CONVOLUTIONAL FRAMES FOR SAMPLING, SIGNAL RECOVERY AND UNCERTAINTY PRINCIPLES

Ph.D. Thesis

By

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CONVOLUTIONAL FRAMES FOR SAMPLING, SIGNAL RECOVERY AND UNCERTAINTY PRINCIPLES

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I hereby certify that the work which is being presented in the thesis entitled **CONVOLUTIONAL FRAMES FOR SAMPLING, SIGNAL RECOVERY AND UNCERTAINTY PRINCIPLES** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DEPARTMENT OF MATHEMATICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from August 2020 to March 2026 under the supervision of Dr. Niraj Kumar Shukla, Professor, Department of Mathematics, Indian Institute of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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To

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Abstract

KEYWORDS: B-spline; Convolutional frame; Derivative sampling; Erasures; Fiberization map; Filter bank; Frame; Fusion frame; Locally compact group; Multi-channel sampling; Multiplication-invariant space; Periodic shift-invariant space; Random sampling; Ramanujan filter bank; Ramanujan subspace; Ramanujan sums; Range function; Signal concentration; Supremum cosine angle; Tight frame; Translation-invariant space; Trigonometric polynomial; Twisted shift-invariant space; Uncertainty principle; Weyl-Zak transform; Zak transform.

Sampling theory addresses the fundamental problem of determining whether a continuous function can be completely reconstructed from a discrete set of its values, commonly referred to as *samples*. The classical Shannon sampling theorem establishes that band-limited functions are entirely determined by their values at integer points and can be reconstructed via *sinc* interpolation. Over the decades, this theory has been generalized to accommodate more realistic and flexible signal models, including nonuniform, derivative, multi-channel, and random sampling, as well as sampling in shift-invariant spaces. These developments have significantly broadened the scope of sampling theory, making it a unifying principle across communication, signal processing, medical imaging, geophysical sensing, machine learning, and quantum signal processing.

In this thesis, we investigate several aspects of sampling theory within the framework of shift-invariant spaces over both abelian and non-abelian groups. We focus on uniform and non-uniform sampling, multi-channel sampling, generalized sampling, and random sampling. The analysis is conducted on various functional settings, including $\ell^2(\mathbb{Z}_N)$, periodic shift-invariant spaces, shift-invariant spaces, and more general locally compact abelian and non-abelian groups, such as the Heisenberg group $\mathbb{H}^n$. A key tool in our study is the theory of frames, which extends the notion of bases to provide stable and robust reconstruction, and convolutional frames associated with multi-channel filter banks, including Ramanujan filter banks. The thesis also employs transform methods, such as the Fourier transform, Zak transform, and Weyl–Zak transform, to bridge spatial and spectral analyses.

We establish a mathematical framework to analyze stability, perfect reconstruction, and uncertainty principles in multi-channel filter bank settings. In particular, we formulate uncertainty principles for convolutional tight frames and demonstrate their implications for signal recovery under erasures and additive noise. For Ramanujan filter banks, we develop a frame-theoretic approach, analyze their stability under channel and frame erasures, and explore their role in enhancing signal reconstruction. Additionally, we study random sampling involving derivatives in periodic shift-invariant spaces and derive probabilistic sampling inequalities that ensure stability with high probability.

Furthermore, we characterize multi-channel stable sampling in shift-invariant systems over locally compact groups using generalized Zak transforms and obtain necessary density conditions for associated sampling sets. In the non-abelian setting, we extend the theory to twisted shift-invariant spaces on the Heisenberg group, providing sampling theorems and stability conditions via the generalized Weyl–Zak transform. These results unify abelian and non-abelian sampling frameworks under a common algebraic and analytic approach.

Finally, this thesis provides a comprehensive perspective on convolutional and multi-channel sampling, offering rigorous mathematical tools for the design of filter banks, signal recovery algorithms, and stable sampling strategies in both finite and infinite-dimensional settings. The work establishes connections between classical and modern sampling theories and extends their applicability to a wide range of structured signal spaces, thereby contributing both theoretical insights and practical methods for stable and robust reconstruction.

LIST OF PUBLICATIONS

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LIST OF NOTATIONS

Symbols

$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{T}$	The unit circle in the complex plane
$\mathcal{I}_m$	The set $\{1, 2, \ldots, m\}$
$\mathcal{J}_d$	The set $\{0, 1, \ldots, d-1\}$
A^c	Complement of a set A
$\#_A$	Cardinality of the set A
$\max A$	Maximum of the set A
$\min A$	Minimum of the set A
$\operatorname{lcm} A$	Least common multiple of elements of A
(a, b)	Greatest common divisor of integers a and b
$a \mid b$	a divides b
$M(d, \mathbb{Z})$	The set of $d \times d$ invertible matrices with integer entries
A^T	Transpose of the matrix A
A^*	Conjugate transpose of the matrix A
$\det(A)$	Determinant of the matrix A
$\operatorname{tr}(A)$	Trace of the matrix A
$\operatorname{rank}(A)$	Rank of the matrix A
$\lambda_{\min}(A)$	Minimum eigenvalue of A
$\lambda_{\max}(A)$	Maximum eigenvalue of A
$\dim(V)$	Dimension of a vector space V
$\phi(q)$	Euler's totient function (number of integers $< q$ coprime to q)

$f^{(p)}$	p-th derivative of f
$\|x\|_p$	ℓ^p-norm of x
$\tilde{x}$	Involution of x defined by $\tilde{x}(n) = \overline{x(-n)}$
$\mathrm{Re}(z)$	Real part of a complex number z
$\mathbb{E}$	Expectation of a random variable
$\mathbb{P}$	Probability of an event

Chapter 1

Introduction

Sampling theory concerns the problem of determining whether a continuous function can be completely recovered from a discrete set of its functional values, commonly referred to as *samples*. The cornerstone of this theory is the classical Shannon sampling theorem [131], which asserts that if a function/signal f contains no frequency components higher than $1/2$ cycles per second, then it is completely determined by its values at integer points and can be reconstructed via the formula

$$f(t) = \sum_{k\in\mathbb{Z}} f(k)\,\frac{\sin \pi(t-k)}{\pi(t-k)}, \quad t \in \mathbb{R}.$$

In other words, the above formula holds for functions band-limited to the interval $[-\pi, \pi]$, that is, for functions whose Fourier transform vanishes outside this interval. Over the decades, Shannon's sampling theorem has been extended in many directions to accommodate more realistic and flexible signal models. Notable generalizations include nonuniform sampling, derivative sampling, random sampling, multi-channel sampling, and sampling in shift-invariant spaces. See, for e.g., [8, 10, 11, 59, 64, 71–73, 77, 79, 113, 143, 144] and references within.

A natural extension of this framework is to reconstruct f not only from its samples, but also from the samples of filtered and decimated versions of the signal f. This approach is referred to as *multi-channel sampling*. This idea traces back to the work of Papoulis [113], which presents a sampling formula for reconstructing a bandlimited function. In the classical Shannon formula, one requires samples at a rate ω per second, the so-called Nyquist rate. In contrast, the Papoulis multi-channel scheme distributes the sampling density equally among N channels, so that each channel contributes ω/N samples per second. Papoulis' result was later extended to general shift-invariant spaces using filter bank techniques (see, e.g., [37, 48, 59, 63, 64, 144]).

These theoretical advances have significantly broadened the scope of sampling theory, extending its applicability far beyond the original band-limited framework. In contemporary research, sampling serves as a unifying mathematical principle connecting diverse domains of science and engineering. From classical signal recovery to modern data-driven paradigms, the concepts of sampling, stability, and reconstruction form the foundation for numerous applied disciplines, including the following:

- *Communication theory:* Sampling provides the theoretical basis for the transmission and recovery of digital signals under channel distortions and data loss, forming the backbone of modern information theory.
- *Signal and image compression:* Transform coding and sparse approximation methods, such as JPEG, MP3, and compressed sensing, rely on sampling-based models for efficient encoding, storage, and recovery of high-dimensional data.
- *Medical imaging:* In modalities such as computed tomography and magnetic resonance imaging, sampling in spatial or frequency domains enables image reconstruction from indirect or incomplete measurements.
- *Astronomical and geophysical sensing:* Observational data in stellar photometry, radar, and seismic imaging are often incomplete or irregularly spaced; sampling theory provides the analytical tools for their stable recovery.
- *Machine learning and data analysis:* Concepts from sampling theory underpin recent advances in data-driven reconstruction, where signals or images are approximated from limited or noisy measurements.
- *Quantum signal processing:* Sampling principles have been extended to quantum systems, enabling discrete representations of quantum states and facilitating stable state reconstruction from finite measurements.

Although Shannon's sampling theory has profoundly influenced signal processing, it faces several fundamental limitations, as emphasized by Unser [143, 144]. In particular, it assumes strict band-limitation which conflicts with finite-duration signals and induces Gibbs oscillations due to the band-limiting process. Moreover, the slow decay of the *sinc* function renders computations in the signal domain inefficient. Additionally, many applications impose specific a priori constraints on the nature of the signals. Consequently, sampling and reconstruction have been extensively studied in alternative functional settings, including spline spaces [9, 136], wavelet spaces [38, 106, 135, 152, 155], and more general shift-invariant spaces [10, 11, 38, 39, 59, 64, 73, 76, 78, 143].

The present thesis investigates several facets of sampling theory, including uniform and non-uniform sampling, multi-channel sampling, generalized sampling, and random sampling. These studies are conducted over different variants of shift-invariant spaces over various locally compact abelian (LCA) groups such as $\mathbb{Z}_N^d$, $\mathbb{T}^d$, $\mathbb{R}^d$, and more general LCA groups. In addition, we extend our analysis to sampling problems on shift-invariant spaces associated with non-abelian structures, including the Heisenberg group and other locally compact groups.

1.1 Motivation and objective

Shift-invariant spaces (SIS) provide a mathematical framework for addressing a wide range of sampling problems due to their rich algebraic and analytic structure. These spaces have been extensively studied in various settings, including Euclidean spaces $\mathbb{R}^n$, tori $\mathbb{T}^n$ (periodic SIS), finite-dimensional Hilbert spaces $\ell^2(\mathbb{Z}_N)$, and more general locally compact groups [10, 11, 23, 24, 26, 28, 32, 39, 42, 78, 90, 155].

An essential concept in the study of sampling theory on shift-invariant spaces is that of a *frame*. A frame extends the notion of a basis by allowing stable, though not necessarily unique, representations of elements in a Hilbert space. While a basis guarantees uniqueness of representation, this condition can be too restrictive in many practical applications. Frames offer a flexible structure that ensures stable and robust reconstruction even in the presence of noise, perturbations, or data loss. Originally introduced by Duffin and Schaeffer [51] in connection with nonharmonic Fourier series, frames have since become central to areas such as signal processing, image analysis, sampling theory, data compression, and filter bank design. For classical references on frame theory, see [36, 42, 82, 83, 154].

Building upon the concept of frames, the study of multi-channel sampling is closely linked to the theory of filter banks, where the perfect reconstruction property is governed by the frame structure of the analysis system. Multi-channel sampling in SIS has been studied extensively using frame and filter bank techniques [59, 64, 65, 143, 144]. Filter banks establish a natural bridge between sampling theory and multiresolution analysis (MRA), enabling scalable representations of signals across multiple resolutions. They play a fundamental role in practical applications such as denoising, signal compression, and feature extraction [27, 29, 42, 83]. Our investigation of multi-channel stable sampling draws on these ideas, aiming to characterize perfect reconstruction through frame-theoretic properties of filter bank outputs.

A related direction explored in this thesis is the connection between uncertainty principles and signal recovery. The classical result of Donoho and Stark [50] establishes a fundamental trade-off between the concentration of a signal and that of its Fourier transform, forming the basis for several modern developments in signal recovery. In this work, we observe that uncertainty principles arise naturally within the framework of filter banks when comparing signal representations with respect to a canonical basis and a tight frame. These principles also relate to signal recovery after erasures, a problem well studied within frame theory. The uncertainty principle for pairs of bases has been extensively investigated and plays a key role in sparse signal representations (see, e.g., [49, 52, 67, 74]).

An interesting example of a filter bank is the Ramanujan filter bank, which employs Ramanujan sums as analysis filters. Introduced by Srinivasa Ramanujan in 1918 [122], these sums have found important applications in modern signal processing due to their ability to represent periodic structures in discrete-time signals [114, 115, 146, 149]. Ramanujan filter banks exhibit distinct algebraic and spectral properties that make them well suited for periodic signal analysis [1, 139, 140, 148]. Motivated by these developments, this thesis investigates the perfect reconstruction capabilities and establishes frame-theoretic formulations for Ramanujan filter banks.

Most existing results on multi-channel sampling rely on the abelian structure of the underlying group. In the abelian setting, the Fourier transform provides a duality between shift-invariant spaces and the corresponding L^2-space, which simplifies the analysis (see, e.g., [59, 64, 65, 95]). In these works, commutativity plays a key role in deriving sampling and reconstruction formulas for shift-invariant spaces generated by continuous Riesz generators. However, many signal models in physics and engineering exhibit inherently non-abelian symmetries, motivating the study of multi-channel stable sampling for shift- and translation-invariant spaces over general locally compact groups, where Fourier duality may no longer apply.

A prototypical example of a non-abelian group is the Heisenberg group, which plays a central role in time-frequency analysis. The twisted convolution defined on the Heisenberg group induces a new form of translation known as the *twisted translation*. In this context, Radha and Saswata [116] introduced *twisted shift-invariant spaces* in $L^2(\mathbb{R}^{2n})$, which incorporate twisted translations, and they characterized the corresponding frame and Riesz sequences for these spaces. Subsequent works [117, 120, 121] extended these results by using the Weyl–Zak transform to fully characterize the system of twisted translates without imposing restrictive conditions on the generators. Building upon this framework, the present work advances the theory of generalized sampling for twisted shift-invariant spaces and establishes conditions for stable multi-channel reconstruction.

1.1.1 Objective of the thesis

The primary objective of this thesis is to develop a mathematical framework based on convolutional frames to address fundamental problems in sampling theory, including multi-channel sampling, signal recovery, and uncertainty principles. The main goals of the thesis are:

1. To formulate uncertainty principles for convolutional tight frames and establish their implications for stable and robust signal recovery.
2. To analyze recovery of signals in the presence of data loss and additive noise, and to demonstrate how uncertainty principles provide quantitative stability guarantees in such settings.
3. To construct a frame-theoretic framework for Ramanujan sums and Ramanujan filter banks, with particular emphasis on their stability under frame and channel erasures, and their role in enhancing the performances of signal recovery algorithms.
4. To investigate random sampling schemes involving derivatives in periodic shift-invariant spaces, and to establish probabilistic frame inequalities ensuring stability with high probability.
5. To characterize multi-channel stable sampling in shift-invariant systems defined over locally compact groups using the generalized Zak transform, and to derive necessary density conditions for the associated sampling sets.
6. To extend the theory of shift-invariant spaces to non-abelian domains, particularly the Heisenberg group $\mathbb{H}^n$, by analyzing twisted shift-invariant spaces, and to develop corresponding sampling theorems via the generalized Weyl–Zak transform.

1.2 Preliminaries

This section establishes the mathematical foundations and notational conventions required throughout the thesis. We begin with a brief review of the fundamental concepts of Hilbert spaces and frame theory, followed by an overview of convolutional systems and

SIS, which form the analytical setting for most of the results developed later. The section also introduces key transform tools such as the Fourier transform, Zak transform, and Weyl–Zak transforms, which serve as bridges between the spatial and spectral domains in both abelian and non-abelian settings. Finally, we recall certain probabilistic inequalities and matrix concentration results that will be employed in the analysis of random sampling schemes.

1.2.0.1 Frames and Riesz Bases

Let $\mathcal{H} \neq \{0\}$ be a separable Hilbert space. A sequence $\{f_k : k \in I\} \subset \mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k\in I} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \qquad \forall f \in \mathcal{H}. \tag{1.1}$$

The constants A and B are called the *frame bounds*. If only the right-hand inequality in (1.1) holds, the sequence is called a *Bessel sequence*. A frame is said to be *tight* if $A = B$, and a *Parseval frame* if $A = B = 1$. When $\{f_k : k \in I\}$ is a frame for its closed linear span, i.e., $\overline{\operatorname{span}\{f_k : k \in I\}}$, it is referred to as a *frame sequence*.

Associated with any frame $\{f_k : k \in I\}$ are two bounded linear operators:

- the *analysis operator* $T : \mathcal{H} \to \ell^2(I)$ defined by

$$Tf = \{\langle f, f_k\rangle\}_{k\in I},$$

- the *synthesis operator* $T^* : \ell^2(I) \to \mathcal{H}$ defined by

$$T^*c = \sum_{k\in I} c_k f_k, \qquad \forall\, c = \{c_k\}_{k\in I} \in \ell^2(I).$$

The composition $S := T^*T$ defines the *frame operator*, given by

$$Sf = \sum_{k\in I} \langle f, f_k\rangle f_k, \qquad f \in \mathcal{H}.$$

The operator S is bounded, self-adjoint, positive, and invertible. The sequence $\{S^{-1}f_k : k \in I\}$ then forms the *canonical dual frame* of $\{f_k : k \in I\}$, with frame operator S^{-1} and bounds B^{-1}, A^{-1}. Two frames $\{f_k\}$ and $\{g_k\}$ for $\mathcal{H}$ are said to be *dual frames* if

$$f = \sum_{k\in I} \langle f, g_k\rangle f_k, \qquad \forall f \in \mathcal{H}.$$

A sequence $\{f_k : k \in I\} \subset \mathcal{H}$ is called a *Riesz basis* for $\mathcal{H}$ if it is the image of an orthonormal basis under a bounded invertible operator $U : \mathcal{H} \to \mathcal{H}$, i.e.,

$$f_k = Ue_k, \qquad \forall k \in I,$$

where $\{e_k : k \in I\}$ is an orthonormal basis for $\mathcal{H}$. Equivalently, $\{f_k : k \in I\}$ is a Riesz basis if it is complete in $\mathcal{H}$ and there exist constants $A, B > 0$ such that for all finite scalar sequences $\{c_k\}_{k\in I}$,

$$A\sum_{k\in I} |c_k|^2 \leq \left\| \sum_{k\in I} c_k f_k \right\|^2 \leq B\sum_{k\in I} |c_k|^2.$$

If $\{f_k : k \in I\}$ is a Riesz basis for $\overline{\operatorname{span}\{f_k : k \in I\}}$, then it is called a *Riesz sequence*. For more details on frames and Riesz bases, see [42, 82].

1.2.1 Filter banks and convolutional frames

Digital filter banks for finite-length signals have been extensively studied by various authors and have proven to be highly useful in digital signal processing and in wavelet theory [20, 27, 81, 145]. Smith and Eddins [132] pioneered their application in image coding and, among other key contributions, proposed replacing linear filtering operations with cyclic (or circular) convolution.

We begin with introducing the group of N-th roots of unity, denoted by

$$\mathbb{Z}(N) := \left\{ e^{2\pi i k/N} : k = 0, 1, \ldots, N-1 \right\}.$$

This set forms a finite cyclic subgroup of the circle group $\mathbb{T} := [0, 1) \subset \mathbb{C}$ under multiplication. The mapping $k \mapsto e^{2\pi i k/N}$ gives a correspondence between the two abelian groups $\mathbb{Z}(N)$ and $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z} = \{0, 1, \ldots, N-1\}$, where $\mathbb{Z}_N$ is the group of integers modulo N. From now on, we write $\mathbb{Z}_N$ but think of either the group of integers modulo N or the group of Nth roots of unity.

Let $\ell^2(\mathbb{Z}_N)$ denote the Hilbert space of N-periodic discrete-time signals,

$$\ell^2(\mathbb{Z}_N) := \left\{ f : \mathbb{Z}_N \to \mathbb{C} \;\middle|\; \|f\|_2^2 = \sum_{n=0}^{N-1} |f(n)|^2 \right\},$$

equipped with the standard inner product

$$\langle f, g \rangle = \sum_{n=0}^{N-1} f(n)\overline{g(n)}, \qquad f, g \in \ell^2(\mathbb{Z}_N).$$

The space $\ell^2(\mathbb{Z}_N)$ serves as a natural model for finite-length or periodic discrete signals, where all operations are understood modulo N. The circular shift (or translation) operator $L_m : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)$ is defined by

$$(L_m f)(n) = f(n-m), \qquad m, n \in \mathbb{Z}_N,$$

where the shift operator is interpreted modulo N.

A filter bank is a collection of filters used to analyze a signal through multiple channels. Each filter extracts specific features of the signal by convolution. For a signal $f \in \ell^2(\mathbb{Z}_N)$ and filters $\{h_k\}_{k \in \mathcal{I}_K}$, the output of the k-th filter, followed by downsampling $\downarrow p$ by a factor p, is given by

$$y_k(n) = \downarrow p\,(f * h_k)(n) = (f * h_k)(pn), \qquad n \in \mathcal{J}_{N/p},$$

where "$*$" denotes circular convolution,

$$(f * h_k)(n) = \sum_{m=0}^{N-1} f(m)\, h_k(n-m), \qquad n \in \mathbb{Z}_N,$$

and the downsampling operator $\downarrow p$ keeps every p-th sample of a sequence, i.e.,

$$(\downarrow p\, y)(n) = y(pn), \qquad n \in \mathcal{J}_{N/p}.$$

The analysis phase of a K-channel filter bank is depicted in Fig. 1.1. In this way, the filter bank defines a *convolutional system* that collects all shifted versions of the filters (possibly downsampled). Mathematically, this system can be written as

$$\mathfrak{F}_{p,K} := \bigcup_{k=1}^{K} \{L_{pm} h_k : m \in \mathbb{Z}_{N/p}\}.$$

The system $\mathfrak{F}_{p,K}$ is said to form a *convolutional frame* for $\ell^2(\mathbb{Z}_N)$ if there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \le \sum_{k=1}^{K} \sum_{m \in \mathbb{Z}_{N/p}} |\langle f, L_{pm} h_k \rangle|^2 \le B\|f\|_2^2, \qquad \forall f \in \ell^2(\mathbb{Z}_N). \tag{1.2}$$

When $A = B$, the system forms a *convolutional tight frame*. The equality case in (1.2) corresponds to the *perfect reconstruction* property of the associated K-channel filter bank.

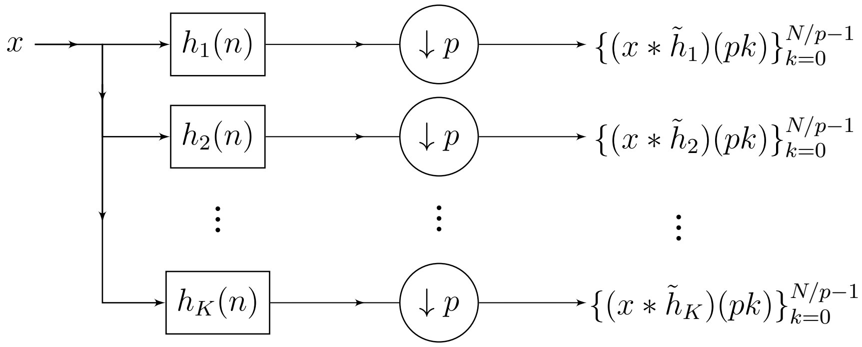

Figure 1.1: Analysis phase of a K-channel filter bank with decimation p.

1.2.2 Shift-invariant and related function spaces

A closed subspace $V(\phi) \subset L^2(\mathbb{R}^d)$ is said to be *shift-invariant* if it satisfies

$$f \in V(\phi) \implies f(\cdot - k) \in V(\phi), \quad \forall k \in \mathbb{Z}^d.$$

If there exists a generator $\phi \in L^2(\mathbb{R}^d)$ such that

$$V(\phi) = \overline{\operatorname{span}}\{\phi(\cdot - k) : k \in \mathbb{Z}^d\},$$

then $V(\phi)$ is called a *principal shift-invariant space.* Multi-generator spaces of the form

$$V(\Phi) = \overline{\operatorname{span}}\{\phi_i(\cdot - k) : k \in \mathbb{Z}^d,\, i = 1, \dots, r\}$$

naturally generalize this notion and frequently appear in the analysis of multi-channel sampling and filter-bank constructions.

Bownik [23] provided a comprehensive characterization of shift-invariant spaces on $\mathbb{R}^n$ through the framework of range functions and established precise conditions under which a collection of translates forms a frame or a Riesz sequence. Building on this foundation, subsequent works extended the theory of shift-invariant spaces to locally compact abelian groups [28, 32] and further to non-abelian groups [26, 88].

The subsequent subsections describe several variants of shift-invariant structures that are central to this thesis, namely, Ramanujan subspaces, periodic shift-invariant spaces, translation-invariant spaces and twisted shift-invariant spaces. These spaces collectively provide a unified platform to study perfect reconstruction property of filter banks, uncertainty principles, multi-channel sampling/generalized sampling, non-uniform/random sampling, and frame theory, in both commutative and non-commutative settings.

1.2.2.1 Ramanujan Subspaces

Ramanujan subspaces naturally arise in the study of integer-periodic signals and possess a rich algebraic and circulant structure governed by Ramanujan sums. In 1918, the Indian mathematician Srinivasa Ramanujan introduced the trigonometric sum [122]

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{2\pi i k n/q}, \tag{1.3}$$

where (k, q) denotes the greatest common divisor of k and q. These sums exhibit several important properties which allow them to serve as building blocks for constructing finite-dimensional subspaces of periodic sequences.

The $\phi(q)$ successive circular shifts $c_q(n),\ c_q(n-1),\ \ldots,\ c_q(n-\phi(q)+1)$ of the q-th Ramanujan sum are linearly independent and span a $\phi(q)$-dimensional subspace [147], known as the *Ramanujan subspace*

$$\mathcal{S}_q = \operatorname{span}\{L_k c_q : 0 \le k \le \phi(q)-1\}. \tag{1.4}$$

The elements of $\mathcal{S}_q$ can be viewed either as length-q vectors or as q-periodic sequences. Moreover, $\mathcal{S}_q$ is *circularly shift-invariant*, i.e., if $f \in \mathcal{S}_q$, then $L_m f \in \mathcal{S}_q$ for all $m \in \mathbb{Z}$, where the shift operator L_m is interpreted modulo q.

For instance, when $q = 10$ and $\phi(10) = 4$, the four shifted sequences

$$\begin{aligned} c_{10}(n) &= \{4, 1, -1, 1, -1, -4, -1, 1, -1, 1\}, \\ c_{10}(n-1) &= \{1, 4, 1, -1, 1, -1, -4, -1, 1, -1\}, \\ c_{10}(n-2) &= \{-1, 1, 4, 1, -1, 1, -1, -4, -1, 1\}, \\ c_{10}(n-3) &= \{1, -1, 1, 4, 1, -1, 1, -1, -4, -1\}, \end{aligned} \tag{1.5}$$

constitute a basis for $\mathcal{S}_{10}$.

Equivalently, $\mathcal{S}_q$ can be viewed as the column space of the $q \times q$ circulant matrix

$$\mathbf{B}_q = \begin{bmatrix} c_q(0) & c_q(q-1) & \cdots & c_q(1) \\ c_q(1) & c_q(0) & \cdots & c_q(2) \\ \vdots & \vdots & \ddots & \vdots \\ c_q(q-1) & c_q(q-2) & \cdots & c_q(0) \end{bmatrix},$$

which has rank $\phi(q)$. Any set of $\phi(q)$ consecutive columns of $\mathbf{B}_q$ forms a basis for $\mathcal{S}_q$. Furthermore, for $q_i \neq q_j$, the subspaces $\mathcal{S}_{q_i}$ and $\mathcal{S}_{q_j}$ are mutually orthogonal.

This orthogonality leads to a natural decomposition of any N-length signal $x \in \ell^2(\mathbb{Z}_N)$ as

$$x = \sum_{q_i|N} \underbrace{\sum_{\ell=0}^{\phi(q_i)-1} \beta_{i,\ell}\, c_{q_i}(\cdot - \ell)}_{x_{q_i}},$$

where the Ramanujan subspaces $\mathcal{S}_{q_i}$, for $q_i \mid N$, are treated as subspaces of $\ell^2(\mathbb{Z}_N)$, and $x_{q_i} \in \mathcal{S}_{q_i}$. Hence, the collection $\{\mathcal{S}_{q_i} : q_i \mid N\}$ provides an orthogonal decomposition of $\ell^2(\mathbb{Z}_N)$ into Ramanujan subspaces. For more details on Ramanujan sums and Ramanujan subspaces, we refer the reader to [146, 147].

This thesis exploits this orthogonal decomposition to develop a frame-theoretic framework based on Ramanujan sums and Ramanujan subspaces. The results are applied to fundamental problems in sampling theory, including period identification, perfect reconstruction property of Ramanujan filter banks, robustness under erasures, and uncertainty principles associated to Ramanujan filter banks for stable signal recovery.

1.2.2.2 Periodic Shift-Invariant Spaces

Periodic shift-invariant (PSI) spaces extend the concept of SIS to the periodic setting, forming a natural framework for discrete and finite-dimensional sampling theory. A closed subspace $V_N \subset L^2(\mathbb{T})$ is said to be *periodic shift-invariant* if

$$f\left(\cdot - \tfrac{k}{N}\right) \in V_N, \quad \forall f \in V_N,\ k \in \mathbb{Z}_N.$$

Here $L^2(\mathbb{T})$ denotes the space of square-integrable 1-periodic functions satisfying

$$\int_0^1 |f(\theta)|^2\, d\theta < \infty.$$

PSI spaces unify a broad class of function systems, including trigonometric polynomials, periodic splines, and periodic wavelets. A prototypical example is the trigonometric polynomial space

$$\mathcal{P}_R = \left\{ f(\theta) = \sum_{n=-R}^{R} c_n e^{2\pi i n\theta} : c_n \in \mathbb{C} \right\},$$

which can be viewed as a PSI space generated by the exponential basis functions $\{e^{2\pi i n \cdot}\}_{|n| \leq R}$. These spaces serve as the periodic analog of bandlimited function spaces and form the backbone of modern discrete-time sampling and reconstruction theories.

The thesis investigates non-uniform sampling involving derivatives in PSI subspaces of $L^2(\mathbb{T})$. In the periodic setting, sampling with derivatives in shift-invariant spaces has been studied in [130]. For the non-periodic case, non-uniform sampling involving derivatives has been explored for bandlimited functions in [68, 71, 124, 125], and more recently in the context of shift-invariant spaces in [2, 77, 79]. PSI spaces provide a unifying framework for a wide range of sampling problems, encompassing trigonometric polynomials, periodic splines, and subspaces generated by periodic wavelets.

1.2.2.3 Translation-Invariant Spaces on Locally Compact Groups

Translation-invariant spaces generalize shift-invariant spaces to the setting of locally compact groups. Let G be a second-countable locally compact group and Γ a closed abelian subgroup of G. A closed subspace $V \subset L^2(G)$ is said to be Γ-*translation-invariant* (Γ-*TI*) if $L_\xi f \in V, \quad \forall f \in V,\ \xi \in \Gamma$, where the left translation operator L_η is defined by

$$(L_\eta f)(\gamma) = f(\eta^{-1}\gamma), \quad \gamma \in G.$$

Early studies on the conditions under which shift-invariant systems form a frame or a Riesz basis were conducted by De Boor et al. [21] and Bownik [23]. Their approach relies on the fiberization map and range functions, the latter of which have a long history in the classification of invariant subspaces, dating back to Helson [84] and Srinivasan [133]. This framework was subsequently extended to locally compact abelian (LCA) groups by Kamyabi Gol and Raisi Tousi [94], Cabrelli and Paternostro [32], and Bownik and Ross [28].

All of these works assume that G is abelian and that the quotient G/Γ is compact. More recent developments by Iverson [88], and Bownik and Iverson [26] relax these assumptions by replacing the fiberization approach with an appropriate Zak-type transform. In particular, they allow G to be any (possibly nonabelian) second-countable locally compact group, requiring only that Γ be closed and abelian. This broader framework includes several fundamental examples that were previously inaccessible, such as $\mathbb{Z}^m \subset \mathbb{R}^n$, corresponding to shifts in $L^2(\mathbb{R}^n)$ by a lattice of less-than-full rank, and $\mathbb{R}^m \subset \mathbb{R}^n$, and it also accommodates more sophisticated instances, including the non-normal copy of $\mathbb{R}$ in the $ax + b$ group.

Building on the tools developed by Bownik and Iverson, the present work investigates multi-channel stable sampling techniques and associated density conditions for

Γ-TI subspaces of $L^2(G)$, where G is not necessarily abelian. This approach enables the characterization of sampling sets that guarantee stable reconstruction, extending classical results from abelian to nonabelian settings and providing access to a rich variety of new examples and applications.

1.2.2.4 Twisted Shift-Invariant Spaces

A simple and natural example of a non-abelian group is the famous Heisenberg group $\mathbb{H}^n$. It is a nilpotent Lie group whose underlying manifold is $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the group operation defined by $(x, y, t)(u, v, s) = (x + u, y + v, t + s + \frac{1}{2}(u \cdot y - v \cdot x))$ and the Haar measure is the Lebesgue measure $dx\, dy\, dt$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. By using the Schrödinger representation π_λ, $\lambda \in \mathbb{R}^*$, given by

$$\pi_\lambda(x, y, t)\phi(\xi) = e^{2\pi i\lambda t}e^{2\pi i\lambda(x\xi+\frac{1}{2}xy)}\phi(\xi + y),\ \phi \in L^2(\mathbb{R}),$$

we define the group Fourier transform of $f \in L^1(\mathbb{H}^n)$ as

$$\widehat{f}(\lambda) = \int_{\mathbb{H}} f(x, y, t)\pi_\lambda(x, y, t)\, dxdydt,\ \text{where } \lambda \in \mathbb{R}^*,$$

which is a bounded operator on $L^2(\mathbb{R}^n)$. It can be extended from $L^1 \cap L^2(\mathbb{H}^n)$ to $L^2(\mathbb{H}^n)$ by using the density argument. Let $f \in L^2(\mathbb{H}^n)$, then we define a function $f^\lambda \in L^1(\mathbb{R}^{2n})$ by $f^\lambda(x, y) = \int_{\mathbb{R}} f(x, y, t)e^{2\pi i\lambda t}\, dt$, which is the inverse Fourier transform of f in the t variable. An essential approach to study the problems on the Heisenberg group is to take the partial Fourier transform in the t variable and reduce the problem to the case of $\mathbb{R}^{2n}$. In fact, for $f, g \in L^1(\mathbb{H}^n)$, the group convolution of f and g is defined by

$$f * g(x, y, t) = \int_{\mathbb{H}^n} f((x, y, t)(u, v, s)^{-1})g(u, v, s)\, dudvds.$$

It is easy to show that $(f * g)^\lambda(x, y) = \int_{\mathbb{R}^{2n}} f^\lambda(x - u, y - v)g^\lambda(u, v)e^{\pi i\lambda(uy-vx)}\, dudv$. This led us to define a new convolution (non-standard convolution) operation on $L^1(\mathbb{R}^{2n})$ in the following way. Let $F, G \in L^1(\mathbb{R}^{2n})$, then we define

$$F *_\lambda G(x, y) = \int_{\mathbb{R}^{2n}} e^{\pi i\lambda(uy-vx)}F(x - u, y - v)G(u, v)\, dudv.$$

These are usually referred to as the λ-twisted convolution. When $\lambda = 1$, it is called the twisted convolution. With respect to the twisted convolution, $L^1(\mathbb{R}^{2n})$ turns out to be a non-commutative Banach algebra. For a detailed study of analysis on the Heisenberg group we refer to [142]. One can notice that the twisted convolution involves the non-standard translation (λ-twisted translation) on $\mathbb{R}^{2n}$.

A closed subspace V of $L^2(\mathbb{R}^{2n})$ is said to be a twisted shift-invariant space if $f \in V$, then $T^t_{(k,l)} f \in V$ for any $k, \ell \in \mathbb{Z}^n$. Here $T^t_{(k,l)}$ denotes the twisted translation given by $T^t_{(k,l)} f(x,y) = e^{\pi i(x \cdot l - y \cdot k)} f(x-k, y-l)$ for $x, y \in \mathbb{R}^n,\ k, l \in \mathbb{Z}^n$. In [116], Radha and Saswata introduced twisted shift-invariant spaces in $L^2(\mathbb{R}^{2n})$ and studied conditions under which a system of twisted translates forms a frame sequence or a Riesz sequence, characterizing them using a certain "Condition C". Later, in [117], these results were extended to shift-invariant spaces on the Heisenberg group. In [119], Radha et al. established a sampling theorem on a subspace of a twisted shift-invariant space. Subsequently, in [121], Radha and Rabeetha introduced the Weyl-Zak transform and provided a characterization of the system of twisted translates $\{T^t_{(k,l)} \varphi : k, l \in \mathbb{Z}^n\}$ for $\varphi \in L^2(\mathbb{R}^{2n})$ to form a frame sequence or a Riesz sequence, eliminating the need for "Condition C" on the generator φ. Recently, in [120], Radha et al. considered generalized twisted translates and derived a necessary and sufficient condition for such a system to form a Parseval frame.

Building on these works, the present thesis aims to extend the framework of generalized sampling to twisted shift-invariant spaces, investigating stable reconstruction using generalized samples and frame-theoretic properties in this non-commutative setting.

1.3 Organization of the thesis

In **Chapter 2**, we develop a Donoho–Stark type uncertainty principle [50] formulated for convolutional tight frames arising from the analysis phase of finite-dimensional filter banks. This principle provides a trade-off between the support of a signal and the support of its frame coefficients, thereby linking the principle to stable signal recovery under erasures. Drawing inspiration from the restriction estimates established by Iosevich and Mayeli [87], the chapter further refines the uncertainty bound through a restriction framework for filter banks. Several applications are discussed, including the recovery of signals from incomplete or noisy frame coefficients and the analysis of Dirac combs with concentrated support.

Chapter 3 explores the perfect reconstruction property of filter banks based on Ramanujan sums and their applications in signal recovery. Originally introduced by Srinivasa

Ramanujan [122], Ramanujan sums serve as powerful tools for extracting periodic components from signals and form the foundation of Ramanujan filter banks. We investigate the perfect reconstruction property of these filter banks and analyze their robustness against erasures for discrete-time signals in a finite-dimensional space $\ell^2(\mathbb{Z}_N)$ (i.e., $\mathbb{C}^N$). The study is further extended to non-uniform Ramanujan filter banks, showcasing their ability to address the limitations of uniform ones. Utilizing the uncertainty principle developed in Chapter 2, we present an uncertainty principle associated with Ramanujan filter banks. This principle establishes representation inequalities in terms of Euler's totient function $\phi(n)$ that provide sufficient conditions for the perfect recovery of signals in scenarios where signal information is lost during transmission or corrupted by noise. Finally, we illustrate that utilizing the signal's periodicity information through Ramanujan filter banks significantly improves the efficiency of signal recovery optimization algorithms, resulting in enhanced signal-to-noise ratio (SNR) gains and more precise reconstruction.

In **Chapter 4**, we study the problem of reconstructing periodic signals from samples of the signal and its finitely many derivatives on a randomly distributed periodic sampling set. Motivated by the work of Antezana, Carbajal, and Romero [14] on probabilistic sampling in shift-invariant spaces on $\mathbb{R}^d$, we establish sampling inequalities for periodic shift-invariant (PSI) spaces in $L^2(\mathbb{T})$ that incorporate derivative information. We show that the sampling inequalities involving derivatives for PSI spaces hold with high probability provided the sampling density is sufficiently large. Our analysis utilizes fundamental tools on the circle group including the Zak transform and the fiberization map to derive sufficient conditions for stable reconstruction. Finally, we present applications of our work to trigonometric polynomials, periodic multi-band signals, wavelet subspaces, and PSI spaces generated by Haar-type scaling functions.

Chapter 5 develops a unified framework for multi-channel stable sampling in shift- and translation-invariant (SI/TI) spaces over locally compact groups, not necessarily abelian. Existing studies on multi-channel sampling primarily rely on the abelian structure of the underlying group. To extend the theory to general locally compact groups, we first establish the framework for multiplication-invariant (MI) spaces in the vector-valued space $L^2(X, \mathcal{H})$, where X is a finite measure space and $\mathcal{H}$ is a separable Hilbert space. The approach is inspired by Iverson [88], who investigated frame properties of TI systems over locally compact groups via the generalized Zak transform. Using the tools developed by Bownik and Iverson [26], we characterize stable sampling in MI spaces and derive corresponding density conditions. The stability of the underlying sampling set is characterized in both global and local (pointwise) forms of the generalized sampling formula employing range functions as a key tool. Finally, employing the generalized Zak transform, these results are transferred to TI spaces over locally compact groups.

