\mathbb{R}^d)$.

It is worth noting that among the function systems mentioned above, SI and GSI systems are defined by translations along uniform lattices. In contrast, TI and GTI systems can be viewed as continuous generalizations of SI and GSI systems, respectively, obtained by allowing translations along co-compact subgroups of an LCA group. The motivation for introducing co-compact subgroups in the study of TI systems [27] and GTI systems [79] arises from the fact that not all LCA groups admit uniform lattices. For example, the p-adic numbers $\mathbb{Q}_p$ have only the trivial discrete subgroup consisting of the neutral element, which is not a uniform lattice. Similarly, the p-adic integers $\mathbb{Z}_p$ possess only trivial uniform lattices but admit a rich family of non-trivial co-compact subgroups. Thus, the works of Bownik and Ross [27] and Jakobson and Lemvig [79] extend the theory of function systems based on translations along uniform lattices, as studied earlier in [29, 88].

In our subsequent work, we investigate applications of pairwise orthogonal Parseval frames in sampling theory. In particular, we focus on orthogonal sampling transforms associated with two unions of co-compact subgroups (not necessarily lattices) defined over two frequency bands. We show that a union of sampling sets is tight if and only if each individual sampling set is tight. Furthermore, we prove that two unions of sampling sets are orthogonal if and only if each corresponding pair of individual sampling sets is orthogonal. This part of our work is motivated by the studies of Weber [115, 116] and Aldroubi et al. [5] on sampling theory.

The above work on GTI systems relies on the technical assumption of the local integrability condition (LIC) and its variants (such as the dual α-LIC), which ensure the boundedness of the associated frame operators. In the context of GTI systems on LCA groups, this condition was introduced by Jakobsen and Lemvig [79]. For a more detailed explanation of the LIC, we refer to [60, 73, 79, 85, 86, 113]. More recently, Führ et al. [53] introduced the unconditional convergence property (UCP) and its variants (such as the dual 1-UCP and dual ∞-UCP), which provide a weaker and more flexible alternative to the classical LIC. These concepts were first developed for GSI systems, a subclass of GTI systems. The 1-UCP condition for a GTI system was introduced in [113].

Motivated by these developments, our next work characterizes pairwise orthogonal frames with GTI structures under the dual 1-UCP. We also obtain a characterization of Parseval GTI frames within this setting. Moreover, using these characterizations, we provide explicit recipe for constructing pairwise orthogonal Parseval frames. This recipe generalizes our earlier construction method by allowing us to relax certain assumptions that were required previously.

In the Euclidean setting, Weber [115] investigated orthogonal frames of translates with various applications. This line of work was extended by Kim et al. [83] to general shift-invariant subspaces of $L^2(\mathbb{R}^d)$, and by Lopez and Han [94], who studied the orthogonality of discrete Gabor frame pairs in $\ell^2(\mathbb{Z}^d)$. Bhatt, Johnson, and Weber [23] developed a technique to construct pairwise orthogonal wavelet Parseval frames in $L^2(\mathbb{R})$, which was later generalized to $L^2(\mathbb{R}^d)$ by Bhatt [22].

Building on these advances, Gumber and Shukla [60] extended the characterization results obtained in the Euclidean case [115], the discrete setting [94], and the uniform lattice case [83], to the framework of co-compact subgroups of LCA groups under the LIC assumption.

The present thesis aims to take this line of research further by characterizing pairwise orthogonal frames with GTI structures under the more flexible UCP framework generalizing the characterization result study in [60]. In addition, we provide explicit construction techniques for pairwise orthogonal Parseval frames with GTI structures, thereby generalizing the wavelet based constructions in $L^2(\mathbb{R})$ [23] and $L^2(\mathbb{R}^d)$ [22]. Finally, we note that the study of frame properties for structured function systems in various settings has received considerable attention in recent years (see, e.g., [11, 13, 33, 46, 48, 60, 57, 80, 94]), which further motivates our investigation of orthogonality for such systems in the LCA-group framework.

The GTI systems can be viewed as special cases of frames generated by unitary representations of LCA groups (see, e.g., [16, 19, 17, 25, 60, 61]). This observation naturally links them to the class of dynamical frames. We study these frames in finite-dimensional Hilbert spaces, which are indispensable for practical applications in digital signal processing, data compression, and communications. For convenience, and without loss of generality, we do work in $\mathbb{C}^d$. Within this setting, we focus on dynamical frames, arising from the paradigm of dynamical sampling introduced by Aldroubi et al. [3, 2, 4]. While much of the existing analysis develops structural properties of such frames, we complement this line of work with an alternative approach that offers a different perspective and extends the theoretical understanding of dynamical frames.

Building on this approach, we introduce and analyze cyclic frames, a concept that is inherently finite-dimensional. Originally studied by Kalra in [82], cyclic frames were already observed to possess attractive features in erasure recovery problems [77, 82, 93, 91]. In this work, we generalize this notion through two distinct characterizations of cyclic frames, from which several fundamental structural properties follow naturally. We further investigate tight cyclic frames and their connections to erasure resilience, demonstrating their potential for robust applications. In this way, our study contributes both to the theoretical foundations of finite-dimensional frame theory and to its relevance for practical scenarios such as error correction and data recovery.

1.1.1. Objective of the thesis

The primary objective of this thesis is to develop techniques for constructing pairs of GTI systems

$$\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,\,p\in P_j} \quad \text{and} \quad \bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,\,p\in P_j},$$

that satisfy both the LIC and the more flexible UCP. The *generalised translation invariant (GTI)* system introduced in [79] is a collection of functions of the form $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p\}_{\lambda\in\Gamma_j,p\in P_j}$, where $\mathcal{J}$ is a countable index set, for any $\lambda\in G$, the *translation operator* T_λ is defined by

$$T_\lambda : L^2(G)\to L^2(G),\ (T_\lambda f)(x) := f(x-\lambda),\ x\in G,$$

P_j is countable or uncountable index set, Γ_j is a closed co-compact (i.e., the quotient group G/Γ_j is compact) subgroups, and $\{g_p\}_{p\in P_j}$ is a subset of $L^2(G)$. If $\Gamma_j=\Gamma$ for each j, then above GTI system is called *translation invariant* (TI) system. If P_j is countable and Γ_j is a uniform lattice, then $\bigcup_{j\in J}\{T_\lambda g_p\}_{\lambda\in\Gamma_j,p\in P_j}$ is called *generalised shift invariant* (GSI) system and if $\Gamma_j=\Gamma$ for each j, then it is called *shift invariant* (SI) system. The GTI systems thus provide a unified framework for analyzing a broad class of structured function systems, including wavelet, Gabor, wave-packet, and shearlet systems (see, [60, 79]).

In addition to characterizing GTI frames under UCP, this thesis aims to develop *explicit construction recipes* for Parseval frames and pairwise orthogonal frames, together with applications in sampling theory. Based on this goal, the specific objectives of this thesis are as follows:

(i) To provide a technique for constructing pairs of GTI systems that satisfy the LIC.

(ii) To develop, as an application of (i), a method for constructing pairwise orthogonal Parseval frames and to relate these techniques to the classical theory of Parseval wavelet frames in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^n)$.

(iii) To characterize orthogonal sampling transforms generated by the action of two co-compact subgroups (not necessarily discrete) of a locally compact abelian group G on distinct bands, and to extend this characterization to unions of such subgroups.

(iv) To characterize pairwise orthogonal GTI Parseval frames under the technical assumption 1-UCP, and to investigate orthogonal systems arising from structured classes such as wavelet, Gabor, and shearlet systems.

(v) To develop new construction techniques (based on UCP) for pairwise orthogonal Parseval frames that relax some of the assumptions required in (ii).

(vi) To study frame theory in finite-dimensional settings, with a focus on dynamical and cyclic frames in finite-dimensional Hilbert spaces, motivated by their applications to error resilience and signal erasure recovery.

1.2. Preliminaries

In this section, we establish the notation and background material required for the subsequent chapters. We begin with a review of fundamental results from Fourier analysis on locally compact abelian (LCA) groups, followed by essential definitions and results from frame theory in Hilbert spaces. Throughout this thesis, we use the standard notation: $\mathbb{N}$ for the set of positive integers, $\mathbb{Z}$ for integers, $\mathbb{R}$ for real numbers, $\mathbb{C}$ for complex numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. These preliminaries will serve as the foundation for the developments presented in later chapters.

1.2.1. Background on LCA Groups

In this subsection, we follow the terminology and results presented in the classical books [33, 50, 75, 76, 105] and research papers [27, 29, 67, 69, 79, 80, 88] for harmonic analysis on locally compact abelian (LCA) groups.

Throughout, let G denote a second countable LCA group with group operation $+$ and identity element 0. Recall that a topological space is called *second countable* if its topology admits a countable basis. Note that the second countable property implies that G is metrizable and σ-compact.

It is well known that every LCA group G admits a unique (up to a positive multiplicative constant) Haar measure, denoted by μ_G. This is a non-negative, regular Borel measure, which is translation invariant, i.e., $\mu_G(E+x) = \mu_G(E)$ for all $x \in G$ and all Borel sets $E \subseteq G$.

Let $\widehat{G}$ denote the set of all continuous characters of G, that is, all continuous homomorphisms from G into the torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Equipped with pointwise multiplication,

$$(\gamma \cdot \gamma')(x) := \gamma(x) \cdot \gamma'(x) \quad \text{for all } \gamma, \gamma' \in \widehat{G},\ x \in G,$$

the set $\widehat{G}$ forms an LCA group with identity element 1, called the *dual group* of G. We equip $\widehat{G}$ with the compact convergence topology. As an LCA group, $\widehat{G}$ also possesses a Haar measure, which we denote by $\mu_{\widehat{G}}$. Furthermore, there exists a topological group isomorphism between G and the dual of its dual group $\widehat{\widehat{G}}$, i.e., $\widehat{\widehat{G}} \cong G$, which is known as the Pontryagin duality theorem [50]. In particular, if G is discrete, then $\widehat{G}$ is compact, and conversely.

Given an LCA group G with Haar measure μ_G, the integral over G is translation invariant in the sense that,

$$\int_G f(x+y) d\mu_G(x) = \int_G f(x) d\mu_G(x)$$

for each element $y \in G$ and for each Borel-measurable function f on G. For $1 \leq p < \infty$, we define the space $L^p(G, \mu_G)$ (simply, $L^p(G)$) as follows:

$$L^p(G) := \left\{ f : G \to \mathbb{C} \text{ is a measurable function and } \int_G |f(x)|^p d\mu_G(x) < \infty \right\}.$$

Since G is a second countable LCA group, $L^p(G)$ is separable, for all $1 \leq p < \infty$. In this thesis, we will focus only on $p = 2$ case. Here, note that $L^2(G)$ is a Hilbert space with inner product given by

$$\langle f, g \rangle = \int_G f(x) \overline{g(x)} d\mu_G(x) \quad \text{for all } f, g \in L^2(G).$$

Let the Fourier transform $\widehat{\ } : L^1(G) \to C_0(\widehat{G})$, $f \mapsto \widehat{f}$, be defined by the operator

$$\mathcal{F} f(\gamma) = \widehat{f}(\gamma) = \int_G f(x) \overline{\gamma(x)} d\mu_G(x) \quad \text{for } \gamma \in \widehat{G},$$

where $C_0(\widehat{G})$ denotes the functions on $\widehat{G}$ vanishing at infinity. If $f \in L^1(G)$, $\widehat{f} \in L^1(\widehat{G})$, and the measures on G and $\widehat{G}$ are normalized appropriately so that the Plancherel theorem holds, then the inverse Fourier transform can be defined by

$$f(x) = \mathcal{F}^{-1}\widehat{f}(x) = \int_{\widehat{G}} \widehat{f}(\xi)\xi(x) d\mu_{\widehat{G}}(\xi) \quad \text{for } x \in G.$$

Note that the Fourier transform $\mathcal{F}$ can be extended from $L^1(G) \cap L^2(G)$ to a surjective isometry between $L^2(G)$ and $L^2(\widehat{G})$ [50, Plancherel theorem]. Thus, the Parseval formula holds and is given by

$$\langle f, g\rangle = \int_G f(x)\overline{g(x)} d\mu_G(x) = \int_{\widehat{G}} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d\mu_{\widehat{G}}(\xi) = \langle \widehat{f}, \widehat{g}\rangle, \quad \text{for all } f, g \in L^2(G).$$

The following definitions will be used in the sequel: Given an LCA group, we call a subgroup Γ in G as *co-compact* if the quotient group G/Γ is compact, whereas Γ in G is said to be a *uniform lattice* if in addition, Γ is discrete.

Let $\Gamma \subseteq G$ be a closed subgroup of an LCA group G. Then, the quotient G/Γ is a regular topological group. Further, we note that it is a second countable LCA group under the quotient topology by using the fact that G is second countable.

Note that for a subgroup Γ of an LCA group G, the symbol $\Gamma^\perp$ denotes the *annihilator* of Γ, which is a subgroup of $\widehat{G}$ defined by

$$\Gamma^\perp := \left\{\gamma \in \widehat{G} : \gamma(x) = 1, \text{ for all } x \in \Gamma\right\}.$$

It follows from the definition of the topology on $\widehat{G}$ that the annihilator $\Gamma^\perp$ is a closed subgroup in $\widehat{G}$. Moreover, if Γ is closed, then $(\Gamma^\perp)^\perp = \Gamma$ and the following hold:

(i) there exists a topological group isomorphism mapping G/Γ onto $\Gamma^\perp$, that is, we have $\widehat{G/\Gamma} \cong \Gamma^\perp$;

(ii) there exists a topological group isomorphism mapping $\widehat{G}/\Gamma^\perp$ onto $\widehat{\Gamma}$, that is, we have $\widehat{G}/\Gamma^\perp \cong \widehat{\Gamma}$.

Let Ω denote the *fundamental domain* associated with $\Gamma^\perp$, i.e. Ω is a Borel measurable set such that

$$\widehat{G} = \bigcup_{w \in \Gamma^\perp} (w + \Omega), \ (w + \Omega) \cap (w' + \Omega) = \varnothing \text{ for } w \neq w', \ w, \ w' \in \Gamma^\perp. \tag{1.2.1}$$

The group Γ and G/Γ carry the Haar measures μ_Γ and $\mu_{G/\Gamma}$, respectively. If we have two out of three Haar measures μ_G, μ_Γ, and $\mu_{G/\Gamma}$, then the third one can be normalized such that for all $f \in L^1(G)$, the following relation holds:

$$\int_G f(x) d\mu_G(x) = \int_{G/\Gamma} \int_\Gamma f(x + \gamma) \ d\mu_\Gamma(\gamma) \ d\mu_{G/\Gamma}(\dot{x}), \tag{1.2.2}$$

where $\dot{x}$ denotes the coset $x+\Gamma$. The measures μ_G, μ_Γ and $\mu_{G/\Gamma}$ are always normalised in such a way that *Weil's integral formula* (1.2.2) holds. If (1.2.2) holds, then the respective dual measures on $\widehat{G}$, $\widehat{G/\Gamma} \cong \Gamma^\perp$, and $\widehat{\Gamma} \cong \widehat{G}/\Gamma^\perp$ satisfy

$$\int_{\widehat{G}} \widehat{f}(\omega) d\mu_{\widehat{G}}(\omega) = \int_{\widehat{G}/\Gamma^\perp} \int_{\Gamma^\perp} \widehat{f}(\omega\xi) d\mu_{\Gamma^\perp}(\xi) d\mu_{\widehat{G}/\Gamma^\perp}(\dot{\omega}). \tag{1.2.3}$$

Since a Haar measure and its dual are selected to satisfy the Plancherel theorem, we have the following uniqueness result: If two of the measures $\mu_G, \mu_\Gamma, \mu_{G/\Gamma}, \mu_{\widehat{G}}, \mu_{\Gamma^\perp}$ and $\mu_{\widehat{G}/\Gamma^\perp}$ are given, and these two are not dual measures, by requiring Weil's formulas (1.2.2) and (1.2.3), all the other measures are uniquely determined. Define the *covolume* $s(\Gamma)$ of Γ by

$$s(\Gamma) := \int_{G/\Gamma} d\mu_{G/\Gamma}(\dot{x}).$$

In the rest of this thesis, unless mentioned otherwise we assume Γ to be a closed and co-compact (not necessarily discrete) subgroup in G. Note that the symbol μ_Γ represents a Haar measure on the subgroup Γ. Since, $\Gamma^\perp$ is topologically isomorphic to the dual of the quotient group G/Γ, that is, $\Gamma^\perp \cong \widehat{(G/\Gamma)}$, therefore, Γ is co-compact in G if, and only if, $\Gamma^\perp$ is a discrete subgroup of $\widehat{G}$ (for more details, see [27]). Thus, $\Gamma^\perp$ will always be discrete in our case, and hence preserves a counting measure.

Note that the dual group $\widehat{G} = \Omega \oplus \Gamma^\perp$, therefore, every $\gamma \in \widehat{G}$ has a unique representation $w+\alpha$ for some $w \in \Omega$ and $\alpha \in \Gamma^\perp$. Here Ω is a $\mu_{\widehat{G}}$-measurable subset of $\widehat{G}$ and represents a Borel section of $\Gamma^\perp$ in $\widehat{G}$, also known as a *fundamental domain* of $\widehat{G}/\Gamma^\perp$, whose existence is guaranteed by [49]. Moreover, it is relevant to note that every element v in $\widehat{\Gamma} \cong \widehat{G}/\Gamma^\perp$ can be thought of as an element in Ω as all cosets in $\widehat{G}/\Gamma^\perp \cong \widehat{\Gamma}$ are of the form $w+\Gamma^\perp$ for some (unique) $w \in \Omega$. For more details, we refer to [27, Section 3].

1.2.2. Theory of frames

In this subsection, we recall some definitions and basic properties about continuous frames for Hilbert spaces. Such frames were introduced independently by Ali et al. [6] and Kaiser [81]. For a brief and self-sufficient introduction to continuous frames, we refer to [54, 97]. For more details on general theory and applications of frames and Bessel sequences, we refer to [6, 11, 27, 33, 29, 46, 57, 58, 71, 80, 81, 101, 110]. For theory of frames generated by unitary actions of LCA groups, we refer to the research articles [19, 18, 78, 104] and various references therein. Note that for basic definitions and theory on finite frames, we use the books by Christensen [33], Casazza et al. [30], and by Han et al. [70].

Definition 1.2.1. Let $\mathcal{H}$ be a complex Hilbert space, and let $(M, \sum_M, \mu_M)$ be a measure space, where $\sum_M$ denotes the σ-algebra and μ_M the non-negative measure. Then, a family of functions $\{f_m\}_{m\in M}$ in $\mathcal{H}$, is called a *continuous frame* for $\mathcal{H}$ with respect to $(M, \sum_M, \mu_M)$ if

(i) $m \mapsto f_m$ is weakly measurable; that is, for all $h \in \mathcal{H}$, the mapping $M \to \mathbb{C}$, $m \mapsto \langle h, f_m\rangle$ is measurable, and

(ii) there exist constants $0 < \alpha_1 \le \alpha_2$ (called continuous frame bounds) such that

$$\alpha_1\|h\|^2 \le \int_M |\langle h, f_m\rangle|^2 d\mu_M(m) \le \alpha_2\|h\|^2, \quad \text{for all } h \in \mathcal{H}. \tag{1.2.4}$$

A continuous frame $\{f_m\}_{m\in M}$ is called *tight* if we can choose $\alpha_1 = \alpha_2$, and *tight frame with frame bound 1* (or *Parseval*) if $\alpha_1 = \alpha_2 = 1$. The family $\{f_m\}_{m\in M}$ is called *Bessel* with constant α_2 as its Bessel constant if the right side of inequality in (1.2.4) holds. In this case, we say that the family $\{f_m\}_{m\in M}$ satisfies the *Bessel condition*.

Since this thesis deals with only separable Hilbert spaces, we can use Petti's theorem to replace weak measurability of $m \mapsto f_m$ with (strong) measurability with respect to the Borel algebra in $\mathcal{H}$.

If μ_M is a counting measure and $M = \mathbb{N}$, then $\{f_m\}_{m\in M}$ reduces to a discrete frame. In this sense, continuous frames can be realized as the generalization of discrete frames. Recall that for a countable index set $\mathfrak{J}$, a sequence $\{f_n\}_{n\in\mathfrak{J}}$ in a separable complex Hilbert space $\mathcal{H}$ is called a *discrete frame* for $\mathcal{H}$ if there exist frame constants $0 < \alpha \le \beta < \infty$ such that for every $f \in \mathcal{H}$, we have

$$\alpha\|f\|^2_{\mathcal{H}} \le \sum_{n\in\mathfrak{J}} |\langle f, f_n\rangle|^2 \le \beta\|f\|^2_{\mathcal{H}}.$$

Here onwards, for the sake of simplicity, we will write continuous/discrete frames as just frames by suppressing the term continuous/discrete.

Given the family of functions $\mathbb{F} := \{f_m\}_{m\in M}$, which is Bessel with respect to a measure space $(M, \sum_M, \mu_M)$, define the *synthesis operator* $\Theta_{\mathbb{F}} : L^2(M, \mu_M) \to \mathcal{H}$ defined by

$$\langle \Theta_{\mathbb{F}}\varphi, h\rangle = \int_M \langle f_m, h\rangle \varphi_m d\mu_M(m), \;\; \varphi = \{\varphi_m\}_{m\in M} \in L^2(M, \mu_M), \;\; h \in \mathcal{H}$$

which is a well-defined, linear and bounded operator [97, Theorem 2.6].

Further, the adjoint of the synthesis operator, known as the *analysis operator* of $\mathbb{F}$, is defined by $\Theta_{\mathbb{F}}^* : \mathcal{H} \to L^2(M, \mu_M)$ with

$$(\Theta_{\mathbb{F}}^* h)(m) = \langle h, f_m \rangle, \quad m \in M.$$

Given two Bessel families $\mathbb{F}$ and $\mathbb{G} := \{g_m\}_{m \in M}$ with respect to the measure space $(M, \sum_M, \mu_M)$ for $\mathcal{H}$, define the *mixed dual Gramian operator* corresponding to $\mathbb{F}$ and $\mathbb{G}$ as

$$\Theta_{\mathbb{G}} \Theta_{\mathbb{F}}^* : \mathcal{H} \to \mathcal{H}; \quad h \mapsto \int\limits_M \langle h, f_m \rangle g_m d\mu_M(m). \tag{1.2.5}$$

Gabardo and Han in [54] defined a dual frame for a continuous frame as follows:

Definition 1.2.2. Let $\mathbb{F}$ and $\mathbb{G}$ be two Bessel families with respect to the measure space $(M, \sum_M, \mu_M)$ for $\mathcal{H}$. We call $\mathbb{G}$ a *dual frame* for $\mathbb{F}$ if the following holds true:

$$\langle h_1, h_2 \rangle = \int_M \langle h_1, f_m \rangle \langle g_m, h_2 \rangle d\mu_M(m), \quad \text{for all } h_1, h_2 \in \mathcal{H}. \tag{1.2.6}$$

Here, $\mathbb{F}$ and $\mathbb{G}$ are actually (continuous) frames, and hence $(\mathbb{F}, \mathbb{G})$ is called a *dual frame pair*. If $\Theta_{\mathbb{F}}$ and $\Theta_{\mathbb{G}}$ denote the synthesis operators of $\mathbb{F}$ and $\mathbb{G}$, respectively, then (1.2.6) is equivalent to $\Theta_{\mathbb{G}} \Theta_{\mathbb{F}}^* = I_{\mathcal{H}}$, that is, an identity operator on $\mathcal{H}$. In this case, we say that the relation

$$h = \int_M \langle h, f_m \rangle g_m d\mu_M(m), \quad \text{for all } f \in \mathcal{H},$$

holds in the weak sense. This relation is generally known as a *reproducing formula* for $f \in \mathcal{H}$.

Next, we define the orthogonality of a pair of Bessel families (frames) as follows:

Definition 1.2.3. Let $\mathbb{F}$ and $\mathbb{G}$ be Bessel families (frames) with respect to $(M, \sum_M, \mu_M)$ for $\mathcal{H}$. Then, if the mixed dual Gramian operator of $\mathbb{F}$ and $\mathbb{G}$ (as defined in (1.2.5)) is zero, that is, $\Theta_{\mathbb{G}} \Theta_{\mathbb{F}}^* = 0$, the Bessel families (frames) are said to be *pairwise orthogonal* (simply, *orthogonal*). In other words, we say that $\mathbb{F}$ and $\mathbb{G}$ satisfy the *orthogonality* property. We say $\mathbb{F}$ and $\mathbb{G}$ are *pairwise orthogonal Parseval frames* if both are Parseval frames and pairwise orthogonal.

1.3. Structure of the thesis

In **Chapter 2**, we construct GTI systems using filters and prove that they satisfy the LIC. Furthermore, we establish that such GTI systems possess a Calderón sum equal to 1. As an illustrative case, we construct an example of a pair of systems generated by B-splines in $L^2(\mathbb{R})$. We demonstrate that these systems satisfy the LIC.

In continuation of the work presented in Chapter 2, **Chapter 3** develops a technique to construct pairwise orthogonal Parseval frames with GTI structure. Additionally, we provide methods to construct two or even N pairwise orthogonal Parseval frames. Next, as an application of our obtained results, we provide an example of pairwise orthogonal Parseval frames using B-splines as the generating function. We also prove that the construction of pairwise orthogonal Parseval frames for $L^2(\mathbb{R}^d)$ with wavelet structure is a special case of our construction technique.

As an application of orthogonal frames in **Chapter 4**, we characterize sampling transforms through the action of two co-compact subgroups of a locally compact abelian group G with orthogonal ranges on two bands. To demonstrate this, we establish necessary and sufficient conditions for the existence of pairwise orthogonal translation invariant systems over distinct co-compact subgroups. As a consequence, we derive pairwise orthogonal co-compact Gabor Bessel systems, which are slightly more general due to different choices of co-compact subgroups. Finally, we extend the notion of orthogonal sampling transforms to unions of compact subgroups.

In **Chapter 5**, we give a characterization of pairwise orthogonal frames with GTI structures. These GTI systems are generated by translating generating functions through a countable family of distinct closed, co-compact subgroups of G. Importantly, the families of subgroups associated with each system may differ from one another. As an application of this characterization, we derive necessary and sufficient conditions for the orthogonality of various structured systems, including Gabor, wavelet, and shearlet systems over LCA groups. Moreover, we also establish a characterization of tight GTI frames.

As an application of the results obtained in Chapter 5, in **Chapter 6** we present explicit constructions of pairs of GTI systems using filter-based methods. Each constructed system is shown to satisfy the ∞-UCP and Calderón sum equal to one. We further establish that these systems form Parseval frames and that the constructed pair of systems is pairwise orthogonal. This framework extends the results of the Chapter 3 by relaxing earlier structural constraints, thereby broadening the class of admissible GTI systems.

In **Chapter 7**, we study the frame theory on finite dimensional Hilbert spaces. In particular we study dynamical frames and cyclic frames. Cyclic frames form a subclass of the dynamical frames introduced and analyzed in detail by Aldroubi et al. in [3] and subsequent papers; they are particularly interesting due to their attractive properties in the context of erasure problems. By applying an alternative approach, we are able to shed new light on general dynamical frames as well as cyclic frames. In particular, we provide a characterization of dynamical frames, which in turn leads to a characterization of cyclic frames.

CHAPTER 2

CONSTRUCTION OF GENERALIZED TRANSLATION INVARIANT SYSTEMS

The purpose of this chapter is to construct a pair of generalized translation invariant (GTI) systems in the separable Hilbert space $L^2(G)$, where G is an LCA group. The construction is based on a nested sequence $\{\Gamma_j\}_{j\in\mathcal{J}}$ of closed co-compact subgroups of G, a countable sequence $\{\Phi_j\}_{j\in\mathcal{J}}$ of generating functions in $L^2(\widehat{G})$, and $\Gamma_j^{\perp}$-periodic filters, where $\widehat{G}$ is a dual group of G and $\Gamma_j^{\perp}$ is the annihilator of Γ_j. For studying the frame properties of GTI systems, such as duality and orthogonality, the local integrability condition (LIC) plays an important role [60, 73, 79, 86, 88]. Sufficient conditions on the generating functions and filters are obtained for the GTI systems satisfying the LIC. In order to establish the LIC, the boundedness of the Calderón sum becomes crucial as it connects to the frame's bounds of the GTI systems. We find that Calderón sum is equal to one for the constructed GTI systems. At the end, we construct an explicit example in $L^2(\mathbb{R})$ generated by B-splines.

2.1. Generalized translation invariant systems

In this section, our goal is to introduce the GTI systems and the LIC. In the context of LCA groups, GTI systems were introduced by Jakobsen and Lemvig in [79]. The GTI framework serves as a bridge between the well-established discrete frame theory of GSI systems and its continuous counterpart. Consequently, GTI systems provide a unified approach for deriving analogous results across several classes of function systems, including the GSI systems studied by Kutyniok and Labate [88] and the TI systems considered by Bownik and Ross [27].

The results of **Chapter 2** are taken from the published article:

Redhu N., Gumber A., Shukla N. K. (2025), *Constructions of pairwise orthogonal Parseval frames generated by filter on LCA groups*, **Applied and computational Harmonic analysis**, **74**, paper No.101708, 27pp, DOI: 10.1016/j.acha.2024.101708.

The function systems defined in this section form the central objects of this thesis. Here, the translation of a function $f \in L^2(G)$ by $x \in G$ is denoted by $T_x f := f(\cdot - x)$.

Definition 2.1.1. Let $\mathcal{J}$ be a countable index set. A *generalized translation invariant (GTI) system* is given by

$$\bigcup_{j \in \mathcal{J}} \{T_\gamma g_p : \gamma \in \Gamma_j,\, p \in P_j\}. \tag{2.1.1}$$

Here, each P_j is a (countable or uncountable) index set, each Γ_j is a closed co-compact subgroup of G, and $\{g_p\}_{p \in P_j} \subset L^2(G)$.

If $\Gamma_j = \Gamma$ for each j, then above GTI system is called *translation invariant* (TI) system. If P_j is countable and Γ_j is a uniform lattice, then $\bigcup_{j \in \mathcal{J}} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$ is called *generalised shift invariant* (GSI) system and if $\Gamma_j = \Gamma$ for each j, then it is called *shift invariant* (SI) system. GTI systems provide a unified framework for analyzing a broad class of function systems, encompassing both discrete and continuous settings such as wavelet, Gabor, wave-packet, and shearlet systems [60, 79].

As in [79], the GTI system introduced in Definition 2.1.1 is assumed to satisfy the following hypotheses.

Standing hypotheses: Before stating the hypotheses, we introduce some notation. For each $j \in \mathcal{J}$, where $\mathcal{J} \subset \mathbb{Z}$ is a countable index set, let $(P_j, \sum_{P_j}, \mu_{P_j})$ denote a measure space. For any topological space X, we denote by $\mathcal{B}_X$ its Borel σ-algebra. The notation $P_j \times G$ denotes the product measurable space obtained by taking the Cartesian product of G with the measure space P_j. The corresponding product σ-algebra is written as $\sum_{P_j} \otimes \mathcal{B}_G$, and the product measure by $\mu_{P_j} \otimes \mu_G$.

We impose the following assumptions for each $j \in \mathcal{J}$:

(I) $(P_j, \sum_{P_j}, \mu_{P_j})$ is a σ-finite measure space;

(II) the mapping $p \mapsto g_{j,p}$ from $(P_j, \sum_{P_j})$ to $(L^2(G), \mathcal{B}_{L^2(G)})$ is measurable;

(III) the mapping $(p, x) \mapsto g_{j,p}(x)$ from $(P_j \times G, \sum_{P_j} \otimes \mathcal{B}_G)$ to $(\mathbb{C}, \mathcal{B}_{\mathbb{C}})$ is measurable.

For several instances in this thesis, we consider countable index sets P_j equipped with the counting measure. With this consideration, the three standing hypotheses (I)–(III) hold automatically (see, [79]).

A GTI system $\bigcup_{j \in \mathcal{J}} \{T_\lambda g_p\}_{\lambda \in \Gamma_j, p \in P_j}$ is called a *GTI frame* for $L^2(G)$ with respect to

$\{L^2(P_j \times \Gamma_j) : j \in \mathcal{J}\}$ if there exist two constants $0 < A, B < \infty$ such that

$$A\,\|f\|^2 \leq \sum_{j\in\mathcal{J}} \int_{P_j} \int_{\Gamma_j} |\langle f, T_\lambda g_p\rangle|^2 \, d\mu_{\Gamma_j}(\lambda)\, d\mu_{P_j}(p) \leq B\,\|f\|^2 \quad \text{for all } f \in L^2(G). \quad (2.1.2)$$

The constants A and B are called the *frame bounds*. If $A = B$, the GTI frame is called a *GTI tight frame*. In particular, when $A = B = 1$, it is called a *GTI Parseval frame*. If only the right-hand side inequality in (2.1.2) holds, then the GTI system defined in (2.1.1) is called a *GTI Bessel system*.

For two GTI Bessel systems $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j}$, we define the *mixed dual Gramian operator* Θ corresponding to these systems as

$$\Theta : L^2(G) \to L^2(G);\ f \mapsto \sum_{j\in\mathcal{J}} \int_{P_j} \int_{\Gamma_j} \langle f, T_\lambda g_p^{(1)}\rangle\, T_\lambda g_p^{(2)} \, d\mu_{\Gamma_j}(\lambda)\, d\mu_{P_j}(p). \quad (2.1.3)$$

Definition 2.1.2. Let $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j}$ be two GTI Bessel (or frame) systems. We say that these systems are *pairwise orthogonal Bessel (frame) systems* if the associated mixed dual Gramian operator satisfies $\Theta = 0$ for all $f \in L^2(G)$. Furthermore, if the GTI systems are Parseval and pairwise orthogonal, then they are called *pairwise orthogonal Parseval frames*.

2.1.1. Local integrability condition

In this subsection, we recall the LIC and its variant, the dual α-LIC, introduced by Jakobsen and Lemvig for GTI systems in the context of LCA groups [79]. Roughly speaking, the LIC imposes certain integrability requirements on the Fourier transforms of the generating functions. Variants of this condition, such as the α-LIC and the dual α-LIC, have been studied in [79, 60] in the context of LCA groups.

In order to formulate these conditions rigorously, we first introduce a dense subset of $L^2(G)$:

$$\mathcal{D}_{\mathcal{B}} := \{f \in L^2(G) : \widehat{f} \in L^\infty(\widehat{G}) \text{ and } \operatorname{supp} \widehat{f} \text{ is compact in } \widehat{G} \smallsetminus \mathcal{B}\},$$

where $\mathcal{B}$ is a Borel set in $\widehat{G}$ with $\mu_{\widehat{G}}(\mathcal{B}) = 0$. The two GTI systems $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j}$ in $L^2(G)$ satisfy the *dual α-local integrability condition* (*dual α-LIC*) if for all $f \in \mathcal{D}_{\mathcal{B}}$,

$$\sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \int_{P_j} \sum_{\alpha\in\Gamma_j^\perp} \int_{\widehat{G}} \left| \widehat{f}(\gamma) \overline{\widehat{f}(\gamma+\alpha)}\, \overline{\widehat{g_p^{(1)}}(\gamma)}\, \widehat{g_p^{(2)}}(\gamma+\alpha) \right| d\mu_{\widehat{G}}(\gamma)\, d\mu_{P_j}(p) < \infty. \quad (2.1.4)$$

If $g_p^{(1)} = g_p^{(2)}$ for each p, then condition (2.1.4) is known as the *α-local integrability condition* (*α-LIC*) for the system $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$.

The system $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j, p\in P_j}$ satisfies the *local integrability condition* (LIC) if

$$\sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \int_{P_j} \sum_{\alpha\in\Gamma_j^\perp} \int_{\operatorname{supp}\widehat{f}} \left|\widehat{f}(\gamma+\alpha)\widehat{g_p^{(i)}}(\gamma)\right|^2 d\mu_{\widehat{G}}(\gamma)\, d\mu_{P_j}(p) < \infty \quad \text{for all } f \in \mathcal{D}_{\mathcal{B}}. \tag{2.1.5}$$

For a more detailed explanation of these conditions, see [73, 79, 86, 113]. According to Lemma 3.9 in [79], if both systems $\cup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ and $\cup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j}$ satisfy the LIC, then they also satisfy the dual α-LIC. Specifically, if a GTI system $\cup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j, p\in P_j}$ satisfies the LIC, it automatically satisfies the α-LIC.

The following results show that the LIC plays a central role in characterizing GTI systems that form Parseval frames and in characterizing pairwise orthogonal Parseval frames. Jakobsen and Lemvig [79] provided the following characterization result for a GTI system to be a Parseval frame with the help of the α-LIC:

Theorem 2.1.3. *Suppose that the GTI system $\cup_{j\in\mathcal{J}}\{T_\lambda g_p\}_{\lambda\in\Gamma_j, p\in P_j}$ satisfies the α-LIC. Then the following assertions are equivalent:*

(i) $\cup_{j\in\mathcal{J}}\{T_\lambda g_p\}_{\lambda\in\Gamma_j, p\in P_j}$ is a Parseval frame for $L^2(G)$.
(ii) For each $\alpha \in \cup_{j\in\mathcal{J}}\Gamma_j^\perp$, we have

$$t_\alpha := \sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp} \frac{1}{s(\Gamma_j)} \int_{P_j} \overline{\widehat{g_p}(\gamma)}\widehat{g_p}(\gamma+\alpha) d\mu_{P_j}(p) = \delta_{\alpha,0} \text{ for a.e. } \gamma \in \widehat{G}. \tag{2.1.6}$$

Gumber and Shukla [60] provided the following characterization result for the pairwise orthogonal GTI Bessel (frame) systems with the help of dual α-LIC:

Theorem 2.1.4. *Let $\cup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ and $\cup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j}$ be two GTI Bessel (frame) systems for $L^2(G)$ satisfying the dual α-LIC. Then, the following assertions are equivalent:*

(i) Both the above GTI Bessel (frame) systems in $L^2(G)$ are pairwise orthogonal.
(ii) For each $\alpha \in \cup_{j\in\mathcal{J}}\Gamma_j^\perp$, we have

$$\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp} \frac{1}{s(\Gamma_j)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(\gamma)}\widehat{g_p^{(2)}}(\gamma+\alpha) d\mu_{P_j}(p) = 0 \text{ for a.e. } \gamma \in \widehat{G}. \tag{2.1.7}$$

In view of Theorem 2.1.4, the dual α-LIC is the essential criterion for establishing both Parseval frames and pairwise orthogonality. Therefore, the remainder of this chapter is devoted to constructing GTI systems and deriving sufficient conditions for them to satisfy the dual α-LIC.

2.2. Construction of GTI systems via filters

The purpose of this section is to construct a pair of GTI systems with the help of filters. The filters are always a key attraction of researchers to construct frames [26, 28, 32, 38, 41, 43, 63, 64, 66, 68, 106]. Frames based on filters play a crucial role in signal processing, image processing, and data analysis. For example, filters can be employed to extract discriminative features from data, enhancing pattern recognition tasks like fingerprint recognition or face detection (see, for example, [31, 100] and references therein). Other applications of filters are that they are often used to construct multiresolution analysis (MRA) system, which allow the representation of signals or data at different levels of detail. MRA is fundamental in signal compression, denoising, and feature extraction tasks. Several researchers have proposed extension principles to construct Parseval frames using MRA associated with filters, such as the unitary extension principle (UEP) [34, 36, 102] and oblique extension principle (OEP) [44].

Here, the general framework of [36] is adopted, along with additional assumptions required for the subsequent analysis in this chapter. *Throughout this chapter and in Chapter 5, the following objects are fixed.*

Let G be an LCA group, and let $\{\Gamma_j\}_{j\in\mathcal{J}}$ denote a countable, nested sequence of closed co-compact subgroups of G, that is,

$$\cdots \subset \Gamma_j \subset \Gamma_{j+1} \subset \Gamma_{j+2} \subset \cdots \tag{2.2.1}$$

such that each quotient group Γ_{j+1}/Γ_j is finite with cardinality $d_j \in \mathbb{N}$. The annihilators $\{\Gamma_j^\perp\}_{j\in\mathcal{J}}$ satisfy $\cdots \supset \Gamma_j^\perp \supset \Gamma_{j+1}^\perp \supset \Gamma_{j+2}^\perp \supset \cdots$, and $|\Gamma_{j+1}/\Gamma_j| = |\Gamma_j^\perp/\Gamma_{j+1}^\perp| = d_j$ (see, e.g., Section 4.2 in [72]). Consequently, for each $j \in \mathcal{J}$, there is a sequence $\{v_{j,\ell}\}_{\ell=1,\dots,d_j}$ in $\Gamma_j^\perp$ such that $v_{j,1} = 0$ and

$$\Gamma_j^\perp = \bigcup_{\ell=1}^{d_j} (v_{j,\ell} + \Gamma_{j+1}^\perp), \;\; (v_{j,\ell} + \Gamma_{j+1}^\perp) \cap (v_{j,\ell'} + \Gamma_{j+1}^\perp) = \varnothing \;\text{ for } \ell \neq \ell'. \tag{2.2.2}$$

The fundamental domain associated with the lattice $\Gamma_j^\perp$ is denoted by Ω_j.

Assume that $\{\Phi_j\}_{j\in\mathcal{J}}$ is a sequence of functions in $L^2(\widehat{G})$, having the property that there exist $\Gamma_{j+1}^{\perp}$-periodic functions $H_{j+1} \in L^\infty(\Omega_{j+1})$ such that

$$\Phi_j(\gamma) = H_{j+1}(\gamma)\Phi_{j+1}(\gamma), \text{ a.e. } \gamma \in \widehat{G}, \tag{2.2.3}$$

for all $j \in \mathcal{J}$. Further for each $i \in \{1,2\}$, we define the functions $\Psi_j^{(i)(m)} \in L^2(\widehat{G})$, by

$$\Psi_j^{(i)(m)}(\gamma) := G_{j+1}^{(i)(m)}(\gamma)\Phi_{j+1}(\gamma),\ \gamma \in \widehat{G} \text{ and } m = 1, 2, \ldots, s_j, \tag{2.2.4}$$

for some $\Gamma_j^{\perp}$-periodic functions $G_j^{(i)(m)} \in L^\infty(\Omega_j)$.

For $j \in \mathcal{J}, m \in \{1, 2, \ldots, s_j\}$ and $i \in \{1,2\}$, we define the functions $g_{(m,j)}^{(i)}$ as inverse Fourier transform of $\Psi_j^{(i)(m)}$ as follows:

$$g_{(m,j)}^{(i)} = \mathcal{F}^{-1}\Psi_j^{(i)(m)}. \tag{2.2.5}$$

The main goal of this chapter is to show that the GTI systems

$$\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j} \text{ and } \bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j} \tag{2.2.6}$$

satisfy the LIC, where $P_j = \{(m,j) : m = 1, 2, \ldots, s_j\}$. Moreover, Chapter 4 establishes that these systems form pairwise orthogonal Parseval frames for $L^2(G)$.

In view of Theorem 2.1.4, our primary objective is to find conditions on *filters* $G_j^{(i)(m)}$ and generating functions Φ_j such that $\cup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ and $\cup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j}$ satisfy the dual α-LIC. Next, define, for $j \in \mathcal{J}$, and $i \in \{1,2\}$, the matrix-valued functions $\mathfrak{B}_j^{(i)}(\gamma)$ and $\widetilde{\mathfrak{B}_j^{(i)}}(\gamma)$ with the help of filters

$$\mathfrak{B}_j^{(i)}(\gamma) := \left(G_{j+1}^{(i)(m)}(\gamma + v_{j,n})\right)_{\substack{0\le m\le s_j \\ 1\le n\le d_j}} \quad \text{for } \ a.e.\ \gamma \in \Omega_j, \tag{2.2.7}$$

and

$$\widetilde{\mathfrak{B}_j^{(i)}}(\gamma) := \left(G_{j+1}^{(i)(m)}(\gamma + v_{j,n})\right)_{\substack{1\le m\le s_j \\ 1\le n\le d_j}} \quad \text{for } \ a.e.\ \gamma \in \Omega_j, \tag{2.2.8}$$

where $G_{j+1}^{(i)(0)} := H_{j+1}$.

The possible index set $\mathcal{J}$ is allowed to be any interval contained in the integers, i.e. either $\mathcal{J} = \mathbb{Z}$ or $\{j\}_{j=j_0}^{\infty}$ or $\{j\}_{j=j_0}^{j_1}$ or $\{j\}_{j=-\infty}^{j_1}$, for $j_0 < j_1$ and $j_0, j_1 \in \mathbb{Z}$. The following calculation is only for $\mathcal{J} = \mathbb{Z}$, the remaining cases can be handled by minor modifications. In the case $\mathcal{J} = \{j\}_{j=j_0}^{j_1}$, it is necessary to assume that j takes values from the set $\{j_0, j_0+1, \cdots, j_1-1\}$, ensuring that (2.2.3), (2.2.4), (2.2.7) and (2.2.8) are defined. Similarly, for the case $\mathcal{J} = \{j\}_{j=-\infty}^{j_1}$, j takes values from the set $\{-\infty, \cdots, j_1 - 1\}$.

2.3. Conditions for the GTI systems satisfying the dual α-LIC

The goal of this section is to establish the conditions under which the GTI systems defined in (2.2.6) satisfy the dual α-LIC. In particular, we prove in Lemma 2.3.6 that such systems have the property that the Calderón sum equal to one. This result serves as an intermediate step toward establishing Theorem 2.3.3, which presents the main result of this section. Theorem 2.3.3 provides sufficient criteria for the GTI systems to satisfy the dual α-LIC. To this end, we now state the following standing assumptions:

($\mathcal{N}_1$) For every compact set S in $\widehat{G} \setminus \mathcal{B}$ and $\epsilon > 0$, there exists $J_1 \in \mathcal{J}$ such that for all $j \geq J_1,\ j \in \mathcal{J}$,

$$\left| \frac{1}{s(\Gamma_j)} |\Phi_j(\gamma)|^2 - 1 \right| \leq \epsilon, \ \ \forall\, \gamma \in S.$$

($\mathcal{N}_2$) For every compact set S in $\widehat{G} \setminus \mathcal{B}$ and $\epsilon > 0$, there exists $J_2 \in \mathcal{J}$ such that for all $j \leq J_2,\ j \in \mathcal{J}$,

$$\frac{1}{\sqrt{s(\Gamma_j)}} |\Phi_j(\gamma)| \leq \epsilon, \ \ \forall\, \gamma \in S.$$

($\mathcal{N}_3$) For every compact set S in $\widehat{G} \setminus \mathcal{B}$, there exists a constant $J_3 > 0$ such that

$$\sum_{\alpha \in \cup_{j \in \mathcal{J}} \Gamma_j^{\perp}} \mu_{\widehat{G}}(S \cap (S - \alpha)) \leq J_3.$$

Remark 2.3.1. In particular, if we consider $\mathcal{J} = \{j_0, j_0+1, j_0+2, \cdots\}$, then assumptions ($\mathcal{N}_3$) holds trivially. This follows by noting the fact that $\Gamma_{j_0}^{\perp} \supset \Gamma_{j_0+1}^{\perp} \supset \cdots$ implies $\alpha \in \cup_{j \in \mathcal{J}} \Gamma_j^{\perp} = \Gamma_{j_0}^{\perp}$. Now to prove ($\mathcal{N}_3$) it is sufficient to show that $\sum\limits_{\alpha \in \Gamma_{j_0}^{\perp}} \mu_{\widehat{G}}(S \cap (S - \alpha))$ is finite. It is easy to see that $\{\alpha \in \Gamma_{j_0}^{\perp} \ : \ S \cap (S - \alpha) \neq \varnothing\} \subseteq \Gamma_{j_0}^{\perp} \cap (-S + S)$. But $\Gamma_{j_0}^{\perp} \cap (-S + S)$ is a finite set. Hence $\{\alpha \in \Gamma_{j_0}^{\perp} \ : \ S \cap (S - \alpha) \neq \varnothing\}$ is also a finite set and therefore there exists some constant $J_3 > 0$ such that $\sum_{\alpha \in \Gamma_{j_0}^{\perp}} \mu_{\widehat{G}}(S \cap (S - \alpha)) \ \leq J_3$.

Remark 2.3.2. In particular, if G is a compact abelian group, then the assumption ($\mathcal{N}_3$) holds trivially. Since the dual group $\widehat{G}$ of a compact abelian group is discrete, the compact set S is finite. Therefore $\{\alpha \in \widehat{G} \ : \ S \cap (S - \alpha) \neq \varnothing\}$ is finite. Since $\bigcup\limits_{j \in \mathcal{J}} \Gamma_j^{\perp} \subset \widehat{G}$, it follows that the set $\left\{\alpha \in \bigcup\limits_{j \in \mathcal{J}} \Gamma_j^{\perp} \ : \ S \cap (S - \alpha) \neq \varnothing \right\}$ is finite. Hence the assumption ($\mathcal{N}_3$) holds trivially.

We are ready to state our result, which outlines sufficient conditions under which the GTI systems defined in (2.2.6) satisfy the LIC.

Theorem 2.3.3. *In addition to the assumptions ($\mathcal{N}_1$) to ($\mathcal{N}_3$), assume that for each $i \in \{1,2\}$ and each $j \in \mathcal{J}$, the matrix-valued function $\mathfrak{B}_j^{(i)}(\gamma)$ defined in (2.2.7) satisfies the following condition:*

$$(\mathfrak{B}_j^{(i)}(\gamma))^* \mathfrak{B}_j^{(i)}(\gamma) = \frac{s(\Gamma_j)}{s(\Gamma_{j+1})} I_{d_j} \text{ for a.e. } \gamma \in \Omega_j, \tag{2.3.1}$$

where $(\mathfrak{B}_j^{(i)}(\gamma))^*$ denotes adjoint of $\mathfrak{B}_j^{(i)}(\gamma)$. Then $\cup_{j \in \mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda \in \Gamma_j, p \in P_j}$ and $\cup_{j \in \mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda \in \Gamma_j, p \in P_j}$ (defined in (2.2.6)) satisfy the LIC. Hence they satisfy dual α-LIC.

To prove Theorem 2.3.3, we require the following results, which will be used extensively in this chapter and the next. The first part of the proposition presented below assists in constructing pairwise orthogonal frames, while the second part contributes to constructing a Parseval frame.

Proposition 2.3.4. *The following statements are true:*

(i) $(\widetilde{\mathfrak{B}_j^{(1)}}(\gamma))^* \widetilde{\mathfrak{B}_j^{(2)}}(\gamma) = O_{d_j}$ *for a.e.* $\gamma \in \Omega_j$, *if and only if for a.e.* $w \in \widehat{G}$, *we have*

$$\sum_{m=1}^{s_j} \overline{G_{j+1}^{(1)(m)}(w + v_{j,\ell})} G_{j+1}^{(2)(m)}(w + v_{j,\ell'}) = 0 \text{ for all } 1 \leq \ell, \ell' \leq d_j.$$

(ii) Let $i \in \{1,2\}$. Then $(\mathfrak{B}_j^{(i)}(\gamma))^* \mathfrak{B}_j^{(i)}(\gamma) = \frac{s(\Gamma_j)}{s(\Gamma_{j+1})} I_{d_j}$ for a.e. $\gamma \in \Omega_j$, if and only if for a.e. $w \in \widehat{G}$, we have

$$\sum_{m=0}^{s_j} \overline{G_{j+1}^{(i)(m)}(w + v_{j,\ell})} G_{j+1}^{(i)(m)}(w + v_{j,\ell'}) = \frac{s(\Gamma_j)}{s(\Gamma_{j+1})} \delta_{\ell,\ell'} \text{ for all } 1 \leq \ell, \ell' \leq d_j.$$

Here, O_{d_j} and I_{d_j} are the zero and identity matrices of order d_j, respectively.