In **Chapter 6**, we establish equivalent conditions for the injectivity of the sampling operator associated with a TI system defined on a union of TI spaces. The study is motivated by the work of [107], which investigates sampling on a unions of finitely generated shift-invariant spaces. Following the setting of MI spaces developed in Chapter 5, we first address the problem for unions of MI spaces. It is shown that the supremum cosine angle between constituent MI spaces provides sufficient conditions for the injectivity of the associated sampling operator, through its relation with the closedness of their sum. These results, valid almost everywhere for the corresponding fiber spaces, are subsequently extended to TI spaces on locally compact groups via the generalized Zak transform under the action of a closed abelian subgroup.

In **Chapter 7**, we extend the results of Chapter 5 on multi-channel stable sampling for TI spaces to the setting of generalized twisted shift-invariant (GTSI) spaces on the Heisenberg group. Using the generalized Weyl–Zak transform [121] as an analytical tool, we establish necessary and sufficient conditions for a system of generalized twisted translates to form a Riesz sequence or a frame for its closed linear span. As a consequence, any function in a GTSI space can be reconstructed from a countable collection of generalized samples obtained through twisted translates of suitably chosen filters. As a special case, selecting an appropriate filter yields a Shannon-type sampling theorem within this non-commutative framework. Finally, we illustrate the theory with the GTSI space generated by a twisted B-spline.

Finally, **Chapter 8** presents the final conclusions and outlook.

Chapter 2

Uncertainty principle for convolutional tight frames and signal recovery

This chapter develops a Donoho–Stark type uncertainty principle [50] for convolutional tight frames arising from the analysis phase of finite-dimensional filter banks. This principle demonstrates a trade-off between the signal support and the support of its frame coefficients, establishing a connection with signal recovery under erasures. Inspired by the restriction framework developed by Iosevich and Mayeli [87], we establish a restriction estimate for filter banks and employ it to refine our uncertainty bounds. Applications to signal recovery from missing or noisy frame coefficients are presented, including recovery conditions for Dirac combs with concentrated support.

2.1 Filter banks over multi-dimensional framework

This section studies the filter banks in a multi-dimensional cyclic framework, specifically on the group

$$\mathbb{Z}_N^d = \mathbb{Z}_N \times \mathbb{Z}_N \times \cdots \times \mathbb{Z}_N \quad (d \text{ times}),$$

where $\mathbb{Z}_N$ denotes the cyclic group of order N, with a primary focus on signal recovery applications. The framework of filter banks over the finite abelian group $\mathbb{Z}_N^d$ has been explored in both theoretical and applied settings. For instance, Frazier [56] investigates filter banks over two-dimensional cyclic groups such as $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$, with applications to perfect reconstruction (see p. 194, Chapter 3, Exercise 3.1.14). Beyond this, several works including [15, 61, 62, 91] have studied filter banks over discrete abelian groups from a practical standpoint, with $\mathbb{Z}_N^d$ arising as a natural special case. These studies validate

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The results presented in this chapter are based on our published work [93]:

Kalra, S. and Shukla, N. K. 2025. Uncertainty principle for convolutional tight frames and its applications in signal recovery. *Sampl. Theory Signal Process. Data Anal.* **23** (2), Paper No. 21, 45.

the mathematical and practical relevance of the filter bank formulation over $\mathbb{Z}_N^d$ adopted in this chapter.

Let $\ell^2(\mathbb{Z}_N^d)$ refer to the N^d-dimensional Hilbert space of functions defined on the cyclic group $\mathbb{Z}_N^d$. This space is equipped with the inner product $\langle x, y\rangle = \sum_{n\in\mathbb{Z}_N^d} x(n)\overline{y(n)}$, where $x, y : \mathbb{Z}_N^d \to \mathbb{C}$ are functions represented as $x = (x(n))_{n\in\mathbb{Z}_N^d}$ and $y = (y(n))_{n\in\mathbb{Z}_N^d}$, $n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}_N^d$, and $\overline{y(n)}$ is the complex conjugate of $y(n)$. For signal transmission, we utilize a K-channel filter bank as depicted in Fig. 2.1. The i-th input channel is based on the analysis filter $h_i \in \ell^2(\mathbb{Z}_N^d)$, for $1 \le i \le K$. For a signal $x \in \ell^2(\mathbb{Z}_N^d)$, its restriction to a subgroup $\mathfrak{B}\mathbb{Z}_N^d \subset \mathbb{Z}_N^d$ for some $d \times d$ matrix $\mathfrak{B}$ is given by

$$(\downarrow_{\mathfrak{B}\mathbb{Z}_N^d} x)(m) = x(m), \quad m \in \mathfrak{B}\mathbb{Z}_N^d.$$

The output from the i-th channel after its restriction to a subgroup $\mathfrak{B}\mathbb{Z}_N^d$ is $\{(x * \tilde{h}_i)(m) : (m, i) \in \mathfrak{B}\mathbb{Z}_N^d \times \{1, 2, \ldots, K\}\}$, where the *discrete circular convolution* $*$ for $x, y : \mathbb{Z}_N^d \to \mathbb{C}$ is given by:

$$(x * y)(k) = \sum_{n\in\mathbb{Z}_N^d} x(n)y(k-n),\ k \in \mathbb{Z}_N^d,$$

and for $1 \le i \le K$, $\tilde{h}_i$ denotes the *involution* of h_i, defined as $\tilde{h}_i(n) = \overline{h_i(-n)}$, $n \in \mathbb{Z}_N^d$.

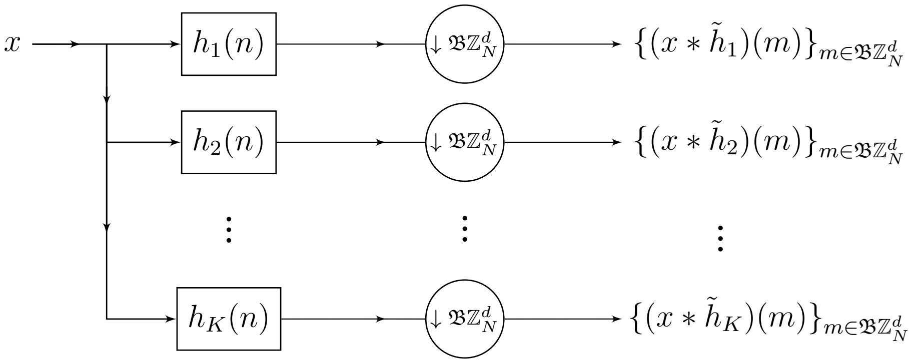


Figure 2.1: Analysis phase of a K-channel multi-dimensional filter bank

To extend the concept of integer downsampling in the setting of $\mathbb{Z}_N^d$, it is required that $\det(\mathfrak{B})$ divides N^d. For this purpose, we choose a $d \times d$ invertible matrix $\mathfrak{B}$ with entries in $\mathbb{Z}$ such that $\mathcal{P} := N\mathfrak{B}^{-1}$ also has entries in $\mathbb{Z}$. This implies that $\det(\mathfrak{B})$ divides N^d since $\det(\mathcal{P})$ is an integer and

$$\det(\mathcal{P}) = \det(N\mathfrak{B}^{-1}) = N^d \det(\mathfrak{B}^{-1}) = \frac{N^d}{\det(\mathfrak{B})}.$$

With this choice of $\mathfrak{B}$, the output from the i-th channel simplifies to:

$$(x * \tilde{h}_i)(m) = \langle x, L_m h_i \rangle, \quad m \in \mathfrak{B}\mathbb{Z}_N^d,\ 1 \leq i \leq K, \tag{2.1}$$

where, for $m \in \mathbb{Z}_N^d$, the *circular shift/translation operator* L_m on $\ell^2(\mathbb{Z}_N^d)$ is defined by $L_m x(n) = x(n - m)$ for all $n \in \mathbb{Z}_N^d$ and $x \in \ell^2(\mathbb{Z}_N^d)$. In view of the earlier discussion in the introduction, the filter bank determined by the analysis filters $h_1, h_2, \ldots, h_K$ admits perfect reconstruction if and only if the associated collection

$$\mathcal{F}_{\mathfrak{B},K} := \{L_m h_1\}_{m \in \mathfrak{B}\mathbb{Z}_N^d} \cup \{L_m h_2\}_{m \in \mathfrak{B}\mathbb{Z}_N^d} \cup \cdots \cup \{L_m h_K\}_{m \in \mathfrak{B}\mathbb{Z}_N^d}, \tag{2.2}$$

forms a frame for $\ell^2(\mathbb{Z}_N^d)$.

In practical applications, a signal is first encoded by computing its frame coefficients, which are then processed according to the requirements of the application. After processing, the original signal x is reconstructed (decoded) from the processed frame coefficients. However, during this process, some coefficients may be lost, disordered, or affected by noise due to various factors. If certain frame coefficients are erased during transmission, a key question arises: How can the original signal be reconstructed as accurately as possible despite these erasures? Furthermore, is perfect recovery still feasible without significantly increasing the computational cost? The problem of signal recovery in the presence of missing or erased frame coefficients has been extensively investigated in the literature [34, 70, 86, 102].

In this chapter, we investigate the conditions under which a signal $x \in \ell^2(\mathbb{Z}_N^d)$ can be uniquely recovered in scenarios when some of its frame coefficients (samples)

$$\{(x * \tilde{h}_i)(m) : (m, i) \in U\}, \tag{2.3}$$

with respect to the tight frame $\mathcal{F}_{\mathfrak{B},K}$, are lost or unobserved for some set $U \subset \mathfrak{B}\mathbb{Z}_N^d \times \{1, 2, \ldots, K\}$ or if the signal becomes noisy.

2.2 Donoho-Stark uncertainty principle and signal recovery

Donoho and Stark, in their seminal paper [50], established fundamental conditions for the unique recovery of signals when certain frequency components are lost or unobserved. Their approach relied on the classical Fourier uncertainty principle, which states that if a signal $x : \mathbb{Z}_N \to \mathbb{C}$ is supported on $B \subset \mathbb{Z}_N$ and its Fourier transform $\widehat{x}$ is supported on

S, then

$$\#_B \#_S \geq N.$$

Various extensions of the Donoho-Stark principle have been widely explored in the literature, establishing connections to classical mathematical topics (see, [108, 109, 112, 138]).

The Donoho–Stark uncertainty principle plays a fundamental role in signal recovery problems, providing a quantitative link between sparsity in the time and frequency domains. Specifically, if a signal $x : \mathbb{Z}_N \to \mathbb{C}$ is supported on a subset $B \subset \mathbb{Z}_N$ and certain frequency components $\{\widehat{x}(m)\}_{m \in M}$ remain unobserved for some $M \subset \mathbb{Z}_N$, then x can be uniquely recovered from the remaining observed frequencies provided $\#_B \, \#_M < \frac{N}{2}$.

Iosevich and Mayeli [87] refined the classical Donoho–Stark uncertainty principle by employing tools from restriction theory. Their goal was to demonstrate that classical restriction estimates can be effectively applied to derive sharper uncertainty principle. The restriction estimate by Iosevich and Mayeli provides a discrete analogue of the classical Fourier restriction theorem. It states that a subset $U \subset \mathbb{Z}_N^d$ satisfies an (a, b) restriction estimate for $1 \leq a \leq b \leq \infty$, if there exists a uniform constant $C_{a,b}$ (independent of N and U) such that for any signal $x \in \ell^2(\mathbb{Z}_N^d)$,

$$\left(\frac{1}{\#_U} \sum_{m \in U} |\hat{x}(m)|^b \right)^{1/b} \leq C_{a,b} N^{-d} \left(\sum_{n \in \mathbb{Z}_N^d} |x(n)|^a \right)^{1/a}. \tag{2.4}$$

The application of restriction estimate (2.4) resulted in an improved Donoho-Stark uncertainty principle, thereby enabling more flexible signal recovery when some of the Fourier coefficients are missing or unobserved.

2.3 Uncertainty principle for filter banks and its sharpness

The signal recovery problem considered in this chapter, where certain samples (see (2.3)) are missing or the signal is corrupted by noise, is fundamentally linked to an uncertainty principle associated with those filter banks whose collection $\mathcal{F}_{\mathfrak{B},K}$ constitutes a tight frame for $\ell^2(\mathbb{Z}_N^d)$. The resulting uncertainty relation imposes restrictions on the sizes of the support of a signal and the support of its frame coefficients. It naturally arises in the context of signal representations in $\ell^2(\mathbb{Z}_N^d)$ with respect to the canonical basis $\{e_n\}_{n \in \mathbb{Z}_N^d}$ and a tight frame $\mathcal{F}_{\mathfrak{B},K}$ (see Theorem 2.1). Similar uncertainty principles for pairs of bases have been extensively studied in the literature and play a fundamental role in sparse signal representations (see, for instance, [49, 52, 67, 74]).

Throughout this chapter, we assume $\mathfrak{B} \in M(d, \mathbb{Z})$ such that $\mathcal{P} := N\mathfrak{B}^{-1} \in M(d, \mathbb{Z})$. For a fixed positive integer K, the collection $\{h_i : i \in \mathcal{I}_K\} \subset \ell^2(\mathbb{Z}_N^d)$ is chosen so that $\mathcal{F}_{\mathfrak{B},K}$ forms a tight frame for $\ell^2(\mathbb{Z}_N^d)$ with frame bound A. For the single-channel case $K = 1$, we write $\mathcal{F}_{\mathfrak{B}} := \{L_m h\}_{m \in \mathfrak{B}\mathbb{Z}_N^d}$ to denote $\mathcal{F}_{\mathfrak{B},1}$ for some $h \in \ell^2(\mathbb{Z}_N^d)$.

For a non-zero signal $x \in \ell^2(\mathbb{Z}_N^d)$, we have the following representations in terms of the canonical basis $\{e_n\}_{n \in \mathbb{Z}_N^d}$ and the tight frame $\mathcal{F}_{\mathfrak{B},K}$ with frame bound A :

$$x = \sum_{n \in \mathbb{Z}_N^d} \alpha_n e_n = \frac{1}{A} \sum_{i \in \mathcal{I}_K} \sum_{m \in \mathfrak{B}\mathbb{Z}_N^d} (x * \tilde{h}_i)(m) L_m h_i. \tag{2.5}$$

Then for any $x \in \ell^2(\mathbb{Z}_N^d)$, we have

$$\|x\|^2 = \sum_{n \in \mathbb{Z}_N^d} |\alpha_n|^2 = \frac{1}{A} \sum_{i \in \mathcal{I}_K} \sum_{m \in \mathfrak{B}\mathbb{Z}_N^d} |(x * \tilde{h}_i)(m)|^2. \tag{2.6}$$

2.3.1 Uncertainty principle associated with filter banks

The following result establishes uncertainty relations associated with pair of representations of signals in $\ell^2(\mathbb{Z}_N^d)$ with respect to the canonical basis and a tight frame $\mathcal{F}_{\mathfrak{B},K}$. The special case when $\mathcal{F}_{\mathfrak{B},K}$ is an orthonormal basis has been addressed in [80, Theorem 3.1]. The proof follows the general strategy of [80, Theorem 3.1]; however, since we work with a tight frame rather than an orthonormal basis, we include a complete proof for the sake of clarity and completeness.

Theorem 2.1. *Let $x \in \ell^2(\mathbb{Z}_N^d)$ be a signal with support B and let its samples, given by $\{(x * \tilde{h}_i)(m) : (m, i) \in \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K\}$, be supported on $U \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$. Consider,*

$$\beta_o := \max\{|\langle e_n, L_m h_i\rangle| : n \in \mathbb{Z}_N^d, m \in \mathfrak{B}\mathbb{Z}_N^d, i \in \mathcal{I}_K\}.$$

Then, the following statements hold:

While presenting this work at a workshop at RWTH Aachen University, Prof. Stefan Kunis pointed out that relations similar to (2.7) appear in Lemma 2 of [47]. That lemma deals with general matrices under certain norm and entry-wise bounds, and our uncertainty relations can, in fact, be derived as a special case of it. We recently became aware of this connection.

(i) The number β_o is bounded by

$$\sqrt{\frac{A}{K|\det(\mathcal{P})|}} \leq \beta_0 \leq \max\{\|h_i\| : i \in \mathcal{I}_K\}.$$

(ii) The representations of $x \in \ell^2(\mathbb{Z}_N^d)$ in terms of tight frame $\mathcal{F}_{\mathfrak{B},K}$ and the canonical basis $\{e_n\}_{n\in\mathbb{Z}_N^d}$ provide the following relations:

$$\#_B + \#_U \geq \frac{2\sqrt{A}}{\beta_o}, \quad \textit{and} \quad \#_B\#_U \geq \frac{A}{\beta_o^2.} \tag{2.7}$$

Proof. By using (2.5), we have

$$\begin{aligned}
\|x\|^2 &= \left|\left\langle \sum_{n\in\mathbb{Z}_N^d} \alpha_n e_n, \frac{1}{A}\sum_{i\in\mathcal{I}_K}\sum_{m\in\mathfrak{B}\mathbb{Z}_N^d} (x * \tilde{h}_i)(m) L_m h_i \right\rangle\right| \\
&= \frac{1}{A}\left|\sum_{n\in\mathbb{Z}_N^d}\sum_{i\in\mathcal{I}_K}\sum_{m\in\mathfrak{B}\mathbb{Z}_N^d} \alpha_n \overline{(x * \tilde{h}_i)(m)} \langle e_n, L_m h_j\rangle\right| \\
&\leq \frac{1}{A}\sum_{n\in\mathbb{Z}_N^d}\sum_{i\in\mathcal{I}_K}\sum_{m\in\mathfrak{B}\mathbb{Z}_N^d} |a_n||\overline{(x * \tilde{h}_i)(m)}||\langle e_n, L_m h_i\rangle| \\
&\leq \frac{1}{A}\sum_{n\in\mathbb{Z}_N^d} |\alpha_n| \left(\sum_{i\in\mathcal{I}_K}\sum_{m\in\mathfrak{B}\mathbb{Z}_N^d} |\overline{(x * \tilde{h}_i)(m)}|\beta_o\right).
\end{aligned}$$

By the application of Cauchy-Schwarz inequality and (2.6), we have

$$\begin{aligned}
\|x\|^2 &\leq \frac{1}{A}\Bigg(\sum_{\substack{n\in\mathbb{Z}_N^d\\ \alpha_n\neq 0}} |\alpha_n|^2\Bigg)^{1/2} \times \Bigg(\sum_{\substack{n\in\mathbb{Z}_N^d\\ \alpha_n\neq 0}} \Big(\sum_{i\in\mathcal{I}_K}\sum_{m\in\mathfrak{B}\mathbb{Z}_N^d} |(x * \tilde{h}_i)(m)|\beta_o\Big)^2\Bigg)^{1/2} \\
&\leq \frac{\beta_o}{A}\|x\| \times \left(\sqrt{\#_B} \times \sum_{i\in\mathcal{I}_K}\sum_{m\in\mathfrak{B}\mathbb{Z}_N^d} 1.|(x * \tilde{h}_i)(m)|\right) \\
&\leq \frac{\beta_o}{A}\|x\| \times \sqrt{\#_B} \times \Bigg(\sum_{i\in\mathcal{I}_K}\sum_{\substack{m\in\mathfrak{B}\mathbb{Z}_N^d\\ (x*\tilde{h}_i)(m)\neq 0}} 1^2\Bigg)^{1/2} \times \Bigg(\sum_{i\in\mathcal{I}_K}\sum_{\substack{m\in\mathfrak{B}\mathbb{Z}_N^d\\ (x*\tilde{h}_i)(m)\neq 0}} |(x * \tilde{h}_i)(\mathfrak{B}m)|^2\Bigg)^{1/2} \\
&\leq \frac{\beta_o}{A}\|x\| \times \sqrt{\#_B} \times \sqrt{\#_U} \times \sqrt{A}\,\|x\| = \frac{\beta_o}{\sqrt{A}}\sqrt{\#_B\#_U}\,\|x\|^2.
\end{aligned}$$

Therefore, $\sqrt{\#_B\#_U} \geq \frac{\sqrt{A}}{\beta_o}$. By using the AM-GM inequality, we get

$$\#_B + \#_U \geq 2\sqrt{\#_B\#_U} \geq \frac{2\sqrt{A}}{\beta_o}.$$

To prove (i), note that by the Cauchy-Schwarz inequality, we have

$$\big|\langle e_n, L_m h_i\rangle\big| \leq \|e_n\|\|L_m h_i\| = 1.\|h_i\| = \|h_i\| \leq \max\{\|h_i\| : i \in \mathcal{I}_K\},$$

for any $n \in \mathbb{Z}_N^d, m \in \mathfrak{B}\mathbb{Z}_N^d, i \in \mathcal{I}_K$. Therefore, $\beta_o \leq \max\{\|h_i\| : i \in \mathcal{I}_K\}$. For the lower bound on β_o, note that for any $n \in \mathbb{Z}_N^d, m \in \mathfrak{B}\mathbb{Z}_N^d, i \in \mathcal{I}_K$, we have $\left|\langle e_n, L_m h_i\rangle\right| \leq \beta_o$. Squaring both sides and summing over $m \in \mathfrak{B}\mathbb{Z}_N^d$ and $i \in \mathcal{I}_K$, we have

$$\sum_{i\in\mathcal{I}_K}\sum_{m\in\mathfrak{B}\mathbb{Z}_N^d}\left|\langle e_n, L_m h_i\rangle\right|^2 \leq \beta_o^2 K|\det(\mathcal{P})|, \quad \text{for any } n \in \mathbb{Z}_N^d, \tag{2.8}$$

where we have used the fact that $\#_{\mathfrak{B}\mathbb{Z}_N^d} = |\det(\mathcal{P})|$ (see [80, Proposition 2.1]). Since the collection $\mathcal{F}_{\mathfrak{B},K}$ is a tight frame for $\ell^2(\mathbb{Z}_N^d)$, the left side of the above inequality is A and after rearranging, (2.8) finally becomes:

$$\sqrt{\frac{A}{K|\det(\mathcal{P})|}} \leq \beta_o.$$

Consequently, (i) is proved. Hence, the claim follows. □

Remark 2.2. If the collection $\mathcal{F}_{\mathfrak{B},K}$ forms an orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$, then it follows that $N^d = K|\det(\mathcal{P})|$ and $A = 1$. Moreover, for any $i \in \mathcal{I}_K$ and $m \in \mathfrak{B}\mathbb{Z}_N^d$, we have $\|h_i\| = \|L_m h_i\| = 1$. In this scenario, the uncertainty relations given in (2.7) simplify to:

$$\#_B + \#_U \geq \frac{2}{\beta_o}, \quad \text{and} \quad \#_B\#_U \geq \frac{1}{\beta_o^2},$$

where β_o satisfies the inequality

$$\sqrt{\frac{1}{N^d}} \leq \beta_o \leq 1.$$

These relations coincide with the uncertainty relations established in [80, Theorem 3.1].

Remark 2.3. It is interesting to observe that $\beta_o = \max\{|h_i(n)| : n \in \mathbb{Z}_N^d, i \in \mathcal{I}_K\}$. To prove this, we consider $\mathcal{K} \subset \mathbb{Z}_N^d$ to be a $\mathcal{P}\mathbb{Z}_N^d$-*tile* of $\mathbb{Z}_N^d$, (i.e., $\{\mathcal{P}\mathbb{Z}_N^d + k : k \in \mathcal{K}\}$ is a disjoint partition of $\mathbb{Z}_N^d$) such that $\mathfrak{B}n_1 = \mathfrak{B}n_2$ implies $n_1 = n_2$ for any $n_1, n_2 \in \mathcal{K}$. Then, by [80, Proposition 2.4], it follows that $\mathfrak{B}\mathbb{Z}_N^d = \mathfrak{B}(\mathcal{K})$. Also, note that by [80, Proposition 2.7], there exists a set $\{\alpha_0, \alpha_1, \ldots \alpha_{|\det(\mathfrak{B})|-1}\} \subset \mathbb{Z}_N^d$ such that $\mathbb{Z}_N^d = \{k + \alpha_\ell : k \in \mathfrak{B}(\mathcal{K}), \ell \in \mathcal{J}_{|\det(\mathfrak{B})|}\}$ and

$$\{L_m e_{\alpha_\ell} : m \in \mathfrak{B}(\mathcal{K}), \ell \in \mathcal{J}_{|\det(B)|}\} = \{e_n\}_{n\in\mathbb{Z}_N^d}.$$

Then, for any $m, k \in \mathfrak{B}(\mathcal{K})$, $i \in \mathcal{I}_K$, and $\ell \in \mathcal{J}_{|\det(\mathfrak{B})|}$, we have

$$\begin{aligned}|\langle L_m h_i, L_k e_{\alpha_\ell}\rangle| = |\langle h_i, L_{(k-m)} e_{\alpha_\ell}\rangle| &= \left| \sum_{n\in\mathbb{Z}_N^d} h_i(n)\overline{L_{m'} e_{\alpha_\ell}(n)}\right| \\ &= \left| \sum_{n\in\mathbb{Z}_N^d} h_i(n)\overline{e_{\alpha_\ell + m'}(n)}\right| = |h_i(e_{\alpha_\ell} + m')|, \quad \text{for some } m' \in \mathfrak{B}(\mathcal{K}).\end{aligned}$$

Using the above relation and the equality $\mathfrak{B}\mathbb{Z}_N^d = \mathfrak{B}(\mathcal{K})$, we get

$$\begin{aligned}\beta_o &= \max\{|\langle L_m h_i, L_k e_{\alpha_\ell}\rangle| : m, k \in \mathfrak{B}(\mathcal{K}), i \in \mathcal{I}_K, \ell \in \mathcal{J}_{|\det(\mathfrak{B})|}\} \\ &= \max\{|h_i(e_{\alpha_\ell} + m)| : m \in \mathfrak{B}(\mathcal{K}), \ell \in \mathcal{J}_{|\det(\mathfrak{B})|}, i \in \mathcal{I}_K\} \qquad (2.9) \\ &= \max\{|h_i(n)| : n \in \mathbb{Z}_N^d, i \in \mathcal{I}_K\}.\end{aligned}$$

Remark 2.4. The uncertainty relations (2.7) are meaningful only when the right-hand sides of both the inequalities are greater than or equal to 1. Indeed, this holds because

$$\beta_o = |\langle e_n, L_m h_i\rangle| \text{ for some } n \in \mathbb{Z}_N^d, m \in \mathfrak{B}\mathbb{Z}_N^d, i \in \mathcal{I}_K.$$

Then, using the fact that the collection $\mathcal{F}_{\mathfrak{B},K}$ forms a tight frame for $\ell^2(\mathbb{Z}_N^d)$, we have

$$\beta_o^2 = |\langle e_n, L_m h_i\rangle|^2 \leq \sum_{i\in\mathcal{I}_K} \sum_{m\in\mathfrak{B}\mathbb{Z}_N^d} |\langle e_n, L_m h_i\rangle|^2 = A \text{ for any } n \in \mathbb{Z}_N^d.$$

Example 2.5. (a). Let $N = 2$, $d = 2$ and $\mathfrak{B} = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$. Consider,

$$h_1 = (h_1(0,0)^T, h_1(0,1)^T, h_1(1,0)^T, h_1(1,1)^T)^T = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right)^T \text{ and } h_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}, 0\right)^T.$$

Then, the collection $\mathcal{F}_{\mathfrak{B},2} = \{h_1, L_{(0,1)^T} h_1, h_2, L_{(0,1)^T} h_2\}$ forms a tight frame for $\ell^2(\mathbb{Z}_2^2)$ where $\mathfrak{B}\mathbb{Z}_2^2 = \{(0,0)^T, (0,1)^T\}$. Indeed, by using $L_{(0,1)^T} h_1 = \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)^T$ and $L_{(0,1)^T} h_2 = \left(0, \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)^T$ for any $x \in \mathbb{Z}_2^2$, we have

$$\begin{aligned}&\sum_{i=1}^{2} \sum_{m\in\mathfrak{B}\mathbb{Z}_2^2} |\langle x, L_m h_i\rangle|^2 = |\langle x, h_1\rangle|^2 + |\langle x, L_{(0,1)^t} h_1\rangle|^2 + |\langle x, h_2\rangle|^2 + |\langle x, L_{(0,1)^t} h_2\rangle|^2 \\ &= \frac{1}{2}|x(0,0) + x(1,0)|^2 + \frac{1}{2}|x(0,1) + x(1,1)|^2 + \frac{1}{2}|x(0,0) - x(1,0)|^2 + \frac{1}{2}|x(0,1) - x(1,1)|^2 \\ &= |x(0,0)|^2 + |x(0,1)|^2 + |x(1,0)|^2 + |x(1,1)|^2 = \|x\|^2,\end{aligned}$$

where we have used the identity $|a+b|^2 + |a-b|^2 = 2(|a|^2 + |b|^2)$, $a, b \in \mathbb{C}$. This proves that $\mathcal{F}_{\mathfrak{B},2}$ is a tight frame for $\ell^2(\mathbb{Z}_2^2)$ with bound $A = 1$. Also, it can be easily verified that $\beta_o = 1/\sqrt{2}$ in this case.

(b). Let $N = 2^n$ for some $n \in \mathbb{N}$ and $d = 1$. Also, let $\mathfrak{B} = 2$ and consider $h_1 = (1, 0, 0, \ldots, 0)$, $h_2 = (-1/2, \sqrt{3}/2, 0, \ldots, 0)$, and $h_3 = (-1/2, -\sqrt{3}/2, 0, \ldots, 0)$. Then, it has been proved in [53] that the collection $\{L_k h_i : k \in 2\mathbb{Z}_{2^n}, i \in \mathcal{I}_3\}$ forms a tight frame for $\ell^2(\mathbb{Z}_{2^n})$ with frame bound $A = 3/2$. Moreover, it can be easily verified that $\beta_o = \sqrt{3}/2$ in this case.

Example 2.6. Let $N = 6$, $d = 1$ and $\mathfrak{B} = 1$. For $n \in \mathbb{Z}_6$, consider $h_1(n) = c_1(n)$, $h_2(n) = c_2(n)$, $h_3(n) = c_3(n)$, $h_4(n) = c_6(n)$, where for a positive integer q, c_q denote the q-th Ramanujan sum as defined in (1.3). Then, the collection $\{L_k h_i : k \in \mathbb{Z}_6, i \in \mathcal{I}_4\}$ forms a tight frame for $\ell^2(\mathbb{Z}_6)$ with frame bound $A = 36$ (see Theorem 3.6), and $\beta_o = \phi(6) = 2$.

2.3.2 Sharpness of uncertainty principle

This subsection aims to demonstrate that the uncertainty relations in (2.7) can be further refined using a restriction estimate for filter banks. Motivated by the work of Iosevich and Mayeli [87], we adopt a similar restriction-theoretic approach to improve the uncertainty relations provided in (2.7).

Specifically, we establish a restriction estimate for filter banks, analogous to (2.4), wherein the Fourier coefficients are replaced by the frame coefficients associated with the tight frame $\mathcal{F}_{\mathfrak{B},K}$. The proposed restriction estimate is then employed to sharpen the associated uncertainty principle. This formulation is relevant in signal recovery scenarios that involve redundant representations, particularly those based on filter bank frames.

Definition 2.7. (Restriction estimate for filter banks). Let $U \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$ and $1 \leq a \leq b \leq \infty$. Then U satisfies an (a, b) *restriction estimate* if there exists a uniform constant $C_{a,b}$ (depending only on a and b) such that for any signal $x \in \ell^2(\mathbb{Z}_N^d)$, we have

$$\left(\frac{1}{\#_U} \sum_{(m,i) \in U} |(x * \tilde{h}_i)(m)|^b\right)^{1/b} \leq C_{a,b} \beta_o \left(\sum_{n \in \mathbb{Z}_N^d} |x(n)|^a\right)^{1/a}. \tag{2.10}$$

Remark 2.8. Note that for any $U \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$, and $a = 1$, the restriction estimate holds for the pair $(1, b)$. Indeed, for any $x \in \ell^2(\mathbb{Z}_N^d)$, we have

$$\begin{aligned}\left(\frac{1}{\#_U} \sum_{(m,i)\in U} |(x * \tilde{h}_i)(m)|^b\right)^{1/b} &\leq \left(\frac{1}{\#_U} \sum_{(m,i)\in U} \Big|\sum_{n\in\mathbb{Z}_N^d} x(n)\tilde{h}_i(m-n)\Big|^b\right)^{1/b} \\ &\leq \max\{|h_i(m+n)| : m \in \mathfrak{B}\mathbb{Z}_N^d, n \in \mathbb{Z}_N^d, i \in \mathcal{I}_K\}\left(\frac{1}{\#_U} \sum_{(m,i)\in U} \Big|\sum_{n\in\mathbb{Z}_N^d} x(n)\Big|^b\right)^{1/b} \\ &\leq \beta_o \sum_{n\in\mathbb{Z}_N^d} |x(n)|.\end{aligned}$$

Thus, the restriction estimate (2.10) holds for U for the pair $(1, b)$ and constant $C_{1,b} = 1$.

A natural and fundamental question is to identify conditions under which a non-trivial restriction estimate holds for a given subset $U \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$. We have the following result in this direction:

Theorem 2.9. *Let the collection $\mathcal{F}_{\mathfrak{B}} = \{L_m h\}_{\mathfrak{B}\mathbb{Z}_N^d}$ form a tight frame for $\ell^2(\mathbb{Z}_N^d)$ for some $h \in \ell^2(\mathbb{Z}_N^d)$ with frame bound A and let $U \subset \mathfrak{B}\mathbb{Z}_N^d$ be such that*

$$\frac{A}{\#_U} \leq \beta_o^2. \tag{2.11}$$

Then the set U satisfies a restriction estimate for the pair (a, b) and constant $C_{a,b} = 1$, where $a = \frac{2k}{2k-1}$, $k \in \mathbb{N}$ and $b = 2$. That is, for every $x \in \ell^2(\mathbb{Z}_N^d)$,

$$\left(\frac{1}{\#_U} \sum_{m\in U} |(x * \tilde{h})(m)|^2\right)^{1/2} \leq \beta_o \left(\sum_{n\in\mathbb{Z}_N^d} |x(n)|^a\right)^{1/a}. \tag{2.12}$$

Proof. Define $y(m) := (x * \tilde{h})(m) \cdot \mathbf{1}_U(m)$ for $x \in \ell^2(\mathbb{Z}_N^d)$ and $m \in \mathbb{Z}_N^d$. Then we write

$$\begin{aligned}\sum_{m\in U} |(x * \tilde{h})(m)|^2 &= \sum_{m\in\mathbb{Z}_N^d} (x * \tilde{h})(m) \cdot \overline{y(m)} \\ &= \sum_{m\in\mathbb{Z}_N^d} \langle x, L_m h\rangle \cdot \overline{y(m)} = \sum_{m\in\mathbb{Z}_N^d} \sum_{n\in\mathbb{Z}_N^d} x(n)\overline{h(n-m)} \cdot \overline{y(m)} \\ &= \sum_{n\in\mathbb{Z}_N^d} x(n) \sum_{m\in\mathbb{Z}_N^d} \overline{y(m)} \cdot \tilde{h}(m-n) = \sum_{n\in\mathbb{Z}_N^d} x(n) \cdot \overline{\langle y, L_n \tilde{h}\rangle}.\end{aligned} \tag{2.13}$$

Note that, if a' is conjugate exponent of $a = \frac{2k}{2k-1}$ for some $k \in \mathbb{N}$, then

$$a' = \frac{a}{a-1} = \frac{\frac{2k}{2k-1}}{\frac{2k}{2k-1} - 1} = 2k.$$

Applying Hölder's inequality to the right hand side of (2.13), we get

$$\sum_{m\in U} |(x*\tilde{h})(m)|^2 \leq \left(\sum_{n\in\mathbb{Z}_N^d} |x(n)|^a\right)^{1/a} \left(\sum_{n\in\mathbb{Z}_N^d} |\langle y, L_n\tilde{h}\rangle|^{2k}\right)^{1/2k}. \tag{2.14}$$

Since for any $n \in \mathbb{Z}_N^d$ and $z \in \ell^2(\mathbb{Z}_N^d)$, we have $\langle \tilde{z}, L_{-n}h\rangle = \overline{\langle z, L_n\tilde{h}\rangle}$. As its consequence, the collection $\mathcal{F}_{\mathfrak{B}}$ being tight frame with frame bound A implies that the collection $\{L_n\tilde{h}\}_{n\in\mathfrak{B}\mathbb{Z}_N^d}$ is also a tight frame with bound A. Using this fact and the norm inequality

$$\sum_{n\in\mathbb{Z}_N^d} |\langle y, L_n\tilde{h}\rangle|^{2k} \leq \left(\sum_{n\in\mathbb{Z}_N^d} |\langle y, L_n\tilde{h}\rangle|^2\right)^k,$$

we get

$$\sum_{n\in\mathbb{Z}_N^d} |\langle y, L_n\tilde{h}\rangle|^{2k} \leq \left(\sum_{n\in\mathbb{Z}_N^d} |\langle y, L_n\tilde{h}\rangle|^2\right)^k = \left(A\|y\|^2\right)^k = A^k\|y\|^{2k}. \tag{2.15}$$

Substituting (2.15) into (2.14), we obtain

$$\begin{aligned}\sum_{m\in U} |(x*\tilde{h})(m)|^2 &\leq \left(\sum_{n\in\mathbb{Z}_N^d} |x(n)|^a\right)^{1/a} \cdot A^{1/2}\|y\| \\ &= \left(\sum_{n\in\mathbb{Z}_N^d} |x(n)|^a\right)^{1/a} \cdot A^{1/2}\left(\sum_{m\in U} |(x*\tilde{h})(m)|^2\right)^{1/2}.\end{aligned}$$

Finally, dividing by $\#_U^{1/2}$ and using the assumption (2.11), we get

$$\left(\frac{1}{\#_U}\sum_{m\in U} |(x*\tilde{h})(m)|^2\right)^{1/2} \leq \frac{A^{1/2}}{\#_U^{1/2}}\left(\sum_{n\in\mathbb{Z}_N^d} |x(n)|^a\right)^{1/a} \leq \beta_o\left(\sum_{n\in\mathbb{Z}_N^d} |x(n)|^a\right)^{1/a},$$

which is precisely (2.12). □

Remark 2.10. Let $\mathfrak{B} = I_2$ (the identity matrix of order 2), $d = 2$, and let N be any positive integer. Consider the Dirac delta generator

$$h(n) = \begin{cases} \sqrt{C}, & \text{if } n = (0,0), \\ 0, & \text{otherwise,} \end{cases}$$

for some positive constant C. In this case, any non-empty subset $U \subset \mathbb{Z}_N^2$ satisfies assumption (2.11). Consequently, by Theorem 2.9, the restriction estimate holds for all non-empty subsets $U \subset \mathbb{Z}_N^2$ corresponding to the pair $(\frac{2k}{2k-1}, 2)$ for any $k \in \mathbb{N}$.

Indeed, for any $x \in \ell^2(\mathbb{Z}_N^2)$, we compute:

$$\sum_{m\in\mathbb{Z}_N^2} |\langle x, L_m h\rangle|^2 = \sum_{m\in\mathbb{Z}_N^2} |\langle x, \sqrt{C}\delta_m\rangle|^2 = \sum_{m\in\mathbb{Z}_N^2} |\sqrt{C}x(m)|^2 = C \sum_{m\in\mathbb{Z}_N^2} |x(m)|^2 = C\|x\|^2.$$

This confirms that the system $\mathcal{F}_{\mathfrak{B}} = \{L_m h\}_{m\in\mathbb{Z}_N^2}$ forms a tight frame for $\ell^2(\mathbb{Z}_N^2)$ with frame bound C. Moreover, in this case, the quantity β_o simplifies to

$$\beta_o = \max\{|h(n)| : n \in \mathbb{Z}_N^2\} = \sqrt{C},$$

so assumption (2.11) is trivially satisfied for any non-empty subset $U \subset \mathbb{Z}_N^2$.

In particular, classical examples of U such as the unit circle $U = \{x \in \mathbb{Z}_N^2 : x_1^2 + x_2^2 = 1\}$, or the parabola $U = \{x \in \mathbb{Z}_N^2 : x_2 = x_1^2\}$, are valid candidates for the application of Theorem 2.9 in this setting.

Following the approach of Iosevich and Mayeli [87], who apply Bourgain's Λ_p theorem [22, Theorem 1] to establish restriction estimate in the Fourier domain, we invoke Bourgain's theorem to derive the $(\frac{b}{b-1}, 2)$ restriction estimate (2.8) for $b > 2$. Bourgain originally proved the following result for a general locally compact abelian group G; however, we present it here specifically for $G = \mathbb{Z}_N^d$.