Proof. (i) First suppose that

$$(\widetilde{\mathfrak{B}_j^{(1)}}(\gamma))^* \widetilde{\mathfrak{B}_j^{(2)}}(\gamma) = O_{d_j} \text{ for a.e. } \gamma \in \Omega_j,$$

which is equivalent to

$$\left(\sum_{m=1}^{s_j} \overline{G_{j+1}^{(1)(m)}(\gamma + v_{j,\ell})} G_{j+1}^{(2)(m)}(\gamma + v_{j,\ell'}) \right)_{\substack{1 \leq \ell \leq d_j \\ 1 \leq \ell' \leq d_j}} = O_{d_j} \text{ for a.e. } \gamma \in \Omega_j, \tag{2.3.2}$$

by using (2.2.8). Now (2.3.2) holds if and only if

$$\sum_{m=1}^{s_j} \overline{G_{j+1}^{(1)(m)}(\gamma + v_{j,\ell})} G_{j+1}^{(2)(m)}(\gamma + v_{j,\ell'}) = 0 \text{ for } 1 \leq \ell, \ell' \leq d_j \text{ and a.e. } \gamma \in \Omega_j. \tag{2.3.3}$$

It is easy to observe that the set

$$\Omega'_{j+1} = \bigcup_{\ell=1}^{d_j} (v_{j,\ell} + \Omega_j) \text{ for } j \in \mathcal{J} \tag{2.3.4}$$

is a fundamental domain associated with the annihilator $\Gamma^{\perp}_{j+1}$. For $\gamma' \in \Omega'_{j+1}$, there exist some $\gamma \in \Omega_j$ and $\tilde{\ell} \in \{1,2,\ldots,d_j\}$ such that $\gamma' = \gamma + v_{j,\tilde{\ell}}$. Next, by observing the fact that $v_{j,\tilde{\ell}} + v_{j,\ell}, v_{j,\tilde{\ell}} + v_{j,\ell'} \in \Gamma^{\perp}_j = \bigcup_{q=1}^{d_j} (v_{j,q} + \Gamma^{\perp}_{j+1})$, there exist $q, q' \in \{1,2,\ldots,d_j\}$ and $w, w' \in \Gamma^{\perp}_{j+1}$ such that $v_{j,\tilde{\ell}} + v_{j,\ell} = v_{j,q} + w$ and $v_{j,\tilde{\ell}} + v_{j,\ell'} = v_{j,q'} + w'$. Then for $\gamma' \in \Omega'_{j+1}$, we have

$$\begin{aligned}
\sum_{m=1}^{s_j} \overline{G^{(1)(m)}_{j+1}(\gamma' + v_{j,\ell})} G^{(2)(m)}_{j+1}(\gamma' + v_{j,\ell'}) &= \sum_{m=1}^{s_j} \overline{G^{(1)(m)}_{j+1}(\gamma + v_{j,\tilde{\ell}} + v_{j,\ell})} G^{(2)(m)}_{j+1}(\gamma + v_{j,\tilde{\ell}} + v_{j,\ell'}) \\
&= \sum_{m=1}^{s_j} \overline{G^{(1)(m)}_{j+1}(\gamma + v_{j,q} + w)} G^{(2)(m)}_{j+1}(\gamma + v_{j,q'} + w') \\
&= \sum_{m=1}^{s_j} \overline{G^{(1)(m)}_{j+1}(\gamma + v_{j,q})} G^{(2)(m)}_{j+1}(\gamma + v_{j,q'}) \\
&= 0,
\end{aligned} \tag{2.3.5}$$

where in the last step we used (2.3.3) and in the second to the last step the $\Gamma^{\perp}_{j+1}$-periodicity of the filters $G^{(i)(m)}_{j+1} \in L^{\infty}(\Omega_{j+1})$ is used. Therefore (2.3.3) is true for a.e. $\gamma \in \Omega'_{j+1}$. Again using periodicity of the filters $G^{(i)(m)}_{j+1} \in L^{\infty}(\Omega_{j+1})$ and (2.3.5), it follows that (2.3.3) holds true for a.e. $w \in \widehat{G}$. The converse part follows immediately. Similarly we can prove (ii). ∎

Lemma 2.3.5. *For each $i \in \{1,2\}$, consider the GTI system $\cup_{j\in\mathcal{J}}\{T_\lambda g^{(i)}_p\}_{\lambda\in\Gamma_j, p\in P_j}$ defined in (2.2.6) and suppose the matrix-valued functions $\mathfrak{B}^{(i)}_j(\gamma)$ given by (2.2.7) satisfy the following condition for each $j_0 \le j \le J$:*

$$(\mathfrak{B}^{(i)}_j(\gamma))^* \mathfrak{B}^{(i)}_j(\gamma) = \frac{s(\Gamma_j)}{s(\Gamma_{j+1})} I_{d_j} \quad \text{for a.e. } \gamma \in \Omega_j,$$

for some $j_0, J \in \mathbb{Z}$. Then for all $\alpha \in \Gamma^{\perp}_J$, we have

$$\begin{aligned}
\sum_{j=j_0}^{J} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \overline{(\widehat{g^{(i)}_p})(w)} (\widehat{g^{(i)}_p})(w+\alpha) &= -\frac{1}{s(\Gamma_{j_0})} \overline{\Phi_{j_0}(w)} \Phi_{j_0}(w+\alpha) \\
&+ \overline{\Phi_{J+1}(w)} \Phi_{J+1}(w+\alpha) \Big(\frac{1}{s(\Gamma_J)} \sum_{m=0}^{s_J} \overline{G^{(i)(m)}_{J+1}(w)} G^{(i)(m)}_{J+1}(w+\alpha) \Big).
\end{aligned} \tag{2.3.6}$$

Proof. By substituting $P_j = \{(m,j) : m = 1,2,\ldots,s_j\}$ in the left-hand side of (2.3.6), we have

$$\mathcal{E} := \sum_{j=j_0}^{J} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \overline{(\widehat{g^{(i)}_p})(w)} (\widehat{g^{(i)}_p})(w+\alpha) = \sum_{j=j_0}^{J} \frac{1}{s(\Gamma_j)} \sum_{m=1}^{s_j} \overline{(\widehat{g^{(i)}_{(m,j)}})(w)} (\widehat{g^{(i)}_{(m,j)}})(w+\alpha).$$

Using the definition of $g^{(i)}_{(m,j)}$ defined in (2.2.5), we get

$$\mathcal{E} = \sum_{j=j_0}^{J} \frac{1}{s(\Gamma_j)} \sum_{m=1}^{s_j} \overline{(\mathcal{F}^{-1}\widehat{\Psi_j^{(i)(m)}})(w)}(\mathcal{F}^{-1}\widehat{\Psi_j^{(i)(m)}})(w+\alpha)$$

$$= \sum_{j=j_0}^{J} \frac{1}{s(\Gamma_j)} \sum_{m=1}^{s_j} \overline{\Psi_j^{(i)(m)}(w)}\Psi_j^{(i)(m)}(w+\alpha).$$

Adding to both sides the quantity $\frac{1}{s(\Gamma_{j_0})}\overline{\Phi_{j_0}(w)}\Phi_{j_0}(w+\alpha)$, and substituting the value of Φ_j and $\Psi_j^{(i)(m)}$ from (2.2.3) and (2.2.4), respectively, then

$$\begin{aligned}
\mathcal{E} + \frac{1}{s(\Gamma_{j_0})}\overline{\Phi_{j_0}(w)}\Phi_{j_0}(w+\alpha) =& \frac{1}{s(\Gamma_{j_0})}\overline{H_{j_0+1}(w)\Phi_{j_0+1}(w)}H_{j_0+1}(w+\alpha)\Phi_{j_0+1}(w+\alpha)\\
&+ \sum_{j=j_0}^{J} \frac{1}{s(\Gamma_j)} \sum_{m=1}^{s_j} \overline{G_{j+1}^{(i)(m)}(w)\Phi_{j+1}(w)}G_{j+1}^{(i)(m)}(w+\alpha)\Phi_{j+1}(w+\alpha)\\
=& \frac{1}{s(\Gamma_{j_0})}\overline{\Phi_{j_0+1}(w)}\Phi_{j_0+1}(w+\alpha)\sum_{m=0}^{s_{j_0}} \overline{G_{j_0+1}^{(i)(m)}(w)}G_{j_0+1}^{(i)(m)}(w+\alpha)\\
&+ \sum_{j=j_0+1}^{J} \frac{1}{s(\Gamma_j)}\overline{\Phi_{j+1}(w)}\Phi_{j+1}(w+\alpha) \sum_{m=1}^{s_j} \overline{G_{j+1}^{(i)(m)}(w)}G_{j+1}^{(i)(m)}(w+\alpha).
\end{aligned}$$

For $\alpha \in \Gamma_J^{\perp}$ and $0 \le m \le s_j$, we have $G_{j+1}^{(i)(m)}(w+\alpha) = G_{j+1}^{(i)(m)}(w)$ for $j \le J-1$, since $G_{j+1}^{(i)(m)}$ are $\Gamma_{j+1}^{\perp}$-periodic and $\Gamma_J^{\perp} \subseteq \Gamma_{j+1}^{\perp}$ for each $j \le J-1$. Therefore, we get

$$\begin{aligned}
\mathcal{E} + \frac{1}{s(\Gamma_{j_0})}\overline{\Phi_{j_0}(w)}\Phi_{j_0}(w+\alpha) =& \frac{1}{s(\Gamma_{j_0})}\overline{\Phi_{j_0+1}(w)}\Phi_{j_0+1}(w+\alpha)\sum_{m=0}^{s_{j_0}} \overline{G_{j_0+1}^{(i)(m)}(w)}G_{j_0+1}^{(i)(m)}(w)\\
&+ \sum_{j=j_0+1}^{J-1} \frac{1}{s(\Gamma_j)}\overline{\Phi_{j+1}(w)}\Phi_{j+1}(w+\alpha) \sum_{m=1}^{s_j} \overline{G_{j+1}^{(i)(m)}(w)}G_{j+1}^{(i)(m)}(w)\\
&+ \frac{1}{s(\Gamma_J)}\overline{\Phi_{J+1}(w)}\Phi_{J+1}(w+\alpha) \sum_{m=1}^{s_J} \overline{G_{J+1}^{(i)(m)}(w)}G_{J+1}^{(i)(m)}(w+\alpha).
\end{aligned}$$

Employing Proposition 2.3.4 (ii), we obtain

$$\begin{aligned}
\mathcal{E} + \frac{1}{s(\Gamma_{j_0})}\overline{\Phi_{j_0}(w)}\Phi_{j_0}(w+\alpha) =& \frac{1}{s(\Gamma_{j_0+1})}\overline{\Phi_{j_0+1}(w)}\Phi_{j_0+1}(w+\alpha)\\
&+ \sum_{j=j_0+1}^{J-1} \frac{1}{s(\Gamma_j)}\overline{\Phi_{j+1}(w)}\Phi_{j+1}(w+\alpha) \sum_{m=1}^{s_j} \overline{G_{j+1}^{(i)(m)}(w)}G_{j+1}^{(i)(m)}(w)\\
&+ \frac{1}{s(\Gamma_J)}\overline{\Phi_{J+1}(w)}\Phi_{J+1}(w+\alpha) \sum_{m=1}^{s_J} \overline{G_{J+1}^{(i)(m)}(w)}G_{J+1}^{(i)(m)}(w+\alpha).
\end{aligned}$$

Repeating above step $(J-1-j_0)$ times, we get

$$\begin{aligned}
\mathcal{E} + \frac{1}{s(\Gamma_{j_0})}\overline{\Phi_{j_0}(w)}\Phi_{j_0}(w+\alpha) =& \frac{1}{s(\Gamma_J)}\overline{\Phi_J(w)}\Phi_J(w+\alpha)\\
&+ \frac{1}{s(\Gamma_J)}\overline{\Phi_{J+1}(w)}\Phi_{J+1}(w+\alpha) \sum_{m=1}^{s_J} \overline{G_{J+1}^{(i)(m)}(w)}G_{J+1}^{(i)(m)}(w+\alpha)\\
=& \frac{1}{s(\Gamma_J)}\overline{\Phi_{J+1}(w)H_{J+1}(w)}\Phi_{J+1}(w+\alpha)H_{J+1}(w+\alpha)
\end{aligned}$$

$$+\overline{\Phi_{J+1}(w)}\Phi_{J+1}(w+\alpha)\frac{1}{s(\Gamma_J)}\sum_{m=1}^{s_J}\overline{G_{J+1}^{(i)(m)}(w)}G_{J+1}^{(i)(m)}(w+\alpha)$$

$$=\overline{\Phi_{J+1}(w)}\Phi_{J+1}(w+\alpha)\frac{1}{s(\Gamma_J)}\sum_{m=0}^{s_J}\overline{G_{J+1}^{(i)(m)}(w)}G_{J+1}^{(i)(m)}(w+\alpha),$$

which completes the proof. ∎

The following lemma illustrates that the Calderón sum of the system $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j,p\in P_j}$ equals 1. The Calderón sum is an important property for frames as its bounds are closely related to the frame's bounds (see [15, 42, 117]).

Lemma 2.3.6. *Under the assumptions of Theorem 2.3.3 [excluding the assumption ($\mathcal{N}_3$)], the Calderón sum of the system $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j,p\in P_j}$ is equal to 1 for each $i\in\{1,2\}$, i.e.*

$$\sum_{j\in\mathcal{J}}\frac{1}{s(\Gamma_j)}\sum_{p\in P_j}\left|\widehat{(g_p^{(i)})}(w)\right|^2=1 \text{ for a.e. } w\in\widehat{G}.$$

Proof. Using the Lemma 2.3.5 for $J>j_0$, we get

$$\sum_{j=j_0}^{J}\frac{1}{s(\Gamma_j)}\sum_{p\in P_j}\left|\widehat{(g_p^{(i)})}(w)\right|^2=-\frac{1}{s(\Gamma_{j_0})}\left|\Phi_{j_0}(w)\right|^2+|\Phi_{J+1}(w)|^2\frac{1}{s(\Gamma_J)}\sum_{m=0}^{s_J}|G_{J+1}^{(i)(m)}(w)|^2.$$

Using the Proposition 2.3.4 (ii) on second term of the right hand side of the above expression, we get

$$\sum_{j=j_0}^{J}\frac{1}{s(\Gamma_j)}\sum_{p\in P_j}\left|\widehat{(g_p^{(i)})}(w)\right|^2=\frac{1}{s(\Gamma_{J+1})}\left|\Phi_{J+1}(w)\right|^2-\frac{1}{s(\Gamma_{j_0})}\left|\Phi_{j_0}(w)\right|^2. \tag{2.3.7}$$

Let S be any compact subset of $\widehat{G}\setminus\mathcal{B}$ then using assumptions ($\mathcal{N}_1$) for every $\epsilon>0$, there exists $J_1\in\mathcal{I}$ such that for all $J\geq J_1-1$, we get

$$(1-\epsilon)-\frac{1}{s(\Gamma_{j_0})}\left|\Phi_{j_0}(w)\right|^2\leq\sum_{j=j_0}^{J}\frac{1}{s(\Gamma_j)}\sum_{p\in P_j}\left|\widehat{(g_p^{(i)})}(w)\right|^2\leq(1+\epsilon)-\frac{1}{s(\Gamma_{j_0})}\left|\Phi_{j_0}(w)\right|^2 \text{ for } w\in S.$$

Now letting $J\to\infty$, we get

$$(1-\epsilon)-\frac{1}{s(\Gamma_{j_0})}\left|\Phi_{j_0}(w)\right|^2\leq\sum_{j=j_0}^{\infty}\frac{1}{s(\Gamma_j)}\sum_{p\in P_j}\left|\widehat{(g_p^{(i)})}(w)\right|^2\leq(1+\epsilon)-\frac{1}{s(\Gamma_{j_0})}\left|\Phi_{j_0}(w)\right|^2 \text{ for } w\in S.$$

Since $\epsilon>0$ is arbitrary, we have

$$\sum_{j=j_0}^{\infty}\frac{1}{s(\Gamma_j)}\sum_{p\in P_j}\left|\widehat{(g_p^{(i)})}(w)\right|^2=1-\frac{1}{s(\Gamma_{j_0})}\left|\Phi_{j_0}(w)\right|^2 \text{ for } w\in S. \tag{2.3.8}$$

Now using equation ($\mathcal{N}_2$) for all $w\in S$, we have

$$\lim_{j_0\to-\infty}\frac{1}{s(\Gamma_{j_0})}|\Phi_{j_0}(w)|^2=0.$$

Therefore taking limit $j_0 \to -\infty$ in (2.3.8), we get

$$\sum_{k=-\infty}^{\infty} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \left|\widehat{(g_p^{(i)})}(w)\right|^2 = 1 \text{ for } w \in S. \tag{2.3.9}$$

As (2.3.9) is true for every $w \in S$, where S can be any compact subset of $\widehat{G} \setminus \mathcal{B}$. Therefore, (2.3.9) holds true for a.e. $w \in \widehat{G}$, i.e.

$$\sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \left|\widehat{(g_p^{(i)})}(w)\right|^2 = 1 \text{ for a.e. } w \in \widehat{G},$$

which completes the proof. ∎

Remark 2.3.7. If we consider the GTI systems

$$\{T_\lambda \mathcal{F}^{-1}\Phi_{j_0}\}_{\lambda\in\Gamma_{j_0}} \cup \bigcup_{j=j_0}^{\infty} \{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j, p\in P_j}, \tag{2.3.10}$$

then in view of (2.3.8) the Calderón sum of this system is equal to 1, i.e.

$$\frac{1}{s(\Gamma_{j_0})} |\Phi_{j_0}(w)|^2 + \sum_{j=j_0}^{\infty} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \left|\widehat{(g_p^{(i)})}(w)\right|^2 = 1 \text{ for a.e. } w \in \widehat{G}.$$

Note that assumption $(\mathcal{N}_2)$ is not required in this case.

With this, we are ready to prove Theorem 2.3.3.

Proof of Theorem 2.3.3. To show for each $i \in \{1,2\}$, $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j, p\in P_j}$ satisfies the LIC, it is sufficient to show for any $f \in \mathcal{D}_{\mathcal{B}}$,

$$\sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \sum_{\alpha\in\Gamma_j^\perp} \int_S |\widehat{f}(\gamma+\alpha)\widehat{g_p^{(i)}}(\gamma)|^2 \, d\mu_{\widehat{G}}(\gamma) < \infty,$$

where $S := \operatorname{supp} \widehat{f}$. Moreover as $\widehat{f} \in L^\infty(\widehat{G})$, we have

$$\begin{aligned}
&\sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \sum_{\alpha\in\Gamma_j^\perp} \int_S |\widehat{f}(\gamma+\alpha)\widehat{g_p^{(i)}}(\gamma)|^2 \, d\mu_{\widehat{G}}(\gamma) \\
&\qquad \le \|\widehat{f}\|_\infty^2 \sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \sum_{\alpha\in\Gamma_j^\perp} \int_{S\cap(S-\alpha)} |\widehat{g_p^{(i)}}(\gamma)|^2 \, d\mu_{\widehat{G}}(\gamma) \\
&\qquad = \|\widehat{f}\|_\infty^2 \sum_{\alpha\in\bigcup_{j\in\mathcal{J}}\Gamma_j^\perp} \int_{S\cap(S-\alpha)} \sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} |\widehat{g_p^{(i)}}(\gamma)|^2 \, d\mu_{\widehat{G}}(\gamma).
\end{aligned}$$

Using Lemma 2.3.6, we get

$$\begin{aligned}
\sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \sum_{\alpha\in\Gamma_j^\perp} \int_S |\widehat{f}(\gamma+\alpha)\widehat{g_p^{(i)}}(\gamma)|^2 \, d\mu_{\widehat{G}}(\gamma) &\le \|\widehat{f}\|_\infty^2 \sum_{\alpha\in\bigcup_{j\in\mathcal{J}}\Gamma_j^\perp} \int_{S\cap(S-\alpha)} 1 \, d\mu_{\widehat{G}}(\gamma) \\
&= \|\widehat{f}\|_\infty^2 \sum_{\alpha\in\bigcup_{j\in\mathcal{J}}\Gamma_j^\perp} \mu_{\widehat{G}}(S\cap(S-\alpha)) \le J_3\|\widehat{f}\|_\infty^2 < \infty,
\end{aligned}$$

where in the last step assumption $(\mathcal{N}_3)$ is used. Thus, both GTI systems satisfy LIC. Hence the systems $\cup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ and $\cup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,p\in P_j}$ satisfy dual α-LIC. ∎

Remark 2.3.8. In view of Remark 2.3.7 and proceeding the same way as in the proof of Theorem 2.3.3, for any $f\in\mathcal{D}_{\mathcal{B}}$, we have

$$\frac{1}{s(\Gamma_{j_0})}\sum_{\alpha\in\Gamma_{j_0}^\perp}\int_S|\widehat{f}(\gamma+\alpha)\Phi_{j_0}(\gamma)|^2\,d\mu_{\widehat{G}}(\gamma)+\sum_{j=j_0}^{\infty}\frac{1}{s(\Gamma_j)}\int_{P_j}\sum_{\alpha\in\Gamma_j^\perp}\int_S|\widehat{f}(\gamma+\alpha)\widehat{g_p^{(i)}}(\gamma)|^2\,d\mu_{\widehat{G}}(\gamma)<\infty.$$

Thus for each $i\in\{1,2\}$, both the GTI systems $\{T_\lambda\mathcal{F}^{-1}\Phi_{j_0}\}_{\lambda\in\Gamma_{j_0}}\cup\bigcup_{j=j_0}^{\infty}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ and $\{T_\lambda\mathcal{F}^{-1}\Phi_{j_0}\}_{\lambda\in\Gamma_{j_0}}\cup\bigcup_{j=j_0}^{\infty}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,p\in P_j}$ satisfy LIC. Hence they satisfy the dual α-LIC. Observe that assumptions $(\mathcal{N}_3)$ holds trivially in this case (see, Remark 2.3.1).

The following observation offers an equivalent formulation of condition $(\mathcal{N}_3)$, which characterizes the stationary behaviour of the sequence of subgroups $(\Gamma_j)_{j<0}$.

Observation (OBS(1)): Condition $(\mathcal{N}_3)$ is equivalent to the condition that the sequence $(\Gamma_j)_{j<0}$ becomes stationary, i.e. there exists j_0 such that $\Gamma_j=\Gamma_{j_0}$ for all $j\le j_0$. (Equivalently: The intersection $\bigcap_{j\in\mathcal{J}}\Gamma_j$ is a co-compact subgroup.

Proof of Observation. The sufficiency of the condition follows from $\bigcup_{j\in\mathcal{J}}\Gamma_j^\perp=\Gamma_{j_0}^\perp$, as observed in Remark 2.3.1 For necessity, we assume that the sequence is not stationary, i.e. there exist a sequence $(j_n)_{n\in\mathbb{N}}$ with $0>j_1>j_2>\dots$ and $\Gamma_{j_1}\supsetneq\Gamma_{j_2}\supsetneq\dots$. Then $\Gamma_{j_1}^\perp\subsetneq\Gamma_{j_2}^\perp\subsetneq\dots$. Let $0\in\widehat{G}$ denote the neutral element. We first claim that 0 is an accumulation point of $\left(\bigcup_{j<0}\Gamma_j^\perp\right)\setminus\{0\}$. To see this, assume the contrary, i.e. the existence of a neighborhood $U\subset\widehat{G}$ such that $U\cap\Gamma_j=\{0\}$, for all $j<0$. Pick a neighborhood $W\subset U$ with $W-W\subset U$. Then one has $W\cap\gamma+W=\varnothing$ for all $\gamma\in\Gamma_j\setminus\{0\}$. This in turn entails the existence of a fundamental domain $V_j\supset W$ modulo $\Gamma_j^\perp$, and therefore the covolume of the annihilators fulfils

$$s(\Gamma_j^\perp)=\widehat{\mu}(V_j)\ge\widehat{\mu}(W)>0. \tag{2.3.11}$$

On the other hand, we have $\Gamma_{j_1}^\perp\subsetneq\Gamma_{j_2}^\perp\dots$, and the covolumes are related by the formula

$$s(\Gamma_{j_{\ell+1}}^\perp)\cdot[\Gamma_{j_\ell}^\perp:\Gamma_{j_{\ell+1}}^\perp]=s(\Gamma_{j_{\ell+1}}^\perp),$$

where $[\Gamma_{j_\ell}^\perp:\Gamma_{j_{\ell+1}}^\perp]$ denotes the *index* of $\Gamma_{j_\ell}^\perp\subset\Gamma_{j_{\ell+1}}^\perp$, which is an integer ≥ 2, by assumption on the sequence $(j_\ell)_{\ell\in\mathbb{N}}$. But then we get

$$s(\Gamma_j^\perp)\to 0 \text{ as } j\to\infty$$

in contradiction to inequality (2.3.11).

Hence 0 is an accumulation point of $\left(\bigcup_{j<0}\Gamma_j^{\perp}\right)\smallsetminus\{0\}$. As a second countable locally compact group, $\widehat{G}$ is metrizable and hence there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}\subset\left(\bigcup_{j<0}\Gamma_j^{\perp}\right)\smallsetminus\{0\}$ converging to 0. Clearly this sequence can be chosen injective. We now use this sequence to show that assumption $(\mathcal{N}_3)$ is violated: Fix a compact subset $S\subset\widehat{G}\smallsetminus\mathcal{B}$ of positive measure. Note that such a set S exists because $\widehat{G}\smallsetminus\mathcal{B}$ has positive Haar measure, and Haar measure is regular. Since the map $\widehat{G}\ni\gamma\mapsto\mu_{\widehat{G}}(S\cap(S-\gamma))$ is continuous, the fact that $\gamma_n\to 0$ guarantees that

$$\mu_{\widehat{G}}(S\cap(S-\gamma_n))\geq\frac{\mu_{\widehat{G}}(S)}{2}>0$$

for all sufficiently large n. But then we get

$$\sum_{\gamma\in\bigcup_{j>0}\Gamma_j^{\perp}}\mu_{\widehat{G}}(S\cap(S-\gamma))\geq\sum_{n\in\mathbb{N}}\mu_{\widehat{G}}(S\cap(S-\gamma_n))=\infty.$$

This completes the proof of the observation. ∎

As in certain systems such as wavelet systems in $L^2(\mathbb{R})$, the sequence $(\Gamma_j)_{j<0}$ may not be stationary. To accommodate such cases where the sequence $(\Gamma_j)_{j<0}$ lacks to be stationary, it becomes necessary to replace condition $(\mathcal{N}_3)$ with an alternative. To address this need, we introduce the following standard assumption:

$(\mathcal{N}_3^*)$ For every compact set S in $\widehat{G}\smallsetminus\mathcal{B}$, there exists a constant $J_3^*>0$ such that for $i\in\{1,2\}$,

$$\sum_{\alpha\in\bigcup_{j\in\mathcal{J}}(\Gamma_j^{\perp}\cap\mathcal{B})}\mu_{\widehat{G}}(S\cap(S-\alpha))\leq J_3^*\text{ and }\sum_{j\in\mathcal{J}}\frac{1}{s(\Gamma_j)}\sum_{p\in P_j}\sum_{\alpha\in(\Gamma_j^{\perp}\smallsetminus\mathcal{B})}\int_{S\cap(S-\alpha)}|\widehat{g_p^{(i)}}(\gamma)|^2\,d\mu_{\widehat{G}}(\gamma)<\infty.$$

Theorem 2.3.9. *In addition to the assumptions $(\mathcal{N}_1)$,$(\mathcal{N}_2)$ and $(\mathcal{N}_3^*)$, assume that for each $i\in\{1,2\}$ and each $j\in\mathcal{J}$, the matrix-valued function $\mathfrak{B}_j^{(i)}(\gamma)$ defined in (2.2.7) satisfies the following condition:*

$$(\mathfrak{B}_j^{(i)}(\gamma))^*\mathfrak{B}_j^{(i)}(\gamma)=\frac{s(\Gamma_j)}{s(\Gamma_{j+1})}I_{d_j}\ \text{ for a.e. }\gamma\in\Omega_j.$$

Then $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,p\in P_j}$ (defined in (2.2.6)) satisfy the LIC. Hence they satisfy dual α-LIC.

Proof. To show for each $i \in \{1,2\}$, $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j, p\in P_j}$ satisfies the LIC, it is sufficient to show for any $f \in \mathcal{D}_{\mathcal{B}}$,

$$\sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \sum_{\alpha\in\Gamma_j^\perp} \int_S |\widehat{f}(\gamma+\alpha)\widehat{g_p^{(i)}}(\gamma)|^2 \, d\mu_{\widehat{G}}(\gamma) < \infty, \text{ where } S := \operatorname{supp}\widehat{f}. \tag{2.3.12}$$

We write left hand side of (2.3.12) is equal to $L_1 + L_2$, where L_1 is the sum in (2.3.12) corresponding to $\alpha \in \Gamma_j^\perp \cap \mathcal{B}$, i.e

$$L_1 = \sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \sum_{\alpha\in(\Gamma_j^\perp\cap\mathcal{B})} \int_S |\widehat{f}(\gamma+\alpha)\widehat{g_p^{(i)}}(\gamma)|^2 \, d\mu_{\widehat{G}}(\gamma),$$

and L_2 is the sum in (2.3.12) corresponding to $\alpha \in \Gamma_j^\perp \setminus \mathcal{B}$, i.e.

$$L_2 = \sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \sum_{\alpha\in(\Gamma_j^\perp\setminus\mathcal{B})} \int_S |\widehat{f}(\gamma+\alpha)\widehat{g_p^{(i)}}(\gamma)|^2 \, d\mu_{\widehat{G}}(\gamma).$$

We first estimate L_2. Note that

$$L_2 \le \|\widehat{f}\|_\infty \sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \sum_{\alpha\in(\Gamma_j^\perp\setminus\mathcal{B})} \int_{S\cap(S-\alpha)} |\widehat{g_p^{(i)}}(\gamma)|^2 \, d\mu_{\widehat{G}}(\gamma) < \infty,$$

by using assumption $(\mathcal{N}_3^*)$. Next we estimate L_1. Now by using Lemma 2.3.6 and $(\mathcal{N}_3^*)$, we have

$$\begin{aligned} L_1 &\le \|\widehat{f}\|_\infty^2 \sum_{\alpha\in\,\cup_{j\in\mathcal{J}}(\Gamma_j^\perp\cap\mathcal{B})} \int_{S\cap(S-\alpha)} 1 \, d\mu_{\widehat{G}}(\gamma) \\ &= \|\widehat{f}\|_\infty^2 \sum_{\alpha\in\,\cup_{j\in\mathcal{J}}(\Gamma_j^\perp\cap\mathcal{B})} \mu_{\widehat{G}}\left(S\cap(S-\alpha)\right) \le J_3^* \|\widehat{f}\|_\infty^2 < \infty. \end{aligned}$$

Thus, both GTI systems satisfy LIC. Hence the systems $\cup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ and $\cup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j}$ satisfy dual α-LIC. ∎

Up to this point, we have constructed GTI systems and proved two main results, Theorems 2.3.3 and 2.3.9, which give conditions ensuring that our constructed GTI systems satisfy the LIC. The next section presents an explicit example demonstrating these results.

2.4. Example in $L^2(\mathbb{R})$ generated by B-spline

In this section using B-spline as a generator, we provide an explicit construction of a pair of a systems having the dual α-LIC in $L^2(\mathbb{R})$. It will better outline the scope of Theorems 2.3.3 and 2.3.9.

Example 2.4.1. For each $j \in \mathcal{J} = \mathbb{Z}$, we define the B-spline of second order at level j as follows:

$$B_j(x) := \frac{1}{(2^{-j})^{3/2}} \chi_{[0,2^{-j}]} * \chi_{[0,2^{-j}]}(x), \;\; x \in \mathbb{R}.$$

Then its Fourier transform is given by

$$\begin{aligned}\Phi_j(\gamma) := \widehat{B_j}(\gamma) &= \frac{1}{(2^{-j})^{3/2}} \frac{(1 - e^{-2\pi i(2^{-j}\gamma)})^2}{-(2\pi\gamma)^2} \\ &= \frac{1}{(2^{-j-1})^{3/2}} \frac{(1 - e^{-2\pi i(2^{-j-1}\gamma)})^2}{-(2\pi\gamma)^2} \frac{(1 + e^{-2\pi i(2^{-j-1}\gamma)})^2}{2^{3/2}} \\ &= \widehat{B_{j+1}}(\gamma) H_{j+1}(\gamma) = H_{j+1}(\gamma)\Phi_{j+1}(\gamma),\end{aligned}$$

where $H_{j+1}(\gamma) = \frac{1}{2^{3/2}}(1 + e^{-2\pi i(2^{-j-1}\gamma)})^2 \in L^\infty[0, 2^{j+1})$ is a $2^{j+1}\mathbb{Z}$-periodic function. Further, we define the functions $\Psi_j^{(i)(m)} \in L^2(\mathbb{R})$, for $i \in \{1,2\}$ and $m \in \{1,2,3,4\}$ by

$$\Psi_j^{(i)(m)}(\gamma) = G_{j+1}^{(i)(m)}(\gamma)\Phi_{j+1}(\gamma),$$

where $G_{j+1}^{(1)(m)}$ and $G_{j+1}^{(2)(m)}$ are given by

$$\begin{pmatrix} G_{j+1}^{(1)(1)}(\gamma) \\ G_{j+1}^{(1)(2)}(\gamma) \\ G_{j+1}^{(1)(3)}(\gamma) \\ G_{j+1}^{(1)(4)}(\gamma) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \frac{1}{2^{3/2}} (1 - e^{-2\pi i(2^{-j-1}\gamma)})^2 \\ \frac{1}{\sqrt{3}} \frac{\sqrt{2}}{2^{3/2}} (1 + e^{-2\pi i(2^{-j-1}\gamma)})(1 - e^{-2\pi i(2^{-j-1}\gamma)}) \\ \frac{1}{\sqrt{3}} \left[\frac{\sqrt{2}}{2^{3/2}} (1 + e^{-2\pi i(2^{-j-1}\gamma)})(1 - e^{-2\pi i(2^{-j-1}\gamma)}) - \frac{1}{2^{3/2}} (1 - e^{-2\pi i(2^{-j-1}\gamma)})^2 \right] \\ \frac{1}{\sqrt{3}} \left[\frac{\sqrt{2}}{2^{3/2}} (1 + e^{-2\pi i(2^{-j-1}\gamma)})(1 - e^{-2\pi i(2^{-j-1}\gamma)}) + \frac{1}{2^{3/2}} (1 - e^{-2\pi i(2^{-j-1}\gamma)})^2 \right] \end{pmatrix},$$

and

$$\begin{pmatrix} G_{j+1}^{(2)(1)}(\gamma) \\ G_{j+1}^{(2)(2)}(\gamma) \\ G_{j+1}^{(2)(3)}(\gamma) \\ G_{j+1}^{(2)(4)}(\gamma) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \left[\frac{\sqrt{2}}{2^{3/2}} (1 + e^{-2\pi i(2^{-j-1}\gamma)})(1 - e^{-2\pi i(2^{-j-1}\gamma)}) + \frac{1}{2^{3/2}} (1 - e^{-2\pi i(2^{-j-1}\gamma)})^2 \right] \\ \frac{1}{\sqrt{3}} \left[\frac{\sqrt{2}}{2^{3/2}} (1 + e^{-2\pi i(2^{-j-1}\gamma)})(1 - e^{-2\pi i(2^{-j-1}\gamma)}) - \frac{1}{2^{3/2}} (1 - e^{-2\pi i(2^{-j-1}\gamma)})^2 \right] \\ \frac{1}{\sqrt{3}} \frac{1}{2^{3/2}} (1 - e^{-2\pi i(2^{-j-1}\gamma)})^2 \\ -\frac{1}{\sqrt{3}} \frac{\sqrt{2}}{2^{3/2}} (1 + e^{-2\pi i(2^{-j-1}\gamma)})(1 - e^{-2\pi i(2^{-j-1}\gamma)}) \end{pmatrix}.$$

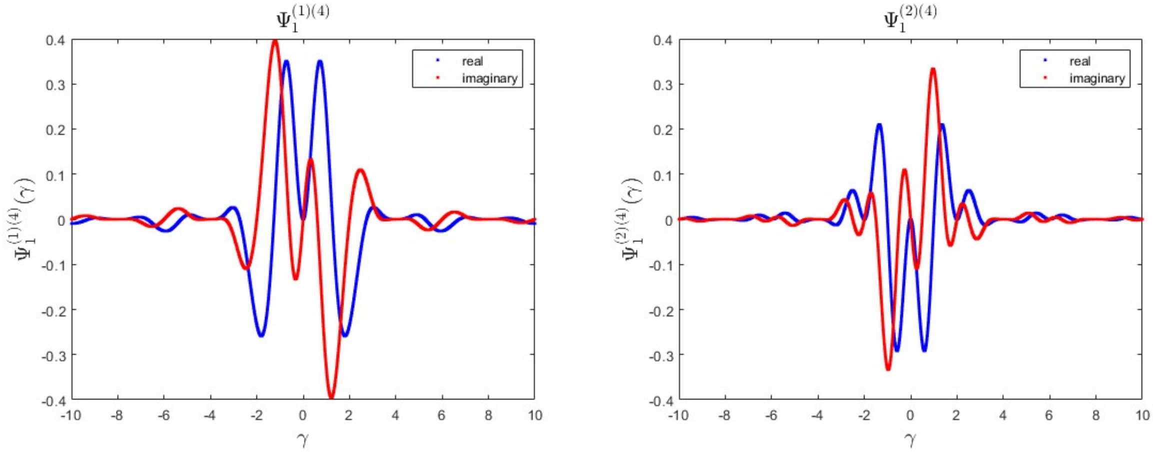

Figure 2.4.1. Plot of the functions $\Psi_1^{(1)(4)}$ and $\Psi_1^{(2)(4)}$ against γ.

We plot the functions $\Psi_1^{(1)(4)}$ and $\Psi_1^{(2)(4)}$ to depict the behaviour of its real and imaginary parts (see, Figure 2.4.1).

In this case, the matrix valued functions for each $i \in \{1,2\}$ defined in (2.2.7) are given below

$$B_j^{(i)}(\gamma) = \begin{pmatrix} G_{j+1}^{(i)(0)}(\gamma) & G_{j+1}^{(i)(1)}(\gamma) & G_{j+1}^{(i)(2)}(\gamma) & G_{j+1}^{(i)(3)}(\gamma) & G_{j+1}^{(i)(4)}(\gamma) \\ G_{j+1}^{(i)(0)}(\gamma+2^j) & G_{j+1}^{(i)(1)}(\gamma+2^j) & G_{j+1}^{(i)(2)}(\gamma+2^j) & G_{j+1}^{(i)(3)}(\gamma+2^j) & G_{j+1}^{(i)(4)}(\gamma+2^j) \end{pmatrix}^T,$$

where $G_{j+1}^{(i)(0)} = H_{j+1}$ and T denotes the transpose of matrix. Now for each $i \in \{1,2\}$ and $j \in \mathcal{J}$, we show that $(\mathfrak{B}_j^{(i)}(\gamma))^* \mathfrak{B}_j^{(i)}(\gamma) = 2I_2$ for a.e. $\gamma \in [0, 2^j)$. For this, it is sufficient to show that for a.e. $\gamma \in [0,2^j)$ and $\ell, \ell' \in \{1,2\}$,

$$\sum_{m=0}^{4} \overline{G_{j+1}^{(i)(m)}(\gamma+\nu_{j,\ell})} G_{j+1}^{(i)(m)}(\gamma+\nu_{j,\ell'}) = 2\delta_{\ell,\ell'}, \text{ where } \nu_{j,1}=0 \text{ and } \nu_{j,2}=2^j. \tag{2.4.1}$$

We demonstrate the proof of (2.4.1) for $i=1$, and the same approach can be applied for $i=2$ as well. First suppose that, $\ell=\ell'=1$, then $\nu_{j,\ell}=\nu_{j,\ell'}=0$. Now

$$\begin{aligned} \sum_{m=0}^{4} \overline{G_{j+1}^{(1)(m)}(\gamma+\nu_{j,\ell})} G_{j+1}^{(1)(m)}(\gamma+\nu_{j,\ell'}) &= |H_{j+1}(\gamma)|^2 + \sum_{m=1}^{4} |G_{j+1}^{(1)(m)}(\gamma)|^2 \\ =& \frac{1}{2^3}|(1+e^{-2\pi i(2^{-j-1}\gamma)})^2|^2 + \frac{1}{3}\frac{1}{2^3}|(1-e^{-2\pi i(2^{-j-1}\gamma)})^2|^2 \\ &+ \frac{1}{3}\frac{2}{2^3}|(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)})|^2 \\ &+ \frac{1}{3}\left|\frac{\sqrt{2}}{2^{3/2}}(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)}) - \frac{1}{2^{3/2}}(1-e^{-2\pi i(2^{-j-1}\gamma)})^2\right|^2 \\ &+ \frac{1}{3}\left|\frac{\sqrt{2}}{2^{3/2}}(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)}) + \frac{1}{2^{3/2}}(1-e^{-2\pi i(2^{-j-1}\gamma)})^2\right|^2. \end{aligned} \tag{2.4.2}$$

Using $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2),\ z_1, z_2 \in \mathbb{C}$ to simplify last two terms of above expression, the right hand side of (2.4.2) becomes

$$\begin{aligned}
&\frac{1}{2^3}|1+e^{-2\pi i(2^{-j-1}\gamma)}|^4 + \frac{1}{3}\frac{1}{2^3}|1-e^{-2\pi i(2^{-j-1}\gamma)}|^4 + \frac{1}{3}\frac{2}{2^3}|(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)})|^2 \\
&+ \frac{2}{3}\left(\frac{2}{2^3}\left|(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)})\right|^2 + \frac{1}{2^3}\left|1-e^{-2\pi i(2^{-j-1}\gamma)}\right|^4\right) \\
=&\frac{1}{2^3}|1+e^{-2\pi i(2^{-j-1}\gamma)}|^4 + \frac{2}{2^3}|1+e^{-2\pi i(2^{-j-1}\gamma)}|^2|1-e^{-2\pi i(2^{-j}\gamma)}|^2 + \frac{1}{2^3}|1+e^{-2\pi i(2^{-j-1}\gamma)}|^4 \\
=&\frac{1}{2^3}\left[|1+e^{-2\pi i(2^{-j-1}\gamma)}|^2 + |1-e^{-2\pi i(2^{-j-1}\gamma)}|^2\right]^2 \\
=&\frac{1}{2^3}4^2 = 2. \qquad (2.4.3)
\end{aligned}$$

When $\ell, \ell' = 2$, we have $\nu_{j,\ell} = \nu_{j,\ell'} = 2^j$. Following a similar approach as we observe for $\ell, \ell' = 1$, we can demonstrate that $\sum_{m=0}^{4}\left|G_{j+1}^{(i)(m)}(\gamma + 2^j)\right|^2 = 2$. Next suppose $\ell = 1$ and $\ell' = 2$, then $\nu_{j,\ell} = 0$ and $\nu_{j,\ell'} = 2^j$. Now

$$\sum_{m=0}^{4}\overline{G_{j+1}^{(1)(m)}(\gamma+\nu_{j,\ell})}G_{j+1}^{(1)(m)}(\gamma+\nu_{j,\ell'}) = \overline{H_{j+1}(\gamma)}H_{j+1}(\gamma+2^j) + \sum_{m=1}^{4}\overline{G_{j+1}^{(1)(m)}(\gamma)}G_{j+1}^{(1)(m)}(\gamma+2^j)$$

$$\begin{aligned}
=&\frac{1}{2^3}\overline{(1+e^{-2\pi i(2^{-j-1}\gamma)})^2}(1+e^{-2\pi i(2^{-j-1}(2^j+\gamma))})^2 + \frac{1}{3}\frac{1}{2^3}\overline{(1-e^{-2\pi i(2^{-j-1}\gamma)})^2}(1-e^{-2\pi i(2^{-j-1}(2^j+\gamma))})^2 \\
&+ \frac{1}{3}\frac{2}{2^3}\overline{(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)})}(1+e^{-2\pi i(2^{-j-1}(2^j+\gamma))})(1-e^{-2\pi i(2^{-j-1}(2^j+\gamma))}) \\
&+ \frac{1}{3}\left(\overline{\frac{\sqrt{2}}{2^{3/2}}(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)}) - \frac{1}{2^{3/2}}(1-e^{-2\pi i(2^{-j-1}\gamma)})^2}\right)\times \\
&\left(\frac{\sqrt{2}}{2^{3/2}}(1+e^{-2\pi i(2^{-j-1}(\gamma+2^j))})(1-e^{-2\pi i(2^{-j-1}(\gamma+2^j))}) - \frac{1}{2^{3/2}}(1-e^{-2\pi i(2^{-j-1}(\gamma+2^j))})^2\right) \\
&+ \frac{1}{3}\left(\overline{\frac{\sqrt{2}}{2^{3/2}}(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)}) + \frac{1}{2^{3/2}}(1-e^{-2\pi i(2^{-j-1}\gamma)})^2}\right)\times \\
&\left(\frac{\sqrt{2}}{2^{3/2}}(1+e^{-2\pi i(2^{-j-1}(\gamma+2^j))})(1-e^{-2\pi i(2^{-j-1}(\gamma+2^j))}) + \frac{1}{2^{3/2}}(1-e^{-2\pi i(2^{-j-1}(\gamma+2^j))})^2\right). \qquad (2.4.4)
\end{aligned}$$

Using $e^{-2\pi i(2^{-j-1}(2^j+\gamma))} = -e^{-2\pi i(2^{-j-1}\gamma)}$ and $(1+\overline{z})(1-z) + (1-\overline{z})(1+z) = 0$ for $|z| = 1$. Then (2.4.4) becomes

$$\begin{aligned}
=&\frac{1}{2^3}\overline{(1+e^{-2\pi i(2^{-j-1}\gamma)})^2}(1+e^{-2\pi i(2^{-j-1}(2^j+\gamma))})^2 + \frac{1}{3}\frac{1}{2^3}\overline{(1-e^{-2\pi i(2^{-j-1}\gamma)})^2}(1-e^{-2\pi i(2^{-j-1}(2^j+\gamma))})^2 \\
&+ \frac{1}{3}\frac{2}{2^3}\overline{(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)})}(1+e^{-2\pi i(2^{-j-1}(2^j+\gamma))})(1-e^{-2\pi i(2^{-j-1}(2^j+\gamma))}) \\
&+ \frac{2}{3}\frac{2}{2^3}\overline{(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)})}(1+e^{-2\pi i(2^{-j-1}(2^j+\gamma))})(1-e^{-2\pi i(2^{-j-1}(2^j+\gamma))})
\end{aligned}$$

$$+\frac{2}{3}\frac{1}{2^3}\overline{(1-e^{-2\pi i(2^{-j-1}\gamma)})^2}(1-e^{-2\pi i(2^{-j-1}(2^j+\gamma))})^2$$

$$=\frac{1}{2^3}\Bigg[\Big((1+\overline{e^{-2\pi i(2^{-j-1}\gamma)}})(1-e^{-2\pi i(2^{-j-1}\gamma)})\Big)^2+\Big((1-\overline{e^{-2\pi i(2^{-j-1}\gamma)}})(1+e^{-2\pi i(2^{-j-1}\gamma)})\Big)^2$$

$$+2\Big((1+\overline{e^{-2\pi i(2^{-j-1}\gamma)}})(1-\overline{e^{-2\pi i(2^{-j-1}\gamma)}})(1-e^{-2\pi i(2^{-j-1}\gamma)})(1+e^{-2\pi i(2^{-j-1}\gamma)})\Big)\Bigg]$$

$$=\frac{1}{2^3}\Big[(1+\overline{e^{-2\pi i(2^{-j-1}\gamma)}})(1-e^{-2\pi i(2^{-j-1}\gamma)})+(1-\overline{e^{-2\pi i(2^{-j-1}\gamma)}})(1+e^{-2\pi i(2^{-j-1}\gamma)})\Big]^2$$

$$=0.$$

Similarly for $\ell=2,\ell'=1$, we have $\sum_{m=0}^{4}\overline{G_{j+1}^{(i)(m)}(\gamma+2^j)}G_{j+1}^{(i)(m)}(\gamma)=0$. Finally, we get

$$(\mathfrak{B}_j^{(i)}(\gamma))^*\mathfrak{B}_j^{(i)}(\gamma)=2I_{d_j}\quad\text{for a.e. }\gamma\in\Omega_j=[0,2^j),\ i\in\{1,2\}\text{ and }j\in\mathbb{Z}.$$

Let $\Gamma_j:=2^{-j}\mathbb{Z}\subset\mathbb{R}$. Then $\Gamma_j^\perp=2^j\mathbb{Z}$ and its fundamental domain is $\Omega_j=[0,2^j)$. Now

$$\frac{1}{s(\Gamma_j)}|\Phi_j(\gamma)|^2=\left|\frac{1}{\sqrt{s(\Gamma_j)}}\Phi_j(\gamma)\right|^2=\left|\frac{1}{(2^{-j})^2}\frac{(1-e^{-2\pi i(2^{-j}\gamma)})^2}{-(2\pi\gamma)^2}\right|^2=\left|\frac{(1-e^{-2\pi i(2^{-j}\gamma)})^2}{(2\pi i(2^{-j}\gamma))^2}\right|^2.$$

Using the fact $\lim_{x\to0}\left(\frac{1-e^{-x}}{x}\right)=1$ and $2^{-j}\to0$ as $j\to\infty$, we have

$$\lim_{j\to\infty}\left(\frac{1-e^{-2\pi i(2^{-j}\gamma)}}{2\pi i(2^{-j}\gamma)}\right)=1.$$

This implies $\lim_{j\to\infty}\frac{1}{s(\Gamma_j)}|\Phi_j(\gamma)|^2=1$. Hence assumption $(\mathcal{N}_1)$ is true. Also

$$\lim_{j\to-\infty}\frac{1}{\sqrt{s(\Gamma_j)}}|\Phi_j(\gamma)|=\left|\lim_{j\to-\infty}\left(\frac{1-e^{-2\pi i(2^{-j}\gamma)}}{2\pi i(2^{-j}\gamma)}\right)^2\right|=0,$$

this implies assumption $(\mathcal{N}_2)$ is true. Hence, by using Lemma 2.3.6 for $i\in\{1,2\}$, we conclude that the Calderón sum of the system $\cup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j,p\in P_j}$ is equal to one.

Case-I: $\mathcal{J}=\{j_0,j_0+1,\cdots\}$.

Since $(\mathcal{N}_3)$ holds trivially in this case (see, Remark 2.3.1), we have for each $i\in\{1,2\}$ the system $\{T_\lambda\mathcal{F}^{-1}\Phi_{j_0}\}_{\lambda\in\Gamma_{j_0}}\cup\bigcup_{j=j_0}^{\infty}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j,p\in P_j}$ satisfies the LIC, in view of Theorem 2.3.3 and Remark 2.3.8. Hence, they satisfy the dual α-LIC.

Case-II: $\mathcal{J}=\mathbb{Z}$.

Since $(\Gamma_j)_{j<0}$ is not stationary, $(\mathcal{N}_3)$ does not hold due to Observation (OBS(1)). Therefore, Theorem 2.3.3 is not applicable in this case. But if we prove that $(\mathcal{N}_3^*)$ holds in this case, then by Theorem 2.3.9, the systems $\cup_{j\in\mathbb{Z}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ and $\cup_{j\in\mathbb{Z}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,p\in P_j}$ satisfy the dual α-LIC. We fix the Borel set $\mathcal{B} = \{0\}$. Let S be any compact set in $\mathbb{R} \setminus \{0\}$. Then,

$$\sum_{\alpha \in \cup_{j\in\mathbb{Z}} (\Gamma_j^{\perp} \cap \{0\})} \mu_{\mathbb{R}}(S \cap (S - \alpha)) = \mu_{\mathbb{R}}(S) < \infty.$$

Therefore in order to prove $(\mathcal{N}_3^*)$ holds, it only remains to prove

$$L_1 := \sum_{j\in\mathbb{Z}} \frac{1}{2^{-j}} \sum_{p\in P_j} \sum_{\alpha\in(2^k\mathbb{Z}\setminus\{0\})} \int_{S\cap(S-\alpha)} |\widehat{g_p^{(i)}}(\gamma)|^2 d\mu_{\mathbb{R}}(\gamma) < \infty.$$

For compact set $S \subset \mathbb{R} \setminus \{0\}$, there exists a $1 < t \in \mathbb{R}$ such that $S \subset (-t, -\frac{1}{t}) \cup (\frac{1}{t}, t) := S(t)$. Now, we estimate the following:

$$\begin{aligned}
L_1 &= \sum_{j\in\mathbb{Z}} \frac{1}{2^{-j}} \sum_{m=1}^{4} \sum_{k\in(\mathbb{Z}\setminus\{0\})} \int_{S\cap(S-2^j k)} |\widehat{g_{(m,j)}^{(i)}}(\gamma)|^2 d\mu_{\mathbb{R}}(\gamma) \\
&\leq \sum_{j\in\mathbb{Z}} \sum_{m=1}^{4} \sum_{k\in(\mathbb{Z}\setminus\{0\})} \int_{S(t)\cap(S(t)-2^j k)} |\Psi_0^{(i)(m)}(2^{-j}\gamma)|^2 d\mu_{\mathbb{R}}(\gamma) \\
&= \sum_{j\in\mathbb{Z}} \sum_{m=1}^{4} \sum_{k\in(\mathbb{Z}\setminus\{0\})} \int_{2^j\xi\in[S(t)\cap(S(t)-2^j k)]} |\Psi_0^{(i)(m)}(\xi)|^2 2^j d\mu_{\mathbb{R}}(\xi),
\end{aligned}$$

as $\widehat{g_{(m,j)}^{(i)}}(\gamma) = 2^{-j/2}\Psi_0^{(i)(m)}(2^{-j}\gamma)$. For $k \in \mathbb{Z} \setminus \{0\}$, if $2^j\xi \in S(t)$ and $2^j\xi + 2^j k \in S(t)$, then, for $j \in \mathbb{Z}$, we get

$$|2^j k| \leq |2^j\xi + 2^j k| + |2^j\xi| < t + t = 2t.$$

Thus $\{k \in \mathbb{Z} \setminus \{0\} : 2^j\xi \in S(t) \text{ and } 2^j\xi + 2^j k \in S(t)\} \subset \{k \in \mathbb{Z} \setminus \{0\} : 2^j k \in (-2t, 2t)\}$ for every $j \in \mathbb{Z}$. It is easy to observe that there exists a C_t such that

$$\#\{k \in \mathbb{Z} \setminus \{0\} : 2^j k \in (-2t, 2t)\} \leq C_t 2^{-j} \text{ for all } j \in \mathbb{Z}.$$

Therefore, we have

$$L_1 \leq C_t \sum_{m=1}^{4} \sum_{j\in\mathbb{Z}} \int_{2^j\xi\in S(t)} |\Psi_0^{(i)(m)}(\xi)|^2 d\mu_{\mathbb{R}}(\xi) \leq C_t N_t \sum_{m=1}^{4} \|\Psi_0^{(i)(m)}\|^2,$$

as there exists a N_t (independent of $\xi \in \mathbb{R}$) such that $\#\{j \in \mathbb{Z} : 2^j\xi \in S(t)\} \leq N_t$. Thus, $(\mathcal{N}_3^*)$ holds. Hence, by Theorem 2.3.9 for each $i \in \{1, 2\}$, the system $\bigcup_{j\in\mathbb{Z}} \{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j, p\in P_j}$ satisfies LIC. Thus they satisfy the dual α-LIC.

In this chapter, we presented a technique for constructing a pair of GTI systems, each satisfying the LIC. We have seen that the LIC plays a crucial role in determining when these systems form Parseval frames (see Theorem 2.1.3) and when they are pairwise orthogonal (see Theorem 2.1.4). Building on these results, the next chapter aims to develop a method for constructing pairwise orthogonal Parseval frames using the GTI systems introduced here.

CHAPTER 3

CONSTRUCTION OF PAIRWISE ORTHOGONAL GTI PARSEVAL FRAMES VIA LIC

This chapter presents a method for constructing pairwise orthogonal Parseval frames with generalized translation invariant (GTI) structures, based on the GTI systems constructed in the Chapter 2. We begin by establishing a condition on the sequence of annihilators $\{\Gamma_j^{\perp}\}_{j\in\mathcal{J}}$ under which the GTI systems becomes Parseval frames. Further, these Parseval frames become pairwise orthogonal by adding suitable conditions on the filters. Explicit techniques are developed to construct two or, more generally, N GTI Parseval frames from a given Parseval frame. Moreover, the resulting frames are pairwise orthogonal. As an application of our result, we illustrate the pairwise orthogonal GTI Parseval frames generated by B-splines. At the end, we deduce a technique for constructing pairwise orthogonal Parseval wavelet frames in $L^2(\mathbb{R}^n)$ as a special case of our result.

3.1. GTI Parseval frames

In Chapter 2, we established sufficient conditions under which the systems $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j,\, p\in P_j}$ (see, (2.2.6)) and $\{T_\lambda \mathcal{F}^{-1}\Phi_{j_0}\}_{\lambda\in\Gamma_{j_0}} \cup \bigcup_{j=j_0}^{\infty}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j,\, p\in P_j}$ (see, (2.3.10)) satisfy the LIC for each $i\in\{1,2\}$. In this chapter, we derive conditions that ensure these systems form pairwise orthogonal Parseval frames, using LIC.

In this section we establish sufficient conditions under which the systems defined in (2.2.6) and (2.3.10) form Parseval frames (see, Theorem 3.1.1). This serves as a first step toward constructing pairwise orthogonal Parseval frames. Theorem 2.1.3, Proposition 2.3.4(ii), and Theorem 2.3.3 play a central role in the proof of Theorem 3.1.1. The same

The results of **Chapter 3** are taken from the published article:

Redhu N., Gumber A., Shukla N. K. (2025), *Constructions of pairwise orthogonal Parseval frames generated by filter on LCA groups*, **Applied and computational Harmonic analysis**, **74**, paper No.101708, 27pp, DOI: 10.1016/j.acha.2024.101708.

result can also be obtained under the assumptions of Theorem 2.3.9 in place of Theorem 2.3.3.

Theorem 3.1.1. *In addition to the assumptions of Theorem 2.3.3, let us assume*

$$\bigcap_{j\in\mathcal{J}} \Gamma_j^{\perp} = \{0\}.$$

Then for each $i \in \{1,2\}$, the following hold true:

(a) $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j, p\in P_j}$ *is a Parseval frame for* $L^2(G)$.

(b) $\{T_\lambda \mathcal{F}^{-1}\Phi_{j_0}\}_{\lambda\in\Gamma_{j_0}} \cup \bigcup_{j=j_0}^{\infty} \{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j, p\in P_j}$ *is a Parseval frame for* $L^2(G)$.

Proof. (a) Since the assumptions of Theorem 2.3.3 are true, it follows from Theorem 2.3.3 that the system $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j, p\in P_j}$ constructed in (2.2.6) satisfy α-LIC. Therefore in view of Theorem 2.1.3, it is sufficient to show that

(i) $\sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \left|\widehat{(g_p^{(i)})}(w)\right|^2 = 1$ for a.e. $w \in \widehat{G}$ and

(ii) for $\alpha \in \cup_{j\in\mathcal{J}} \Gamma_j^{\perp} \smallsetminus \{0\}$,

$$t_\alpha = \sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^{\perp}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \overline{\widehat{(g_p^{(i)})}(w)}\widehat{(g_p^{(i)})}(w+\alpha) = 0 \text{ for a.e. } w \in \widehat{G}.$$

We need to prove (ii) because (i) is already confirmed by Lemma 2.3.6. The assumptions $\cdots \supset \Gamma_{j_0}^{\perp} \supset \Gamma_{j_0+1}^{\perp} \supset \cdots$ and $\bigcap_{j\in\mathcal{J}} \Gamma_j^{\perp} = \{0\}$ imply $\bigcup_{j\in\mathcal{J}} \Gamma_j^{\perp} \smallsetminus \{0\} = \bigcup_{j\in\mathcal{J}} \left(\Gamma_j^{\perp} \smallsetminus \Gamma_{j+1}^{\perp}\right)$. Thus for $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^{\perp} \smallsetminus \{0\}$, there exists $J \in \mathcal{J}$ such that $\alpha \in \Gamma_J^{\perp} \smallsetminus \Gamma_{J+1}^{\perp}$. Hence we obtain

$$\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^{\perp}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \overline{\widehat{(g_p^{(i)})}(w)}\widehat{(g_p^{(i)})}(w+\alpha) = \sum_{j=-\infty}^{J} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \overline{\widehat{(g_p^{(i)})}(w)}\widehat{(g_p^{(i)})}(w+\alpha), \quad (3.1.1)$$

since $\alpha \in \Gamma_j^{\perp}$ for $j \le J$ and $\alpha \notin \Gamma_j^{\perp}$ for $j > J$, by observing the fact $\cdots \supset \Gamma_j^{\perp} \supset \Gamma_{j+1}^{\perp} \supset \cdots$. Further by using Lemma 2.3.5, the right hand side of (3.1.1) becomes

$$\begin{aligned}
&\lim_{j_0\to-\infty} \sum_{j=j_0}^{J} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \overline{\widehat{(g_p^{(i)})}(w)}\widehat{(g_p^{(i)})}(w+\alpha) \\
&\qquad = -\lim_{j_0\to-\infty} \frac{1}{s(\Gamma_{j_0})} \overline{\Phi_{j_0}(w)}\Phi_{j_0}(w+\alpha) \\
&\qquad\quad + \overline{\Phi_{J+1}(w)}\Phi_{J+1}(w+\alpha)\frac{1}{s(\Gamma_J)} \sum_{m=0}^{s_J} \overline{G_{J+1}^{(i)(m)}(w)}G_{J+1}^{(i)(m)}(w+\alpha) \\
&\qquad = \overline{\Phi_{J+1}(w)}\Phi_{J+1}(w+\alpha)\frac{1}{s(\Gamma_J)} \sum_{m=0}^{s_J} \overline{G_{J+1}^{(i)(m)}(w)}G_{J+1}^{(i)(m)}(w+\alpha), \qquad (3.1.2)
\end{aligned}$$

where the first term becomes zero because of assumption $(\mathcal{N}_2)$. Since $\alpha \in \Gamma_J^\perp \smallsetminus \Gamma_{J+1}^\perp$ and $\Gamma_J^\perp = \bigcup_{\ell=1}^{d_J}(v_{J,\ell} + \Gamma_{J+1}^\perp)$, $(v_{J,\ell} + \Gamma_{J+1}^\perp) \cap (v_{J,\ell'} + \Gamma_{J+1}^\perp) = \varnothing$ for $\ell \neq \ell'$ with $v_{J,1} = 0$ (using (2.2.2)), it follows that $\alpha = v_{J,\ell} + \alpha'$ for some $\alpha' \in \Gamma_{J+1}^\perp$, and $\ell \in \{2, 3, \cdots, d_J\}$. Here $\ell = 1$ is not possible, because if $\ell = 1$, then $\alpha = \alpha' \in \Gamma_{J+1}^\perp$, which is a contradiction. Substituting $\alpha = v_{J,\ell} + \alpha'$ in right hand side of (3.1.2) and then using the periodicity of $G_{J+1}^{(i)(m)}$, i.e. $G_{J+1}^{(i)(m)}(w + v_{J,\ell} + \alpha') = G_{J+1}^{(i)(m)}(w + v_{J,\ell})$, we get

$$\begin{aligned}\sum_{j \in \mathcal{J}: \alpha \in \Gamma_j^\perp} \frac{1}{s(\Gamma_j)} \sum_{p \in P_j} \overline{\widehat{(g_p^{(i)})}(w)} \widehat{(g_p^{(i)})}(w+\alpha) &= \overline{\Phi_{J+1}(w)} \Phi_{J+1}(w+\alpha) \\ &\quad \times \frac{1}{s(\Gamma_J)} \sum_{m=0}^{s_J} \overline{G_{J+1}^{(i)(m)}(w)} G_{J+1}^{(i)(m)}(w + v_{J,\ell}) \\ &= \overline{\Phi_{J+1}(w)} \Phi_{J+1}(w+\alpha) \frac{1}{s(\Gamma_J)}(0) = 0,\end{aligned}$$

using Proposition 2.3.4(ii). Hence the result follows.

(b) In view of Remark 2.3.8, for each $i \in \{1, 2\}$, the system $\{T_\lambda \mathcal{F}^{-1}\Phi_{j_0}\}_{\lambda \in \Gamma_{j_0}} \cup \bigcup_{j=j_0}^{\infty} \{T_\lambda g_p^{(i)}\}_{\lambda \in \Gamma_j, p \in P_j}$ satisfies LIC. Also by Remark 2.3.7, we have

$$\frac{1}{s(\Gamma_{j_0})} |\Phi_{j_0}(w)|^2 + \sum_{j=j_0}^{\infty} \frac{1}{s(\Gamma_j)} \sum_{p \in P_j} \left| \widehat{(g_p^{(i)})}(w) \right|^2 = 1.$$

Therefore in view of Theorem 2.1.3, it is sufficient to show that for $\alpha \in \cup_{j=j_0}^{\infty} \Gamma_j^\perp \smallsetminus \{0\}$,

$$t_\alpha = \frac{1}{s(\Gamma_{j_0})} \overline{\Phi(w)} \Phi(w+\alpha) + \sum_{j=j_0: \alpha \in \Gamma_j^\perp}^{\infty} \frac{1}{s(\Gamma_j)} \sum_{p \in P_j} \overline{\widehat{(g_p^{(i)})}(w)} \widehat{(g_p^{(i)})}(w+\alpha) = 0 \text{ for a.e. } w \in \widehat{G}.$$

By doing similar calculation as in (3.1.1), we have

$$\begin{aligned}t_\alpha &= \frac{1}{s(\Gamma_{j_0})} \overline{\Phi(w)} \Phi(w+\alpha) + \sum_{j=j_0}^{J} \frac{1}{s(\Gamma_j)} \sum_{p \in P_j} \overline{\widehat{(g_p^{(i)})}(w)} \widehat{(g_p^{(i)})}(w+\alpha) \\ &= \overline{\Phi_{J+1}(w)} \Phi_{J+1}(w+\alpha) \frac{1}{s(\Gamma_J)} \sum_{m=0}^{s_J} \overline{G_{J+1}^{(i)(m)}(w)} G_{J+1}^{(i)(m)}(w+\alpha),\end{aligned}$$

by using Lemma 2.3.5. Now t_α is same as right hand side of (3.1.2). Thus similar to (a), we get $t_\alpha = 0$. ∎

Remark 3.1.2. Observe that the index set $\mathcal{J} = \{j_0, j_0 + 1, \cdots\}$ in Theorem 3.1.1 (b), and hence assumption $(\mathcal{N}_3)$ holds trivially (see, Remark 2.3.1). Also the assumption $(\mathcal{N}_2)$ is not required in this case (see Remark 2.3.7). Theorem 3.1.1 (b) was already proved by Christensen and Goh in [36, Theorem 3.3]. Our approach of proving Theorem 3.1.1 (a) and (b) is different by taking advantage of LIC and t_α-equations. One of the main benefits of this approach is that it provides a more generalized version of the unitary extension principle (UEP) given by Christensen and Goh [36], as demonstrated in Theorem 3.1.1 (a). To construct sets of pairwise orthogonal Parseval frames, we will utilize the systems specified in Theorem 3.1.1 (a).

Example 3.1.3. Recall that GTI systems $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,p\in P_j}$ defined in Example 2.4.1, satisfies the α-LIC. Here $\Gamma_j^\perp = 2^j\mathbb{Z}$ and $\cap_{j\in\mathbb{Z}}2^j\mathbb{Z} = \{0\}$, therefore by Theorem 3.1.1, it follows that $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,p\in P_j}$ are Parseval frames for $L^2(\mathbb{R})$.

3.2. Pairwise orthogonal Parseval frames

In Subsection 3.1, we have obtained conditions that ensure the systems $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma j,p\in P_j}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma j,p\in P_j}$ (defined in (2.2.6)) are Parseval frames. Also these systems satisfy the dual α-LIC as observed in Chapter 2. In this section, using these result together with Theorem 2.1.4, we are ready to present our main results about constructing pairwise orthogonal Parseval frames.

Theorem 3.2.1 establishes the conditions required for pairwise orthogonal Parseval frames with GTI structures. Additionally, this section provides two other significant results: Theorem 3.2.2 and Theorem 3.2.3. These results outline the construction technique for generating pairwise orthogonal Parseval frames based on a given Parseval frame. Theorems 3.2.2 and 3.2.3 are motivated by the work of [23] and [22], respectively. The statements of all three results are provided here, while their respective proofs are presented in Subsections 3.2.1, 3.2.2, and 3.2.3.

To establish the following result, Theorem 2.3.3, Proposition 2.3.4(i), and Theorem 3.1.1(a) play an important role.

Theorem 3.2.1. *In addition to the assumptions of Theorem 2.3.3, let us assume* $\bigcap_{j\in\mathcal{J}}\Gamma_j^\perp = \{0\}$ *and for each* $j\in\mathcal{J}$, *the matrix valued functions* $\widetilde{\mathfrak{B}_j^{(1)}}(\gamma)$ *and* $\widetilde{\mathfrak{B}_j^{(2)}}(\gamma)$ *(defined in (2.2.8)) satisfy*

$$(\widetilde{\mathfrak{B}_j^{(1)}}(\gamma))^*\widetilde{\mathfrak{B}_j^{(2)}}(\gamma) = O_{d_j} \text{ for a.e. } \gamma\in\Omega_j,$$

where O_{d_j} *is the zero matrix of order* d_j. *Then* $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ *and* $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,p\in P_j}$ *(defined in (2.2.6)) are pairwise orthogonal Parseval frames in* $L^2(G)$.

Theorem 3.2.1 can also be developed under the assumptions of Theorem 2.3.9 instead of Theorem 2.3.3. It can be used for deducing construction results for pairwise orthogonal Parseval frames with various structured systems including wavelet and Gabor systems. For pairwise orthogonal wavelet system the result is deduced in Corollary 3.3.2 in Section 3.3.

Next, we provide techniques for constructing two or N-Parseval frames which are pairwise orthogonal based on a given Paresval frame. For any $i \in \{1, 2, \cdots, N\}$, similar to (2.2.6), we define the GTI systems

$$\bigcup_{j \in \mathcal{J}} \{T_\lambda g_p^{(i)}\}_{\lambda \in \Gamma_j, p \in P_j}, \tag{3.2.1}$$

where $p \in P_j = \{(m, j) : m = 1, 2, \ldots, s_j\}$ and $g_p^{(i)} = g_{(m,j)}^{(i)} = \mathcal{F}^{-1}\Psi_j^{(i)(m)}$. The function $\Psi_j^{(i)(m)}(\gamma)$ is defined as in (2.2.4)

$$\Psi_j^{(i)(m)}(\gamma) := G_{j+1}^{(i)(m)}(\gamma)\Phi_{j+1}(\gamma), \quad \text{and } \Phi_j(\gamma) = H_{j+1}(\gamma)\Phi_{j+1}(\gamma) \ a.e.\ \gamma \in \widehat{G}.$$

Further, for each $i \in \{1, 2, \ldots, N\}$ and $j \in \mathcal{J}$, we define the matrix-valued functions $\mathfrak{B}_j^{(i)}(\gamma)$ and $\widetilde{\mathfrak{B}_j^{(i)}}(\gamma)$ analogously to (2.2.7) and (2.2.8), respectively.

Next, Theorem 3.2.2 provides a technique for the construction of pairwise orthogonal Parseval frames when one Parseval frame is given.

Theorem 3.2.2. *Consider the GTI system $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ (defined in (3.2.1)) satisfying the assumptions of Theorem 3.1.1. For each $i \in \{2, 3\}$, $j \in \mathcal{J}$ and $m \in \{0, 1, \cdots, 2s_j\}$, assume the $\Gamma_j^\perp$-periodic functions $G_{j+1}^{(i)(m)}$ satisfy the relations $G_{j+1}^{(i)(0)}(\gamma) = H_{j+1}(\gamma)$ and*

$$\begin{pmatrix} G_{j+1}^{(i)(1)}(\gamma) \\ G_{j+1}^{(i)(2)}(\gamma) \\ \vdots \\ G_{j+1}^{(i)(2s_j)}(\gamma) \end{pmatrix} = \left(K^j_{1+(i-2)s_j}(\gamma)\, K^j_{2+(i-2)s_j}(\gamma) \cdots K^j_{s_j+(i-2)s_j}(\gamma)\right) \begin{pmatrix} G_{j+1}^{(1)(1)}(\gamma) \\ G_{j+1}^{(1)(2)}(\gamma) \\ \vdots \\ G_{j+1}^{(1)(s_j)}(\gamma) \end{pmatrix}, \ \gamma \in \widehat{G}, \tag{3.2.2}$$

where $K^j(\gamma) = \left(K_1^j(\gamma)\ K_2^j(\gamma) \cdots\ K_{2s_j}^j(\gamma)\right)$ is a $2s_j \times\ 2s_j$ unitary matrix with $\Gamma_j^\perp$-periodic entries. Then $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j'}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(3)}\}_{\lambda\in\Gamma_j, p\in P_j'}$ (defined in (3.2.1)) are pairwise orthogonal Parseval frames, where $P_j' = \{(m, j) : m = 1, 2, \cdots, 2s_j\}$.

Filters play an important role in the construction of tight frames [28]. In the next result, we generate N-sets of filters using a given set of filters with the help of unitary matrices. Then, using these N-sets of filters, we construct N-Parseval frames, which are pairwise orthogonal Parseval frames.

Theorem 3.2.3. *Consider the GTI system* $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ *(defined in (3.2.1)) satisfying the assumptions of Theorem 3.1.1 for* $P_j=\{(1,j)\}$*. For each* $i\in\{2,3,\cdots,N+1\}$*,* $j\in\mathcal{J}$ *and* $m\in\{1,2,\cdots,N\}$*, assume the* $\Gamma_j^\perp$*-periodic functions* $G_{j+1}^{(i)(m)}$ *satisfy the relations* $G_{j+1}^{(i)(0)}(\gamma)=H_{j+1}(\gamma)$ *and*

$$\begin{pmatrix} G_{j+1}^{(i)(1)}(\gamma)\\ G_{j+1}^{(i)(2)}(\gamma)\\ \vdots\\ G_{j+1}^{(i)(N)}(\gamma)\end{pmatrix}=G_{j+1}^{(1)(1)}(\gamma)\begin{pmatrix} U_{i-1,1}^{j}(\gamma)\\ U_{i-1,2}^{j}(\gamma)\\ \vdots\\ U_{i-1,N}^{j}(\gamma)\end{pmatrix},\ \gamma\in\widehat{G}, \tag{3.2.3}$$

where the entries of unitary matrix $\left(U_{m,n}^j(\gamma)\right)_{1\le m,\ n\le N}$ *are* $\Gamma_j^\perp$*-periodic. Then the GTI systems* $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j,p\in P_j''}$ *and* $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i')}\}_{\lambda\in\Gamma_j,p\in P_j''}$ *are pairwise orthogonal Parseval frames for* $L^2(G)$*, where* $P_j''=\{(m,j):m=1,2,\cdots,N\}$*,* $i\neq i'$ *and* $i,i'\in\{2,3,\cdots,N+1\}$*.*

3.2.1. Proof of Theorem 3.2.1

The following result is essential in establishing Theorem 3.2.1.

Lemma 3.2.4. *Under the assumptions of Theorem 3.2.1, the following holds true:*

(i) For a.e. $w\in\widehat{G}$*,*

$$\sum_{j\in\mathcal{J}}\frac{1}{s(\Gamma_j)}\sum_{p\in P_j}\overline{\widehat{(g_p^{(1)})}(w)}\widehat{(g_p^{(2)})}(w)=\sum_{j\in\mathcal{J}}\overline{\Phi_{j+1}(w)}\Phi_{j+1}(w)\left(\frac{1}{s(\Gamma_j)}\sum_{m=1}^{s_j}\overline{G_{j+1}^{(1)(m)}(w)}G_{j+1}^{(2)(m)}(w)\right).$$

(ii) For a.e. $w\in\widehat{G}$ *and* $\alpha\in\cup_{j\in\mathcal{J}}\Gamma_j^\perp\setminus\{0\}$*, there exists a* $J\in\mathcal{J}$ *such that*

$$\begin{aligned}\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp}\frac{1}{s(\Gamma_j)}\sum_{p\in P_j}\overline{\widehat{(g_p^{(1)})}(w)}\widehat{(g_p^{(2)})}(w+\alpha)=&\sum_{j\in\mathcal{J}:j\le J-1}\overline{\Phi_{j+1}(w)}\Phi_{j+1}(w+\alpha)\\ &\times\left(\frac{1}{s(\Gamma_j)}\sum_{m=1}^{s_j}\overline{G_{j+1}^{(1)(m)}(w)}G_{j+1}^{(2)(m)}(w)\right)\\ &+\overline{\Phi_{J+1}(w)}\Phi_{J+1}(w+\alpha)\\ &\times\left(\frac{1}{s(\Gamma_J)}\sum_{m=1}^{s_J}\overline{G_{J+1}^{(i)(m)}(w)}G_{J+1}^{(1)(m)}(w+\alpha)\right).\end{aligned}$$

Proof. (i) Since $P_j = \{(m,j) : m = 1,2,\cdots,s_j\}$, we get the following expression for a.e. $w \in \widehat{G}$,

$$\sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \overline{\widehat{(g_p^{(1)})}(w)}\widehat{(g_p^{(2)})}(w) = \sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{m=1}^{s_j} \overline{\widehat{(g_{(m,j)}^{(1)})}(w)}\widehat{(g_{(m,j)}^{(2)})}(w)$$
$$= \sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{m=1}^{s_j} \overline{(\widehat{\mathcal{F}^{-1}\Psi_j^{(1)(m)}})(w)}(\widehat{\mathcal{F}^{-1}\Psi_j^{(2)(m)}})(w),$$

by substituting the value of $g_{(m,j)}^{(i)}$ given in (2.2.5). Further, we substitute the value of $\Psi_j^{(i)(m)}$ from (2.2.4) and then we obtain the required expression as follows:

$$\sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \overline{\widehat{(g_p^{(1)})}(w)}\widehat{(g_p^{(2)})}(w) = \sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{m=1}^{s_j} \overline{G_{j+1}^{(1)(m)}(w)\Phi_{j+1}(w)}G_{j+1}^{(2)(m)}(w)\Phi_{j+1}(w)$$
$$= \sum_{j\in\mathcal{J}} \overline{\Phi_{j+1}^{(1)}(w)}\Phi_{j+1}^{(2)}(w)\left(\frac{1}{s(\Gamma_j)} \sum_{m=1}^{s_j} \overline{G_{j+1}^{(1)(m)}(w)}G_{k+1}^{(2)(m)}(w)\right).$$

(ii) For $\alpha \in \cup_{j\in\mathcal{J}}\Gamma_j^{\perp} \smallsetminus \{0\}$, there exists a $J \in \mathcal{J}$ such that $\alpha \in \Gamma_J^{\perp} \smallsetminus \Gamma_{J+1}^{\perp}$ by noting $\Gamma_j^{\perp} \supset \Gamma_{j+1}^{\perp}$ and $\cup_{j\in\mathcal{J}}\Gamma_j^{\perp} \smallsetminus \{0\} = \cup_{j\in\mathcal{J}}\left(\Gamma_j^{\perp} \smallsetminus \Gamma_{j+1}^{\perp}\right)$. Therefore, we get $\alpha \in \Gamma_j^{\perp}$ for $j \le J$, and $\alpha \notin \Gamma_j^{\perp}$ for $j > J$. Hence we can write

$$\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^{\perp}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \overline{\widehat{(g_p^{(1)})}(w)}\widehat{(g_p^{(2)})}(w+\alpha) = \sum_{j\le J} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \overline{\widehat{(g_p^{(1)})}(w)}\widehat{(g_p^{(2)})}(w+\alpha)$$
$$= \sum_{j\le J} \overline{\Phi_{j+1}^{(1)}(w)}\Phi_{j+1}^{(2)}(w+\alpha)\left(\frac{1}{s(\Gamma_j)} \sum_{m=1}^{s_j} \overline{G_{j+1}^{(1)(m)}(w)}G_{j+1}^{(2)(m)}(w+\alpha)\right)$$
$$= \sum_{j\le J-1} \overline{\Phi_{j+1}^{(1)}(w)}\Phi_{j+1}^{(2)}(w+\alpha)\left(\frac{1}{s(\Gamma_j)} \sum_{m=1}^{s_j} \overline{G_{j+1}^{(1)(m)}(w)}G_{j+1}^{(2)(m)}(w+\alpha)\right)$$
$$+ \overline{\Phi_{J+1}^{(1)}(w)}\Phi_{J+1}^{(2)}(w+\alpha)\left(\frac{1}{s(\Gamma_J)} \sum_{m=1}^{s_J} \overline{G_{J+1}^{(1)(m)}(w)}G_{J+1}^{(2)(m)}(w+\alpha)\right),$$

by following the same steps as used in the proof of first part (i). Since $G_{j+1}^{(i)(m)}$ is a $\Gamma_{j+1}^{\perp}$-periodic function and $\Gamma_J^{\perp} \subset \Gamma_{j+1}^{\perp}$ for each $j \le J-1$, we have $G_{j+1}^{(i)(m)}(w+\alpha) = G_{j+1}^{(i)(m)}(w)$, $w \in \widehat{G}$, $\alpha \in \Gamma_J^{\perp}$ and for each $j \le J-1$. Hence we get required result by using this periodicity argument in the last expression. ∎

Proof of Theorem 3.2.1. From Theorem 3.1.1 (a), we have the systems $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,p\in P_j}$ are Parseval frames for $L^2(G)$, and the systems satisfy the dual α-LIC from Theorem 2.3.3. For proving the systems are pairwise orthogonal Parseval frames, it suffices to show the following in view of Theorem 2.1.4:

$$\sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \overline{\widehat{(g_p^{(1)})}(w)}\widehat{(g_p^{(2)})}(w) = 0 \text{ for a.e. } w \in \widehat{G}, \tag{3.2.4}$$

and for $\alpha \in \cup_{j\in\mathcal{J}}\Gamma_j^{\perp} \smallsetminus \{0\}$,

$$\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \overline{\widehat{(g_p^{(1)})}(w)}\widehat{(g_p^{(2)})}(w+\alpha) = 0 \text{ for a.e. } w \in \widehat{G}. \tag{3.2.5}$$

Using Lemma 3.2.4 (i), we have

$$\sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \overline{\widehat{(g_p^{(1)})}(w)}\widehat{(g_p^{(2)})}(w) = \sum_{j\in\mathcal{J}} \overline{\Phi_{j+1}^{(1)}(w)}\Phi_{j+1}^{(2)}(w)\left(\frac{1}{s(\Gamma_j)}\sum_{m=1}^{s_j} \overline{G_{j+1}^{(1)(m)}(w)}G_{j+1}^{(2)(m)}(w)\right). \tag{3.2.6}$$

From Proposition 2.3.4 (i), we have the following for all $j \in \mathcal{J}$:

$$\sum_{m=1}^{s_j} \overline{G_{j+1}^{(1)(m)}(w+v_{j,\ell})}G_{j+1}^{(2)(m)}(w+v_{j,\ell'}) = 0 \text{ for all } 1 \le \ell, \ell' \le d_j, \tag{3.2.7}$$

due to the assumption $(\widetilde{\mathfrak{B}_j^{(1)}})^*(\gamma)\widetilde{\mathfrak{B}_j^{(2)}}(\gamma) = O_{d_j}$ for a.e. $\gamma \in \Omega_j$. Substituting $\ell = \ell' = 1$ in (3.2.7), we get $\sum_{m=1}^{s_j} \overline{G_{j+1}^{(1)(m)}(w)}G_{j+1}^{(2)(m)}(w) = 0$, since $v_{j,1} = 0$ from (2.2.2). Thus by substituting $\sum_{m=1}^{s_j} \overline{G_{j+1}^{(1)(m)}(w)}G_{j+1}^{(2)(m)}(w) = 0$ in (3.2.6), we get (3.2.4).

Next for proving (3.2.5), let $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^{\perp} \smallsetminus \{0\}$. By Lemma 3.2.4 (ii), there exists a $J \in \mathcal{J}$ such that the following expression holds for $\alpha \in \Gamma_J^{\perp} \smallsetminus \Gamma_{J+1}^{\perp}$:

$$\begin{aligned}
&\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^{\perp}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \overline{\widehat{(g_p^{(1)})}(w)}\widehat{(g_p^{(2)})}(w+\alpha) \\
&= \sum_{j\in\mathcal{J}:j\le J-1} \overline{\Phi_{j+1}(w)}\Phi_{j+1}(w+\alpha)\left(\frac{1}{s(\Gamma_j)}\sum_{m=1}^{s_j} \overline{G_{j+1}^{(1)(m)}(w)}G_{j+1}^{(2)(m)}(w)\right) \\
&\quad + \overline{\Phi_{J+1}(w)}\Phi_{J+1}(w+\alpha)\left(\frac{1}{s(\Gamma_J)}\sum_{m=1}^{s_J} \overline{G_{J+1}^{(1)(m)}(w)}G_{J+1}^{(2)(m)}(w+\alpha)\right) \\
&= \overline{\Phi_{J+1}(w)}\Phi_{J+1}(w+\alpha)\left(\frac{1}{s(\Gamma_J)}\sum_{m=1}^{s_J} \overline{G_{J+1}^{(1)(m)}(w)}G_{J+1}^{(2)(m)}(w+\alpha)\right),
\end{aligned} \tag{3.2.8}$$

in view of (3.2.7). From (2.2.2), we have $\Gamma_J^{\perp} = \bigcup_{n=1}^{d_J}(v_{J,n} + \Gamma_{J+1}^{\perp})$, $(v_{J,n} + \Gamma_{J+1}^{\perp}) \cap (v_{J,n'} + \Gamma_{J+1}^{\perp}) = \varnothing$ for $n \ne n'$, with $v_{J,1} = 0$. Then $\alpha \in \Gamma_J^{\perp} \smallsetminus \Gamma_{J+1}^{\perp}$ can be written as $\alpha = v_{J,n} + \alpha'$ for some $\alpha' \in \Gamma_{J+1}^{\perp}$, and $n \in \{2, 3, \cdots, d_J\}$. Here $n = 1$ is not possible, otherwise $\alpha = \alpha' \in \Gamma_{J+1}^{\perp}$.

Substituting $\alpha = v_{J,n} + \alpha'$ in (3.2.8), we get

$$\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^{\perp}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \overline{\widehat{(g_p^{(1)})}(w)}\widehat{(g_p^{(2)})}(w+\alpha)$$

$$= \overline{\Phi_{J+1}(w)}\Phi_{J+1}(w+\alpha)\left(\frac{1}{s(\Gamma_J)}\sum_{m=1}^{s_J}\overline{G_{J+1}^{(1)(m)}(w)}G_{J+1}^{(2)(m)}(w+v_{J,n}+\alpha')\right)$$

$$= \overline{\Phi_{J+1}(w)}\Phi_{J+1}(w+\alpha)\left(\frac{1}{s(\Gamma_J)}\sum_{m=1}^{s_J}\overline{G_{J+1}^{(1)(m)}(w)}G_{J+1}^{(2)(m)}(w+v_{J,n})\right)$$

$$= 0,$$

by noting (3.2.7) for $\ell = 1,\ \ell' = n$ and $\Gamma_{J+1}^{\perp}$-periodicity of $G_{J+1}^{(2)(m)}$. Hence (3.2.5) follows. ∎

Example 3.2.5. Recall $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,p\in P_j}$ defined in Example 2.4.1, which are Parseval frames for $L^2(\mathbb{R})$. Additionally, they are pairwise orthogonal Parseval frames from Theorem 3.2.1 since for all $j\in\mathcal{J}$,

$$(\widetilde{\mathfrak{B}_j^{(1)}})^*(\gamma)\widetilde{\mathfrak{B}_j^{(2)}}(\gamma) = O_{d_j} \text{ for a.e. } \gamma\in[0,2^j),$$

equivalently for a.e. $\gamma\in[0,2^{-j})$ and $\ell,\ell'\in\{1,2\}$,

$$\sum_{m=1}^{4}\overline{G_{j+1}^{(1)(m)}(\gamma+\nu_{j,\ell})}G_{j+1}^{(2)(m)}(\gamma+\nu_{j,\ell'}) = 0, \text{ where } \nu_{j,1}=0 \text{ and } \nu_{j,2}=2^j. \tag{3.2.9}$$

Assume first that $\ell=\ell'=1$, which implies $\nu_{j,\ell}=\nu_{j,\ell'}=0$. We now compute

$$\sum_{m=1}^{4}\overline{G_{j+1}^{(1)(m)}(\gamma+\nu_{j,\ell})}G_{j+1}^{(2)(m)}(\gamma+\nu_{j,\ell'}) = \sum_{m=1}^{4}\overline{G_{j+1}^{(1)(m)}(\gamma)}G_{j+1}^{(2)(m)}(\gamma)$$

$$=\Bigg(\frac{1}{3}\frac{\sqrt{2}}{2^3}\overline{(1-e^{-2\pi i(2^{-j-1}\gamma)})^2}(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)})$$

$$+\frac{1}{3}\frac{1}{2^3}\overline{(1-e^{-2\pi i(2^{-j-1}\gamma)})^2}(1-e^{-2\pi i(2^{-j-1}\gamma)})^2\Bigg)$$

$$+\Bigg(\frac{1}{3}\frac{\sqrt{2}}{2^3}\overline{(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)})}(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)})$$

$$-\frac{1}{3}\frac{1}{2^3}\overline{(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)})}(1-e^{-2\pi i(2^{-j-1}\gamma)})^2\Bigg)$$

$$+\Bigg(\frac{1}{3}\frac{\sqrt{2}}{2^3}(1-e^{-2\pi i(2^{-j-1}\gamma)})^2\overline{(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)})}$$

$$-\frac{1}{3}\frac{1}{2^3}(1-e^{-2\pi i(2^{-j-1}\gamma)})^2\overline{(1-e^{-2\pi i(2^{-j-1}\gamma)})^2}\Bigg)$$

$$+\Bigg(-\frac{1}{3}\frac{\sqrt{2}}{2^3}(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)})\overline{(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)})}$$

$$- \frac{1}{3}\frac{\sqrt{2}}{2^3}(1+e^{-2\pi i(2^{-j-1}\gamma)})(1-e^{-2\pi i(2^{-j-1}\gamma)})\overline{(1-e^{-2\pi i(2^{-j-1}\gamma)})^2}\Big)$$

$$=0.$$

Similarly, we can show that (3.2.9) is true for the other values of ℓ and ℓ'.

3.2.2. Proof of Theorem 3.2.2

Proof of Theorem 3.2.2. First, we use Theorem 3.1.1 to prove that the systems $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,p\in P_j'}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(3)}\}_{\lambda\in\Gamma_j,p\in P_j'}$ are Parseval frames for $L^2(G)$. For this, it is sufficient to show that for each $j\in\mathcal{J}$ and $i\in\{2,3\}$, we have

$$(\mathfrak{B}_j^{(i)}(\gamma))^*\mathfrak{B}_j^{(i)}(\gamma) = \frac{s(\Gamma_j)}{s(\Gamma_{j+1})}I_{d_j} \text{ for a.e. } \gamma\in\Omega_j,$$

where the matrix $\mathfrak{B}_j^{(i)}(\gamma) := \Big(G_{j+1}^{(i)(m)}(\gamma+v_{j,n})\Big)_{\substack{0\le m\le 2s_j\\ 1\le n\le d_j}}$ (defined similar to (2.2.7)) is of order $(1+2s_j)\times d_j$ since $G_{j+1}^{(i)(0)} = H_{j+1}$ and for $1\le m\le 2s_j$, $G_{j+1}^{(i)(m)}$ is defined in (3.2.2). Note that the matrix $\mathfrak{B}_j^{(1)}(\gamma)$ is of order $(s_j+1)\times d_j$. For $i\in\{2,3\}$, the matrix $\mathfrak{B}_j^{(i)}(\gamma)$ can be expressed as

$$\mathfrak{B}_j^{(i)}(\gamma) = A_j^{(i)}(\gamma)\mathfrak{B}_j^{(1)}(\gamma),$$

where

$$A_j^{(i)}(\gamma) = \begin{pmatrix} 1 & 0 & 0 & \cdot\ \cdot\ \cdot & 0 \\ 0 & K^j_{1+(i-2)s_j}(\gamma) & K^j_{2+(i-2)s_j}(\gamma) & \cdot\ \cdot\ \cdot & K^j_{s_j+(i-2)s_j}(\gamma) \end{pmatrix},$$

since the entries of $K_i^j(\gamma)$ are $\Gamma_j^\perp$-periodic. The columns $K_1^j(\gamma), K_2^j(\gamma)\cdots, K_{2s_j}^j(\gamma)$ are orthonormal implies $(A_j^{(i)}(\gamma))^*A_j^{(i)}(\gamma) = I_{s_j+1}$. Using (2.3.1), we obtain

$$\begin{aligned}(\mathfrak{B}_j^{(i)}(\gamma))^*\mathfrak{B}_j^{(i)}(\gamma) &= (\mathfrak{B}_j^{(1)}(\gamma))^*(A_j^{(i)}(\gamma))^*A_j^{(i)}(\gamma)\mathfrak{B}_j^{(1)}(\gamma)\\ &= (\mathfrak{B}_j^{(1)}(\gamma))^*I_{s_j+1}\mathfrak{B}_j^{(1)}(\gamma)\\ &= (\mathfrak{B}_j^{(1)}(\gamma))^*\mathfrak{B}_j^{(1)}(\gamma) = \frac{s(\Gamma_j)}{s(\Gamma_{j+1})}I_{d_j}.\end{aligned}$$

Therefore for each $i\in\{2,3\}$, the GTI system $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j,p\in P_j'}$ is a Parseval frame. Now, it remains to show that the GTI systems $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,p\in P_j'}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(3)}\}_{\lambda\in\Gamma_j,p\in P_j'}$ are pairwise orthogonal Parseval frames. For this it suffices to prove

$$(\widetilde{\mathfrak{B}_j^{(2)}})^*(\gamma)\widetilde{\mathfrak{B}_j^{(3)}}(\gamma) = O_{d_j} \text{ for a.e. } \gamma\in\Omega_j \text{ and for each } j\in\mathcal{J},$$

in view of Theorem 3.2.1, where for $i \in \{2,3\}$, the matrix $\widetilde{\mathfrak{B}_j^{(i)}}(\gamma) := \left(G_{j+1}^{(i)(m)}(\gamma + v_{j,n})\right)_{\substack{1 \le m \le 2s_j \\ 1 \le n \le d_j}}$ is of order $2s_j \times d_j$. Note that the matrix $\widetilde{\mathfrak{B}_j^{(1)}}(\gamma)$ is of order $s_j \times d_j$. Using the periodicity of entries of $K^j(\gamma)$, we can express

$$\widetilde{\mathfrak{B}_j^{(2)}}(\gamma) = \left(K_1^j(\gamma)\ K_2^j(\gamma),\ \cdots,\ K_{s_j}^j(\gamma)\right)\widetilde{\mathfrak{B}_j^{(1)}}(\gamma)$$

and

$$\widetilde{\mathfrak{B}_j^{(3)}}(\gamma) = \left(K_{s_j+1}^j(\gamma)\ K_{s_j+2}^j(\gamma),\ \cdots,\ K_{2s_j}^j(\gamma)\right)\widetilde{\mathfrak{B}_j^{(1)}}(\gamma),$$

and hence, we get

$$\begin{aligned}
(\widetilde{\mathfrak{B}_j^{(2)}})^*(\gamma)\widetilde{\mathfrak{B}_j^{(3)}}(\gamma) =& \left(\left(K_1^j(\gamma)\ K_2^j(\gamma),\ \cdots,\ K_{s_j}^j(\gamma)\right)\widetilde{\mathfrak{B}_j^{(1)}}(\gamma)\right)^* \\
&\times \left(K_{s_j+1}^j(\gamma)\ K_{s_j+2}^j(\gamma),\ \cdots,\ K_{2s_j}^j(\gamma)\right)\widetilde{\mathfrak{B}_j^{(1)}}(\gamma) \\
=&(\widetilde{\mathfrak{B}_j^{(1)}})^*(\gamma)\left(K_1^j(\gamma)\ K_2^j(\gamma),\ \cdots,\ K_{s_j}^j(\gamma)\right)^* \\
&\times \left(K_{s_j+1}^j(\gamma)\ K_{s_j+2}^j(\gamma),\ \cdots,\ K_{2s_j}^j(\gamma)\right)\widetilde{\mathfrak{B}_j^{(1)}}(\gamma) \\
=&O_{d_j},
\end{aligned}$$

since $\left(K_1^j(\gamma)\ K_2^j(\gamma),\ \cdots,\ K_{s_j}^j(\gamma)\right)^*\left(K_{s_j+1}^j(\gamma)\ K_{s_j+2}^j(\gamma),\ \cdots,\ K_{2s_j}^j(\gamma)\right) = O_{s_j}$. Thus the result follows. ∎

Remark 3.2.6. From the application point of view, Theorem 3.2.2 can function as an algorithm that generates pairwise orthogonal Parseval frames. The input for this algorithm requires one Parseval frame of GTI structure, and $K^j(\gamma)$ is a $2s_j \times 2s_j$ unitary matrix with $\Gamma_j^{\perp}$-periodic entries. The output is a pair of Parseval frames of GTI structure, which are orthogonal.

Next we apply the Theorem 3.2.2 in Example 2.4.1 to produce pairwise orthogonal Parseval frames.

Example 3.2.7. Recall the GTI system $\bigcup_{j \in \mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda \in \Gamma_j, p \in P_j}$ as defined in Example 2.4.1, which is a Parseval frame for $L^2(\mathbb{R})$. Thus, in order to use Theorem 3.2.2, we have all the necessary input data except the matrix $K_j(\gamma)$. For each $j \in \mathcal{J}$, we can choose $K_j(\gamma)$ as any orthogonal matrix of order 8×8 with entries from $\mathbb{R}$. Since all entries of $K_j(\gamma)$ are independent of γ, they are $\Gamma_j^{\perp}$-periodic. Then the systems $\cup_{j \in \mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda \in \Gamma_j, p \in P_j'}$ and $\cup_{j \in \mathcal{J}}\{T_\lambda g_p^{(3)}\}_{\lambda \in \Gamma_j, p \in P_j'}$ (as defined in (3.2.1)) are pairwise orthogonal Parseval frames in $L^2(\mathbb{R})$, where $P_j' = \{(m,j) : m = 1, 2, \cdots, 8\}$, in view of Theorem 3.2.2.

3.2.3. Proof of Theorem 3.2.3

Proof of Theorem 3.2.3. By Theorem 3.1.1, the GTI system $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j,p\in P_j''}$ forms a Parseval frame for $L^2(G)$ if we get

$$(\mathfrak{B}_j^{(i)})^*(\gamma)\mathfrak{B}_j^{(i)}(\gamma)=\frac{s(\Gamma_j)}{s(\Gamma_{j+1})}I_{d_j}\ \text{ for a.e. } \gamma\in\Omega_j.$$

Equivalently, from (2.2.7) we have

$$\overline{G_{j+1}^{(i)(0)}(\gamma+\nu_{j,\ell})}G_{j+1}^{(i)(0)}(\gamma+\nu_{j,\ell'})+\sum_{m=1}^{N}\overline{G_{j+1}^{(i)(m)}(\gamma+\nu_{j,\ell})}G_{j+1}^{(i)(m)}(\gamma+\nu_{j,\ell'})=\frac{s(\Gamma_j)}{s(\Gamma_{j+1})}\delta_{\ell,\ell'}$$

for a.e. $\gamma\in\Omega_j$ and $\ell,\ell'=1,2,\dots,d_j$. Using the definition of $G_{j+1}^{(i)(m)}$, the above expression becomes

$$\overline{H_{j+1}(\gamma+\nu_{j,\ell})}H_{j+1}(\gamma+\nu_{j,\ell'})+\sum_{m=1}^{N}\overline{U_{m,i}^{j}(\gamma+\nu_{j,\ell})G_{j+1}^{(1)(1)}(\gamma+\nu_{j,\ell})}U_{m,i}^{j}(\gamma+\nu_{j,\ell'})G_{j+1}^{(1)(1)}(\gamma+\nu_{j,\ell'})$$
$$=\frac{s(\Gamma_j)}{s(\Gamma_{j+1})}\delta_{\ell,\ell'}.$$

The periodicity of the entries of the matrix $\left(U_{m,n}^{j}(\gamma)\right)_{1\le m,\ n\le N}$ gives

$$\overline{H_{j+1}(\gamma+\nu_{j,\ell})}H_{j+1}(\gamma+\nu_{j,\ell'})+\sum_{m=1}^{N}|U_{m,i}^{j}(\gamma)|^2\overline{G_{j+1}^{(1)(1)}(\gamma+\nu_{j,\ell})}G_{j+1}^{(1)(1)}(\gamma+\nu_{j,\ell'})=\frac{s(\Gamma_j)}{s(\Gamma_{j+1})}\delta_{\ell,\ell'}.$$

Since the norm of the vector columns of matrix $\left(U_{m,n}^{j}(\gamma)\right)_{1\le m,\ n\le N}$ is 1, we get

$$\overline{H_{j+1}(\gamma+\nu_{j,\ell})}H_{j+1}(\gamma+\nu_{j,\ell'})+\overline{G_{j+1}^{(1)(1)}(\gamma+\nu_{j,\ell})}G_{j+1}^{(1)(1)}(\gamma+\nu_{j,\ell'})=\frac{s(\Gamma_j)}{s(\Gamma_{j+1})}\delta_{\ell,\ell'}.$$

It is true in view Theorem 3.1.1 since the system $\cup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ is a Parseval frame with $s_j=1$. Thus the system $\cup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j,p\in P_j''}$ forms a Parseval frame for $L^2(G)$.

Now it remains to show that $\cup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j,p\in P_j''}$ and $\cup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i')}\}_{\lambda\in\Gamma_j,p\in P_j''}$ are pairwise orthogonal. To demonstrate this, according to Theorem 3.2.1, it suffices to prove that for each $j\in\mathcal{J}$, and $i,i'\in\{2,3,\cdots,N+1\}$,

$$(\widetilde{\mathfrak{B}_j^{(i)}})^*(\gamma)\widetilde{\mathfrak{B}_j^{(i')}}(\gamma)=O_{d_j},\ \text{ a.e. } \gamma\in\Omega_j \text{ and } i\neq i',$$

where the matrix $\widetilde{\mathfrak{B}_j^{(i)}}(\gamma) = \left(G_{j+1}^{(i)(m)}(\gamma + v_{j,n})\right)_{\substack{1\le m\le N\\1\le n\le d_j}}$ is of order $N \times d_j$, by (3.2.3). Using the periodicity of $U_{m,n}^j(\gamma)$, the matrix $\widetilde{\mathfrak{B}_j^{(i)}}(\gamma)$ can be expressed as

$$\widetilde{\mathfrak{B}_j^{(i)}}(\gamma) = \left(G_{j+1}^{(i)(m)}(\gamma + v_{j,n})\right)_{\substack{1\le m\le s_j\\1\le n\le d_j}} = \begin{pmatrix} U_{1,i}^j(\gamma) \\ U_{2,i}^j(\gamma) \\ \vdots \\ U_{N,i}^j(\gamma) \end{pmatrix} \left(G_{j+1}^{(1)(1)}(\gamma+\nu_{j,1}) \ldots G_{j+1}^{(1)(1)}(\gamma+\nu_{j,d_j})\right),$$

and hence, we get

$$\begin{aligned}(\widetilde{\mathfrak{B}_j^{(i)}}))^*(\gamma)\widetilde{\mathfrak{B}_j^{(i')}}(\gamma) =& \begin{pmatrix} \overline{G_{j+1}^{(1)(1)}(\gamma+\nu_{j,1})} \\ \vdots \\ \overline{G_{j+1}^{(1)(1)}(\gamma+\nu_{j,d_j})} \end{pmatrix} \begin{pmatrix} U_{1,i}^j(\gamma) \\ U_{2,i}^j(\gamma) \\ \vdots \\ U_{N,i}^j(\gamma) \end{pmatrix}^* \begin{pmatrix} U_{1,i'}^j(\gamma) \\ U_{2,i'}^j(\gamma) \\ \vdots \\ U_{N,i'}^j(\gamma) \end{pmatrix} \\ &\times \left(G_{j+1}^{(1)}(\gamma+\nu_{j,1}) \ldots G_{j+1}^{(1)}(\gamma+\nu_{j,d_j})\right) \\ =& O_{d_j},\end{aligned}$$

by using orthogonality of columns of $\left(U_{m,n}^j(\gamma)\right)_{1\le m,n\le N}$. Thus the result follows. ∎

Remark 3.2.8. One limitation of Theorem 3.2 is its ability to generate only two Parseval frames at a time. However, having multiple orthogonal frames becomes crucial in scenarios such as multiple access communication channels, where several users need to share a single channel (e.g., radio or television broadcasting). To address this limitation, we can propose an algorithm utilizing Theorem 3.3. This algorithm constructs N Parseval frames that are pairwise orthogonal (where N is a natural number) by providing inputs of one Parseval frame and a suitable unitary matrix.

3.3. Applications

The purpose of this section is to explore the applications of our main results (that is Theorem 3.2.1, Theorem 3.2.2 and Theorem 3.2.3 in Section 3.2). We discuss a pairwise orthogonal Parseval frames in $L^2(G)$, using B-spline as a generating function. Then, we observe that the GTI system $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j, p\in P_j}$ (defined in (2.2.6)) is equivalent to a wavelet system when specific window functions are chosen. Further by using Theorem 3.2.1, we provide sufficient conditions to obtain pairwise orthogonal Parseval wavelet frames in $L^2(\mathbb{R}^n)$.

3.3.1. Pairwise orthogonal Parseval frames generated by B-spline

Consider a sequence of nested lattices $\cdots \subset \Gamma_j \subset \Gamma_{j+1} \subset \cdots$ in G satisfying $\cap_{j\in\mathbb{Z}}\Gamma_j^{\perp} = \{0\}$, and $|\Gamma_{j+1}/\Gamma_j| = 2$. Let Q_j and Ω_j be fundamental domain associated with lattices Γ_j and $\Gamma_j^{\perp}$, respectively. For $N \in \mathbb{N}$, we define the B-spline of N-th order at level j by the N-fold convolution,

$$B_{j,N}(x) := \frac{1}{(\mu_G(Q_j))^{N-\frac{1}{2}}}\chi_{Q_j} * \chi_{Q_j} * \cdots * \chi_{Q_j}(x),\ x \in G.$$

Consider the functions $\Phi_{j,N},\ j \in \mathbb{Z}$, defined by

$$\Phi_{j,N}(\gamma) := \widehat{B_{j,N}}(\gamma) = \frac{1}{(\mu_G(Q_j))^{N-\frac{1}{2}}}\Big(\int_{Q_j}(-x,\gamma)dx\Big)^N,\ \gamma \in \widehat{G},$$

which satisfy $\Phi_{j,N}(\gamma) = H_{j+1}(\gamma)\Phi_{j+1,N}(\gamma)$, a.e. $\gamma \in \widehat{G}$, where $H_{j+1} \in L^\infty(\Omega_{j+1})$ is given by

$$H_{j+1}(\gamma) = \frac{1}{2^{N-\frac{1}{2}}}\left[1 + (-\eta_j,\gamma)\right]^N,\ \gamma \in \widehat{G} \text{ and } \eta_j \in \Gamma_{j+1} \smallsetminus \Gamma_j.$$

Assume that for every $\epsilon \in (0,1)$ and any compact set S in $\widehat{G} \smallsetminus \mathcal{B}$ there exists $j \in \mathbb{Z}$ such that

$$|(-x,\gamma) - 1| \le \epsilon,\ \forall\ x \in Q_j \text{ and } \gamma \in S. \tag{3.3.1}$$

Then assumption $(\mathcal{N}_1)$ is true for proof one can see the proof of Lemma 4.1 (ii) in [35]. If $\mu_G(Q_j) \to \infty$ as $j \to -\infty$, then it is easy to show that the assumption $(\mathcal{N}_2)$ holds. We assume that the assumption $(\mathcal{N}_3^*)$ holds.

In case of $N = 1$, i.e. the B-spline of order 1, we define the function $G_{j+1}^{(1)(1)} \in L^\infty(\Omega_{j+1})$ by

$$G_{j+1}^{(1)(1)}(\gamma) = \frac{1}{\sqrt{2}}\left[1 - (-\eta_j,\gamma)\right],\ \gamma \in \widehat{G} \text{ and } \eta_j \in \Gamma_{j+1} \smallsetminus \Gamma_j.$$

Then $(\mathfrak{B}_j^{(1)}(\gamma))^*\mathfrak{B}_j^{(1)}(\gamma) = \frac{s(\Gamma_j)}{s(\Gamma_{j+1})}I_2$ for a.e. $\gamma \in \Omega_j$ (for proof, see, Proposition 4.6 (i) in [35]). Hence by Theorem 3.1.1, $\cup_{j\in\mathbb{Z}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ is a Parseval frame. Further employing Theorem 3.2.3, we construct pairwise orthogonal Parseval frames. Consider a 4×4 unitary matrix $U^j(\gamma)$ with entries from $\mathbb{C}$.

For each $i \in \{2,3,4,5\},\ j \in \mathbb{Z}$ and $m \in \{0,1,2,3,4\}$, we define $G_j^{(i)(m)}$ (same as (3.2.3)) as follows :

$$\begin{pmatrix} G_{j+1}^{(i)(1)}(\gamma) \\ G_{j+1}^{(i)(2)}(\gamma) \\ G_{j+1}^{(i)(3)}(\gamma) \\ G_{j+1}^{(i)(4)}(\gamma) \end{pmatrix} = G_{j+1}^{(1)(1)}(\gamma)\begin{pmatrix} U_{i-1,1}^j(\gamma) \\ U_{i-1,2}^j(\gamma) \\ U_{i-1,3}^j(\gamma) \\ U_{i-1,4}^j(\gamma) \end{pmatrix} \text{ and } G_{j+1}^{(i)(0)} = H_{j+1}.$$

Using Theorem 3.2.3 we conclude that the systems $\bigcup_{j\in\mathbb{Z}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j,p\in P''_j}$ and $\bigcup_{j\in\mathbb{Z}}\{T_\lambda g_p^{(i')}\}_{\lambda\in\Gamma_j,p\in P''_j}$ (defined in (3.2.1)) are pairwise orthogonal Parseval frames for each $i,i'\in\{2,3,4,5\}$ and $i\neq i'$.

For the case $N=2M$ $(M\in\mathbb{N})$, i.e. B-spline of order $N=2M$, define the filter

$$G_{j+1}^{(1)(m)}(\gamma)=\frac{1}{2^{2M}}\sqrt{\binom{2M}{m}}\left[1+(-\eta_j,\gamma)\right]^{2M-m}\left[1-(-\eta_j,\gamma)\right]^{m},\ \gamma\in\widehat{G}\text{ and }\eta_j\in\Gamma_{j+1}\setminus\Gamma_j,$$

for each $j\in\mathbb{Z}$ and $m\in\{1,2,\cdots 2M\}$. Then $(\mathfrak{B}_j^{(1)}(\gamma))^*\mathfrak{B}_j^{(1)}(\gamma)=\frac{s(\Gamma_j)}{s(\Gamma_{j+1})}I_2$ for a.e. $\gamma\in\Omega_j$ (for proof, see, Proposition 4.6 (ii) in [35]). Hence $\bigcup_{j\in\mathbb{Z}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ is a Parseval frame by Theorem 3.1.1.

Fix $M=2$ and choose any unitary matrix $K^j(\gamma)$ of order 8×8 with entries from $\mathbb{C}$ and for $i\in\{2,3\}$,

$$\begin{pmatrix}G_{j+1}^{(i)(1)}(\gamma)\\ G_{j+1}^{(i)(2)}(\gamma)\\ \vdots\\ G_{j+1}^{(i)(8)}(\gamma)\end{pmatrix}=\left(K^j_{1+(i-2)4}(\gamma)\,K^j_{2+(i-2)4}(\gamma)\cdots K^j_{4+(i-2)4}(\gamma)\right)\begin{pmatrix}G_{j+1}^{(1)(1)}(\gamma)\\ G_{j+1}^{(1)(2)}(\gamma)\\ G_{j+1}^{(1)(3)}(\gamma)\\ G_{j+1}^{(1)(4)}(\gamma)\end{pmatrix},\ \gamma\in\widehat{G}.$$

Then the GTI systems $\bigcup_{j\in\mathbb{Z}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,p\in P_j}$ and $\bigcup_{j\in\mathbb{Z}}\{T_\lambda g_p^{(3)}\}_{\lambda\in\Gamma_j,p\in P_j}$ (defined as in (3.2.2)) are pairwise orthogonal Parseval frames using Theorem 3.2.2.