Theorem 2.11. *[22] Let $\{\gamma_i : i \in \mathcal{I}_s\} \subset \ell^2(\mathbb{Z}_N^d)$ be a sequence of s mutually orthogonal functions such that $\max\{|\gamma_i(x)| : x \in \mathbb{Z}_N^d\} \leq 1$ for each $i \in \mathcal{I}_s$. Then, for a given $2 < b < \infty$, there exists $V \subset \mathcal{I}_s$ with $\#_V > s^{2/b}$, such that for all scalar sequences (a_i)*

$$\left\| \sum_{i\in V} a_i\gamma_i \right\|_{\ell^b(\mathbb{Z}_N^d)} \leq C(b) \left(\sum_{i\in V} |a_i|^2 \right)^{\frac{1}{2}}.$$

The constant $C(b)$ depends only on b and the estimate above holds for a generic set of size $s^{2/b}$.

Remark 2.12. In Theorem 2.11, a generic set V of size $s^{2/b}$ refers to a randomly chosen subset of $\mathcal{I}_s$, where each element is included with probability $p = s^{2/b-1}$. Theorem 2.11 then holds for such a set V with probability close to 1. For a detailed explanation of generic sets, we refer to [22] and the references therein.

Based on Theorem 2.11, we obtain the following result, which compares the b-norm of $x \in \ell^2(\mathbb{Z}_N^d)$ for $b > 2$ with the 2-norm of its samples $\{(x * \tilde{h}_i)(k) : (k,i) \in \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K\}$ supported on a generic set of size greater than $(\mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K)^{2/b}$. It uses the requirement that the collection $\mathcal{F}_{\mathfrak{B},K}$ forms an orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$. One such example is explained in Example 2.5.

Theorem 2.13. *Let the collection $\mathcal{F}_{\mathfrak{B},K}$ form an orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$. Further, let $x \in \ell^2(\mathbb{Z}_N^d)$ be a signal whose samples $\{(x * \tilde{h}_i)(m) : (m,i) \in \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K\}$ are supported on $U \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$, where U is a generic set of size greater than $(\mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K)^{2/b}$ for some $b > 2$. Then,*

$$\left(\sum_{n \in \mathbb{Z}_N^d} |x(n)|^b\right)^{1/b} \leq C(b)\left(\sum_{(m,i)\in U} |(x * \tilde{h}_i)(m)|^2\right)^{1/2}. \tag{2.16}$$

Proof. Since $\mathcal{F}_{\mathfrak{B},K}$ forms an orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$, it follows that $A = 1$. Using (2.5) and the fact that the samples $\{(x * \tilde{h}_i)(m) : (m,i) \in \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K\}$ are supported on U, we obtain

$$x = \sum_{(m,i)\in U} (x * \tilde{h}_i)(m) L_m h_i. \tag{2.17}$$

Applying Theorem 2.11 to U, together with (2.17), completes the proof. $\square$

The following result uses Theorem 2.13 to provide $(\frac{b}{b-1}, 2)$ restriction estimate for $b > 2$.

Theorem 2.14. *Under the standing assumptions of Theorem 2.13, we have*

$$\left(\frac{1}{\#_U}\sum_{(m,i)\in U} |(x * \tilde{h}_i)(m)|^2\right)^{1/2} \leq \beta_o \delta C(b)\left(\sum_{n\in\mathbb{Z}_N^d} |x(n)|^{\frac{b}{b-1}}\right)^{\frac{b-1}{b}}, \tag{2.18}$$

where δ is a positive real number such that $\beta_o \geq \frac{1}{\delta}$ i.e., (2.18) implies that the $(\frac{b}{b-1}, 2)$ restriction theorem holds with the constant $\delta C(b)$.

Proof. Let $b' = \frac{b}{b-1}$. Using the fact that $\mathcal{F}_{\mathfrak{B},K}$ forms an orthonormal basis for $\ell^2(\mathbb{Z}_N^d)$, applying Hölder's inequality and then using (2.16) we get

$$\begin{aligned}\frac{1}{\#_U}\sum_{(m,i)\in U}|(x*\tilde{h}_i)(m)|^2 &= \frac{1}{\#_U}\sum_{n\in\mathbb{Z}_N^d}|x(n)|^2 = \frac{1}{\#_U}\sum_{n\in\mathbb{Z}_N^d}x(n)\overline{x(n)}\\ &\leq \frac{1}{\#_U}\Big(\sum_{n\in\mathbb{Z}_N^d}|x(n)|^{b'}\Big)^{1/b'}\Big(\sum_{n\in\mathbb{Z}_N^d}|x(n)|^{b}\Big)^{1/b}\\ &\leq \frac{1}{\#_U}\Big(\sum_{n\in\mathbb{Z}_N^d}|x(n)|^{b'}\Big)^{1/b'}C(b)\Big(\sum_{(m,i)\in U}|(x*\tilde{h}_i)(m)|^2\Big)^{1/2}\\ &= \Big(\sum_{n\in\mathbb{Z}_N^d}|x(n)|^{b'}\Big)^{1/b'}C(b)\Big(\frac{1}{\#_U}\sum_{(m,i)\in U}|(x*\tilde{h}_i)(m)|^2\Big)^{1/2}.\end{aligned} \tag{2.19}$$

By Theorem 2.1, we have $\beta_0 \leq \max\{\|h_i\| : i \in \mathcal{I}_K\} = \max\{\|L_k h_i\| : k \in \mathfrak{B}\mathbb{Z}_N^d, i \in \mathcal{I}_K\} = 1$. Choosing $\delta > 0$ such that $\delta\beta_0 \geq 1$ and using this in (2.19), we obtain

$$\Big(\frac{1}{\#_U}\sum_{(m,i)\in U}|(x*\tilde{h}_i)(m)|^2\Big)^{1/2} \leq C(b)\Big(\sum_{n\in\mathbb{Z}_N^d}|x(n)|^{b'}\Big)^{1/b'} \leq \beta_0(\delta C(b))\Big(\sum_{n\in\mathbb{Z}_N^d}|x(n)|^{b'}\Big)^{1/b'}.$$

This completes the proof. ☐

We now provide an improved version of Theorem 2.1 by using the restriction estimate (2.10).

Theorem 2.15. *In addition to the standing assumptions of Theorem 2.1, assume that the restriction estimate (2.10) holds for U for a pair (a,b) with $1 \leq a \leq b < \infty$. Then*

$$\#_B^{1/a}\#_U \geq \frac{A}{\beta_o^2 C_{a,b}}. \tag{2.20}$$

Proof. Using the tight frame property of $\mathcal{F}_{\mathfrak{B},K}$, Hölder's inequality and restriction estimate (2.10) for U for the pair (a,b), we have

$$\begin{aligned}|x(n)| &= \frac{1}{A}\Big|\sum_{(m,i)\in U}(x*\tilde{h}_i)(m)L_m h_i(n)\Big| = \frac{1}{A}\Big|\sum_{(m,i)\in U}(x*\tilde{h}_i)(m)h_i(n-m)\Big|\\ &\leq \frac{\beta_o}{A}\sum_{(m,i)\in U}|(x*\tilde{h}_i)(m)| \leq \frac{\beta_o}{A}\#_U\left(\frac{1}{\#_U}\sum_{(m,i)\in U}|(x*\tilde{h}_i)(m)|^b\right)^{1/b}\\ &\leq \frac{\beta_o^2}{A}\#_U C_{a,b}\Big(\sum_{n\in\mathbb{Z}_N^d}|x(n)|^a\Big)^{1/a} \leq \frac{\beta_o^2}{A}\#_U C_{a,b}\Big(\sum_{n\in B}|x(n)|^a\Big)^{1/a}.\end{aligned}$$

Raising the power a on both sides of the above equation, summing over B and dividing both sides by $\sum_{n\in\mathbb{Z}_N^d} |x(n)|^a$, we get

$$\#_B \#_U^a \geq \frac{A^a}{\beta_o^{2a} C_{a,b}^a},$$

or equivalently,

$$\#_B^{1/a} \#_U \geq \frac{A}{\beta_o^2 C_{a,b}}.$$

Hence the claim follows. $\square$

We have another improved version of Theorem 2.1. For a set $S \subset \mathbb{Z}_N^d$, we define the norm $\|x\|_{\ell^a(S)} := \left(\sum_{n\in S} |x(n)|^a\right)^{1/a}$. We have the following result.

Theorem 2.16. *In addition to the standing assumptions of Theorem 2.1, assume that the restriction estimate (2.10) holds for U for a pair (a,b) with $1 \leq a \leq 2 \leq b < \infty$. Then, we have*

$$\#_B^{\frac{2-a}{a}} \#_U \geq \frac{A}{\beta_o^2 C_{a,b}^2}. \tag{2.21}$$

Proof. Note that if $1 \leq a \leq b$, then by using Hölder's inequality, we have

$$\left(\frac{1}{\#_B}\sum_{n\in B}|x(n)|^a\right)^{1/a} \leq \left(\frac{1}{\#_B}\sum_{n\in B}|x(n)|^b\right)^{1/b}.$$

As its consequence, we have

$$\begin{aligned}\|x\|_{\ell^a(B)} = \left(\sum_{n\in B}|x(n)|^a\right)^{1/a} &= \#_B^{1/a}\left(\frac{1}{\#_B}\sum_{n\in B}|x(n)|^a\right)^{1/a} \\ &\leq \#_B^{1/a}\left(\frac{1}{\#_B}\sum_{n\in B}|x(n)|^b\right)^{1/b} = \#_B^{1/a-1/b}\left(\sum_{n\in B}|x(n)|^b\right)^{1/b} = \#_B^{1/a-1/b}\|x\|_{\ell^b(B)}.\end{aligned} \tag{2.22}$$

Similarly, by using Hölder's inequality for $1 \leq a \leq b$, we have

$$\left(\frac{1}{\#_U}\sum_{(m,i)\in U}|(x*\tilde{h}_i)(m)|^a\right)^{1/a} \leq \left(\frac{1}{\#_U}\sum_{(m,i)\in U}|(x*\tilde{h}_i)(m)|^b\right)^{1/b}.$$

Consequently, for $b \geq 2$, we have

$$\left(\frac{1}{\#_U}\sum_{(m,i)\in M}|(x*\tilde{h}_i)(m)|^2\right)^{1/2} \leq \left(\frac{1}{\#_U}\sum_{(m,i)\in U}|(x*\tilde{h}_i)(m)|^b\right)^{1/b}. \tag{2.23}$$

This implies that if restriction estimate (2.10) holds for U for the pair (a, b) with constant $C_{a,b}$ then it also hold for the pair $(a, 2)$ with the same constant. Moreover, we have

$$\left(\frac{1}{\#_U}\sum_{(m,i)\in U}|(x*\tilde{h}_i)(m)|^2\right)^{1/2} = \#_U^{-1/2}\left(\sum_{(m,i)\in U}|(x*\tilde{h}_i)(m)|^2\right)^{1/2} = \#_U^{-1/2}A^{1/2}\|x\|_{\ell^2(B)}. \tag{2.24}$$

By using (2.24), (2.23), (2.10) and (2.22) in the same order, we obtain

$$\#_U^{-1/2}A^{1/2}\|x\|_{\ell^2(B)} \leq \beta_o C_{a,b}\|x\|_{\ell^a(B)} \leq \beta_o C_{a,b}\#_B^{1/a-1/2}\|x\|_{\ell^2(B)}. \tag{2.25}$$

This gives,

$$\#_B^{1/a-1/2}\#_U^{1/2} \geq \frac{A^{1/2}}{\beta_o C_{a,b}}.$$

Squaring both sides gives the required result. □

Remark 2.17. If $0 < C_{a,b} < 1$, Theorem 2.16 imposes a stricter condition than Theorem 2.15, since the squared term $C_{a,b}^2$ amplifies the right-hand side of (2.21). Furthermore, for $a > 1$, the inequality $\frac{1}{a} > \frac{2-a}{a}$ implies that the product $\#_B^{\frac{2-a}{a}}\#_U$ is smaller than $\#_B^{1/a}\#_U$, leading to a sharper bound in Theorem 2.16 compared to Theorem 2.15.

2.4 Signal recovery

In this section, we demonstrate that the uncertainty relations established in Theorem 2.1 and Theorem 2.15 provide conditions ensuring the perfect and unique reconstruction of $x \in \ell^2(\mathbb{Z}_N^d)$ when some samples of a given signal are lost or when it is affected by noise.

2.4.1 Recovery from missing data and noisy observations

We focus on the following signal recovery problems:

Problem 1: Recovery in case of missing samples

Suppose for a given signal $x \in \ell^2(\mathbb{Z}_N^d)$, a subset $\{(x*\tilde{h}_i)(k) : (k,i) \in U\}$ of its samples are lost for some $U \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$. We aim to establish conditions under which $x \in \ell^2(\mathbb{Z}_N^d)$ can be reconstructed uniquely from the truncated sum:

$$\mathcal{D}_{U^c}x := \frac{1}{A}\sum_{(k,i)\in U^c}(x * \tilde{h}_i)(k)L_k h_i, \tag{2.26}$$

assuming that $\mathcal{F}_{\mathfrak{B},K}$ is a tight frame for $\ell^2(\mathbb{Z}_N^d)$ with frame bound A, where $U^c := \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K \backslash U$. Indeed, we have the following result:

Theorem 2.18. *Let $x \in \ell^2(\mathbb{Z}_N^d)$ be a signal with support $B \subset \mathbb{Z}_N^d$ and assume that a subset $\{(x * \tilde{h}_i)(k) : (k,i) \in U\}$ of its samples are lost for some $U \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$.*

(*i*) *Then x can be uniquely recovered from $\mathcal{D}_{U^c}x$ if*

$$\#_B \#_U < \frac{A}{2\beta_o^2}. \tag{2.27}$$

(*ii*) *Suppose that the restriction estimate* (2.10) *holds for U for a pair $(a,b), 1 \leq a \leq b < \infty$. Then x can be uniquely recovered from $\mathcal{D}_{U^c}x$ if*

$$\#_B^{1/a} \#_U < \frac{A}{2\beta_o^2 C_{a,b}}. \tag{2.28}$$

Problem 2: Recovery in presence of noise

Suppose the signal x is sent to a receiver that captures it accurately except over a set $\mathcal{T} \subset \mathbb{Z}_N^d$, where it is corrupted by noise η. Then, the received signal r at $k \in \mathbb{Z}_N^d$ is given by

$$r(k) = \begin{cases} x(k) + \eta(k), & k \in \mathcal{T}, \\ x(k), & k \in \mathcal{T}^c, \end{cases} \tag{2.29}$$

where $\mathcal{T}^c := \mathbb{Z}_N^d \backslash \mathcal{T}$. Our goal is to reconstruct x from the noisy signal y, assuming that the noise vector η is sparse, meaning the noise vector η is zero outside a given set.

In many practical situations, such as sensor failures, impulsive disturbances, or transmission errors, the noise typically affects only a few locations in the signal. This motivates modeling the noise vector η as sparse. See, for example, [40], where recovery under sparse noise assumptions is systematically analyzed.

The following result serves as a solution to Problem 2.

Theorem 2.19. *Let $x \in \ell^2(\mathbb{Z}_N^d)$ be a signal whose samples $\{(x * \tilde{h}_i)(k) : (k,i) \in \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K\}$ are supported on U for some $U \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$ and assume that the noise vector η is supported on $\mathcal{T} \subset \mathbb{Z}_N^d$.*

(*i*) *Then x can be uniquely recovered from r if*

$$\#_{\mathcal{T}} \#_U < \frac{A}{2\beta_o^2}. \tag{2.30}$$

(*ii*) *Suppose that the restriction estimate* (2.10) *holds for U for a pair $(a,b), 1 \leq a \leq b < \infty$. Then x can be uniquely recovered from r if*

$$\#_{\mathcal{T}}^{1/a} \#_U < \frac{A}{2\beta_o^2 C_{a,b}}. \tag{2.31}$$

Remark 2.20. Donoho and Stark employed a ℓ_2-minimization method to recover sparse signals with missing frequencies using the *frequency-limiting operator*. Specifically, if $x \in \ell^2(\mathbb{Z}_N^d)$ is a signal with support S and the frequencies $\{\hat{x}(n)\}_{n\in\mathbb{Z}_N^d}$ are supported on U, the least squares problem

$$\min\{\|\mathcal{F}_U x - \mathcal{F}_U s'\| : P_{\mathcal{W}} s' = s'\}$$

is solved for each $\mathcal{W} \subset \mathbb{Z}_N^d$ with $\#_{\mathcal{W}} = \#_S$, where for a set $U \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$, the *frequency-limiting operator* $\mathcal{F}_U : \ell^2(\mathbb{Z}_N^d) \to \ell^2(\mathbb{Z}_N^d)$ is a projection map onto the space of all signals with frequency support U, given by

$$(\mathcal{F}_U x)(m) = \sum_{n\in U} \hat{x}(n) e^{2\pi i n.m/N},$$

and for a set $\mathcal{W} \subset \mathbb{Z}_N^d$, the *time-limiting operator* $P_{\mathcal{W}}$ is a projection map onto the space of all signals supported on $\mathcal{W}$, given by

$$(P_{\mathcal{W}} x)(n) = \begin{cases} x(n), & \text{if } n \in \mathcal{W}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.32}$$

Let $s_{\mathcal{W}}$ be the signal that solves the problem for a given $\mathcal{W}$. After calculating $s_{\mathcal{W}}$, we then compute

$$\tilde{s} = \arg \min_{s_{\mathcal{W}}, \#_{\mathcal{W}} = \#_S} \|\mathcal{F}_U x - \mathcal{F}_U s_{\mathcal{W}}\|$$

and conclude that $x = \tilde{s}$. This algorithm is impractical for large N, as the runtime grows exponentially with increasing N, but it ensures that the original signal is recovered exactly.

For the exact recovery, we utilize the ℓ_1-reconstruction algorithm which provides a much better approach than the combinatorial approach, as the latter becomes impractical for large N. For a better explanation, we refer to [50]. For Theorem 2.18(i), we solve

$$\tilde{x} = \arg\min_{x'} \{\|x'\|_1 \ : \ \mathcal{D}_{U^c} x' = \mathcal{D}_{U^c} x\} \tag{2.33}$$

and for Theorem 2.19(i), we solve

$$\tilde{x} = \arg \min_{x' \in \mathcal{S}_U} \|r - x'\|_1, \tag{2.34}$$

while for the exact reconstruction in the cases of Theorem 2.18(ii) and Theorem 2.19(ii), we employ the combinatorial approach similar to that discussed in Remark 2.20.

The following result guarantees that optimization problems (2.33) and (2.34) provides exact recovery. This is a version of discrete Logan's phenomenon for our setting, originally presented in the context of bandlimited functions [50, Lemma 4].

Theorem 2.21. *Let $x \in \ell^2(\mathbb{Z}_N^d)$ be a signal with support $B \subset \mathbb{Z}_N^d$. Then, for $U \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$, the best approximation to x by signals in*

$$\mathcal{S}_U = \left\{ x \in \ell^2(\mathbb{Z}_N^d) | \ (x * \tilde{h}_i)(k) = 0 \text{ for } (k,i) \in U^c \right\} \tag{2.35}$$

is the zero signal, provided

$$\#_B \#_U < \frac{A}{2\beta_o^2}. \tag{2.36}$$

Proof. Note that if $\#_B = 0$, then $x = 0$. Then the result trivially holds as $0 \in \mathcal{S}_U$.

Now assume, $\#_B \neq 0$. By an application of Theorem 2.1, we first prove $x \notin \mathcal{S}_U$. For that, assume that the set of samples $\{(x * \tilde{h}_i)(k) : k \in \mathfrak{B}\mathbb{Z}_N^d, i \in \mathcal{I}_K\}$ of x has at most $\#_\mathcal{L}$ non-zero samples for some $\mathcal{L} \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$ i.e., $(x * \tilde{h}_i)(k) = 0$ for $(k,i) \in \mathcal{L}^c$. By Theorem 2.1 and (2.36), we have

$$\#_\mathcal{L} \#_B \geq \frac{A}{\beta_o^2} > 2\#_U\#_B.$$

This implies, $\#_\mathcal{L} > 2\#_U$. Since the signals in $\mathcal{S}_U$ have at most $\#_U$ non-zero samples, thus $x \notin \mathcal{S}_U$.

To prove that the zero signal is the best approximation to x, consider an arbitrary $y \in \mathcal{S}_U$ with $y \neq 0$. Since $\mathcal{F}_{\mathfrak{B},K}$ is a tight frame for $\ell^2(\mathbb{Z}_N^d)$ with frame bound A, then for $n \in \mathbb{Z}_N^d$, we can write

$$\begin{aligned} |y(n)| &= \frac{1}{A}\left| \sum_{(k,i)\in U} (y * \tilde{h}_i)(k) L_k h_i(n) \right| = \frac{1}{A}\left| \sum_{(k,i)\in U} \sum_{\ell \in \mathbb{Z}_N^d} y(\ell)\overline{L_k h_i(\ell)} L_k h_i(n) \right| \\ &\leq \frac{1}{A} \sum_{\ell\in\mathbb{Z}_N^d} |y(\ell)| \left(\sum_{(k,i)\in U} \left| \overline{h_i(\ell-k)} h_i(n-k) \right| \right) \leq \frac{1}{A} \sum_{\ell\in\mathbb{Z}_N^d} |y(\ell)| \, \beta_o^2 \#_U, \end{aligned}$$

where the last inequality is due to (2.9). This implies, $\max_{n\in\mathbb{Z}_N^d} |y(n)| \leq \frac{\beta_o^2}{A}\#_U \|y\|_1$. Then, by using (2.36), we obtain

$$\|P_B y\|_1 = \sum_{n\in B} |y(n)| \leq \max_{n\in\mathbb{Z}_N^d} |y(n)| \#_B \leq \frac{\beta_o^2}{A} \#_U \#_B \|y\|_1 < \frac{1}{2}\|y\|_1.$$

where for $B \subset \mathbb{Z}_N^d$, the projection operator P_B is defined in (2.32). From this, we get $\|(I-P_B)y\|_1 \geq \|y\|_1 - \|P_B y\|_1 > \frac{1}{2}\|y\|_1 > \|P_B y\|_1$. Consequently, we get

$$\begin{aligned}\|x-y\|_1 &= \|P_B(x-y)\|_1 + \|(I-P_B)y\|_1 \\ &\geq \|P_B x\|_1 - \|P_B y\|_1 + \|(I-P_B)y\|_1 > \|P_B x\|_1 = \|x\|_1.\end{aligned}$$

Thus, any nonzero signal in $\mathcal{S}_U$ does not approximate x well. Hence the claim follows. ☐

We are now ready to provide the proof of our results, Theorem 2.18 and Theorem 2.19. The idea of their proofs is adopted from [50].

Proof of Theorem 2.18

We first prove (i). Suppose, for the sake of contradiction, that there exists $y \in \ell^2(\mathbb{Z}_N^d)$ satisfying (2.27) and $\mathcal{D}_{U^c}x = \mathcal{D}_{U^c}y$. Define $h := y - x$. Then, $\mathcal{D}_{U^c}h = 0$. This implies that $h \in \mathcal{S}_U$. Moreover, $\#_{\text{supp}(h)} \leq \#_{\text{supp}(y)} + \#_{\text{supp}(x)} = 2\#_B$, i.e, h has at most $2\#_B$ nonzero elements. Then by using Theorem 2.1, we obtain $(2\#_B).\#_U \geq \frac{A}{\beta_o^2}$, contradicting (2.27). Hence x is the unique signal satisfying (2.27) and generating $\mathcal{D}_{U^c}x$.

To reconstruct x from $D_{U^c}x$, we show that $\tilde{x} = x$ in (2.33). First note that any $u \in \mathcal{S}_U$ can be written as $u = x' - x$, where x' satisfies $\mathcal{D}_{U^c}x' = \mathcal{D}_{U^c}x$. Indeed, setting $x' = u + x$ for some $u \in \mathcal{S}_U$, we obtain $\mathcal{D}_{U^c}x' = \mathcal{D}_{U^c}(x+u) = \mathcal{D}_{U^c}x$. Since it is given that $2\#_B\#_U < \frac{A}{\beta_o^2}$, Theorem 2.21 implies that the best approximation to x by signals in $\mathcal{S}_U$ is the the zero signal. In other words, we have $\min\limits_{u \in \mathcal{S}_U} \|x+u\|_1 = \|x\|_1$ or $\min\{\|x + (x'-x)\|_1 : \mathcal{D}_{U^c}x' = \mathcal{D}_{U^c}x\} = \|x\|_1$ or

$$\arg\min_{x'}\{\|x + (x'-x)\|_1 : \mathcal{D}_{U^c}x' = \mathcal{D}_{U^c}x\} = x.$$

Now we prove (ii). The uniqueness part follows along the same lines of proof of (i) by noting that Theorem 2.15 yields $(2\#_B)^{1/a}\#_U \geq \frac{A}{\beta_o^2 C_{a,b}}$, which contradicts (2.28) as $2\#_B^{1/a} \geq (2\#_B)^{1/a}$ for any $a \geq 1$. For the reconstruction part, we solve

$$\min\{\|\mathcal{D}_{U^c}x - \mathcal{D}_{U^c}x'\| : P_{\mathcal{W}}x' = x'\}$$

for every $\mathcal{W} \subset \mathbb{Z}_N^d$ with $\#_{\mathcal{W}} = \#_B$. Let $x_{\mathcal{W}}$ denotes the solution for a given $\mathcal{W}$. After calculating $x_{\mathcal{W}}$, we compute

$$\tilde{x} = \arg\min_{x_{\mathcal{W}}, \#_{\mathcal{W}} = \#_B} \|\mathcal{D}_{U^c}x - \mathcal{D}_{U^c}x_{\mathcal{W}}\|$$

and conclude that $\tilde{x} = x$. ☐

Proof of Theorem 2.19

We first prove (i). To prove the uniqueness part, assume x and y are two signals such that the samples $\{(x * \tilde{h}_i)(k) : k \in \mathfrak{B}\mathbb{Z}_N^d, i \in \mathcal{I}_K\}$ and $\{(y * \tilde{h}_i)(k) : k \in \mathfrak{B}\mathbb{Z}_N^d, i \in \mathcal{I}_K\}$ of x and y, respectively, are supported on U. Since it is given that $\eta = r - x$ is supported on $\mathcal{T}$, thus we also assume that $r - y$ is also supported on $\mathcal{T}$. Then, consider, $h = (r - x) - (r - y) = y - x$. Note that, the signal h is supported on a set of size at most $2\#_{\mathcal{T}}$. Also, for $(k, i) \in U^c$, we have

$$\big((y - x) * \tilde{h}_i\big)(k) = \langle y - x, L_k h_i\rangle = \langle y, L_k h_i\rangle - \langle x, L_k h_i\rangle = 0.$$

This implies that h has atmost $\#_U$ nonzero elements. Then by using Theorem 2.1, we have $(2\#_{\mathcal{T}})\#_U \geq \frac{A}{\beta_o^2}$, which contradicts (2.30). Hence, x is the unique signal that can be recovered from r when its samples supported on U.

To achieve exact reconstruction, we prove that the solution to (3.38) is x, i.e., $\tilde{x} = x$. For this purpose, let $x' \in \mathcal{S}_U$, so that $\|r - x^{'}\|_1 = \|x + \eta - x^{'}\|_1 = \|\eta + (x - x^{'})\|_1$. Moreover, since $x - x' \in \mathcal{S}_U$ and $\#_{\mathcal{T}}\#_U \leq \frac{A}{2\beta_o^2}$, applying Theorem 2.21 for η, we obtain $\min\limits_{x' \in \mathcal{S}_U} \|r - x'\|_1 = \min\limits_{x' \in \mathcal{S}_U} \|\eta + (x - x')\|_1 = \|\eta\|_1$ or $\arg\min\limits_{x' \in \mathcal{S}_U} \|r - x'\|_1 = \arg\min\limits_{x' \in \mathcal{S}_U} \|\eta + (x - x')\|_1 = x$. This proves (i).

Now we prove (ii). The uniqueness part follows along the same lines as the proof of (i) by noting that Theorem 2.15 yields $(2\#_{\mathcal{T}})^{1/a}\#_U \geq \frac{A}{\beta_o^2 C_{a,b}}$, which contradicts (2.31) since $2\#_{\mathcal{T}}^{1/a} \geq (2\#_{\mathcal{T}})^{1/a}$ holds for any $a \geq 1$. For the reconstruction part, we solve the optimization problem

$$\min\{\|\mathcal{D}_U x - \mathcal{D}_U x'\| : P_{\mathcal{W}}(r - x') = r - x'\}$$

for every $\mathcal{W} \subset \mathbb{Z}_N^d$ with $\#_{\mathcal{W}} = \#_{\mathcal{T}}$. Let $x_{\mathcal{W}}$ be the signal that solves the problem for a given $\mathcal{W}$. After calculating $x_{\mathcal{W}}$, we determine $\tilde{x} = \arg\min\limits_{x_{\mathcal{W}}, \#_{\mathcal{W}} = \#_{\mathcal{T}}} \|\mathcal{D}_U x - \mathcal{D}_U x_{\mathcal{W}}\|$ and conclude that $x = \tilde{x}$. $\square$

2.4.2 Exact recovery of binary signals

The authors in [87] provided an algorithm for the exact recovery of binary (0 - 1) signals in the scenario when their Fourier coefficients are missing on a given set. We apply the same algorithm for the exact recovery of binary signals in cases where their samples are lost or unobserved outside a given set. We begin with presenting the algorithm for the exact recovery of binary signals.

Algorithm. Let $x \in \ell^2(\mathbb{Z}_N^d)$ be a binary (0-1) signal supported on $B \subset \mathbb{Z}_N^d$. Assume that a subset

$$\{(x * \tilde{h}_i)(k) : (k, i) \in U\}$$

of its samples are missing for some index set $U \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$. Under the recovery conditions (2.27) or (2.28), it follows that

$$\mathrm{Re}(\mathcal{D}_U x(n)) < \tfrac{1}{2}, \qquad \forall\, n \in \mathbb{Z}_N^d,$$

where $\mathrm{Re}(z)$ denotes the real part of $z \in \mathbb{C}$ and $\mathcal{D}_{U^c}x$ is defined in (2.26). Since

$$x(n) = \frac{1}{A} \sum_{(m,i)\in U} (x * \tilde{h}_i)(m) L_m h_i(n) + \frac{1}{A} \sum_{(m,i)\in U^c} (x * \tilde{h}_i)(m) L_m h_i(n) = \mathcal{D}_U x(n) + \mathcal{D}_{U^c} x(n),$$

we conclude that, for each $n \in \mathbb{Z}_N^d$, $x(n) = 1$ only if $\mathrm{Re}(\mathcal{D}_{U^c}x(n)) \geq \frac{1}{2}$.

We have the following result:

Theorem 2.22. *In addition to the standing assumptions of Theorem 2.18, assume that $x \in \ell^2(\mathbb{Z}_N^d)$ is a binary signal.*

(i) Then, x can be uniquely recovered from $\mathcal{D}_{U^c}x$ if

$$\#_B \#_U < \frac{A}{2\beta_o^2}. \tag{2.37}$$

(ii) Suppose that the restriction estimate (2.10) holds for U for a pair (a, b) with $1 \leq a \leq b < \infty$. Then x can be recovered uniquely from $\mathcal{D}_{U^c}x$ if

$$\#_B^{1/a} \#_U \leq \frac{A}{2\beta_o^2 C_{a,b}}. \tag{2.38}$$

Moreover, in both cases, the signal x is given by:

$$x(n) = \begin{cases} 1, & \textit{if } \mathrm{Re}(\mathcal{D}_{U^c}x(n)) \geq 1/2, \\ 0, & \textit{otherwise}, \end{cases} \qquad n \in \mathbb{Z}_N^d. \tag{2.39}$$

Proof. We first prove part (i). For any $n \in \mathbb{Z}_N^d$, we have

$$\begin{aligned}
\mathrm{Re}(\mathcal{D}_U x(n)) \leq |\mathcal{D}_U x(n)| &= \frac{1}{A} \left| \sum_{(k,i)\in U} (x * \tilde{h}_i)(k) L_k h_i(n) \right| \\
&\leq \frac{1}{A} \sum_{(k,i)\in U} \left| \sum_{m \in \mathbb{Z}_N^d} x(m) \overline{h_i(k-m)} \right| |h_i(n-k)| \\
&\leq \frac{\beta_o^2}{A} \#_U \sum_{m\in B} |x(m)| \leq \frac{\beta_o^2}{A} \#_U \#_B < \frac{1}{2}.
\end{aligned}$$

Hence, $\mathrm{Re}(\mathcal{D}_U x(n)) < \frac{1}{2}$ for all $n \in \mathbb{Z}_N^d$. Applying the recovery algorithm, we reconstruct x by setting

$$x(n) = \begin{cases} 1, & \text{if } \mathrm{Re}(\mathcal{D}_{U^c} x(n)) \geq \frac{1}{2}, \\ 0, & \text{if } \mathrm{Re}(\mathcal{D}_{U^c} x(n)) < \frac{1}{2}. \end{cases}$$

This completes the proof of (i).

We now prove part (ii). Using Hölder's inequality together with the restriction estimate (2.10), and noting that $\|x\|_{\ell^a(B)} = \#_B^{1/a}$ since x is binary, we obtain for any $n \in \mathbb{Z}_N^d$,

$$\begin{aligned}
\mathrm{Re}(\mathcal{D}_U x(n)) \leq |\mathcal{D}_U x(n)| &= \frac{1}{A} \left| \sum_{(k,i)\in U} (x * \tilde{h}_i)(k) L_k h_i(n) \right| \\
&\leq \frac{\beta_o}{A} \sum_{(k,i)\in U} |(x * \tilde{h}_i)(k)| \\
&\leq \frac{\beta_o}{A} \#_U \left(\frac{1}{\#_U} \sum_{(k,i)\in U} |(x * \tilde{h}_i)(k)|^b \right)^{1/b} \\
&\leq \frac{\beta_o^2}{A} \#_U C_{a,b} \|x\|_{\ell^a(B)} = \frac{\beta_o^2}{A} \#_U C_{a,b} \#_B^{1/a} < \frac{1}{2},
\end{aligned}$$

where the final inequality follows from (2.38). Following the same argument as in part (i), the claim follows. □

Example 2.23. Consider the filter bank setting as discussed in Example 2.6 and a binary signal x as shown in Fig. 2.2(a). From the figure, it is clear that $\#_B = 2$. We assume that for $U = \{(1,4),(2,3)\}$, the samples $\{(x * \tilde{h}_i)(k) : k \in \mathbb{Z}_6, i \in \mathcal{I}_K\}$ are lost. Under this assumption, the condition (2.37) is satisfied since

$$\#_B \#_U = 4 < 4.5 = \frac{36}{2 \times 4} = \frac{A}{2\beta_o^2}.$$

The signal $\mathcal{D}_{U^c}x$ is shown in Fig. 2.2(b). From the figure, it is clear that $\mathrm{Re}(\mathcal{D}_{U^c}x(n)) \geq 1/2$ only for $n = 0$ and $n = 3$. Therefore, x takes the value of 1 at $n = 0, 3$, as confirmed by Fig. 2.2(a).

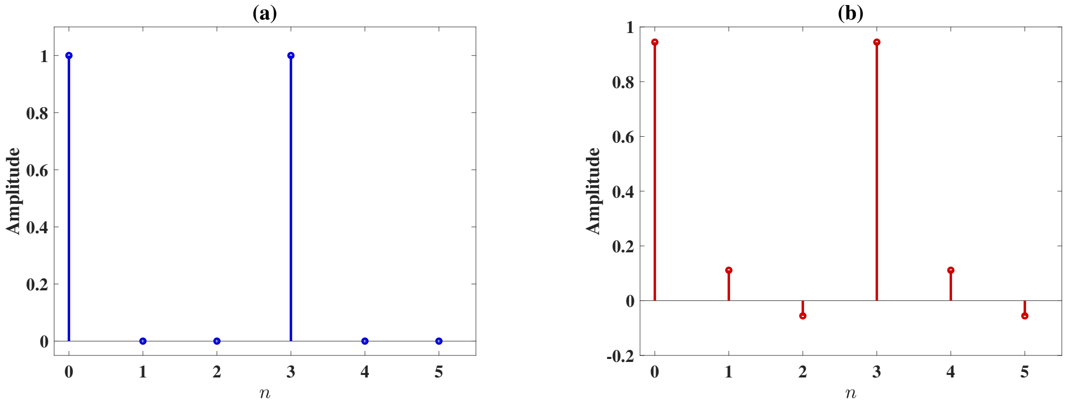


Figure 2.2: (a) The original signal of length $N = 6$. (b) The signal $\mathcal{D}_{U^c}x$ corresponding to the missing sample set $U = \{(1, 4), (2, 3)\}$.

2.5 Recovery of signals concentrated on a given set

In the previous section, we utilized the support conditions, namely, that the signal is supported on a set $B \subset \mathbb{Z}_N^d$, and its samples are supported on a set $U \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$. In contrast, the authors in [66] established uncertainty relations for a specific class of signals, focusing on their concentration properties rather than explicitly considering their support. They further applied these uncertainty principles to derive exact recovery conditions for such signals when certain frequency components are missing.

2.5.1 Uncertainty principle and recovery conditions with concentration

In this subsection, we assume that x is concentrated in B in an appropriate sense and derive uncertainty relations based on its concentration property. We then apply our uncertainty relations to establish conditions for the unique recovery of a signal in scenarios where some of its samples are lost or it is corrupted by noise. We now provide a precise definition.

Definition 2.24. We say that $x \in \ell^2(\mathbb{Z}_N^d)$ is ℓ^p-concentrated on $E \subset \mathbb{Z}_N^d$, if there exists a constant $\mathcal{C}_{E,p} \geq 1$ such that

$$\left(\sum_{n \in \mathbb{Z}_N^d} |x(n)|^p \right)^{\frac{1}{p}} \leq \mathcal{C}_{E,p} \left(\sum_{n \in E} |x(n)|^p \right)^{\frac{1}{p}}.$$

Our uncertainty relations stated in Theorem 2.1 and Theorem 2.15 will typically hold at the cost of a constant if $x \in \ell^2(\mathbb{Z}_N^d)$ is merely ℓ^p-concentrated on $B \subset \mathbb{Z}_N^d$ rather than properly supported on B. However, if x and y are concentrated on $B \subset \mathbb{Z}_N^d$, then $x - y$ may fail to be concentrated on any set of size at most $2\#_B$. Thus, the same signal recovery techniques may not apply, as the support condition is crucial in proving the uniqueness results in Theorem 2.18 and Theorem 2.19.

It is important to note that the concentration condition makes it more difficult to apply restriction inequalities, as it leads to more restrictive signal recovery conditions (2.28) and (2.31). Indeed, if $x \in \ell^2(\mathbb{Z}_N^d)$ is ℓ^p-concentrated on a subset $U \subset \mathbb{Z}_N^d$ with concentration constant λ, rather than being fully supported on U as in Theorem 2.15, then the inequality (2.20) takes the form

$$\#_B^{1/a} \#_U \geq \frac{A}{\beta_o^2 C_{a,b} \lambda}.$$

As a consequence, the signal recovery condition (2.28) in Theorem 2.18 for the case of missing samples becomes

$$\#_B^{1/a} \#_U < \frac{A}{2\beta_o^2 C_{a,b} \lambda},$$

and similarly, the recovery condition (2.31) in Theorem 2.19 for the case of noisy signals becomes

$$\#_{\mathcal{T}}^{1/a} \#_U < \frac{A}{2\beta_o^2 C_{a,b} \lambda}.$$

As λ increases, the right-hand side of the recovery conditions decreases, making them harder to satisfy. Consequently, under the concentration condition, the effectiveness of the restriction estimate is reduced. Therefore, the restriction estimate is more useful when signals are supported on a set, while it is less effective under concentration assumptions.