In particular, for the case $G=\mathbb{R}^d$, in the below example, we demonstrate the fulfilment of all technical assumptions in this context to construct pairwise orthogonal Parseval frames for $L^2(\mathbb{R}^d)$ using B-spline.

Example 3.3.1. Let $G=\mathbb{R}^d$, with dual group $\widehat{G}=\mathbb{R}^d$. Let A be a $d\times d$ matrix with integer entries, eigenvalues outside the unit circle and $|\det A|=2$. For $j\in\mathbb{Z}$, consider the lattice $\Gamma_j=(A^{\#})^j\mathbb{Z}^d$ in $\mathbb{R}^d$, where $A^{\#}:=(A^T)^{-1}$, i.e the inverse of transpose of A. Since the matrix A has integer entries, $A^T\mathbb{Z}^d\subset\mathbb{Z}^d$, which implies that $\mathbb{Z}^d\subset A^{\#}\mathbb{Z}^d$, and therefore $\Gamma_j=(A^{\#})^j\mathbb{Z}^d\subset(A^{\#})^{j+1}\mathbb{Z}^d=\Gamma_{j+1}$ for all $j\in\mathbb{Z}$. The annihilator $\Gamma_j^\perp$ of Γ_j is $\Gamma_j^\perp=A^j\mathbb{Z}^d$. Further, we can see easily $|\Gamma_{j+1}/\Gamma_j|=|(A^{\#})^{j+1}\mathbb{Z}^d/(A^{\#})^j\mathbb{Z}^d|=2$ and $|\Gamma_j^\perp/\Gamma_{j+1}^\perp|=|A^j\mathbb{Z}^d/A^{j+1}\mathbb{Z}^d|=2$ for all $j\in\mathbb{Z}$. Also note that as the eigenvalues of A are outside the unique circle, it follows $\bigcap_{j\in\mathbb{Z}}\Gamma_j^\perp=\{0\}$.

Next, we consider the fundamental domain $Q_j=(A^{\#})^j[0,1)^d$ associated with the lattice $\Gamma_j=(A^{\#})^j\mathbb{Z}^d$. Then the Lebesgue measure $\mu_{\mathbb{R}^d}(Q_j)=\mu_{\mathbb{R}^d}((A^{\#})^j[0,1)^d)=|\det(A^{\#})^j|=2^{-j}\to\infty$ as $j\to-\infty$. To verify the condition (3.3.1), assume a compact set $S\subset\mathbb{R}^d\setminus\mathcal{B}$, where the Borel set $\mathcal{B} = \{0\}$. For $j \in \mathbb{Z}$ and $x \in Q_j$, we express $x = (A^{\#})^j q$ for some $q \in [0,1)^d$. Then

$$|(-x,\gamma) - 1| = |e^{-2\pi i (A^{\#})^j q.\gamma} - 1|, \gamma \in S. \tag{3.3.2}$$

Since S is compact, there exists a $C > 0$ such that $\|\gamma\|_2 \le C$ and hence we estimate the following due to $\|q\|_2 < 1$:

$$|-2\pi i (A^{\#})^j q.\gamma| \le 2\pi \|(A^{\#})^j q\|_2 \|\gamma\|_2 \le 2\pi \|(A^{\#})^j\|_2 \|q\|_2 \|\gamma\|_2 \le 2\pi C \|(A^{\#})^j\|_2.$$

We can find a suitable norm $\|\cdot\|$ such that $\|(A^T)^{-1}\| < 1$ as the eigenvalues of matrix A are outside the unit circle. As all norms on a finite-dimensional space are equivalent, there exists a constant $C_1 > 0$ such that

$$\|(A^{\#})^j\|_2 = \|(A^T)^{-j}\|_2 \le C_1 \|(A^T)^{-j}\| \le C_1 \|(A^T)^{-1}\|^j,$$

and hence $\|(A^{\#})^j\|_2 \to 0$ as $j \to \infty$ since $\|(A^T)^{-1}\|^j \to 0$ as $j \to \infty$. Thus (3.3.1) follows by observing above arguments.

For the condition $(\mathcal{N}_3^*)$, consider the compact set $S \subset \mathbb{R}^d \setminus \mathcal{B}$, with the Borel set $\mathcal{B} = \{0\}$. Since the set $\cup_{j\in\mathbb{Z}}(\Gamma_j^{\perp} \cap \mathcal{B}) = \cup_{j\in\mathbb{Z}}(A^j\mathbb{Z}^d \cap \{0\}) = \{0\}$, it follows that

$$\sum_{\alpha \in \cup_{j\in\mathbb{Z}}(A^j\mathbb{Z}^d \cap \{0\})} \mu_{\mathbb{R}^d}(S \cap (S - \alpha)) = \mu_{\mathbb{R}^d}(S) < \infty.$$

Next, we show that

$$L_1 = \sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \sum_{p\in P_j} \sum_{\alpha\in(A^j\mathbb{Z}^d\setminus\{0\})} \int_{S\cap(S-\alpha)} |\widehat{g_p^{(1)}}(\gamma)|^2 \, d\mu_{\mathbb{R}^d}(\gamma) < \infty,$$

where S is any compact set in $\mathbb{R}^d \setminus \{0\}$. Choose a suitable $t \in (1,\infty)$ such that

$$S \subset \{x \in \mathbb{R}^n \setminus \{0\} : \frac{1}{t} < \|x\|_2 < t\} := S(t).$$

By substituting the values of $P_j = \{(m,j) : m = 1,2\cdots 2M\}$ and $\widehat{g_p^{(1)}} = \Psi_j^{(1)(m)}(\gamma) := G_{j+1}^{(1)(m)}(\gamma)\Phi_{j+1}(\gamma)$ in the above expression, we get

$$\begin{aligned} L_1 &\le \sum_{j\in\mathbb{Z}} \sum_{m=1}^{2M} \sum_{k\in(\mathbb{Z}^n\setminus\{0\})} \int_{S(t)\cap(S(t)-A^jk)} |\Psi_0^{(1)(m)}(A^{-j}\gamma)|^2 d\mu_{\mathbb{R}^d}(\gamma) \\ &= \sum_{j\in\mathbb{Z}} \sum_{m=1}^{2M} \sum_{k\in(\mathbb{Z}^n\setminus\{0\})} \int_{A^j\xi\in[S(t)\cap(S(t)-A^jk)]} |\Psi_0^{(1)(m)}(\xi)|^2 |\det A|^j d\mu_{\mathbb{R}^d}(\xi) \end{aligned}$$

which is finite using the same steps as outlined in the proof of Proposition 5.12 in [73]. Therefore $(\mathcal{N}_3^*)$ holds. Hence, $\bigcup_{j\in\mathbb{Z}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ is a Parseval frame for $L^2(\mathbb{R}^d)$ by Theorem 3.1.1. If we choose any unitary matrix $K^j(\gamma)$ of order $4M \times 4M$ with entries from

$\mathbb{C}$, then by Theorem 3.2.2 the GTI systems $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j'}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(3)}\}_{\lambda\in\Gamma_j, p\in P_j'}$ (defined in (3.2.1)) are pairwise orthogonal Parseval frames generated by B-splines, where $P_j' = \{(m,j) : m = 1,2,\cdots,4M\}$.

3.3.2. Pairwise orthogonal Parseval wavelet frames in $L^2(\mathbb{R}^n)$

Let $A \in GL_n(\mathbb{Z})$ be a matrix having all its eigenvalues outside the unit circle. We define the *dilation matrix* $D_A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$D_A f(\gamma) = |\det A|^{1/2} f(A\gamma) \text{ for all } \gamma \in \mathbb{R}^n \text{ and } f \in L^2(\mathbb{R}^n).$$

The system $\cup_{j\in\mathbb{Z}}\{D_A^j T_\lambda \psi^{(m)}\}_{\lambda\in\mathbb{Z}^n, m=1,\dots,s}$ is known as *wavelet system* [33], where $\psi^{(m)} \in L^2(\mathbb{R}^n)$. Since $D_A^j T_\lambda \psi^{(m)} = T_{A^{-j}\lambda} D_A^j \psi^{(m)}$, the wavelet system can be written in the form of GTI system $\cup_{j\in\mathbb{Z}}\{T_\lambda D_A^j \psi^{(m)}\}_{\lambda\in A^{-j}\mathbb{Z}^n, m=1,\dots,s}$. In this case, $\Gamma_j := A^{-j}\mathbb{Z}^n$, $s(\Gamma_j) = |\det A|^{-j}$ and $\Gamma_j^\perp = B^j\mathbb{Z}^n$, where $B = A^T$ (transpose of A). Since $\Gamma_j^\perp \supset \Gamma_{j+1}^\perp$ and $|A^{-j-1}\mathbb{Z}^n / A^{-j}\mathbb{Z}^n| = |B^j\mathbb{Z}^n / B^{j+1}\mathbb{Z}^n| = |\det(A)|$, we have a sequence $\{\nu_{j,\ell}\}_{\ell=1,2,\dots,|\det(A)|}$ satisfying $\Gamma_j^\perp = \cup_{\ell=1}^a (\nu_{j,\ell} + \Gamma_{j+1}^\perp)$, $(\nu_{j,\ell} + \Gamma_{j+1}^\perp) \cap (\nu_{j,\ell'} + \Gamma_{j+1}^\perp) = \varnothing$ for $\ell \neq \ell'$, $a = |\det(A)|$.

Let $\Phi_j := \widehat{D_A^j \varphi}$, $\varphi \in L^2(\mathbb{R}^n)$ satisfying $\widehat{\varphi}(\gamma) \to 1$ as $\gamma \to 0$. Consider the scaling relation

$$\Phi_{-1}(\gamma) = H_0(\gamma)\Phi_0(\gamma) \text{ for a.e. } \gamma \in \mathbb{R}^n \text{ and } H_0 \in L^\infty([0,1)^n). \tag{3.3.3}$$

This equation means that $(\widehat{D_A^{-1}\varphi})(\gamma) = H_0(\gamma)\widehat{\varphi}(\gamma)$, i.e. $|\det B|^{1/2}\widehat{\varphi}(B\gamma) = H_0(\gamma)\widehat{(\varphi)}(\gamma)$ for $\gamma \in \mathbb{R}^n$. It follows that for any $j \in \mathbb{Z}$, we have $|\det B|^{1/2}\widehat{\varphi}(B^{-j}\gamma) = H_0(B^{-j-1}\gamma)\widehat{(\varphi)}(B^{-j-1}\gamma)$, or

$$\Phi_j(\gamma) = H_0(B^{-j-1}\gamma)\Phi_{j+1}(\gamma) \quad \text{for} \quad \gamma \in \mathbb{R}^n.$$

That is, the scaling equation (3.3.3) implies the refinement equation $\Phi_j(\gamma) = H_{j+1}(\gamma)\Phi_{j+1}(\gamma)$ for all levels, with $H_{j+1}(\gamma) = H_0(B^{-j-1}\gamma)$. The assumptions $(\mathcal{N}_1)$ and $(\mathcal{N}_2)$ can be proved using the following calculation

$$\frac{1}{s(\Gamma_j)}|\Phi_j(\gamma)|^2 = \frac{1}{|\det A|^{-j}}\left||\det B|^{-j/2}\widehat{\varphi}(B^{-j}\gamma)\right|^2 = |\varphi(B^{-j}\gamma)|^2.$$

For $i = 1,2$ and $m = 1,2,\dots,s$, choose the functions $G_0^{(m)(i)} \in L^\infty(\Omega_0)$, where $s \geq a-1$ and $\Omega_j = B^j([0,1])^n$ is a fundamental domain associated with lattice $\Gamma_j^\perp = B^j\mathbb{Z}^n$.

Define the functions $G_{j+1}^{(m)(i)}(\gamma) = G_0^{(m)(i)}(B^{-j-1}\gamma)$, $\gamma \in \mathbb{R}^n$ and $\Psi_j^{(i)(m)}(\gamma) = G_{j+1}^{(i)(m)}(\gamma)\Phi_{j+1}(\gamma)$. By considering $g_{(m,j)}^{(i)} = \mathcal{F}^{-1}\Psi_j^{(i)(m)} = D_A^j g_{(m,0)}^{(i)}$, the system

$\cup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j, p\in P_j}$ (defined in (2.2.6)) becomes

$$\bigcup_{j\in\mathbb{Z}}\{T_\lambda D_A^j g_{(m,0)}^{(i)}\}_{\lambda\in A^{-j}\mathbb{Z}^n, m=1,\ldots,s} = \bigcup_{j\in\mathbb{Z}}\{D_A^j T_\lambda g_{(m,0)}^{(i)}\}_{\lambda\in\mathbb{Z}^n, m=1,\ldots,s},$$

which is a wavelet system for $i = 1, 2$. Further we observe $H_{j+1}(\gamma + \nu_{j,\ell}) = H_0(B^{-j-1}(\gamma + \nu_{j,\ell})) = H_0(B^{-j-1}\gamma + B^{-j-1}\nu_{-1,\ell})$ and, similarly for $G_j^{(m)(i)}, m = 1, 2, \ldots, s$. Then we conclude $\mathfrak{B}_j^{(i)}(\gamma) = \mathfrak{B}_{-1}^{(i)}(B^{-j-1}\gamma)$ and $\widetilde{\mathfrak{B}_j^{(i)}}(\gamma) = \widetilde{\mathfrak{B}_{-1}^{(i)}}(B^{-j-1}\gamma)$, for a.e. $\gamma \in \Omega_j = B^j([0,1])^n$. If we assume $(\mathfrak{B}_{-1}^{(i)}(\gamma))^*\mathfrak{B}_{-1}^{(i)}(\gamma) = |\det A| I_{|\det A|}$ for a.e. $\gamma \in B^{-1}([0,1])^n$, then Calderón sum of the system $\cup_{j\in\mathbb{Z}}\{D_A^j T_\lambda g_{(m,0)}^{(i)}\}_{\lambda\in\mathbb{Z}^n, m=1,\ldots,s}$ is 1 (see, Lemma 2.3.6). Furthermore, conditions $(\mathcal{N}_3^*)$ is true by following the same steps as in Example 3.3.1. Additionally, it is easy to note that $\bigcap_{j\in\mathbb{Z}} B^j\mathbb{Z}^n = \{0\}$. Hence, we deduce the following result for wavelet systems using the above arguments and Theorem 3.2.1:

Corollary 3.3.2. *Let $A \in GL_n(\mathbb{Z})$ be a matrix such that its all eigenvalues are outside the unit circle and $\varphi \in L^2(\mathbb{R}^n)$ such that $\widehat{\varphi}(\gamma) \to 1$ as $\gamma \to 0$. Define $\Phi_j := \widehat{D_A^j\varphi}$ and $G_{j+1}^{(m)(i)}(\gamma) = G_0^{(m)(i)}(B^{-j}\gamma)$ for some function $G_0^{(m)(i)} \in L^\infty(([0,1])^n)$, where $i \in \{1,2\}, m = 1, \ldots, s$ and $B = A^T$. Suppose the matrix-valued functions satisfy the following conditions:*

(i) $(\mathfrak{B}_{-1}^{(i)}(\gamma))^*\mathfrak{B}_{-1}^{(i)}(\gamma) = |\det A| I_{|\det A|}$ *for a.e.* $\gamma \in B^{-1}([0,1])^n$,
(ii) $(\widetilde{\mathfrak{B}_{-1}^{(1)}}(\gamma))^*\widetilde{\mathfrak{B}_{-1}^{(2)}}(\gamma) = O_{|\det A|}$ *for a.e.* $\gamma \in B^{-1}([0,1])^n$.

Then the systems $\bigcup_{j\in\mathbb{Z}}\{D_A^j T_\lambda g_{(m,0)}^{(1)}\}_{\lambda\in\mathbb{Z}^n, m=1,\ldots,s}$ and $\bigcup_{j\in\mathbb{Z}}\{D_A^j T_\lambda g_{(m,0)}^{(2)}\}_{\lambda\in\mathbb{Z}^n, m=1,\ldots,s}$ are pairwise orthogonal Parseval wavelet frames in $L^2(\mathbb{R}^n)$.

When $n = 1$ and $A = [2]$, the above corollary confirms the same outcome as Bhatt et al. [23]. Bhatt has also discussed this result for $L^2(\mathbb{R}^n)$ in [22].

The subsequent chapter explores applications of orthogonal frames to sampling theory, demonstrating that two unions of samples are orthogonal precisely when each corresponding pair of individual samples is orthogonal.

CHAPTER 4

THE GEOMETRY OF SAMPLING ON UNIONS OF CO-COMPACT SUBGROUPS

The purpose of this chapter is to present applications of the pairwise orthogonal frames discussed in Chapter 3 within the context of sampling theory. In particular, we characterize sampling transforms defined through the actions of two co-compact subgroups of a LCA group G, whose ranges are orthogonal on distinct bands. To this end, we establish necessary and sufficient conditions for the existence of pairwise orthogonal TI systems over distinct co-compact subgroups. As a consequence, a characterization for pairwise orthogonal co-compact Gabor Bessel systems associated with different co-compact subgroups is obtained. Finally, we extend the notion of orthogonal sampling transforms to unions of compact subgroups.

4.1. Introduction

In **Chapter 3**, we discussed techniques for constructing pairwise orthogonal frames. This chapter focuses on their applications in sampling theory, particularly the study of orthogonal sampling transforms associated with two unions of co-compact subgroups (not necessarily lattices) over two frequency bands. The analysis is studied in the separable Hilbert space $L^2(G)$.

In many applications, the orthogonal ranges of the sampling transforms has direct applications in areas such as signal denoising [21] and multiple-access communication systems [5], where effective control of the sampling operator's range ensures accurate and stable reconstruction. In particular, the orthogonal tight samples allow multiplexing and recovery of signals from summed samples, a method extensively used in radio, television, and computer networking [13, 14]. Weber and Al-Sa'di provide a recipe to

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The results of **Chapter 4** are from the following article:

Redhu N., Gumber A., Shukla N. K. (2025), *The geometry of sampling on unions of co-compact subgroups*, Under review.

recover signals in de Branges spaces from multiplexed samples [7]. For more details, we refer to [5, 12, 24, 56, 59, 71, 115, 116].

We begin with the *translation invariant subspace*

$$V_E := \{f \in L^2(G) : \mathrm{supp}(\widehat{f}) \subset E\}$$

of $L^2(G)$, where $E \subset \widehat{G}$ (dual group of G) is a band, i.e., a measurable subset of finite Haar measure. It is immediate that $T_\lambda V_E = V_E$ for all $\lambda \in G$, where T_λ denotes the translation operator. Since V_E is a reproducing kernel Hilbert space, every element $f \in V_E$ satisfies

$$f(\lambda) = \langle f, T_\lambda \psi \rangle, \tag{4.1.1}$$

where $\widehat{\psi}_E = \chi_E$ is the indicator function of the set E.

Let Γ be a fixed co-compact subgroup of G, and let $\mathrm{Aut}(G)$ denote the set of all bi-continuous group automorphisms of G. For $\eta \in \mathrm{Aut}(G)$, we define the associated *sampling transform* by

$$\mathcal{T}_\eta : V_E \to L^2(\Gamma), \quad f \mapsto \big(f(\eta(\lambda))\big)_{\lambda \in \Gamma},$$

provided $\mathcal{T}_\eta$ is bounded. Boundedness requires the existence of $B > 0$ such that

$$\int_\Gamma |\langle f, T_{\eta\lambda}\psi_E \rangle|^2 \, d\mu_\Gamma(\lambda) \le B \, \|f\|^2 \qquad \text{for } f \in V_E, \tag{4.1.2}$$

which is equivalent to the condition that $\{T_\lambda \psi_E : \lambda \in \eta\Gamma\}$ forms a Bessel sequence for V_E. We say that the samples in the set $\mathcal{M}_\eta := \{\eta(\lambda) : \lambda \in \Gamma\}$ forms a *set of sampling* for the band E if $\mathcal{T}_\eta$ is bounded above and below. Equivalently, the system $\{T_\lambda \psi_E : \lambda \in \eta\Gamma\}$ is a frame for V_E. The set of sampling is *tight* (or, *samples is tight*) if there exist a constant K such that $\|\mathcal{T}_\eta f\|^2_{L^2(\Gamma)} = K\|f\|^2$, for all $f \in V_E$. For more details on continuous sampling, we refer to [55].

Given $\eta, \zeta \in \mathrm{Aut}(G)$, we consider two bands E and F with associated bounded sampling transforms $\mathcal{T}_\eta$ and $\mathcal{T}_\zeta$. We say that these sampling transforms are *orthogonal on E and F* (or *samples $\mathcal{M}_\eta$ and $\mathcal{M}_\zeta$ are orthogonal*) if and only if

$$\mathcal{T}_\eta(V_E) \perp \mathcal{T}_\zeta(V_F) \quad \text{in } L^2(\Gamma),$$

which is equivalent to the condition $\mathcal{T}_\zeta^* \mathcal{T}_\eta = 0$, equivalently, the TI systems $\{T_\lambda \psi_E : \lambda \in \eta\Gamma\}$, and $\{T_\lambda \psi_F : \lambda \in \zeta\Gamma\}$, are *pairwise orthogonal* Bessel (frame) systems.

In the remainder of this chapter, we assume that $\eta, \zeta \in \mathrm{Aut}(G)$, and Γ is a co-compact subgroup of G with annihilator $\Gamma^{\perp} \subseteq \widehat{G}$. Moreover, we have $(\eta\Gamma)^{\perp} = \eta^{*}\Gamma^{\perp}$, where $\eta^{*} := (\eta')^{-1}$ denotes the inverse of the adjoint of η.

In the present chapter, our first main result is a characterization of pairwise orthogonal samples associated with two co-compact subgroups. We state our first main characterization result, that is, Theorem 4.2.1 (along with Examples) in Section 4.2, while the proof for the result is discussed in Subsection 4.2.3.

To establish Theorem 4.2.1, we first provide a characterization of pairwise orthogonal translation-invariant (TI) frames of the form $\{T_{\lambda} g_p^{(1)}\}_{\lambda \in \eta\Gamma,\, p \in P}$ and $\{T_{\lambda} g_p^{(2)}\}_{\lambda \in \zeta\Gamma,\, p \in P}$. This characterization, stated as Theorem 4.2.5 in Subsection 4.2.1, extends the result studied in [115] for shift-invariant systems in $L^2(\mathbb{R}^n)$. As a special case of Theorem 4.2.5, Subsection 4.2.1 presents Theorem 4.2.6, which provides a characterization of pairwise orthogonal Gabor Bessel (frame) systems with different translation parameters in each system.

Furthermore, our second main result extends these findings to the setting of unions of co-compact subgroups. In Section 4.3, Theorem 4.3.1 shows that a union of samples is tight if and only if each individual sample is tight, and Theorem 4.3.2 shows that two unions of samples are orthogonal if and only if each corresponding pair of individual samples is orthogonal. This part of our work generalizes classical results in $L^2(\mathbb{R}^n)$ with $\Gamma = \mathbb{Z}^n$, as discussed in [5, 115, 116].

Before proceeding to the next section, we present a characterization result for pairwise orthogonal TI systems that share the same translation subgroups. This result follows directly from Theorem 2.1.4, since TI systems are a special case of GTI systems.

Theorem 4.1.1. *Let $\bigcup_{p \in P} \{T_{\lambda} g_p^{(1)}\}_{\lambda \in \Gamma, p \in P}$ and $\bigcup_{p \in P} \{T_{\lambda} g_p^{(2)}\}_{\lambda \in \Gamma, p \in P}$ be two TI Bessel (frame) systems for $L^2(G)$. Then, the following assertions are equivalent:*

(i) Both the above TI Bessel (frame) systems in $L^2(G)$ are pairwise orthogonal.

(ii) For each $\alpha \in \Gamma^{\perp}$, we have

$$\int_{P} \frac{1}{s(\Gamma)} \overline{\widehat{g_p^{(1)}}(\gamma)}\, \widehat{g_p^{(2)}}(\gamma + \alpha) = 0 \text{ for a.e. } \gamma \in \widehat{G}. \tag{4.1.3}$$

4.2. Orthogonal sampling transforms

The main objective of this section is to study the orthogonality of two sampling systems. To establish this, we analyze orthogonality of a pair of TI systems of the form

$$\{T_\lambda g_p^{(1)}\}_{\lambda\in\eta\Gamma,\, p\in P} \quad \text{and} \quad \{T_\lambda g_p^{(2)}\}_{\lambda\in\zeta\Gamma,\, p\in P}.$$

In contrast, the existing literature typically focuses on TI systems defined over the same co-compact subgroup, that is, systems of the form

$$\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma,\, p\in P} \quad \text{and} \quad \{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma,\, p\in P}$$

as discussed in Theorem 4.1.1. For further studies on TI systems of this type, we refer to [27, 60, 79]. The key distinction in our setup is that the co-compact subgroups $\eta\Gamma$ and $\zeta\Gamma$ may differ depending on the η and ζ. In the special case $\eta = \zeta$, our framework reduces to the classical case considered in the earlier works. The advantage of studying TI systems with distinct subgroups lies in their direct relevance to sampling theory, where different sampling grids naturally arise.

As an application of our results, we also present a characterization of pairwise orthogonal Bessel Gabor systems whose translations act on different co-compact subgroups.

Theorem 4.2.1. *For $\eta, \zeta \in \mathrm{Aut}(G)$, let us consider the sampling sets*

$$\mathcal{M}_\eta := \{\eta(\lambda) : \lambda \in \Gamma\} \qquad \text{and} \qquad \mathcal{M}_\zeta := \{\zeta(\lambda) : \lambda \in \Gamma\}, \tag{4.2.1}$$

with associated sampling transforms $\mathcal{T}_\eta$ and $\mathcal{T}_\zeta$, which are bounded on the subspaces V_E and V_F, respectively. Then the samples $\mathcal{M}_\eta$ and $\mathcal{M}_\zeta$ are orthogonal on the frequency bands E and F if and only if

$$\sum_{\alpha\in\Gamma^\perp} \chi_E\big(\eta^*(\gamma+\alpha)\big) \sum_{\beta\in\Gamma^\perp} \chi_F\big(\zeta^*(\gamma+\beta)\big) = 0, \quad \textit{for a.e. } \gamma \in \widehat{G}.$$

In particular, if we fix $G = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$, then Theorem 4.2.1 specializes to [5, Theorem 1(ii)]. To illustrate the theorem concretely, we now present a simple example in the familiar Euclidean setting.

Example 4.2.2. Let $G = \mathbb{R}^n$ with $\Gamma = \mathbb{Z}^n$, and let I_n denote the $n \times n$ identity matrix. Consider the automorphisms $\eta = I_n$ and $\zeta = 2I_n$. Define the frequency bands

$$E = \left[0, \frac{1}{8}\right]^n \quad \text{and} \quad F = \left[\frac{1}{2}, \frac{5}{8}\right]^n.$$

Direct calculation shows that for almost every $\gamma \in \mathbb{R}^n$,

$$\sum_{\alpha\in\mathbb{Z}^n} \chi_E(\gamma+\alpha) \cdot \sum_{\beta\in\mathbb{Z}^n} \chi_F\left(\tfrac{1}{2}(\gamma+\beta)\right) = 0.$$

Thus, the sampling transforms associated with the sets

$$\mathcal{M}_\eta = \{z : z \in \mathbb{Z}^n\} \quad \text{and} \quad \mathcal{M}_\zeta = \{2z : z \in \mathbb{Z}^n\}$$

are orthogonal on E and F.

Before turning to the proof, we illustrate Theorem 4.2.1 in a concrete setting. Many important LCA groups admit explicit descriptions, for example $G = \mathbb{R}^n$, $G = \mathbb{Z}^n$, and the compact case $G = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. As a motivating example, we focus on $G = \mathbb{T}^n$, where the structure of co-compact subgroups and their annihilators can be described in terms of finite lattices. This case demonstrates how the general orthogonality condition naturally specialises to a familiar and highly structured group.

Recall that the n-torus $\mathbb{T}^n$ can be identified as the quotient $\mathbb{R}^n/\mathbb{Z}^n$. A *lattice* Γ in $\mathbb{T}^n$ is a finite subgroup isomorphic to

$$\Gamma = \left\{\left(\frac{k_1}{m_1}, \frac{k_2}{m_2}, \dots, \frac{k_n}{m_n}\right) \bmod \mathbb{Z}^n : k_i = 0, \dots, m_i - 1 \text{ and } 1 \le i \le n\right\}$$

for some fixed positive integers $m_1, \dots, m_n$.

The group of (automorphisms) $\mathrm{Aut}(\mathbb{T}^n)$ is isomorphic to the group of invertible integer matrices $GL_n(\mathbb{Z})$, where each automorphism η corresponds to a matrix $A \in GL_n(\mathbb{Z})$ acting by

$$\eta : x + \mathbb{Z}^n \mapsto Ax + \mathbb{Z}^n.$$

As a special case of the Theorem 4.2.1, we present the following example in $\mathbb{T}^n$.

Example 4.2.3. Let Γ be a lattice in $\mathbb{T}^n$ as above, and let $A, B \in GL_n(\mathbb{Z})$ induce automorphisms $\eta, \zeta \in \mathrm{Aut}(\mathbb{T}^n)$. Define the sampling sets

$$\mathcal{M}_\eta := \{\eta(\lambda) : \lambda \in \Gamma\} \quad \text{and} \quad \mathcal{M}_\zeta := \{\zeta(\lambda) : \lambda \in \Gamma\}.$$

Let $E, F \subseteq \mathbb{Z}^n$ be frequency bands with associated translation invariant subspaces V_E, V_F, respectively. The sampling transforms $\mathcal{T}_\eta$ and $\mathcal{T}_\zeta$ are bounded on V_E and V_F, respectively. Then the sampling systems $\mathcal{M}_\eta$ and $\mathcal{M}_\zeta$ are orthogonal on the bands E and F if and only if

$$\sum_{\alpha\in\Gamma^\perp} \chi_E\big((A^T)^{-1}(\gamma+\alpha)\big) \sum_{\beta\in\Gamma^\perp} \chi_F\big((B^T)^{-1}(\gamma+\beta)\big) = 0, \quad \text{for a.e. } \gamma \in \mathbb{Z}^n.$$

The proof of Theorem 4.2.1 is based on the characterization of pairwise orthogonal frames; we prove it in Subsection 3.3. In the following subsection, we present a characterization of pairwise orthogonal frames.

4.2.1. Pairwise Orthogonal Frames

The purpose of this subsection is to characterize when two translation invariant Bessel (or frame) systems are pairwise orthogonal. We begin by recalling the definition of a frame in $L^2(G)$. A TI sequence $\{T_\lambda g_p^{(1)}\}_{\lambda\in\eta\Gamma, p\in P}$ is called a *frame* if there exist constants $A, B > 0$ such that for all $f \in L^2(G)$,

$$A\|f\|^2 \leq \int_{p\in P}\int_{\Gamma} |\langle f, T_{\eta\lambda} g_p^{(1)}\rangle|^2 \, d\mu_\Gamma(\lambda) \, d\mu_P(p) \leq B\|f\|^2.$$

Here, P is an index set. If only the upper bound holds (i.e., the right-hand inequality), then the sequence is called a *Bessel system.*

Let $\mathcal{G}_1 := \{T_\lambda g_p^{(1)}\}_{\lambda\in\eta\Gamma, p\in P}$ and $\mathcal{G}_2 := \{T_\lambda g_p^{(2)}\}_{\lambda\in\zeta\Gamma, p\in P}$ be two Bessel (frame) system in $L^2(G)$. We define the mixed dual Gramian operator $\Theta_{g^{(1)}, g^{(2)}} : L^2(G) \to L^2(G)$ by

$$\Theta_{g^{(1)}, g^{(2)}} f := \int_{p\in P}\int_{\Gamma} \langle f, T_{\eta\lambda} g_p^{(1)}\rangle \, T_{\zeta\lambda} g_p^{(2)} \, d\mu_\Gamma(\lambda) \, d\mu_P(p).$$

When $\Theta_{g^{(1)}, g^{(2)}} = 0$, we say that the two systems are *pairwise orthogonal Bessel (frame) systems.*

The following theorem characterizes when two translation invariant Bessel (or frame) systems are pairwise orthogonal. It highlights the key role of the mixed Gramian operator and the condition that it commutes with translations. This result generalizes [115, Proposition 2.4] from the classical setting of $L^2(\mathbb{R})$ to the broader framework of $L^2(G)$.

Theorem 4.2.4. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be TI Bessel (frame) systems in $L^2(G)$. Suppose that the mixed Gramian operator $\Theta_{g^{(1)}, g^{(2)}}$ commutes with the translation operators $T_{\eta\gamma}$ for every $\gamma \in \Gamma$, and $\eta \neq \zeta$. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ form a pairwise orthogonal Bessel (frame) systems.*

Proof. First, we observe that for any $\gamma \in \Gamma$,

$$\begin{aligned}
\Theta_{g^{(1)},g^{(2)}} T_{\eta\gamma} &= \int_P \int_\Gamma \langle T_{\eta\gamma}\cdot, T_{\eta\lambda} g_p^{(1)} \rangle \, T_{\zeta\lambda} g_p^{(2)} \, d\mu_\Gamma(\lambda) \, d\mu_P(p) \\
&= \int_P \int_\Gamma \langle \cdot, T_{\eta(\lambda-\gamma)} g_p^{(1)} \rangle \, T_{\zeta\lambda} g_p^{(2)} \, d\mu_\Gamma(\lambda) \, d\mu_P(p) \\
&= \int_P \int_\Gamma \langle \cdot, T_{\eta\lambda} g_p^{(1)} \rangle \, T_{\zeta(\gamma+\lambda)} g_p^{(2)} \, d\mu_\Gamma(\lambda) \, d\mu_P(p) \\
&= T_{\zeta\gamma} \, \Theta_{g^{(1)},g^{(2)}}. \qquad (4.2.2)
\end{aligned}$$

By the hypothesis that $\Theta_{g^{(1)},g^{(2)}}$ commutes with $T_{\eta\gamma}$, we also have

$$\Theta_{g^{(1)},g^{(2)}} T_{\eta\gamma} = T_{\eta\gamma} \Theta_{g^{(1)},g^{(2)}}. \qquad (4.2.3)$$

Combining equations (4.2.2) and (4.2.3) gives

$$T_{\eta\gamma} \, \Theta_{g^{(1)},g^{(2)}} = T_{\zeta\gamma} \, \Theta_{g^{(1)},g^{(2)}}.$$

Now, we assume for contradiction that the systems $\mathcal{G}_1$ and $\mathcal{G}_2$ are not pairwise orthogonal; thus

$$\Theta_{g^{(1)},g^{(2)}} \neq 0.$$

Then there exists some non-zero $f \in L^2(G)$ such that $T_{\eta\gamma} f = T_{\zeta\gamma} f$, which implies $T_{\eta\gamma-\zeta\gamma} f = f$. However, it is a well-known fact that if $\eta\gamma - \zeta\gamma \neq 0$, the translation operator $T_{\eta\gamma-\zeta\gamma}$ has purely continuous spectrum and therefore possesses no non-zero eigenvectors[115]. This contradicts the existence of such an $f \neq 0$.

Hence, the equality $T_{\eta\gamma} = T_{\zeta\gamma}$ must hold for all $\gamma \in \Gamma$, which contradicts the assumption that $\eta \neq \zeta$. Therefore, $\Theta_{g^{(1)},g^{(2)}} = 0$, and the systems $\mathcal{G}_1$ and $\mathcal{G}_2$ are pairwise orthogonal Bessel. ■

Next, we provide a characterization results for the pairwise orthogonal Bessel (frame) systems. Before stating the result for $\eta \in \mathrm{Aut}(G)$, we define the isometric *dilation operator* $D_\eta : L^2(G) \to L^2(G)$ associated with η by

$$D_\eta f(x) = \Delta(\eta)^{-1/2} f(\eta(x)) \quad \text{for all } x \in G.$$

Theorem 4.2.5. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be TI Bessel (frame) systems in $L^2(G)$. Then the following statements are equivalent:*

(i) $\mathcal{G}_1$ and $\mathcal{G}_2$ are pairwise orthogonal Bessel (frame) systems.

(ii) For each $\alpha \in \Gamma^{\perp}$,

$$\int_{P} \overline{\widehat{g_p^{(1)}}(\eta^*\gamma)}\widehat{g_p^{(2)}}(\zeta^*(\gamma+\alpha))\ d_{\mu_P}(p) = 0 \ \textit{for a.e. } \gamma \in \widehat{G}.$$

Proof. Let D_η and D_ζ be the (unitary) dilation operators associated with η and ζ, respectively. By polarisation identity, $\mathcal{G}_1$ and $\mathcal{G}_2$ are pairwise orthogonal if and only if for every $f \in \mathcal{D}_B$, we have

$$\langle D_\zeta \Theta_{g^{(1)},g^{(2)}} D_\eta^{-1} f, f\rangle = 0,$$

as $\mathcal{D}_B$ is dense in $L^2(G)$. We can rewrite the left-hand side of the above equation as follows:

$$\begin{aligned}\langle D_\zeta \Theta_{g^{(1)},g^{(2)}} D_\eta^{-1} f, f\rangle &= \langle \Theta_{g^{(1)},g^{(2)}} D_\eta^{-1} f, D_\zeta^{-1} f\rangle \\ &= \int_P \int_\Gamma \langle D_\eta^{-1} f, T_{\eta\lambda} g_p^{(1)}\rangle \langle T_{\zeta\lambda} g_p^{(2)}, D_\zeta^{-1} f\rangle\, d\mu_\Gamma(\lambda)\, d\mu_P(p) \\ &= \int_P \int_\Gamma \langle f, D_\eta T_{\eta\lambda} g_p^{(1)}\rangle \langle D_\zeta T_{\zeta\lambda} g_p^{(2)}, f\rangle\, d\mu_\Gamma(\lambda)\, d\mu_P(p).\end{aligned}$$

Using the intertwining relation $D_\eta T_\lambda = T_{\eta^{-1}(\lambda)} D_\eta$ for $\lambda \in \Gamma$, the above expression becomes

$$\langle D_\zeta \Theta_{g^{(1)},g^{(2)}} D_\eta^{-1} f, f\rangle = \int_P \int_\Gamma \langle f, T_\lambda D_\eta g_p^{(1)}\rangle \langle T_\lambda D_\zeta g_p^{(2)}, f\rangle\, d\mu_\Gamma(\lambda)\, d\mu_P(p).$$

Thus, $\mathcal{G}_1$ and $\mathcal{G}_2$ are pairwise orthogonal if and only if the systems $\bigcup_{p\in P}\{T_\lambda D_\eta g_p^{(1)}\}_{\lambda\in\Gamma}$ and $\bigcup_{p\in P}\{T_\lambda D_\zeta g_p^{(2)}\}_{\lambda\in\Gamma}$ are pairwise orthogonal. By Theorem 4.1.1, these systems are pairwise orthogonal if and only if for each $\alpha \in \Gamma^{\perp}$,

$$\int_P \overline{\widehat{D_\eta g_p^{(1)}}(\gamma)}\widehat{D_\zeta g_p^{(2)}}(\gamma+\alpha)\, d\mu_P(p) = 0 \text{ for a.e. } \gamma \in \widehat{G}.$$

Since $\widehat{D_\eta f}(\gamma) = \Delta(\eta)^{1/2}\widehat{f}(\eta^*(\gamma))$, it follows that, for each $\alpha \in \Gamma^{\perp}$, the orthogonality condition above can be rewritten as

$$\int_P \overline{\widehat{g_p^{(1)}}(\eta^*\gamma)}\widehat{g_p^{(2)}}(\zeta^*(\gamma+\alpha))\, d\mu_P(p) = 0 \text{ for a.e. } \gamma \in \widehat{G}.$$

This completes the proof. ∎

Theorem 4.2.5 is broadly applicable to TI systems on LCA groups G, with flexibility in the choice of G (e.g., $\mathbb{R}$, $\mathbb{T}$, $\mathbb{Z}^n$), co-compact subgroups Γ, and automorphisms η, ζ. By adjusting these, the theorem unifies classical and modern results across diverse settings, including cases $G = \mathbb{R}^n$, $\Gamma = \mathbb{Z}^n$ in [115].

4.2.2. Applications to Pairwise Orthogonal Gabor Systems with Different Co-Compact Subgroups

In this subsection, we apply Theorem 4.2.5 to study pairwise orthogonality of Gabor systems generated by functions translated along different co-compact subgroups. The flexibility in choosing these co-compact subgroups allows a broader framework than those previously considered, generalizing classical results in harmonic analysis and frame theory.

For a character $\chi \in \widehat{G}$, we define the *modulation operator* M_χ on $L^2(G)$ by $M_\chi(f)(x) = \chi(x)f(x)$ for all $x \in G$. This operator satisfies the following Fourier domain identity:

$$(\widehat{M_\chi f})(\gamma) = \int_G \chi(x)f(x)\overline{\gamma(x)}d\mu_G(x) = \int_G f(x)\overline{(\gamma - \chi)(x)}d\mu_G(x) = T_\chi \widehat{f}(\gamma) \quad (4.2.4)$$

for all $f \in L^2(G)$ and for a.e. $\gamma \in \widehat{G}$. Let $\Gamma \subset G$ and $\Lambda \subset \widehat{G}$ be co-compact subgroups. The co-compact Gabor system generated $g_p^{(1)}$ is given by

$$\{T_\lambda M_\chi g_p^{(1)}\}_{\lambda\in\Gamma,\chi\in\Lambda,p\in P}. \quad (4.2.5)$$

We are interested in the orthogonality properties of two such Gabor systems, possibly associated with different translation subgroups $\eta\Gamma$ and $\zeta\Gamma$.

Orthogonality of co-compact Gabor systems of the form

$$\{T_\lambda M_\chi g_p^{(1)}\}_{\lambda\in\Gamma,\chi\in\Lambda,p\in P} \text{ and } \{T_\lambda M_\chi g_p^{(2)}\}_{\lambda\in\Gamma,\chi\in\Lambda,p\in P}$$

has been studied previously in the literature, for example in [60, 61], and the discrete setting $\ell^2(\mathbb{Z}^n)$ is addressed in [94]. Our approach generalizes these by permitting translations along different co-compact subgroups, thus allowing a richer and more flexible class of Gabor systems.

Applying Theorem 4.2.5, we obtain the following necessary and sufficient conditions for the pairwise orthogonality of the two co-compact Gabor systems

$$\{T_\lambda M_\chi g_p^{(1)}\}_{\lambda\in\eta\Gamma,\chi\in\Lambda,p\in P} \text{ and } \{T_\lambda M_\chi g_p^{(2)}\}_{\lambda\in\zeta\Gamma,\chi\in\Lambda,p\in P}. \quad (4.2.6)$$

This provides a clear characterization of orthogonality in terms of their Fourier transforms and the structure of the subgroups $\eta\Gamma$ and $\zeta\Gamma$.

Theorem 4.2.6. *Let* $\{T_\lambda M_\chi g_p^{(1)}\}_{\lambda\in\eta\Gamma,\chi\in\Lambda,p\in P}$ *and* $\{T_\lambda M_\chi g_p^{(2)}\}_{\lambda\in\zeta\Gamma,\chi\in\Lambda,p\in P}$ *be (co-compact) Gabor Bessel (frame) systems in* $L^2(G)$*. These two systems are pairwise orthogonal if and* *only if, for every $\alpha \in \Gamma^\perp$, the following condition holds a.e. for $\gamma \in \widehat{G}$:*

$$\int_P \int_\Lambda \overline{\widehat{g_p^{(1)}}(\eta^*(\gamma+\chi))}\,\widehat{M_\chi g_p^{(2)}}(\zeta^*(\gamma+\chi+\alpha))\,d\mu_\Lambda(\chi)\,d\mu_P(p) = 0.$$

Proof. Define the index set $\mathcal{J} := P \times \Lambda$, and for each $j = (p,\chi) \in \mathcal{J}$, set

$$h_j^{(1)} := M_\chi g_p^{(1)} \quad \text{and} \quad h_j^{(2)} := M_\chi g_p^{(2)}.$$

Then, the given Gabor systems $\{T_\lambda M_\chi g_p^{(1)}\}_{\lambda\in\eta\Gamma,\chi\in\Lambda,p\in P}$ and $\{T_\lambda M_\chi g_p^{(2)}\}_{\lambda\in\zeta\Gamma,\chi\in\Lambda,p\in P}$, can be rewritten as

$$\{T_\lambda h_j^{(1)}\}_{\lambda\in\eta\Gamma,j\in\mathcal{J}} \quad \text{and} \quad \{T_\lambda h_j^{(2)}\}_{\lambda\in\zeta\Gamma,j\in\mathcal{J}}.$$

By applying Theorem 4.2.5, these two systems are pairwise orthogonal if and only if for every $\alpha \in \Gamma^\perp$,

$$\int_{\mathcal{J}} \overline{\widehat{h_j^{(1)}}(\eta^*\gamma)}\,\widehat{h_j^{(2)}}(\zeta^*(\gamma+\alpha))\,d\mu_{\mathcal{J}}(j) = 0 \quad \text{for a.e. } \gamma \in \widehat{G}.$$

Substituting the values of $\mathcal{J}$, $h_j^{(1)}$, and $h_j^{(2)}$, the above integral condition becomes

$$\int_P \int_\Lambda \overline{\widehat{M_\chi g_p^{(1)}}(\eta^*\gamma)}\,\widehat{M_\chi g_p^{(2)}}(\zeta^*(\gamma+\alpha))\,d\mu_\Lambda(\chi)\,d\mu_P(p) = 0 \quad \text{for a.e. } \gamma \in \widehat{G}.$$

Now the result follows by applying $\widehat{M_\chi f}(\gamma) = T_\chi \widehat{f}(\gamma)$ given in (4.2.4). ■

If we take $\eta = \zeta = I$, where $I \in \mathrm{Aut}(G)$ is the identity automorphism, then Theorem 4.2.6 reduces to the characterization result given in [60, Proposition 3.11].

To illustrate this result concretely, we present the following example in the discrete setting $G = \mathbb{Z}^n$.

Example 4.2.7. Let $G = \mathbb{Z}^n$, and let A, B, C, D be some invertible $n \times n$ matrices with integer entries. Define the uniform lattices $\Gamma := C\mathbb{Z}^n$ and $\Lambda := D\mathbb{Z}^n$ in $\mathbb{Z}^n$. Set $\eta := A$ and $\zeta := B$ in $\mathrm{Aut}(\mathbb{Z}^n)$, and denote by $A^* := (A^T)^{-1}$ the inverse transpose of A. In this setting, the Gabor systems (4.2.6) reduce to the collections:

$$\{T_\lambda M_\chi g_p^{(1)}\}_{\lambda\in AC\mathbb{Z}^n,\chi\in D\mathbb{Z}^n,p\in P} \quad \text{and} \quad \{T_\lambda M_\chi g_p^{(2)}\}_{\lambda\in BC\mathbb{Z}^n,\chi\in D\mathbb{Z}^n,p\in P}.$$

These Gabor systems are pairwise orthogonal if and only if, for every $k \in \mathbb{Z}^n$,

$$\int_P \sum_{m\in\mathbb{Z}^n} \overline{\widehat{g_p^{(1)}}(A^*(\gamma+Dm))}\,\widehat{g_p^{(2)}}(B^*(\gamma+Dm+C^*k))\,d\mu_P(p) = 0 \quad \text{for a. e. } \gamma \in \widehat{\mathbb{T}}.$$

Remark 4.2.8. In Example 4.2.7, if we assume $A = B = I$ (the identity matrix) and that P is a singleton set, then the result reduces to the characterization given by Lopez and Han in [94, Theorem 1.4(ii)].

4.2.3. Proof of Theorem 4.2.1:

To prove Theorem 4.2.1, we first establish the following auxiliary result, which is a special case of Theorem 4.2.5 where the index set P contains a single element. This simplification provides a more clear characterization of pairwise orthogonal translation-invariant Bessel (or frame) systems generated by individual functions. The corollary below states this result explicitly.

Corollary 4.2.9. *Let $\{T_\lambda g^{(1)} : \lambda \in \eta\Gamma\}$ and $\{T_\lambda g^{(2)} : \lambda \in \zeta\Gamma\}$ be TI Bessel (frame) systems in $L^2(G)$. Then the following statements are equivalent:*

(i) These are pairwise orthogonal Bessel (frame) systems.

(ii) For almost every $\gamma \in \widehat{G}$,

$$\sum_{\alpha \in \Gamma^\perp} |\widehat{g^{(1)}}(\eta^*(\gamma+\alpha))|^2 \sum_{\beta \in \Gamma^\perp} |\widehat{g^{(2)}}(\zeta^*(\gamma+\beta)) = 0.$$

Proof. In Theorem 4.2.5, suppose that for every $p \in P$, we have $g_p^{(1)} = g^{(1)}$ and $g_p^{(2)} = g^{(2)}$. Then the systems $\mathcal{G}_1$ and $\mathcal{G}_2$ reduce to

$$\{T_\lambda g^{(1)} : \lambda \in \eta\Gamma\} \text{ and } \{T_\lambda g^{(2)} : \lambda \in \zeta\Gamma\},$$

respectively. Again, by Theorem 4.2.5, these systems are pairwise orthogonal if and only if for each $\alpha \in \Gamma^\perp$,

$$\overline{\widehat{g^{(1)}}(\eta^*\gamma)}\,\widehat{g^{(2)}}(\zeta^*(\gamma+\alpha)) = 0 \quad \text{for a.e. } \gamma \in \widehat{G}.$$

From this, it follows that for each $\beta \in \Gamma^\perp$ and for a.e. $\gamma \in \widehat{G}$,

$$|\widehat{g^{(1)}}(\eta^*(\gamma+\beta))|^2 \cdot |\widehat{g^{(2)}}(\zeta^*(\gamma+\beta+\alpha))|^2 = 0.$$

Summing over β and α, we get

$$\begin{aligned} 0 &= \sum_{\beta \in \Gamma^\perp} \sum_{\alpha \in \Gamma^\perp} |\widehat{g^{(1)}}(\eta^*(\gamma+\beta))|^2 \cdot |\widehat{g^{(2)}}(\zeta^*(\gamma+\beta+\alpha))|^2 \\ &= \sum_{\beta \in \Gamma^\perp} |\widehat{g^{(1)}}(\eta^*(\gamma+\beta))|^2 \cdot \sum_{\alpha \in \Gamma^\perp} |\widehat{g^{(2)}}(\zeta^*(\gamma+\beta+\alpha))|^2 \\ &= \sum_{\beta \in \Gamma^\perp} |\widehat{g^{(1)}}(\eta^*(\gamma+\beta))|^2 \cdot \sum_{\alpha \in \Gamma^\perp} |\widehat{g^{(2)}}(\zeta^*(\gamma+\alpha))|^2 \end{aligned}$$

for a. e. $\gamma \in \widehat{G}$. The converse direction follows by reversing these steps. ∎

With these preparations, we are now ready to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. The samples $\mathcal{M}_\eta := \{\eta(\lambda) : \gamma \in \Gamma\}$, and $\mathcal{M}_\zeta := \{\zeta(\lambda) : \gamma \in \Gamma\}$, are orthogonal on the band E and F if and only if the corresponding TI Bessel systems $\{T_\lambda \psi_E : \lambda \in \eta\Gamma\}$, and $\{T_\lambda \psi_F : \lambda \in \zeta\Gamma\}$, form pairwise orthogonal frames. By Corollary 4.2.9, this is further equivalent to

$$\sum_{\alpha \in \Gamma^\perp} \chi_E\big(\eta^*(\gamma+\alpha)\big) \sum_{\beta \in \Gamma^\perp} \chi_F\big(\zeta^*(\gamma+\beta)\big) = 0 \quad \text{for a.e. } \gamma \in \widehat{G}.$$

This finishes the proof. ∎

4.3. Sampling on union of co-compact subgroups

In this section, we extend the orthogonality results of Theorem 4.2.1 to the case of sampling over the union of co-compact subgroups. Such unions naturally arise in applications where multiple sampling patterns are combined, and it is important to understand how frame or orthogonality properties behave under this extension.

For $\eta_1, \ldots, \eta_n \in \mathrm{Aut}(G)$, we define the *sampling transform* associated with the union of samples $\{\eta_j\lambda : 1 \le j \le n, \lambda \in \Gamma\}$ by

$$\mathcal{T}_{\eta_j,n} : V_E \to \bigoplus_{j=1}^{n} L^2(\Gamma) : f \mapsto (f(\eta_1\lambda), \cdots, f(\eta_n\lambda))$$

provided $\mathcal{T}_{\eta_j,n}$ is bounded. The operator $\mathcal{T}_{\eta_j,n}$ is bounded if there exists $B > 0$ such that

$$\sum_{j=1}^{n} \int_{\eta_j\Gamma} \left|\langle f, T_\lambda \psi_E\rangle\right|^2 d\mu_{\eta_j\Gamma}(\lambda) \;\le\; B\|f\|^2 \qquad \text{for } f \in V_E. \tag{4.3.1}$$

In particular, this holds if and only if the system $\bigcup_{j=1}^{n} \{T_\lambda \psi_E : \lambda \in \eta_j\Gamma\}$ is a Bessel sequence for V_E. We say that the samples $\{\eta_j\lambda : 1 \le j \le n, \lambda \in \Gamma\}$ form a *set of sampling* for the band E if $\mathcal{T}_{\eta_j,n}$ is bounded above and below, equivalently, if $\bigcup_{j=1}^{n} \{T_\lambda \psi_E : \lambda \in \eta_j\Gamma\}$ is a frame for V_E. The system is called *tight* with constant K if $\|\mathcal{T}_{\eta_j,n} f\|^2 = K\|f\|^2,\ f \in V_E$.

Throughout the remainder of this section we fix $\eta_j, \zeta_j \in \mathrm{Aut}(G)$ *for each* $j \in \{1, 2, \cdots, n\}$. Suppose that the sampling sets $\{\eta_j\lambda : 1 \le j \le n, \lambda \in \Gamma\}$ and $\{\zeta_j\lambda : 1 \le j \le n, \lambda \in \Gamma\}$ are given, with associated sampling transforms $\mathcal{T}_{\eta_j,n}$ and $\mathcal{T}_{\zeta_j,n}$ that are bounded operators on bands E and F, respectively. We say that the samples $\{\eta_j\lambda : 1 \le j \le n, \lambda \in \Gamma\}$ and $\{\zeta_j\lambda : 1 \le j \le n, \lambda \in \Gamma\}$ are *orthogonal* on bands E and F if and only if the image of V_E under $\mathcal{T}_{\eta_j,n}$ is orthogonal to the image of V_F under $\mathcal{T}_{\zeta_j,n}$ in $\bigoplus_{j=1}^{n} L^2(\Gamma)$, which is equivalent to $\mathcal{T}^*_{\zeta_j,n}\mathcal{T}_{\eta_j,n} = 0$.