To make use of the restriction estimate under a concentration assumption, we consider signals that can be expressed as linear combinations of indicator functions. Notice that any signal $x \in \ell^2(\mathbb{Z}_N^d)$ can be written in the form

$$x = \sum_{i=1}^{\ell} b_i 1_{B_i}.$$

where, for $i \in \mathcal{I}_\ell$, $b_i \in \mathbb{C}$, and B_i are disjoint subsets of $\ell^2(\mathbb{Z}_N^d)$. Then, we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}_N^d} |x(n)|^a &= \sum_{n\in\mathbb{Z}_N^d} |b_1 1_{B_1} + b_2 1_{B_2} + \ldots b_\ell 1_{B_\ell}(n)|^a \\ &= \sum_{i=1}^{\ell} |b_i|^a \#_{B_i} \leq \max_{i\in\mathcal{I}_\ell} |b_i|^a \#_{B_i} \ell \leq \ell \sum_{n\in B_m} |b_m|^a = \ell \sum_{n\in B_m} |x(n)|^a,\end{aligned} \tag{2.40}$$

where $|b_m|^a \#_{B_m} = \max_{i\in\mathcal{I}_\ell} |b_i|^a \#_{B_i}$. This implies,

$$\left(\sum_{n\in\mathbb{Z}_N^d} |x(n)|^a\right)^{\frac{1}{a}} \leq \ell^{1/a} \left(\sum_{n\in B_m} |x(n)|^a\right)^{\frac{1}{a}}. \tag{2.41}$$

Therefore, x is ℓ^a-concentrated on B_m with constant $\ell^{1/a}$. Our uncertainty principles (Theorem 2.1 and Theorem 2.15) with concentration takes the following form:

Theorem 2.25. *Let $x \in \ell^2(\mathbb{Z}_N^d)$ be a signal of the form*

$$x = \sum_{i=1}^{\ell} b_i 1_{B_i},$$

where, for $i \in \mathcal{I}_\ell$, $b_i \in \mathbb{C}$, and B_i are disjoint subsets of $\ell^2(\mathbb{Z}_N^d)$ and let its samples $\{(x * \tilde{h}_i)(k) : (k, i) \in \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K\}$ be supported on U for some $U \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$ Suppose that the restriction estimate (2.10) holds for U for a pair $(a, b), 1 \leq a \leq b < \infty$. Then,

$$\#_{B_m}^{1/a} \#_U \geq \frac{A}{\ell \beta_o^2 C_{a,b}},$$

where $|b_m|^a \#_{B_m} = \max_{i\in\mathcal{I}_\ell} |b_i|^a \#_{B_i}$.

Proof. The proof follows along the same lines as the proof of Theorem 2.15 utilizing (2.41). □

2.5.2 Condition for signal recovery in case of concentration

The unique recoveries for the case of missing samples (Theorem 2.18) and the case of a noisy signal (Theorem 2.19) are established using the supports of the original signal and the noisy signal, respectively. Here, the similar arguments do not follow under the concentration condition, as discussed at the beginning of this section.

We first discuss the condition for the unique recovery of a given signal $x \in \ell^2(\mathbb{Z}_N^d)$ from the noisy signal r (given in (2.29)) such that its samples $\{(x * \tilde{h}_i)(k) : (k,i) \in \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K\}$ are supported on U for some $U \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$. To obtain a condition for the unique recovery from r, assume x and y are two signals such that their samples $\{(x * \tilde{h}_i)(k) : k \in \mathfrak{B}\mathbb{Z}_N^d, i \in \mathcal{I}_K\}$ and $\{(y * \tilde{h}_i)(k) : k \in \mathfrak{B}\mathbb{Z}_N^d, i \in \mathcal{I}_K\}$ are supported on U.

Assume that $r - x = \sum_{i=1}^{\ell} b_i B_i$ and $r - y = \sum_{j=1}^{q} c_j C_j$, where for $i \in \mathcal{I}_\ell$ and $j \in \mathcal{I}_q$, the coefficients $b_i, c_j \in \mathbb{C}$, and B_i, C_j are disjoint subsets of $\ell^2(\mathbb{Z}_N^d)$. Moreover, the coefficients $\{b_i\}_{i\in\mathcal{I}_\ell}$ and $\{c_j\}_{j\in\mathcal{I}_q}$ are chosen from a given set of coefficients $\{e_i\}_{i\in I}$. We define $h = (r-y) - (r-x) = x - y$. Then, h takes the form:

$$h(n) = \begin{cases} b_i - c_j, & \text{if } n \in B_i \cap C_j, \quad i \in \mathcal{I}_\ell, j \in \mathcal{I}_q, \\ b_i, & \text{if } n \in B_i \setminus (C_1 \cup C_2 \cup \ldots \cup C_q), \quad i \in \mathcal{I}_\ell, \\ -c_j, & \text{if } n \in C_j \setminus (B_1 \cup B_2 \cup \ldots \cup B_\ell), \quad j \in \mathcal{I}_q, \\ 0, & \text{otherwise.} \end{cases} \tag{2.42}$$

Then, the collection

$$\mathcal{E} := \{B_i \cap C_j, B_i \setminus (C_1 \cup \ldots \cup C_q), C_j \setminus (B_1 \cup \ldots \cup B_\ell) : i \in \mathcal{I}_\ell, j \in \mathcal{I}_q\}$$

consists of mutually disjoint sets. Therefore, if h is ℓ^a-concentrated on $\mathcal{W}$ for some $\mathcal{W} \subset \mathbb{Z}_N^d$, then from (2.42), it follows that $\mathcal{W} \in \mathcal{E}$. Now, if $h(n) = w$ for all $n \in \mathcal{W}$, then $|w|^a \#_{\mathcal{W}}$ takes one of the following forms:

$$|b_i - c_j|^a \#_{B_i \cap C_j}, \quad |b_i|^a \#_{B_i \setminus (C_1 \cup \ldots \cup C_q)}, \quad \text{or} \quad |c_j|^a \#_{C_j \setminus (B_1 \cup B_2 \cup \ldots \cup B_\ell)}. \tag{2.43}$$

Suppose it is known that $|b_m|^a \#_{B_m} = \max_{i\in\mathcal{I}_\ell} |b_i|^a \#_{B_i}$, then we can assume that

$$|b_m|^a \#_{B_m} = |c_m|^a \#_{C_m}, \tag{2.44}$$

where $|c_m|^a \#_{C_m} = \max_{j\in\mathcal{I}_q} |c_j|^a \#_{C_j}$. Note that if $|w|^a \#_{\mathcal{W}}$ corresponds to the second or third form in (2.43), then it is clearly bounded above by $|b_m|^a \#_{B_m}$ after using (2.44). If $|w|^a \#_{\mathcal{W}}$ corresponds to the first form in (2.43), then using the property

$$|b_i - c_j|^a \le (|b_i| + |c_j|)^a \le 2^{a-1}(|b_i|^a + |c_j|^a), \quad \text{for any } i \in \mathcal{I}_\ell, j \in \mathcal{I}_q \text{ and } a \ge 1,$$

along with (2.44), we obtain that $|w|^a \#_{\mathcal{W}}$ is bounded above by $2^a |b_m|^a \#_{B_m}$. Consequently, $|w|^a \#_{\mathcal{W}}$ is ultimately bounded above by $2^a |b_m|^a \#_{B_m}$.

Also, for $(k,i) \in U^c$, we have $\big((y-x)*\tilde{h}_i\big)(k) = \langle y-x, L_k h_i\rangle = \langle y, L_k h_i\rangle - \langle x, L_k h_i\rangle = 0$. This implies that the size of the set of non-zero samples of h is at most U. Thus if we assume that an (a,b) restriction estimate holds for U, then by Theorem 2.25, we have

$$2|b_m|\#_{B_m}^{1/a}\#_U \geq |w|\#_{\mathcal{W}}^{1/a}\#_U \geq \frac{A|w|}{(\ell q+\ell+q)^{1/a}C_{a,b}\beta_o^2}.$$

This gives,

$$\#_{B_m}^{1/a}\#_U \geq \frac{A|w|}{2(\ell q+\ell+q)^{1/a}C_{a,b}\beta_o^2|b_m|}. \tag{2.45}$$

Furthermore, if we assume that set of coefficients $\{e_i\}_{i\in I}$ satisfies $|e_i - e_j| \geq \epsilon > 0$ for each $i \neq j$ and $|e_i| \leq M$ for each i, then it is clear that $|w| \geq \epsilon$ and $|b_m| \leq M$. Finally by using $\ell, q \leq N^d$, (2.45) gives

$$\#_{B_m}^{1/a}\#_U \geq \frac{A\epsilon}{2N^{d/a}(N^d+2)^{1/a}C_{a,b}\beta_o^2 M}.$$

Thus, the signal x can be uniquely recovered if

$$\#_{B_m}^{1/a}\#_U < \frac{A\epsilon}{2N^{d/a}(N^d+2)^{1/a}C_{a,b}\beta_o^2 M}. \tag{2.46}$$

Remark 2.26. It can be easily verified that if $x = \sum_{i=1}^{\ell} b_i B_i \in \ell^2(\mathbb{Z}_N^d)$ is a signal such that its samples $\{(x * \tilde{h}_i)(k) : k \in \mathfrak{B}\mathbb{Z}_N^d, i \in \mathcal{I}_K\}$ are lost or unobserved for some set $S \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$, then under the condition

$$\#_{B_m}^{1/a}\#_S < \frac{A\epsilon}{2N^{d/a}(N^d+2)^{1/a}C_{a,b}\beta_o^2 M} \tag{2.47}$$

the signal x can be recovered uniquely from $\mathcal{D}_{U^c}x$ (defined in (2.26)).

Conditions (2.46) and (2.47) are derived under specific assumptions. To ensure these assumptions hold, we must carefully select signals concentrated on a given set. To do so, we focus on Dirac combs and adopt the terminology from [66].

2.5.3 Recovery of Dirac combs without a support condition

We now provide the formal definition of Dirac comb signals as stated in [66].

Definition 2.27. [66] Let $\alpha \in \mathbb{N}\cup\{0\}$ and let $\epsilon, M > 0$. We say $x \in \ell^2(\mathbb{Z}_N^d)$ is a Dirac comb of complexity α with parameters ϵ and M if it is of the form

$$x = \sum_{i=1}^{\alpha} b_i 1_{B_i}, \tag{2.48}$$

where $B_1, \ldots, B_\alpha$ are disjoint subsets of $\mathbb{Z}_N^d$ and for each $i \in \mathcal{I}_\alpha$, the coefficient $b_i \in \{e_j : j \in \mathcal{J}\} \subset \mathbb{C}$ that satisfies the following properties:

(*i*) $0 \in \{e_j : j \in \mathcal{J}\}$,

(*ii*) $|e_i - e_j| \geq \epsilon$ for each $i \neq j$,

(*iii*) $|e_j| \leq M$ for each $j \in \mathcal{J}$.

Moreover, α is chosen to be the smallest number so that these properties are satisfied.

Definition 2.28. [66] Let $x \in \ell^2(\mathbb{Z}_N^d)$ be a Dirac comb of complexity α as given in (2.48) with parameters ϵ and M and $1 \leq a < \infty$ be given. The quantities$|b_1|^a \#_{B_1}, |b_2|^a \#_{B_2}, \ldots,$ $|b_\alpha|^a \#_{B_\alpha}$ can be reordered as follows

$$|b_1| \#_{B_1} \geq |b_2| \#_{B_2} \geq \cdots \geq |b_\alpha| \#_{B_\alpha},$$

with the additional rule

(*i*) If $|b_i| \#_{B_i} = |b_j| \#_{B_j}$, then i precedes j if $\#_{B_i} \leq \#_{B_j}$.

(*ii*) If $|b_i| \#_{B_i} = |b_j| \#_{B_j}$ and $\#_{B_i} = \#_{B_j}$, then place i before j if $\arg(b_i) \leq \arg(b_j)$.

We refer to b_1, B_1, and the quantity $|b_1|^a \#_{B_1}$, the a-effective weight, a-effective support, and a-effective mass of x, respectively.

From (2.40), it is clear that a Dirac comb of the form given in (2.48) with a-effective support B_1 is ℓ^a-concentrated on B_1 with constant $\alpha^{1/a}$. Also note that the a-effective support of x can vary depending on the value of a as shown in the example below.

Example 2.29. Consider the Dirac comb $x \in \ell^2(\mathbb{Z}_8)$, defined as $x = 3.\, 1_{\{0,1,2,3\}} + 5.\, 1_{\{4,5\}} + 1.\, 1_{\{6,7\}}$, as shown in Figure 2.3. Then, its 1-effective support is $\{0, 1, 2, 3\}$, while its 2-effective support is $\{4, 5\}$, also depicted in Figure 2.3.

The following result establishes an uncertainty principle for Dirac combs, expressed in terms of their 2-effective support, the support of their samples, and their complexity. This can be seen as an alternative formulation of [66, Theorem 3.7], which presents an uncertainty principle for Dirac combs based on their 2-effective support, the support of their Fourier transform, and their complexity.

Theorem 2.30. *Let* $x \in \ell^2(\mathbb{Z}_N^d)$ *be a Dirac comb of complexity* α *with parameters* ϵ *and* M*. Suppose that for some* $1 \leq a < \infty$*,* B *is the* a*-effective support of* x*, and the samples*

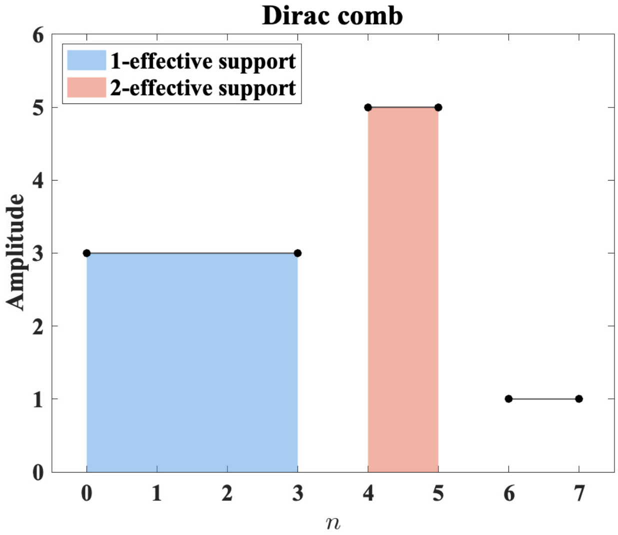


Figure 2.3: A Dirac comb $x \in \ell^2(\mathbb{Z}_8)$ with 1-effective support $\{0,1,2,3\}$ and 2-effective support $\{4,5\}$.

$\{(x * \tilde{h}_i)(k) : (k,i) \in \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K\}$ *are supported on* $U \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$*. Then, the following statements hold:*

(i) *For* $a = 1$ *or* $a = 2$*, we have*

$$\#_B \#_U \geq \frac{A}{\beta_o^2 \alpha}. \tag{2.49}$$

(ii) *Suppose an* (a,b) *restriction estimate holds for* U *for some* $1 \leq b < \infty$*. Then,*

$$\#_B^{1/a} \#_U \geq \frac{A}{\alpha^{1/a} \beta_o^2 C_{a,b}}. \tag{2.50}$$

Proof. We first prove (i). For $a = 1$, the proof follows similarly to Theorem 2.15, noting that the restriction estimate (2.10) holds for U for the pair $(1,b)$ (see Remark 2.8) and that the 1-effective support B satisfies

$$\sum_{n \in \mathbb{Z}_N^d} |x(n)| \leq \alpha \sum_{n \in B} |x(n)|.$$

For $a = 2$, using the fact that the collection $\mathcal{F}_{\mathfrak{B},K}$ forms a tight frame for $\ell^2(\mathbb{Z}_N^d)$ with frame bound A, along with the inequality $(\sum_{n \in \mathbb{Z}_N^d} |x(n)|^2)^{1/2} \leq \alpha^{1/2} (\sum_{n \in B} |x(n)|^2)^{1/2}$, we have

$$\begin{aligned}|x(n)| &= \frac{1}{A}\left|\sum_{(m,i)\in U}(x*\tilde{h}_i)(m)L_m h_i(n)\right| = \frac{1}{A}\left|\sum_{(m,i)\in U}(x*\tilde{h}_i)(m)h_i(n-m)\right| \\ &\leq \frac{\beta_o}{A}\sum_{(m,i)\in U}|(x*\tilde{h}_i)(m)| \leq \frac{\beta_o}{A}\#_U^{1/2}\left(\sum_{(m,i)\in U}|(x*\tilde{h}_i)(m)|^2\right)^{1/2} \\ &\leq \frac{\beta_o}{A}\#_U^{1/2}A^{1/2}\left(\sum_{n\in\mathbb{Z}_N^d}|x(n)|^2\right)^{1/2} \leq \frac{\beta_o}{\sqrt{A}}\#_U^{1/2}\alpha^{1/2}\left(\sum_{n\in B}|x(n)|^2\right)^{1/2}.\end{aligned}$$

Squaring both sides, summing over B and dividing both sides by $\sum_{n\in B}|x(n)|^2$, we get,

$$\#_B\#_U \geq \frac{A}{\beta_o^2\alpha}. \tag{2.51}$$

The proof of (ii) follows similarly to that of Theorem 2.15, after noting that a-effective support B satisfies

$$\left(\sum_{n\in\mathbb{Z}_N^d}|x(n)|^a\right)^{1/a} \leq \alpha^{1/a}\left(\sum_{n\in B}|x(n)|^a\right)^{1/a}.$$

This completes the proof. □

2.5.4 Signal recovery with Dirac combs

In this subsection, we establish conditions for recovering a Dirac comb when some of its samples are lost or unobserved, using its effective support. Additionally, we provide conditions for recovering a signal from its noisy version, assuming that the noise vector is a Dirac comb. General recovery conditions for a signal with concentration information were discussed in Subsection 2.5.2. Here, we refine that discussion to derive specific recovery conditions for Dirac combs. Similar conditions for Dirac comb recovery in the case of missing or unobserved Fourier coefficients are presented in [66, Theorem 3.12].

We have the following results:

Theorem 2.31. *Let $x\in\ell^2(\mathbb{Z}_N^d)$ be a Dirac comb of complexity α with parameters ϵ and M whose samples $\{(x*\tilde{h}_i)(k):(k,i)\in\mathfrak{B}\mathbb{Z}_N^d\times\mathcal{I}_K\}$ are lost or unobserved for some subset $U\subset\mathfrak{B}\mathbb{Z}_N^d\times\mathcal{I}_K$. Suppose for some $1\leq a<\infty$, B is the a-effective support of x and the a-effective mass of x is known, then x can be uniquely recovered from $\mathcal{D}_{U^c}x$ (defined in (2.26)) if,* (*i*) $a = 1$ *and*

$$\#_B \#_U < \frac{A\epsilon}{2(\alpha^2 + 2\alpha)\beta_o^2 M}.$$

(*ii*) $a = 2$ *and*

$$\#_B \#_U < \frac{A\epsilon}{4(\alpha^2 + 2\alpha)\beta_o^2 M}.$$

(*iii*) *an* (a, b) *restriction estimate holds for* U *for some* $1 \leq a \leq b < \infty$ *and*

$$\#_B^{1/a} \#_U < \frac{A\epsilon}{2C_{a,b}(\alpha^2 + 2\alpha)^{1/a}\beta_o^2 M}.$$

Theorem 2.32. *Let* $x \in \ell^2(\mathbb{Z}_N^d)$ *be a signal whose samples* $\{(x * \tilde{h}_i)(k) : (k, i) \in \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K\}$ *are supported on* U *for some* $U \subset \mathfrak{B}\mathbb{Z}_N^d \times \mathcal{I}_K$*. Suppose the noise vector* $\eta \in \mathbb{Z}_N^d$ *is a Dirac comb of complexity* α *with parameters* ϵ *and* M *such that* $\mathcal{T}$ *is the a-effective support of* η *and the a-effective mass of* η *is known. Then* x *can be uniquely recovered from* r *(defined in* (2.29)*) if,*

(*i*) $a = 1$ *and*

$$\#_{\mathcal{T}} \#_U < \frac{A\epsilon}{2(\alpha^2 + 2\alpha)\beta_o^2 M}. \tag{2.52}$$

(*ii*) $a = 2$ *and*

$$\#_{\mathcal{T}} \#_U < \frac{A\epsilon}{4(\alpha^2 + 2\alpha)\beta_o^2 M}.$$

(*iii*) *an* (a, b) *restriction estimate holds for* U *for some* $1 \leq a \leq b < \infty$ *and*

$$\#_{\mathcal{T}}^{1/a} \#_U < \frac{A\epsilon}{2C_{a,b}(\alpha^2 + 2\alpha)^{1/a}\beta_o^2 M}.$$

We first prove Theorem 2.32.

Proof of Theorem 2.32. We first prove (i). Assume that x and y are two signals such that their samples are supported on U. Since it is given that $\mathcal{T}$ is the 1-effective support of $\eta = r - x$, thus we also assume that $\mathcal{T}$ is the 1-effective support of $r - y$.

Furthermore, since α is the complexity of $\eta = r - x$, we assume that $r - x = \sum_{i=1}^{\alpha} b_i B_i$ and $r - y = \sum_{j=1}^{\alpha} c_j C_j$, where for $i \in \mathcal{I}_\ell$ and $j \in \mathcal{I}_q$, $b_i, c_j \in \mathbb{C}$ and B_i, C_j are disjoint subsets of $\ell^2(\mathbb{Z}_N^d)$. If $|b|\#_{\mathcal{T}}$ is the 1-effective mass of $\eta = r - x$ for some $b \in \{b_i : i \in \mathcal{I}_\ell\}$, then we also assume that $|b|\#_{\mathcal{T}}$ is also the 1-effective mass of $r - y$. Then by using Theorem 2.30 (i) and following the same arguments as in Subsection 2.5.2, we get

$$\#_{\mathcal{T}} \#_U \geq \frac{A\epsilon}{2\beta_o^2(\alpha^2 + 2\alpha)M}, \tag{2.53}$$

which contradicts our assumption (2.52). This proves the uniqueness.

The proofs of uniqueness parts of (ii) and (iii) follow similarly to (i) after using Theorem 2.30 (i) and Theorem 2.30 (ii), respectively.

For the exact recovery, we modify the combinatorial approach (see Remark 2.20) to incorporate the effective support information of the noise vector. Specifically, the least square problem

$$\min \left\{ \|\mathcal{D}_U x - \mathcal{D}_U x'\| : \|r - x'\|_{\ell^a(\mathbb{Z}_N^d)} \leq \alpha^{1/a} \|r - x'\|_{\ell^a(\mathcal{W})} \right\} \tag{2.54}$$

is solved for every $\mathcal{W} \subset \mathbb{Z}_N^d$ with $\#_{\mathcal{W}} = \#_{\mathcal{T}}$. Let $x_{\mathcal{W}}$ be the signal that solves the problem for a given $\mathcal{W}$. After calculating $x_{\mathcal{W}}$, we compute

$$\tilde{x} = \arg \min_{x_{\mathcal{W}}, \#_{\mathcal{W}} = \#_{\mathcal{T}}} \|\mathcal{D}_U x - \mathcal{D}_U x_{\mathcal{W}}\| \tag{2.55}$$

and conclude that $\tilde{x} = x$. □

Proof of Theorem 2.31. The proof follows similarly to that of Theorem 2.32 after noting Remark 2.26. The approach for the exact recovery also follows from the one followed in Theorem 2.32 after replacing $r - x'$ in (2.54) with x' and $\mathcal{T}$ with the support B. □

In the following chapter, we apply the uncertainty relations established here to the context of Ramanujan sums. Through numerical experiments, we further demonstrate that employing Ramanujan sums as analysis filters within the filter bank framework enhances signal recovery, even when the theoretical signal recovery conditions are not satisfied, thereby utilizing the number-theoretic properties of Ramanujan sums in signal processing applications.

Chapter 3

Ramanujan sums in signal recovery and uncertainty principle inequalities

In this chapter, we examine the perfect reconstruction property of filter banks based on Ramanujan sums and their role in signal recovery utilizing the uncertainty relations developed in Chapter 2. Originally proposed by Srinivasa Ramanujan [122], these sums extract periodic components from signals and form the basis of Ramanujan filter banks. We analyze the robustness of these filter banks against erasures in discrete-time spaces $\ell^2(\mathbb{Z}_N)$ (or $\mathbb{C}^N$). Through numerical experiments, we further demonstrate that even when the recovery conditions developed in Chapter 2 are not met, a stable reconstruction can still be achieved by exploiting the properties of Ramanujan filter banks.

3.1 Ramanujan sums in signal processing

We begin with recalling the definition of Ramanujan sums. For a positive integer q, the *q-th Ramanujan sum* c_q has the following representation [122]:

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{2\pi i k n/q}, \quad n \geq 1, \tag{3.1}$$

where (k, q) denotes the greatest common divisor of k and q. For example, if $q = 5$, then $(k, q) = 1$ for $k \in \{1, 2, 3, 4\}$, thus

$$c_5(n) = e^{2\pi i n/5} + e^{2\pi i 2n/5} + e^{2\pi i 3n/5} + e^{2\pi i 4n/5}, \; n \geq 1.$$

Ramanujan sums have found increasing relevance in the field of signal processing due to their various properties, particularly, to capture periodicities in discrete-time signals.

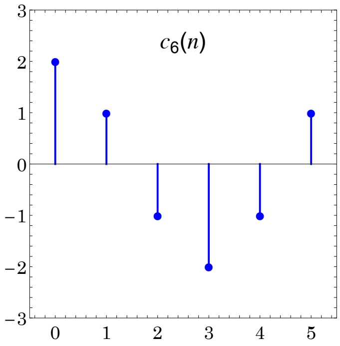
Note that c_q is q-periodic, i.e., $c_q(n + q) = c_q(n)$, $n \geq 1$ and $c_q(0) = \phi(q)$. For more properties of Ramanujan sums, we refer to [122, 146, 147].

The results presented in this chapter are based on our revised submission:

Kalra, S. and Shukla, N. K. 2025. Ramanujan sums in signal recovery and uncertainty principle inequalities, *Applied and Computational Harmonic Analysis*, Revision submitted.

The first few Ramanujan sums are listed below:

$$\begin{aligned} c_1 &= \{1, 1, \ldots\}; \\ c_2 &= \{1, -1, 1, -1, \ldots\}; \\ c_3 &= \{2, -1, -1, 2, \ldots\}; \\ c_4 &= \{2, 0, -2, 0, 2, \ldots\}; \\ c_5 &= \{4, -1, -1, -1, -1, 4, \ldots\}; \\ c_6 &= \{2, 1, -1, -2, -1, 1, 2, \ldots\}. \end{aligned}$$



The collection $\{c_q, c_q(\cdot - 1), \ldots, c_q(\cdot - \phi(q) + 1)\}$, consisting of $\phi(q)$ consecutive circular shifts of the q-th Ramanujan sum c_q, is linearly independent and thus spans a $\phi(q)$-dimensional subspace, referred to as the *Ramanujan subspace* and denoted by S_q. These subspaces enable the development of Ramanujan periodicity dictionaries for representing and extracting periodic components of signals [147].

The following result summarizes some essential properties of Ramanujan sums.

Proposition 3.1. *[146] (Some properties of the Ramanujan sums). For any positive integer q, the Ramanujan sum c_q, as defined in (3.1), satisfies the following properties:*

- *(i)* $c_q(n)$ *is an integer and* $c_q(n) \leq \phi(q)$ *for any* n.
- *(ii)* $c_q(n) = c_q(-n)$ *for any* n.
- *(iii) (Autocorrelation)* $\sum_{n=0}^{q-1} c_q(n)c_q(n-l) = qc_q(l) \quad$ *for any* $l \in \mathbb{Z}$.
- *(iv) (Orthogonality)* $\sum_{n=0}^{\mathrm{lcm}(p,q)-1} c_{q'}(n)c_q(n-l) = 0 \quad$ *for any* $l \in \mathbb{Z}$ *when* $q' \neq q$.
- *(v) (Sum)* $\sum_{n=0}^{q-1} c_q(n) = 0$ *for* $q > 1$.
- *(vi) (Sum of squares)* $\sum_{n=0}^{q-1} c_q^2(n) = q\phi(q)$.

A significant application of Ramanujan sums is in the construction of *Ramanujan filter banks*. These filter banks use Ramanujan sums as analysis filters, providing a mechanism for detecting and tracking periodic patterns in time series data [114, 115, 149]. In the following subsection, we demonstrate how these filter banks are used to effectively identify the period of an unknown signal.

3.1.1 Period identification using Ramanujan sums

In this section, we investigate the structure and properties of Ramanujan filter banks in the finite-dimensional setting of $\ell^2(\mathbb{Z}_N)$, and employ them for the task of period identification of discrete signals.

The use of Ramanujan filter banks (without downsampling) for identifying the period of a signal was developed by Vaidyanathan and Tenneti in [148]. Specifically, Theorem 1 in [148] identifies which filters in the bank will respond to a periodic input signal.

Theorem 3.2 ([148, Theorem 1])**.** *Consider a filter bank based on Ramanujan sums* $c_1, c_2, \ldots, c_M$ *and let* x *be a period-*P *input signal with* $1 \leq P \leq M$*. Then nonzero outputs can only be produced by those filters* c_q*,* $1 \leq q \leq M$*, such that the filter index* q *is a divisor of* P*, that is,* $q \mid P$*.*

Furthermore, Theorem 3 of [148] establishes how the period can be precisely recovered from the set of responding filters.

Theorem 3.3 ([148, Theorem 3])**.** *Consider a filter bank based on Ramanujan sums* $c_1, c_2, \ldots, c_M$ *and let* x *be a period-*P *input signal with* $1 \leq P \leq M$*. Let nonzero outputs be produced by the subset of filters* c_{q_i} *with periods* $q_1, q_2, \ldots, q_K$*. Then the period* P *is given by* $P = \operatorname{lcm}\{q_1, q_2, \ldots, q_K\}$*.*

These results show that a Ramanujan filter bank isolates the components corresponding to the divisors of the signal period, and the original period can be recovered as the least common multiple of the indices of the non-zero filters. However, when the period P is large, considering all Ramanujan sums c_q for $1 \leq q \leq M$ with $1 \leq P \leq M$ may increase computational complexity and reduce the efficiency of the period identification process. In the special case where the signals are cosidered in $\ell^2(\mathbb{Z}_N)$ and the period P of a given signal is a divisor of N, the period identification process can be made significantly simpler by considering the filter bank based on Ramanujan sums $\{c_{q_1}, c_{q_2}, \ldots, c_{q_K}\} \subset \ell^2(\mathbb{Z}_N)$ corresponding to all the divisors $q_1, q_2, \ldots, q_K$ of N.

To this end, let N be a positive integer and let $q_1, q_2, \ldots, q_K$ be all the divisors of N. For each $1 \leq i \leq K$, we consider the q_i-th Ramanujan sum c_{q_i} as an element of $\ell^2(\mathbb{Z}_N)$, extended periodically over $\mathbb{Z}_N$. For a signal $x \in \ell^2(\mathbb{Z}_N)$, the outputs of the Ramanujan filter bank with analysis filters $\{c_{q_1}, c_{q_2}, \ldots, c_{q_K}\} \subset \ell^2(\mathbb{Z}_N)$ and unit decimation ratio are obtained as the discrete circular convolutions

$$y_{q_i}(k) = (x * \tilde{c}_{q_i})(k), \quad k \in \mathcal{J}_N,\, i \in \mathcal{I}_K, \tag{3.2}$$

where $\tilde{c}_q(n) = \overline{c_q(-n)}$.

As a consequence of Theorem 3.2 and Theorem 3.3, we obtain the following result as a corollary for a special case. In particular, under the assumption that the period P of a signal $x \in \ell^2(\mathbb{Z}_N)$ divides N, it follows that a Ramanujan filter bank with analysis filters $\{c_{q_1}, c_{q_2}, \dots, c_{q_K}\} \subset \ell^2(\mathbb{Z}_N)$, corresponding to the divisors of N, is sufficient to identify P.

Corollary 3.4. *Let $x \in \ell^2(\mathbb{Z}_N)$ be a period-P input signal such that P divides N, and let $q_1, q_2, \dots, q_K$ be all the divisors of N. Consider the Ramanujan filter bank based on Ramanujan sums $\{c_{q_1}, c_{q_2}, \dots, c_{q_K}\} \subset \ell^2(\mathbb{Z}_N)$. Then, the nonzero outputs y_{q_i} (as defined in (3.2)), $1 \leq i \leq K$, must be generated by a subset of the Ramanujan sums $c_{q_1}, c_{q_2}, \dots, c_{q_m}$ with $m \leq K$. Moreover, the period P is given by $P = \operatorname{lcm}(q_1, q_2, \dots, q_m)$.*

Proof. The output $y_{q_i}(n)$ corresponding to the q_i-th Ramanujan sum c_{q_i}, for $1 \leq i \leq K$ and $0 \leq n \leq N-1$, can be expressed as

$$y_{q_i}(n) = (x * \tilde{c}_{q_i})(n) = \sum_{m=0}^{N-1} x(m)\tilde{c}_{q_i}(n-m) = \sum_{m=0}^{N-1} x(m)\overline{L_n c_{q_i}(m)} = \langle x, L_n c_{q_i} \rangle.$$

It follows Theorem 3.2 that the output y_{q_i}, $1 \leq i \leq K$, can be non-zero only if $q_i \mid P$. Since P divides N, then clearly $\{p_1, p_2, \dots, p_m\} \subset \{q_1, q_2, \dots, q_K\}$, where $p_1, p_2, \dots, p_m$ are all the divisors of P. Therefore, the filter bank based on $c_{q_1}, c_{q_2}, \dots, c_{q_K}$ contains all those Ramanujan sums, i.e., $c_{p_1}, c_{p_2}, \dots, c_{p_m}$ for which y_{p_i}, $1 \leq i \leq m$, can be non-zero. Then by using Theorem 3.3, P is known precisely and is given by $\operatorname{lcm}(p_1, p_2, \dots, p_m)$. ☐

We provide an example to illustrate Corollary 3.4.

Example 3.5. Consider a signal $x \in \ell^2(\mathbb{Z}_{30})$, depicted in Fig. 3.1(a), with period $P \leq N = 30$. The divisors of $N = 30$ are given by $q_1 = 1, q_2 = 2, q_3 = 3, q_4 = 5, q_5 = 6, q_6 = 10, q_7 = 15, q_8 = 30$, so that $K = 8$. We consider the filter bank based on the Ramanujan sums $c_1, c_2, c_3, c_5, c_6, c_{10}, c_{15}$, and c_{30}, in accordance with Corollary 3.4.

Fig. 3.1(b) illustrates the energy outputs produced by each channel after the signal x is passed through it, where the energy E_i corresponding to the q_i-th Ramanujan sum is given by:

$$E_i = \sum_{n=0}^{N-1} |x * \tilde{c}_{q_i}(n)|^2, \ 1 \leq i \leq K.$$

From Fig. 3.1(b), it is clear that the outputs corresponding to the Ramanujan sums c_3, c_5 and c_{15} are non-zero. therefore by using Corollary 3.4, we get $P = \operatorname{lcm}(3, 5, 15) = 15$.

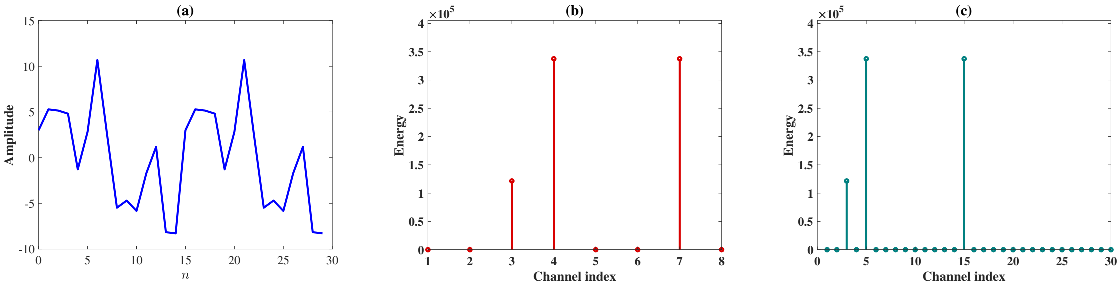


Figure 3.1: (a) Signal of length $N = 30$ with periodic components 3, 5, and 15, (b) energy outputs of the signal x using Ramanujan sums $c_1, c_2, c_3, c_5, c_6, c_{10}, c_{15}$ and c_{30}, (according to Corollary 3.4) and (c) energy outputs of the signal x using Ramanujan sums c_1 to c_{30} (according to Theorem 3.2).

On the other hand, identifying P, as described in Theorem 3.2, requires a filter bank based on Ramanujan sums $c_1, c_2, c_3, \ldots, c_{30}$, whose energy outputs are shown in Fig. 3.1(c). However, Corollary 3.4 shows that the filter bank constructed using only the Ramanujan sums $c_1, c_2, c_3, c_5, c_6, c_{10}, c_{15}$, and c_{30} is sufficient to identify P.

Corollary 3.4 provides a practical framework for identifying the period P of an input signal using a filter bank constructed from Ramanujan sums corresponding to the divisors of N. This underlines the importance of such filter banks in signal processing. Motivated by this, we study the frame properties of the collection

$$\mathcal{R}_{p,N} := \{L_{pk}c_{q_1}\}_{k=0}^{d-1} \cup \{L_{pk}c_{q_2}\}_{k=0}^{d-1} \cup \ldots \cup \{L_{pk}c_{q_K}\}_{k=0}^{d-1}, \tag{3.3}$$

which corresponds to the decimated Ramanujan filter bank shown in Fig. 3.2, where $N = pd$ for some $p, d \in \mathbb{N}$, K denotes the number of divisors of N, and $c_{q_i} \in \ell^2(\mathbb{Z}_N)$ for $1 \leq i \leq K$.

Throughout the rest of this chapter, we assume that $N = pd$ *for some* $p, d \in \mathbb{N}$. *Let* $q_1, q_2, \ldots, q_K$ *denote all the divisors of* N, *where* K *is the number of such divisors. In the following discussions, we consider the Ramanujan filter bank based on the Ramanujan sums* $\{c_{q_1}, c_{q_2}, \ldots, c_{q_K}\} \subset \ell^2(\mathbb{Z}_N)$.

3.1.2 Tight frames involving Ramanujan sums and erasures

In this subsection, we prove that $\mathcal{R}_{1,N}$, in general, and $\mathcal{R}_{2,N}$, under certain conditions, form a tight frame for $\ell^2(\mathbb{Z}_N)$ (Theorem 3.6). The same result also establishes that $\mathcal{R}_{p,N}$, for $p > 2$, does not form a frame for $\ell^2(\mathbb{Z}_N)$.

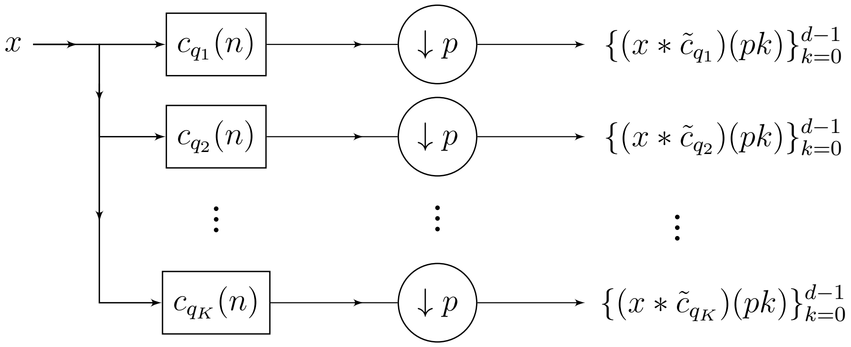


Figure 3.2: Analysis phase of a K-channel Ramanujan filter bank

We prove the following main result of this section:

Theorem 3.6. *Let the system*

$$\mathcal{R}_{p,N} := \{L_{pk}c_{q_1}\}_{k=0}^{d-1} \cup \{L_{pk}c_{q_2}\}_{k=0}^{d-1} \cup \cdots \cup \{L_{pk}c_{q_K}\}_{k=0}^{d-1} \tag{3.4}$$

be generated from a filter bank based on the Ramanujan sums $c_{q_1}, c_{q_2}, \ldots, c_{q_K}$, with decimation ratio p. Then, under the assumption $K \geq p$, the following statements hold:

- *(i) The system $\mathcal{R}_{1,N}$ forms a tight frame for $\ell^2(\mathbb{Z}_N)$ with bound N^2.*
- *(ii) For odd d, the system $\mathcal{R}_{2,N}$ forms a tight frame for $\ell^2(\mathbb{Z}_N)$ with bound $2d^2$. Moreover, in the case of even d, $\mathcal{R}_{2,N}$ does not form a frame for $\ell^2(\mathbb{Z}_N)$.*
- *(iii) For any $p > 2$, the collection $\mathcal{R}_{p,N}$ does not form a frame for $\ell^2(\mathbb{Z}_N)$.*

The condition $K \geq p$ in the hypothesis of Theorem 3.6 is justified by the fact that the total number of elements in $\mathcal{R}_{p,N}$ is Kd. According to frame theory, for $\mathcal{R}_{p,N}$ to constitute a frame for $\ell^2(\mathbb{Z}_N)$, it is necessary that $Kd \geq N$, which implies $K \geq p$.