Theorem 4.3.1. *The samples $\{\eta_j\lambda : 1 \le j \le n, \lambda \in \Gamma\}$ are tight on band E with constant K if and only if, for each $j \in \{1, 2, \cdots, n\}$, the samples $\{\eta_j\lambda : \lambda \in \Gamma\}$ are tight on E with constant K_j. In this case, we have $K_j = \frac{1}{s(\eta_j\Gamma)}$ and $K = \sum_{j=1}^{n} K_j$.*

Theorem 4.3.2. *Suppose that the sampling sets $\{\eta_j\lambda : 1 \le j \le n, \lambda \in \Gamma\}$ and $\{\zeta_j\lambda : 1 \le j \le n, \lambda \in \Gamma\}$ are given, with associated sampling transforms $\mathcal{T}_{\eta_j,n}$ and $\mathcal{T}_{\zeta_j,n}$ that are bounded operators on bands E and F, respectively. The samples*

$$\{\eta_j\lambda : 1 \le j \le n, \lambda \in \Gamma\} \text{ and } \{\zeta_j\lambda : 1 \le j \le n, \lambda \in \Gamma\}$$

are orthogonal on the bands E and F if and only if for $j \in \{1, 2, \cdots, n\}$,

$$\sum_{\alpha \in \Gamma^\perp} \chi_E(\eta_j^*(\gamma + \alpha)) \sum_{\beta \in \Gamma^\perp} \chi_F(\zeta_j^*(\gamma + \beta)) = 0 \text{ for a.e. } \gamma \in \widehat{G}.$$

Equivalently, the samples above are orthogonal if and only if the samples $\{\eta_j\lambda\}_{\lambda\in\Gamma}$ and $\{\zeta_j\lambda\}_{\lambda\in\Gamma}$ are orthogonal on the bands E and F for each $j \in \{1, 2, \cdots, n\}$.

As a special case, when $G = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$, the above results reduce to those in [115].

4.3.1. Proof of Theorem 4.3.1

In this subsection we prove Theorem 4.3.1, To prove this we first recall the following auxiliary result, which is a special case of [79, Theorem 3.4] and characterizes when a GTI system forms a tight frame.

Theorem 4.3.3. *Suppose that the GTI system $\bigcup_{p\in P}\{T_\lambda g_p^{(1)}\}_{\lambda\in\eta_p\Gamma}$ satisfies the LIC, $\eta_p \in$ Aut(G). Then the following assertions are equivalent:*

(i) $\bigcup_{p\in P}\{T_\lambda g_p^{(1)}\}_{\lambda\in\eta_p\Gamma, p\in P}$ is a tight frame for $L^2(G)$.
(ii) For each $\alpha \in \bigcup_{p\in\mathcal{P}} \eta_p^*\Gamma^\perp$, we have

$$\sum_{p\in P:\alpha\in\eta_p^*\Gamma^\perp} \frac{1}{s(\eta_p\Gamma)} \overline{\widehat{g_p^{(1)}}(\gamma)} \widehat{g_p}(\gamma + \alpha) d\mu_{P_j}(p) = K\delta_{\alpha,0} \quad \textit{for a.e. } \gamma \in \widehat{G}. \tag{4.3.2}$$

The next corollary is an application of this result. In order to present this, we recall the analysis operator $\theta_g^* : L^2(G) \to L^2(\Gamma \times \mathcal{J})$ for the Bessel system $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_j\}_{\lambda\in\eta_j\Gamma}$, is given by

$$\theta_g^* : L^2(G) \to L^2(\Gamma \times \mathcal{J}); f \mapsto (\langle f, T_{\eta_j\lambda}\rangle)_{j\in\mathcal{J},\lambda\in\Gamma}.$$

If $\|\theta_g f\|^2 = K\|f\|^2$ for some constant K, then $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_j\}_{\lambda\in\eta_j\Gamma}$ is a *tight frame.*

Corollary 4.3.4. *Suppose* $\{g_j\}_{j\in\mathcal{J}} \subset V_E$ *and* Θ_g^* *be the analysis operator of the system* $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_j\}_{\lambda\in\eta_j\Gamma}$ *satisfying the LIC. Then* $\|\theta_g f\|^2 = K\|f\|^2$ *if and only if for all* $\alpha \in \bigcup_{j\in\mathcal{J}} \eta_j^*\Gamma^\perp$, *we have*

$$\sum_{j\in\mathcal{J}:\alpha\in\eta_j^*\Gamma^\perp} \frac{1}{s(\eta_j\Gamma)} \overline{\widehat{g_j^{(1)}}(\gamma)}\widehat{g_j^{(1)}}(\gamma+\alpha) = K\delta_{\eta,0}\chi_E(\gamma) \textit{ for a.e. } \gamma \in \widehat{G}.$$

Proof. Suppose $\|\theta_g f\|^2 = K\|f\|^2$, equivalently, $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_j\}_{\lambda\in\eta_j\Gamma}$ is a tight frame. By hypothesis supp $\widehat{g_j} \subset E$, and using $P = \mathcal{J}$ in Theorem 4.3.3, the result follows. ■

Remark 4.3.5. If the sampling transform $\mathcal{T}_{\eta_j,n}$ is bounded on V_E, then the corresponding GTI system $\bigcup_{j=1}^{n} \{T_\lambda \psi_E\}_{\lambda\in\eta_j\Gamma}$ is Bessel; hence satisfies the LIC. Because it is well known that Bessel TI systems satisfy the LIC; thus, the finite union of such TI systems also satisfies the LIC.

Now we are ready to prove our result.

Proof of Theorem 4.3.1 : Let $\mathcal{J} = \{1, 2, \dots, n\}$. By Remark 4.3.5, the GTI system $\bigcup_{j=1}^{n} \{T_\lambda \psi_E\}_{\lambda\in\eta_j\Gamma}$ satisfies the LIC. From Corollary 4.3.4, we obtain

$$\|\mathcal{T}_{\eta_j,n} f\|^2 = K\|f\|^2$$

if and only if, for all $\alpha \in \bigcup_{j\in\mathcal{J}} \eta_j^*\Gamma^\perp$,

$$\begin{aligned} K\delta_{\alpha,0}\chi_E(\gamma) &= \sum_{j\in\mathcal{J}:\,\alpha\in\eta_j^*\Gamma^\perp} \tfrac{1}{s(\eta_j\Gamma)} \overline{\widehat{g_j^{(1)}}(\gamma)}\widehat{g_j^{(1)}}(\gamma+\alpha) \quad \text{for a.e. } \gamma \in \widehat{G} \\ &= \sum_{j\in\mathcal{J}:\,\alpha\in\eta_j^*\Gamma^\perp} \tfrac{1}{s(\eta_j\Gamma)} \overline{\chi_E(\gamma)}\chi_E(\gamma+\alpha) \quad \text{for a.e. } \gamma \in \widehat{G} \\ &= \sum_{j\in\mathcal{J}:\,\alpha\in\eta_j^*\Gamma^\perp} \tfrac{1}{s(\eta_j\Gamma)} \chi_E(\gamma)\chi_E(\gamma+\alpha) \quad \text{for a.e. } \gamma \in \widehat{G}. \end{aligned} \tag{4.3.3}$$

Thus, (4.3.3) holds precisely when, for each $j \in \mathcal{J}$ and all $\alpha \in \eta_j^* \Gamma^\perp$,

$$K_j \delta_{\alpha,0} \chi_E(\gamma) = \tfrac{1}{s(\eta_j \Gamma)} \chi_E(\gamma) \chi_E(\gamma + \alpha) \quad \text{for a.e. } \gamma \in \widehat{G}, \tag{4.3.4}$$

where $K_j = \frac{1}{s(\eta_j \Gamma)}$. Equivalently, for each $j \in \{1, 2, \ldots, n\}$, the samples set $\{\eta_j \lambda\}_{\lambda \in \Gamma}$ is tight on E with constant K_j. This completes the proof. ∎

4.3.2. Proof of Theorem 4.3.2

In this subsection we prove Theorem 4.3.2 and the next lemma will be helpful in proving Theorem 4.3.2. Define the bracket function

$$[f, g](x, \Gamma) = \int_\Gamma f(x + \lambda) \overline{g(x + \lambda)} d\mu_\Gamma(\lambda).$$

Lemma 4.3.6. *Let the systems $\bigcup_{j \in \mathcal{J}} \{T_\lambda g_j^{(1)}\}_{\lambda \in \eta_j \Gamma}$ and $\bigcup_{j \in \mathcal{J}} \{T_\lambda g_j^{(2)}\}_{\lambda \in \zeta_j \Gamma}$ be Bessel systems, both satisfying the LIC. Define $\Theta_{g^{(1)}, g^{(2)}}$ analogous to mixed dual gramian operator:*

$$\Theta_{g^{(1)}, g^{(2)}} : L^2(G) \to L^2(G) : \; f \mapsto \sum_{j \in \mathcal{J}} \int_\Gamma \langle f, T_{\eta_j \lambda} g_j^{(1)} \rangle T_{\zeta_j \lambda} g_j^{(2)} d\mu_\Gamma(\lambda).$$

Then for all $f, g \in \mathcal{D}_B$, we have

$$\langle \Theta_{g^{(1)}, g^{(2)}} f, g \rangle = \sum_{j \in \mathcal{J}} \frac{1}{\Delta(\eta_j) \Delta(\zeta_j)} \int_{\widehat{G}/\Gamma^\perp} \left[\widehat{f}, \widehat{g_j^{(1)}} \right] \left(\eta_j^* w, \eta_j \Gamma^\perp \right) \left[\widehat{g_j^{(2)}}, \widehat{g} \right] \left(\zeta_j^* w, \zeta_j \Gamma^\perp \right) \; d\mu_{\widehat{G}/\Gamma^\perp}(\dot{w}),$$

where $\dot{w}$ denotes the coset in $\widehat{G}/\Gamma^\perp$.

Proof. We compute $\langle \Theta_{g^{(1)}, g^{(2)}} f, g \rangle$ explicitly in the Fourier domain. Since both systems are Bessel and satisfy the LIC, the operator $\Theta_{g^{(1)}, g^{(2)}}$ is well defined, and all sums and integrals below converge absolutely.

By definition,

$$\begin{aligned} \langle \Theta_{g^{(1)}, g^{(2)}} f, g \rangle &= \sum_{j \in \mathcal{J}} \int_\Gamma \langle f, T_{\eta_j \lambda} g_j^{(1)} \rangle \langle T_{\zeta_j \lambda} g_j^{(2)}, g \rangle \, d\mu_\Gamma(\lambda) \\ &= \sum_{j \in \mathcal{J}} \int_\Gamma \langle \widehat{f}, M_{-\eta_j \lambda} \widehat{g_j^{(1)}} \rangle \, \overline{\langle \widehat{g}, M_{-\zeta_j \lambda} \widehat{g_j^{(2)}} \rangle} \, d\mu_\Gamma(\lambda). \end{aligned} \tag{4.3.5}$$

Consider the first factor in (4.3.5). By the substitution $\xi = (\eta_j')^{-1} w := \eta_j^* w$ we obtain

$$I_1 := \int_{\widehat{G}} \widehat{f}(\xi) \, \overline{(\xi, \eta_j \lambda)} \, \overline{\widehat{g_j^{(1)}}(\lambda)} \, d\mu_{\widehat{G}}(\xi) = \frac{1}{\Delta(\eta_j)} \int_{\widehat{G}} \widehat{f}(\eta_j^* w) \, \overline{\widehat{g_j^{(1)}}(\eta_j^* w)} \, \overline{(w, \lambda)} \, d\mu_{\widehat{G}}(w).$$

Applying Weil's formula (1.2.3) gives

$$I_1 = \frac{1}{\Delta(\eta_j)} \int_{\widehat{G}/\Gamma^\perp} \sum_{\alpha\in\Gamma^\perp} \widehat{f}(\eta_j^*(w+\alpha))\, \overline{\widehat{g_j^{(1)}}(\eta_j^*(w+\alpha))}\, \overline{(w,\lambda)}\, d\mu_{\widehat{G}/\Gamma^\perp}(\dot{w}).$$

Since $(\alpha,\lambda) = 1$ for all $\alpha \in \Gamma^\perp$, the factor reduces to (w,λ), where $\dot{w}$ denotes the coset in $\widehat{G}/\Gamma^\perp$. Hence

$$I_1 = \frac{1}{\Delta(\eta_j)} \int_{\widehat{G}/\Gamma^\perp} [\widehat{f}, \widehat{g_j^{(1)}}](\eta_j^* w, \eta_j^*\Gamma^\perp)\, \overline{(w,\lambda)}\, d\mu_{\widehat{G}/\Gamma^\perp}(\dot{w}).$$

A similar computation with $\xi = (\zeta_j^*)^{-1} w$ yields

$$I_2 := \int_{\widehat{G}} \widehat{g}(\xi)\, \overline{(\xi,\zeta_i\gamma)}\, \overline{\widehat{g_j^{(2)}}(\xi)}\, d\mu_{\widehat{G}}(\xi) = \frac{1}{\Delta(\zeta_j)} \int_{\widehat{G}/\Gamma^\perp} [\widehat{g}, \widehat{g_j^{(2)}}](\zeta_j^* w, \zeta_j^*\Gamma^\perp)\, \overline{(w,\gamma)}\, d\mu_{\widehat{G}/\Gamma^\perp}(\dot{w}).$$

Substituting I_1 and I_2 into (4.3.5) gives

$$\begin{aligned} \langle \Theta_{g^{(1)},g^{(2)}} f, g\rangle &= \sum_{j\in\mathcal{J}} \frac{1}{\Delta(\eta_j)\Delta(\zeta_j)} \int_{\Gamma} \left(\int_{\widehat{G}/\Gamma^\perp} [\widehat{f}, \widehat{g_j^{(1)}}](\eta_j^* w, \eta_j^*\Gamma^\perp)\, \overline{(w,\gamma)}\, d\mu_{\widehat{G}/\Gamma^\perp}(\dot{w}) \right) \\ &\qquad \times \overline{\left(\int_{\widehat{G}/\Gamma^\perp} [\widehat{g}, \widehat{g_j^{(2)}}](\zeta_j^* w, \zeta_j^*\Gamma^\perp)\, \overline{(w,\gamma)}\, d\mu_{\widehat{G}/\Gamma^\perp}(\dot{w}) \right)}\, d\mu_\Gamma(\lambda). \end{aligned}$$

Define, for each $j \in \mathcal{J}$, the functions on the quotient $\widehat{G}/\Gamma^\perp$

$$A_j(\dot{w}) := \left[\widehat{f}, \widehat{g}_j^{(1)}\right](\eta_j^* w, \eta_j^*\Gamma^\perp) \quad \text{and} \quad B_j(\dot{\xi}) := \left[\widehat{g}, \widehat{g}_j^{(2)}\right](\zeta_j^* w, \zeta_j^*\Gamma^\perp).$$

For each fixed j, the inner integrals in the displayed formula above can be written as

$$F_j(\gamma) = \int_{\widehat{G}/\Gamma^\perp} A_j(\dot{\xi})\, \overline{(\xi,\gamma)}\, d\mu_{\widehat{G}/\Gamma^\perp}(\dot{\xi}) \quad \text{and} \quad G_j(\gamma) = \int_{\widehat{G}/\Gamma^\perp} B_j(\dot{w})\, \overline{(\xi,\gamma)}\, d\mu_{\widehat{G}/\Gamma^\perp}(\dot{\xi}),$$

which are precisely the inverse Fourier transforms of A_j and B_j when we identify $\widehat{\Gamma} \cong \widehat{G}/\Gamma^\perp$. Hence $F_j, G_j \in L^2(\Gamma)$ and by Plancherel on Γ (or equivalently on $\widehat{\Gamma}$), we have

$$\int_{\Gamma} F_j(\lambda)\, \overline{G_j(\lambda)}\, d\mu_\Gamma(\lambda) = \int_{\widehat{G}/\Gamma^\perp} A_j(\dot{w})\, \overline{B_j(\dot{w})}\, d\mu_{\widehat{G}/\Gamma^\perp}(\dot{w}).$$

Applying this identity to the earlier double integral collapses the integration over Γ and yields

$$\langle \Theta_{g^{(1)},g^{(2)}} f, g\rangle = \sum_{j\in\mathcal{J}} \frac{1}{\Delta(\eta_j)\Delta(\zeta_j)} \int_{\widehat{G}/\Gamma^\perp} \left[\widehat{f}, \widehat{g}_j^{(1)}\right](\eta_j^* w, \eta_j^*\Gamma^\perp)\, \overline{\left[\widehat{g}, \widehat{g}_j^{(2)}\right](\zeta_j^* w, \zeta_j^*\Gamma^\perp)}\, d\mu_{\widehat{G}/\Gamma^\perp}(\dot{w}),$$

which is the desired formula. ∎

Proof of Theorem 4.3.2. The samples $\{\eta_j\lambda : 1 \le j \le n, \lambda \in \Gamma\}$ and $\{\zeta_j\lambda : 1 \le j \le n, \lambda \in \Gamma\}$ are bounded on the band E and F. Thus the corresponding GTI system $\bigcup_{j=1}^{n}\{T_\lambda\psi_E\}_{\lambda\in\eta_j\Gamma}$ and $\bigcup_{j=1}^{n}\{T_\lambda\psi_F\}_{\lambda\in\zeta_j\Gamma}$ are Bessel on subspaces V_E and V_F, respectively; hence, in view of Remark 4.3.5, both the system $\bigcup_{j=1}^{n}\{T_\lambda\psi_E\}_{\lambda\in\eta_j\Gamma}$ and $\bigcup_{j=1}^{n}\{T_\lambda\psi_F\}_{\lambda\in\zeta_j\Gamma}$ satisfy LIC. Suppose the samples $\{\eta_j\lambda : 1 \le j \le n, \lambda \in \Gamma\}$ and $\{\zeta_j\lambda : 1 \le j \le n, \lambda \in \Gamma\}$ are orthogonal, equivalently the $\bigcup_{j=1}^{n}\{T_\lambda\psi_E\}_{\lambda\in\eta_j\Gamma}$ and $\bigcup_{j=1}^{n}\{T_\lambda\psi_F\}_{\lambda\in\zeta_j\Gamma}$ are pairwise orthogonal. Thus by Lemma 4.3.6, we have

$$\langle \Theta_{g^{(1)},g^{(2)}} f, g\rangle = 0 \text{ for all } f \in \mathcal{D}_{\mathcal{B}} \cap V_E \text{ and } g \in \mathcal{D}_{\mathcal{B}} \cap V_F.$$

which further gives

$$\sum_{j=1}^{n}\frac{1}{\Delta(\eta_j)\Delta(\zeta_j)}\int_{\widehat{G}/\Gamma^\perp}\left[\widehat{f},\widehat{g}_j^{(1)}\right](\eta_j^* w, \eta_j^*\Gamma^\perp)\,\overline{\left[\widehat{g},\widehat{g}_j^{(2)}\right](\zeta_j^* w, \zeta_j^*\Gamma^\perp)}\,d\mu_{\widehat{G}/\Gamma^\perp}(\dot{w}) = 0. \tag{4.3.6}$$

Define $g_0 \in V_E$ and $f_0 \in V_F$ by $\widehat{g_0} = \chi_{E_0}$ and $\widehat{f_0} = \chi_{F_0}$. Replacing f and g by f_0 and g_0 in (4.3.6) and $\widehat{g_j^{(2)}} = \chi_F$, $\widehat{g_j^{(1)}} = \chi_E$, we get

$$\sum_{j=1}^{n}\frac{1}{\Delta(\eta_j)\Delta(\zeta_j)}\int_{\widehat{G}/\Gamma^\perp}\left[\chi_{E_0},\chi_E\right](\eta_j^* w, \eta_j^*\Gamma^\perp)\,\overline{\left[\chi_F,\chi_{F_0}\right](\zeta_j^* w, \zeta_j^*\Gamma^\perp)}\,d\mu_{\widehat{G}/\Gamma^\perp}(\dot{w}) = 0. \tag{4.3.7}$$

Therefore for each $j \in \{1, 2, \cdots, n\}$, we have

$$\int_{\widehat{G}/\Gamma^\perp}\left[\chi_{E_0},\chi_E\right](\eta_j^* w, \eta_j^*\Gamma^\perp)\,\overline{\left[\chi_F,\chi_{F_0}\right](\zeta_j^* w, \zeta_j^*\Gamma^\perp)}\,d\mu_{\widehat{G}/\Gamma^\perp}(\dot{w}) = 0$$
$$\implies \sum_{\alpha\in\Gamma^\perp}\chi_{E_0}(\eta_j^*(w+\alpha))\sum_{\beta\in\Gamma^\perp}\chi_{F_0}(\zeta_j^*(w+\beta))) = 0.$$

Since $E_0 \subset E$ and $F_0 \subset F$ are arbitrary measurable sets. Therefore the above formula holds for E and F, i.e.

$$\sum_{\gamma\in\Gamma^\perp}\chi_E(\eta_j^*(w+\gamma))\sum_{\lambda\in\Gamma^\perp}\chi_F(\zeta_j^*(w+\lambda)) = 0 \quad \text{for a.e. } w \in \widehat{G}. \tag{4.3.8}$$

Conversely, suppose that for each $j \in \mathcal{J}$ (4.3.8) holds. Now it is easy to observe that for any arbitrary $E_0 \subset E$ and $F_0 \subset F$, also holds true.

$$\begin{aligned}
0 &= \sum_{\alpha\in\Gamma^\perp}\chi_{E_0}(\eta_j^*(w+\gamma))\sum_{\beta\in\Gamma^\perp}\chi_{F_0}(\zeta_j^*(w+\lambda)) \\
&= \sum_{\alpha\in\Gamma^\perp}\chi_{E_0}(\eta_j^*(w+\alpha))\chi_E(\eta_j^*(w+\alpha))\sum_{\beta\in\Gamma^\perp}\chi_{F_0}(\zeta_j^*(w+\beta))\chi_F(\zeta_j^*(w+\beta)) \\
&= [\chi_{E_0},\chi_E](\eta_j^* w, \eta_j^*\Gamma^\perp)[\chi_F,\chi_{F_0}](\zeta^* w, \zeta_j^*\Gamma^\perp),
\end{aligned}$$

which further gives for each $j \in \mathcal{J}$

$$\frac{1}{s(\eta_j\Gamma)s(\zeta_j\Gamma)} \int_{\widehat{G}/\Gamma^\perp} [\chi_{E_0}, \chi_E](\eta_j^* w, \eta_j^* \Gamma^\perp)[\chi_F, \chi_{F_0}](\zeta_j^* w, \zeta_j^* \Gamma^\perp) \; d\mu_{\widehat{G}/\Gamma^\perp}(\dot{w}) = 0.$$

Thus $\langle \Theta_{g^{(1)},g^{(2)}} f_0, g_0 \rangle = 0$ for $\widehat{f_0} = \chi_{E_0}$ and $\widehat{g} = \chi_{F_0}$. Since E_0 and F_0 are arbitrary subset of E and F, it follows that $\langle \Theta_{g^{(1)},g^{(2)}} f, g \rangle = 0$ for all $f \in V_E$ and $g \in V_F$. Indeed, any function in V_E has compact support contained in E and can therefore be approximated by finite linear combinations of indicator functions.

In view of Theorem 4.3.2, it is clear that the samples above are orthogonal if and only if the samples $\{\eta_j \lambda\}_{\gamma \in \Gamma}$ and $\{\zeta_j \lambda\}_{\lambda \in \Gamma}$ are orthogonal on the bands E and F for each $j \in \{1, 2, \cdots, n\}$. This finishes the proof. ■

In this chapter, we examined orthogonal sampling transforms arising from the actions of two co-compact subgroups of an LCA group G and extended the framework to unions of compact subgroups. The results revealed that the orthogonality of translation invariant systems plays a crucial role, even when the underlying sequences of translation subgroups differ. These observations motivate the study of more general GTI systems in which the families of translation subgroups may differ. This forms the main focus of the next chapter.

CHAPTER 5

CHARACTERIZATION OF PAIRWISE ORTHOGONAL FRAME SYSTEMS VIA UCP

In this chapter, , submitting soon.we give a characterization of pairwise orthogonal frames with GTI structures. These GTI systems are generated by translating generating functions through a countable family of closed, co-compact subgroups of G. Importantly, the families of subgroups associated with each system may differ from one another. As an application of this characterization, we derive necessary and sufficient conditions for the orthogonality of various structured systems, including Gabor, wavelet, and shearlet systems over LCA groups. Moreover, we also establish a characterization of GTI tight frames.

5.1. Introduction

In **Chapter 4**, we demonstrated that the orthogonality of systems of the form

$$\bigcup_{j\in\mathcal{J}}\{T_\lambda g_j^{(1)}\}_{\lambda\in\eta_j\Gamma} \text{ and } \bigcup_{j\in\mathcal{J}}\{T_\lambda g_j^{(2)}\}_{\lambda\in\zeta_j\Gamma}$$

has direct applications in sampling theory. By allowing nontrivial automorphisms η_j, ζ_j in the GTI construction, we obtain a richer class of translation invariant systems that still retain the possibility of pairwise orthogonality. This generalization is not only natural from a theoretical viewpoint but also useful in sampling theory, where flexibility in the underlying group actions enables more versatile designs.

Motivated by these developments, **Chapters 5–6** extend the study of orthogonal systems beyond the case where both GTI systems share the same sequence of translation

The results of **Chapter 5** are from the following manuscript:
Redhu N., Gumber A., Führ H., Shukla N. K., *Characterization and explicit construction of pairwise orthogonal Parseval frames in LCA groups*, submitting soon.

subgroups. In particular, we consider GTI systems of the form

$$\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\eta_j\Gamma_j,\,p\in P_j} \qquad \text{and} \qquad \bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\zeta_j\Gamma_j,\,p\in P_j}, \tag{5.1.1}$$

where the sequences of translation subgroups associated with the two systems may differ due to the choices of automorphisms η_j and ζ_j. The applications outlined above suggest the following problems for investigation:

Problem 1. Determine the necessary and sufficient conditions under which the GTI Bessel (frame) systems in (5.1.1) form pairwise orthogonal Bessel (frame) systems (in the sense of Definition 5.1.1) for $L^2(G)$.

Problem 2. Develop explicit constructions of pairwise orthogonal Parseval GTI frames as defined in (5.1.1).

Existing research works have addressed these problems primarily under the restrictive assumption that $\eta_j = \zeta_j = I$ (the identity automorphism) for all $j \in \mathcal{J}$ [60, 73, 79, 98]. In particular, Gumber and Shukla addressed Problem 1 under the local integrability condition (LIC) in [60], and while we proposed construction methods for Problem 2 in Chapter 3, as part of our published work [98]. To the best of our knowledge, the generalization to nontrivial automorphisms η_j, ζ_j based on the weaker assumptions (than LIC) remains open.

The main objective of this chapter is to address Problem 1 under the framework where $\eta_j = \eta$ and $\zeta_j = \zeta$ for all $j \in \mathcal{J}$, assuming the unconditional convergence property (UCP), which is weaker than LIC. As an application of GTI systems, we also derive the necessary and sufficient conditions for the pairwise orthogonality of Gabor, wavelet, and shearlet systems.

In the remainder of this section, we recall the definition of orthogonal GTI systems introduced in chapter 2. Section 5.2 then discusses the dual 1-UCP and its properties. Finally, in Section 5.3, we present our main characterization result, Theorem 5.3.1, which provides necessary and sufficient conditions for pairwise orthogonal GTI Bessel (frame) systems, based on the dual 1-UCP. This result generalizes [60, Theorem 3.5] in two ways. First, it applies to a slightly more general form of GTI systems. Second, it holds under the dual 1-UCP, which is weaker than the dual α-LIC condition used in [60, Theorem 3.5]. Additionally, Theorem 5.3.3 characterizes when a GTI system forms a Parseval frame. In Subsection 5.4, we apply Theorem 5.3.2 to derive characterization results for pairwise orthogonal Bessel systems with various structures, including Gabor, composite wavelet, and cone-adapted shearlet systems.

For this chapter, the GTI system introduced in Definition 2.1.1 is assumed to satisfy the standing hypotheses (I)–(III) introduced in Chapter 2.

In the remainder of this chapter, we assume that $\eta, \zeta \in \mathrm{Aut}(G)$*, and* Γ_j *is a co-compact subgroup of* G *with annihilator* $\Gamma_j^{\perp} \subseteq \widehat{G}$*. Moreover, we have* $(\eta\Gamma_j)^{\perp} = \eta^{*}\Gamma_j^{\perp}$*, where* $\eta^{*} := (\eta')^{-1}$ *denotes the inverse of the adjoint of* η*.*

We study the orthogonality of two GTI systems of the form

$$\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda\in\eta\Gamma_j, p\in P_j} \qquad \text{and} \qquad \bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(2)}\}_{\lambda\in\zeta\Gamma_j, p\in P_j}, \tag{5.1.2}$$

where $\eta, \zeta \in \mathrm{Aut}(G)$.

Suppose both the GTI systems defined in (5.1.2) are Bessel, then we define the *mixed dual Gramian operator* Θ corresponding to these GTI systems, is given by

$$\Theta : L^2(G) \to L^2(G); f \mapsto \sum_{j\in\mathcal{J}} \int_{P_j} \int_{\Gamma_j} \langle f, T_{\eta\lambda} g_p^{(1)} \rangle T_{\zeta\lambda} g_p^{(2)} \, d\mu_{\Gamma_j}(\lambda) \, d\mu_{P_j}(p). \tag{5.1.3}$$

Definition 5.1.1. Let $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda\in\eta\Gamma_j, p\in P_j}$ and $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(2)}\}_{\lambda\in\zeta\Gamma_j, p\in P_j}$ be two GTI Bessel (or frame) systems. We say that these systems are *pairwise orthogonal Bessel (frame) systems* if the associated mixed dual Gramian operator satisfies $\Theta = 0$ for all $f \in L^2(G)$. Furthermore, if the GTI systems are Parseval and pairwise orthogonal, then they are called *pairwise orthogonal Parseval frames*.

5.2. Unconditional convergence property

In this subsection, we introduce UCP, which play a central role in analyzing the frame properties of GTI systems. Führ et al. collaborators [53] introduced the UCP and its variants (such as the dual 1-UCP and dual ∞-UCP), which provide a weaker and more flexible alternative to the classical LIC. These concepts were first developed for GSI systems, a subclass of GTI systems. The 1-UCP condition for a GTI system was introduced in [113]. We begin by recalling the set

$$\mathcal{D}_B := \{f \in L^2(G) : \widehat{f} \in L^\infty(\widehat{G}) \text{ and supp } \widehat{f} \text{ is compact in } \widehat{G} \setminus B\},$$

where $B \subset \widehat{G}$ is a Borel set with $\mu_{\widehat{G}}(B) = 0$, referred to as a *blind spot*. The space $\mathcal{D}_B$ is both translation invariant and dense in $L^2(G)$. For any $f \in \mathcal{D}_B$, we define a family of functions $w_{f;g_p^{(1)},g_p^{(2)},j} : G \to \mathbb{C}$ by

$$w_{f;g_p^{(1)},g_p^{(2)},j}(x) = \int_{P_j} \int_{\Gamma_j} \langle T_x f, T_{\eta\lambda} g_p^{(1)} \rangle \langle T_{\zeta\lambda} g_p^{(2)}, T_x f \rangle \; d\mu_{\Gamma_j}(\lambda) d\mu_{P_j}(p) \text{ for } x \in G. \tag{5.2.1}$$

Under standing assumptions (I)–(III), for each $j \in \mathcal{J}$, the above series converges pointwise to a continuous function (see [53, Lemma 3.1]). Next, we define the function $w_{f;g_p^{(1)},g_p^{(2)}} : G \to \mathbb{C}$ by

$$\begin{aligned} w_{f;g_p^{(1)},g_p^{(2)}}(x) &= \sum_{j\in\mathcal{J}} w_{f;g_p^{(1)},g_p^{(2)}}(x) \\ &= \sum_{j\in\mathcal{J}} \int_{P_j} \int_{\Gamma_j} \langle T_x f, T_{\eta\lambda} g_p^{(1)} \rangle \langle T_{\zeta\lambda} g_p^{(2)}, T_x f \rangle \; d\mu_{\Gamma_j}(\lambda) d\mu_{P_j}(p) \end{aligned} \tag{5.2.2}$$

provided the series on the right-hand side converges. If $g_p^{(1)} = g_p^{(2)}$ for all $p \in P_j, j \in \mathcal{J}$, we write $w_{f;g_p^{(1)},g_p^{(2)}} = w_{f;g_p^{(1)}}$ and $w_{f;g_p^{(1)},g_p^{(2)},j} = w_{f;g_p^{(i)},j}$.

Remark 5.2.1.

(i) Under standing assumptions (I)–(III), the series $\sum_{j\in\mathcal{J}} w_{f;g_p^{(i)},j}$ converges in $[0,\infty]$, ensuring that $w_{f;g_p^{(i)}}$ is well-defined, although $w_{f;g_p^{(1)},g_p^{(2)}}$ may not be.

(ii) If each GTI system forms a Bessel family, then $w_{f;g_p^{(1)},g_p^{(2)},j}$ converges absolutely and uniformly on compact subsets of G; see [53, Lemma 3.3].

We are now ready to introduce the notion of unconditional convergence, motivated by [53].

Definition 5.2.2. Let $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda\in\eta\Gamma_j, p\in P_j}$ and $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(2)}\}_{\lambda\in\zeta\Gamma_j, p\in P_j}$ be two GTI systems in $L^2(G)$.

(i) They satisfy the *dual* 1-*unconditional convergence property (dual 1-UCP)* with respect to $B \in \mathcal{B}$, whenever for all $f \in \mathcal{D}_B(G)$, the function $w_{f;g_p^{(1)},g_p^{(2)}}$ is almost periodic and the series

$$w_{f;g_p^{(1)},g_p^{(2)}} = \sum_{j\in\mathcal{J}} w_{f;g_p^{(1)},g_p^{(2)},j} \tag{5.2.3}$$

converges unconditionally with respect to the mean $M(|\cdot|)$, i.e. for every $\epsilon > 0$, there exist a finite set $\mathcal{J}' \subset \mathcal{J}$ such that for all finite set $\mathcal{J}'' \supset \mathcal{J}'$,

$$M\left(\left|w_{f;g_p^{(1)},g_p^{(2)}} - \sum_{k\in\mathcal{I}''} w_{f;g_p^{(1)},g_p^{(2)},j}\right|\right) < \epsilon,$$

where M is the mean, and its exact definition can be found in [53, Theorem 3.6].

(ii) They satisfy the *dual* ∞*-UCP* if the series in (5.2.3) converges uniformly on G.

In case $g_p^{(1)} = g_p^{(2)}$ for all $p \in P_j$, we refer to the above conditions collectively as the α*-UCP* for the system $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda\in\eta\Gamma_j, p\in P_j}$, where $\alpha \in \{1, \infty\}$.

Note that $w_{f;g_p^{(1)},g_p^{(2)}}$ is almost periodic is assumed in dual 1-UCP, whereas in the dual ∞-UCP, it follows from uniform convergence of the series. We now describe the relationships between various forms of the UCP and the LIC for GTI systems. For the definition of LIC, we refer to [79, 60].

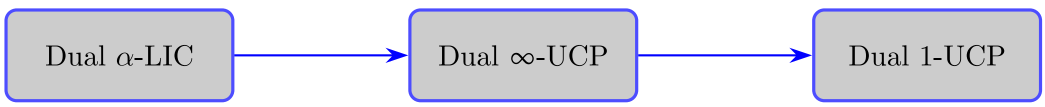
Remark 5.2.3. The following relationships hold among the UCP and LIC properties for GTI systems:

(i) If the dual ∞-UCP holds for the given two GTI systems, then the dual 1-UCP also holds.

(ii) If the dual α-LIC holds for the given two GTI systems, then the dual ∞-UCP holds, and consequently, the dual 1-UCP holds as well [53, Remark 6].

(iii) If the α-LIC (or LIC) holds for a given GTI system, then the ∞-UCP holds, and hence the 1-UCP also holds for that GTI system.

The logical implications among these properties can be visualized as follows:



Remark 5.2.4. The Bessel property and the ∞-UCP are generally independent of each other, neither implies the other. However, following [53, Lemma 3.9], if one GTI system is Bessel and the other satisfies the ∞-UCP, then together they satisfy the dual ∞-UCP. The same holds for the 1-UCP.

For further details on the connection between Bessel systems and the UCP, we refer the reader to [53, 90]. With the unconditional convergence property in place, we are now in a position to state our main result: a characterization of when two GTI Bessel systems are pairwise orthogonal under the dual 1-UCP condition.

5.3. Characterization result on GTI orthogonal frame pairs via UCP

In this section, we provide a characterization for two GTI Bessel (frame) systems to be pairwise orthogonal under the dual 1-UCP condition. Such a characterization has not been established previously, even in classical settings such as $L^2(\mathbb{R}^n)$. In this context, Theorem 5.3.1 gives necessary and sufficient conditions for GTI Bessel (frame) systems in $L^2(G)$ to be pairwise orthogonal.

Theorem 5.3.1. *Let* $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\eta\Gamma_j,p\in P_j}$ *and* $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\zeta\Gamma_j,p\in P_j}$ *be two GTI Bessel (frame) systems in* $L^2(G)$ *such that one of the systems is satisfying the* 1*-UCP. Then the following assertions are equivalent:*

(i) The systems $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\eta\Gamma_j,p\in P_j}$ *and* $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\zeta\Gamma_j,p\in P_j}$ *are pairwise orthogonal Bessel (frame).*

(ii) For every $\alpha\in\bigcup_{j\in\mathcal{J}}\Gamma_j^\perp\setminus\{0\}$*, we have*

$$t_\alpha(\gamma):=\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp}\frac{1}{s(\Gamma_j)}\int_{P_j}\overline{\widehat{g_p^{(1)}}(\eta^*\gamma)}\widehat{g_p^{(2)}}(\zeta^*(\gamma+\alpha))d\mu_{P_j}(p)=0 \text{ for a.e. } \gamma\in\widehat{G},$$

and

$$t_0(\gamma):=\sum_{j\in\mathcal{J}}\frac{1}{s(\Gamma_j)}\int_{P_j}\overline{\widehat{g_p^{(1)}}(\eta^*\gamma)}\widehat{g_p^{(2)}}(\zeta^*\gamma)d\mu_{P_j}(p)=0 \text{ for a.e. } \gamma\in\widehat{G}.$$

In view of Remark 5.2.3, a similar result holds under the dual ∞-UCP condition. To prove 5.3.1, we first establish the following auxiliary result.

Theorem 5.3.2. *Let* $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ *and* $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,p\in P_j}$ *be two GTI Bessel (frame) systems in* $L^2(G)$ *satisfying the dual* 1*-UCP. Then the following assertions are equivalent:*

(i) For every $\alpha\in\bigcup_{j\in\mathcal{J}}\Gamma_j^\perp\setminus\{0\}$*, we have*

$$t_\alpha(\gamma):=\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp}\frac{1}{s(\Gamma_j)}\int_{P_j}\overline{\widehat{g_p^{(1)}}(\gamma)}\widehat{g_p^{(2)}}(\gamma+\alpha)d\mu_{P_j}(p)=0 \text{ for a.e. } \gamma\in\widehat{G},$$

and

$$t_0(\gamma) := \sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(\gamma)} \widehat{g_p^{(2)}}(\gamma) d\mu_{P_j}(p) = 0 \text{ for a.e. } \gamma \in \widehat{G}.$$

(ii) The systems $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ *and* $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j}$ *are pairwise orthogonal.*

The following result provides a necessary and sufficient condition under which a GTI system $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$, satisfying the 1-UCP, forms a tight frame.

Theorem 5.3.3. *Let the GTI system* $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ *satisfy the 1-UCP. Then the following assertions are equivalent:*

(i) The system $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ *is a tight frame for* $L^2(G)$ *with frame bound* K.

(ii) For each $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^\perp$,

$$\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp} \frac{1}{s(\Gamma_j)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(\gamma)} \widehat{g_p^{(1)}}(\gamma+\alpha) d\mu_{P_j}(p) = \delta_{\alpha,0} K \text{ for a.e. } \gamma \in \widehat{G}.$$

It is worth mentioning that if $K = 1$, the above result provide necessary and sufficient condition under which a GTI system $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$, satisfying the 1-UCP, forms a Parseval frame.

Further it can be observed that Theorem 5.3.3 extends [53, Theorem 3.12], which provides characterization for Parseval frames in the special case $P_j = \{j\}$ and Γ_j taken as lattices. Moreover, our theorem generalizes [79, Theorem 3.5], where a similar Parseval frame characterization was obtained under the α-LIC.

5.3.1. Proof of Theorems 5.3.1, 5.3.2, and 5.3.3

In order to prove Theorems 5.3.1, 5.3.2 and 5.3.3, we first provide Proposition 5.3.4, which identifies a key structural property of the mixed dual Gramian operator. Specifically, it shows that the vanishing of the components t_α, for $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^\perp \setminus \{0\}$, is equivalent to the mixed dual Gramian operator commuting with translations. Moreover, it characterizes such operators as Fourier multipliers.

Proposition 5.3.4. *Let* $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ *and* $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j}$ *be two GTI Bessel systems satisfying the dual* 1*-UCP, and let* Θ*, defined in* (5.1.3)*, be the mixed dual Gramian operator associated with these systems. Then the following statements are equivalent:*

(i) The operator Θ *commutes with the family of translation operators* $\{T_x\}_{x\in G}$*, i.e.,* $\Theta T_x = T_x \Theta$ *for all* $x \in G$*.*

(ii) For every $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^\perp \setminus \{0\}$*, the function*

$$t_\alpha(\gamma) = \sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp} \frac{1}{s(\Gamma_j)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(\gamma)}\widehat{g_p^{(2)}}(\gamma+\alpha) d\mu_{P_j}(p) = 0 \text{ for a.e. } \gamma \in \widehat{G}.$$

Furthermore, if either (i) or (ii) holds, then Θ *is a Fourier multiplier operator with symbol*

$$s(\gamma) = \sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(\gamma)}\, \widehat{g_p^{(2)}}(\gamma)\, d\mu_{P_j}(p),$$

that is, $\widehat{\Theta f}(\gamma) = s(\gamma)\,\widehat{f}(\gamma)$ *for all* $f \in L^2(G)$*.*

The above proposition may be viewed as a generalization of [116, Lemma 1], which addresses the classical Euclidean case $L^2(\mathbb{R}^d)$, and [60, Proposition 3.7], which is formulated for LCA groups. The following lemma will be used in the proof of Proposition 5.3.4.

Lemma 5.3.5. *Suppose the GTI systems* $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ *and* $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,p\in P_j}$ *satisfy the assumptions of Proposition 5.3.4. Then the multiplication operator* $M_{\bar{t}_\alpha} : L^2(\widehat{G}) \to L^2(\widehat{G}),\ f \mapsto f.\bar{t}_\alpha$ *is well defined and bounded. Moreover, for all* $f \in \mathcal{D}_B$*, the following identity holds:*

$$\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp} d_{P_j,\alpha} = \left\langle \widehat{f}, M_{\bar{t}_\alpha} T_{-\alpha}\widehat{f}\right\rangle, \tag{5.3.1}$$

where

$$d_{P_j,\alpha} = \frac{1}{s(\Gamma_j)} \int_{\widehat{G}} \int_{P_j} \widehat{f}(w)\overline{\widehat{f}(w+\alpha)}\overline{\widehat{g_p^{(1)}}(w)}\widehat{g_p^{(2)}}(w+\alpha)\ d\mu_{\widehat{G}}(w) d\mu_{P_j}(p). \tag{5.3.2}$$

Proof. We have

$$t_\alpha(\gamma) = \sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp} \frac{1}{s(\Gamma_j)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(\gamma)}\widehat{g_p^{(2)}}(\gamma+\alpha) d\mu_{P_j}(p).$$

Note that the right-hand side converges absolutely, as seen from the following chain of inequalities:

$$\begin{aligned}\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp} \frac{1}{s(\Gamma_j)} \int_{P_j} \left|\overline{\widehat{g_p^{(1)}}(\gamma)}\widehat{g_p^{(2)}}(\gamma+\alpha)\right| d\mu_{P_j}(p) &\le \sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \int_{P_j} \left|\overline{\widehat{g_p^{(1)}}(\gamma)}\widehat{g_p^{(2)}}(\gamma+\alpha)\right| d\mu_{P_j}(p) \\ &\le \left(\sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \int_{P_j} \left|\overline{\widehat{g_p^{(1)}}(\gamma)}\right|^2 d\mu_{P_j}(p)\right)^{1/2} \left(\sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \int_{P_j} \left|\widehat{g_p^{(2)}}(\gamma+\alpha)\right|^2 d\mu_{P_j}(p)\right)^{1/2}\end{aligned}$$

$$\leq B_{g^{(1)}}^{1/2} B_{g^{(2)}}^{1/2},$$

where the last inequality follows from [79, Proposition 3.3], and $B_{g^{(1)}}$ and $B_{g^{(2)}}$ are the Bessel constants associated with $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j}$, respectively. Hence, the multiplication operator $M_{\bar{t}_\alpha}: L^2(\widehat{G}) \to L^2(\widehat{G}),\ f \mapsto f.\bar{t}_\alpha$ is well defined and bounded. Also note that for $f \in \mathcal{D}_B$,

$$\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp} \int_{\widehat{G}} \left| \widehat{f}(w)\overline{\widehat{f}(w+\alpha)} \frac{1}{s(\Gamma_j)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(w)} \widehat{g_p^{(2)}}(w+\alpha) d\mu_{P_j}(p) \right| d\mu_{\widehat{G}}(w) < \infty. \qquad (5.3.3)$$

Now, for $f \in \mathcal{D}_B$, we compute:

$$\begin{aligned}\langle \widehat{f}, M_{\bar{t}_\alpha} T_{-\alpha}\widehat{f}\rangle &= \int_{\widehat{G}} \widehat{f}(w)\overline{M_{\bar{t}_\alpha} T_{-\alpha}\widehat{f}(w)}\ d\mu_{\widehat{G}}(w) = \int_{\widehat{G}} \widehat{f}(w) t_\alpha(w)\overline{\widehat{f}(w+\alpha)}\ d\mu_{\widehat{G}}(w) \\ &= \int_{\widehat{G}} \widehat{f}(w)\overline{\widehat{f}(w+\alpha)} \sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp} \frac{1}{s(\Gamma_j)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(w)} \widehat{g_p^{(2)}}(w+\alpha) d\mu_{P_j}(p)\ d\mu_{\widehat{G}}(w).\end{aligned}$$

Applying Fubini's theorem, we obtain:

$$\begin{aligned}\langle \widehat{f}, M_{\bar{t}_\alpha T_{-\alpha}}\widehat{f}\rangle &= \sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp} \frac{1}{s(\Gamma_j)} \int_{P_j}\int_{\widehat{G}} \widehat{f}(w)\overline{\widehat{f}(w+\alpha)}\, \overline{\widehat{g_p^{(1)}}(w)} \widehat{g_p^{(2)}}(w+\alpha)\ d\mu_{\widehat{G}}(w) d\mu_{P_j}(p) \\ &= \sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp} d_{P_j,\alpha},\end{aligned}$$

which completes the proof. ∎

With the help of the previous lemma, we now proceed to prove Proposition 5.3.4.

Proof of Proposition 5.3.4. It is easy to observe that Θ commutes with the family of translations $\{T_x\}_{x\in G}$ if and only if $w_{f;g_p^{(1)},g_p^{(2)}}$ is constant for all $f \in \mathcal{D}_B$. Next, we show that $w_{f;g_p^{(1)},g_p^{(2)}}$ is constant if and only if, for each $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^\perp \setminus \{0\}$, we have

$$t_\alpha(\gamma) = \sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp} \frac{1}{s(\Gamma_j)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(\gamma)} \widehat{g_p^{(2)}}(\gamma+\alpha) d\mu_{P_j}(p) = 0 \quad \text{for a.e. } \gamma \in \widehat{G}.$$

Since the GTI systems under consideration satisfy the dual 1-UCP, we follow an approach similar to that used in [53, Proposition 3.10]. For $x \in G$, the function $w_{f;g_p^{(1)},g_p^{(2)}}(x)$ can be expressed as follows:

$$w_{f;g_p^{(1)},g_p^{(2)}}(x) = \sum_{\alpha\in \bigcup_{j\in\mathcal{J}}\Gamma_j^\perp} \alpha(x) \widehat{w_{f;g_p^{(1)},g_p^{(2)}}}(\alpha),$$

where for $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^{\perp}$,

$$\widehat{w_{f;g_p^{(1)},g_p^{(2)}}}(\alpha) = \sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^{\perp}} d_{P_j,\alpha}, \tag{5.3.4}$$

and $d_{P_j,\alpha}$ is given in (5.3.2). Now for $x \in G$, we consider the function

$$z_{f,g_p^{(1)},g_p^{(2)}}(x) = w_{f;g_p^{(1)},g_p^{(2)}}(x) - w_{f;g_p^{(1)},g_p^{(2)}}(0),$$

which is continuous due to the continuity of $w_{f;g_p^{(1)},g_p^{(2)}}$. The generalized Fourier series of $z_{f,g_p^{(1)},g_p^{(2)}}(x)$ is given by

$$z_{f,g_p^{(1)},g_p^{(2)}}(x) = \sum_{\alpha\in \bigcup_{j\in\mathcal{J}} \Gamma_j^{\perp}} \alpha(x)\, \widehat{z_{f,g_p^{(1)},g_p^{(2)}}}(\alpha),$$

with generalized Fourier coefficients

$$\widehat{z_{f;g_p^{(1)},g_p^{(2)}}}(\alpha) = \begin{cases} \widehat{w_{f;g_p^{(1)},g_p^{(2)}}}(\alpha) - w_{f;g_p^{(1)},g_p^{(2)}}(0) & \text{if } \alpha = 0, \\ \widehat{w_{f;g_p^{(1)},g_p^{(2)}}}(\alpha) & \text{if } \alpha \neq 0. \end{cases} \tag{5.3.5}$$

The function $w_{f;g_p^{(1)},g_p^{(2)}}$ is constant if and only if $z_{f,g_p^{(1)},g_p^{(2)}}(x) \equiv 0$. By the uniqueness theorem for generalized Fourier series, this holds if and only if

$$\widehat{z_{f,g_p^{(1)},g_p^{(2)}}}(\alpha) = 0 \quad \text{for all } \alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^{\perp}.$$

This is equivalent to

$$\widehat{w_{f;g_p^{(1)},g_p^{(2)}}}(\alpha) = \delta_{\alpha,0} w_{f;g_p^{(1)},g_p^{(2)}}(0) \quad \text{for all } \alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^{\perp}. \tag{5.3.6}$$

For the forward implication, assume $w_{f;g_p^{(1)},g_p^{(2)}}$ is constant. Then, by (5.3.6), we have

$$\widehat{w_{f;g_p^{(1)},g_p^{(2)}}}(\alpha) = 0 \quad \text{for all } \alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^{\perp} \setminus \{0\}. \tag{5.3.7}$$

Next, by Lemma 5.3.5, for $f \in \mathcal{D}_B$, we have

$$\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^{\perp}} d_{P_j,\alpha} = \left\langle \widehat{f},\, M_{\bar{t}_\alpha} T_{-\alpha} \widehat{f} \right\rangle,$$

and by (5.3.4), it follows that

$$\left\langle \widehat{f},\, M_{\bar{t}_\alpha} T_{-\alpha} \widehat{f} \right\rangle = \widehat{w_{f;g_p^{(1)},g_p^{(2)}}}(\alpha). \tag{5.3.8}$$

Therefore, using (5.3.7), we obtain

$$\left\langle \widehat{f},\, M_{\bar{t}_\alpha} T_{-\alpha} \widehat{f} \right\rangle = 0 \quad \text{for all } \alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^{\perp} \setminus \{0\}.$$

Since $\mathcal{D}_B$ is dense in $L^2(\widehat{G})$, it follows that $M_{\bar{t}_\alpha} T_{-\alpha} \widehat{f} = 0$ for all $\widehat{f} \in L^2(\widehat{G})$ and all $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^\perp \setminus \{0\}$. Equivalently,

$$M_{\bar{t}_\alpha} T_{-\alpha} = 0 \quad \text{for all } \alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^\perp \setminus \{0\}.$$

Hence for all $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^\perp \setminus \{0\}$, we have

$$M_{\bar{t}_\alpha} T_{-\alpha} \widehat{g}(\gamma) = \bar{t}_\alpha(\gamma) \widehat{g}(\gamma + \alpha) = 0 \quad \text{a.e. } \gamma \in \widehat{G},\ \forall \widehat{g} \in L^2(\widehat{G}),$$

which implies $t_\alpha = 0$ for all $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^\perp \setminus \{0\}$. Conversely, we assume $t_\alpha = 0$ for all $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^\perp \setminus \{0\}$. Then,

$$\left\langle \widehat{f}, M_{\bar{t}_\alpha} T_{-\alpha} \widehat{f} \right\rangle = 0 \quad \forall f \in \mathcal{D}_B,$$

and by (5.3.8), it follows that $w_{\widehat{f;g_p^{(1)},g_p^{(2)}}}(\alpha) = 0$ for $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^\perp \setminus \{0\}$. Moreover, from (5.3.5), we have $w_{\widehat{f;g_p^{(1)},g_p^{(2)}}}(0) = w_{f;g_p^{(1)},g_p^{(2)}}(0)$, so

$$w_{\widehat{f;g_p^{(1)},g_p^{(2)}}}(\alpha) = \delta_{\alpha,0} w_{f;g_p^{(1)},g_p^{(2)}}(0) \quad \text{for all } \alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^\perp.$$

Hence, by (5.3.6), we conclude that $w_{f;g_p^{(1)},g_p^{(2)}}$ is constant. Without loss of generality, assume that (i) holds. It is well known that if the mixed dual Gramian operator Θ commutes with the family of translation operators $\{T_x\}_{x\in G}$, then Θ is a Fourier multiplier [89, Theorem 4.1.1]. Hence, there exists a unique function $s \in L^\infty(\widehat{G})$ such that $\widehat{\Theta f}(w) = s(w)\,\widehat{f}(w)$, where $s(w)$ denotes the symbol corresponding to Θ. Now, using the definition of $w_{f;g_p^{(1)},g_p^{(2)}}$, we have

$$\begin{aligned} w_{f;g_p^{(1)},g_p^{(2)}}(0) &= \langle \Theta f, f \rangle = \left\langle \widehat{\Theta f}, \widehat{f} \right\rangle \\ &= \int_{\widehat{G}} \widehat{\Theta f}(w)\, \overline{\widehat{f}(w)}\, d\mu_{\widehat{G}}(w) \\ &= \int_{\widehat{G}} s(w)\, \widehat{f}(w)\, \overline{\widehat{f}(w)}\, d\mu_{\widehat{G}}(w), \end{aligned} \tag{5.3.9}$$

where the final equality follows from the Fourier multiplier representation of Θ. Next, using equation (5.3.4) and simplifying, we obtain

$$\begin{aligned} w_{\widehat{f;g_p^{(1)},g_p^{(2)}}}(\alpha) &= \sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp} d_{P_j,\alpha} \\ &= \sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp} \frac{1}{s(\Gamma_j^\perp)} \int_{\widehat{G}} \int_{P_j} \widehat{f}(w)\, \overline{\widehat{f}(w+\alpha)}\, \overline{\widehat{g_p^{(1)}}(w)}\, \widehat{g_p^{(2)}}(w+\alpha)\, d\mu_{\widehat{G}}(w)\, d\mu_{P_j}(p) \end{aligned}$$

$$= \sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp} \int_{\widehat{G}} \widehat{f}(w)\,\overline{\widehat{f}(w+\alpha)} \left(\frac{1}{s(\Gamma_j^\perp)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(w)}\,\widehat{g_p^{(2)}}(w+\alpha)\, d\mu_{P_j}(p) \right) d\mu_{\widehat{G}}(w)$$

$$= \int_{\widehat{G}} \widehat{f}(w)\,\overline{\widehat{f}(w+\alpha)} \left(\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp} \frac{1}{s(\Gamma_j^\perp)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(w)}\,\widehat{g_p^{(2)}}(w+\alpha)\, d\mu_{P_j}(p) \right) d\mu_{\widehat{G}}(w). \tag{5.3.10}$$

Also, using equation (5.3.6), and applying (5.3.4) for $\alpha = 0$, we get

$$w_{f;g_p^{(1)},g_p^{(2)}}(0) = \widehat{w_{f;g_p^{(1)},g_p^{(2)}}}(0) = \sum_{j\in\mathcal{J}} d_{P_j,0}.$$

By setting $\alpha = 0$ in equation (5.3.10) and substituting into the expression above, we obtain:

$$w_{f;g_p^{(1)},g_p^{(2)}}(0) = \int_{\widehat{G}} \left(\sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(w)}\,\widehat{g_p^{(2)}}(w)\, d\mu_{P_j}(p) \right) \widehat{f}(w)\,\overline{\widehat{f}(w)}\, d\mu_{\widehat{G}}(w). \tag{5.3.11}$$

Comparing (5.3.9) and (5.3.11), we conclude that

$$s(w) = \sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(w)}\,\widehat{g_p^{(2)}}(w)\, d\mu_{P_j}(p).$$

This completes the proof. ∎

Proof of Theorem 5.3.2: **(i) $\implies$ (ii):** Let Θ be the mixed dual Gramian operator associated with the given GTI systems. Since $t_\alpha(\gamma) = 0$ for all $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^\perp \setminus \{0\}$, it follows from Proposition 5.3.4, Θ is a Fourier multiplier with symbol s, that is, $\widehat{\Theta f}(w) = s(w)\,\widehat{f}(w)$. Moreover,

$$s(w) = \sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(w)}\,\widehat{g_p^{(2)}}(w)\, d\mu_{P_j}(p) = t_0(w) = 0$$

for almost every $w \in \widehat{G}$. Thus, $\widehat{\Theta f} = 0$, which implies $\Theta f = 0$ for all $f \in \mathcal{D}_B$. Therefore, $\Theta = 0$ and hence the GTI systems $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j}$ are pairwise orthogonal Bessel families (and frames). This proves (ii).

(ii) $\implies$ (i): Assume that the GTI systems $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j}$ are pairwise orthogonal frames, i.e., $\Theta = 0$. Then $\Theta T_x f = 0$ for all $x \in G$ and $f \in \mathcal{D}_B$. By definition of $w_{f;g_p^{(1)},g_p^{(2)}}$, we have

$$w_{f;g_p^{(1)},g_p^{(2)}}(x) = \langle \Theta T_x f, T_x f\rangle \quad \text{for } f \in \mathcal{D}_B,\ x \in G.$$

Therefore, $w_{f;g_p^{(1)},g_p^{(2)}} \equiv 0$. By the uniqueness theorem for generalized Fourier series, for each $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^{\perp}$, we have $\widehat{w_{f;g_p^{(1)},g_p^{(2)}}}(\alpha) = 0$. Using (5.3.4) and Lemma 5.3.5, we obtain

$$\widehat{w_{f;g_p^{(1)},g_p^{(2)}}}(\alpha) = \sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^{\perp}} d_{j,\alpha} = \left\langle \widehat{f}, M_{\bar{t}_\alpha} T_{-\alpha}\widehat{f} \right\rangle = 0 \tag{5.3.12}$$

for all $f \in \mathcal{D}_B$ and $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^{\perp}$. Since $\mathcal{D}_B$ is dense in $L^2(\widehat{G})$, it follows that $M_{\bar{t}_\alpha} T_{-\alpha}\widehat{f} = 0$ for all $\widehat{f} \in L^2(\widehat{G})$. Hence,

$$M_{\bar{t}_\alpha} T_{-\alpha}\widehat{g}(\gamma) = \bar{t}_\alpha(\gamma)\widehat{g}(\gamma+\alpha) = 0 \quad \text{for a.e. } \gamma \in \widehat{G},\ \forall\, \widehat{g} \in L^2(\widehat{G})$$

which implies $t_\alpha = 0$ for all $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^{\perp}$. This proves (i). ∎

Following Remark 5.2.3, if the so-called dual α-LIC holds for the two GTI systems, then these systems satisfy the dual ∞-UCP. Consequently, [60, Theorem 3.5] becomes a corollary of Theorem 5.3.2, as stated below:

Corollary 5.3.6. *Let $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j}$ be two GTI Bessel (frame) systems for $L^2(G)$ which satisfy the dual α-LIC. Then the following assertion are equivalent:*

(i) Both the above GTI Bessel (frame) systems in $L^2(G)$ are pairwise orthogonal.

(ii) For each $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^{\perp} \setminus \{0\}$, we have

$$t_\alpha(\gamma) := \sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^{\perp}} \frac{1}{s(\Gamma_j)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(\gamma)}\,\widehat{g_p^{(2)}}(\gamma+\alpha)\, d\mu_{P_j}(p) = 0 \textit{ for a.e. } \gamma \in \widehat{G},$$

and

$$t_0(\gamma) := \sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(\gamma)}\,\widehat{g_p^{(2)}}(\gamma)\, d\mu_{P_j}(p) = 0 \textit{ for a.e. } \gamma \in \widehat{G}.$$

With this we are ready to prove Thoerem 5.3.1.

Proof of Theorem 5.3.1 Let D_η and D_ζ be the (unitary) dilation operators associated with η and ζ, respectively. By polarisation identity, $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\eta\Gamma_j, p\in P_j}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\zeta\Gamma_j, p\in P_j}$ are pairwise orthogonal if and only if for every $f \in \mathcal{D}_B$, we have

$$\langle D_\zeta \Theta D_\eta^{-1} f, f\rangle = 0,$$

as $\mathcal{D}_B$ is dense in $L^2(G)$. We can rewrite the left-hand side of the above equation as follows:

$$\begin{aligned}\langle D_\zeta \Theta D_\eta^{-1} f, f\rangle &= \langle \Theta D_\eta^{-1} f, D_\zeta^{-1} f\rangle \\ &= \sum_{j\in\mathcal{J}} \int_{P_j} \int_{\Gamma_j} \langle D_\eta^{-1} f, T_{\eta\lambda} g_p^{(1)}\rangle \langle T_{\zeta\lambda} g_p^{(2)}, D_\zeta^{-1} f\rangle \, d\mu_{\Gamma_j}(\lambda)\, d\mu_{P_j}(p) \\ &= \sum_{j\in\mathcal{J}} \int_{P_j} \int_{\Gamma_j} \langle f, D_\eta T_{\eta\lambda} g_p^{(1)}\rangle \langle D_\zeta T_{\zeta\lambda} g_p^{(2)}, f\rangle \, d\mu_{\Gamma_j}(\lambda)\, d\mu_{P_j}(p).\end{aligned}$$

Using the intertwining relation $D_\eta T_\lambda = T_{\eta^{-1}(\lambda)} D_\eta$ for $\gamma \in \Gamma_j$, the above expression becomes

$$\langle D_\zeta \Theta D_\eta^{-1} f, f\rangle = \sum_{j\in\mathcal{J}} \int_{P_j} \int_{\Gamma_j} \langle f, T_\lambda D_\eta g_p^{(1)}\rangle \langle T_\lambda D_\zeta g_p^{(2)}, f\rangle \, d\mu_{\Gamma_j}(\lambda)\, d\mu_{P_j}(p).$$

Thus $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda\in\eta\Gamma_j, p\in P_j}$ and $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(2)}\}_{\lambda\in\zeta\Gamma_j, p\in P_j}$ are pairwise orthogonal if and only if the systems $\bigcup_{j\in\mathcal{J}} \{T_\lambda D_\eta g_p^{(1)}\}_{\gamma\in\Gamma_j}$ and $\bigcup_{j\in\mathcal{J}} \{T_\lambda D_\zeta g_p^{(2)}\}_{\lambda\in\Gamma_j}$ are pairwise orthogonal. Next, we prove that the GTI systems $\bigcup_{j\in\mathcal{J}} \{T_\lambda D_\eta g_p^{(1)}\}_{\gamma\in\Gamma_j}$ and $\bigcup_{j\in\mathcal{J}} \{T_\lambda D_\zeta g_p^{(2)}\}_{\lambda\in\Gamma_j}$ satisfy dual 1-UCP. By hypothesis, without loss of generality we assume that $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda\in\eta\Gamma_j, p\in P_j}$ satisfy 1-UCP. Thus, for each $f \in \mathcal{D}_{\mathcal{B}}$, $w_{f;g_p^{(1)}}$ is almost periodic and

$$w_{f;g_p^{(1)}} = \sum_{j\in\mathcal{J}} w_{f;g_p^{(i)},j}$$

converges with respect to M.

Now for $f \in \mathcal{D}_B$ and $x \in G$, we have

$$\begin{aligned} w_{D_\eta^{-1} f, g_p^{(1)}, j} &= \int_{P_j} \int_{\Gamma_j} \left|\langle T_x D_\eta^{-1} f, T_{\eta\lambda} g_p^{(1)}\rangle\right|^2 \; d\mu_{\Gamma_j}(\lambda) d\mu_{P_j}(p) \\ &= \int_{P_j} \int_{\Gamma_j} \left|\langle D_\eta^{-1} T_{\eta^{-1}x} f, T_{\eta\lambda} g_p^{(1)}\rangle\right|^2 \; d\mu_{\Gamma_j}(\gamma) d\mu_{P_j}(p) \\ &= \int_{P_j} \int_{\Gamma_j} \left|\langle T_{\eta^{-1}x} f, D_\eta T_{\eta\lambda} g_p^{(1)}\rangle\right|^2 \; d\mu_{\Gamma_j}(\lambda) d\mu_{P_j}(p) \\ &= \int_{P_j} \int_{\Gamma_j} \left|\langle T_{\eta^{-1}x} f, T_\lambda D_\eta g_p^{(1)}\rangle\right|^2 \; d\mu_{\Gamma_j}(\lambda) d\mu_{P_j}(p). \end{aligned}$$

Thus for $g := D_\eta^{-1} f$ and $y := \alpha^{-1} x$, we have

$$w_{g, g_p^{(1)}, j} = \int_{P_j} \int_{\Gamma_j} \left|\langle T_y g, T_\lambda D_\eta g_p^{(1)}\rangle\right|^2 \; d\mu_{\Gamma_j}(\gamma) d\mu_{P_j}(p). \tag{5.3.13}$$

Hence, using (5.3.13) and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\eta\Gamma_j,p\in P_j}$ satisfy UCP, we can say that $\bigcup_{j\in\mathcal{J}}\{T_\lambda D_\eta g_p^{(1)}\}_{\lambda\in\Gamma_j}$ satisfy the 1-UCP. Similarly, since $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\zeta\Gamma_j,p\in P_j}$ is a Bessel, it follows that $\bigcup_{j\in\mathcal{J}}\{T_\lambda D_\zeta g_p^{(2)}\}_{\lambda\in\Gamma_j}$ is Bessel system. Now, in view of Remark 5.2.4, the systems $\bigcup_{j\in\mathcal{J}}\{T_\lambda D_\eta g_p^{(1)}\}_{\lambda\in\Gamma_j}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda D_\zeta g_p^{(2)}\}_{\lambda\in\Gamma_j}$ satisfy the dual 1-UCP. By Theorem 5.3.2, these systems are pairwise orthogonal if and only if for each $\alpha\in\bigcup_{j\in\mathcal{J}}\Gamma_j^\perp\smallsetminus\{0\}$, we have

$$\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp}\frac{1}{s(\Gamma_j)}\int_{P_j}\overline{\widehat{D_\eta g_p^{(1)}}(\gamma)}\widehat{\mathcal{D}_\zeta g_p^{(2)}}(\gamma+\alpha)d\mu_{P_j}(p)=0\quad\text{for a. e. }\gamma\in\widehat{G},$$

and

$$\sum_{j\in\mathcal{J}}\frac{1}{s(\Gamma_j)}\int_{P_j}\overline{\widehat{D_\eta g_p^{(1)}}(\gamma)}\widehat{\mathcal{D}_\zeta g_p^{(2)}}(\gamma)d\mu_{P_j}(p)=0\quad\text{for a. e. }\gamma\in\widehat{G}.$$

Since $\widehat{D_\eta f}(\gamma)=(\Delta(\eta))^{1/2}\widehat{f}(\eta^*(\gamma)))$, this orthogonality condition is equivalent to, for each $\alpha\in\bigcup_{j\in\mathcal{J}}\Gamma_j^\perp\smallsetminus\{0\}$, we have

$$\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp}\frac{1}{s(\Gamma_j)}\int_{P_j}\overline{\widehat{g_p^{(1)}}(\eta^*\gamma)}\widehat{g_p^{(2)}}(\zeta^*(\gamma+\alpha))d\mu_{P_j}(p)=0,\quad\text{for a. e. }\gamma\in\widehat{G},$$

and

$$\sum_{j\in\mathcal{J}}\frac{1}{s(\Gamma_j)}\int_{P_j}\overline{\widehat{g_p^{(1)}}(\eta^*\gamma)}\widehat{\mathcal{D}_\beta g_p^{(2)}}(\zeta^*\gamma)d\mu_{P_j}(p)=0\quad\text{for a. e. }\gamma\in\widehat{G}.$$

This completes the proof. ■

Proof of Theorem 5.3.3. Suppose (ii) holds, therefore $t_\alpha(\gamma)=0$ for all $\alpha\in\bigcup_{j\in\mathcal{J}}\Gamma_j^\perp\smallsetminus\{0\}$, it follows from Proposition 5.3.4, Θ is a Fourier multiplier with symbol s, that is, $\widehat{\Theta f}(w)=s(w)\,\widehat{f}(w)$, $w\in\widehat{G}$. Moreover,

$$s(w)=\sum_{j\in\mathcal{J}}\frac{1}{s(\Gamma_j)}\int_{P_j}\overline{\widehat{g_p^{(1)}}(w)}\,\widehat{g_p^{(1)}}(w)\,d\mu_{P_j}(p)=t_0(w)=K,\ \text{for }w\in\widehat{G}.$$

by hypothesis. Thus $\left\|\widehat{\Theta f}(w)\right\|=K\left\|\widehat{f}\right\|$ and hence, the system $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ is a tight frame with frame bound K. The converse part follows directly by observing the following formula:

$$w_{\widehat{f;g_p^{(1)},g_p^{(2)}}}(\alpha)=\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp}d_{j,\alpha}=\left\langle\widehat{f},M_{\bar{t}_\alpha}T_{-\alpha}\widehat{f}\right\rangle=K\delta_{\alpha,0}.$$

This completes the proof. ■

As an application of the GTI framework, we provide explicit criteria for the pairwise orthogonality of structured systems such as wavelets, Gabor, and shearlets.

5.4. Applications to Gabor, wavelet, and shearlet systems

The purpose of this section is to present applications of our first main result, Theorem 5.3.1, to Bessel families with TI, Gabor, wavelet, and cone-adapted shearlet structures, which are special cases of GTI systems.

This section is organized as follows.

In Corollary 5.4.2, we establish necessary and sufficient conditions for the pairwise orthogonality of Gabor systems. Subsequently, Proposition 5.4.3 provides a corresponding characterization for pairwise orthogonal wavelet systems in $L^2(G)$. As a consequence, we obtain results for composite wavelet and classical shearlet systems in Proposition 5.4.4 and Proposition 5.4.5, respectively. Finally, we present a characterization for cone-adapted shearlet systems (see, Proposition 5.4.6), which are often more effective in applications due to their ability to represent directional information more uniformly. Moreover, cone-adapted shearlet systems can be viewed as finite unions of shift-invariant and wavelet systems with composite dilations.

5.4.1. Translation invariant systems

Theorem 5.3.1 immediately yields the following application:

Proposition 5.4.1. *Let the TI systems $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\eta\Gamma,p\in P}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\zeta\Gamma,p\in P}$ be Bessel (frame) systems. Then, these systems are pairwise orthogonal if and only if for each $\alpha\in\Gamma^\perp$, we have*

$$t_\alpha(\gamma) := \sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(\eta^*\gamma)}\widehat{g_p^{(2)}}(\zeta^*(\gamma+\alpha))\, d\mu_{P_j}(p) = 0 \text{ for a.e. } \gamma\in\widehat{G}.$$

5.4.2. Gabor systems

For a character $\chi\in\widehat{G}$, we define the *modulation operator* M_χ on $L^2(G)$ by $M_\chi(f)(x) = \chi(x)f(x)$ for all $x\in G$. This operator satisfies the following Fourier domain identity:

$$(\widehat{M_\chi f})(\gamma) = \int_G \chi(x) f(x)\overline{\gamma(x)} d\mu_G(x) = \int_G f(x)\overline{(\gamma-\chi)(x)} d\mu_G(x) = T_\chi \widehat{f}(\gamma) \quad (5.4.1)$$

for all $f \in L^2(G)$ and a.e. $\gamma \in \widehat{G}$. Let Γ and Λ be co-compact subgroups of G and $\widehat{G}$, respectively. Consider a measure space Λ, Σ_Λ and μ_Λ satisfying the standing hypothesis, where Σ_Λ is a σ-algebra and μ_Λ is a regular Borel measure. For some index set $\mathcal{J} \subset \mathbb{Z}$, let $\Psi := \{\psi_j : j \in \mathcal{J}\} \subset L^2(G)$ be a set of functions. Then the collection

$$\{T_\lambda M_\chi \psi_j\}_{\lambda \in \eta\Gamma, \chi \in \Lambda, j \in \mathcal{J}} \quad (5.4.2)$$

is called the *Gabor system* generated by Ψ. We can express the Gabor system as

$$\{T_\lambda M_\chi \psi_j\}_{\lambda \in \eta\Gamma, \chi \in \Lambda, j \in J} = \bigcup_{j \in \mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda \in \eta\Gamma, p \in P_j},$$

where $P_j = \{(j,\chi) : \chi \in \Lambda\}$ and $g_p^{(1)} = g_{j,\chi}^{(1)} := M_\chi \psi_j$. Thus, the Gabor system is a special case of a TI system. Consequently, the following result is a direct corollary of Proposition 5.3.2.

Corollary 5.4.2. *Let $\{T_\lambda M_\chi \psi_j\}_{\lambda \in \eta\Gamma, \chi \in \Lambda, j \in J}$ and $\{T_\lambda M_\chi \phi_j\}_{\lambda \in \zeta\Gamma, \chi \in \Lambda, j \in J}$ be two Gabor Bessel (frame) systems in $L^2(G)$. Then these two systems are pairwise orthogonal Gabor Bessel (frame) systems in $L^2(G)$ if and only if, for each $\alpha \in \Gamma^\perp$, the following condition is satisfied:*

$$\sum_{j \in J} \int_\Lambda \overline{\widehat{\phi}_j(\eta^*(\gamma-\chi))}\widehat{\psi}_j(\zeta^*(\gamma+\alpha-\chi)) d\mu_\Lambda(\chi) = 0 \quad \textit{for a.e. } \gamma \in \widehat{G}.$$

Note that if we assume $\eta = \zeta = I$ (identity automorphism), then the above result coincide with [60, Proposition 4.3]. Furthermore if we assume $G = \mathbb{Z}^n$, and $\eta = \zeta = I$ (the identity matrix) and that P is a singleton set, then the result reduces to the characterization given by Lopez and Han in [94, Theorem 1.4(ii)].

5.4.3. Wavelet systems

Let $\mathrm{Aut}(G)$ denote the set of all bi-continuous group automorphisms of G. For each $\eta \in \mathrm{Aut}(G)$, the *modular function* $\Delta : \mathrm{Aut}(G) \to (0,\infty)$ is a semigroup homomorphism characterized uniquely by the condition

$$\int_G (g \circ \eta)(x)\, d\mu_G(x) = \Delta(\eta) \int_G g(x)\, d\mu_G(x)$$

for all integrable functions g on G with respect to the Haar measure μ_G (see [27, Theorem 6.2]). We define the isometric *dilation operator* $D_\eta : L^2(G) \to L^2(G)$ associated with η by

$$D_\eta f(x) = \Delta(\eta)^{-1/2} f(\eta(x)) \quad \text{for all } x \in G.$$

Let $\mathcal{J} \subset \mathbb{Z}$ be an index set and consider a family of automorphisms $\mathcal{A} = \{\eta_j : j \in \mathcal{J}\} \subset \mathrm{Aut}(G)$. Let Γ be a co-compact subgroup of G. For a countable index set $\mathcal{I}$, let $\Psi := \{\psi_i : i \in \mathcal{I}\} \subset L^2(G)$ be a collection of functions. Then, the system

$$\{D_{\eta_j} T_\lambda \psi_i\}_{j\in\mathcal{J},\, \lambda\in\eta\Gamma,\, i\in\mathcal{I}}$$

is called the *wavelet system* generated by Ψ. Using the intertwining relation $D_{\eta_j} T_\lambda = T_{\eta_j^{-1}(\lambda)} D_{\eta_j}$ for $\eta_j \in \mathcal{A}$ and $\lambda \in \Gamma$, the wavelet system can be expressed in the form of a GTI system as

$$\{D_{\eta_j} T_\lambda \psi_i\}_{j\in\mathcal{J},\, \lambda\in\Gamma,\, i\in\mathcal{I}} = \bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,\, p\in P_j},$$

where $\Gamma_j = \eta_j^{-1}\Gamma$ for each $\eta_j \in \mathcal{A}$, the functions $g_p^{(1)} = g_{i,j}^{(1)} := D_{\eta_j}\psi_i$, and the index set $P_j = \{(i,j) : i \in \mathcal{I}\}$. The adjoint of an automorphism $\eta \in \mathrm{Aut}(G)$ is the automorphism $\eta' : \widehat{G} \to \widehat{G}$. Using this notion, the annihilators $\Gamma_j^\perp$ of Γ_j for $j \in \mathcal{J}$ can be expressed as $\Gamma_j^\perp = ((\eta_j')^{-1}\Gamma)^\perp = \eta_j^*(\Gamma^\perp)$, [27, Proposition 6.5], where $\eta^* := (\eta')^{-1}$. As an application of Theorem 5.3.1, we obtain the following result.

Proposition 5.4.3. *Let $\{D_{\eta_j} T_\lambda \psi_i\}_{j\in\mathcal{J},\, \lambda\in\Gamma,\, i\in\mathcal{I}}$ and $\{D_{\eta_j} T_\lambda \phi_i\}_{j\in\mathcal{J},\, \lambda\in\Gamma,\, i\in\mathcal{I}}$ be two wavelet Bessel (frame) systems in $L^2(G)$ satisfying the corresponding dual 1-UCP. Then the following assertions are equivalent:*

(i) $\{D_{\eta_j} T_\lambda \psi_i\}_{j\in\mathcal{J},\, \lambda\in\Gamma,\, i\in\mathcal{I}}$ and $\{D_{\eta_j} T_\lambda \phi_i\}_{j\in\mathcal{J},\, \lambda\in\Gamma,\, i\in\mathcal{I}}$ are pairwise orthogonal wavelet Bessel (frame) systems in $L^2(G)$.

(ii) For each $\alpha \in \bigcup_{j\in\mathcal{J}} \eta_j^*\Gamma^\perp$, we have

$$\sum_{j\in\mathcal{J}:\alpha\in\eta_j^*(\Gamma^\perp)} \frac{1}{s(\Gamma_j)} \sum_{i\in\mathcal{I}} \overline{\widehat{\psi}_i(\gamma)}\widehat{\phi}_i((\gamma+\alpha))\, d\mu_{P_j}(p) = 0 \quad \textit{for a.e. } \gamma \in \widehat{G}.$$

5.4.4. Composite wavelets and shearlet systems

Consider the Cartesian product $\mathcal{K} \times \mathcal{J}$ for two countable index sets $\mathcal{K}$ and $\mathcal{J}$. Let $A_k, B_j \in \mathrm{GL}_d(\mathbb{R})$ for $k \in \mathcal{K}$ and $j \in \mathcal{J}$. Let $\Gamma = C\mathbb{Z}^d$ be a full-rank lattice in $\mathbb{R}^d$. The wavelet system associated with the pair $\left(\{A_k B_j\}_{(k,j)\in\mathcal{K}\times\mathcal{J}}, \Gamma\right)$ is the collection of functions of the form

$$\{D_{A_k B_j} T_\gamma \psi_i\}_{k\in\mathcal{K},\, j\in\mathcal{J},\, \gamma\in\Gamma,\, i\in\mathcal{I}}$$

and is referred to as a *wavelet system with composite dilations* in $L^2(\mathbb{R}^d)$ (see, [65]). One usually assumes that one of the two families of matrices, say $\{A_k\}_{k\in\mathcal{K}}$, is volume preserving. In our setting, we assume that the transposes A_k^{T} for $k \in \mathcal{K}$, act invariantly on $\Gamma^\perp$, that is,

$A_k^{\mathrm{T}}\Gamma^\perp = \Gamma^\perp$. For example, if $\Gamma = \mathbb{Z}^d$, this assumption corresponds to $A_i \in \mathrm{SL}_d(\mathbb{Z})$. Under this assumption, for each $(k,j) \in \mathcal{K} \times \mathcal{J}$, the annihilator subgroup can be written as $\Gamma^\perp_{(k,j)} = B_j^{\mathrm{T}} A_k^{\mathrm{T}} \Gamma^\perp = B_j^{\mathrm{T}}\Gamma^\perp$. Thus the following result is a application of Proposition 5.4.3.

Proposition 5.4.4. *Let* $\{D_{A_i B_j} T_\gamma \psi_\ell\}_{i\in\mathcal{K}, j\in\mathcal{J}, \gamma\in\Gamma, \ell\in\mathcal{I}}$ *and* $\{D_{A_i B_j} T_\gamma \phi_\ell\}_{i\in\mathcal{K}, j\in\mathcal{J}, \gamma\in\Gamma, \ell\in\mathcal{I}}$ *be two wavelet with composite dilations Bessel (frame) systems in* $L^2(\mathbb{R}^d)$ *satisfying the corresponding dual* 1*-UCP are pairwise orthogonal if and only if for each* $\alpha \in \bigcup_{j\in\mathcal{J}} B_j^T(C^\sharp \mathbb{Z}^d)$*, we have*

$$\sum_{j\in\mathcal{J}:\alpha\in B_j^T(C^\sharp\mathbb{Z}^d)} \frac{1}{s(\Gamma_j)} \sum_{i\in\mathcal{I}} \overline{\widehat{\psi}_i\left((A_k^\sharp B_j^\sharp \gamma\right)} \widehat{\phi}_i\left(A_k^\sharp B_j^\sharp(\gamma+\alpha)\right) d\mu_{P_j}(p) = 0 \quad \text{for a.e. } \gamma \in \widehat{G},$$

where for a matrix $A \in \mathrm{GL}_d(\mathbb{R})$*, we denote its inverse transpose by* $A^\sharp := (A^T)^{-1}$*.*

The classical shearlet system can be modelled as a special case of wavelets with composite dilations. For clarity and simplicity, we focus our discussion on the space $L^2(\mathbb{R}^2)$; however, we refer the reader to [65, Section 3.4] for a detailed treatment of shearlet systems in the more general setting of $L^2(\mathbb{R}^d)$. We define

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let $\Gamma = C\mathbb{Z}^2$ for $C \in GL_2(\mathbb{R})$. The wavelet system associated with the pair $(\{S^k A^j\}_{k,j\in\mathbb{Z}}, \Gamma)$ is the collection of functions of the form

$$\{D_{S^k A^j} T_\gamma \psi_i\}_{k\, j\in\mathbb{Z}, \gamma\in\Gamma, i\in\mathcal{I}}$$

and is referred to as a *classical shearlet system* in $L^2(\mathbb{R}^2)$. Since every shearlet system that satisfies the CC-condition also fulfills the α-LIC, the following result can be seen as a direct application of Proposition 5.4.3.

Proposition 5.4.5. *Let* $\{D_{S^k A^j} T_\gamma \psi_i\}_{k\, j\in\mathbb{Z}, \gamma\in\Gamma, i\in\mathcal{I}}$ *and* $\{D_{S^k A^j} T_\gamma \phi_i\}_{k\, j\in\mathbb{Z}, \gamma\in\Gamma, i\in\mathcal{I}}$ *be two classical shearlet Bessel systems (frames) in* $L^2(\mathbb{R}^2)$*. Then, these systems are pairwise orthogonal if and only if for each* $m \in \mathbb{Z}$ $q \in (C^\sharp\mathbb{Z}^2 \setminus AC^\sharp\mathbb{Z}^2)$*, we have*

$$\frac{1}{s(\Gamma_j)} \sum_{n=0}^{\infty} \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{I}} \overline{\widehat{\psi}_i\left((S^k)^\sharp A^{n+m}\gamma\right)} \widehat{\phi}_i\left((S^k)^\sharp A^n(A^m\gamma + q)\right) = 0 \text{ for a.e. } \gamma \in \mathbb{R}^2$$

and

$$\frac{1}{s(\Gamma_j)} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{I}} \overline{\widehat{\psi}_i\left((S^k)^\sharp A^{-j}\gamma\right)} \widehat{\phi}_i\left((S^k)^\sharp A^{-j}\gamma\right) = 0 \text{ for a.e. } \gamma \in \mathbb{R}^2.$$

We now turn our attention to cone-adapted shearlet systems, which will be the focus of the remainder of this subsection. To introduce these systems, we define $A_1 = A$, $S_1 = S$,

$$A_2 = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

For generators $\phi, \psi_i \in L^2(\mathbb{R}^2)$, $i = 1,2$, and full-rank lattices $\Gamma_i = C_i\mathbb{Z}^2$, $i = 0,1,2$, the cone-adapted shearlet system is given as:

$$\left\{T_\gamma \phi\right\}_{\gamma\in\Gamma_0} \cup \left\{D_{S_i^k A_i^j} T_\gamma \psi_i\right\}_{j\in\mathbb{N}_0,\, k\in\{-K_j,\ldots,K_j\},\, \gamma\in\Gamma_i,\, i\in\{1,2\}},$$

where $K_j \in \mathbb{N}_0$ for $j \in \mathbb{N}_0$, and usually one takes $K_j = 2^j$ or $K_j = 2^j \pm 1$. For simplicity, we assume $\Gamma_i = \Gamma = C\mathbb{Z}^2$ for all $i = 0,1,2$, where $C \in \mathrm{GL}_2(\mathbb{R})$ is chosen such that the matrix $C^T A_i C^T$ has integer entries for each $i \in \{1,2\}$. As local integrability conditions can be ignored for shearlet systems, we can directly apply Proposition 5.4.3.

Proposition 5.4.6. *Let*

$$\left\{T_\gamma \phi^{(1)}\right\}_{\gamma\in\Gamma_0} \cup \left\{D_{S_i^k A_i^j} T_\gamma \psi_i^{(1)}\right\}_{j\in\mathbb{N}_0,\, k\in\{-K_j,\ldots,K_j\},\, \gamma\in\Gamma_i,\, i\in\{1,2\}}$$

and

$$\left\{T_\gamma \phi^{(2)}\right\}_{\gamma\in\Gamma_0} \cup \left\{D_{S_i^k A_i^j} T_\gamma \psi_i^{(2)}\right\}_{j\in\mathbb{N}_0,\, k\in\{-K_j,\ldots,K_j\},\, \gamma\in\Gamma_i,\, i\in\{1,2\}}$$

be two cone-adapted shearlet Bessel systems (frames) in $L^2(\mathbb{R}^2)$. Then, these are pairwise orthogonal if and only if, we have

$$\overline{\widehat{\phi^{(1)}}(\gamma)}\widehat{\phi^{(2)}}(\gamma) + \sum_{i\in\{1,2\}} \sum_{j=0}^{\infty} \sum_{k=-K_j}^{K_j} \overline{\widehat{\psi_i^{(1)}}((S_i^\sharp)^k A_i^{-j}\gamma)}\widehat{\psi_i^{(2)}}((S_i^\sharp)^k A_i^{-j}\gamma) = 0 \textit{ for a.e. } \gamma \in \mathbb{R}^2$$

and

$$\overline{\widehat{\phi^{(1)}}(\gamma)}\widehat{\phi^{(2)}}(\gamma+\alpha) + \sum_{i\in\{1,2\}} \sum_{j=0}^{m_i} \sum_{k=-K_j}^{K_j} \overline{\widehat{\psi_i^{(1)}}((S_i^\sharp)^k A_i^{-j}\gamma)}\widehat{\psi_i^{(2)}}((S_i^\sharp)^k A_i^{-j}\gamma+\alpha) = 0 \textit{ for a.e. } \gamma \in \mathbb{R}^2,$$

where $\alpha \in \Gamma^* \setminus \{0\}$, for each $i \in \{1,2\}$, is written as $\alpha = A_i^{m_i} q_i$ for unique $m_i \geq 0$ and $q_i \in \Gamma^* \setminus A_i\Gamma^*$.

This chapter provides a characterization of pairwise orthogonal frames with GTI structures, generated by translations over families of closed, co-compact subgroups of G. The subgroup families associated with each system may differ. As an application, we derive necessary and sufficient conditions for the orthogonality of structured systems such as Gabor, wavelet, and shearlet frames, and we also characterize GTI tight frames. Building upon these characterization results, the next chapter focuses on the explicit construction of pairwise orthogonal frames.

CHAPTER 6

EXPLICIT CONSTRUCTION OF PAIRWISE ORTHOGONAL PARSEVAL FRAMES VIA UCP

As an application of the results established in Chapter 5, this chapter presents explicit constructions of pairs of GTI systems using filter based methods. Each constructed system satisfies the ∞-UCP and has a Calderón sum equal to one. Furthermore, we show that these systems form Parseval frames as well as pairwise orthogonal. In Chapter 3, while establishing that GTI systems form pairwise orthogonal Parseval frames, several assumptions were required. However, due to the ∞-UCP considered in this chapter, some of those assumptions are no longer necessary.

6.1. Construction of GTI systems via filters

The construction method for pairs of GTI systems presented in this chapter follows the same framework as in Chapter 2, but introduces several key refinements and improvements. Instead of restating those details here, we briefly recall that in Section 2.2, a pair of GTI systems of the form

$$\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j} \text{ and } \bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,p\in P_j}$$

were constructed using filters defined in (2.2.6). Under suitable assumptions, it was shown in Theorem 2.3.3 that each GTI system satisfies the LIC and further, Chapter 3 established that these systems form pairwise orthogonal Parseval frames. However, in Chapter 3, the construction method relied on specific assumption on the sequence of co-compact subgroups $\{\Gamma_j\}_{j\in\mathcal{J}}$, requiring it to become stationary as $j\to-\infty$. In contrast, the approach developed in this chapter works under the weaker ∞-UCP, which allows us to relax this stationary assumption and thereby broaden the class of admissible GTI systems.

The results of **Chapter 6** are from the following manuscript:

Redhu N., Gumber A., Führ H., Shukla N. K., *Characterization and explicit construction of pairwise orthogonal Parseval frames in LCA groups*, submitting soon.

Throughout this chapter, we adopt the notations and assumptions established in Sections 2.2 and 2.3 without restating them here.

6.2. Conditions for the GTI systems satisfying the ∞-UCP

The following result provides sufficient conditions under which the GTI system $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j,p\in P_j}$, defined in (2.2.6), satisfies the ∞-UCP, forms a Parseval frame, and possesses a Calderón sum equal to one.

Theorem 6.2.1. *In addition to the assumptions ($\mathcal{N}_1$) to ($\mathcal{N}_2$) (see, Section 2.3), we assume that for every compact set S, there exist a $J\in\mathcal{J}$, such that $\mu_G\big((\omega+S)\cap(\omega'+S)\big)=0$ for $\omega\neq\omega'$ and $\omega,\omega'\in\Gamma_J^\perp$. Furthermore, for each $j\in\mathcal{J}$, suppose the matrix valued function $\mathfrak{B}_j^{(i)}(\gamma)$, defined in (2.2.7), satisfies the following condition:*

$$(\mathfrak{B}_j^{(i)}(\gamma))^*\mathfrak{B}_j^{(i)}(\gamma)=\frac{s(\Gamma_j)}{s(\Gamma_{j+1})}I_{d_j}\ \text{ for a.e. } \gamma\in\Omega_j, \tag{6.2.1}$$

where $(\mathfrak{B}_j^{(i)}(\gamma))^*$ denotes adjoint of $\mathfrak{B}_j^{(i)}(\gamma)$. Then, for each $i\in\{1,2\}$, the GTI system $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j,p\in P_j}$ (defined in (2.2.6))

(i) satisfies the ∞-UCP,
(ii) is a Parseval frame for $L^2(G)$,
(iii) has the Calderón sum 1, i.e., $\sum_{j\in\mathcal{J}}\frac{1}{s(\Gamma_j)}\sum_{p\in P_j}\big|\widehat{g_p^{(i)}}(w)\big|^2=1$ for a.e. $w\in\widehat{G}$.

Before we proceed to prove the above theorem, we first require the following two Lemmas 6.2.2 and 6.2.3, which are essential for establishing our results. These lemmas are inspired by the proof of the unitary extension principle for $L^2(\mathbb{R})$. Specifically, a similar versions of Lemmas 6.2.2 and 6.2.3(ii) also appear in [36], while part (i) and (iii) of Lemma 6.2.3 are presented here for the first time in the setting of LCA groups, as a generalization from the $L^2(\mathbb{R})$ setting.

Before stating the lemmas, we define the function $w_{f;\Phi_j}$ by

$$w_{f;\Phi_j}(x):=\int_{\Gamma_j}\big|\langle T_xf,T_\lambda\mathcal{F}^{-1}\Phi_j\rangle\big|^2\ d\mu_{\Gamma_j}(\lambda)$$

for each $j\in\mathcal{J}$ and $x\in G$.

Lemma 6.2.2. *Let $i \in \{1,2\}$, and consider the GTI system $\bigcup_{j \in \mathcal{J}} \{T_\lambda g_p^{(i)}\}_{\gamma \in \Gamma_j, p \in P_j}$ defined in (2.2.6). Suppose that, for some integers j_0, J with $j_0 \leq J$, the matrix valued functions $\mathfrak{B}_j^{(i)}(\gamma)$, defined in (2.2.7), satisfy*

$$(\mathfrak{B}_j^{(i)}(\gamma))^* \mathfrak{B}_j^{(i)}(\gamma) = \frac{s(\Gamma_j)}{s(\Gamma_{j+1})} I_{d_j} \quad \textit{for a.e. } \gamma \in \Omega_j,$$

for all $j = j_0, \dots, J$. Then the following identity holds:

$$\sum_{j=j_0}^{J} w_{f;g_p^{(i)},j}(x) = w_{f;\Phi_{J+1}}(x) - w_{f;\Phi_{j_0}}(x) \quad \textit{for } x \in G,$$

where $w_{f;g_p^{(i)},j}$ is defined in (5.2.2).

Proof. Since the Fourier transform preserves norms, we have

$$\begin{aligned} w_{f;g_p^{(i)},j_0}(x) &= \int_{P_{j_0}} \int_{\Gamma_{j_0}} \left| \langle T_x f, T_\lambda g_p^{(i)} \rangle \right|^2 \, d\mu_{\Gamma_{j_0}}(\lambda) d\mu_{P_{j_0}}(p) \\ &= \int_{P_{j_0}} \int_{\Gamma_{j_0}} \left| \langle \mathcal{F}(T_x f), \mathcal{F}(T_\lambda g_p^{(i)}) \rangle \right|^2 \, d\mu_{\Gamma_{j_0}}(\lambda) d\mu_{P_{j_0}}(p) \\ &= \int_{P_{j_0}} \int_{\Gamma_{j_0}} \left| \langle \mathcal{F}(T_x f), \mathcal{M}_\lambda \widehat{g_p^{(i)}} \rangle \right|^2 \, d\mu_{\Gamma_{j_0}}(\lambda) d\mu_{P_{j_0}}(p). \end{aligned} \tag{6.2.2}$$

Substituting $P_{j_0} = \{(m, j_0) : m = 1, 2, \dots, s_{j_0}\}$ and using the expression for $\widehat{g_p^{(i)}}$ in the right-hand side of (6.2.2), we obtain

$$w_{f;g_p^{(i)},j_0}(x) = \sum_{m=1}^{s_{j_0}} \int_{\Gamma_{j_0}} \left| \langle \mathcal{F}(T_x f), \mathcal{M}_\lambda \Psi_{j_0}^{(i)(m)} \rangle \right|^2 \, d\mu_{\Gamma_{j_0}}(\lambda). \tag{6.2.3}$$

Similarly, the functions $w_{f;\Phi_{j_0}}(x)$ and $w_{f;\Phi_{j_0+1}}(x)$ can be expressed as

$$w_{f;\Phi_{j_0}}(x) = \int_{\Gamma_{j_0}} |\langle \mathcal{F}(T_x f), \mathcal{M}_\lambda \Phi_{j_0} \rangle|^2 \, d\mu_{\Gamma_{j_0}}(\lambda) \tag{6.2.4}$$

and

$$w_{f;\Phi_{j_0+1}}(x) = \int_{\Gamma_{j_0+1}} |\langle \mathcal{F}(T_x f), \mathcal{M}_\lambda \Phi_{j_0+1} \rangle|^2 \, d\mu_{\Gamma_{j_0+1}}(\lambda). \tag{6.2.5}$$

Now using the matrix condition $(\mathfrak{B}_{j_0}^{(i)}(\gamma))^*\mathfrak{B}_{j_0}^{(i)}(\gamma) = \frac{s(\Gamma_{j_0})}{s(\Gamma_{j_0+1})} I_{d_{j_0}}$ for a.e. $\gamma \in \Omega_{j_0}$ and following a similar argument as in [36, Lemma 3.1], we obtain

$$\sum_{m=1}^{s_{j_0}} \int_{\Gamma_{j_0}} \left|\langle \mathcal{F}(T_x f), \mathcal{M}_\lambda \Psi_{j_0}^{(i)(m)} \rangle\right|^2 \, d\mu_{\Gamma_{j_0}}(\lambda) = \int_{\Gamma_{j_0+1}} |\langle \mathcal{F}(T_x f), \mathcal{M}_\lambda \Phi_{j_0+1}\rangle|^2 \, d\mu_{\Gamma_{j_0+1}}(\lambda) - \int_{\Gamma_{j_0}} |\langle \mathcal{F}(T_x f), \mathcal{M}_\lambda \Phi_{j_0}\rangle|^2 \, d\mu_{\Gamma_{j_0}}(\lambda).$$

Therefore, we conclude that $w_{f;g_p^{(i)},j_0}(x) = w_{f;\Phi_{j_0+1}}(x) - w_{f;\Phi_{j_0}}(x)$. Applying the same reasoning recursively for $j \in \{j_0+1,\dots,J\}$, we obtain

$$\begin{aligned}\sum_{j=j_0}^{J} w_{f;g_p^{(i)},j}(x) &= \left(w_{f;\Phi_{j_0+1}}(x) - w_{f;\Phi_{j_0}}(x)\right) + \left(w_{f;\Phi_{j_0+2}}(x) - w_{f;\Phi_{j_0+1}}(x)\right) \\ &\quad + \cdots + \left(w_{f;\Phi_{J+1}}(x) - w_{f;\Phi_J}(x)\right) \\ &= w_{f;\Phi_{J+1}}(x) - w_{f;\Phi_{j_0}}(x).\end{aligned}$$

This completes the proof. ∎

Lemma 6.2.3. *Under the assumptions of Theorem 6.2.1, for any $f \in D_B$, $x \in G$ and given $\epsilon > 0$, the following statements hold:*

(i) There exists an integer $J_2 \in \mathbb{Z}$ such that for all $j \in \mathcal{J}$ with $j \le J_2$, we have

$$w_{f;\Phi_j}(x) \le \epsilon \|f\|^2.$$

(ii) There exists an integer $J_1 \in \mathbb{Z}$ such that for all $j \in \mathcal{J}$ with $j \ge J_1$, we have

$$(1-\epsilon)\|f\|^2 \le w_{f;\Phi_j}(x) \le (1+\epsilon)\|f\|^2.$$

(iii) Moreover, the following identity holds:

$$\sum_{j\in\mathbb{Z}} w_{f;g_p^{(i)},j}(x) = \|f\|^2.$$

Proof. By following a similar approach as in Lemma 6.2.2, we can express $w_{f;\Phi_j}(x)$ as

$$w_{f;\Phi_j}(x) = \int_{\Gamma_j} |\langle \mathcal{F}(T_x f), \mathcal{M}_\lambda \Phi_j\rangle|^2 \, d\mu_{\Gamma_j}(\lambda). \tag{6.2.6}$$

Let us define $S := \operatorname{supp} \mathcal{F}(T_x f)$. For $j \in \mathcal{J}$ and $\omega \in \Gamma_j^\perp$, we define

$$S_{j,\omega} := \{\gamma \in \Omega_j : \omega + \gamma \in S\}.$$

Since $S_{j,\omega} = \Omega_j \cap (S - \omega)$, the set $S_{j,\omega}$ is measurable. Furthermore, we have the disjoint (up to a set of measure zero) decomposition $S = \bigcup_{\omega\in\Gamma_j^\perp}(\omega + S_{j,\omega})$, which follows from the tiling property in (1.2.1). Now, by [36, Proposition 2.2], the right-hand side of (6.2.6) can be rewritten as:

$$w_{f;\Phi_j}(x) = \frac{1}{s(\Gamma_j)} \int_{\Omega_j} \left| \sum_{\omega \in \Gamma_j^\perp} \mathcal{F}(T_x f)(\omega + \gamma)\overline{\Phi_j(\omega + \gamma)} \right|^2 d\mu_G(\gamma).$$

Note that in the integral above, we only get non-zero contributions for $\gamma \in \Omega_j$ when there exists $\omega' \in \Gamma_j^\perp$ such that $\omega' + \gamma \in S$, i.e., we only get contributions for $\gamma \in \bigcup_{\omega' \in \Gamma_j^\perp} S_{j,\omega'}$. Hence, we can write

$$\begin{aligned} w_{f;\Phi_j}(x) &= \frac{1}{s(\Gamma_j)} \int_{\left[\bigcup_{\omega' \in \Gamma_j^\perp} S_{j,\omega'}\right]} \left| \sum_{\omega \in \Gamma_j^\perp} \mathcal{F}(T_x f)(\omega + \gamma)\overline{\Phi_j(\omega + \gamma)} \right|^2 d\mu_G(\gamma) \\ &\leq \frac{1}{s(\Gamma_j)} \sum_{\omega' \in \Gamma_j^\perp} \int_{S_{k,\omega'}} \left| \sum_{\omega \in \Gamma_j^\perp} \mathcal{F}(T_x f)(\omega + \gamma)\overline{\Phi_j(\omega + \gamma)} \right|^2 d\mu_G(\gamma). \qquad (6.2.7) \end{aligned}$$

We observe that in (6.2.7), the summation over ω, for a fixed $\omega' \in \Gamma_j^\perp$, only the term with $\omega = \omega'$ contributes within $S_{j,\omega'}$. Thus, the expression simplifies to

$$\begin{aligned} w_{f;\Phi_j}(x) &\leq \frac{1}{s(\Gamma_j)} \sum_{\omega' \in \Gamma_j^\perp} \int_{S_{k,\omega'}} \left| \mathcal{F}(T_x f)(\omega' + \gamma)\overline{\Phi_j(\omega' + \gamma)} \right|^2 d\mu_G(\gamma) \\ &= \frac{1}{s(\Gamma_j)} \sum_{\omega' \in \Gamma_j^\perp} \int_{\omega' + S_{k,\omega'}} \left| \mathcal{F}(T_x f)(\gamma)\overline{\Phi_j(\gamma)} \right|^2 d\mu_G(\gamma) \\ &= \frac{1}{s(\Gamma_j)} \int_S \left| \mathcal{F}(T_x f)(\gamma)\Phi_j(\gamma) \right|^2 d\mu_{\widehat{G}}(\gamma). \qquad (6.2.8) \end{aligned}$$

Now, using the assumption $(\mathcal{N}_2)$, for a given $\epsilon > 0$ there exists $J_2 \in \mathcal{J}$ such that for all $j \in \mathcal{J}$ with $j \leq J_2$, we have

$$w_{f;\Phi_j}(x) \leq \epsilon \int_S \left| \mathcal{F}(T_x f)(\gamma) \right|^2 d\mu_{\widehat{G}}(\gamma) = \epsilon \left\| \mathcal{F}(T_x f) \right\|^2 = \epsilon \left\| f \right\|^2.$$

This completes the proof of part (i).

By assumption, for compact set S in $\widehat{G}$ there exist a $J \in \mathbb{Z}$ such that

$$\mu_G\big((\omega + S) \cap (\omega' + S)\big) = 0 \quad \text{for } \omega \neq \omega' \text{ and } \omega, \omega' \in \Gamma_J^\perp.$$

Choose $J_1 \in \mathbb{Z}$ such that $J_1 > J$ and the assumption $(\mathcal{N}_1)$ is satisfied for all $j \geq J_1$. Then, following similar steps as in the proof of [36, Lemma 3.2], we obtain

$$w_{f;\Phi_j}(x) = \frac{1}{s(\Gamma_j)} \int_S \left| \mathcal{F}(T_x f)(\gamma)\Phi_j(\gamma) \right| d\mu_{\widehat{G}}(\gamma) \text{ for all } j \geq J_1.$$

Now, using the choice of J_1 and the assumption $(\mathcal{N}_1)$, it follows that

$$(1-\epsilon)\left\|\mathcal{F}(T_x f)\right\|^2 \le w_{f;\Phi_j}(x) \le (1+\epsilon)\left\|\mathcal{F}(T_x f)\right\|^2 \text{ for all } j \ge J_1.$$

Since $\|\mathcal{F}(T_x f)\| = \|T_x f\| = \|f\|$, the proof of part (ii) is complete.

Now, by Lemma 6.2.2, we have

$$\sum_{j=j_0}^{J} w_{f;g_p^{(i)},j}(x) = w_{f;\Phi_{J+1}}(x) - w_{f;\Phi_{j_0}}(x).$$

Taking the limit as $j_0 \to -\infty$ on both sides and using part (i), which gives $w_{f;\Phi_{j_0}}(x) \to 0$ as $j_0 \to -\infty$, we obtain

$$\sum_{j=-\infty}^{J} w_{f;g_p^{(i)},j}(x) = w_{f;\Phi_{J+1}}(x).$$

Next, applying part (ii), we know that for all $J \ge J_1$, we have

$$(1-\epsilon)\|f\|^2\, d\mu_G(\gamma) \le \sum_{j=-\infty}^{J} w_{f;g_p^{(i)},j}(x) \le (1+\epsilon)\|f\|^2\, d\mu_G(\gamma).$$

Since this estimate holds for all $J \ge J_1$ and $\epsilon > 0$ is arbitrary, we conclude that

$$\sum_{j\in\mathbb{Z}} w_{f;g_p^{(i)},j}(x) = \|f\|^2$$

which completes the proof of part (iii). ■

Proof of Theorem 6.2.1. Let $\epsilon > 0$ be arbitrary. By Lemma 6.2.3((i)–(ii)), there exist integers $J_1, J_2 \in \mathbb{Z}$ such that

$$w_{f;\Phi_j}(x) \le \epsilon\left\|f\right\|^2 \quad \text{for all } j \le J_2 \text{ and } (1-\epsilon)\|f\|^2 \le w_{f;\Phi_j}(x) \le (1+\epsilon)\|f\|^2 \quad \text{for all } j \ge J_1. \tag{6.2.9}$$

Define the interval $\mathcal{J}' = [J_2, J_1] \cap \mathbb{Z}$ and choose a set $\mathcal{J}'' \supset \mathcal{J}'$ of the form $\mathcal{J}'' = [j_0, J]$ for some integers $j_0, J \in \mathbb{Z}$ such that $j_0 \le J_2$ and $J \ge J_1$. We now estimate the error between the full sum and the partial sum

$$\begin{aligned}\left|\sum_{j\in\mathbb{Z}} w_{f;g_p^{(i)},j}(x) - \sum_{j\in\mathcal{J}''} w_{f;g_p^{(i)},j}(x)\right| &= \sum_{j\in\mathbb{Z}} w_{f;g_p^{(i)},j}(x) - \sum_{j=j_0}^{J} w_{f;g_p^{(i)},j}(x)\\ &= \|f\|^2 - \left[w_{f;\Phi_{J+1}}(x) - w_{f;\Phi_{j_0}}(x)\right],\end{aligned}$$

where the last equality follows from Lemma 6.2.2. Since $j_0 \leq J_2$ and $J \geq J_1$, applying (6.2.9) on the above expression gives

$$\begin{aligned}\left|\sum_{j\in\mathbb{Z}} w_{f;g_p^{(i)},j}(x) - \sum_{j\in\mathcal{J}''} w_{f;g_p^{(i)},j}(x)\right| &= \|f\|^2 - w_{f;\Phi_{J+1}}(x) + w_{f;\Phi_{j_0}}(x)\\ &\leq \|f\|^2 - (1-\epsilon)\|f\|^2 + \epsilon\|f\|^2\\ &= 2\epsilon\,\|f\|^2.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude that $\sum_{j\in\mathbb{Z}} w_{f;g_p^{(i)},j}$ converges uniformly to $w_{f;g_p^{(i)}}$. This proves (i).

By Lemma 6.2.3 (iii), we have

$$\sum_{j\in\mathbb{Z}} w_{f;g_p^{(i)},j}(0) = \sum_{j\in\mathbb{Z}} \int_{\Gamma_j} |\langle \mathcal{F}(T_x f), \mathcal{M}_\lambda \Phi_j\rangle|^2 \, d\mu_{\Gamma_j}(\gamma) = \|f\|^2.$$

Thus the GTI system $\bigcup_{j\in\mathbb{Z}} \{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j, p\in P_j}$ is Parseval frame for $L^2(G)$. Hence, (ii) is proved.