We now simplify the outputs using the Zak transform. The *Zak transform* on $\ell^2(\mathbb{Z}_N)$ is the map $\mathcal{Z} : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_d \times \mathbb{Z}_p)$ and is defined by the formula:

$$(\mathcal{Z}x)(m,n) = \frac{1}{\sqrt{d}} \sum_{\ell=0}^{d-1} x(p\ell + n)\, e^{-2\pi i m\ell/d} \quad \text{for all } x \in \ell^2(\mathbb{Z}_N),\ m \in \mathbb{Z}_d,\ n \in \mathbb{Z}_p. \tag{3.5}$$

It is also worth noting that the Zak transform (3.5) can be obtained as a particular case of the Zak transform for LCA groups (see, for instance, Refs. [16, 96]). The Zak transform $\mathcal{Z} : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_d \times \mathbb{Z}_p)$, as defined in (3.5), is a unitary map on $\ell^2(\mathbb{Z}_N)$ and satisfies the intertwining relation:

$$\mathcal{Z}(L_{pk}x) = \frac{1}{\sqrt{d}} e^{-2\pi i k \cdot /d}\, \mathcal{Z}x \quad \text{for all } x \in \ell^2(\mathbb{Z}_N) \text{ and } k \in \mathcal{J}_d, \tag{3.6}$$

Let us now define a $K \times p$ matrix $\mathcal{U}(m)$ for each $m \in \mathcal{J}_d$, given by

$$\mathcal{U}(m) = \sqrt{d} \begin{pmatrix} \overline{\mathcal{Z}c_{q_1}(m,0)} & \overline{\mathcal{Z}c_{q_1}(m,1)} & \dots & \overline{\mathcal{Z}c_{q_1}(m,p-1)} \\ \overline{\mathcal{Z}c_{q_2}(m,0)} & \overline{\mathcal{Z}c_{q_2}(m,1)} & \dots & \overline{\mathcal{Z}c_{q_2}(m,p-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\mathcal{Z}c_{q_K}(m,0)} & \overline{\mathcal{Z}c_{q_K}(m,1)} & \dots & \overline{\mathcal{Z}c_{q_K}(m,p-1)} \end{pmatrix}. \tag{3.7}$$

We call the matrix $\mathcal{U}(m)$ the *analysis polyphase matrix.* We denote

$$A := \min_{k \in \mathcal{J}_d} \lambda_{\min}[\mathcal{U}^*(k)\mathcal{U}(k)] \quad \text{and} \quad B := \max_{k \in \mathcal{J}_d} \lambda_{\max}[\mathcal{U}^*(k)\mathcal{U}(k)]. \tag{3.8}$$

The following lemma characterizes the frame property of the collection $\mathcal{R}_{p,N}$ in terms of the rank of the polyphase matrix (3.7) and the sampling expansion of any signal $x \in \ell^2(\mathbb{Z}_N)$. Various versions of the following result can be found in [20, 45]. For the sake of simplicity, we omit its proof.

Lemma 3.7. *The following statements are equivalent:*

(*i*) *The system* $\mathcal{R}_{p,N}$*, defined in* (3.4)*, forms a frame for* $\ell^2(\mathbb{Z}_N)$ *with frame bounds* A *and* B *as given in* (3.8)*.*

(*ii*) *There exist* $g_i \in \ell^2(\mathbb{Z}_N), i \in \mathcal{I}_K$ *such that the collection* $\{L_{pk}g_i : k \in \mathcal{J}_d,\ i \in \mathcal{I}_K\}$ *is a frame for* $\ell^2(\mathbb{Z}_N)$ *and the following sampling formula holds for any* $x \in \ell^2(\mathbb{Z}_N)$:

$$x = \sum_{i \in \mathcal{I}_K} \sum_{k \in \mathcal{J}_d} (x * \tilde{c}_{q_i})(pk) L_{pk} g_i.$$

(*iii*) *There exist functions* $h_i \in \ell^2(\mathbb{Z}_N)$ *for each* $i \in \mathcal{I}_K$ *such that*

$$\begin{bmatrix} \mathcal{Z}h_1(m,\cdot) & \mathcal{Z}h_2(m,\cdot) & \cdots & \mathcal{Z}h_K(m,\cdot) \end{bmatrix} \mathcal{U}(m) = I_p, \quad m \in \mathcal{J}_d,$$

where I_p *denotes the identity matrix of order* p*.*

(*iv*) *rank* $\mathcal{U}(m) = p$ *for each* $m \in \mathcal{J}_d$*.*

The following lemma is required in the sequel.

Lemma 3.8. *Let* $m \in \{1, 2, \dots, N-1\}$*. Then, there exists a divisor* q *of* N *such that* m *can be uniquely expressed as* $m = kN/q$*, where* k *is coprime to* q*.*

Proof. The least common multiple of m and N can be written as $\operatorname{lcm}(m, N) = qm = kN$, where q and k are positive, coprime integers. Since q divides kN and $(q, k) = 1$, q divides N. Thus $m = kN/q$ is an expression of the desired form. The uniqueness of this expression follows from the fact that k/q is the unique representation of the rational number m/N in lowest terms. $\square$

Proposition 3.9. *Let $N = 2d$ be a positive even integer, where d is odd. For $m \in \mathcal{J}_N$ and a divisor q of N, the following statements hold for the Ramanujan sum $c_q \in \ell^2(\mathbb{Z}_N)$:*

(i) $\sum_{n=0}^{N-1} c_q(n)e^{-2\pi imn/N} = \begin{cases} N, & \text{if } m = \frac{kN}{q},\ (k,q) = 1, \\ 0, & \text{otherwise.} \end{cases}$

(ii) $c_q\left(\frac{N}{2}+n\right)e^{-2\pi im\left(\frac{N}{2}+n\right)/N} = c_q(n)e^{-2\pi imn/N},\ n \in \mathbb{Z}_N.$

(iii) $c_{q/2}(n) = (-1)^n c_q(n)$ *for* $n \in \mathbb{Z}_N$*, provided* q *is even.*

Proof. We first prove (i). Simplifying the left hand side and using (3.1), we have

$$\sum_{n=0}^{N-1} c_q(n)e^{-2\pi imn/N} = \sum_{n=0}^{N-1}\sum_{\substack{\ell=1\\(\ell,q)=1}}^{q} e^{2\pi in\ell/q}e^{-2\pi imn/N} = \sum_{\substack{\ell=1\\(\ell,q)=1}}^{q}\left(\sum_{n=0}^{N-1} e^{2\pi in(\ell/q-m/N)}\right). \tag{3.9}$$

Since q divides N, then we can write $N = kq$ for some $k \in \mathbb{Z}$. Then, $e^{2\pi in(\ell/q-m/N)} = e^{2\pi in(\ell k-m)/N}$. If $\ell k \neq m$ or $m \neq \ell N/q$ for any ℓ with $(\ell,q) = 1$, then $e^{2\pi in(\ell k-m)/N} \neq 1$ for $1 \leq \ell k, m \leq N-1$ since $-N < \ell k - m < N$. Therefore the inner sum is the partial sum of a geometric series, so

$$\sum_{n=0}^{N-1} e^{2\pi in(\ell k-m)/N} = \frac{1-e^{2\pi i(\ell k-m)N/N}}{1-e^{2\pi i(\ell k-m)/N}} = 0,$$

since $\ell k - m$ is an integer. Now if $\ell k = m$ or $m = \ell N/q$ for some ℓ with $(\ell,q) = 1$, then the inner sum is N. By Lemma 3.8, for a given m and q, the choice of ℓ is unique. Thus the value of the expression in (3.9) becomes: $\sum_{n=0}^{N-1} c_q(n)e^{-2\pi imn/N} = \sum_{n=0}^{N-1} 1 = N$.

Now we prove (ii). By using Lemma 3.8 for a given $m \in \{1,\ldots,N-1\}$, there exist a divisor q of N and a number k coprime to q such that $m = \frac{kN}{q}$. On simplifying the left-hand side, we have

$$c_q\left(\frac{N}{2}+n\right)e^{-2\pi im\left(\frac{N}{2}+n\right)/N} = \sum_{\substack{\ell=1\\(\ell,q)=1}}^{q} e^{2\pi i\ell(\frac{N}{2}+n)/q}e^{-2\pi im\left(\frac{N}{2}+n\right)/N} = \sum_{\substack{\ell=1\\(\ell,q)=1}}^{q} e^{2\pi i(\frac{N}{2}+n)(\frac{\ell}{q}-\frac{m}{N})}$$

$$= \sum_{\substack{\ell=1\\(\ell,q)=1}}^{q} e^{2\pi i\frac{N}{2}(\frac{\ell N-mq}{qN})}e^{2\pi in\ell/q}e^{-2\pi imn/N} = \sum_{\substack{\ell=1\\(\ell,q)=1}}^{q} e^{\pi i(\ell N/q-m)}e^{2\pi in\ell/q}e^{-2\pi imn/N}. \tag{3.10}$$

Now if q is even, then N/q being the product of odd primes is odd and k being coprime to q is odd. Therefore m being the product of two odd numbers is odd. Similarly ℓ is also odd as ℓ is coprime to q. Then for any ℓ with $(\ell, q) = 1$, the term $\ell N/q - m$ is always even being the difference of two odd numbers. Finally, (3.10) becomes

$$c_q\left(\frac{N}{2}+n\right)e^{-2\pi i m\left(\frac{N}{2}+n\right)/N} = (-1)^{m+1}c_q(n)e^{-2\pi i m n/N}.$$

Similarly, it can be easily shown that if q is odd, then for any ℓ with $(\ell, q) = 1$, the term $\ell N/q - m$ is always even being the difference of two even numbers. In this case also (3.10) becomes:

$$c_q\left(\frac{N}{2}+n\right)e^{-2\pi i m\left(\frac{N}{2}+n\right)/N} = c_q(n)e^{-2\pi i m n/N}.$$

Finally, we prove (iii). Splitting the sum $c_{q/2}(n) = \sum_{\substack{k=1\\(k,q/2)=1}}^{q/2} e^{\frac{2\pi i k n}{q/2}}$ into odd and even values of k and simplifying, we get

$$c_{q/2}(n) = \sum_{\substack{k=1\\(k,q/2)=1\\k\text{ is odd}}}^{q/2} e^{\frac{2\pi i k n}{q/2}} + \sum_{\substack{k=1\\(k,q/2)=1\\k\text{ is even}}}^{q/2} e^{\frac{2\pi i k n}{q/2}} = \sum_{\substack{k=1\\(k,q/2)=1\\k\text{ is odd}}}^{q/2} e^{\frac{2\pi i k n}{q/2}} + \sum_{\substack{k=1\\(k,q/2)=1\\k\text{ is even}}}^{q/2} e^{\frac{2\pi i n(q/2+k)}{q/2}}. \tag{3.11}$$

We first show that the set

$$B := \{k : (k, q/2) = 1 \text{ and } k \text{ is odd}\} \cup \{q/2 + k : (k, q/2) = 1 \text{ and } k \text{ is even}\}$$

consists of all the numbers coprime to q. If k is odd with $(k, q/2) = 1$ and $(k, q) = r > 1$, then, $r \mid k$ and $r \mid q$. Since k is odd and $r \mid k$, implies r is odd. Therefore, $r \mid q$ implies $r \mid q/2$ as well. This contradicts the fact that $(k, q/2) = 1$. Hence $r = 1$. Similarly, if k is even with $(k, q/2) = 1$ and $(q/2 + k, q) = r' > 1$. Then $r' \mid (q/2 + k)$ and $r' \mid q$. Also note that $q/2$ is odd since N contains a single factor of 2. As k is even and $q/2$ is odd, then $q/2 + k$ is odd. Therefore, $r' \mid (q/2 + k)$ implies r' is odd and hence $r' \mid q$ implies $r' \mid q/2$. This leads to a contradiction, since $(k, q/2) = 1$. Thus, $r' = 1$.

Since $q = 2p_1p_2 \cdots p_n$ for some primes $p_i, 1 \leq i \leq n$ other than 2, therefore $\phi(q) = \phi(q/2) = |B|$. This implies that the set B contains all the numbers coprime to q. Using this fact in (3.11), we get $c_{q/2}(n) = c_q(2n)$. We further prove $c_q(2n) = (-1)^n c_q(n)$ for which the left hand side simplifies to

$$c_q(2n) = \sum_{\substack{k=1\\(k,q)=1}}^{q} e^{2\pi i k(2n)/q} = \sum_{\substack{k=1\\(k,q)=1}}^{q} e^{2\pi i k n/q} e^{\frac{\pi i k n}{q/2}} = (-1)^n \sum_{\substack{k=1\\(k,q)=1}}^{q} e^{2\pi i n(2k-q/2)/q}. \tag{3.12}$$

We now show that the set $B' := \{2k - q/2 : (k, q) = 1\}$ contains all the numbers coprime to q. For this purpose, assume that for any given k with $(k, q) = 1$, we have $(2k - q/2, q) = \ell > 1$. Then, $\ell \mid (2k - q/2)$ and $\ell \mid q$. We can easily check that $2k - q/2$ is odd which eventually implies ℓ is odd. Therefore, $\ell \mid q$ implies $\ell \mid q/2$. As a consequence, we get $\ell \mid 2k$. Again as ℓ is odd, we get $\ell \mid k$. Thus, we arrive at a contradiction, since $(k, q) = 1$. Therefore, B' contains all the numbers coprime to q and then by using this in (3.12), we finally get $c_q(2n) = (-1)^n c_q(n)$. Hence the claim follows. □

Proposition 3.10. *Let $N = 2d$ be a positive even integer, where d is odd. Then, for any $i, j \in \mathcal{I}_K$, $n \in \{0, 1\}$, and any positive integer k such that $(k, q_i) = 1$, we have*

$$\mathcal{Z}c_{q_j}\left(\frac{kN}{q_i}, n\right) = e^{2\pi ikn/q_i} \begin{cases} \sqrt{N/2}, & \text{if } q_j = q_i, \\ (-1)^n\sqrt{N/2}, & \text{if } q_j = q_i/2 \text{ or } q_j = 2q_i, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We divide the proof into the three cases: (i) $q_j = q_i$, (ii) $q_j = q_i/2$ or $2q_i$, and (iii) $q_j \neq q_i, q_i/2, 2q_i$.

Case 1: $q_j = q_i$. By using Proposition 3.9 (ii) and then Proposition 3.9 (i), we get

$$\sum_{\ell=0}^{d-1} c_{q_i}(2\ell)e^{-2\pi i2\ell k/q_i} = \sum_{\ell=0}^{d-1} c_{q_i}(2\ell+1)e^{-2\pi i(2\ell+1)k/q_i} = N/2. \tag{3.13}$$

Consequently, by using (3.5), (3.13), and $N = 2d$, we get the following equality for $n \in \{0, 1\}$:

$$\begin{aligned} \mathcal{Z}c_{q_i}(kN/q_i, n) &= \frac{1}{\sqrt{d}}\sum_{\ell=0}^{d-1} c_{q_i}(2\ell+n)e^{-2\pi i2\ell k/q_i} \\ &= \frac{e^{2\pi ikn/q_i}}{\sqrt{d}}\sum_{\ell=0}^{d-1} c_{q_i}(2\ell+n)e^{-2\pi i(2\ell+n)k/q_i} \\ &= e^{2\pi ikn/q_i}N/(2\sqrt{d}) = e^{2\pi ikn/q_i}\sqrt{N/2}. \end{aligned}$$

Case 2: $q_j = q_i/2$ or $2q_i$. If q_i is even, then $q_i/2$ is a divisor of N. Using Proposition 3.9 (i) and (iii), we get

$$\sum_{n=0}^{N-1}(-1)^n c_{q_i/2}(n)e^{-2\pi ikn/q_i} = \sum_{n=0}^{N-1} c_{q_i}(n)e^{-2\pi ikn/q_i} = N. \tag{3.14}$$

Again, using Proposition 3.9 (i), we have $\sum_{n=0}^{N-1} c_{q_i/2}(n)e^{-2\pi ikn/q_i} = 0$. This equality and (3.14) gives

$$\sum_{\ell=0}^{d-1} c_{q_i/2}(2\ell)e^{-2\pi i2\ell k/q_i} = N/2 \quad \text{and} \quad \sum_{\ell=0}^{d-1} c_{q_i/2}(2\ell+1)e^{-2\pi i(2\ell+1)k/q_i} = -N/2. \tag{3.15}$$

Consequently, by using (3.5), (3.15), and $N = 2d$, we get the following equality for $n \in \{0,1\}$:

$$\mathcal{Z}c_{q_i/2}\left(\frac{kN}{q_i}, n\right) = \frac{e^{2\pi ikn/q_i}}{\sqrt{d}} \sum_{\ell=0}^{d-1} c_{q_i/2}(2\ell+n)e^{-2\pi i(2\ell+n)k/q_i} = (-1)^n e^{2\pi ikn/q_i}\sqrt{N/2}.$$

Also, if q_i is odd, then $2q_i$ is a divisor of N and the equality

$$\mathcal{Z}c_{2q_i}(kN/q_i, n) = (-1)^n e^{2\pi ikn/q_i}\sqrt{N/2}$$

holds similarly for $n \in \{0,1\}$.

Case 3: $q_j \neq q_i, q_i/2, 2q_i$. Using (3.5) and $N = 2d$, we get

$$\begin{aligned}
\mathcal{Z}c_{q_j}\left(\frac{kN}{q_i}, 0\right) &= \frac{1}{\sqrt{d}} \sum_{\ell=0}^{d-1} c_{q_j}(2\ell)e^{-2\pi i(2\ell)k/q_i} = \frac{1}{\sqrt{d}} \sum_{\ell=0}^{d-1} \left(\sum_{\substack{r=1 \\ (r,q_i)=1}}^{q_j} e^{2\pi i(2\ell)r/q_j} \right) e^{-2\pi i(2\ell)k/q_i} \\
&= \frac{1}{\sqrt{d}} \sum_{\substack{r=1 \\ (r,q_j)=1}}^{q_j} \left(\sum_{\ell=0}^{d-1} e^{2\pi i\left(\frac{r}{q_j}-\frac{k}{q_i}\right)(2\ell)} \right) = \frac{1}{\sqrt{d}} \sum_{\substack{r=1 \\ (r,q_j)=1}}^{q_j} \left(\sum_{\ell=0}^{d-1} e^{2\pi i(rt_1-kt_2)\ell/d} \right),
\end{aligned}$$

where $N = t_1q_j = t_2q_i$ for some positive integers t_1 and t_2. Note that the inner sum $\sum_{\ell=0}^{d-1} e^{2\pi i(rt_1-kt_2)\ell/d}$, is zero unless $rt_1 - kt_2$ is a multiple of d. However,

$$rt_1 - kt_2 = N\left(\frac{r}{q_j} - \frac{k}{q_i}\right)$$

can never be an integer since $(r, q_j) = (k, q_i) = 1$. This gives, $\mathcal{Z}c_{q_j}(kN/q_i, 0) = 0$. Similarly, we can show $\mathcal{Z}c_{q_j}(kN/q_i, 1) = 0$. This completes the proof. □

We are now ready to prove Theorem 3.6.

Proof of Theorem 3.6. First, we prove (ii). Note that for $m \in \mathcal{J}_d$, $\mathcal{U}(m)$ (defined in (3.7)) is a $K \times 2$ matrix as $p = 2$ in this case. Then for $m \in \mathcal{J}_d$, we have

$$\mathcal{U}^*(m)\mathcal{U}(m) = d \begin{pmatrix} \sum_{i=1}^{K} |\mathcal{Z}c_{q_i}(m,0)|^2 & \sum_{i=1}^{K} \mathcal{Z}c_{q_i}(m,0)\,\overline{\mathcal{Z}c_{q_i}(m,1)} \\ \sum_{i=1}^{K} \overline{\mathcal{Z}c_{q_i}(m,0)}\,\mathcal{Z}c_{q_i}(m,1) & \sum_{i=1}^{K} |\mathcal{Z}c_{q_i}(m,1)|^2 \end{pmatrix}. \tag{3.16}$$

Let $m = 0$. We show that $\mathcal{Z}c_{q_i}(0, n) = 0$ for $q_i \neq 1, 2$ and $n \in \{0, 1\}$. Indeed, we have

$$\begin{aligned}\mathcal{Z}c_{q_i}(0, n) &= \frac{1}{\sqrt{d}} \sum_{\ell=0}^{d-1} c_{q_i}(2\ell + n) = \frac{1}{\sqrt{d}} \sum_{\ell=0}^{d-1} \left(\sum_{\substack{k=1\\(k,q_i)=1}}^{q_i} e^{2\pi i(2\ell+n)k/qi} \right) \\ &= \frac{1}{\sqrt{d}} \sum_{\substack{k=1\\(k,q_i)=1}}^{q_i} \left(\sum_{\ell=0}^{d-1} e^{2\pi ikt\ell/d} \right) e^{2\pi ikn/q_i},\end{aligned}$$

where $N = tq_i$ for some positive integer t. Note that for $q_i \neq 1, 2$, the inner sum is the partial sum of a geometric series with common ratio $e^{2\pi ikt\ell/d} \neq 1$ since $kt\ell/d = kN\ell/(q_i d) = 2k\ell/q_i \notin \mathbb{Z}$ for any k with $(k, q_i) = 1$. Therefore,

$$\sum_{\ell=0}^{d-1} e^{2\pi ikt\ell/d} = \frac{1 - e^{2\pi ikt\ell}}{1 - e^{2\pi ikt\ell/d}} = 0.$$

This implies, $\mathcal{Z}c_{q_i}(0, n) = 0$ for $q_i \neq 1, 2$ and $n \in \{0, 1\}$. Also, it is easy to check that $\mathcal{Z}c_1(0, n) = \sqrt{d}$ and $\mathcal{Z}c_2(0, n) = (-1)^n\sqrt{d}$ for $n \in \{0, 1\}$. Combining everything and using (3.16), we get

$$\mathcal{U}^*(0)\mathcal{U}(0) = d \begin{pmatrix} 2d & 0 \\ 0 & 2d \end{pmatrix} = 2d^2 I_2. \tag{3.17}$$

Now let $m \in \{1, \ldots, d-1\}$ be fixed. Then by Lemma 3.8, there exists a divisor q_i of N and a number k coprime to q_i such that $m = kN/q_i$. First assume that q_i is even. Then by Proposition 3.10, we get

$$\sum_{j=1}^{K} |\mathcal{Z}c_{q_j}(kN/q_i, n)|^2 = |\mathcal{Z}c_{q_i}(kN/q_i, n)|^2 + |\mathcal{Z}c_{q_i/2}(kN/q_i, n)|^2 = N, \tag{3.18}$$

for any $n \in \{0, 1\}$ and

$$\begin{aligned}\sum_{j=1}^{K} \mathcal{Z}c_{q_j}(kN/q_i, 0)\overline{\mathcal{Z}c_{q_j}(kN/q_i, 1)} &= \mathcal{Z}c_{q_i}(kN/q_i, 0)\overline{\mathcal{Z}c_{q_i}(kN/q_i, 1)} \\ &+ \mathcal{Z}c_{q_i/2}(kN/q_i, 0)\overline{\mathcal{Z}c_{q_i/2}(kN/q_i, 1)} = N/2\, e^{-2\pi ik/q_i} + N/2(-1)e^{-2\pi ik/q_i} = 0.\end{aligned} \tag{3.19}$$

Then by using (3.16), (3.18) and (3.19), we get

$$\mathcal{U}^*(m)\mathcal{U}(m) = d \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix} = 2d^2 I_2, \;\; m \in \mathcal{J}_d.$$

This, along with (3.17) shows that $\mathcal{U}^*(m)\mathcal{U}(m)$ has the same eigenvalues for $m \in \{0, 1, \dots, d-1\}$. This implies rank $\mathcal{U}(m) = 2$ for $m \in \mathcal{J}_d$ and moreover $A = B = 2d^2$ where A and B are defined in (3.8). Consequently, $\mathcal{R}_{2,N}$ is a tight frame with frame bound $2d^2$.

In order to prove the latter claim in (ii), we first show that $\mathcal{Z}c_{q_i}(1,1) = 0$ for $i \in \mathcal{I}_K$. To this end, consider

$$\begin{aligned}\mathcal{Z}c_{q_i}(1,1) &= \frac{1}{\sqrt{d}}\sum_{\ell=0}^{d-1} c_{q_i}(2\ell+1)e^{-2\pi i\ell/d} = \frac{1}{\sqrt{d}}\sum_{\ell=0}^{d-1}\Bigg(\sum_{\substack{k=1\\(k,q_i)=1}}^{q_i} e^{2\pi i(2\ell+1)k/qi}\Bigg)e^{-2\pi i\ell/d} \\ &= \frac{1}{\sqrt{d}}\sum_{\substack{k=1\\(k,q_i)=1}}^{q_i}\left(\sum_{\ell=0}^{d-1} e^{2\pi i\left(\frac{2k}{q_i}-\frac{1}{d}\right)\ell}\right)e^{2\pi ik/q_i} = \frac{1}{\sqrt{d}}\sum_{\substack{k=1\\(k,q_i)=1}}^{q_i}\left(\sum_{\ell=0}^{d-1} e^{2\pi i(rk-1)\ell/d}\right)e^{2\pi ik/q_i},\end{aligned} \tag{3.20}$$

where $N = rq_i$ for some positive integer r. Note that the inner sum in the last term of the above equality is zero unless for some k such that $(k, q_i) = 1$, $rk - 1$ is a multiple of d. Thus, if $rk - 1 = td$ for some $t \in \mathbb{Z}$, then we get

$$\frac{k}{q_i} = \frac{td+1}{N}. \tag{3.21}$$

Since d is even, thus $td + 1$ is odd and hence $td + 1$ and N do not share any factor of 2. Also, if $s \neq 2$ is a prime such that $s \mid N$ and $s \mid td+1$, then $s \mid N = 2d$ implies $s \mid d$. This gives $s \mid td$. Consequently, we get $s \mid 1$, which is not possible. Therefore, $td + 1$ and N do not share any prime factor as well. This implies $(td + 1, N) = 1$. Then, the equality (3.21) is possible if and only if $q_i = N$ and $k = 1, d + 1$. In these cases, the common ratio $e^{2\pi i(rk-1)/d} = 1$ and hence (3.20) becomes:

$$\mathcal{Z}c_N(1,1) = \sqrt{d}\sum_{k=1,d+1} e^{2\pi ik/N} = \sqrt{d}\left(e^{2\pi i/N} + e^{2\pi i(d+1)/N}\right) = \sqrt{d}\left(e^{2\pi i/N} - e^{2\pi i/N}\right) = 0.$$

On combining everything, we get $\mathcal{Z}c_{q_i}(1,1) = 0$ for $i \in \mathcal{I}_K$. This implies that the second column of $\mathcal{U}(1)$ is zero, and hence rank $\mathcal{U}(1) = 1 \neq 2$, violating Lemma 3.7.

Now we prove (i). In this case, $p = 1$, so $\mathcal{U}(m)$ is a $K \times 1$ matrix and thus similar to the proof of (ii), we show that $\mathcal{U}^*(m)\mathcal{U}(m) = AI_1$ for $m \in \mathcal{J}_N$. where $A = \min_{k\in\mathcal{J}_N}\lambda_{\min}[\mathcal{U}^*(m)\mathcal{U}(m)]$. Let $m \in \{1, 2, \dots, N-1\}$ be fixed. Then by Lemma 3.8, we have $m = kN/q_i$ for some $q_i|N$ and $(k, q_i) = 1$. Consequently, by Proposition 3.9 (i), we get

$$\mathcal{U}^*(m)\mathcal{U}(m) = N\sum_{j=1}^{K}|\mathcal{Z}c_{q_i}(m,0)|^2 = N\sum_{j=1}^{K}\left|\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}c_{q_j}(n)e^{-2\pi ikn/q_i}\right|^2 = N^2.$$

Also, for $m = 0$, we have

$$\mathcal{U}^*(0)\mathcal{U}(0) = N\sum_{j=1}^{K} |\mathcal{Z}c_{q_j}(0,0)|^2 = N\sum_{j=1}^{K}\left|\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} c_{q_j}(n)\right|^2 = N\left|\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} c_1(n)\right|^2 = N^2,$$

where we have used the sum property of the Ramanujan sums (see Proposition 3.1(v)). On Combining, we get $A = N^2$ and therefore $\mathcal{R}_{1,N}$ is a tight frame with frame bound N^2.

Finally, we prove (iii). We first prove that $\mathcal{Z}c_{q_i}(0, j) = \mathcal{Z}c_{q_i}(0, p - j)$ for $j \in \mathcal{J}_p$ and $i \in \mathcal{I}_K$. To this end, we solve

$$\mathcal{Z}c_{q_i}(0, p-j) = \frac{1}{\sqrt{d}}\sum_{\ell=0}^{d-1} c_{q_i}(p\ell + p - j) = \frac{1}{\sqrt{d}}\sum_{\ell=0}^{d-1} c_{q_i}(p(\ell+1) - j).$$

Substituting $t = \ell + 1$ and using the fact that c_{q_i} is N-periodic, we obtain

$$\begin{aligned}\mathcal{Z}c_{q_i}(0, p-j) &= \frac{1}{\sqrt{d}}\sum_{t=1}^{d} c_{q_i}(pt - j) = \frac{1}{\sqrt{d}}\sum_{t=1}^{d-1} c_{q_i}(pt-j) + \frac{1}{\sqrt{d}}c_{q_i}(N-j)\\ &= \frac{1}{\sqrt{d}}\sum_{t=1}^{d-1} c_{q_i}(pt-j) + \frac{1}{\sqrt{d}}c_{q_i}(0-j) = \frac{1}{\sqrt{d}}\sum_{t=0}^{d-1} c_{q_i}(pt-j).\end{aligned} \tag{3.22}$$

Expanding the quantity $\sum_{t=0}^{d-1} c_{q_i}(pt-j)$ and using $(q_i - k, q_i) = 1$ for any k with $(k, q_i) = 1$, we get

$$\begin{aligned}\sum_{t=0}^{d-1} c_{q_i}(pt-j) &= \sum_{t=0}^{d-1}\sum_{\substack{k=1\\(k,q_i)=1}}^{q_i} e^{2\pi i(pt-j)k/q_i} = \sum_{\substack{k=1\\(k,q_i)=1}}^{q_i}\left(\sum_{t=0}^{d-1} e^{2\pi iptk/q_i}\right)e^{-2\pi ikj/q_i}\\ &= \sum_{\substack{k=1\\(k,q_i)=1}}^{q_i}\left(\sum_{t=0}^{d-1} e^{-2\pi ipt(q_i-k)/q_i}\right)e^{2\pi i(q_i-k)j/q_i} = \sum_{\substack{k=1\\(q_i-k,q_i)=1}}^{q_i}\left(\sum_{t=0}^{d-1} e^{-2\pi ipt(q_i-k)/q_i}\right)e^{2\pi i(q_i-k)j/q_i}\\ &= \sum_{\substack{h=1\\(h,q_i)=1}}^{q_i}\left(\sum_{t=0}^{d-1} e^{-2\pi ipth/q_i}\right)e^{2\pi ihj/q_i} = \sum_{\substack{h=1\\(h,q_i)=1}}^{q_i}\left(\sum_{t=0}^{d-1} e^{2\pi ipth/q_i}\right)e^{2\pi ihj/q_i}\\ &= \sum_{\substack{h=1\\(h,q_i)=1}}^{q_i}\left(\sum_{t=0}^{d-1} e^{2\pi i(pt+j)h/q_i}\right) = \sum_{t=0}^{d-1} c_{q_i}(pt+j),\end{aligned} \tag{3.23}$$

where we have used the fact that the sum $\sum_{t=0}^{d-1} e^{-2\pi ipth/q_i} = \sum_{t=0}^{d-1} e^{-2\pi irth/d}$ is real. By using (3.22) and (3.23), we get

$$\mathcal{Z}c_{q_i}(0, p-j) = \frac{1}{\sqrt{d}}\sum_{t=0}^{d-1} c_{q_i}(pt+j) = \mathcal{Z}c_{q_i}(0, j),\ i \in \mathcal{I}_K,\ j \in \mathcal{J}_p.$$

This shows that the j-th and $(p\text{-}j)$-th column of $\mathcal{U}(0)$ are equal for $j \in \mathcal{J}_p$. Therefore, we get

$$\operatorname{rank} \mathcal{U}(0) \leq \begin{cases} \frac{p-1}{2} + 1, & p \text{ is odd,} \\ (p/2) + 1, & p \text{ is even.} \end{cases}$$

In both cases, $\operatorname{rank} \mathcal{U}(0) < p$ since $p \neq 1, 2$, violating Lemma 3.7. Hence the claim follows. □

Example 3.11. Let $N = 6, p = 2$. Then, in view of Theorem 3.6, $d = 3$, $K = 4$ and the collection

$$\mathcal{R}_{2,6} = \{L_{2k}c_1\}_{k=0}^{2} \cup \{L_{2k}c_2\}_{k=0}^{2} \cup \{L_{2k}c_3\}_{k=0}^{2} \cup \{L_{2k}c_6\}_{k=0}^{2},$$

forms a tight frame for $\ell^2(\mathbb{Z}_6)$. The 4×2 matrix $\mathcal{U}(m)$ for $m \in \{0, 1, 2\}$ is given by:

$$\mathcal{U}(m) = \sqrt{3} \begin{pmatrix} \sum_{\ell=0}^{2} c_1(2\ell)e^{2\pi i m\ell/3} & \sum_{\ell=0}^{2} c_1(2\ell+1)e^{2\pi i m\ell/3} \\ \sum_{\ell=0}^{2} c_2(2\ell)e^{2\pi i m\ell/3} & \sum_{\ell=0}^{2} c_2(2\ell+1)e^{2\pi i m\ell/3} \\ \sum_{\ell=0}^{2} c_3(2\ell)e^{2\pi i m\ell/3} & \sum_{\ell=0}^{2} c_3(2\ell+1)e^{2\pi i m\ell/3} \\ \sum_{\ell=0}^{2} c_6(2\ell)e^{2\pi i m\ell/3} & \sum_{\ell=0}^{2} c_6(2\ell+1)e^{2\pi i m\ell/3} \end{pmatrix}.$$

After calculating these matrices, we obtain

$$\mathcal{U}(0) = \begin{pmatrix} 3 & 3 \\ 3 & -3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{U}(1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 3 & -1.5 + i\,2.5981 \\ 3 & 1.5 - i\,2.5981 \end{pmatrix}, \quad \mathcal{U}(2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 3 & -1.5 - i\,2.5981 \\ 3 & 1.5 + i\,2.5981 \end{pmatrix}.$$

It can be verified that $\mathcal{U}^*(m)\mathcal{U}(m) = 18I_2$ for $m = 0, 1, 2$. Thus, $\mathcal{R}_{2,6}$ is a tight frame with bound 18.

Example 3.12. Let $N = 8$, $p = 1$. Then, in view of Theorem 3.6, $d = 8$, $K = 4$ and the collection

$$\mathcal{R}_{1,8} = \{L_k c_1\}_{k=0}^{7} \cup \{L_k c_2\}_{k=0}^{7} \cup \{L_k c_4\}_{k=0}^{7} \cup \{L_k c_8\}_{k=0}^{7},$$

forms a tight frame for $\ell^2(\mathbb{Z}_8)$. The 4×1 matrix $\mathcal{U}(m)$ for $m \in \{0, 1, \ldots, 7\}$ is given by:

$$\mathcal{U}(m) = \sqrt{8} \begin{pmatrix} \sum_{\ell=0}^{7} c_1(\ell)e^{2\pi i m\ell/8} \\ \sum_{\ell=0}^{2} c_2(\ell)e^{2\pi i m\ell/8} \\ \sum_{\ell=0}^{7} c_4(\ell)e^{2\pi i m\ell/8} \\ \sum_{\ell=0}^{7} c_8(\ell)e^{2\pi i m\ell/8} \end{pmatrix}.$$

After calculating these matrices, we get

$$\mathcal{U}(m) = \begin{pmatrix} 8 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \, m = 0, \quad \mathcal{U}(m) = \begin{pmatrix} 0 \\ 8 \\ 0 \\ 0 \end{pmatrix}, \, m = 4,$$

$$\mathcal{U}(m) = \begin{pmatrix} 0 \\ 0 \\ 8 \\ 0 \end{pmatrix}, \, m = \{2, 6\}, \quad \mathcal{U}(m) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 8 \end{pmatrix}, \, m = \{1, 3, 5, 7\}.$$

Therefore, $\mathcal{U}^*(m)\mathcal{U}(m) = 64I_1$ for $m \in \{0, 1, \ldots, 7\}$. Thus, $\mathcal{R}_{1,8}$ is a tight frame with bound 64.

Example 3.13. Let $N = 12$, $p = 2$. Then, in view of Theorem 3.6, $d = 6$ and $K = 6$ and the collection

$$\mathcal{R}_{2,12} = \{L_{2k}c_1\}_{k=0}^{5} \cup \{L_{2k}c_2\}_{k=0}^{5} \cup \{L_{2k}c_3\}_{k=0}^{5} \cup \{L_{2k}c_4\}_{k=0}^{5} \cup \{L_{2k}c_6\}_{k=0}^{5} \cup \{L_{2k}c_{12}\}_{k=0}^{5},$$

does not form a frame for $\ell^2(\mathbb{Z}_{12})$. The 6×2 matrix $\mathcal{U}(m)$, $m \in \{0, 1, \ldots, 5\}$ is given by:

$$\mathcal{U}(m) = \sqrt{5} \begin{pmatrix} \sum_{\ell=0}^{5} c_1(2\ell)e^{2\pi i m\ell/12} & \sum_{\ell=0}^{5} c_1(2\ell+1)e^{2\pi i m\ell/12} \\ \sum_{\ell=0}^{5} c_2(2\ell)e^{2\pi i m\ell/12} & \sum_{\ell=0}^{5} c_2(2\ell+1)e^{2\pi i m\ell/12} \\ \sum_{\ell=0}^{5} c_3(2\ell)e^{2\pi i m\ell/12} & \sum_{\ell=0}^{5} c_3(2\ell+1)e^{2\pi i m\ell/12} \\ \sum_{\ell=0}^{5} c_4(2\ell)e^{2\pi i m\ell/12} & \sum_{\ell=0}^{5} c_4(2\ell+1)e^{2\pi i m\ell/12} \\ \sum_{\ell=0}^{5} c_6(2\ell)e^{2\pi i m\ell/12} & \sum_{\ell=0}^{5} c_6(2\ell+1)e^{2\pi i m\ell/12} \\ \sum_{\ell=0}^{5} c_{12}(2\ell)e^{2\pi i m\ell/12} & \sum_{\ell=0}^{5} c_{12}(2\ell+1)e^{2\pi i m\ell/12} \end{pmatrix}.$$

It can be verified that rank $\mathcal{U}(m) = 1 \neq 2$ for $m \in \{1, 3, 5\}$. Then, by Lemma 3.7, the collection $\mathcal{R}_{2,12}$ is not a frame for $\ell^2(\mathbb{Z}_{12})$.

In our analysis, we demonstrated that $\mathcal{R}_{p,N}$ for $p > 2$ does not span the full space $\ell^2(\mathbb{Z}_N)$, as it does not form a frame for $\ell^2(\mathbb{Z}_N)$. This is because, for $p > 2$, certain individual collections $\{L_{pk}c_{q_i}\}_{k \in \mathcal{J}_d}$ within $\mathcal{R}_{p,N}$ fail to span the specific subspaces of $\ell^2(\mathbb{Z}_N)$ to which they are associated. In the following section, we show that by implementing a non-uniform filter bank structure, where decimation ratios in each channel are chosen appropriately according to the properties of the Ramanujan sums and Ramanujan subspaces, the collection $\mathcal{R}_{p,N}$ can indeed be modified to form a frame for $\ell^2(\mathbb{Z}_N)$.

3.1.3 Non-uniform filter banks based on Ramanujan sums

In this subsection, we show that updating the decimation ratios of selected channels in a Ramanujan filter bank, initially designed with a fixed decimation ratio $p > 2$, yields a non-uniform configuration that forms a frame. Non-uniform filter banks are signal processing frameworks that allow variable decimation rates across channels, unlike traditional filter banks which use a uniform rate. See, [85, 104] for more details.

The need for a non-uniform filter bank arises from the fact that, for $p > 2$, certain collections within $\mathcal{R}_{p,N}$ fail to span their respective subspaces, preventing $\mathcal{R}_{p,N}$ from forming a frame for $\ell^2(\mathbb{Z}_N)$. Notably, as shown in Theorem 3.14, any two collections $\{L_{pk}c_{q_i}\}_{k=0}^{d-1}$ and $\{L_{pk}c_{q_j}\}_{k=0}^{d-1}$ with $j \neq i$ lie in distinct orthogonal subspaces of $\ell^2(\mathbb{Z}_N)$, yielding an orthogonal decomposition in terms of Ramanujan subspaces. This decomposition allows us to adjust the decimation ratios for non-spanning channels, leading to a non-uniform configuration. We demonstrate that modified form of $\mathcal{R}_{p,N}$, arising from the resulting non-uniform filter bank with $p > 2$, forms a frame for $\ell^2(\mathbb{Z}_N)$ (Theorem 3.18).