Now, since (i) and (ii) hold, then applying Theorem 5.3.3 yields that for each $\alpha \in \bigcup_{j\in\mathcal{J}} \Gamma_j^\perp \setminus \{0\}$,

$$\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j^\perp} \frac{1}{s(\Gamma_j)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(\gamma)}\widehat{g_p^{(1)}}(\gamma+\alpha)\, d\mu_{P_j}(p) = 0 \text{ for a.e. } \gamma \in \widehat{G}$$

and

$$\sum_{j\in\mathcal{J}} \frac{1}{s(\Gamma_j)} \int_{P_j} \overline{\widehat{g_p^{(1)}}(\gamma)}\widehat{g_p^{(1)}}(\gamma)\, d\mu_{P_j}(p) = 1 \text{ for a.e. } \gamma \in \widehat{G}. \tag{6.2.10}$$

Therefore, by (6.2.10), the Calderón sum of the system $\bigcup_{j\in\mathbb{Z}} \{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j, p\in P_j}$ equals 1 for each $i \in \{1,2\}$, proving (iii). ■

Remark 6.2.4. As a special case $\mathcal{J} = \{j\}_{j=j_0}^{\infty}$, the GTI system defined in Theorem 6.2.1 take the form

$$\left\{T_\lambda \mathcal{F}^{-1}\Phi_{j_0}\right\}_{\lambda\in\Gamma_{j_0}} \cup \bigcup_{j=j_0}^{\infty} \left\{T_\lambda g_p^{(i)}\right\}_{\lambda\in\Gamma_j,\, p\in P_j}, \tag{6.2.11}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Christensen and Goh established in [36, Theorem 3.3] that system of the form (6.2.11) constitute Parseval frames. The result in Theorem 6.2.1 also establishes that the Calderón sum equals one, together with the ∞-UCP property. Moreover, we extend the framework by allowing the index set $\mathcal{J}$ to take the forms $\{j\}_{j=-\infty}^{j_1}$ or $\mathbb{Z}$, which were not considered in [36].

Using Theorem 6.2.1, we construct a pair of GTI Parseval frames that satisfy the ∞-UCP, with B-splines serving as generating functions. We return to these GTI systems in Example 6.4.2, where we demonstrate that these GTI systems are also pairwise orthogonal.

Example 6.2.5. Let $G = \mathbb{Z}$ be the LCA group. For some $k \in \mathbb{Z}$, we let $\mathcal{J} = \{-\infty, \cdots, -1, 0, 1, \cdots, k\}$. For each $j \in \mathcal{J}$, we define the B-spline of third order at level j as follows:

$$B_j(x) := \frac{1}{(2^{k-j})^{5/2}} \chi_{\{0,1,\cdots,2^{k-j}-1\}} * \chi_{\{0,1,\cdots,2^{k-j}-1\}} * \chi_{\{0,1,\cdots,2^{k-j}-1\}} * \chi_{\{0,1,\cdots,2^{k-j}-1\}}(x), \quad x \in \mathbb{Z}.$$

Then its Fourier transform is given by

$$\begin{aligned}\Phi_j(\gamma) := \widehat{B_j}(\gamma) &= \frac{1}{(2^{k-j})^{7/2}} \frac{(1 - e^{-2\pi i(2^{k-j}\gamma)})^4}{(1 - e^{-2\pi i\gamma})^4} \\ &= \frac{1}{(2^{k-j-1})^{7/2}} \frac{(1 - e^{-2\pi i(2^{k-j-1}\gamma)})^4}{(1 - e^{-2\pi i\gamma})^4} \frac{(1 + e^{-2\pi i(2^{k-j-1}\gamma)})^4}{2^{7/2}} \\ &= \widehat{B_{j+1}}(\gamma) H_{j+1}(\gamma) = H_{j+1}(\gamma)\Phi_{j+1}(\gamma),\end{aligned}$$

where $H_{j+1}(\gamma) = \frac{(1+e^{-2\pi i(2^{k-j-1}\gamma)})^4}{2^{7/2}} \in L^\infty(\{0, 1, \cdots, 2^{k-j}\})$ is a $2^{-k+j+1}\mathbb{Z}$-periodic function. Further, we define the functions $\Psi_j^{(i)(m)} \in L^2(\mathbb{R})$ for $i \in \{1, 2\}$ and $m \in \{1, 2, \cdots, 8\}$ by $\Psi_j^{(i)(m)}(\gamma) = G_{j+1}^{(i)(m)}(\gamma)\Phi_{j+1}(\gamma)$, where $G_{j+1}^{(1)(m)}$ and $G_{j+1}^{(2)(m)}$ are given by

$$\begin{aligned}G_{j+1}^{(1)(m)}(\gamma) =& a_{1m}(1 + e^{-2\pi i(2^{k-j-1}\gamma)})^3(1 - e^{-2\pi i(2^{k-j-1}\gamma)}) + a_{2m}(1 + e^{-2\pi i(2^{k-j-1}\gamma)})^2(1 - e^{-2\pi i(2^{k-j-1}\gamma)})^2 \\ &+ a_{3m}(1 + e^{-2\pi i(2^{k-j-1}\gamma)})(1 - e^{-2\pi i(2^{k-j-1}\gamma)})^3 + a_{4m}(1 - e^{-2\pi i(2^{k-j-1}\gamma)})^4\end{aligned}$$

and

$$\begin{aligned}G_{j+1}^{(2)(m)}(\gamma) =& b_{1m}(1 + e^{-2\pi i(2^{k-j-1}\gamma)})^3(1 - e^{-2\pi i(2^{k-j-1}\gamma)}) + b_{2m}(1 + e^{-2\pi i(2^{k-j-1}\gamma)})^2(1 - e^{-2\pi i(2^{k-j-1}\gamma)})^2 \\ &+ b_{3m}(1 + e^{-2\pi i(2^{k-j-1}\gamma)})(1 - e^{-2\pi i(2^{k-j-1}\gamma)})^3 + b_{4m}(1 - e^{-2\pi i(2^{k-j-1}\gamma)})^4,\end{aligned}$$

where

$$(a_{nm}) = \begin{pmatrix} \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} \\ \frac{3^{1/2}}{2^{9/2}} & \frac{3^{1/2}}{2^5} + i\frac{3^{1/2}}{2^5} & i\frac{3^{1/2}}{2^{9/2}} & -\frac{3^{1/2}}{2^5} + i\frac{3^{1/2}}{2^5} & -\frac{3^{1/2}}{2^{9/2}} & -\frac{3^{1/2}}{2^5} - i\frac{3^{1/2}}{2^5} & -\frac{3^{1/2}}{2^{9/2}} & \frac{3^{1/2}}{2^5} - i\frac{3^{1/2}}{2^5} \\ \frac{1}{2^4} & i\frac{1}{2^4} & -\frac{1}{2^4} & -i\frac{1}{2^4} & \frac{1}{2^4} & i\frac{1}{2^4} & -\frac{1}{2^4} & -i\frac{1}{2^4} \\ \frac{1}{2^5} & -\frac{1}{2^{11/2}} + i\frac{1}{2^{11/2}} & -i\frac{1}{2^5} & \frac{1}{2^{11/2}} + i\frac{1}{2^{11/2}} & -\frac{1}{2^5} & \frac{1}{2^{11/2}} - i\frac{1}{2^{11/2}} & i\frac{1}{2^5} & -\frac{1}{2^{11/2}} - i\frac{1}{2^{11/2}} \end{pmatrix}$$

and

$$(b_{nm}) = \begin{pmatrix} \frac{1}{2^4} & -\frac{1}{2^4} & \frac{1}{2^4} & -\frac{1}{2^4} & \frac{1}{2^4} & -\frac{1}{2^4} & \frac{1}{2^4} & -\frac{1}{2^4} \\ \frac{3^{1/2}}{2^3} & -\frac{3^{1/2}}{2^{7/2}} - i\frac{3^{1/2}}{2^{7/2}} & \frac{3^{1/2}}{2^{7/2}} + i\frac{3^{1/2}}{2^{7/2}} & \frac{3^{1/2}}{2^{7/2}} - i\frac{3^{1/2}}{2^{7/2}} & -\frac{3^{1/2}}{2^3} & \frac{3^{1/2}}{2^{7/2}} + i\frac{3^{1/2}}{2^{7/2}} & -i\frac{3^{1/2}}{2^3} & -\frac{3^{1/2}}{2^{7/2}} + i\frac{3^{1/2}}{2^{7/2}} \\ \frac{1}{2^4} & -i\frac{1}{2^4} & -\frac{1}{2^4} & i\frac{1}{2^4} & \frac{1}{2^4} & -i\frac{1}{2^4} & -\frac{1}{2^4} & i\frac{1}{2^4} \\ \frac{1}{2^5} & \frac{1}{2^{11/2}} - i\frac{1}{2^{11/2}} & -i\frac{1}{2^5} & -\frac{1}{2^{11/2}} - i\frac{1}{2^{11/2}} & -\frac{1}{2^5} & -\frac{1}{2^{11/2}} + i\frac{1}{2^{11/2}} & i\frac{1}{2^5} & \frac{1}{2^{11/2}} + i\frac{1}{2^{11/2}} \end{pmatrix}$$

for $1 \le n \le 4$ and $1 \le m \le 8$. In this case, the matrix valued functions for each $i \in \{1,2\}$ defined in (2.2.7) are given as

$$B_j^{(i)}(\gamma) = \begin{pmatrix} G_{j+1}^{(i)(0)}(\gamma) & G_{j+1}^{(i)(1)}(\gamma) & \cdot & \cdot & \cdot & G_{j+1}^{(i)(8)}(\gamma) \\ G_{j+1}^{(i)(0)}(\gamma+2^j) & G_{j+1}^{(i)(1)}(\gamma+2^j) & \cdot & \cdot & \cdot & G_{j+1}^{(i)(8)}(\gamma+2^j) \end{pmatrix}^T,$$

where $G_{j+1}^{(i)(0)} = H_{j+1}$. Now for each $i \in \{1,2\}$ and $j \in \mathcal{J}$, we show that $(\mathfrak{B}_j^{(i)}(\gamma))^* \mathfrak{B}_j^{(i)}(\gamma) = 2I_2$ for a.e. $\gamma \in [0, 2^{j-k})$. For this, it is sufficient to show that for a.e. $\gamma \in [0, 2^{j-k})$ and $\ell, \ell' \in \{1,2\}$,

$$\sum_{m=0}^{8} \overline{G_{j+1}^{(i)(m)}(\gamma+\nu_{j,\ell})} G_{j+1}^{(i)(m)}(\gamma+\nu_{j,\ell'}) = 2\delta_{\ell,\ell'}, \text{ where } \nu_{j,1}=0 \text{ and } \nu_{j,2}=2^{j-k}. \tag{6.2.12}$$

We demonstrate the proof of (6.2.12) for $i=1$, and the same approach can be applied for $i=2$ as well. First suppose that, $\ell = \ell' = 1$, then $\nu_{j,\ell} = \nu_{j,\ell'} = 0$. Now

$$\sum_{m=0}^{8} \overline{G_{j+1}^{(1)(m)}(\gamma+\nu_{j,\ell})} G_{j+1}^{(1)(m)}(\gamma+\nu_{j,\ell'}) = |H_{j+1}(\gamma)|^2 + \sum_{m=1}^{8} |G_{j+1}^{(1)(m)}(\gamma)|^2. \tag{6.2.13}$$

Substituting the values of H_{j+1} and $G_{j+1}^{(1)(m)}$, simplifying the calculations, the right hand side of (6.2.13) becomes

$$\begin{aligned}
=&\frac{1}{2^7}\left|(1+e^{-2\pi i(2^{k-j-1}\gamma)})^4\right|^2 + \sum_{m=1}^{8}|a_{1m}|^2\left|(1+e^{-2\pi i(2^{k-j-1}\gamma)})^3(1-e^{-2\pi i(2^{k-j-1}\gamma)})\right|^2 \\
&+\sum_{m=1}^{8}|a_{2m}|^2\left|(1+e^{-2\pi i(2^{k-j-1}\gamma)})^2(1-e^{-2\pi i(2^{k-j-1}\gamma)})^2\right|^2 \\
&+\sum_{m=1}^{8}|a_{3m}|^2\left|(1+e^{-2\pi i(2^{k-j-1}\gamma)})^1(1-e^{-2\pi i(2^{k-j-1}\gamma)})^3\right|^2 + \sum_{m=1}^{8}|a_{4m}|^2\left|(1-e^{-2\pi i(2^{k-j-1}\gamma)})^4\right|^2 \\
=&\frac{1}{2^7}\left[\left|(1+e^{-2\pi i(2^{k-j-1}\gamma)})\right|^2\right]^4 + \frac{1}{2^5}\left|(1+e^{-2\pi i(2^{k-j-1}\gamma)})\right|^6\left|(1-e^{-2\pi i(2^{k-j-1}\gamma)})\right|^2 + \frac{3}{2^6}\left|(1+e^{-2\pi i(2^{k-j-1}\gamma)})\right|^4 \\
&\times\left|(1-e^{-2\pi i(2^{k-j-1}\gamma)})\right|^4 + \frac{1}{2^5}\left|(1+e^{-2\pi i(2^{k-j-1}\gamma)})\right|^2\left|(1-e^{-2\pi i(2^{k-j-1}\gamma)})^6\right|^2 + \left[\frac{1}{2^7}\left|(1-e^{-2\pi i(2^{k-j-1}\gamma)})\right|^2\right]^4
\end{aligned}$$

since $\sum_{m=1}^{8}|a_{1m}|^2 = \frac{1}{2^5}$, $\sum_{m=1}^{8}|a_{2m}|^2 = \frac{3}{2^3}$, $\sum_{m=1}^{8}|a_{2m}|^2 = \frac{1}{2^5}$ and $\sum_{m=1}^{8}|a_{2m}|^2 = \frac{1}{2^7}$. This is further equivalent to

$$= \frac{1}{2^7}\left[\left|(1+e^{-2\pi i(2^{k-j-1}\gamma)})\right|^2 + \left|(1-e^{-2\pi i(2^{k-j-1}\gamma)})\right|^2\right]^4 = \frac{1}{2^7}4^4 = 2.$$

When ℓ and $\ell' = 2$, we have $\nu_{j,\ell} = \nu_{j,\ell'} = 2^{j-k}$. Following a similar approach as we observe for ℓ and $\ell' = 1$, we can demonstrate that $\sum_{m=0}^{8}\left|G_{j+1}^{(i)(m)}(\gamma+2^{j-k})\right|^2 = 2$. Next, we suppose

$\ell = 1$ and $\ell' = 2$, then $\nu_{j,\ell} = 0$ and $\nu_{j,\ell'} = 2^{j-k}$. Now,

$$\sum_{m=0}^{4} \overline{G_{j+1}^{(1)(m)}(\gamma+\nu_{j,\ell})} G_{j+1}^{(1)(m)}(\gamma+\nu_{j,\ell'}) = \overline{H_{j+1}(\gamma)} H_{j+1}(\gamma+2^{j-k}) + \sum_{m=1}^{4} \overline{G_{j+1}^{(1)(m)}(\gamma)} G_{j+1}^{(1)(m)}(\gamma+2^{j-k}). \tag{6.2.14}$$

Next, using $e^{-2\pi i(2^{k-j-1}(2^{j-k}+\gamma))} = -e^{-2\pi i(2^{k-j-1}\gamma)}$ and by following a similar steps as in $\ell, \ell' = 1$, the right hand side of (6.2.14) can be expressed as

$$\begin{aligned}
=&\frac{1}{2^7}\left[(1+\overline{e^{-2\pi i(2^{k-j-1}\gamma)}})(1-e^{-2\pi i(2^{k-j-1}\gamma)})\right]^4 + \frac{1}{2^7}\left[(1-\overline{e^{-2\pi i(2^{k-j-1}\gamma)}})(1+e^{-2\pi i(2^{k-j-1}\gamma)})\right]^4 \\
&+ \frac{1}{2^5}\left[(1+\overline{e^{-2\pi i(2^{k-j-1}\gamma)}})(1-e^{-2\pi i(2^{k-j-1}\gamma)})\right]^3\left[(1-\overline{e^{-2\pi i(2^{k-j-1}\gamma)}})(1+e^{-2\pi i(2^{k-j-1}\gamma)})\right] \\
&+ \frac{3}{2^6}\left[(1+\overline{e^{-2\pi i(2^{k-j-1}\gamma)}})(1-e^{-2\pi i(2^{k-j-1}\gamma)})\right]^2\left[(1-\overline{e^{-2\pi i(2^{k-j-1}\gamma)}})(1+e^{-2\pi i(2^{k-j-1}\gamma)})\right]^2 \\
&+ \frac{1}{2^5}\left[(1+\overline{e^{-2\pi i(2^{k-j-1}\gamma)}})(1-e^{-2\pi i(2^{k-j-1}\gamma)})\right]\left[(1-\overline{e^{-2\pi i(2^{k-j-1}\gamma)}})(1+e^{-2\pi i(2^{k-j-1}\gamma)})\right]^3 \\
=&\frac{1}{2^7}\left[(1-\overline{e^{-2\pi i(2^{k-j-1}\gamma)}})(1+e^{-2\pi i(2^{k-j-1}\gamma)}) + (1-\overline{e^{-2\pi i(2^{k-j-1}\gamma)}})(1+e^{-2\pi i(2^{k-j-1}\gamma)})\right]^4 = 0
\end{aligned}$$

using $(1+\overline{z})(1-z) + (1-\overline{z})(1+z) = 0$ for $|z| = 1$. Similarly, for $\ell = 2$ and $\ell' = 1$, we have $\sum_{m=0}^{8} \overline{G_{j+1}^{(i)(m)}(\gamma+2^{j-k})} G_{j+1}^{(i)(m)}(\gamma) = 0$. Finally, we get

$$(\mathfrak{B}_j^{(i)}(\gamma))^* \mathfrak{B}_j^{(i)}(\gamma) = 2I_2 \ \text{ for a.e. } \gamma \in \Omega_j = [0, 2^{j-k}),\ i \in \{1,2\} \text{ and } j \in \mathbb{Z}.$$

Let $\Gamma_j := 2^{k-j}\mathbb{Z} \subset \mathbb{Z}$. Then $\Gamma_j^{\perp} = 2^{-k+j}\mathbb{Z}$ and its fundamental domain is $\Omega_j = [0, 2^{j-k})$. Now

$$\frac{1}{s(\Gamma_j)}|\Phi_j(\gamma)|^2 = \left|\frac{1}{\sqrt{s(\Gamma_j)}}\Phi_j(\gamma)\right|^2 = \left|\frac{1}{(2^{k-j})^4}\frac{(1-e^{-2\pi i(2^{k-j}\gamma)})^4}{(1-e^{-2\pi i\gamma})^4}\right|^2.$$

For $j = k$, we have

$$\frac{1}{s(\Gamma_k)}|\Phi_k(\gamma)|^2 = \left|\frac{1}{(2^{k-k})^4}\frac{(1-e^{-2\pi i(2^{k-k}\gamma)})^4}{(1-e^{-2\pi i\gamma})^4}\right|^2 = \left|\frac{(1-e^{-2\pi i\gamma})^4}{(1-e^{-2\pi i\gamma})^4}\right|^2 = 1.$$

Hence assumption $(\mathcal{N}_1)$ is true. Also

$$\lim_{j\to-\infty}\frac{1}{\sqrt{s(\Gamma_j)}}|\Phi_j(\gamma)| = \left|\lim_{j\to-\infty}\left(\frac{1}{(2^{k-j})}\frac{(1-e^{-2\pi i(2^{k-j}\gamma)})}{(1-e^{-2\pi i\gamma})}\right)^4\right| = 0$$

using the fact $\lim_{x\to\infty}\left(\frac{1-e^{-ix}}{x}\right) = 0$. This implies assumption $(\mathcal{N}_2)$ is true. Thus all the assumptions of Theorem 6.2.1 are true and hence, for each $i \in \{1,2\}$, the system $\bigcup_{j=-\infty}^{j_0}\{T_\lambda g_p^{(i)}\}_{\lambda\in 2^{-j+k}\mathbb{Z},\, p\in\{(m,j):m=1,2,\ldots,8\}}$ is a Parseval frame and satisfies ∞-UCP.

Theorem 6.2.1(ii) is known as the Unitary Extension Principle (UEP) for LCA groups. In the following subsection, our goal is to introduce a more flexible generalization of the UEP, called the Oblique Extension Principle. For a detailed motivation and background on the OEP in $L^2(\mathbb{R})$, we refer the reader to Chapter 18, Subsection 18.4 in [33].

6.3. Oblique Extension Principle

The result stated below is called *oblique extension principle* for LCA groups.

Theorem 6.3.1. *Let* $\{\Phi_j, H_j, G_j^{(i)(m)}\}_{j\in\mathcal{J}, m\in\{0,1,\cdots,s_j\}}$ *be defined as in general setup in Section 2.2. Suppose there exists a sequence of strictly positive* $\Gamma_j^{\perp}$*-periodic functions* $\theta_j \in L^\infty(\Omega_j)$ *such that for every compact set* S*, there exist a* $J_1' \in J$ *such that for all* $j \geq J_1'$*, we have*

$$|\theta_j(\gamma) - 1| \leq \epsilon \quad \text{for all } \gamma \in S, \tag{6.3.1}$$

and

$$(\mathfrak{R}_j^{(i)}(\gamma))^* \mathfrak{R}_j^{(i)}(\gamma) = \frac{s(\Gamma_j)}{s(\Gamma_{j+1})} \theta_{j+1}(\gamma) I_{d_j}, \tag{6.3.2}$$

where the matrix $\mathfrak{R}_j^{(i)}(\gamma)$ *is defined as follows:*

$$\mathfrak{R}_j^{(i)}(\gamma) := \begin{pmatrix} H_{j+1}(\gamma+\nu_{j,1})\sqrt{\theta_j(\gamma)} & \cdots & H_{j+1}(\gamma+\nu_{j,d_j})\sqrt{\theta_j(\gamma)} \\ G_{j+1}^{(1)(i)}(\gamma+\nu_{j,1}) & \cdots & G_{j+1}^{(1)(i)}(\gamma+\nu_{j,d_j}) \\ \vdots & \ddots & \vdots \\ G_{j+1}^{(s_j)(i)}(\gamma+\nu_{j,1}) & \cdots & G_{j+1}^{(s_j)(i)}(\gamma+\nu_{j,d_j}) \end{pmatrix} \quad \text{for } \gamma \in \Omega_j. \tag{6.3.3}$$

Then, for $i \in \{1,2\}$*, the GTI system* $\bigcup_{j\in\mathcal{J}} \{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j, p\in P_j}$ *is a Parseval frame for* $L^2(G)$*.*

Proof. Assume that the conditions in Theorem 6.3.1 and define the $\{\widetilde{\Phi_j}\}_{j\in\mathcal{J}}$ is a sequence of functions in $L^2(\widehat{G})$, by

$$\widetilde{\Phi_j}(\gamma) = \sqrt{\theta_j(\gamma)}\Phi_j(\gamma).$$

Define the $\Gamma_j^{\perp}$-periodic functions $\widetilde{H_j}(\gamma)$ and $\widetilde{G_j^{(i)(m)}}(\gamma)$ in $L^\infty(\Omega_j)$, by

$$\widetilde{H_j}(\gamma) = \sqrt{\frac{\theta_{j-1}(\gamma)}{\theta_j(\gamma)}} H_j(\gamma) \text{ and } \widetilde{G_j^{(i)(m)}}(\gamma) = \sqrt{\frac{1}{\theta_j(\gamma)}} G_j^{(i)(m)}(\gamma). \tag{6.3.4}$$

Also define the matrices $\widetilde{\mathfrak{Z}_j^{(i)}}(\gamma) := \left(\widetilde{G_{j+1}^{(i)(m)}}(\gamma + \upsilon_{j,n})\right)_{\substack{1\leq m\leq s_j \\ 1\leq n\leq d_j}}$ for *a.e.* $\gamma \in \Omega_j$. The idea of the proof is to apply Theorem 6.2.1 to $\widetilde{\Phi_j}, \widetilde{H_j}, \widetilde{\Psi_j^{(i)(m)}}$ and therefore obtain the GTI

Parseval frame $\bigcup_{j\in\mathcal{J}}\{T_\lambda \widetilde{g_p^{(i)}}\}_{\lambda\in\Gamma_j, p\in P_j}$ for $L^2(G)$, where $P_j = \{(m,j) : m = 1,2,\cdots,s_j\}$ and $\mathcal{F}(\widetilde{g_p^{(i)}}) = \widetilde{\Psi_j^{(i)(m)}}(\gamma) = \widetilde{G_{j+1}^{(i)(m)}}$. Finally it turns out that $\widetilde{g_p^{(i)}} = g_p^{(i)}$. We now prove that $\widetilde{\Phi_j}, \widetilde{H_j}$ and $\widetilde{\Psi_j^{(i)(m)}}$ satisfy the conditions in general setup in Section 2.2. First,

$$\begin{aligned}\widetilde{\Phi_j}(\gamma) = \sqrt{\theta_j(\gamma)}\Phi_j(\gamma) &= \sqrt{\theta_j(\gamma)}H_{j+1}(\gamma)\Phi_{j+1}(\gamma)\\ &= \sqrt{\frac{\theta_j(\gamma)}{\theta_{j+1}(\gamma)}}H_{j+1}(\gamma)\widetilde{\Phi_{j+1}}(\gamma)\\ &= \widetilde{H_{j+1}}(\gamma)\widetilde{\Phi_{j+1}}(\gamma).\end{aligned}$$

Next, Let S be any compact set in $\widehat{G}\smallsetminus\mathcal{B}$, then for $\gamma\in S$, we have

$$\begin{aligned}\left|\frac{1}{s(\Gamma_j)}\left|\widetilde{\Phi_j}(\gamma)\right|^2 - 1\right| &= \left|\frac{1}{s(\Gamma_j)}\left|\Phi_j(\gamma)\right|^2\theta_j(\gamma) - 1\right|\\ &= \left|\frac{1}{s(\Gamma_j)}\left|\Phi_j(\gamma)\right|^2\theta_j(\gamma) - \frac{1}{s(\Gamma_j)}\left|\Phi_j(\gamma)\right|^2 + \frac{1}{s(\Gamma_j)}\left|\Phi_j(\gamma)\right|^2 - 1\right|\\ &\le \frac{1}{s(\Gamma_j)}\left|\Phi_j(\gamma)\right|^2\left|\theta_j(\gamma)-1\right| + \left|\frac{1}{s(\Gamma_j)}\left|\Phi_j(\gamma)\right|^2 - 1\right|.\end{aligned}$$

Now, let $\epsilon > 0$ and choose $J := \max\{J_1, J_1'\}$, then using assumption $(\mathcal{N}_1)$ and (6.3.1), we obtain

$$\left|\frac{1}{s(\Gamma_j)}\left|\widetilde{\Phi_j}(\gamma)\right|^2 - 1\right| \le \frac{1}{s(\Gamma_j)}\left|\Phi_j(\gamma)\right|^2\epsilon + \epsilon = \left(\frac{1}{s(\Gamma_j)}\left|\Phi_j(\gamma)\right|^2 + 1\right)\epsilon \le (M+1)\epsilon$$

for all $j \ge J$ and $\gamma\in S$, where $M > (1+\epsilon)$. Also, since sequence of functions $\theta_j(\gamma)$ is uniformly bounded, and using $(\mathcal{N}_2)$, it follows that for every compact set $S\in\widehat{G}\smallsetminus\mathcal{B}$, there exists $J_2\in\mathcal{J}$ such that for all $j\le J_2,\ j\in\mathcal{J}$,

$$\frac{1}{\sqrt{s(\Gamma_j)}}|\widetilde{\Phi_j}(\gamma)| \le \epsilon,\ \ \forall\,\gamma\in S.$$

Next, in order to show that

$$\left(\widetilde{\mathfrak{Z}_j^{(i)}}(\gamma)\right)^*\widetilde{\mathfrak{Z}_j^{(i)}}(\gamma) = \frac{s(\Gamma_j)}{s(\Gamma_{j+1})}I_{d_j}\ \text{for a.e. } \gamma\in\Omega_j,$$

it is equivalent to show that

$$\left(\overline{\widetilde{H_{j+1}}(\gamma+v_{j,\ell})}\widetilde{H_{j+1}}(\gamma+v_{j,\ell'}) + \sum_{m=1}^{s_j}\overline{\widetilde{G_{j+1}^{(i)(m)}}(\gamma+v_{j,\ell})}\widetilde{G_{j+1}^{(i)(m)}}(\gamma+v_{j,\ell'})\right)_{\substack{1\le\ell\le d_j\\ 1\le\ell'\le d_j}} = \frac{s(\Gamma_j)}{s(\Gamma_{j+1})}I_{d_j}$$

for a.e. $\gamma\in\Omega_j$. Equivalently,

$$\overline{\widetilde{H_{j+1}}(\gamma+v_{j,\ell})}\widetilde{H_{j+1}}(\gamma+v_{j,\ell'}) + \sum_{m=1}^{s_j}\overline{\widetilde{G_{j+1}^{(i)(m)}}(\gamma+v_{j,\ell})}\widetilde{G_{j+1}^{(i)(m)}}(\gamma+v_{j,\ell'}) = \delta_{\ell,\ell'}\frac{s(\Gamma_j)}{s(\Gamma_{j+1})} \qquad (6.3.5)$$

for $1 \le \ell, \ell' \le d_j$ and for a.e. $\gamma \in \Omega_j$. Now, by substituting $\widetilde{H_{j+1}}$ and $\widetilde{G_{j+1}^{(i)(m)}}$, we get

$$\begin{aligned}
&\overline{\widetilde{H_{j+1}}(\gamma+v_{j,\ell})}\widetilde{H_{j+1}}(\gamma+v_{j,\ell'}) + \sum_{m=1}^{s_j} \overline{\widetilde{G_{j+1}^{(i)(m)}}(\gamma+v_{j,\ell})}\widetilde{G_{j+1}^{(i)(m)}}(\gamma+v_{j,\ell'}) \\
&= \overline{\sqrt{\frac{\theta_j(\gamma+v_{j,\ell})}{\theta_{j+1}(\gamma+v_{j,\ell})}}H_{j+1}(\gamma+v_{j,\ell})}\sqrt{\frac{\theta_j(\gamma+v_{j,\ell'})}{\theta_{j+1}(\gamma+v_{j,\ell'})}}H_{j+1}(\gamma+v_{j,\ell'}) \\
&\quad + \sum_{m=1}^{s_j} \overline{\sqrt{\frac{1}{\theta_j(\gamma+v_{j,\ell})}}G_{j+1}^{(i)(m)}(\gamma+v_{j,\ell})}\sqrt{\frac{1}{\theta_j(\gamma+v_{j,\ell'})}}G_{j+1}^{(i)(m)}(\gamma+v_{j,\ell'}) \\
&= \frac{\theta_j(\gamma)}{\sqrt{\theta_{j+1}(\gamma+v_{j,\ell})\theta_{j+1}(\gamma+v_{j,\ell'})}}\overline{H_{j+1}(\gamma+v_{j,\ell})}H_{j+1}(\gamma+v_{j,\ell'}) \\
&\quad + \sum_{m=1}^{s_j} \frac{1}{\sqrt{\theta_{j+1}(\gamma+v_{j,\ell})\theta_{j+1}(\gamma+v_{j,\ell'})}}\overline{G_{j+1}^{(i)(m)}(\gamma+v_{j,\ell})}G_{j+1}^{(i)(m)}(\gamma+v_{j,\ell'}) \\
&= \frac{1}{\sqrt{\theta_{j+1}(\gamma+v_{j,\ell})\theta_{j+1}(\gamma+v_{j,\ell'})}}\Bigg[\theta_j(\gamma)\overline{H_{j+1}(\gamma+v_{j,\ell})}H_{j+1}(\gamma+v_{j,\ell'}) \\
&\quad + \sum_{m=1}^{s_j} \overline{G_{j+1}^{(i)(m)}(\gamma+v_{j,\ell})}G_{j+1}^{(i)(m)}(\gamma+v_{j,\ell'})\Bigg]
\end{aligned}$$

since θ_j is a $\Gamma_j^\perp$, that is, $\theta_j(\gamma+v_{j,\ell}) = \theta_j(\gamma)$ as $v_{j,\ell} \in \Gamma_j^\perp$ for each $\ell \in \{1,2,\cdots,d_j\}$. Therefore, (6.3.5), is further equivalent to

$$\theta_j(\gamma)\overline{H_{j+1}(\gamma)}H_{j+1}(\gamma) + \sum_{m=1}^{s_j} \overline{G_{j+1}^{(i)(m)}(\gamma)}G_{j+1}^{(i)(m)}(\gamma) = \theta_{j+1}(\gamma)\frac{s(\Gamma_j)}{s(\Gamma_{j+1})} \tag{6.3.6}$$

and

$$\theta_j(\gamma)\overline{H_{j+1}(\gamma+v_{j,\ell})}H_{j+1}(\gamma+v_{j,\ell'}) + \sum_{m=1}^{s_j} \overline{G_{j+1}^{(i)(m)}(\gamma+v_{j,\ell})}G_{j+1}^{(i)(m)}(\gamma+v_{j,\ell'}) = 0. \tag{6.3.7}$$

Now, equations (6.3.6) and (6.3.7) are true by the assumption (6.3.2). Now, we define the functions $\widetilde{\Psi_j^{(i)(m)}}$ for $j \in \mathcal{J}$ and $m = 1,2,\dots,s_j$ as follows:

$$\widetilde{\Psi_j^{(i)(m)}}(\gamma) := \widetilde{G_{j+1}^{(i)(m)}}(\gamma)\widetilde{\Phi_{j+1}}(\gamma), \ \gamma \in \widehat{G} \text{ and } m = 1,2,\dots,s_j. \tag{6.3.8}$$

For $j \in \mathcal{J}, m \in \{1,2,\dots,s_j\}$ and $i \in \{1,2\}$, we define the functions $g_{(m,j)}^{(i)}$ as inverse Fourier transform of $\widetilde{\Psi_j^{(i)(m)}}$ by

$$\widetilde{g_{(m,j)}^{(i)}} = \mathcal{F}^{-1}\widetilde{\Psi_j^{(i)(m)}}. \tag{6.3.9}$$

Thus by Theorem 6.2.1, GTI system $\bigcup_{j\in\mathcal{J}}\{T_\lambda \widetilde{g_p^{(i)}}\}_{\lambda\in\Gamma_j,p\in P_j}$ is a Parseval frames, where $P_j = \{(m,j) : m = 1,2,\cdots,s_j\}$. Now, observe that for $m = 1,2,\cdots,s_j$,

$$\Psi_j^{(i)(m)}(\gamma) = G_{j+1}^{(i)(m)}(\gamma)\Phi_{j+1}(\gamma) = \sqrt{\frac{1}{\theta_{j+1}(\gamma)}}G_{j+1}^{(i)(m)}(\gamma)\sqrt{\theta_{j+1}(\gamma)}\Phi_{j+1}(\gamma)$$
$$= \widetilde{G_{j+1}^{(i)(m)}}(\gamma)\widetilde{\Phi_{j+1}}(\gamma) = \widetilde{\Psi_j^{(i)(m)}}(\gamma).$$

This completes the proof. ∎

6.4. Construction of Pairwise orthogonal Parseval frames

The following theorem provides a technique to construct pairwise orthogonal Parseval frames.

Theorem 6.4.1. *Assume all the hypotheses of Theorem 6.2.1 hold. Further, suppose that for each $j\in\mathcal{J}$, the matrix valued functions $\widetilde{\mathfrak{B}_j^{(1)}}(\gamma)$ and $\widetilde{\mathfrak{B}_j^{(2)}}(\gamma)$, defined in (2.2.8), satisfy*

$$(\widetilde{\mathfrak{B}_j^{(1)}}(\gamma))^*\widetilde{\mathfrak{B}_j^{(2)}}(\gamma) = O_{d_j} \text{ for a.e. } \gamma\in\Omega_j,$$

where O_{d_j} is the zero matrix of order d_j. Then $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ and $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,p\in P_j}$ (defined in (2.2.6)) are pairwise orthogonal Parseval frames in $L^2(G)$.

Proof. For each $i\in\{1,2\}$, the GTI system $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\gamma\in\Gamma_j,p\in P_j}$ satisfies ∞-UCP, using Theorem 6.2.1(i) and also Parseval frame by Theorem 6.2.1(ii). Now the GTI system $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ is Parseval frame and the GTI system $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j,p\in P_j}$ satisfies the ∞-UCP. Thus, in view of Remark 5.2.4, together both the GTI systems satisfy the dual ∞-UCP and hence dual 1-UCP. For proving the systems are pairwise orthogonal Parseval frames, it is sufficent to show the following in view of Theorem 5.3.2:

$$\sum_{j\in\mathcal{J}}\frac{1}{s(\Gamma_j)}\sum_{p\in P_j}\overline{(\widehat{g_p^{(1)}})(w)}(\widehat{g_p^{(2)}})(w) = 0 \text{ for a.e. } w\in\widehat{G}, \tag{6.4.1}$$

and for $\alpha\in\bigcup_{j\in\mathcal{J}}\Gamma_j^\perp\setminus\{0\}$,

$$\sum_{j\in\mathcal{J}:\alpha\in\Gamma_j}\frac{1}{s(\Gamma_j)}\sum_{p\in P_j}\overline{(\widehat{g_p^{(1)}})(w)}(\widehat{g_p^{(2)}})(w+\alpha) = 0 \text{ for a.e. } w\in\widehat{G}. \tag{6.4.2}$$

To prove (6.4.1) and (6.4.2), we refer to the proof of [98, Theorem 3.1]. However, [98, Theorem 3.1] relies on the assumption that $\bigcap_{j\in\mathcal{J}}\Gamma_j^\perp = 0$. We demonstrate that this condition follows from the hypothesis already stated in Theorem 6.2.1, i.e., for a compact set S in $\widehat{G}$, there exists $J \in \mathbb{Z}$ such that

$$\mu_{\widehat{G}}\big((\omega + S) \cap (\omega' + S)\big) = 0 \quad \text{for } \omega \neq \omega',\ \ \omega, \omega' \in \Gamma_J^{\perp}. \tag{6.4.3}$$

To see this, we assume the contrary, i.e., $\bigcap_{j \in \mathcal{J}} \Gamma_j^{\perp} \neq \{0\}$. Then there exists a $w \in \bigcap_{j \in \mathcal{J}} \Gamma_j^{\perp}$ such that $w \neq 0$. Let Ω_j be a fundamental domain with $\mu_{\widehat{G}}(\Omega_j) \neq 0$. Choose a compact set $T \subset \Omega_j$ such that $\mu_{\widehat{G}}(T) \neq 0$ and define $S := T \cup (T + w)$. Then $S \cap (S + w) \supset T + w$, so we have

$$\mu_{\widehat{G}}(S \cap S + w) \geq \mu_{\widehat{G}}(T + w) \neq 0, w \in \Gamma_j^{\perp} \text{ for each } j \in \mathcal{J}.$$

Hence, for this compact set S, there is no $j \in \mathcal{J}$ such that $\mu_{\widehat{G}}\big((w + S) \cap (0 + S)\big) = 0$ for $w \in \Gamma_j^{\perp}$. This contradicts (6.4.3). Therefore $\bigcap_{j \in \mathcal{J}} \Gamma_j^{\perp} = 0$. ∎

In contrast to [98], where the construction requires the subgroups $\{\Gamma_j\}_{j \in J}$ to become stationary as $j \to -\infty$, Theorem 6.4.1 provides pairwise orthogonal Parseval GTI systems without this assumption.

Example 6.4.2. Recall the GTI systems $\bigcup_{j=-\infty}^{j_0} \{T_\lambda g_p^{(i)}\}_{\lambda \in 2^{-j+k}\mathbb{Z},\, p \in \{(m,j): m=1,2,\ldots,8\}}$ for $i \in \{1, 2\}$ constructed in Example 6.2.5. Since all the assumptions of Theorem 6.2.1 are satisfied, it follows that both are Parseval frame. Next, we show that for each $j \in \mathcal{J}$,

$$(\widetilde{\mathfrak{B}_j^{(1)}}(\gamma))^* \widetilde{\mathfrak{B}_j^{(2)}}(\gamma) = O_{d_j} \text{ for a.e. } \gamma \in \Omega_j.$$

Equivalently to show that for a.e. $\gamma \in [0, 2^{j-k})$ and $\ell, \ell' \in \{1, 2\}$,

$$\sum_{m=1}^{4} \overline{G_{j+1}^{(1)(m)}(\gamma + \nu_{j,\ell})} G_{j+1}^{(2)(m)}(\gamma + \nu_{j,\ell'}) = 0, \text{ where } \nu_{j,1} = 0 \text{ and } \nu_{j,2} = 2^{j-k}. \tag{6.4.4}$$

First suppose $\ell = \ell' = 1$, then $\nu_{j,\ell} = \nu_{j,\ell'} = 0$. Now, we calculate

$$\begin{aligned}
&\sum_{m=1}^{8} \overline{G_{j+1}^{(1)(m)}(\gamma + \nu_{j,\ell})} G_{j+1}^{(2)(m)}(\gamma + \nu_{j,\ell'}) = \sum_{m=1}^{8} \overline{G_{j+1}^{(1)(m)}(\gamma)} G_{j+1}^{(2)(m)}(\gamma) \\
&= \sum_{m=1}^{8} \overline{a_{1m}} b_{1m} \left| (1 + e^{-2\pi i (2^{k-j-1}\gamma)})^3 (1 - e^{-2\pi i (2^{k-j-1}\gamma)}) \right|^2 \\
&\quad + \sum_{m=1}^{8} \overline{a_{2m}} b_{2m} \left| (1 + e^{-2\pi i (2^{k-j-1}\gamma)})^2 (1 - e^{-2\pi i (2^{k-j-1}\gamma)})^2 \right|^2 \\
&\quad + \sum_{m=1}^{8} \overline{a_{3m}} b_{3m} \left| (1 + e^{-2\pi i (2^{k-j-1}\gamma)})^1 (1 - e^{-2\pi i (2^{k-j-1}\gamma)})^3 \right|^2 + \sum_{m=1}^{8} \overline{a_{4m}} b_{4m} \left| (1 - e^{-2\pi i (2^{k-j-1}\gamma)})^4 \right|^2 \\
&\quad = 0,
\end{aligned}$$

since $\sum\limits_{m=1}^{8} \overline{a_{nm}} b_{nm} = 0$ for each $n \in \{1,2,3,4\}$. Similarly, we can show that (6.4.4) is true for the other values of ℓ and ℓ'. Hence, using Theorem 6.4.1, the GTI systems $\bigcup\limits_{j=-\infty}^{j_0} \{T_\gamma g_p^{(1)}\}_{\gamma\in 2^{-j+k}\mathbb{Z},\, p\in\{(m,j):m=1,2,\ldots,8\}}$ and $\bigcup\limits_{j=-\infty}^{j_0} \{T_\lambda g_p^{(2)}\}_{\lambda\in 2^{-j+k}\mathbb{Z},\, p\in\{(m,j):m=1,2,\ldots,8\}}$ are pairwise orthogonal Parseval frames.

The following application of Theorem 6.4.1 provides a recipe to construct pairwise orthogonal Parseval frames of the form

$$\bigcup_{j\in\mathcal{J}} \{T_\lambda h_p^{(1)}\}_{\lambda\in\eta\Gamma_j, p\in P_j} \text{ and } \bigcup_{j\in\mathcal{J}} \{T_\lambda h_p^{(2)}\}_{\lambda\in\zeta\Gamma_j, p\in P_j}.$$

Theorem 6.4.3. *Let* $\bigcup\limits_{j\in\mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ *and* $\bigcup\limits_{j\in\mathcal{J}} \{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j}$ *are obtained as in Theorem 6.4.1. Then the systems*

$$\bigcup_{j\in\mathcal{J}} \{T_\lambda h_p^{(1)}\}_{\lambda\in\eta\Gamma_j, p\in P_j} \text{ and } \bigcup_{j\in\mathcal{J}} \{T_\lambda h_p^{(2)}\}_{\lambda\in\zeta\Gamma_j, p\in P_j}$$

are pairwise orthogonal Parseval frames, where $h_p(1) := D_\eta^{-1} g_p^{(1)}$ *and* $h_p(2) := D_\zeta^{-1} g_p^{(2)}$.

Proof. Let Θ be the mixed dual Gramian operator for the systems $\bigcup\limits_{j\in\mathcal{J}} \{T_\lambda h_p^{(1)}\}_{\lambda\in\eta\Gamma_j, p\in P_j}$ and $\bigcup\limits_{j\in\mathcal{J}} \{T_\lambda h_p^{(2)}\}_{\lambda\in\zeta\Gamma_j, p\in P_j}$. By polarisation identity, these systems are pairwise orthogonal if and only if for all $f \in L^2(G)$, we have

$$\langle D_\zeta \Theta D_\eta^{-1} f, f\rangle = 0.$$

By following the similar steps as in the proof of Theorem 5.3.1, the above condition is equivalent to the systems $\bigcup\limits_{j\in\mathcal{J}} \{T_\lambda D_\eta h_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ and $\bigcup\limits_{j\in\mathcal{J}} \{T_\lambda D_\zeta h_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j}$ are pairwise orthogonal. By substituting values of $h_p^{(1)}$ and $h_p^{(2)}$, it is equivalent to $\bigcup\limits_{j\in\mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ and $\bigcup\limits_{j\in\mathcal{J}} \{T_\lambda g_p^{(2)}\}_{\lambda\in\Gamma_j, p\in P_j}$ are pairwise orthogonal. This is true by Theorem 6.4.1.

Note that the system

$$\bigcup_{j\in\mathcal{J}} \{T_\lambda h_p^{(1)}\}_{\lambda\in\eta\Gamma_j, p\in P_j} = \bigcup_{j\in\mathcal{J}} \{T_{\eta\gamma} D_\eta^{-1} g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j} = \bigcup_{j\in\mathcal{J}} \{D_\eta^{-1} T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$$

which is a Parseval frame, since $\bigcup\limits_{j\in\mathcal{J}} \{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j, p\in P_j}$ is a Parseval frame by Theorem 6.4.1 and D_η^{-1} is a unitary operator. A similar arguments shows that $\bigcup\limits_{j\in\mathcal{J}} \{T_\lambda h_p^{(2)}\}_{\lambda\in\zeta\Gamma_j, p\in P_j}$ is also a Parseval frame. This finishes the proof. ∎

In [98], the author presents a method for constructing two orthogonal frames and N Parseval frames from a given Parseval frame. However, the construction of N Parseval frames is possible only when the original frame has a single filter $G^{(1)(1)}$, that is, when $s_j = 1$. If the given frame contains more than one filter, only two new Parseval frames can be generated using the method from [98].

In contrast, the following theorem addresses this limitation: it enables the construction of N Parseval frames from a single given Parseval frame, regardless of the number of filters present. This advancement generalizes [98, Theorems 3.2 and 3.3], offering a broader, more powerful construction framework.

Theorem 6.4.4. *Consider the GTI system* $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(1)}\}_{\lambda\in\Gamma_j,p\in P_j}$ *(defined in (2.2.6)) satisfying the assumptions of Theorem 6.2.1. For each* $i \in \{2,3,\cdots,N+1\}$, $j \in \mathcal{J}$ *and* $m \in \{0,1,\cdots,Ns_j\}$, *assume the* $\Gamma_j^\perp$*-periodic functions* $G_{j+1}^{(i)(m)}$ *satisfy the relations* $G_{j+1}^{(i)(0)}(\gamma) = H_{j+1}(\gamma)$ *and*

$$\begin{pmatrix} G_{j+1}^{(i)(1)}(\gamma) \\ G_{j+1}^{(i)(2)}(\gamma) \\ \vdots \\ G_{j+1}^{(i)(2s_j)}(\gamma) \end{pmatrix} = \left(K^j_{1+(i-2)s_j}(\gamma)\, K^j_{2+(i-2)s_j}(\gamma) \cdots K^j_{s_j+(i-2)s_j}(\gamma)\right) \begin{pmatrix} G_{j+1}^{(1)(1)}(\gamma) \\ G_{j+1}^{(1)(2)}(\gamma) \\ \vdots \\ G_{j+1}^{(1)(s_j)}(\gamma) \end{pmatrix}, \ \gamma \in \widehat{G}, \tag{6.4.5}$$

where $K^j(\gamma) = \left(K_1^j(\gamma)\ K_2^j(\gamma) \cdots\ K^j_{Ns_j}(\gamma)\right)$ *is a* $Ns_j \times\ Ns_j$ *unitary matrix with* $\Gamma_j^\perp$*-periodic entries. Then* $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j,p\in P_j'}$ *and* $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i')}\}_{\lambda\in\Gamma_j,p\in P_j'}$ *(defined in (2.2.6)) are pairwise orthogonal Parseval frames, where* $P_j' = \{(m,j) : m = 1,2,\cdots,Ns_j\}$, *and* $i,i' \in \{2,3,\cdots,N+1\}$.

Proof of Theorem 6.4.4. First, we use Theorem 6.2.1 (ii) to prove that the system $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\gamma\in\Gamma_j,p\in P_j}$ is a Parseval frame for $L^2(G)$ for each $i \in \{2,\cdots,N+1\}$. For this, it is sufficient to show that for each $j \in \mathcal{J}$ and $i \in \{2,3,\cdots,N+1\}$, we have

$$(\mathfrak{B}_j^{(i)}(\gamma))^*\mathfrak{B}_j^{(i)}(\gamma) = \frac{s(\Gamma_j)}{s(\Gamma_{j+1})} I_{d_j} \text{ for a.e. } \gamma \in \Omega_j,$$

where the matrix $\mathfrak{B}_j^{(i)}(\gamma) := \left(G_{j+1}^{(i)(m)}(\gamma + v_{j,n})\right)_{\substack{0\le m\le Ns_j \\ 1\le n\le d_j}}$ (defined similar to (2.2.7)) is of order $(1+Ns_j)\times d_j$ since $G_{j+1}^{(i)(0)} = H_{j+1}$ and for $1 \le m \le Ns_j$, $G_{j+1}^{(i)(m)}$ is defined in (6.4.5). Note that the matrix $\mathfrak{B}_j^{(1)}(\gamma)$ is of order $(s_j+1)\times d_j$. For $i \in \{2,3,\cdots,N+1\}$, the matrix

$\mathfrak{B}_j^{(i)}(\gamma)$ can be expressed as

$$\mathfrak{B}_j^{(i)}(\gamma) = A_j^{(i)}(\gamma)\mathfrak{B}_j^{(1)}(\gamma), \text{ where } A_j^{(i)}(\gamma) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & K^j_{1+(i-2)s_j}(\gamma) & K^j_{2+(i-2)s_j}(\gamma) & \cdots & K^j_{s_j+(i-2)s_j}(\gamma) \end{pmatrix},$$

since the entries of $K_i^j(\gamma)$ are $\Gamma_j^{\perp}$-periodic. The columns $K_1^j(\gamma), K_2^j(\gamma)\cdots, K^j_{Ns_j}(\gamma)$ are orthonormal implies $(A_j^{(i)}(\gamma))^* A_j^{(i)}(\gamma) = I_{s_j+1}$. Using (6.2.1), we obtain

$$\begin{aligned}(\mathfrak{B}_j^{(i)}(\gamma))^*\mathfrak{B}_j^{(i)}(\gamma) &= (\mathfrak{B}_j^{(1)}(\gamma))^*(A_j^{(i)}(\gamma))^* A_j^{(i)}(\gamma)\mathfrak{B}_j^{(1)}(\gamma) \\ &= (\mathfrak{B}_j^{(1)}(\gamma))^* I_{s_j+1}\mathfrak{B}_j^{(1)}(\gamma) \\ &= (\mathfrak{B}_j^{(1)}(\gamma))^*\mathfrak{B}_j^{(1)}(\gamma) = \frac{s(\Gamma_j)}{s(\Gamma_{j+1})} I_{d_j}.\end{aligned}$$

Therefore for each $i \in \{2,3,\cdots,N+1\}$, the GTI system $\bigcup_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j, p\in P'_j}$ is a Parseval frame. Now it remains to show that $\bigcup\limits_{j\in\mathcal{J}}\{T_\lambda g_p^{(i)}\}_{\lambda\in\Gamma_j,p\in P''_j}$ and $\bigcup\limits_{j\in\mathcal{J}}\{T_\lambda g_p^{(i')}\}_{\lambda\in\Gamma_j,p\in P''_j}$ are pairwise orthogonal. To demonstrate this, according to Theorem 6.4.1, it suffices to prove that for each $j \in \mathcal{J}$, and $i, i' \in \{2,3,\cdots,N+1\}$, $(\widetilde{\mathfrak{B}_j^{(i)}})^*(\gamma)\widetilde{\mathfrak{B}_j^{(i')}}(\gamma) = O_{d_j}$, a.e. $\gamma \in \Omega_j$ and $i \neq i'$, where for $i \in \{2,3,\cdots,N+1\}$, the matrix $\widetilde{\mathfrak{B}_j^{(i)}}(\gamma) := \left(G_{j+1}^{(i)(m)}(\gamma + v_{j,n})\right)_{\substack{1\le m\le 2s_j\\ 1\le n\le d_j}}$ is of order $Ns_j \times d_j$. Note that the matrix $\widetilde{\mathfrak{B}_j^{(1)}}(\gamma)$ is of order $s_j \times d_j$. Using the periodicity of entries of $K^j(\gamma)$, we can express

$$\widetilde{\mathfrak{B}_j^{(i)}}(\gamma) = \begin{pmatrix} K^j_{1+(i-2)s_j}(\gamma) & K^j_{2+(i-2)s_j}(\gamma) & \cdots & K^j_{s_j+(i-2)s_j}(\gamma)\end{pmatrix}\widetilde{\mathfrak{B}_j^{(1)}}(\gamma) \text{ and}$$
$$\widetilde{\mathfrak{B}_j^{(i')}}(\gamma) = \begin{pmatrix} K^j_{1+(i'-2)s_j}(\gamma) & K^j_{2+(i'-2)s_j}(\gamma) & \cdots & K^j_{s_j+(i'-2)s_j}(\gamma)\end{pmatrix}\widetilde{\mathfrak{B}_j^{(1)}}(\gamma)$$

and hence, we get

$$\begin{aligned}(\widetilde{\mathfrak{B}_j^{(i)}})^*(\gamma)\widetilde{\mathfrak{B}_j^{(i')}}(\gamma) =& \left(\begin{pmatrix} K^j_{1+(i-2)s_j}(\gamma) & K^j_{2+(i-2)s_j}(\gamma) & \cdots & K^j_{s_j+(i-2)s_j}(\gamma)\end{pmatrix}\widetilde{\mathfrak{B}_j^{(1)}}(\gamma)\right)^* \\ &\times \begin{pmatrix} K^j_{1+(i'-2)s_j}(\gamma) & K^j_{2+(i'-2)s_j}(\gamma) & \cdots & K^j_{s_j+(i'-2)s_j}(\gamma)\end{pmatrix}\widetilde{\mathfrak{B}_j^{(1)}}(\gamma) \\ =&(\widetilde{\mathfrak{B}_j^{(1)}})^*(\gamma)\begin{pmatrix} K^j_{1+(i-2)s_j}(\gamma) & K^j_{2+(i-2)s_j}(\gamma) & \cdots & K^j_{s_j+(i-2)s_j}(\gamma)\end{pmatrix}^* \\ &\times \begin{pmatrix} K^j_{1+(i'-2)s_j}(\gamma) & K^j_{2+(i'-2)s_j}(\gamma) & \cdots & K^j_{s_j+(i'-2)s_j}(\gamma)\end{pmatrix}\widetilde{\mathfrak{B}_j^{(1)}}(\gamma) \\ =&O_{d_j},\end{aligned}$$

since

$$\begin{pmatrix} K^j_{1+(i-2)s_j}(\gamma) & \cdots & K^j_{s_j+(i-2)s_j}(\gamma)\end{pmatrix}^*\begin{pmatrix} K^j_{1+(i'-2)s_j}(\gamma) & \cdots & K^j_{s_j+(i'-2)s_j}(\gamma)\end{pmatrix} = O_{s_j}.$$

Thus the result follows. ■

CHAPTER 7

CYCLIC FRAMES IN FINITE-DIMENSIONAL HILBERT SPACES

Generalizing a definition by Kalra [82], the purpose of this chapter is to analyze cyclic frames in finite-dimensional Hilbert spaces. Cyclic frames form a subclass of the dynamical frames introduced and analyzed in detail by Aldroubi et al. in [3] and subsequent papers; they are particularly interesting due to their attractive properties in the context of erasure problems. By applying an alternative approach, we are able to shed new light on general dynamical frames as well as cyclic frames. In particular, we provide a characterization of dynamical frames, which in turn leads to a characterization of cyclic frames.