For our further analysis, we first discuss the spanning properties of the Ramanujan subspace S_q. The authors in [146, Theorem 4] proved that any $\phi(q)$ consecutive shifts of the Ramanujan sum c_q are linearly independent. In particular, the collection $\{c_q, L_1c_q, \ldots, L_{\phi(q)-1}c_q\}$ is a basis for the Ramanujan subspace S_q.

Since $c_{q_i} \in \ell^2(\mathbb{Z}_N)$, the space S_{q_i} is a $\phi(q_i)$-dimensional subspace of $\ell^2(\mathbb{Z}_N)$ for $i \in \mathcal{I}_K$. It can be observed that the subspaces S_{q_i} and S_{q_j} are orthogonal for $q_i, q_j | N$ and $q_i \neq q_j$ due to the orthogonality property of the Ramanujan sums (see, Proposition 3.1(iv)). Now by using Gauss's theorem on sums of Euler's totient functions, we have $\dim \ell^2(\mathbb{Z}_N) = N = \sum_{i=1}^{K} \phi(q_i) = \dim\, S_{q_1} + \dim\, S_{q_2} + \cdots + \dim\, S_{q_K}$. This gives the orthogonal decomposition

$$\ell^2(\mathbb{Z}_N) = S_{q_1} \oplus S_{q_2} \oplus \cdots \oplus S_{q_K}.$$

As a consequence, any $x \in \ell^2(\mathbb{Z}_N)$ admits the representation

$$x = \sum_{i=1}^{K} \sum_{\ell=0}^{\phi(q_i)-1} \alpha_{i,\ell}\, c_{q_i}(\cdot - \ell), \tag{3.24}$$

for suitable coefficients $\alpha = \{\alpha_{i,\ell} : 0 \leq \ell < \phi(q_i) - 1, i \in \mathcal{I}_K\}$. This expansion serves as the foundation of the Ramanujan Periodic Transform (RPT) proposed in [147]. The RPT is well-suited for extracting the periodic structure of finite-length signals and identifying hidden periods. It is also useful for denoising periodic signals by utilizing their underlying periodic information captured in (3.24).

The next result establishes that the space $\ell^2(\mathbb{Z}_N)$ admits another orthogonal decomposition into Ramanujan subspaces generated by even consecutive shifts of Ramanujan sums corresponding to the divisors of N. To this end, for each pair of divisors p and q_i of N, where $i \in \mathcal{I}_K$, we define the Ramanujan subspace

$$S_{p,q_i} := \operatorname{span}\left\{L_{pk}c_{q_i} : 0 \leq k \leq \phi(q_i) - 1\right\}. \tag{3.25}$$

Note that for $p = 1$, $S_{1,q_i} = S_{q_i}$ for $i \in \mathcal{I}_K$. For $p = 2$, the following result holds.

Theorem 3.14. *Let p_i, $i \in \mathcal{I}_n$, be primes and α_i, $i \in \mathcal{I}_n$, be positive integers such that $N = 2p_1^{\alpha_1}p_2^{\alpha_2}\dots p_n^{\alpha_n}, p_i \neq 2$. Let S_{2,q_i} denote the Ramanujan subspace corresponding to the divisor q_i of N, for $i \in \mathcal{I}_K$, as defined in (3.25). Then, for any $\ell \in \mathbb{Z}$ and $i \in \mathcal{I}_K$, the collection $\{L_{2k}c_{q_i}\}_{k=\ell}^{\ell+\phi(q_i)-1}$ is a basis of S_{2,q_i} and thus dim $S_{2,q_i} = \phi(q_i)$. Moreover, the system*

$$\mathcal{B}_N := \{L_{2k}c_{q_1}\}_{k=0}^{\phi(q_1)-1} \cup \{L_{2k}c_{q_2}\}_{k=0}^{\phi(q_2)-1} \cup \{L_{2k}c_{q_3}\}_{k=0}^{\phi(q_3)-1} \cup \dots \cup \{L_{2k}c_{q_K}\}_{k=0}^{\phi(q_K)-1}$$

forms a basis for $\ell^2(\mathbb{Z}_N)$ and the orthogonal decomposition of $\ell^2(\mathbb{Z}_N)$ is given by:

$$\ell^2(\mathbb{Z}_N) = S_{2,q_1} \oplus S_{2,q_2} \oplus \cdots \oplus S_{2,q_K}.$$

Remark 3.15. The collection $\mathcal{B}_N$ is written as a union of shifted Ramanujan sums for notational convenience. However, we implicitly view each vector $L_{2k}c_{q_i}$ as a block vector of the form $(0, \dots, L_{2k}c_{q_i}, \dots, 0)$, where the nonzero component appears in the ith summand of the orthogonal decomposition $\ell^2(\mathbb{Z}_N) = S_{2,q_1} \oplus S_{2,q_2} \oplus \cdots \oplus S_{2,q_K}$. From now on, all such unions in an orthogonal decomposition of a Hilbert space will be implicitly interpreted as block-wise unions within the corresponding direct sum decomposition.

The following lemma, concerning frames and bases in orthogonal subspaces, will be used in the sequel. While the idea that the union of frames (or bases) in mutually orthogonal subspaces yields a frame (or basis) for the direct sum is discussed in [151, Section 5.1] and [83, Chapter 1, page 12], we include a precise formulation here for completeness and omit the proof for brevity.

Lemma 3.16. *Let $\{\mathcal{H}_i\}_{i\in\mathcal{I}_J}$ be mutually orthogonal subspaces of a finite-dimensional Hilbert space $\mathcal{K}$, i.e., $\langle x,y\rangle_{\mathcal{K}} = 0$ for all $x \in \mathcal{H}_i$, $y \in \mathcal{H}_j$ with $i \neq j$. Assume that $\mathcal{K} = \bigoplus_{i\in\mathcal{I}_J} \mathcal{H}_i$, and for each $i \in \mathcal{I}_J$, let $\mathcal{D}_i$ be a finite collection of vectors in $\mathcal{H}_i$.*

Then, the union $\mathcal{D} := \bigcup_{i\in\mathcal{I}_J} \mathcal{D}_i$ forms a frame (respectively, a basis) for $\mathcal{K}$ if and only if each $\mathcal{D}_i$ is a frame (respectively, a basis) for $\mathcal{H}_i$. Moreover, $\mathcal{D}$ is a tight frame for $\mathcal{K}$ if and only if each $\mathcal{D}_i$ is a tight frame for $\mathcal{H}_i$ with the same frame bound.

We now prove Theorem 3.14. The arguments of its proof are adopted from [146, Theorem 4].

Proof of Theorem 3.14. Let q_i, for $i \in \mathcal{I}_K$, be fixed. We begin by proving the basis part of S_{2,q_i}. For a given $\ell \in \mathbb{Z}$, let $M^\ell_{q_i}$ denote the matrix formed by the elements of the collection $\{L_{2k}c_{q_i}\}_{k=\ell}^{\ell+\phi(q_i)-1}$, that is, $M^\ell_{q_i} = \begin{bmatrix} L_{2\ell}c_{q_i} & L_{2(\ell+1)}c_{q_i} & \dots & L_{2(\ell+\phi(q_i)-1)}c_{q_i}\end{bmatrix}$. Then $M^\ell_{q_i}$ has the form:

$$M^\ell_{q_i} = \begin{bmatrix} U_{q_i}V^\ell_{q_i} \\ \hline U_{q_i}V^\ell_{q_i} \\ \hline \vdots \\ \hline U_{q_i}V^\ell_{q_i} \end{bmatrix}_{N\times\phi(q_i)} \qquad (N/q_i \,\text{times}),$$

where

$$U_{q_i} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ e^{2\pi i/q_i} & e^{2\pi i k_2/q_i} & \dots & e^{2\pi i k_{\phi(q_i)}/q_i} \\ e^{2\pi i 2k_1/q_i} & e^{2\pi i 2k_2/q_i} & \dots & e^{2\pi i 2k_{\phi(q_i)}/q_i} \\ \vdots & \vdots & \ddots & \vdots \\ e^{2\pi i(q_i-1)k_1/q_i} & e^{2\pi i(q_i-1)k_2/q_i} & \dots & e^{2\pi i(q_i-1)k_{\phi(q_i)}/q_i} \end{bmatrix}_{q_i\times\phi(q_i)} \tag{3.26}$$

and

$$V^\ell_{q_i} = \begin{bmatrix} e^{-2\pi i 2k_1\ell/q_i} & e^{-2\pi i 2k_1(\ell+1)/q_i} & \dots & e^{-2\pi i 2k_1(\ell+\phi(q_i)-1)/q_i} \\ e^{-2\pi i 2k_2\ell/q_i} & e^{-2\pi i 2k_2(\ell+1)/q_i} & \dots & e^{-2\pi i 2k_2(\ell+\phi(q_i)-1)/q_i} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-2\pi i 2k_{\phi(q_i)}\ell/q_i} & e^{-2\pi i 2k_{\phi(q_i)}(\ell+1)/q_i} & \dots & e^{-2\pi i 2k_{\phi(q_i)}(\ell+\phi(q_i)-1)/q_i} \end{bmatrix}_{\phi(q_i)\times\phi(q_i)},$$

where $k_j \in \mathbb{Z}_{q_i}$ with $1 \leq j \leq \phi(q_i)$ is such that $(k_j, q_i) = 1$. It can be observed that U_{q_i} is a row-Vandermonde matrix with distinct elements in its second row. Indeed, if $e^{-2\pi i\ell_1/q_i} = e^{-2\pi i\ell_2/q_i}$ for $1 \leq \ell_1 < \ell_2 < \phi(q_i)$, then $e^{-2\pi i(\ell_1-\ell_2)/q_i} = 1$. This gives $\ell_1 - \ell_2 =$

mq_i for some integer m. Since $|\ell_1 - \ell_2| < \phi(q_i)$, the last equality is possible only if $m = 0$, which contradicts $\ell_1 \neq \ell_2$. Therefore, each element in the second row of U_{q_i} is distinct, and hence U_{q_i} has rank $\phi(q_i)$. Also notice that $V_{q_i}^{\ell}$ can be written in the form: $V_{q_i}^{\ell} = DV_{q_i}^{0}$, where D is a diagonal matrix with the first column of $V_{q_i}^{\ell}$ as its diagonal entries. We now prove that the column-Vandermonde matrix $V_{q_i}^{0}$ has distinct elements in its second column. Indeed, if $e^{-2\pi i 2k_j/q_i} = e^{-2\pi i 2k_m/q_i}$ for $1 \leq k_j < k_m < q_i$, then $e^{-2\pi i 2(k_j - k_m)/q_i} = 1$. This gives $2(k_j - k_m) = nq_i$ for some $n \in \mathbb{Z}$.

If $q_i = 2$, then the equality $2(k_j - k_m) = nq_i$ gives $(k_j - k_m) = n$, which is possible only when $n = 0$ since $|k_j - k_m| < q_i$. If $q_i \neq 2$, then since $N = 2p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}, p_i \neq 2$ implies that q_i is odd. Consequently, $k_j - k_m = nq_i/2$ with $(q_i, 2) = 1$. Again, this is possible only when $n = 0$. Thus, both the cases leads to the equality $k_j = k_m$, which is a contradiction. Therefore each element in the second column of $V_{q_i}^{0}$ is distinct and hence $V_{q_i}^{0}$ has rank $\phi(q_i)$. Since D is a diagonal matrix with non-zero elements on the diagonal, $V_{q_i}^{\ell}$ has rank $\phi(q_i)$. Consequently, the product $U_{q_i}V_{q_i}^{\ell}$ has rank $\phi(q_i)$. Since each block in $M_{q_i}^{\ell}$ is a duplicate of the first block, increasing blocks does not increase the number of linearly independent rows in the matrix. Thus, the total number of linearly independent rows in $M_{q_i}^{\ell}$ remains $\phi(q_i)$, and consequently, $M_{q_i}^{\ell}$ has rank $\phi(q_i)$. In particular, for $\ell = 0$, the collection $\{L_{2k}c_{q_i}\}_{k=0}^{\phi(q_i)-1}$ is linearly independent and hence $\dim S_{2,q_i} = \phi(q_i)$. Since shifts are interpreted modulo q_i, the space S_{2,q_i} is invariant under shifts and hence $\{L_{2k}c_{q_i}\}_{k=\ell}^{\ell+\phi(q_i)-1} \subset S_{2,q_i}$. Consequently, $\{L_{2k}c_{q_i}\}_{k=\ell}^{\ell+\phi(q_i)-1}$ is a basis for S_{2,q_i} for any $q_i|N$ and $\ell \in \mathbb{Z}$.

We now turn to the latter claim. The fact that $\mathcal{B}_N$ forms a basis follows directly from the previous discussion together with Lemma 3.16. The orthogonal decomposition follows by arguments analogous to those used for the case $p = 1$. This completes the proof. □

Remark 3.17. Whenever $N = 2^m$ for some m, it can be shown that the collection $\mathcal{B}_{1,N}$ forms an orthogonal basis for $\ell^2(\mathbb{Z}_N)$, and the collection

$$\left\{\frac{1}{\sqrt{N\phi(1)}}L_k c_{2^0}\right\}_{k=0}^{\phi(1)-1} \cup \left\{\frac{1}{\sqrt{N\phi(2)}}L_k c_{2^1}\right\}_{k=0}^{\phi(2)-1} \cup \cdots \cup \left\{\frac{1}{\sqrt{N\phi(2^m)}}L_k c_{2^m}\right\}_{k=0}^{\phi(2^m)-1}$$

forms an orthonormal basis for $\ell^2(\mathbb{Z}_N)$. Given the orthogonal decomposition of $\ell^2(\mathbb{Z}_N)$, it is enough to show that the collection $\{L_k c_{2^j}\}_{k=0}^{\phi(2^j)-1}$ for $0 \leq \ell \leq m$ forms an orthogonal basis of $S_{1,2^j}$. The proof of this fact can be found in [146, Lemma 2] and orthonormality follows from the norm equality: $\|c_{2^\ell}\|_{\ell^2(\mathbb{Z}_N)}^2 = \sum_{n=0}^{N-1} c_{2^\ell}^2(n) = \frac{N}{2^\ell}2^\ell\phi(2^\ell) = N\phi(2^\ell),\ 0 \leq \ell \leq m$.

We now move our discussion to non-uniform filter banks. The following result demonstrates that for a prime divisor $p > 2$ of N, a non-uniform filter bank constructed using Ramanujan sums $c_{q_i}, i \in \mathcal{I}_K$, exhibits frame properties under specific conditions. Specifically, when the decimation ratio is set to p for c_{q_i} if $p \nmid q_i$ or $p > q_i$, and to 1 if $p \mid q_i$ and $p \leq q_i$, the collection $\mathcal{R}_{p,N}$ is modified to form a frame for $\ell^2(\mathbb{Z}_N)$.

Theorem 3.18. *Let p be a prime divisor of N, and let $\{q_1, q_2, \dots, q_K\}$ denote the set of all positive divisors of N. For $p > 2$ and $r \in \{1, 2\}$, associate to each $i \in \mathcal{I}_K$ the integer*

$$p_i := \begin{cases} p, & \text{if } q_i \notin \mathfrak{D}_p, \\ r, & \text{if } q_i \in \mathfrak{D}_p, \end{cases}$$

where $\mathfrak{D}_p := \{q_i \mid N : p \mid q_i \text{ and } p \leq q_i\}$. Consider the system

$$\mathcal{R}^r_{p,N} := \{L_{p_1 k} c_{q_1}\}_{k \in \mathcal{J}_{N/p_1}} \cup \{L_{p_2 k} c_{q_2}\}_{k \in \mathcal{J}_{N/p_2}} \cup \dots \cup \{L_{p_K k} c_{q_K}\}_{k \in \mathcal{J}_{N/p_K}}, \tag{3.27}$$

generated by a non-uniform Ramanujan filter bank. The following assertions hold:

(i) *For $r = 1$, the system $\mathcal{R}^1_{p,N}$ forms a frame for $\ell^2(\mathbb{Z}_N)$ for all positive integers N.*

(ii) *For $r = 2$, assume $N = 2p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ with $p_i \neq 2$ for all $i \in \mathcal{I}_n$. Then, the system $\mathcal{R}^2_{p,N}$ is a frame for $\ell^2(\mathbb{Z}_N)$.*

To prove Theorem 3.18, we require the following supporting result, which characterizes the set $\mathfrak{D}_p$ as the collection of those divisors q_i of N for which the associated Ramanujan sum c_{q_i} requires an update in its decimation ratio.

Proposition 3.19. *Let p be a prime divisor of N and let $\mathcal{Q}_{q_i}$ be the matrix formed by the columns $\{L_{pk} c_{q_i}\}_{k \in \mathcal{J}_d}$, i.e., $\mathcal{Q}_{q_i} = \begin{bmatrix} c_{q_i} & L_p c_{q_i} & L_{2p} c_{q_i} & \cdots & L_{p(d-1)} c_{q_i} \end{bmatrix}$. Then,*

$$\operatorname{rank} \mathcal{Q}_{q_i} = \begin{cases} \phi(q_i), & \text{if } p \nmid q_i \text{ or } p > q_i, \\ \phi(q_i/p), & \text{if } p \mid q_i \text{ and } p \leq q_i. \end{cases}$$

Proof. Let $q_i, i \in \mathcal{I}_K$, be fixed. Consider the column-Vandermonde matrix

$$W_{q_i} = \begin{bmatrix} 1 & e^{-2\pi i p k_1/q_i} & \cdots & e^{-2\pi i (d-1) p k_1/q_i} \\ 1 & e^{-2\pi i p k_2/q_i} & \cdots & e^{-2\pi i (d-1) p k_2/q_i} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-2\pi i p k_{\phi(q_i)}/q_i} & \cdots & e^{-2\pi i (d-1) p k_{\phi(q_i)}/q_i} \end{bmatrix}_{\phi(q_i) \times d}, \tag{3.28}$$

where $k_j \in \mathbb{Z}_{q_i}$, with $1 \leq j \leq \phi(q_i)$, are such that $(k_j, q_i) = 1$ for each j. Then, it can be observed that the matrix $\mathcal{Q}_{q_i}$ can be written as follows:

$$\mathcal{Q}_{q_i} = \begin{bmatrix} U_{q_i} W_{q_i} \\ \hline U_{q_i} W_{q_i} \\ \hline \vdots \\ \hline U_{q_i} W_{q_i} \end{bmatrix}_{N \times d} \qquad (N/q_i \,\text{times}),$$

where U_{q_i} is the matrix defined in (3.26).

We divide the remaining proof into two cases depending on whether $q_i \in \mathfrak{D}_p$ or $q_i \notin \mathfrak{D}_p$.

Case 1: $q_i \notin \mathfrak{D}_p$. In this case, we show that all entries in the second column of W_{q_i} are distinct. Suppose, for contradiction, that $e^{-2\pi i p k_j / q_i} = e^{-2\pi i p k_m / q_i}$ for some $1 \leq j < m \leq \phi(q_i)$. This implies $e^{-2\pi i p (k_j - k_m)/q_i} = 1$, and hence $p(k_j - k_m) = n q_i$ for some integer n. This gives $k_j - k_m = \frac{n q_i}{p}$. Since p is prime, and $p \nmid q_i$ or $p > q_i$, we have $(p, q_i) = 1$. Therefore, $n q_i / p \in \mathbb{Z}$ only if $p \mid n$, which forces $n = 0$ because $|k_j - k_m| < q_i$. Thus, $k_j = k_m$, a contradiction. Hence, all entries in the second column of W_{q_i} are distinct. Using the same argument as in the proof of Theorem 3.14, it follows that rank $\mathcal{Q}_{q_i} = \phi(q_i)$ in this case.

Case 2: $q_i \in \mathfrak{D}_p$. Let $q_i = p \cdot r$. Then the entries in the second column of the Vandermonde matrix W_{q_i} are of the form $e^{-2\pi i p k_j / q_i} = e^{-2\pi i k_j / r}, \quad$ where $(k_j, q_i) = 1, \quad 1 \leq j \leq \phi(q_i)$. Since $(k_j, q_i) = 1$ and $q_i = p \cdot r$, it follows that $(k_j, r) = 1$ as well. Consequently, the values $e^{-2\pi i k_j / r}, \quad 1 \leq j \leq \phi(q_i)$, are $\phi(r)$ distinct rth roots of unity corresponding to residue classes coprime to r. Therefore, the second column of W_{q_i} contains only $\phi(r)$ distinct values, and so rank $W_{q_i} = \phi(r)$. As a result, the matrix $\mathcal{Q}_{q_i}$ also has rank $\phi(r) = \phi(q_i/p)$, as claimed. □

Finally, we prove Theorem 3.18.

Proof of Theorem 3.18. We first prove (i). Consider the case when $q_i \in \mathfrak{D}_p$ and $r = 1$. Then the corresponding individual collection $\{L_k c_{q_i}\}_{k=0}^{N-1}$ contains $\phi(q_i)$ consecutive elements since $N \geq \phi(q_i)$ for any $i \in \mathcal{I}_K$. Then by [146, Theorem 4], the collection $\{L_k c_{q_i}\}_{k=0}^{N-1}$ contains a basis of S_{1,q_i} and hence forms a frame for S_{1,q_i}. Next, consider the case when $q_i \notin \mathfrak{D}_p$ and the decimation ratio is p. Since S_{1,q_i} is invariant under shifts, thus each of the individual collection $\{L_{pk}c_{q_i}\}_{k=0}^{d-1}$ for any p lies inside S_{1,q_i}. Furthermore, $\{L_{pk}c_{q_i}\}_{k=0}^{d-1}$ contains a basis of S_{1,q_i} since rank $\mathcal{Q}_{q_i} = \phi(q_i) = \dim S_{1,q_i}$ and hence forms a frame for S_{1,q_i}. Thus for each $i \in \mathcal{I}_K$, the corresponding collection from the given non-uniform filter bank forms a frame for S_{1,q_i}. Then by using the orthogonal decomposition of $\ell^2(\mathbb{Z}_N)$ in terms of Ramanujan subspaces S_{1,q_i} for $i \in \mathcal{I}_K$ and Lemma 3.16, it follows that $\mathcal{R}_{p,K}^1$ forms a frame for $\ell^2(\mathbb{Z}_N)$.

The proof of (ii) follows analogously by using Theorem 3.14 and Lemma 3.16. This completes the proof. □

Example 3.20. Let $N = 12$ and choose $p = 3$, so that $d = 4$. In this case, the set $\mathfrak{D}_3$, as defined in Theorem 3.18, is given by $\mathfrak{D}_3 = \{3, 6, 12\}$. Then, by Theorem 3.18, the collection

$$\mathcal{R}_{3,12}^1 = \{L_{3k}c_1\}_{k=0}^{3} \cup \{L_{3k}c_2\}_{k=0}^{3} \cup \{L_k c_3\}_{k=0}^{11} \cup \{L_{3k}c_4\}_{k=0}^{3} \cup \{L_k c_6\}_{k=0}^{11} \cup \{L_k c_{12}\}_{k=0}^{11}$$

generated from a non-uniform filter bank with Ramanujan sums c_1, c_2, c_3, c_4, c_6, and c_{12}, and with respective decimation ratios 3, 3, 1, 3, 1, 1, forms a frame for $\ell^2(\mathbb{Z}_{12})$.

Remark 3.21. It is worth noting that a non-uniform filter bank based on Ramanujan sums can also be constructed when d is even, such that the resulting collection $\mathcal{R}_{2,N}$ can be modified to form a frame for $\ell^2(\mathbb{Z}_N)$. In particular, consider $p = 2$ and a divisor $q_i \mid N$ satisfying $2 \mid q_i$ but $2^2 \nmid q_i$. In this case, it can be observed that $\varphi(q_i) = \varphi\left(q_i/2\right)$, which implies that the decimation ratio for c_{q_i} need not be modified, as the matrix $\mathcal{Q}_{q_i}$ already has rank $\varphi(q_i)$. Therefore, the set $\mathfrak{D}_2$ consists precisely of those divisors $q_i \mid N$ for which $2^2 \mid q_i$. With this characterization of $\mathfrak{D}_2$, a non-uniform filter bank can be constructed.

3.1.4 Erasures in the context of Ramanujan sums

In this subsection, we investigate the conditions for the robustness of the filter bank and the associated frame systems, $\mathcal{R}_{p,N}$, under erasures in the context of Ramanujan sums. *Filter bank erasures* refer to the removal of one or more channels from a filter bank when certain channels become noisy and fail to preserve the original signal's information. Many researchers have studied the effect of such erasures on filter banks and obtained conditions under which the underlying collection $\{L_{pk}\tilde{h}_j\}_{k\in\mathcal{J}_d}^{j\in\mathcal{I}_s\setminus\{j_1,j_2,\ldots,j_n\}}$ (after removing the channels corresponding to $h_{j_1}, h_{j_2}, \ldots, h_{j_n}$) remains a frame, where $h_i, i \in \mathcal{I}_s$ are the signals representing the channels of the given filter bank. When this condition holds, the filter bank is said to be *robust to n filter bank erasures.* See, for example, [34, 86, 102].

In practical scenarios, it may happen that a subset of frame elements $\{L_{pk}\tilde{h}_i\}_{(k,i)\in\mathcal{I}}$ is lost due to erasures, where $\mathcal{I}$ denotes the corresponding index set and $\#_{\mathcal{I}} = e$. For stable reconstruction in the presence of such e erasures, it is essential that the remaining collection $\{L_{pk}\tilde{h}_i\}_{(k,i)\in\mathcal{I}^c}$ still forms a frame. If this condition holds, the original frame $\{L_{pk}\tilde{h}_i\}_{(k,i)\in\mathcal{I}\cup\mathcal{I}^c}$ is said to be *robust to e erasures.* It is shown in [70] that uniform tight frames are robust to 1 erasure, whereas general tight frames may not possess this property. In our work, we show that the tight frames $\mathcal{R}_{1,N}$ and $\mathcal{R}_{2,N}$ are robust to 1 erasure, despite not being uniform. Furthermore, we establish conditions under which these tight frames are robust to 2 erasures.

3.1.4.1 Filter bank erasures

Let $\mathcal{U}_j(m)$, for $m \in \mathcal{J}_d$ and $j \in \mathcal{I}_K$, denote the analysis polyphase matrix obtained by removing the q_j-th Ramanujan sum c_{q_j} from the filter bank based on the Ramanujan sums $c_{q_1}, c_{q_2}, \ldots, c_{q_K}$. This is a $(K-1)\times p$ matrix obtained by removing the j-th row from $\mathcal{U}(m)$. The following result provides an equivalent condition under which the filter bank based on the Ramanujan sums $c_{q_1}, c_{q_2}, \ldots, c_{q_K}$ is robust to 1 filter bank erasure, under the assumption that it forms a tight frame.

Proposition 3.22. *Let the collection $\mathcal{R}_{p,N}$ (defined in (3.4)) be a tight frame for $\ell^2(\mathbb{Z}_N)$ with frame bound A. Then, the filter bank based on Ramanujan sums $c_{q_1}, c_{q_2}, \ldots, c_{q_K}$ is robust to 1 filter bank erasure if and only if, for all $m \in \mathcal{J}_d$ and $j \in \mathcal{I}_K$, the following condition holds:*

$$1 - \frac{d}{A}\left(|\mathcal{Z}c_{q_j}(m,0)|^2 + |\mathcal{Z}c_{q_j}(m,1)|^2 + \cdots + |\mathcal{Z}c_{q_j}(m,p-1)|^2\right) \neq 0.$$

Proof. By Lemma 3.7, it is clear that the filter bank, after the removal of the channel corresponding to c_{q_j}, exhibits a frame only if rank $\mathcal{U}_j(m) = p$. We therefore show that rank $\mathcal{U}_j(m)^*\mathcal{U}_j(m) =$ rank $\mathcal{U}_j(m) = p$ for $j \in \mathcal{I}_K$. Note that, for any matrix M with rows $R_i, i \in \mathcal{I}_L$, we can write $M^*M = \sum_{i=1}^{L} R_i^* R_i$. Thus, if $C_{q_i}(m)$ denotes the i-th row of $\mathcal{U}(m)$, i.e.,

$$C_{q_i}(m) = \sqrt{d}\left[\overline{\mathcal{Z}c_{q_i}(m,0)} \quad \overline{\mathcal{Z}c_{q_i}(m,0)} \quad \cdots \quad \overline{\mathcal{Z}c_{q_i}(m,p-1)}\right], \quad m \in \mathcal{J}_d,$$

then for $k \in \mathcal{I}_K$ and $m \in \mathcal{J}_d$, we have

$$\mathcal{U}_j(m)^*\mathcal{U}_j(m) = \sum_{i=1}^{p} C_{q_i}(m)^*C_{q_i}(m) - C_{q_j}(m)^*C_{q_j}(m) \quad = \mathcal{U}(m)^*\mathcal{U}(m) - C_{q_j}(m)^*C_{q_j}(m).$$

Taking the inverse on both sides of the above equation and using the identity $(P-QS)^{-1} = P^{-1} + P^{-1}Q(1-SP^{-1}Q)^{-1}SP^{-1}$ along with $\mathcal{U}^*(m)\mathcal{U}(m) = AI_p$ for $m \in \mathcal{J}_d$, we get

$$\begin{aligned}(\mathcal{U}_j(m)^*\mathcal{U}_j(m))^{-1} = (\mathcal{U}(m)^*\mathcal{U}(m) - C_{q_j}(m)^*C_{q_j}(m))^{-1} &= \big(AI_p - C_{q_j}(m)^*C_{q_j}(m)\big)^{-1} \\ &= \frac{1}{A}I_p + \frac{1}{A}I_pC_{q_j}(m)^*\left(1 - C_{q_j}(m)\frac{1}{A}I_pC_{q_j}(m)^*\right)^{-1}C_{q_j}(m)\frac{1}{A}I_p \\ &= \frac{1}{A}I_p + \frac{1}{A^2}\left(1 - \frac{1}{A}C_{q_j}(m)C_{q_j}(m)^*\right)^{-1}C_{q_j}(m)^*C_{q_j}(m),\end{aligned} \tag{3.29}$$

$j \in \mathcal{I}_K$ and $m \in \mathcal{J}_d$. Note that $\mathcal{U}_j(m)^*\mathcal{U}_j(m)$ is invertible if and only if $1-\frac{1}{A}C_{q_j}(m)C_{q_j}(m)^* \neq 0$, which is equivalent to $1 - \frac{d}{A}\sum_{n=0}^{p-1}|\mathcal{Z}c_{q_j}(m,n)|^2 \neq 0$ for $j \in \mathcal{I}_K$ and $m \in \mathcal{J}_d$. Hence the claim follows. □

We have the following result.

Theorem 3.23. *The filter bank based on Ramanujan sums $c_{q_1}, c_{q_2}, \ldots, c_{q_K}$ is not robust to one filter bank erasure for any decimation ratio.*

Proof. Note that for $p > 2$, and for $p = 2$ with even d, $\mathcal{R}_{p,N}$, as shown in Theorem 3.6, does not form a frame for $\ell^2(\mathbb{Z}_N)$, and hence the corresponding filter bank is trivially not robust to one filter bank erasure for these cases.

Let $p = 2$ with d odd. Note that for $q_i = 1$ and $m = 0$, we have

$$|\mathcal{Z}c_1(0,0)|^2 + |\mathcal{Z}c_1(0,1)|^2 = \frac{1}{d}\Big(\big|\sum_{\ell=0}^{d-1} c_1(2\ell)\big|^2 + \big|\sum_{\ell=0}^{d-1} c_1(2\ell+1)\big|^2\Big) = \frac{2d^2}{d} = 2d = N,$$

where we have used $c_1 = (1, 1, \ldots, 1)$. The above equation implies

$$N\big(1 - \frac{d}{2d^2}\sum_{n=0}^{1}|\mathcal{Z}c_1(0,n)|^2\big) = 0.$$

Similarly, for $p = 1, q_i = 1$, and $m = 0$, we have

$$|\mathcal{Z}c_1(0,0)|^2 = \frac{1}{N}\big|\sum_{\ell=0}^{N-1} c_1(\ell)\big|^2 = \frac{N^2}{N} = N.$$

Then by Proposition 3.22, the corresponding filter banks are not robust 1 filter bank erasure for $p = 2$ with d odd, and for $p = 1$. On combining everything, the claim follows. ☐

Theorem 3.23 suggests that filter banks associated with the tight frames $\mathcal{R}_{1,N}$ and $\mathcal{R}_{2,N}$ do not allow perfect reconstruction after the erasure of even one channel. This raises the question: if instead of one channel, several vectors are deleted from $\mathcal{R}_{1,N}$ or $\mathcal{R}_{2,N}$, does the remaining collection still retain its frame properties? We address this question in the following subsection.

3.1.4.2 Frame erasures

In this subsection, we prove that the tight frames $\mathcal{R}_{1,N}$ and $\mathcal{R}_{2,N}$ are robust to 1 erasure. Moreover, we have also obtained the conditions under which tight frames $\mathcal{R}_{1,N}$ and $\mathcal{R}_{2,N}$ are robust to 2 erasures.

Theorem 3.24. *Let $N \geq 2$. Then, the following statements hold:*

- *(i) The system $\mathcal{R}_{1,N}$ is robust to 1 erasure.*
- *(ii) For odd d, the system $\mathcal{R}_{2,N}$ is robust to 1 erasure.*

Proof. We begin by proving part (i). For any $(\ell, j) \in \mathcal{J}_N \times \mathcal{I}_K$, define

$$\mathcal{R}_{1,N}^{(\ell,j)} := \mathcal{R}_{1,N} \setminus \{L_\ell c_{q_j}\}.$$

The system $\mathcal{R}_{1,N}$ is robust to one erasure if and only if $\mathcal{R}_{1,N}^{(\ell,j)}$ forms a frame for $\ell^2(\mathbb{Z}_N)$ for every such index pair (ℓ, j). By Lemma 3.16, it suffices to verify that the collection $\{L_k c_{q_j}\}_{\substack{k\in\mathcal{J}_N \\ k\neq\ell}}$ forms a frame for the subspace S_{1,q_j}, since for all $i \neq j$, the collections $\{L_k c_{q_i}\}_{k\in\mathcal{J}_N}$ continue to be frames for S_{1,q_i} by Theorem 3.6 and Lemma 3.16.

Observe that for any $N \geq 2$, we have $N - 1 \geq \phi(q_i)$ for all $i \in \mathcal{I}_K$. Thus, after one erasure, the collection $\{L_k c_{q_j}\}_{\substack{k\in\mathcal{J}_N \\ k\neq\ell}}$ contains at least $\phi(q_j)$ elements. Moreover, due to periodicity, it must include a set of $\phi(q_j)$ consecutive shifts of c_{q_j}, i.e., $\{L_k c_{q_j}\}_{k=\ell+1}^{\ell+\phi(q_j)} \subseteq \{L_k c_{q_j}\}_{\substack{k\in\mathcal{J}_N \\ k\neq\ell}}$. Therefore, this collection contains a basis of S_{1,q_j}, and hence forms a frame for S_{1,q_j}. Consequently, by Lemma 3.16, the system $\mathcal{R}_{1,N}^{(\ell,j)}$ is a frame for $\ell^2(\mathbb{Z}_N)$ for every $(\ell, j) \in \mathcal{J}_N \times \mathcal{I}_K$, completing the proof of (i).

The proof of part (ii) follows analogously, by applying Theorem 3.14 and Lemma 3.16. ☐

Theorem 3.25. *The following statements hold:*

- *(i) If $N-1 \geq 2\phi(N)$, then the system $\mathcal{R}_{1,N}$ is robust to 2 erasures.*
- *(ii) If d is odd and $d-1 \geq 2\phi(N)$, then the system $\mathcal{R}_{2,N}$ is robust to 2 erasures.*

Proof. We first prove (i). For indices $(k_i, i), (k_j, j) \in \mathcal{J}_N \times \mathcal{I}_K$ and $p \in \{1, 2\}$, we define $\mathcal{R}_{p,N}^{\{(k_i,i),(k_j,j)\}} := \mathcal{R}_{p,N} \backslash \{L_{k_i} c_{q_i}, L_{k_j} c_{q_j}\}$. If $i \neq j$, then the claim follows from Theorem 3.24 (i) as there is at most one erasure in each of the individual collections of $\mathcal{R}_{1,N}$. We divide the case $i = j$ into two subcases: $|k_i - k_j| > \phi(q_i)$ and $|k_i - k_j| \leq \phi(q_i)$. Now if $|k_i - k_j| > \phi(q_i)$, then $\{L_{k_i+1} c_{q_i}, L_{k_i+2} c_{q_i}, \ldots, L_{k_i-1} c_{q_i}\}$ is a subcollection of $\{L_k c_{q_i} : k \in \mathcal{J}_N, k \neq k_i, k_j\}$, containing at least $\phi(q_i)$ consecutive elements and hence contains a basis of S_{1,q_i}. Also if $|k_i - k_j| \leq \phi(q_i)$, then $\{L_{k_j+1} c_{q_i}, L_{k_j+2} c_{q_i}, \ldots, L_{N-1} c_{q_i}, \ldots, L_{k_i-1} c_{q_i}\}$ is a subcollection of $\{L_k c_{q_i} : k \in \mathcal{J}_N, k \neq k_i, k_j\}$, containing at least $\phi(q_i)$ consecutive elements since the total number of elements in this subcollection is:

$$(N-1) - (k_j + 1) + 1 + k_i = N - 1 + k_i - k_j \geq N - 1 - \phi(q_i) \geq \phi(q_i),$$

where the last inequality follows from the hypothesis, using $\phi(q_i) \leq \phi(N)$ for $i \in \mathcal{I}_K$. Therefore this subcollection also contains a basis of S_{1,q_i}. Thus, in both the cases $\{L_k c_{q_i} : k \in \mathcal{J}_N, k \neq k_i, k_j\}$ is a frame for S_{1,q_i} and hence each of the individual collections in $\mathcal{R}_{1,N}^{\{(k_i,i),(k_j,j)\}}$ is a frame for their corresponding subspaces. Then, the claim follows from Lemma 3.16.

We now prove (ii). As in the proof of (i), if $i \neq j$, the claim follows from Theorem 3.24 (ii). For $i = j$, we divide the proof into two subcases: $|k_i - k_j| > \phi(q_i)$, and $|k_i - k_j| \leq \phi(q_i)$. For the first case, the subcollection $\{L_{2(k_i+1)} c_{q_i}, L_{2(k_i+2)} c_{q_i}, \ldots, L_{2(k_j-1)} c_{q_i}\}$ of the collection $\{L_{2k} c_{q_i} : k \in \mathcal{J}_N, k \neq k_i, k_j\}$, contains at least $\phi(q_i)$ consecutive elements and hence contains a basis of S_{2,q_i} by using Theorem 3.14. For the second case, the subcollection $\{L_{2(k_j+1)} c_{q_i}, L_{2(k_j+2)} c_{q_i}, \ldots, L_{2(d-1)} c_{q_i}, \ldots, L_{2(k_i-1)} c_{q_i}\}$ of the collection $\{L_{2k} c_{q_i} : k \in \mathcal{J}_d, k \neq k_i, k_j\}$ contains at least $\phi(q_i)$ consecutive elements since

$$(d-1) - (k_i + 1) + 1 + k_i = d - 1 + k_i - k_j > d - 1 - \phi(q_i) \geq \phi(q_i),$$

using reasoning similar to that in (i). Consequently, each collection in $\mathcal{R}_{2,N}^{\{(k_i,i),(k_j,j)\}}$ is a frame for its corresponding subspace. The result then follows from Lemma 3.16. □

From the proofs of Theorems 3.24 and 3.25, it follows that a tight frame system $\mathcal{R}_{p,N}$ (for $p \in \{1, 2\}$) is robust to k erasures if and only if each of its constituent tight frames

$\{L_{pk}c_{q_i}\}_{k=0}^{d-1}$ is robust to k erasures, for all $i \in \mathcal{I}_K$. Thus, the problem of determining the robustness of $\mathcal{R}_{p,N}$ reduces to analyzing the robustness properties of these individual collections. The next remark shows that, for $p \in \{1,2\}$, each such collection can be interpreted both as a dynamical frame and as a harmonic frame. Such frames have a wide range of applications and are studied widely in the literature; see, for example, [7, 43, 150].

Remark 3.26. Let $p \in \{1,2\}$. Then, the tight frame $\{L_{pk}c_{q_i}\}_{k=0}^{d-1}$ can be realized as the orbit of a single vector under the action of a unitary operator. Indeed, let $T : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)$ be defined by

$$(Tx)(n) = x\big((n-p) \bmod N\big), \quad n \in \mathbb{Z}_N,$$

so that $T = L_p$ is the circular shift by p. Then

$$\{L_{pk}c_{q_i}\}_{k=0}^{d-1} = \{T^n c_{q_i}\}_{n=0}^{d-1},$$

showing that it is a dynamical frame generated by T.

Furthermore, the same frame can also be interpreted as a harmonic frame for the Ramanujan subspace S_{p,q_i}. For that, let $U(S_{p,q_i})$ denote the group of unitary operators on S_{p,q_i}. Since S_{p,q_i} is invariant under circular shifts, the restriction $L_m|_{S_{p,q_i}}$ belongs to $U(S_{p,q_i})$ for all $m \in \mathbb{Z}_N$. Define the subgroup

$$G_p := \{L_{pk}\,|_{S_{p,q_i}} : k = 0, 1, \ldots, d-1\} \subset U(S_{p,q_i}),$$

which is abelian because circular shifts commute. The collection $\{L_{pk}c_{q_i}\}_{k=0}^{d-1}$ is precisely the G_p–orbit of c_{q_i},

$$\{L_{pk}c_{q_i}\}_{k=0}^{d-1} = \{g\, c_{q_i} : g \in G_p\},$$

and satisfies $\|L_{pk}c_{q_i}\| = \|c_{q_i}\|$ for all k. By [150, Theorem 5.4], such a frame is harmonic.

Remark 3.27. Note that for 2 erasures, it is necessary that for any $q_i|N$, $N-2 \geq \phi(q_i)$ in case of the erasures from $\mathcal{R}_{1,N}$ and $d-2 \geq \phi(q_i)$ in case of erasures from $\mathcal{R}_{2,N}$ since to accommodate a basis after erasures, we need at least $\phi(q_i)$ elements in each of the individual collections of the erased versions of $\mathcal{R}_{1,N}$ and $\mathcal{R}_{2,N}$. Both of these conditions are implied by the conditions in the hypotheses of Theorem 3.25 (i) and Theorem 3.25 (ii), respectively. Indeed, we have $N-1 \geq 2\phi(N)$ from Theorem 3.25 (i). This implies $N-2 \geq 2\phi(N)-1 \geq \phi(N) \geq \phi(q_i)$ for any $q_i|N$. Similarly, the necessary condition $d-2 \geq \phi(q_i)$ for $q_i|N$ also follows from the condition in Theorem 3.25 (ii).

Example 3.28. (a). Let $N = 4$. The matrix M representing $\mathcal{R}_{1,4}$ is:

$$M = \begin{bmatrix} L_0c_1 & \cdots & L_3c_1 & L_0c_2 & \cdots & L_3c_2 & L_0c_4 & \cdots & L_3c_4 \end{bmatrix}_{4\times 12},$$

where $c_1 = (1, 1, 1, 1)^T$, $c_2 = (1, -1, 1, -1)^T$, and $c_4 = (2, 0, -2, 0)^T$. Due to the orthogonal decomposition of $\ell^2(\mathbb{Z}_4)$, the set of columns $D_{q_i} := \{L_0c_{q_i}, L_1c_{q_i}, L_2c_{q_i}, L_3c_{q_i}\}$ is orthogonal to $D_{q_j} := \{L_0c_{q_j}, L_1c_{q_j}, L_2c_{q_j}, L_3c_{q_j}\}$ for $q_i, q_j \in \{1, 2, 4\}$, and $j \neq i$. After removing one column from some D_{q_i}, each D_{q_i} still contains at least 3 columns, preserving $\phi(q_i)$ consecutive columns for each divisor $q_i = 1, 2, 4$, since $\phi(q_i) < 3$ holds for each q_i. Then, by Lemma 3.16, $R_{1,4}$ is robust to 1 erasure. But note that after the erasure of columns L_0c_4 and L_2c_4 from D_4, it will be left with $\{L_1c_4, L_3c_4\}$, which are linearly dependent and thus not able to span $S_{1,4}$ since $\dim S_{1,4} = \phi(4) = 2 > 1$. This shows $\mathcal{R}_{1,4}$ is not robust to 2 erasures. It is due to the fact that the necessary condition $N - 2 \geq \phi(q_i)$ for $q_i|4$ (discussed in Remark 3.27) fails here for $q_i = 4$, since $2 \ngeq 2 \times \phi(4) = 4$.

(b). Let $N = 210$. Then for $p = 2, d = 105$ is odd. Also, $105 - 1 > 2 \times \phi(210) = 96$. Thus, both the conditions given in Theorem 3.24 (ii) and Theorem 3.25 (ii) are satisfied and hence $\mathcal{R}_{2,N}$ is robust to 1 erasure and 2 erasures.

3.1.5 Fusion frames based on Ramanujan subspaces

Fusion frames, also known as frames of subspaces, arise naturally in the context of distributed processing. Unlike classical frames, which analyze signals via one-dimensional projections, fusion frames perform analysis by projecting signals onto multidimensional subspaces. For further details on fusion frames, we refer the reader to [33, 36].

In this subsection, we show that the tight frames $\mathcal{R}_{1,N}$ and $\mathcal{R}_{2,N}$ admit an interpretation as tight fusion frames in terms of Ramanujan subspaces.

To that end, we begin with the definition of a fusion frame adapted to our setting.

Definition 3.29. Let S_{p,q_i} denote the Ramanujan subspace corresponding to the divisors p and q_i of N, for $i \in \mathcal{I}_K$, as defined in (3.25), and let $(w_i)_{i\in\mathcal{I}_K} \subseteq \mathbb{R}^+$ be a corresponding set of positive weights. Then the family $\{(S_{p,q_i}, w_i)\}_{i\in\mathcal{I}_K}$ is called a *fusion frame* for $\ell^2(\mathbb{Z}_N)$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i\in\mathcal{I}_K} w_i^2 \|P_{S_{p,q_i}}(x)\|^2 \leq B\|x\|^2 \quad \text{for all } x \in \ell^2(\mathbb{Z}_N),$$

where for a subspace $W \subseteq \ell^2(\mathbb{Z}_N)$, P_W denotes the orthogonal projection onto W.

The constants A and B are referred to as the *lower* and *upper fusion frame bounds*, respectively. If $A = B$, the fusion frame is said to be a *tight fusion frame with bound* A. When $A = B = 1$, the fusion frame is called a *Parseval fusion frame*. If the weights satisfy $w_i = 1$ for all $i \in \mathcal{I}_K$, we simply write $\{S_{p,q_i}\}_{i\in\mathcal{I}_K}$.

We have the following result.

Theorem 3.30. *The following statements hold:*

(i) The family $\{S_{1,q_i}\}_{i\in\mathcal{I}_K}$ is a Parseval fusion frame.

(ii) For odd d, the family $\{S_{2,q_i}\}_{i\in\mathcal{I}_K}$ is a Parseval fusion frame.

The following proposition, adapted from [36, Corollary 13.2] to our setting, establishes a connection between the tight frame property of $\mathcal{R}_{p,N}$ and the tight fusion frame structure of the Ramanujan subspaces S_{p,q_i}, for $i \in \mathcal{I}_K$. For brevity, the proof is omitted.

Proposition 3.31. *Let the collection $\{L_{pk}c_{q_i}\}_{k\in\mathcal{J}_d}$ be a tight frame for S_{p,q_i} with frame bound A for each $i \in \mathcal{I}_K$. Then, the following conditions are equivalent.*

(i) The family $\{S_{p,q_i}\}_{i\in\mathcal{I}_K}$ is a tight fusion frame for $\ell^2(\mathbb{Z}_N)$ with bound C.

(ii) The system $\mathcal{R}_{p,N}$ is a tight frame for $\ell^2(\mathbb{Z}_N)$ with bound AC.

We now prove Theorem 3.30.

Proof of Theorem 3.30. We first prove part (i). By Theorem 3.6(i), the system $\mathcal{R}_{1,N}$ forms a tight frame for $\ell^2(\mathbb{Z}_N)$ with frame bound N^2. Consequently, Lemma 3.16 implies that each individual collection $\{L_k c_{q_i}\}_{k\in\mathcal{J}_N}$ forms a tight frame for the subspace S_{1,q_i} with the same frame bound N^2. Since $AC = N^2$ and $A = N^2$, Proposition 3.31 ensures that the family $\{S_{1,q_i}\}_{i\in\mathcal{I}_K}$ forms a tight fusion frame for $\ell^2(\mathbb{Z}_N)$ with frame bound $C = 1$. Hence, $\{S_{1,q_i}\}_{i\in\mathcal{I}_K}$ is a Parseval fusion frame for $\ell^2(\mathbb{Z}_N)$, completing the proof of part (i).

The proof of part (ii) follows analogously by applying Theorem 3.6(ii), Lemma 3.16, and Proposition 3.31. □

As previously discussed, if the frame system $\mathcal{R}_{p,N}$ is robust to k erasures, then each individual collection $\{L_{pk}c_{q_i}\}_{k\in\mathcal{J}_d}$, for $i \in \mathcal{I}_K$, must also be robust to k erasures. In light of this, Theorems 3.24 and 3.25 (for $p \in \{1, 2\}$) provide sufficient conditions under which the collection $\{L_{pk}c_{q_i}\}_{k\in\mathcal{J}_d}$ remain robust to one or two erasures. The following result, motivated by [35, Theorem 4.1], builds upon these conditions and shows that the deletion of a prescribed number of local frame vectors from each individual collection preserves the fusion frame property.

Theorem 3.32. *Let* $L_i \subset \mathcal{J}_d$ *for each* $i \in \mathcal{I}_K$*. For the divisors* p *and* q_i *of* N*, define the subspaces*

$$\widetilde{S}_{p,q_i} := \operatorname{span}\{L_{pk}c_{q_i} : k \in \mathcal{J}_d \setminus L_i\}, \quad i \in \mathcal{I}_K.$$

Then the following statements hold:

(a) Single erasure case: Assume $\# L_i \leq 1$ *for all* $i \in \mathcal{I}_K$*.*

- *(i) If* $p = 1$ *and* $N \geq 2$*, then the family* $\{\widetilde{S}_{1,q_i}\}_{i\in\mathcal{I}_K}$ *forms a fusion frame for* $\ell^2(\mathbb{Z}_N)$*.*
- *(ii) If* $p = 2$*,* $N \geq 2$*,* d *is odd, then the family* $\{\widetilde{S}_{2,q_i}\}_{i\in\mathcal{I}_K}$ *forms a fusion frame for* $\ell^2(\mathbb{Z}_N)$*.*

(b) Double erasure case: Assume $\# L_i \leq 2$ *for all* $i \in \mathcal{I}_K$*.*

- *(i) If* $p = 1$ *and* $N - 1 \geq 2\phi(N)$*, then the family* $\{\widetilde{S}_{1,q_i}\}_{i\in\mathcal{I}_K}$ *forms a fusion frame for* $\ell^2(\mathbb{Z}_N)$*.*
- *(ii) If* $p = 2$*,* d *is odd, and* $d - 1 \geq 2\phi(N)$*, then the family* $\{\widetilde{S}_{2,q_i}\}_{i\in\mathcal{I}_K}$ *forms a fusion frame for* $\ell^2(\mathbb{Z}_N)$*.*

Moreover, in all cases above, the collections $\{L_{pk}c_{q_i}\}_{k\in\mathcal{J}_d\setminus L_i}$ *form frames for* S_{p,q_i} *for* $i \in \mathcal{I}_K$ *for respective values of* p*, and the associated fusion frame bounds are given by*

$$\frac{\min\limits_{i\in\mathcal{I}_K} A_{p,i}}{pd^2} \quad \textit{and} \quad 1,$$

where $A_{p,i}$ *denotes the lower frame bound of* $\{L_{pk}c_{q_i}\}_{k\in\mathcal{J}_d\setminus L_i}$*.*

Proof. We begin with proving (a). Let $x \in \ell^2(\mathbb{Z}_N)$ be arbitrary. For each $i \in \mathcal{I}_K$, set $y_i := P_{S_{1,q_i}}x$. Since $\widetilde{S}_{1,q_i} \subseteq S_{1,q_i}$, we decompose

$$y_i = u_i + v_i, \quad \text{where } u_i := P_{\widetilde{S}_{1,q_i}} y_i \in \widetilde{S}_{1,q_i}, \quad v_i \in S_{1,q_i} \ominus \widetilde{S}_{1,q_i}.$$

For any $k \in \mathcal{J}_N \setminus L_i$, we have $L_k c_{q_i} \in \widetilde{S}_{1,q_i}$, and thus

$$\langle y_i, L_k c_{q_i}\rangle = \langle u_i, L_k c_{q_i}\rangle,$$

since $v_i \perp \widetilde{S}_{1,q_i}$. Using the fact that the collection $\{L_k c_{q_i}\}_{k\in\mathcal{J}_N}$ forms a tight frame for S_{1,q_i} with bound N^2 (due to Theorem 3.6(i) and Lemma 3.16), we estimate:

$$\sum_{k\in\mathcal{J}_N\setminus L_i} |\langle y_i, L_k c_{q_i}\rangle|^2 = \sum_{k\in\mathcal{J}_N\setminus L_i} |\langle u_i, L_k c_{q_i}\rangle|^2 \leq \sum_{k\in\mathcal{J}_N} |\langle u_i, L_k c_{q_i}\rangle|^2 = N^2\|u_i\|^2.$$

Summing over $i \in \mathcal{I}_K$ and using the fact that for $N \geq 2$ and $\#_{L_i} \leq 1$, the collection $\{L_k c_{q_i}\}_{k \in \mathcal{J}_N \setminus L_i}$ forms a frame for S_{1,q_i} for $i \in \mathcal{I}_K$ due to Theorem 3.24(i) with bound $A_{1,i}$, we obtain:

$$\begin{aligned} N^2 \sum_{i=1}^{K} \|P_{\widetilde{S}_{1,q_i}} x\|^2 = N^2 \sum_{i=1}^{K} \|u_i\|^2 &\geq \sum_{i=1}^{K} \sum_{k \in \mathcal{J}_N \setminus L_i} |\langle y_i, L_k c_{q_i} \rangle|^2 \\ &\geq \sum_{i=1}^{K} A_{1,i} \|P_{S_{1,q_i}} x\|^2 \geq \left(\min_{i \in \mathcal{I}_K} A_{1,i} \right) \sum_{i=1}^{K} \|P_{S_{1,q_i}} x\|^2. \end{aligned}$$

Finally, since $\{S_{1,q_i}\}_{i \in \mathcal{I}_K}$ forms a Parseval fusion frame for $\ell^2(\mathbb{Z}_N)$ by Theorem 3.30(i), we conclude that $\sum_{i=1}^{K} \|P_{S_{1,q_i}} x\|^2 = \|x\|^2$, and thus

$$\sum_{i=1}^{K} \|P_{\widetilde{S}_{1,q_i}} x\|^2 \geq \frac{\min_{i \in \mathcal{I}_K} A_{1,i}}{N^2} \|x\|^2.$$

The upper frame bound 1 is immediate, since projections are norm-contracting and $\widetilde{S}_{1,q_i} \subseteq S_{1,q_i}$. This proves (a)(i).

When $p = 2$, the same argument applies, as the collection $\{L_{2k} c_{q_i}\}_{k \in \mathcal{J}_d}$ forms a tight frame for S_{2,q_i} with frame bound $2d^2$ (Theorem 3.6(ii) and Lemma 3.16). Moreover, under the assumptions $N \geq 2$, odd d, and $\#_{L_i} \leq 1$, the collection $\{L_{2k} c_{q_i}\}_{k \in \mathcal{J}_d \setminus L_i}$ also forms a frame for S_{2,q_i} for all $i \in \mathcal{I}_K$, by Theorem 3.24(ii). This establishes part (a)(ii).

The proof of part (b) follows analogously by applying Theorem 3.6, Lemma 3.16, Theorem 3.25, and Proposition 3.30. $\square$

In the following section, we introduce an uncertainty principle for signal representations in $\ell^2(\mathbb{Z}_N)$, comparing the canonical basis $\{e_n\}_{n \in \mathbb{Z}_N}$ with a tight frame $\mathcal{R}_{p,N}$. We further explore the utility of this principle in addressing practical challenges, such as recovering signals with missing information and signal denoising.

3.2 Uncertainty principle and signal recovery

In this section, we utilize Theorem 2.1, developed in the previous chapter, to establish an uncertainty principle in the context of Ramanujan sums. Specifically, we consider the one-dimensional case by setting $d = 1$, $\mathcal{B} = p$, and choosing the analysis functions as $h_i = c_{q_i}$, where K denotes the number of divisors of N. Before applying Theorem 2.1 to derive the corresponding uncertainty relations, we first present a few auxiliary results that will be instrumental in this development.

Proposition 3.33. *Let the collection $\mathcal{R}_{p,N}$ (defined in (3.4)) form a tight frame for $\ell^2(\mathbb{Z}_N)$ with frame bound A. Then $A = pd^2$.*

Proof. In view of (3.7), the trace of the matrix $(\mathcal{U}^*(m)\mathcal{U}(m))$ can be simplified as follows:

$$tr(\mathcal{U}^*(m)\mathcal{U}(m)) = d\sum_{i=1}^{K}\sum_{n=0}^{p-1}|\mathcal{Z}c_{q_i}(m,n)|^2 = \sum_{n=0}^{p-1}\lambda_n(m), \tag{3.30}$$

where for each m, $\{\lambda_n(m)\}_{n=0}^{p-1}$ is the set of eigenvalues of the matrix $\mathcal{U}^*(m)\mathcal{U}(m)$. Since $\mathcal{R}_{p,N}$ is a tight frame with frame bound A, therefore $\mathcal{U}^*(m)\mathcal{U}(m) = AI_p$ for $m \in \mathcal{J}_d$. From (3.30), we get $\frac{pA}{d} = \sum_{i=1}^{K}\sum_{n=0}^{p-1}|\mathcal{Z}c_{q_i}(m,n)|^2$. Taking summation from $m = 0$ to $d-1$ on both sides of the last equality, the left side becomes pA and the right quantity becomes $\sum_{i=1}^{K}\|\mathcal{Z}c_{q_i}\|^2 = \sum_{i=1}^{K}\|c_{q_i}\|^2$. Finally we have,

$$A = \frac{1}{p}\sum_{i=1}^{K}\|c_{q_i}\|^2 = \frac{N}{p}\sum_{i=1}^{K}\phi(q_i) = \frac{N^2}{p} = pd^2,$$

where the second equality follows using Proposition 3.1(v), according to which $\|c_{q_i}\|^2 = N \times \phi(q_i)$. Hence the claim follows. ☐

Lemma 3.34. *Let N be any positive integer. Then, for any divisor q of N, we have $|c_q(n)| \le \phi(N)$, $n \in \mathbb{Z}_N$, and hence* $\max\{|c_q(n)| : n \in \mathbb{Z}_N,\ q \text{ divides } N\} = \phi(N)$.

Proof. By the definition of $c_q(n)$, we have

$$|c_q(n)| = \left|\sum_{\substack{k=1\\(k,q)=1}}^{q} e^{2\pi ikn/q}\right| \le \phi(q). \tag{3.31}$$

Note that $\phi(N) = \phi(k\cdot q) \ge \phi(k)\phi(q) \ge \phi(q)$, where $N = k.q$ for some integer $k \ge 1$ and the inequality $\phi(k.q) \ge \phi(k)\phi(q)$ holds since for any positive integers m and n, $\phi(mn) = \phi(m)\phi(n)\ell/\phi(\ell)$ with $\ell = (m,n)$. Consequently from (3.31), we get $|c_q(n)| \le \phi(N)$ for any divisor q of N. Since $c_N(0) = \phi(N)$, therefore $\max\{|c_q(n)| : n \in \mathbb{Z}_N,\ q \text{ divides } N\} = \phi(N)$. Hence the claim follows. ☐

From Remark 2.3 and Lemma 3.34, it is clear that $\beta_o = \phi(N)$ in this case. Utilizing this and Proposition 3.33, we obtain the following version of Theorem 2.1 in this case.

Theorem 3.35. *Let the collection $\mathcal{R}_{p,N}$ form a tight frame for $\ell^2(\mathbb{Z}_N)$. Then, for any signal $x \in \ell^2(\mathbb{Z}_N)$, its representations with respect to the tight frame $\mathcal{R}_{p,N}$ and the canonical* *basis* $\{e_n\}_{n\in\mathbb{Z}_N}$ *yield the following uncertainty inequalities:*

$$S_x + B_x \geq \frac{2d\sqrt{p}}{\phi(N)}, \quad \text{and} \quad S_x B_x \geq p\left(\frac{d}{\phi(N)}\right)^2, \tag{3.32}$$

where $S_x = \#\{(k,i) \in \mathcal{J}_d \times \mathcal{I}_K \mid (x * c_{q_i})(pk) \neq 0\}$ *and* $B_x = \#\{n \in \mathcal{J}_N \mid \alpha_n \neq 0\}$.

Remark 3.36. Note that, if $N = kq$ for some positive integer k, then by using Proposition (3.1)(ii), we get

$$\tilde{c_q}(n) = c_q(-n) = c_q(N-n) = c_q(kq-n) = c_q(n), \;\; n \in \mathbb{Z}_N.$$

Thus the collection $\{(x * c_{q_i})(pk) : (k,i) \in \mathcal{J}_d \times \mathcal{I}_\mathcal{K}\}$, without the involution of $c_{q_i}, i \in \mathcal{I}_K$, is the output from the filter bank based on Ramanujan sums $c_{q_1}, c_{q_2}, \ldots, c_{q_K}$.

We now turn to the problem of signal recovery in the framework of Ramanujan sums utilizing 3.35. For this purpose, we restate Problems 1 and 2 from Subsection 2.4 in the context of Ramanujan sums.

We define $M^c := (\mathcal{J}_d \times \mathcal{I}_K) \setminus M$ for any $M \subset \mathcal{J}_d \times \mathcal{I}_K$. We focus on the following two problems:

Problem 1: Suppose a signal $x \in \ell^2(\mathbb{Z}_N)$ has a subset of its samples $\{(x * c_{q_i})(pk) : (k,i) \in \mathcal{J}^c\}$ missing or unavailable, for some $\mathcal{J} \subset \mathcal{J}_d \times \mathcal{I}_K$. We ask under what conditions x can be uniquely reconstructed from the truncated sum

$$\mathcal{T}_\mathcal{J} x := \frac{1}{pd^2} \sum_{(k,i)\in\mathcal{J}} (x * c_{q_i})(pk)\, L_{pk} c_{q_i}, \tag{3.33}$$

assuming that $\mathcal{R}_{p,N}$ forms a tight frame for $\ell^2(\mathbb{Z}_N)$.

The following result is a particular case of Theorem 2.18 for the case of Ramanujan sums and serves as a solution to Problem 1.

Theorem 3.37. *Let* $x \in \ell^2(\mathbb{Z}_N)$ *be a signal supported on a set* $\mathcal{C} \subseteq \mathbb{Z}_N$*, and let* $\mathcal{J} \subset \mathcal{J}_d \times \mathcal{I}_K$ *be such that a subset* $\{(x * c_{q_i})(pk) : (k,i) \in \mathcal{J}\}$ *of the samples are lost or unavailable. Suppose that the collection* $\mathcal{R}_{p,N}$ *forms a tight frame for* $\ell^2(\mathbb{Z}_N)$*. If*

$$2\#_{\mathcal{J}^c}\#_\mathcal{C} < p\left(\frac{d}{\phi(N)}\right)^2, \tag{3.34}$$

then x *can be uniquely recovered from* $\mathcal{T}_\mathcal{J} x$ *(defined in (3.33)). Moreover, the unique reconstruction* $\tilde{x}$ *is given by*

$$\tilde{x} = \arg\min_{x'} \{\|x'\|_1 \,:\, \mathcal{T}_\mathcal{J} x' = \mathcal{T}_\mathcal{J} x\}, \tag{3.35}$$

Problem 2: Given a noisy observation $y = x + \eta$, where $x \in \ell^2(\mathbb{Z}_N)$ is an unknown signal and $\eta \in \ell^2(\mathbb{Z}_N)$ is a sparse noise vector, under what conditions can x be uniquely reconstructed?

For $\mathcal{M} \subset \mathcal{J}_d \times \mathcal{I}_K$, we define

$$\mathcal{S}_{\mathcal{M}} := \left\{ h \in \ell^2(\mathbb{Z}_N) \mid (h * c_{q_i})(pk) = 0 \text{ for } (k, i) \in \mathcal{M}^c \right\}. \tag{3.36}$$

The following result is a particular case of Theorem 2.19 for the case of Ramanujan sums and serves as a solution to Problem 2.

Theorem 3.38. *Let $x \in \mathcal{S}_{\mathcal{M}}$, as defined in (3.36), for some fixed set $\mathcal{M} \subset \mathcal{J}_d \times \mathcal{I}_K$, and let $\mathcal{N} = \{m \in \mathbb{Z}_N : \eta(m) \neq 0\}$ denote the support of the noise vector η. Suppose that the collection $\mathcal{R}_{p,N}$ forms a tight frame for $\ell^2(\mathbb{Z}_N)$. If*

$$2\#_{\mathcal{M}}\#_{\mathcal{N}} < p \left(\frac{d}{\phi(N)} \right)^2, \tag{3.37}$$

then the signal x can be exactly recovered from the noisy observation $y = x + \eta$. Moreover, the unique reconstruction $\tilde{x}$ is given by

$$\tilde{x} = \arg \min_{x' \in \mathcal{S}_{\mathcal{M}}} \|y - x'\|_1. \tag{3.38}$$

3.3 Applications

The signal recovery conditions stated in Theorem 3.37 and Theorem 3.38 may not always be met in practice, raising two fundamental questions: Do the optimization problems (3.35) and (3.38) yield exact recovery when the proposed signal recovery conditions do not hold? If not exact recovery, can we still ensure stable and accurate approximation? Intuitively, as the number of missing samples increases or the noise level worsens, the recovery quality deteriorates.

In this section, we address these questions using Ramanujan sums. Instead of relying on the support of the signal and its samples, this approach exploits the inherent *periodic structure* of the signal. This is achieved using Ramanujan filter banks, where the analysis filters are given by the Ramanujan sums corresponding to the divisors of N.

We show that incorporation of the periodicity information of signals using the properties of Ramanujan filter bank leads to greater signal-to-noise ratio (SNR) gains for each of the optimization problems (3.35) and (3.38).

The SNR gain metric measures the improvement in the signal after applying a particular signal processing technique, such as filtering, noise reduction, or amplification. It compares the quality of the signal before and after processing, in terms of how much the signal's power is boosted relative to the noise. SNR is defined as the ratio of the power of the signal (P_{signal}) to the power of the noise and is typically expressed in decibels (dB), using the formula: $\text{SNR (dB)} = 10\log_{10}\left(P_{\text{signal}}/P_{\text{noise}}\right)$, where the power P of a signal $x \in \ell^2(\mathbb{Z}_N)$ is given by the formula $P = \frac{1}{N}\|x\|^2$. The SNR gain is the difference between the SNR after processing (e.g., filtering) and the SNR before processing:

$$\text{SNR gain (dB)} = \text{SNR after (dB)} - \text{SNR before (dB)}.$$

3.3.1 Application in signal recovery from missing data

The condition (3.34) is impractical for applications for two main reasons. First, the value of N_B is often unavailable, and second, the bound on the right-hand side is typically too small to be meaningful. For instance, consider $N = 38$ and $p = 2$. Here, $d = 19$, and the bound evaluates to $p\left(\frac{d}{\phi(N)}\right)^2 = 2\left(\frac{19}{\phi(38)}\right)^2 = 2.111$. This implies that $\#_{\mathcal{J}^c}\#_{\mathcal{C}} < 1.055$, which is not meaningful in practical scenarios and thus it is reasonable to expect that the value $2\#_{\mathcal{J}^c}\#_{\mathcal{C}}$ will surpass the bound $p\left(\frac{d}{\phi(N)}\right)^2$.

The condition (3.34) can be regarded as ideal in nature. In particular, the closer the quantity $2\#_{\mathcal{J}^c}\#_{\mathcal{C}}$ is to the bound $p\left(\frac{d}{\phi(N)}\right)^2$, the more accurately the signal x can be recovered from $\mathcal{T}_{\mathcal{J}}x$ via the optimization problem (3.35). Therefore, the reconstruction from $\mathcal{T}_{\mathcal{J}}x$ can be expected to be worse if a significant number of samples are missing from the signal. In such cases, the problem (3.35) admits further refinement under the assumption that the period of the signal (say, P) is known and satisfies $P \mid N$. Suppose the periodic components of the signal x are $P_1, P_2, \ldots, P_m$. Since $P = \operatorname{lcm}(P_1, P_2, \ldots, P_m)$, it follows that each P_i also divides N. Consequently, $(x * c_{q_j})(pk) = 0$ for $q_j \notin \{P_1, P_2, \ldots, P_m\}$. This information can be incorporated into the optimization problem (3.35), modifying it as follows:

$$\tilde{x} = \arg\min_{x'}\left\{\|x'\|_1 \ : \ \mathcal{T}_{\mathcal{J}}x' = \mathcal{T}_{\mathcal{J}}x \text{ and } (x' * c_q) = 0 \text{ for all } q \mid N \text{ and } q \notin \{P_1, P_2, \ldots, P_m\}\right\}. \tag{3.39}$$

The solution provided by (3.39) achieves a significantly higher SNR gain from $\mathcal{T}_{\mathcal{J}}x$ compared to the SNR gain obtained from the solution of (3.35).

In order to illustrate this, we consider an example of a signal x with length $N = 70$ and periodic components $P_1 = 5$ and $P_2 = 7$, represented by solid blue line in Fig. 3.3. The divisors of $N = 70$ are $q_1 = 1, q_2 = 2, q_3 = 5, q_4 = 7, q_5 = 10, q_6 = 14, q_7 = 35,$ and $q_8 = 70$, yielding $K = 8$. We consider the filter bank with decimation ratio $p = 2$, whose i-th channel corresponds to the q_i-th Ramanujan sum c_{q_i} for $1 \leq i \leq 8$. For $p = 2$, Theorem 3.6 guarantees that $\mathcal{R}_{2,70}$ forms a tight frame for $\ell^2(\mathbb{Z}_{70})$, satisfying the tight-frame condition of Theorem 3.37. Now, suppose the following samples of the signal x are missing:

$$\{(x * c_{q_3})(2k)\}_{k=0}^{10} \cup \{(x * c_{q_5})(2k)\}_{k=21}^{34} \cup \{(x * c_{q_7})(2k)\}_{k=10}^{34}. \tag{3.40}$$

The set $\mathcal{J}$ for the truncated sum $\mathcal{T}_{\mathcal{J}}x$ in this case is the complement of the set

$$\{(k,3)\}_{k=0}^{10} \cup \{(k,5)\}_{k=21}^{34} \cup \{(k,7)\}_{k=10}^{34},$$

within $\{0, 1, \ldots, 34\} \times \{1, 2, \ldots, 8\}$. The solution to the optimization problem (3.35) for this case is depicted by the green line with triangular markers in Fig. 3.3. It is clear from the figure that the present solution deviates significantly from the original signal, yielding a low SNR gain of -8.4547 dB. It is due to the reason that the value $2\#_{\mathcal{J}^c}\#_{\mathcal{C}}$ is large compared to the bound $p(d/\phi(N)^2)$ in this case. By incorporating the information that the original signal x has periodic components $P_1 = 5$ and $P_2 = 7$, the modified optimization problem (3.39) becomes:

$$\tilde{x} = \arg\min_{x'} \{\|x'\|_1 \,:\, \mathcal{T}_{\mathcal{J}}x' = \mathcal{T}_{\mathcal{J}}x \text{ and } (x' * c_q) = 0 \text{ for } q|70 \text{ and } q \neq 5, 7\}.$$

The solution for the missing sample set (3.40) is illustrated using red square markers in Fig. 3.3. It is evident that this solution perfectly aligns with the original signal and achieves an SNR gain of 197.3205 dB.

The SNR gains obtained from optimization problems (3.35) and (3.39) for various cases of missing samples of the signal x are compared in Table 3.1. It is clear from the table that the solutions obtained from problem (3.39) consistently outperform those obtained from problem (3.35) for all cases. This indicates that even when a significant number of samples are missing—or equivalently, when the value of $2\#_{\mathcal{J}^c}\#_{\mathcal{C}}$ deviates substantially from the bound $p(d/\phi(N))^2$—the signal x can still be recovered with high accuracy from $\mathcal{T}_{\mathcal{J}}x$, provided that the periodicity information of the signal is known in advance.

S.No.	**Missing sample locations** (k,i)	$2\#_{\mathcal{J}^c}\#_{\mathcal{C}}$	**SNR gain (dB)** **(3.35)**	**SNR gain (dB)** **(3.39)**
1.	$\{(k,2)\}_{k=0}^{2}, \{(k,3)\}_{k=17}^{20}, \{(k,5)\}_{k=27}^{29}$	$20\#_{\mathcal{C}}$	174.8449	193.6376
2.	$\{(k,4)\}_{k=15}^{34}$	$40\#_{\mathcal{C}}$	163.9832	223.7594
3.	$\{(k,3)\}_{k=0}^{24}, \{(k,8)\}_{k=0}^{24}$	$100\#_{\mathcal{C}}$	2.7150	210.6610
4.	$\{(k,3)\}_{k=0}^{10}, \{(k,5)\}_{k=21}^{34}, \{(k,7)\}_{k=10}^{34}$	$100\#_{\mathcal{C}}$	-8.4547	197.3205
5.	$\{(k,4)\}_{k=6}^{34}, \{(k,5)\}_{k=0}^{34}, (12,6), \{(k,7)\}_{k=0}^{34}$	$200\#_{\mathcal{C}}$	-12.3595	207.8186
6.	$\{(k,1)\}_{k=0}^{9}, \{(k,2)\}_{k=5}^{14}, \{(k,3)\}_{k=11}^{30}$ $\{(k,4)\}_{k=21}^{34}, \{(k,5)\}_{k=17}^{34}, \{(k,7)\}_{k=6}^{34}$	$200\#_{\mathcal{C}}$	-6.1835	224.2178

Table 3.1: Comparison of SNR gains obtained from the solutions of optimization problems (3.35) and (3.39) across various cases of missing samples.

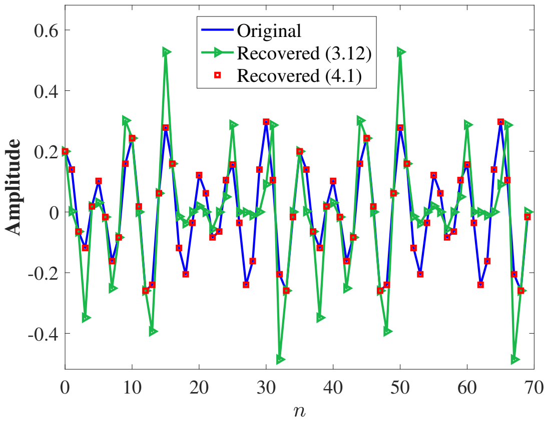


Figure 3.3: Original signal (blue line) of length $N = 70$ with periodic components 5 and 7, the solution of the optimization problem (3.35) (green line with triangular markers) for missing sample set (3.40) with SNR gain -8.4547 dB, and the solution of the optimization problem (3.39) for missing sample set (3.40) (red square markers) with SNR gain 197.3205 dB.

In some scenarios, there may be no a priori knowledge of the periodic components of a signal. This raises the question: *Is it still possible to recover the original signal x from $\mathcal{T}_{\mathcal{J}}x$ with high-accuracy if a significant amount of samples are missing?* The answer to this question is "yes". In such cases, the periodic components can be extracted from $\mathcal{T}_{\mathcal{J}}x$, provided the available samples in $\mathcal{T}_{\mathcal{J}}x$ are sufficient to accurately predict the periodic components of the original signal and the period of the signal is a divisor of

N. The energy outputs for the truncated sum $\mathcal{T}_{\mathcal{J}}x$, corresponding to the missing sample sets listed in Table 3.1, are shown in Fig. 3.4. It is evident from the figure that $\mathcal{T}_{\mathcal{J}}x$

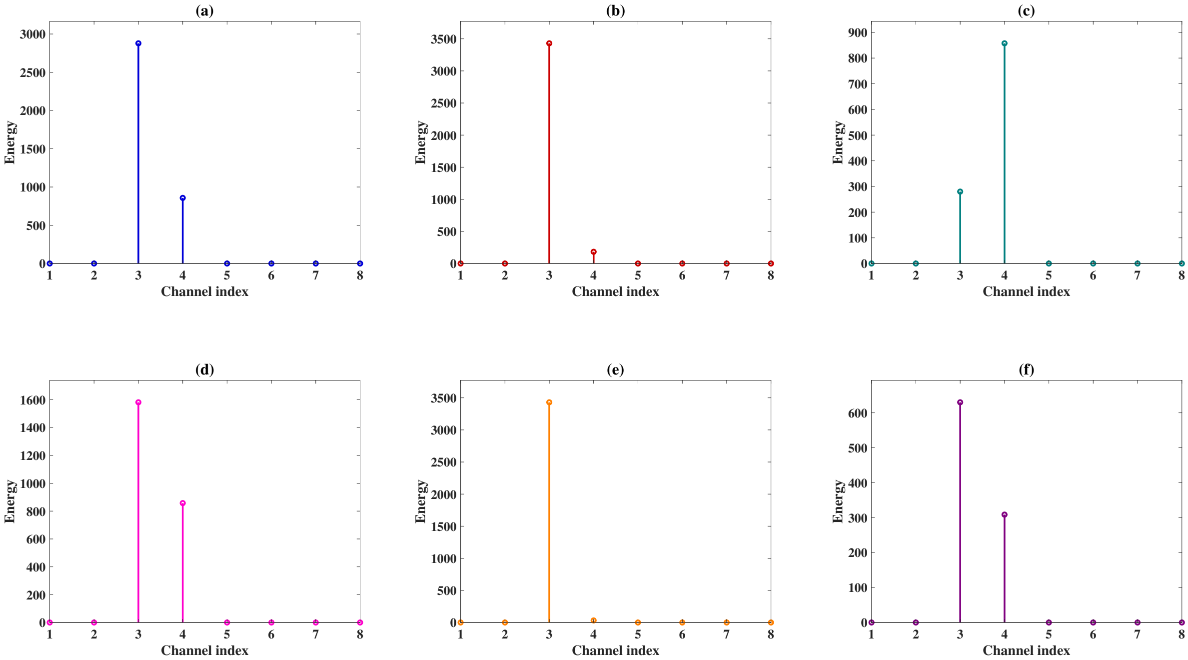


Figure 3.4: Energy outputs for $\mathcal{T}_{\mathcal{J}}x$ for various choices of $\mathcal{J}$ associated with missing sample sets listed in Table 3.1.

accurately identifies the periodic components $q_3 = 5$ and $q_5 = 7$ of the original signal in all cases listed in Table 3.1, except for case 5. In this instance, the energy from the fourth channel, corresponding to c_7, is notably low, as shown in Fig. 3.4(e). This occurs because the missing sample set for case 5, specified in Table 3.1, includes a significant portion of the output $\{(k,4)\}_{k=6}^{34}$, from the Ramanujan sum c_7. Consequently, $\mathcal{T}_{\mathcal{J}}x$ cannot reliably predict a periodic component P if a substantial portion of the output from the corresponding Ramanujan sum c_P is missing.

In summary, when more samples are missing, utilizing information about the periodic components of the original signal can enhance the SNR gain.

3.3.2 Application in denoising

In this subsection, we explore the recovery of the original signal $x \in \ell^2(\mathbb{Z}_N)$ from the noisy signal $y = x + \eta$ using a filter bank based on Ramanujan sums corresponding to the divisors of N. Additionally, we examine the relationship between the condition (3.37) in our denoising result, Theorem 3.38, and the SNR gain for various signals.