7.1. Introduction

In Chapters 2–6, we studied the frame properties of generalized translation invariant (GTI) frames and their various structural subclasses, such as wavelet and Gabor frames. These systems can be viewed as special cases of frames generated by unitary representations of LCA groups. This observation naturally links them to the class of dynamical frames.

Let ρ be a unitary representation of an LCA group Γ on a separable Hilbert space $\mathcal{H}$. For a family of vectors $\{\psi_p : p \in P\} \subset \mathcal{H}$, a fundamental problem in harmonic analysis is to determine when the orbit system

$$\{\rho(\lambda)\psi_p : \lambda \in \Gamma, p \in P\}$$

forms a Bessel sequence, a Riesz basis, or a frame (see, e.g., [16, 19, 17, 25, 61]). This formulation encompasses several well-known structured systems. For instance, if $\mathcal{H} = L^2(G)$ and $\rho(\lambda)$ acts as translation by λ on G, the system reduces to

$$\{T_\lambda\psi_p : \lambda \in \Gamma,\, p \in P\},$$

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known as a translation invariant (TI) system (see, [27, 61]). A countable union of such TI systems yields a GTI system, as studied in the earlier chapters. On the other hand, if $\Gamma = \mathbb{Z}$ and $\rho(k)$ is given by the iterates of a unitary operator T on $\mathcal{H}$, i.e., $\rho(k) = T^k$, then the corresponding orbit system

$$\{T^k \psi_p : k \in \mathbb{Z},\, p \in P\}$$

is known as a dynamical system. Such systems naturally arise in the context of dynamical sampling, where the goal is to recover a signal $f \in \mathcal{H}$ from the samples $\{\langle f, T^k \psi_p \rangle : k \in \mathbb{Z}, p \in P\}$.

Dynamical sampling was introduced around 2013–2014 by Aldroubi et al. in a series of papers [4, 3, 2]. Discarding the applied context in which dynamical sampling was introduced, the mathematical issue is how to construct frames in a Hilbert space by iterating the action of a bounded operator on a collection of vectors; in the literature such frames are called *dynamical frames.* Very fast, dynamical sampling became an active research area, mainly with contributions dealing with infinite-dimensional spaces; note, however, that already the paper [3] characterized the frame property for such iterated systems in the finite-dimensional setting. Other theoretical contributions are contained in the paper [10] by Ashbrock and Powell.

In the current chapter we focus on dynamical frames in finite-dimensional spaces; for convenience, and without loss of generality, we formulate all results for frames in $\mathbb{C}^d$. We will complement the analysis in [3] with an alternative approach, which shed light on dynamical frames from a different angle. Based on this approach, we will analyze *cyclic frames,* a concept that is strictly restricted to the finite-dimensional setting. Cyclic frames were introduced in the paper [82] by Kalra in 2006; since the terminology has not been used in later papers, we will allow ourself to use the same name for a slightly more general concept, see Section 7.3 for details. Already Kalra noticed that cyclic frames have attractive features in the context of erasure problems. For more details on erasures, we refer to [77, 82, 93, 91]. The main purpose of this chapter is to characterize (our generalized version of) cyclic frames and discuss their key features.

Our alternative approach to dynamical frames is presented in Section 7.2. In Section 7.3 we state the formal definition of cyclic frames and two ways of characterization of such frames. Furthermore we prove a number of properties that follow directly from our approach in Section 7.2. In Section 7.4 we consider tight cyclic frames and their connection to erasure problems. Finally, the appendix contains proofs of a number of more technical results.

7.2. Dynamical frames

A frame $\{f_k\}_{k=1}^n$ for $\mathbb{C}^d$ is called a *dynamical frame* for $\mathbb{C}^d$ if there exists a linear operator $T : \mathbb{C}^d \to \mathbb{C}^d$ such that

$$f_k = T^{k-1} f_1 \text{ for } 1 \le k \le n. \tag{7.2.1}$$

In that case we can write

$$\{f_k\}_{k=1}^n = \{T^{k-1} f_1\}_{k=1}^n = \{f_1, Tf_1, \ldots, T^{n-1} f_1\}. \tag{7.2.2}$$

Note that in [10], a dynamical frame is defined as a frame of the form

$$\{f_k\}_{k=1}^n = \{T^k f\}_{k=1}^n = \{Tf, T^2 f, \ldots, T^n f\}, f \in \mathbb{C}^d; \tag{7.2.3}$$

clearly, this is a subclass of the dynamical frames considered in the current chapter.

Our main purpose is to consider a special class of dynamical frames, to be introduced in Section 7.3. For this purpose we first state a number of properties of general dynamical frames. The following lemma is well-known; it implies in particular that every basis for $\mathbb{C}^d$ is a dynamical frame.

Lemma 7.2.1. *Assume that $\{f_k\}_{k=1}^n$ is a collection of linearly independent vectors in $\mathbb{C}^d$. Then there exists a linear operator $T : \mathbb{C}^d \to \mathbb{C}^d$ such that*

$$\{f_k\}_{k=1}^n = \{T^{k-1} f_1\}_{k=1}^n. \tag{7.2.4}$$

Proof. The assumption of linear independence of $\{f_k\}_{k=1}^n$ implies that $n \le d$. Due to linear independence of the vectors $\{f_k\}_{k=1}^n$, we can clearly define the operator T by $Tf_k = f_{k+1}$ for $k = 1, \ldots, n-1$. Note that this only defines T on $\text{span}\{f_1, \ldots, f_{n-1}\}$. The operator can be extended to a linear operator on $\mathbb{C}^d$ by defining it in an arbitrary way on a basis for $\text{span}\{f_1, \ldots, f_{n-1}\}^\perp$. ■

In general, the key feature of a frame is that it might contain more elements than a basis. The following result states that if $\{T^{k-1} f_1\}_{k=1}^n$ is a dynamical frame for $\mathbb{C}^d$, then the set of the *first* d consecutive vectors, i.e., the set $\{T^{k-1}f_1\}_{k=1}^d$, forms a basis for $\mathbb{C}^d$. The similar result for dynamical frames of the form (7.2.3) was proved in [10].

Lemma 7.2.2. *Assume that* $\{T^{k-1}f_1\}_{k=1}^n$ *is a frame for* $\mathbb{C}^d$. *Then the first* d *elements, i.e.,* $\{T^{k-1}f_1\}_{k=1}^d$, *form a basis for* $\mathbb{C}^d$.

Proof. Assume that $\{T^{k-1}f_1\}_{k=1}^d$ is not a basis for $\mathbb{C}^d$. Then $\{T^{k-1}f_1\}_{k=1}^d$ is linearly dependent. There are now two possibilities:

1) If $\{T^{k-1}f_1\}_{k=1}^{d-1}$ are linearly independent, then $T^{d-1}f_1 \in \text{span}(\{T^{k-1}f_1\}_{k=1}^{d-1})$. But then the subspace $\text{span}(\{T^{k-1}f_1\}_{k=1}^{d-1})$ is invariant under the operator T, and, regardless of the choice of $n \in \mathbb{N}$, a family $\{T^{k-1}f_1\}_{k=1}^n$ cannot be a frame for $\mathbb{C}^d$. This contradicts the assumptions in the lemma.

2) If $\{T^{k-1}f_1\}_{k=1}^{d-1}$ are linearly dependent, the same argument as in 1) implies that $\text{span}(\{T^{k-1}f_1\}_{k=1}^{d-1})$ is invariant under T. Hence, again the subspace $\text{span}(\{T^{k-1}f_1\}_{k=1}^{d-1})$ is invariant under the operator T, and, regardless of the choice of $n \in \mathbb{N}$, a family $\{T^{k-1}f_1\}_{k=1}^n$ cannot be a frame for $\mathbb{C}^d$. This contradicts the assumptions in the lemma.

Altogether, we conclude that $\{T^{k-1}f_1\}_{k=1}^d$ is a basis for $\mathbb{C}^d$. ∎

Note that Lemma 7.2.2 does not guarantee that any other set of d-consecutive vectors in $\{T^{k-1}f_1\}_{k=1}^n$ will form a basis for $\mathbb{C}^d$; see Corollary 7.2.5 for a more detailed discussion.

Our approach in the current chapter is based on the next result, which shows that the class of all finite dynamical frames is parametrized by a choice of d linearly independent vectors $\{f_k\}_{k=1}^d$, a vector $\varphi \in \mathbb{C}^d$ and an integer $n \geq d$.

Theorem 7.2.3.

(i) Consider a collection of linearly independent vectors $\{f_k\}_{k=1}^d$ *in* $\mathbb{C}^d$, *an arbitrary vector* $\varphi \in \mathbb{C}^d$, *and an integer* $n \geq d$. *Define the linear map* $T : \mathbb{C}^d \to \mathbb{C}^d$ *by*

$$Tf_k = f_{k+1},\ k = 1, \ldots, d-1,\ Tf_d = \varphi. \tag{7.2.5}$$

Then $\{T^{k-1}f_1\}_{k=1}^n$ *is a frame for* $\mathbb{C}^d$.

(ii) Conversely, any dynamical frame $\{T^{k-1}f_1\}_{k=1}^n$ *for* $\mathbb{C}^d$ *corresponds to the setup in (i), with* $f_k = T^{k-1}f_1,\ k = 2, \ldots, d,\ \varphi = Tf_d$.

Proof. The statement in (i) is precisely the construction in Lemma 7.2.1; already in the proof of the lemma it was highlighted that the construction works for arbitrary choices of the vector $\varphi \in \mathbb{C}^d$. Clearly $\{f_k\}_{k=1}^d = \{T^{k-1}f_1\}_{k=1}^d$ is a basis for $\mathbb{C}^d$, and hence $\{T^{k-1}f_1\}_{k=1}^n$ is an (overcomplete) frame. Concerning (ii), Lemma 7.2.2 shows that if $\{T^{k-1}f_1\}_{k=1}^n$ is a dynamical frame, then the first d elements, $\{T^{k-1}f_1\}_{k=1}^d$, form a basis for $\mathbb{C}^d$, i.e., a linearly independent set of vectors. Now the result follows from Lemma 7.2.1. ■

The next result concerns the special case of $d+1$ vectors in $\mathbb{C}^d$. We return to this setting in Lemma 7.3.4 and Proposition 7.4.4.

Corollary 7.2.4. *A set of $d+1$ vectors $\{f_k\}_{k=1}^{d+1}$ is a dynamical frame for $\mathbb{C}^d$ if and only if $\{f_k\}_{k=1}^d$ is a basis for $\mathbb{C}^d$.*

Proof. Suppose $\{f_1, \dots, f_{d+1}\}$ is a dynamical frame for $\mathbb{C}^d$. Then by Lemma 7.2.2, the first d elements, i.e., $\{f_k\}_{k=1}^d$, forms a basis for $\mathbb{C}^d$. The converse follows from Theorem 7.2.3(i) by choosing $n = d+1$ and $\varphi = f_{d+1}$. ■

Note that in the setup of Theorem 7.2.3(i), the operator T in (7.2.5) is either surjective (if $\varphi \notin \operatorname{span}\{f_2, \dots, f_d\}$), or its range has codimension one (if $\varphi \in \operatorname{span}\{f_2, \dots, f_d\}$). The resulting frame has different properties in these two cases:

Corollary 7.2.5. *Consider the setup in Theorem 7.2.3(i). Then the following hold:*

(i) If the operator T in (7.2.5) is surjective, each collection of d consecutive vectors $\{T^{k-1}f_1\}_{k=\ell+1}^{\ell+d}$ is a basis for $\mathbb{C}^d$, for any $\ell = 0, 1, \dots$.

(ii) If the operator T in (7.2.5) is not surjective, $\{T^{k-1}f_1\}_{k=\ell+1}^{\ell+d}$ is not a basis for $\mathbb{C}^d$, for any $\ell = 1, 2 \dots$.

Proof. For the proof of (i), given any $\ell \in \{0, 1, 2, \dots\}$, the vectors $\{T^{k-1}f_1\}_{k=\ell+1}^{\ell+d}$ are the images of the basis vectors $\{T^{k-1}f_1\}_{k=1}^d$ under the bijective operator T^ℓ, and hence itself a basis for $\mathbb{C}^d$. Concerning (ii), if T is not surjective, the vectors $\{T^{k-1}f_1\}_{k=\ell+1}^{\ell+d}$ cannot span $\mathbb{C}^d$, and hence not be a basis (or frame) for $\mathbb{C}^d$. ■

Note that for dynamical frames of the form (7.2.3), it was already observed in [10] that every collection of d consecutive vectors form a basis for $\mathbb{C}^d$.

Let us end this section with an application of Theorem 7.2.3. Using a measure-theoretic approach, Ashbrock and Powell showed in [10] that every frame $\{f_k\}_{k=1}^n$ (dynamical or not) for $\mathbb{C}^d$ has a dynamical dual frame; see [9, Theorem 3.6] for an alternative approach. If we allow ourself to reorder the elements in the frame $\{f_k\}_{k=1}^n$, we can provide a short and constructive proof of this:

Proposition 7.2.6. *Let $\{f_k\}_{k=1}^n$ be a frame for $\mathbb{C}^d$, ordered such that $\{f_k\}_{k=1}^d$ is linearly independent. Then $\{f_k\}_{k=1}^n$ has a dual frame of the form $\{T^{k-1}g_1\}_{k=1}^n$.*

Proof. Every frame $\{f_k\}_{k=1}^n$ contains a basis, so by a reordering we can assume that the first d elements $\{f_k\}_{k=1}^d$ is a basis for $\mathbb{C}^d$. Let $\{g_k\}_{k=1}^d$ denote the dual basis, and define according to Theorem 7.2.3 the operator $T : \mathbb{C}^d \to \mathbb{C}^d$ by

$$Tg_k := g_{k+1},\ k = 1, \dots, d-1,\ Tg_k := 0.$$

Then $\{g_k\}_{k=1}^d = \{T^{k-1}g_1\}_{k=1}^d$; furthermore, by construction,

$$\{T^{k-1}g_1\}_{k=1}^n = \{g_1, g_2, \dots, g_d, 0, 0, \dots 0\},$$

which is clearly a dual frame of $\{f_k\}_{k=1}^n$. ∎

7.3. Cyclic frames

We now move to the cyclic frames. The term *cyclic frame* was coined in the paper [82] by Kalra in 2006. The terminology has apparently not been used in the literature since this chapter; we will use the name in a more general sense, as follows.

Definition 7.3.1. A dynamical frame $\{f_k\}_{k=1}^n = \{T^{k-1}f_1\}_{k=1}^n$ for $\mathbb{C}^d$ is called a cyclic frame if $T^n = I$; if n is the minimal choice of a positive integer such that $T^n = I$, we call $\{T^{k-1}f_1\}_{k=1}^n$ a minimal cyclic frame.

The difference between the terminology in [82] and Definition 7.3.1 is that Kalra requires the operator T to be unitary; furthermore, the distinction between cyclic frames and minimal cyclic frames does not occur in [82].

Cyclicity of a frame $\{f_k\}_{k=1}^n = \{T^{k-1}f_1\}_{k=1}^n$ means that if we consider iterates $\{T^{k-1}f_1\}_{k=1}^N$ for some $N > n$, the resulting system of vectors would simply repeat (some of) the vectors in $\{f_k\}_{k=1}^n$; and $\{T^{k-1}f_1\}_{k=1}^n$ being minimal means that there are no repetitions at all. In order to have an intuitive feeling for this (and for the subsequent results), we recommend the reader to think about the so-called *Mercedes-Benz frame* $\{f_k\}_{k=1}^3$ for $\mathbb{R}^2$, which is a minimal cyclic frame with T being a rotation of $\frac{2\pi}{3}$ rad; see Figure 7.3.1. We return to this frame in Example 7.4.5.

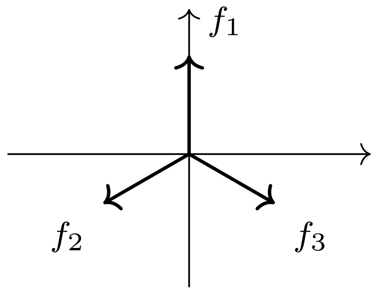

Figure 7.3.1. Mercedes-Benz frame $\{f_k\}_{k=1}^3$ in $\mathbb{R}^2$.

The approach in Section 7.2 has a number of immediate consequences for cyclic frames:

Corollary 7.3.2. *A basis* $\{f_k\}_{k=1}^d$ *for* $\mathbb{C}^d$ *is a cyclic frame.*

Proof. This follows from Theorem 7.2.3(i), with $n = d$ and $\varphi = f_1$. ■

A cyclic frame for $\mathbb{C}^d$ has the particular property that each collection of d consecutive elements form a basis for $\mathbb{C}^d$; as we saw in Corollary 7.2.5(ii) general dynamical frames do not have this property.

Corollary 7.3.3. *Let* $\{T^{k-1}f_1\}_{k=1}^n$ *be a cyclic frame for* $\mathbb{C}^d$*, with* $n > d$*. Then, for every* $\ell \in \{1, \cdots, n-d\}$*, the set* $\{T^{n-1}f_1\}_{n=\ell+1}^{\ell+d}$ *is a basis for* $\mathbb{C}^d$*.*

Proof. Since T is a cyclic operator, T is invertible, and hence surjective. Now the result follows from Corollary 7.2.5(i). ■

We will now address the question of how to construct cyclic frames $\{T^{k-1}f_1\}_{k=1}^n$ for $\mathbb{C}^d$, with $n > d$. For this purpose we refer to Theorem 7.2.3(i), which shows that the construction of *any* dynamical frame $\{f_k\}_{k=1}^n = \{T^{k-1}f_1\}_{k=1}^n$, $n > d$, must be based on a choice of a basis $\{f_k\}_{k=1}^d \subset \mathbb{C}^d$ and a vector $\varphi \in \mathbb{C}^d$; the linear map $T : \mathbb{C}^d \to \mathbb{C}^d$ must then be defined by

$$Tf_k = f_{k+1},\ k = 1, \ldots, d-1,\ Tf_d = \varphi. \tag{7.3.1}$$

Writing the vector $\varphi \in \mathbb{C}^d$ as

$$\varphi = c_1 f_1 + c_2 f_2 + \cdots + c_d f_d, \ c_1, \dots, c_d \in \mathbb{C},$$

we see that to construct a cyclic frame is a question of choosing $c_1, \dots, c_d$ such that $T^n = I$ for some $n > d$. This can easily be done in the case $n = d+1$; we return to the construction below in Proposition 7.4.4.

Lemma 7.3.4. *Let $\{f_k\}_{k=1}^d$ be a basis for $\mathbb{C}^d$. Then, with $f_{d+1} := -f_1 - f_2 - \cdots - f_d$, the family $\{f_k\}_{k=1}^{d+1}$ is a cyclic frame.*

Proof. Following the construction in Theorem 7.2.3(i), and taking $n = d+1$ and $\varphi := f_{d+1} = -f_1 - f_2 - \cdots - f_d$, we have that

$$\begin{aligned} T f_{d+1} &= -T f_1 - T f_2 - \cdots - T f_d \\ &= -f_2 - f_3 - \cdots - f_d - (-f_1 - f_2 - \cdots - f_d) = f_1. \end{aligned} \tag{7.3.2}$$

From the definition of T in (7.2.5), we have $f_{d+1} = T^d f_1$. Therefore,

$$T^{d+1} f_1 = T f_{d+1} = f_1. \tag{7.3.3}$$

Applying T^{k-1} to both sides of (7.3.3) for each $k \in \{1, 2, \dots, d\}$, and using the definition of T from (7.2.5), we obtain

$$T^{d+1} f_k = f_k, \quad \forall\, k \in \{1, 2, \dots, d\}.$$

Since $\{f_k\}_{k=1}^d$ forms a basis of $\mathbb{C}^d$, it follows that $T^{d+1} = I$. Hence, $\{f_k\}_{k=1}^{d+1}$ is a cyclic frame.
∎

Notice that for $n = 2$ and f_1, f_2 being the two first vectors in the Mercedes-Benz frame, the construction in Lemma 7.3.4 yields precisely the third vector f_3. It is clear that by looking at the nth roots of unity in $\mathbb{C}$, we can easily construct cyclic frames for $\mathbb{R}^2$ (and hereby for $\mathbb{C}^2$) for an arbitrary $n > 2$. We will now state a construction of a cyclic frame $\{T^{k-1} f_1\}_{k=1}^n$ for $\mathbb{C}^d$, for *any* choice of $d \in \mathbb{N}$ and $n > d$. The basic insight in Lemma 7.3.5 (iii) below goes back to Zimmermann [118], who proved the result for the case where $f_1 = (1, \dots, 1)$; also, Kalra [82] noticed that the obtained frame is cyclic. The result is based on the nth roots of unity, which we write in the form

$$\omega = e^{\frac{2\pi i m}{n}}, \, m = 1, \dots, n. \tag{7.3.4}$$

Recall that if $gcd(m,n)=1$, ω is said to be a *primitive* root of order n; this means that while $\omega^n=1$, we also have that $\omega^k\neq 1$ for $k=1,\dots,n-1$.

Lemma 7.3.5. *Fix any $n>d$, and let $\omega_1,\omega_2,\dots,\omega_d$ denote nth roots of unity. Consider the $d\times d$ matrix*

$$T:=\begin{pmatrix}\omega_1 & 0 & 0 & \cdot & \cdot & 0\\ 0 & \omega_2 & 0 & \cdot & \cdot & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & \cdot & \cdot & 0 & \omega_d\end{pmatrix},$$

and let $f_1\in\mathbb{C}^d$. Then the following hold:

(i) If the vector $f_1\in\mathbb{C}^d$ has a coordinate which is zero, then $\{T^{k-1}f_1\}_{k=1}^n$ is not a frame.

(ii) If two or more of the roots $\omega_1,\omega_2,\dots,\omega_d$ are identical, then $\{T^{k-1}f_1\}_{k=1}^n$ is not a frame.

(iii) If $\omega_1,\omega_2,\dots,\omega_d$ are distinct and all d coordinates of the vector f_1 are nonzero, then $\{T^{k-1}f_1\}_{k=1}^n$ is a cyclic frame for $\mathbb{C}^d$.

(iv) Under the assumptions in (iii), if at least one of the ω_j, $j=1,\dots,d$, is primitive, then $\{T^{k-1}f_1\}_{k=1}^n$ is a minimal cyclic frame.

Proof. First, writing

$$f_1=\begin{pmatrix}d_1\\ d_2\\ \vdots\\ d_d\end{pmatrix}=D\begin{pmatrix}1\\ 1\\ \vdots\\ 1\end{pmatrix},\quad D:=\begin{pmatrix}d_1 & 0 & 0 & \cdot & \cdot & 0\\ 0 & d_2 & 0 & \cdot & \cdot & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & \cdot & \cdot & 0 & d_d\end{pmatrix}, \tag{7.3.5}$$

we have

$$T^{k-1}f_1=DT^{k-1}\begin{pmatrix}1\\ 1\\ \vdots\\ 1\end{pmatrix}. \tag{7.3.6}$$

Thus it is clear that $\{T^{k-1}f_1\}_{k=1}^n$ can only be a frame if D is invertible, which proves (i).

Now, ignoring for the moment the matrix D and looking at the first d vectors in (7.3.6), we have

$$\left\{T^{k-1}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}\right\}_{k=1}^{d}=\left\{\begin{pmatrix}1\\1\\\cdot\\\cdot\\1\end{pmatrix},\begin{pmatrix}\omega_1\\\omega_2\\\cdot\\\cdot\\\omega_d\end{pmatrix},\begin{pmatrix}\omega_1^2\\\omega_2^2\\\cdot\\\cdot\\\omega_d^2\end{pmatrix},\begin{pmatrix}\omega_1^3\\\omega_2^3\\\cdot\\\cdot\\\omega_d^3\end{pmatrix},\cdots,\begin{pmatrix}\omega_1^{d-1}\\\omega_2^{d-1}\\\cdot\\\cdot\\\omega_d^{d-1}\end{pmatrix}\right\}. \tag{7.3.7}$$

The $d\times d$ matrix with the vectors $\{T^{k-1}f_1\}_{k=1}^d$ as columns is a Vandermonde matrix; if $\omega_j=\omega_\ell$ for some $j\neq\ell$, the determinant vanishes, implying that the first d vectors are not a basis for $\mathbb{C}^d$; hence, applying again (7.3.6) also the first d vectors in $\{T^{k-1}f_1\}_{k=1}^n$ can not be a basis. Applying Lemma 7.2.2 this proves (ii).

On the other hand, in the case (iii) the Vandermonde matrix clearly has nonzero determinant; hence the vectors in (7.3.7) are linearly independent, and therefore a basis for $\mathbb{C}^d$. Hence $\{T^{k-1}f_1\}_{k=1}^n$ is a frame - and it is cyclic because $T^n=I$, due to the choice of $\omega_1,\dots,\omega_d$. If at least one of the $\omega_j, j=1,\dots,n$, has order n, clearly $T^\ell\neq I$ for all $\ell<n$; this proves (iv). ∎

Recall that a $d\times d$ matrix T with d distinct eigenvalues, $\omega_1,\omega_2,\dots,\omega_d$, is diagonalizable, i.e., there exists an invertible $d\times d$ matrix U such that

$$T=U\begin{pmatrix}\omega_1&0&0&\cdot&\cdot&0\\0&\omega_2&0&\cdot&\cdot&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&\cdot&\cdot&0&\omega_d\end{pmatrix}U^{-1}. \tag{7.3.8}$$

Based on Lemma 7.3.5 we will now prove the following characterization of cyclic frames.

Theorem 7.3.6. *Fix $d,n\in\mathbb{N}$, and let T denote a $d\times d$ matrix. Then the following hold:*

(i) Assume that T has d distinct eigenvalues $\omega_1,\omega_2,\dots,\omega_d$, each being an nth root of unity. Let $f_1\in\mathbb{C}^d$ denote an arbitrary vector with nonzero entries. Then, using the notation in (7.3.8) and letting $\varphi:=Uf_1$, the sequence

$$\{T^{n-1}\varphi\}_{k=1}^n \tag{7.3.9}$$

is a cyclic frame for $\mathbb{C}^d$.

(ii) Assume that $\{T^{k-1}\varphi\}_{k=1}^n$ is a cyclic frame for some $\varphi\in\mathbb{C}^d$. Then T is diagonalizable and has d distinct eigenvalues $\omega_1,\omega_2,\dots,\omega_d$, each being an nth root of unity.

Furthermore, with the notation in (7.3.8), the vector φ has the form $\varphi = Uf_1$ for a vector f_1 with nonzero coordinates.

Proof. Under the assumptions in (i), direct calculation using (7.3.8) and applying again the notation in (7.3.5),

$$T^{k-1}\varphi = \left[U \begin{pmatrix} \omega_1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \omega_2 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & \omega_d \end{pmatrix} U^{-1} \right]^{k-1} UD \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{7.3.10}$$

$$= U \begin{pmatrix} \omega_1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \omega_2 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & \omega_d \end{pmatrix}^{k-1} D \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{7.3.11}$$

$$= UD \begin{pmatrix} \omega_1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \omega_2 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & \omega_d \end{pmatrix}^{k-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Thus, the sequence $\{T^{n-1}\varphi\}_{k=1}^{n}$ is the image of the frame in Lemma 7.3.5 under the bijective map UD, and hence a frame. Applying (7.3.8) again, it follows immediately that the frame is cyclic.

In order to prove (ii), notice that if T is not diagonalizable, its Jordan decomposition contains blocks of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \cdot & \cdot & 0 \\ 0 & \lambda & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \lambda & 1 \\ 0 & 0 & \cdot & \cdot & 0 & \lambda \end{pmatrix};$$

since such blocks are noncyclic, this excludes that $T^n = I$. Thus T is indeed diagonalizable. The property $T^n = I$ immediately implies that all eigenvalues of T are nth roots of unity; thus, we only have to prove that all eigenvalues have multiplicity one.

Considering again (7.3.8), we have

$$T^{k-1} = U \begin{pmatrix} \omega_1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \omega_2 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & \omega_d \end{pmatrix}^{k-1} U^{-1};$$

assuming now that $\{T^{k-1}\varphi\}_{k=1}^n$ is a frame, it follows that

$$\left\{ \begin{pmatrix} \omega_1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \omega_2 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & \omega_d \end{pmatrix}^{k-1} U^{-1}\varphi \right\}$$

is a frame. By Lemma 7.3.5 (ii)+(i) this immediately implies that all the eigenvalues ω_j are distinct; furthermore, all coordinates of $U^{-1}\varphi$ are nonzero, meaning that $\varphi = UU^{-1}\varphi = Uf_1$, for a vector $f_1 := U^{-1}\varphi$ with nonzero coordinates. ■

We now provide an alternative characterization of cyclic frames. The result is motivated by the infinite dimensional case given in [39, Theorem 2.1], but for technical reasons the proof (which we give in the appendix) is different. First, define the (cyclic) *right shift operator* $R : \mathbb{C}^n \to \mathbb{C}^n$ by

$$Rx = R(x(1), x(2), \dots, x(n)) = (x(n), x(1), \cdots, x(n-1)), \tag{7.3.12}$$

where $x = (x(1), \dots, x(n)) \in \mathbb{C}^n$.

Theorem 7.3.7. *Let $\{f_k\}_{k=1}^n$ be a frame in $\mathbb{C}^d$ with synthesis operator Θ. Then $\{f_k\}_{k=1}^n$ is a cyclic frame if and only if* $\ker(\Theta)$ *is invariant under the right shift operator R.*

The following result presents an alternative method for constructing cyclic frames; we provide the proof in the appendix.

Theorem 7.3.8. *Consider $d, n \in \mathbb{N}$ with $n > d$. Assume that $a := (a(1), \cdots, a(n)) \in \mathbb{C}^n$ has exactly $n - d$ non-zero coordinates. Let $c := \check{a}$ denote the inverse discrete Fourier transform of a, and consider the circulant matrix*

$$\mathcal{C}_n(c) = \begin{pmatrix} c(1) & c(n) & c(n-1) & \cdot & c(2) \\ c(2) & c(1) & \cdot & \cdot & c(3) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c(n) & c(n-1) & \cdot & c(2) & c(1) \end{pmatrix}_{n\times n}. \tag{7.3.13}$$

Define

$$M := \{\mathcal{C}_n(c)z \ : \ z \in \mathbb{C}^n\}.$$

Then, choosing the set $\{v_1, v_2, \cdots, v_d\}$ as a basis for the orthogonal complement $M^\perp$ of M in $\mathbb{C}^n$ and letting $V := [v_1, v_2, \cdots, v_d]$, the columns of the transposed matrix V^t forms a cyclic frame for $\mathbb{C}^d$.

The following application of Theorem 7.3.8 uses an auxiliary result from the appendix, see Lemma 7.4.8.

Example 7.3.9. Suppose $d = 3$, $n = 4$ and $a = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \in \mathbb{C}^4$. Define $\mathcal{C}_4(c)$ to be a circulant matrix by taking $c = \check{a}$ as follows

$$\mathcal{C}_4(c) = \begin{pmatrix} 1/4 & -i/4 & -1/4 & i/4 \\ i/4 & 1/4 & -i/4 & -1/4 \\ -1/4 & i/4 & 1/4 & -i/4 \\ -i/4 & -1/4 & i/4 & 1/4 \end{pmatrix}.$$

Define $M := \text{range}(\mathcal{C}_4(c))$. Using Lemma 7.4.8, only one eigenvalue of $\mathcal{C}$ is non-zero and hence $\dim(M) = 1$. Therefore, we have $V^t = \begin{pmatrix} -i & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}$. Thus columns of V^t forms a cyclic frame $\mathbb{C}^3$, using Theorem 7.3.8. Moreover, if we consider

$$\{T^{k-1}f_1\}_{k=1}^4 = \left\{ \begin{pmatrix} -i \\ 1 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

then

$$T = \begin{pmatrix} 0 & 0 & -i \\ 1 & 0 & 1 \\ 0 & 1 & i \end{pmatrix} \text{ and } f_1 = \begin{pmatrix} -i \\ 1 \\ i \end{pmatrix}.$$

We end this section with a number of observations that are needed in Section 7.4. We first prove a relationship between the frame bounds for a cyclic frame $\{T^{k-1}f_1\}_{k=1}^n$ and the norm of the operators T and T^{-1}.

Lemma 7.3.10. *Let $\{T^{k-1}f_1\}_{k=1}^n$ be a cyclic frame for $\mathbb{C}^d$, with frame bounds A, B. Then*

$$1 \le \|T\| \le \sqrt{\frac{B}{A}} \text{ and } 1 \le \|T^{-1}\| \le \sqrt{\frac{B}{A}}.$$

Proof. The proof of the estimate on $\|T\|$ follows the same steps as the proof for frames of the form $\{T^k\varphi\}_{k=-\infty}^\infty$ in an infinite-dimensional Hilbert space, see [37, Theorem 2.3]. Since $\{(T^{-1})^{k-1}[T^{n-1}f_1]\}_{k=1}^n$ consists of precisely the same vectors as $\{T^{k-1}f_1\}_{k=1}^n$ (just "in the opposite order), the result about the norm of T^{-1} is an immediate consequence. ■

It is well-known that the canonical dual frame of a dynamical frame $\{T^{k-1}f_1\}_{k=1}^n$ is again a dynamical frame; indeed, denoting the frame operator for $\{T^{k-1}f_1\}_{k=1}^n$ by S, we have that

$$\{S^{-1}T^{k-1}f_1\}_{k=1}^n = \{(S^{-1}TS)^{k-1}S^{-1}f_1\}_{k=1}^n. \tag{7.3.14}$$

For a cyclic frame $\{T^{k-1}f_1\}_{k=1}^n$, the operator $S^{-1}TS$ equals the inverse of the adjoint T^* :

Lemma 7.3.11. *Let $\{T^{k-1}f_1\}_{k=1}^n$ be a cyclic frame with frame operator S. Then $S^{-1}TS = (T^*)^{-1}$.*

Proof. Using that $T^n = I$, a direct calculation shows that for any $f \in \mathbb{C}^d$,

$$TSf = T\sum_{k=1}^n \langle f, T^{k-1}f_1\rangle T^{k-1}f_1 = \sum_{k=1}^n \langle (T^*)^{-1}f, T^k f_1\rangle T^k f_1 = S(T^*)^{-1}f. \tag{7.3.15}$$

Hence $TS = S(T^*)^{-1}$, and the result follows. ■

7.4. Tight cyclic frames and erasures

In this section we show that tight cyclic frames have desirable properties in the context of erasure problems. First, recall that a frame $\{f_k\}_{k=1}^n$ is called an *equal-norm* or *uniform frame* if $\|f_k\| = c$ for each $k \in \{1, \dots, n\}$. A frame $\{f_k\}_{k=1}^n$ for $\mathbb{C}^d$ is called *equiangular* or 2*-uniform frame* if it is uniform and $|\langle f_{k_1}, f_{k_2}\rangle| = c$ for all $k_1 \neq k_2$. These frames play an important role in the case of erasures. For example, in [77] it is proved that a Parseval frame $\{f_k\}_{k=1}^n$ is optimal for 1-erasures if and only if it is a uniform frame; further, if a frame $\{f_k\}_{k=1}^n$ is optimal for 1-erasures and it is equiangular, then it is optimal for 2 erasures. We refer to [77] for the exact definition of these concepts.

In order to prove our main results about tight cyclic frames, we need the next result, which is a direct consequence of Lemma 7.3.11.

Lemma 7.4.1. *If $\{T^{k-1}f_1\}_{k=1}^n$ is tight cyclic frame for $\mathbb{C}^d$, then T is unitary.*

We now prove that tight cyclic frames are uniform, and provide a characterization of equiangularity.

Proposition 7.4.2. *Let $\{T^{k-1}f_1\}_{k=1}^n$ be a tight cyclic frame $\mathbb{C}^d$. Then the following hold:*

(i) $\{T^{k-1}f_1\}_{k=1}^n$ is uniform.
(ii) $\{T^{k-1}f_1\}_{k=1}^n$ is equiangular if and only if $|\langle T^{\ell}f_1, f_1\rangle|$ is constant for $\ell \in \{1, 2, \dots, n-1\}$.

Proof. Concerning (i), it follows directly from Lemma 7.4.1 that the operator T is unitary; thus $||T^{k-1}f_1|| = ||f_1||$. Concerning (ii), since the operator T is unitary

$$|\langle T^i f_1, T^j f_1\rangle| = |\langle T^{i-j} f_1, f_1\rangle|, \; i.j = 1, \dots, n;$$

thus the result follows from (i). ∎

Let us phrase a concrete version of the results in Lemma 7.4.1 and Proposition 7.4.2 for the canonical tight frame associated with any cyclic frame:

Corollary 7.4.3. *Given any cyclic frame $\{T^{k-1}f_1\}_{k=1}^n$ with frame operator S, the following holds:*

(i) The sequence $\left\{\left(S^{-1/2}TS^{1/2}\right)^{k-1} S^{-1/2}f_1\right\}_{k=1}^n$ is a cyclic, tight, and equal-norm frame.

(ii) The operator $S^{-1/2}TS^{1/2}$ *is unitary.*

(iii) The sequence $\left\{\left(S^{-1/2}TS^{1/2}\right)^{k-1} S^{-1/2}f_1\right\}_{k=1}^{n}$ *is equiangular if and only if* $|\langle T^{\ell} f_1, S^{-1}f_1\rangle|$ *is constant for* $\ell \in \{1, 2, \ldots, n-1\}$.

Proof. It is well known that if $\{T^{k-1}f_1\}_{k=1}^{n}$ is frame, then $\{S^{-1/2}(T^{k-1}f_1)\}_{k=1}^{n}$ is a tight frame. Moreover, observe that

$$\left\{\left(S^{-1/2}TS^{1/2}\right)^{k-1} S^{-1/2}f_1\right\}_{k=1}^{n} = \left\{S^{-1/2}(T^{k-1}f_1)\right\}_{k=1}^{n}.$$

Therefore, $\left\{\left(S^{-1/2}TS^{1/2}\right)^{k-1} S^{-1/2}f_1\right\}_{k=1}^{n}$ is a tight frame for $\mathbb{C}^d$. Next, we prove that this frame is cyclic. A direct calculation gives

$$\left(S^{-1/2}TS^{1/2}\right)^{n} = S^{-1/2}T^{n}S^{1/2} = S^{-1/2}S^{1/2} = I,$$

where we have used that $T^n = I$. Hence the sequence is indeed a cyclic tight frame for $\mathbb{C}^d$. Now the result follows from Lemma 7.4.1 and Proposition 7.4.2. ∎

Lemma 7.3.4 and Corollary 7.4.3 lead to the following construction of an equiangular cyclic frame $\{T^{k-1}f_1\}_{k=1}^{n}$ for $\mathbb{C}^d$ with $n = d+1$.

Proposition 7.4.4. *Let* $\{f_k\}_{k=1}^{d}$ *be the standard orthnormal basis for* $\mathbb{C}^d$, *and consider the cyclic frame* $\{T^{k-1}f_1\}_{k=1}^{d+1} = \{f_k\}_{k=1}^{d+1}$, *where* $f_{d+1} = -f_1 - f_2 - \cdots - f_d$. *Then, letting* S *denote the frame operator, the sequence*

$$\left\{\left(S^{-1/2}TS^{1/2}\right)^{k-1} S^{-1/2}f_1\right\}_{k=1}^{d+1}$$

is a cyclic, tight, equal-norm and equiangular frame for $\mathbb{C}^d$.

Proof. A direct calculation shows that the frame operator S and its inverse S^{-1} are given by

$$S = \begin{pmatrix} 2 & 1 & 1 & \cdot & \cdot & 1 \\ 1 & 2 & 1 & \cdot & \cdot & 1 \\ 1 & 1 & 2 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & 2 \end{pmatrix}_{d\times d} \quad \text{and } S^{-1} = \begin{pmatrix} \frac{d}{d+1} & -\frac{1}{d+1} & -\frac{1}{d+1} & \cdot & \cdot & -\frac{1}{d+1} \\ -\frac{1}{d+1} & \frac{d}{d+1} & -\frac{1}{d+1} & \cdot & \cdot & -\frac{1}{d+1} \\ -\frac{1}{d+1} & -\frac{1}{d+1} & \frac{d}{d+1} & \cdot & \cdot & -\frac{1}{d+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\frac{1}{d+1} & -\frac{1}{d+1} & \cdot & \cdot & -\frac{1}{d+1} & \frac{d}{d+1} \end{pmatrix}_{d\times d}. \quad (7.4.1)$$

Therefore

$$S^{-1}f_1 = \begin{bmatrix} \frac{d}{d+1} \\ -\frac{1}{d+1} \\ \cdot \\ \cdot \\ -\frac{1}{d+1} \end{bmatrix} = f_1 + \begin{bmatrix} -\frac{1}{d+1} \\ -\frac{1}{d+1} \\ \cdot \\ \cdot \\ -\frac{1}{d+1} \end{bmatrix} = f_1 - \frac{1}{d+1}(f_1 + f_2 + \ldots + f_d).$$

Thus, for $\ell \in \{1, \ldots, d-1\}$, we have

$$|\langle T^\ell f_1, S^{-1}f_1\rangle| = \left|\frac{-1}{d+1}\langle f_{\ell+1}, f_{\ell+1}\rangle\right| = \frac{1}{d+1},$$

and with $\ell = d$,

$$\begin{aligned} |\langle T^d f_1, S^{-1}f_1\rangle| &= \left|-\langle f_1, f_1\rangle + \frac{1}{d+1}(\langle f_1, f_1\rangle + \ldots + \langle f_d, f_d\rangle)\right| \qquad (7.4.2) \\ &= \left|-1 + \frac{d}{d+1}\right| = \frac{1}{d+1}. \end{aligned}$$

Thus, $|\langle T^\ell f_1, S^{-1}f_1\rangle| = \frac{1}{d+1}$ for each $\ell \in \{1, \ldots, d\}$. Now the result follows from Corollary 7.4.3 (i)+(iii). ∎

The following example demonstrates that applying the construction in Proposition 4.4 in the case of $\mathbb{R}^2$ results in a frame which is a rotation of the Mercedes-Benz frame.

Example 7.4.5. Consider the cyclic frame $\{T^{k-1}f_1\}_{k=1}^3 = \{f_k\}_{k=1}^3$ in $\mathbb{R}^2$ with frame operator S, where $\{f_1, f_2\}$ are standard orthonormal basis for $\mathbb{R}^2$, and $f_3 = -f_1 - f_2$. The operators S and S^{-1} are the same as those defined in equation (7.4.1) for $d = 2$. Thus, a direct calculation gives the operator $S^{1/2}$ and its inverse $S^{-1/2}$ as follows:

$$S^{1/2} = \begin{pmatrix} \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \\ \frac{\sqrt{3}+1}{2} & \frac{\sqrt{3}-1}{2} \end{pmatrix} \text{ and } S^{-1/2} = \begin{pmatrix} \frac{1-\sqrt{3}}{2\sqrt{3}} & \frac{1+\sqrt{3}}{2\sqrt{3}} \\ \frac{1+\sqrt{3}}{2\sqrt{3}} & \frac{1-\sqrt{3}}{2\sqrt{3}} \end{pmatrix}.$$

Hence, by Proposition 4.4, the sequence

$$\left\{\left(S^{-1/2}TS^{1/2}\right)^{k-1} S^{-1/2}f_1\right\}_{k=1}^3 = \left\{\begin{pmatrix} \frac{1-\sqrt{3}}{2\sqrt{3}} \\ \frac{1+\sqrt{3}}{2\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1+\sqrt{3}}{2\sqrt{3}} \\ \frac{1-\sqrt{3}}{2\sqrt{3}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}\right\}$$

is a tight equiangular frame for $\mathbb{R}^2$. An elementary calculation reveals that the frame is a rotation with $\frac{-\pi}{12}$ rad of the Mercedes Benz frame in Figure 1.

Example 7.4.6. Recall the cyclic frame constructed in Example 7.3.9

$$\{T^{k-1}f_1\}_{k=1}^4 = \left\{\begin{pmatrix}-i\\1\\i\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\},$$

for $\mathbb{C}^3$, where

$$T = \begin{pmatrix}0 & 0 & -i\\1 & 0 & 1\\0 & 1 & i\end{pmatrix} \text{ and } f_1 = \begin{pmatrix}-i\\1\\i\end{pmatrix}.$$

Let S be the frame operator of $\{T^{k-1}f_1\}_{k=1}^4$. Then

$$S = \begin{pmatrix}2 & -i & -1\\i & 2 & -i\\-1 & i & 2\end{pmatrix} \text{ and } S^{-1} = \begin{pmatrix}\frac{3}{4} & \frac{i}{4} & \frac{1}{4}\\-\frac{i}{4} & \frac{3}{4} & \frac{i}{4}\\\frac{1}{4} & -\frac{i}{4} & \frac{3}{4}\end{pmatrix}.$$

Now $S^{-1}f_1 = \begin{pmatrix}-\frac{i}{4}\\\frac{1}{4}\\\frac{i}{4}\end{pmatrix}$. A simple calculations shows that for each $\ell \in \{1,2,3\}$ we have

$$|\langle T^\ell f_1, S^{-1}f_1\rangle| = \frac{1}{4}.$$

Hence, by Corollary 7.4.3 (iii), the frame $\left\{\left(S^{-1/2}TS^{1/2}\right)^{k-1}S^{-1/2}f_1\right\}_{k=1}^4$ is equiangular. In this case

$$S^{\frac{1}{2}} = \begin{pmatrix}\frac{4}{3} & -\frac{i}{3} & -\frac{1}{3}\\\frac{i}{3} & \frac{4}{3} & -\frac{i}{3}\\-\frac{1}{3} & \frac{i}{3} & \frac{4}{3}\end{pmatrix} \text{ and } S^{-\frac{1}{2}} = \begin{pmatrix}\frac{5}{6} & \frac{i}{6} & \frac{1}{6}\\-\frac{i}{6} & \frac{5}{6} & \frac{i}{6}\\\frac{1}{6} & -\frac{i}{6} & \frac{5}{6}\end{pmatrix}.$$

Therefore,

$$\{(S^{-\frac{1}{2}}TS^{\frac{1}{2}})^{k-1}S^{-\frac{1}{2}}f_1\}_{k=1}^4 = \left\{\begin{pmatrix}-\frac{i}{2}\\\frac{1}{2}\\\frac{i}{2}\end{pmatrix},\begin{pmatrix}\frac{5}{6}\\-\frac{i}{6}\\\frac{1}{6}\end{pmatrix},\begin{pmatrix}\frac{i}{6}\\\frac{5}{6}\\-\frac{i}{6}\end{pmatrix},\begin{pmatrix}\frac{1}{6}\\\frac{i}{6}\\\frac{5}{6}\end{pmatrix}\right\}.$$

Appendix: Proof of Thm. 7.3.7 and Thm. 7.3.8

Proof of Theorem 7.3.7: If $\{f_k\}_{k=1}^n$ is a cyclic frame for $\mathbb{C}^d$, a direct calculation shows that if $x \in \ker(\Theta)$, then also $R(x) \in \ker(\Theta)$. Conversely assume that $R[\ker(\Theta)] \subseteq \ker(\Theta)$. Define the operator $T : \mathbb{C}^d \to \mathbb{C}^d$ by

$$T\left(\sum_{k=1}^n \alpha(k)f_k\right) = \alpha(n)f_1 + \sum_{k=1}^{n-1}\alpha(k)f_{k+1}.$$

First, we show that T is well-defined. Suppose $\sum_{k=1}^{n} \alpha(k) f_k = \sum_{k=1}^{n} \beta(k) f_k$; then, letting $\gamma(k) = \alpha(k) - \beta(k)$ for $1 \le k \le n$, we have that $\gamma \in \ker(\Theta)$ and thus $R(\gamma) \in \ker(\Theta)$; that is,

$$0 = \gamma(n) f_1 + \sum_{k=2}^{n} \gamma(k-1) f_k = (\alpha(n) - \beta(n)) f_1 + \sum_{k=2}^{n-1} (\alpha(k) - \beta(k)) f_{k+1}.$$

It immediately follows that $\alpha(n) f_1 + \sum_{k=1}^{n-1} \alpha(k) f_{k+1} = \beta(n) f_1 + \sum_{k=1}^{n-1} \beta(k) f_{k+1}$, and hence that T is well defined. It is now easy to verify that $T^n f_1 = f_1$, implying that $\{f_k\}_{k=1}^n$ is a cyclic frame. ∎

In order to prove Theorem 7.3.8 we first state a number of auxiliary results about circulant matrices. For these results we consider an arbitrary circulant matrix $\mathcal{C}_n(c)$, i.e., we do not assume that the vector c has the special form in Theorem 7.3.8.

Lemma 7.4.7. *If $\mathcal{C}_n(c)$ is a circulant matrix, then range$(\mathcal{C}_n(c))$ is invariant under the right shift operator R.*

Proof. Let $y \in \text{range}(\mathcal{C}_n(c))$; then, by the definition of a circulant matrix, as defined in (7.3.13), $y = x(1)c + x(2)Rc + \ldots + x(n)R^{n-1}c$ for some $x = (x(1), \ldots, x(n)) \in \mathbb{C}^n$. Therefore

$$Ry = x(1)Rc + x(2)R^2c + \ldots + x(n-1)R^{n-1}c + x(n)c \in \text{range}(\mathcal{C}_n(c)),$$

as claimed. ∎

The following result and its proof can be found in [95, Corollary 3.33].

Lemma 7.4.8. *If $\mathcal{C}_n(c)$ is a circulant matrix, then the set of eigenvalues is given by $\{\widehat{c}(1), \cdots, \widehat{c}(n)\}$, where $\widehat{c}$ denotes the discrete Fourier transform of c .*

Proof of Theorem 7.3.8: By Lemma 7.4.7, M is invariant under right shift operator R. Using Lemma 7.4.8, the set of eigenvalues $\mathcal{C}_n(c)$ is $\{\widehat{c}(1), \cdots, \widehat{c}(n)\}$. Since $\widehat{c}(k) \neq 0$ for $n-d$ values of $k \in \{1, \cdots, n\}$, it follows that $\dim(M) = n - d$. Now, the matrix $V = [v_1, v_2, \cdots, v_d]$, has rank d. Then the transposed matrix V^t is of order $d \times n$ having the property

$$V^t x = 0 \text{ for all } x \in M.$$

This implies that $\ker(V^t) \supset M$. Indeed $\ker(V^t) = M$, follows by observing that $\dim(M) = n - d$, and the dimension of $\ker(V^t)$ is $n - d$ since the rank of V^t is d. Thus, V^t is a matrix of size $d \times n$ with kernel M, which is invariant under the right shift operator. Hence by Theorem 7.3.7, the columns of V^t forms a cyclic frame for $\mathbb{C}^d$. ■

For the sake of convenient application, we note that Theorem 7.3.8 can be phrased as an algorithm, as follows.

Algorithm 1: Construction of cyclic frame

Input:

$d, n \in \mathbb{N}$

A vector $a \in \mathbb{C}^n$ has $n - d$ entry non-zero.

Output: V^t is a matrix of order $d \times n$ having kernel M.

Set $c = \check{a}$

Set $\mathcal{C}_n(c)$ as a circulant matrix having first column c.

Set $M = \text{Range}(\mathcal{C}_n(c))$

Set $M^\perp$ orthogonal complement of M

Set V is a matrix of order $n \times d$ such that columns vector form a basis of $M^\perp$

Set V^t transpose of V

The final output is V^t. Then by Theorem 7.3.8, the columns vector of V^t forms a cyclic frame for $\mathbb{C}^d$.

CHAPTER 8

SUMMARY AND FUTURE DIRECTIONS

Chapter 1 provides an introduction to the research area along with a review of the available literature, including the necessary preliminaries for the subsequent chapters. In **Chapter 2**, we construct GTI systems and present sufficient conditions under which they satisfy the LIC. **Chapter 3** develops a technique to construct Parseval GTI frames and establishes conditions under which they are pairwise orthogonal. **Chapter 4** focuses on applications of pairwise orthogonal frames to orthogonal sampling transforms. In particular, we characterize conditions under which two sampling transforms have orthogonal ranges.

In **Chapter 5**, we characterize when GTI systems form pairwise orthogonal Parseval frames under the technical assumption UCP. As an application of these characterizations, **Chapter 6** introduces a technique to construct Parseval frames as well as (our generalized version of) GTI pairwise orthogonal Parseval frames. **Chapter 7** discusses dynamical frames and cyclic dynamical frames in finite-dimensional Hilbert spaces. Finally, **Chapter 8** contains concluding remarks and outlines potential directions for future work.

It would be of particular interest to characterize pairwise orthogonal Parseval frames with different co-compact subgroups in full generality, as they have direct applications in sampling theory. Another promising direction is to extend these results to the setting of locally compact groups. Since finite fields have significant importance in practical applications, it would also be valuable to study dynamical and cyclic frames over finite fields.

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