For recovery of the original signal x from noisy signal y using Theorem 3.38, $x \in \mathcal{S}_{\mathcal{M}}$ for some $\mathcal{M} \subset \mathcal{J}_d \times \mathcal{I}_K$. In practice, it is possible that $\mathcal{S}_{\mathcal{M}}$ is unknown. In such scenarios, a filter bank based on Ramanujan sums $c_{q_1}, c_{q_2}, \ldots, c_{q_K}$, where K denotes the number of divisors of N, can be employed to recover the support set $\mathcal{S}_{\mathcal{M}}$ from the observation y under the assumption that the (unknown) period P of the signal satisfies $P \mid N$. To this end, consider a signal $x \in \ell^2(\mathbb{Z}_N)$, with the assumption that its (unknown) period P is a divisor of N. Let the noisy signal $y = x + \eta$ be input to the filter bank with channels based on Ramanujan sums $c_{q_i}, i \in \mathcal{I}_K$. Under moderate noise conditions, it is reasonable to expect that the channels corresponding to divisors of P, say, $q_1, q_2, \ldots, q_m$ will produce outputs with significantly higher energies compared to the remaining channels in the filter bank due to Theorem 3.4. The small non-zero outputs produced by other channels are primarily attributable to the presence of noise. Since $P \mid N$, it follows that each of the divisors $q_1, q_2, \ldots, q_m$ of P also divides N. To filter out the low energies, we apply a threshold to the output energies. An appropriately chosen threshold should ideally preserve energies from those channels $c_{q_1}, c_{q_2}, \ldots, c_{q_m}$, which would have yielded non-zero outputs if the noiseless signal x had been passed through the filter bank instead of the noisy signal y. Once the divisors $q_i, 1 \leq i \leq m$ are known, the set $\mathcal{M}$ is given by $\mathcal{M} = \{(k, i) : k \in \mathcal{J}_d,\ q_i \in \{q_1, q_2, \ldots, q_m\}\}$ and the set $\mathcal{S}_{\mathcal{M}}$ can be computed as follows:

$$\mathcal{S}_{\mathcal{M}} = \left\{x \in \ell^2(\mathbb{Z}_N) \mid (x' * c_q)(pk) = 0,\ k \in \mathcal{J}_d, \text{ for } q \mid N \text{ with } q \neq q_1, q_2, \ldots, q_m\right\}.$$

Finally, the denoised signal $\tilde{x}$ is given by:

$$\tilde{x} = \arg \min_{x' \in \mathcal{S}_{\mathcal{M}}} \|y - x'\|_1. \tag{3.41}$$

Table 3.2 lists signals of length $N = 30$ with different periodic components, which are processed using a filter bank with filters $c_1, c_2, c_3, c_5, c_6, c_{10}, c_{15}, c_{30}$ and decimation ratio $p = 1$. The table demonstrates that as the number of periodic components in the signal increases, the SNR gain from the noisy signals (with varying SNR's) decreases. This drop in SNR gain is attributed to the increasing value of $\#_{\mathcal{M}}$, which leads to $2\#_{\mathcal{M}}\#_{\mathcal{T}}$ deviating from the bound $p\left(\frac{d}{\phi(N)}\right)^2$, as shown in Table 3.2.

We explain the denoising process using an example.

S.No.	**True Periodic Components**	**Estimated Components (via Filter Energies)**	$2\#_{\mathcal{M}}\#_{\mathcal{T}}$	**SNR (dB) (Noisy Signal)**	**SNR gain (dB) (3.41)**
1.	1,3	3	$60\#_{\mathcal{T}}$	0.0007	11.1245
2.	3,5	3,5	$120\#_{\mathcal{T}}$	0.0006	9.4778
3.	2,15	2,15	$120\#_{\mathcal{T}}$	0.0009	6.7987
4.	3,5,10	5,10	$120\#_{\mathcal{T}}$	0.0004	6.6875
5.	1,2,3,6	3,6	$120\#_{\mathcal{T}}$	0.0008	6.3936
6.	1,3,5,6,10	5,6,10,30	$240\#_{\mathcal{T}}$	0.0005	1.4456
7.	2,3,5,6,10,15	5,6,10,15,30	$300\#_{\mathcal{T}}$	0.0008	0.3772

Table 3.2: Comparison of SNR gains from optimization problem (3.41) with the values $2\#_{\mathcal{M}}\#_{\mathcal{T}}$ corresponding to the signals with various periodic components.

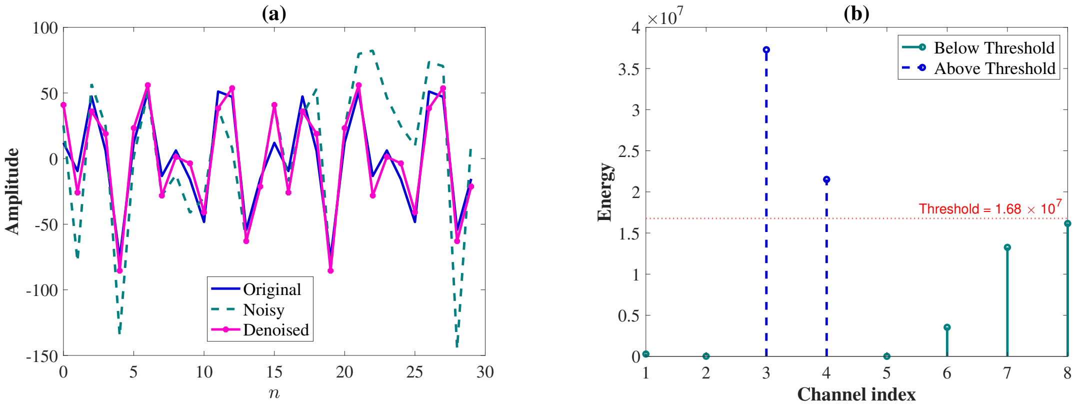


Figure 3.5: (a) Original signal (solid blue line) with length $N = 30$, the noisy signal (dashed green line) obtained by adding white Gaussian noise with SNR 6×10^{-4} dB and the denoised signal (solid magenta line with circular markers), i.e., the solution of the optimization problem (3.41) with SNR 9.4778 dB. (b) Energy output from each channel of the filter bank and the corresponding threshold value.

Consider a signal x with length $N = 30$ depicted by the blue solid line in Fig. 3.5(a). The divisors of $N = 30$ are $q_1 = 1, q_2 = 2, q_3 = 3, q_4 = 5, q_5 = 6, q_6 = 10, q_7 = 15$, and $q_8 = 30$, resulting in $K = 8$. We consider a filter bank with decimation ratio $p = 1$, where the i-th channel corresponds to the q_i-th Ramanujan sum c_{q_i} for $1 \leq i \leq 8$. For $p = 1$, Theorem 3.6 ensures that $\mathcal{R}_{1,30}$ forms a tight frame for $\ell^2(\mathbb{Z}_{30})$, thus satisfying the tight-frame condition of Theorem 3.38. The green dashed line in Fig. 3.5(a) shows the noisy signal obtained by adding white Gaussian noise with SNR 6×10^{-4} dB. The noisy signal is then passed through the filter bank and the energy of each channel is recorded. The threshold value is set to 0.45 times the maximum energy output of the filter bank. The energy output from each channel of the filter bank is illustrated in Fig. 3.5(b). It is evident from the figure that the channels corresponding to c_3 and c_5 (indicated by the dashed blue lines) exhibit energies above the threshold whereas the channels corresponding to $c_1, c_2, c_6, c_{10}, c_{15}$, and c_{30} (indicated by the solid green lines) are below the threshold value 1.68×10^7. Consequently, the set $\mathcal{M}$ in this case is given by: $\mathcal{M} = \{(k,i) : 0 \leq k \leq 29,\ i = 3,4\}$ and the solution of the optimization problem (3.41) after taking $\mathcal{S}_{\mathcal{M}} = \{x \in \ell^2(\mathbb{Z}_N) \mid (x' * c_q)(k) = 0,\ 0 \leq k \leq 29,\ \text{for } q \mid 30 \text{ with } q \neq 3, 5\}$ is depicted in Fig. 3.5(a) using solid magenta line with circular markers.

From the figure, it is clear that the solution from the optimization problem (3.41) resembles the original signal, achieving an SNR gain of 9.4778 dB. However, the solution is not perfect as the value $2\#_{\mathcal{M}}\#_{\mathcal{T}} = 120\#_{\mathcal{T}}$ exceeds the bound $p(d/\phi(N))^2 = 7.031$ for any noise vector n with $\#_{\mathcal{T}} > 1$.

Chapter 4

Random sampling involving derivatives for periodic shift-invariant spaces

In this chapter, we establish probabilistic sampling inequalities for periodic shift-invariant (PSI) spaces in $L^2(\mathbb{T})$, incorporating random derivative samples of the underlying function. Motivated by the work of Antezana, Carbajal, and Romero [14] on random sampling in shift-invariant spaces over $\mathbb{R}^d$, we show that such inequalities for PSI spaces hold with high probability when the sampling density is sufficiently large. The analysis employs harmonic tools on the circle group, including the Zak transform and fiberization map to obtain stability conditions for reconstruction. Applications are discussed for trigonometric polynomials, periodic multi-band signals, wavelet subspaces, and PSI spaces generated by Haar-type scaling functions.

4.1 Introduction to the problem

We consider a PSI space $V \subset L^2(\mathbb{T})$ generated by a finite set $\Phi \subset L^2(\mathbb{T})$ consisting of r-times differentiable functions. To formulate the sampling problem, we introduce a periodic non-uniform sampling set $\mathcal{Q} \subset \mathbb{T}$ of the form

$$\mathcal{Q} = \frac{1}{N}\mathbb{Z}_N + \{\lambda_1, \lambda_2, \dots, \lambda_L\},$$

and study whether stable recovery is possible from samples of the function and its derivatives taken on $\mathcal{Q}$. Specifically, we analyze the validity of the sampling inequality

$$A\|f\|^2 \leq \sum_{p=0}^{r-1}\sum_{j=1}^{L}\sum_{\ell\in\mathbb{Z}_N} \left| f^{(p)}\left(\lambda_j + \frac{\ell}{N}\right)\right|^2 \leq B\|f\|^2, \quad \forall f \in V, \tag{4.1}$$

The results presented in this chapter are based on our work:

Kalra, S. and Shukla, N. K. 2025. Random sampling involving derivatives for periodic shift-invariant spaces, *Under preparation.*

for some constants $A, B > 0$, where $f^{(p)}$ denotes the p-th derivative of f. The goal is to derive sufficient conditions under which (4.1) holds, thereby ensuring stable reconstruction from non-uniform derivative samples in V.

Unlike classical sampling schemes that rely solely on point evaluations of the signal, incorporating derivative samples in (4.1) provides access to additional local information, which further enhances the stability and accuracy of reconstruction. In the periodic setting, sampling with derivatives in shift-invariant spaces has been studied in [130]. For the non-periodic case, non-uniform sampling involving derivatives has been explored for bandlimited functions in [68, 71, 124, 125], and more recently in the context of shift-invariant spaces in [2, 77, 79]. PSI spaces provide a unifying framework for a wide range of sampling problems, encompassing trigonometric polynomials, periodic splines, and subspaces generated by periodic wavelets. A prototypical example is the space $\mathcal{P}_R$ of trigonometric polynomials of degree at most R, which arises naturally as a PSI space. These spaces play a central role in harmonic analysis and have broad applications in the theory of periodic wavelets, finite frames, and approximation theory.

Our main result (see Theorem 4.2) establishes probabilistic sampling inequalities for signals in PSI subspaces of $L^2(\mathbb{T})$. Specifically, we demonstrate that if the sampling nodes $\{\lambda_1, \lambda_2, \ldots, \lambda_L\} \subset \mathbb{T}$ are chosen independently at random, then the sampling inequalities (4.1) hold with high probability, provided the number of samples L are sufficiently high. The sampling strategy carried out in this chapter is inspired by the work of Antezana, Carbajal, and Romero [14], who established similar result for shift-invariant spaces on $\mathbb{R}^d$ without derivatives. In contrast, our approach utilizes the framework of PSI spaces and incorporates derivative information in the sampling process. The novelty of the periodic framework lies in its compact structure, where the boundedness of continuous functions and the inherent Reproducing Kernel Hilbert Space (RKHS) property of PSI subspaces simplifies the analysis and reconstruction.

Our methodology for finding the probability of the sampling inequalities (4.1) is based on the study of random matrices arising from the Zak transform on $\mathbb{T}$, evaluated at the random sampling nodes $\{\lambda_1, \ldots, \lambda_L\}$. The use of random matrices for random sampling of bandlimited functions is facilitated in [18, 19] and extended to several other signal models including shift-invariant spaces [58, 69, 89, 153] and more general reproducing kernel Hilbert spaces [105]. To derive a probabilistic bound for (4.1), we reformulate the problem using the fiberization map and the associated range function, reducing it to a spectral estimate for a random matrix defined via the Zak transform. This reformulation enables the application of matrix Bernstein inequality to control the deviation of eigenvalues, thereby yielding high-probability bounds for the sampling inequalities. The literature on random sampling is extensive. For instance, [41, 128] investigate sampling for bandlimited signals by checking that deterministic conditions for sampling apply to certain random perturbations of the integer grid. In addition, [17, 103, 123] investigates random sampling of frequency-sparse multivariate trigonometric polynomials.

4.2 Analytic framework for random sampling in PSI spaces

We begin by recalling the definition of PSI spaces. Let $N \in \mathbb{N}$. A closed subspace $V_N \subseteq L^2(\mathbb{T})$ is called a *peridodic shift-invariant (PSI) space* if

$$f\big(\cdot - \tfrac{k}{N}\big) \in V_N \quad \text{for all } f \in V_N \text{ and } k \in \mathbb{Z}_N.$$

Given a finite collection $\Phi = \{\phi_1, \phi_2, \dots, \phi_M\} \subset L^2(\mathbb{T})$, the associated finitely generated PSI system and space are defined by

$$E_N(\Phi) := \big\{\phi_i\big(\cdot - \tfrac{k}{N}\big) : i = 1, \dots, M,\ k \in \mathbb{Z}_N\big\}, \qquad V_N(\Phi) := \overline{\operatorname{span}}\, E_N(\Phi). \tag{4.2}$$

A fundamental tool for analyzing PSI spaces is the fiberization map, which provides a bridge between the spatial and frequency domains. It enables the study of translation-invariant structures through their frequency localization properties.

For a function $f \in L^1(\mathbb{T})$, let its Fourier coefficients be defined by

$$\widehat{f}(n) := \int_0^1 f(\theta)\, e^{-2\pi i n\theta}\, d\theta, \quad n \in \mathbb{Z}.$$

Using these coefficients, the *fiberization map*

$$\mathcal{T} : L^2(\mathbb{T}) \to L^2(\mathbb{Z}_N \times \mathbb{Z}) \tag{4.3}$$

is defined by $(\mathcal{T} f)(\ell, m) := \widehat{f}(mN + \ell)$, $\ell \in \mathbb{Z}_N$, $m \in \mathbb{Z}$. This map is an isometric isomorphism that preserves inner products, i.e.,

$$\langle \mathcal{T} f, \mathcal{T} g\rangle = \langle f, g\rangle, \quad \text{for } f, g \in L^2(\mathbb{T}), \tag{4.4}$$

and intertwines periodic translations with frequency modulations as

$$\mathcal{T}\big(f\big(\cdot - \tfrac{k}{N}\big)\big)\,(\ell, m) = e^{-2\pi i k\ell/N}\, \mathcal{T} f(\ell, m), \qquad k, \ell \in \mathbb{Z}_N,\ m \in \mathbb{Z}. \tag{4.5}$$

The fiberization map thus decomposes each function into its frequency fibers indexed by $\ell \in \mathbb{Z}_N$. This representation allows one to describe PSI spaces in terms of range functions, as established by Helson's theorem [84]. Specifically, every PSI space $V \subseteq L^2(\mathbb{T})$ admits a range function representation of the form

$$V = \left\{ f \in L^2(\mathbb{T}) : \mathcal{T}f(\ell,\cdot) \in J(\ell),\ \ell \in \mathbb{Z}_N \right\},$$

where J assigns to each $\ell \in \mathbb{Z}_N$, a closed subspace of $L^2(\mathbb{Z})$. In the finitely generated case, when $V = V_N(\Phi)$ with $\Phi = \{\phi_1, \dots, \phi_M\}$, the range function is explicitly given by

$$J(\ell) = \overline{\operatorname{span}}\{\mathcal{T}\phi_i(\ell,\cdot) : i = 1, \dots, M\}. \tag{4.6}$$

Since $V_N(\Phi)$ is finite-dimensional, it forms a RKHS. Consequently, for each $\theta \in \mathbb{T}$, there exists a unique element $K_\theta \in V_N(\Phi)$ satisfying

$$f(\theta) = \langle f, K_\theta \rangle, \quad f \in V_N(\Phi),$$

with the reproducing kernel defined as

$$K(\theta,\delta) := \langle K_\delta, K_\theta \rangle, \quad \theta,\delta \in \mathbb{T}.$$

If, in particular, $E_N(\Phi)$ forms an orthonormal basis of $V_N(\Phi)$, then the kernel admits the explicit representation

$$K_\theta(\delta) = \sum_{i=1}^{M} \sum_{\ell \in \mathbb{Z}_N} \overline{\phi_i\big(\theta + \tfrac{\ell}{N}\big)}\, \phi_i\big(\delta + \tfrac{\ell}{N}\big), \quad \theta,\delta \in \mathbb{T}. \tag{4.7}$$

For more details, we refer the reader to [28, 42].

It is important to note that the derivation of bounds in (4.1) relies on the family of fiber systems

$$\mathcal{G}_\ell = \left\{ \mathcal{T}\phi_i^{(p)}(\ell,\cdot) : 1 \le i \le M,\ 0 \le p \le r-1 \right\},$$

which naturally lies in the range space $J(\ell)$ for each $\ell \in \mathbb{Z}_N$. Indeed, for every generator ϕ_i, with $1 \le i \le M$, and for all $\ell \in \mathbb{Z}_N$, $m \in \mathbb{Z}$, and $0 \le p \le r-1$, we have

$$\mathcal{T}\phi_i^{(p)}(\ell,m) = \widehat{\phi_i^{(p)}}(mN+\ell) = (2\pi i(mN+\ell))^p\, \widehat{\phi}_i(mN+\ell) = (2\pi i(mN+\ell))^p\, \mathcal{T}\phi_i(\ell,m).$$

Hence, for each fixed $\ell \in \mathbb{Z}_N$, the fiber

$$\mathcal{T}\phi_i^{(p)}(\ell,\cdot) = \big((2\pi i(mN+\ell))^p\, \mathcal{T}\phi_i(\ell,m)\big)_{m\in\mathbb{Z}}$$

is obtained by pointwise multiplication of $\mathcal{T}\phi_i(\ell,\cdot)$ with the scalar sequence $(2\pi i(mN+\ell))^p$, confirming that $\mathcal{G}_\ell \subset J(\ell)$.

Moreover, since the subcollection $\{\mathcal{T}\phi_i(\ell,\cdot) : 1 \le i \le M\} \subset \mathcal{G}_\ell$ forms an orthonormal basis for $J(\ell)$, it follows that $\mathcal{G}_\ell$ constitutes a frame for $J(\ell)$ for every $\ell \in \mathbb{Z}_N$. Let u_ℓ and v_ℓ denote its corresponding lower and upper frame bounds, respectively. Then, for any $c \in J(\ell)$, we have

$$U\|c\|^2 \le u_\ell\|c\|^2 \le \|\mathcal{A}_{\Phi,\ell}\, c\|^2 \le v_\ell\|c\|^2 \le V\|c\|^2, \tag{4.8}$$

where $U := \min_{\ell\in\mathbb{Z}_N} u_\ell$ and $V := \max_{\ell\in\mathbb{Z}_N} v_\ell$.

4.2.1 Probability bounds

For a matrix $X \in \mathbb{C}^{n\times n}$, we define the norm

$$\|X\| := \sup\left\{\|Xa\|_2 : a \in \mathbb{C}^n,\ \|a\|_2 = 1\right\}.$$

For two Hermitian matrices $A, B \in \mathbb{C}^{n\times n}$, we write $A \le B$ to denote that $B - A$ is a positive semi-definite matrix. This relation is known as the *Loewner partial order.*

In the sequel, we use the matrix Bernstein inequality, stated below, to establish probabilistic bounds.

Lemma 4.1. *Let $\{X_1, \ldots, X_\ell\}$ be a sequence of independent, random self-adjoint $n \times n$ matrices. Suppose that*

$$\mathbb{E}[X_i] = 0 \quad \text{and} \quad \|X_i\| \le B \quad \text{almost surely,} \quad \text{for all } i = 1, \ldots, \ell,$$

and define

$$\sigma^2 := \left\|\sum_{i=1}^{\ell} \mathbb{E}[X_i^2]\right\|.$$

Then, for all $m \ge 0$, we have

$$\mathbb{P}\left(\lambda_{\max}\left(\sum_{i=1}^{\ell} X_i\right) \ge m\right) \le n \exp\left(-\frac{m^2/2}{\sigma^2 + Bm/3}\right).$$

Throughout this chapter, we assume that the collection $\Phi = \{\phi_1, \phi_2, \ldots, \phi_M\} \subset L^2(\mathbb{T})$ consists of functions that are r-times differentiable on $\mathbb{T}$ such that $E_N(\Phi)$, defined in (4.2), is an orthonormal set in $L^2(\mathbb{T})$. We further assume that J, as given in (4.6), denotes the range function associated with $V_N(\Phi)$, and K_θ represents the reproducing kernel associated with $V_N(\Phi)$ at point $\theta \in \mathbb{T}$. Additionally, we fix the set $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_L\}$

to be composed of L independent random variables that are uniformly distributed over $\mathbb{T}$. For any $f \in L^2(\mathbb{T})$, the notation $f^{(p)}$ is used to indicate its p-th derivative.

4.3 Random periodic sampling with derivatives

We begin by presenting our main result, which establishes probabilistic sampling inequalities for PSI subspaces of $L^2(\mathbb{T})$. This result is inspired by the framework developed in [14, Theorem 2.1]. We show that if the sampling points $\{\lambda_1, \lambda_2, \dots, \lambda_L\} \subset \mathbb{T}$ are chosen independently at random, then stable reconstruction of any signal from a given PSI subspace holds with high probability, provided that the number of samples L exceeds a threshold that depends on the dimension and the cardinality of the generating set of the PSI subspace.

We now state our main result.

Theorem 4.2. *Let $\Phi = \{\phi_1, \dots, \phi_M\} \subset L^2(\mathbb{T})$ be a collection of r-times differentiable functions such that $E_N(\Phi)$ (defined in (4.2)) forms an orthonormal system and Φ satisfies the bounded frequency periodization condition*

$$\max_{1\le i\le M} \max_{\ell\in\mathbb{Z}_N} \sum_{m\in\mathbb{Z}} |\widehat{\phi}_i(mN+\ell)| < \infty. \tag{4.9}$$

Assume that the sampling set $\Lambda = \{\lambda_1, \dots, \lambda_L\}$ consists of L independent random variables uniformly distributed over $\mathbb{T}$, and that for some $\epsilon > 0$ and $0 < \alpha < 1$,

$$L \ge \frac{3\,(NMC+1)}{\alpha^2} \cdot \log\left(\frac{2NM}{\epsilon}\right),$$

where

$$C := \max_{1\le i\le M} \sup_{\theta\in\mathbb{T}} \sum_{k\in\mathbb{Z}_N} \left|\phi_i\left(\theta + \tfrac{k}{N}\right)\right|^2. \tag{4.10}$$

Then, with probability at least $1-\epsilon$, the following sampling inequalities hold for every $f \in V_N(\Phi)$:

$$LN\,V(1-\alpha)\,\|f\|^2 \le \sum_{p=0}^{r-1}\sum_{j=1}^{L}\sum_{\ell\in\mathbb{Z}_N} \left|f^{(p)}\left(\lambda_j + \tfrac{\ell}{N}\right)\right|^2 \le LN\,U(1+\alpha)\,\|f\|^2, \tag{4.11}$$

where $U, V > 0$ denote, respectively, the uniform lower and upper frame bounds corresponding to the family of fiber systems

$$\mathcal{G}_\ell = \left\{\mathcal{T}\phi_i^{(p)}(\ell,\cdot) : 1 \le i \le M,\ 0 \le p \le r-1\right\}, \qquad \ell \in \mathbb{Z}_N.$$

Remark 4.3. Observe that the value C defined in (4.10) is finite since the mapping $\theta \mapsto \sum_{k\in\mathbb{Z}_N} \left|\phi_i\left(\theta+\frac{k}{N}\right)\right|^2$ is continuous on $\mathbb{T}$ and due to its compactness, we have

$$\max_{1\le i\le M} \sup_{\theta\in\mathbb{T}} \sum_{k\in\mathbb{Z}_N} \left|\phi_i\left(\theta+\frac{k}{N}\right)\right|^2 < \infty.$$

Remark 4.4. The bounded frequency periodization condition, described in (4.9), guarantees the absolute convergence (and hence uniform convergence) of the Fourier series

$$\phi_i(\theta) = \sum_{n\in\mathbb{Z}} \widehat{\phi}_i(n)\, e^{2\pi i n\theta}, \quad \theta\in\mathbb{T},\ i=1,2,\ldots,M, \tag{4.12}$$

since as a consequence of (4.9), we get $\sum_{n\in\mathbb{Z}} |\hat{\phi}_i(n)| = \sum_{\ell\in\mathbb{Z}_N}\sum_{m\in\mathbb{Z}} |\hat{\phi}_i(mN+\ell)| < \infty$.

It is important to observe that if the collection Φ consists of continuously differentiable functions on $\mathbb{T}$, then by [134, Exercise 14, Chapter 3], the Fourier series representation (4.12) converges absolutely for each $i=1,2,\ldots,M$. Thus, for $r\ge 2$, the condition (4.9) can be relaxed.

Remark 4.5. An important feature of the sampling inequalities (4.11) is its *snugness*, which refers to the fact that the ratio between the upper and lower stability bounds remains close to one. Specifically, when the number of random samples L satisfies the logarithmic lower bound in Theorem 4.2, the constants $1-\alpha$ and $1+\alpha$ appearing in the inequalities are close to each other. This implies that the sampling operator is nearly an isometry on the space $V_N(\Phi)$, thereby ensuring stable and computationally efficient recovery of functions from their random derivative samples.

4.3.1 Analysis of the main result via Zak matrix

We define the Zak transform in the setting of the circle group, which serves as a crucial tool for analyzing the structure of PSI spaces.

Definition 4.6. The *Zak transform* is the mapping $\mathcal{Z} : L^2(\mathbb{T}) \to L^2(\mathbb{Z}_N\times\mathbb{T})$ defined by

$$\mathcal{Z}f\,(\ell,\theta) := \frac{1}{\sqrt{N}} \sum_{k\in\mathbb{Z}_N} f\left(\theta+\tfrac{k}{N}\right) e^{-2\pi i k\ell/N}, \quad \ell\in\mathbb{Z}_N,\ \theta\in\mathbb{T},\ f\in L^2(\mathbb{T}).$$

For the random set $\Lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_L\}$, we define the Zak matrix $\mathfrak{Z}_{\ell,\Lambda}$ of size $rM\times rL$ consisting of repeating $M\times L$ blocks. The first $M\times L$ block, denoted by $\mathcal{B}_{\ell,\Lambda}$, is given as:

$$\mathcal{B}_{\ell,\Lambda} = \sqrt{N} \begin{bmatrix} \mathcal{Z}\phi_1(\ell,\lambda_1) & \mathcal{Z}\phi_1(\ell,\lambda_2) & \cdots & \mathcal{Z}\phi_1(\ell,\lambda_L) \\ \mathcal{Z}\phi_2(\ell,\lambda_1) & \mathcal{Z}\phi_2(\ell,\lambda_2) & \cdots & \mathcal{Z}\phi_2(\ell,\lambda_L) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Z}\phi_M(\ell,\lambda_1) & \mathcal{Z}\phi_M(\ell,\lambda_2) & \cdots & \mathcal{Z}\phi_M(\ell,\lambda_L) \end{bmatrix}, \quad \ell \in \mathbb{Z}_N. \tag{4.13}$$

Now, the full matrix $\mathfrak{Z}_{\ell,\Lambda}$ is formed by repeating $\mathcal{B}_{\ell,\Lambda}$ along the diagonal:

$$\mathfrak{Z}_{\ell,\Lambda} = \begin{bmatrix} \mathcal{B}_{\ell,\Lambda} & 0 & 0 & \cdots & 0 \\ 0 & \mathcal{B}_{\ell,\Lambda} & 0 & \cdots & 0 \\ 0 & 0 & \mathcal{B}_{\ell,\Lambda} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathcal{B}_{\ell,\Lambda} \end{bmatrix}, \quad \ell \in \mathbb{Z}_N.$$

It can be noted that the matrices $\mathcal{B}_{\ell,\Lambda}$ and $\mathfrak{Z}_{\ell,\Lambda}$ are random for each $\ell \in \mathbb{Z}_N$ due to their dependence on the random set Λ.

The following result characterizes the sampling inequalities (4.11) through equivalent formulations in terms of frame property linked with reproducing kernel K and the Zak matrix $\mathfrak{Z}_{\ell,\Lambda}$. The result is motivated from the work of [14], who also utilize similar equivalences (without involving derivatives) in their framework to prove their main result. While similar results already exists in the literature (see, for instance, [23, 32]), the key novelty here is the inclusion of derivatives in the frame system and its associated system in the fiber space given by the range function.

We have the following result:

Theorem 4.7. *The following statements are equivalent for some constants* $0 < A \leq B < \infty$:

(i) The sampling inequalities

$$A\|f\|^2 \leq \sum_{p=0}^{r-1} \sum_{j=1}^{L} \sum_{\ell \in \mathbb{Z}_N} \left| f^{(p)}\left(\lambda_j + \frac{\ell}{N}\right) \right|^2 \leq B\|f\|^2$$

hold for all $f \in V_N(\Phi)$.

(ii) The set of translates of the reproducing kernel and its derivatives

$$\left\{ K_{\lambda_j}^{(p)}\left(\cdot - \frac{\ell}{N}\right) : 1 \leq j \leq L,\ \ell \in \mathbb{Z}_N,\ 0 \leq p \leq r-1 \right\}$$

is a frame for $V_N(\Phi)$ *with frame bounds* A, B.

(iii) For all $\ell \in \mathbb{Z}_N$, the collection $\{\mathcal{T}K^{(p)}_{\lambda_j}(\ell,.) : 1 \le j \le L, 0 \le p \le r-1\}$ is a frame for $J(\ell)$ with uniform frame bounds $\frac{A}{N}$, $\frac{B}{N}$.

(iv) For all $\ell \in \mathbb{Z}_N$ and all $a \in \mathcal{A}_{\Phi,\ell}(J(\ell))$,

$$\frac{A}{N}\|a\|^2 \le \langle \overline{\mathfrak{Z}_{\ell,\Lambda}}\mathfrak{Z}^t_{\ell,\Lambda}a, a\rangle \le \frac{B}{N}\|a\|^2, \tag{4.14}$$

where for each $\ell \in \mathbb{Z}_N$, $\mathfrak{Z}^t_{\ell,\Lambda}$ denotes the transpose of the matrix $\mathfrak{Z}_{\ell,\Lambda}$ and $\mathcal{A}_{\Phi,\ell}$ denotes the analysis operator associated with the frame $\mathcal{G}_\ell$.

Proof. First, we claim that for any $f \in V_N(\Phi)$ and $p = 0, 1, \dots, r-1$, the derivatives $f^{(p)}$ belong to $L^2(\mathbb{T})$. Indeed, by the definition of the PSI space $V^t_m(\phi)$, $f^{(p)}$ admits the representation

$$f^{(p)}(\theta) = \sum_{i=1}^{M}\sum_{\ell\in\mathbb{Z}_N} c_{\ell,i}\,\phi_i^{(p)}\left(\theta - \frac{\ell}{N}\right), \quad \theta \in \mathbb{T},$$

for some sequence $(c_{\ell,i})^{\ell\in\mathbb{Z}_N}_{i=1,\dots,M} \in \ell^2(\mathbb{Z}_N \times \{1, 2, \dots, M\})$. Since $\phi_i^{(p)} \in C(\mathbb{T}) \subseteq L^2(\mathbb{T})$ for all $p = 0, 1, \dots, r-1$ and $i = 1, 2, \dots, M$, it follows that $f^{(p)} \in L^2(\mathbb{T})$, completing the claim.

(i) $\Longleftrightarrow$ (ii). Note that by using (4.7) and the orthonormal expansion of f in $V_N(\Phi)$ in terms of the orthonormal basis $E_N(\Phi)$, we obtain

$$f^{(p)}(\theta) = \langle f, K^{(p)}_\theta\rangle, \quad \theta \in \mathbb{T},\ p = 0, 1, \dots, r-1. \tag{4.15}$$

Again using (4.7), we compute

$$\begin{aligned} K_{\theta+\frac{k}{N}}(\omega) &= \sum_{i=1}^{M}\sum_{\ell\in\mathbb{Z}_N} \overline{\phi_i\left(\theta + \frac{k}{N} + \frac{\ell}{N}\right)}\phi_i\left(\omega + \frac{\ell}{N}\right) \\ &= \sum_{i=1}^{M}\sum_{\ell\in\mathbb{Z}_N} \overline{\phi_i\left(\theta + \frac{\ell}{N}\right)}\phi_i\left(\omega + \frac{\ell}{N} - \frac{k}{N}\right) = K_\theta\left(\omega - \frac{k}{N}\right), \quad \theta, \omega \in \mathbb{T},\ k \in \mathbb{Z}_N. \end{aligned} \tag{4.16}$$

Then by substituting $\theta = \lambda_j + \frac{k}{N}$ into (4.15) and using (4.16), we obtain

$$f^{(p)}\left(\lambda_j + \frac{k}{N}\right) = \left\langle f, K^{(p)}_{\lambda_j+\frac{k}{N}}\right\rangle = \left\langle f, K^{(p)}_{\lambda_j}\left(. - \frac{k}{N}\right)\right\rangle \tag{4.17}$$

for $k \in \mathbb{Z}_N$, $p = 0, 1, \dots, r-1$ and $j = 1, 2, \dots, L$. Then (i) $\Longleftrightarrow$ (ii) is a consequence of (4.17).

(ii) $\iff$ (iii). By using this fact along with (4.4), (4.5) and Parseval's identity, we obtain

$$\begin{aligned}\sum_{j=1}^{L}\sum_{p=0}^{r-1}\sum_{k\in\mathbb{Z}_N}\left|\left\langle f, K_{\lambda_j}^{(p)}\left(.-\frac{k}{N}\right)\right\rangle\right|^2 &= \sum_{j=1}^{L}\sum_{p=0}^{r-1}\sum_{k\in\mathbb{Z}_N}\left|\left\langle \mathcal{T}f, \mathcal{T}K_{\lambda_j}^{(p)}\left(.-\frac{k}{N}\right)\right\rangle\right|^2\\ &= \sum_{j=1}^{L}\sum_{p=0}^{r-1}\sum_{k\in\mathbb{Z}_N}\left|\sum_{\ell\in\mathbb{Z}_N} e^{2\pi i\ell k/N}\left\langle \mathcal{T}f(\ell,.), \mathcal{T}K_{\lambda_j}^{(p)}(\ell,.)\right\rangle\right|^2 \\ &= N\sum_{j=1}^{L}\sum_{p=0}^{r-1}\sum_{\ell\in\mathbb{Z}_N}\left|\left\langle \mathcal{T}f(\ell,.), \mathcal{T}K_{\lambda_j}^{(p)}(\ell,.)\right\rangle\right|^2, \quad \forall f\in V_m^t(\phi).\end{aligned} \tag{4.18}$$

Suppose (iii) holds. Note that by using (4.6), we have $\mathcal{T}f(\ell,.)\in J(\ell)$ for all $\ell\in\mathbb{Z}_N$. Then by (iii), there exists constants $0<A\leq B<\infty$ such that

$$\frac{A}{N}\|\mathcal{T}f(\ell,.)\|^2 \leq \sum_{j=1}^{L}\sum_{p=0}^{r-1}\left|\left\langle \mathcal{T}f(\ell,.), \mathcal{T}K_{\lambda_j}^{(p)}(\ell,.)\right\rangle\right|^2 \leq \frac{B}{N}\|\mathcal{T}f(\ell,.)\|^2, \quad \ell\in\mathbb{Z}_N.$$

Summing over $\ell=0,\ldots,N-1$ and using (4.18) along with the isometry property of $\mathcal{T}$, we get

$$A\|f\|^2 \leq \sum_{j=1}^{L}\sum_{p=0}^{r-1}\sum_{k\in\mathbb{Z}_N}\left|\left\langle f, K_{\lambda_j}^{(p)}\left(.-\frac{k}{N}\right)\right\rangle\right|^2 \leq B\|f\|^2, \quad \forall f\in L^2(\mathbb{T}).$$

Hence (ii) holds.

Conversely, assume (ii) holds with frame bounds A, B. Suppose (iii) is false. Then there exists $\ell\in\mathbb{Z}_N$ and $u\in J(\ell)$ such that at least one of the following two happens:

$$\sum_{j=1}^{L}\sum_{p=0}^{r-1}\left\langle \mathcal{T}K_{\lambda_j}^{(p)}(\ell,.), u\right\rangle > \frac{B}{N}\|u\|^2 \quad \text{or} \quad \sum_{j=1}^{L}\sum_{p=0}^{r-1}\left\langle \mathcal{T}K_{\lambda_j}^{(p)}(\ell,.), u\right\rangle < \frac{A}{N}\|u\|^2. \tag{4.19}$$

Suppose the first inequality of (4.19) is true. Define $g\in L^2(\mathbb{T})$ such that for all $k\in\mathbb{Z}_N$,

$$(\mathcal{T}g)(k,.) = \begin{cases} u, & \text{if } k=\ell\\ 0, & \text{otherwise.}\end{cases}$$

Since $\mathcal{T}g(k,.) \in J(k)$ for all $k \in \mathbb{Z}_N$, then by the definition of $V_m^t(\phi)$, we have $g \in V_N(\Phi)$. Finally by using (4.18) for g, and (4.4), we get

$$\sum_{j=1}^{L}\sum_{p=0}^{r-1}\sum_{k\in\mathbb{Z}_N}\left|\left\langle g, K_{\lambda_j}^{(p)}\left(. - \tfrac{k}{N}\right)\right\rangle\right|^2 = N\sum_{j=1}^{L}\sum_{p=0}^{r-1}\sum_{k\in\mathbb{Z}_N}\left|\left\langle (\mathcal{T}g)(k,.), \mathcal{T}K_{\lambda_j}^{(p)}(k,.)\right\rangle\right|^2$$
$$> B\|u\|^2 = B\sum_{k\in\mathbb{Z}_N}\|(\mathcal{T}g)(k,.)\|^2 = B\|\mathcal{T}g\|^2 = B\|g\|^2,$$

which is a contradiction to (ii). Similarly if the second inequality of (4.19) holds, it also contradicts condition (ii). Hence (iii) is true.

(iii) $\Longleftrightarrow$ (iv). For $\alpha = (\alpha_{j,p})_{\substack{j=1,2,\ldots,L\\ p=0,1,\ldots,r-1}} \in \mathbb{C}^{rL}$, we compute

$$\begin{aligned}
\mathcal{A}^*_{\Phi,\ell}\left(\mathfrak{Z}_{\ell,\Lambda}\,\alpha\right) &= \sqrt{N}\sum_{p=0}^{r-1}\sum_{i=1}^{M}\left(\sum_{j=1}^{L}\overline{\mathcal{Z}\phi_i(\ell,\lambda_j)}\alpha_{j,p}\right)\mathcal{T}\phi_i^{(p)}(\ell,.)\\
&= \sum_{p=0}^{r-1}\sum_{i=1}^{M}\sum_{j=1}^{L}\alpha_{j,p}\left(\sum_{k\in\mathbb{Z}_N}\overline{\phi_i\left(\lambda_j+\tfrac{k}{N}\right)}e^{2\pi ik\ell/N}\right)\left(\widehat{\phi_i^{(p)}}(mN+\ell)\right)_{m\in\mathbb{Z}}\\
&= \left(\sum_{p=0}^{r-1}\sum_{i=1}^{M}\sum_{j=1}^{L}\sum_{k\in\mathbb{Z}_N}\alpha_{j,p}\,\overline{\phi_i\left(\lambda_j+\tfrac{k}{N}\right)}\int_0^1\phi_i^{(p)}(\theta)e^{-2\pi i(mN+\ell)(\theta-k/N)}d\theta\right)_{m\in\mathbb{Z}}\\
&= \left(\sum_{p=0}^{r-1}\sum_{i=1}^{M}\sum_{j=1}^{L}\sum_{k\in\mathbb{Z}_N}\alpha_{j,p}\,\overline{\phi_i\left(\lambda_j+\tfrac{k}{N}\right)}\int_0^1\phi_i^{(p)}\left(\theta+\frac{k}{N}\right)e^{-2\pi i(mN+\ell)\theta}d\theta\right)_{m\in\mathbb{Z}}\\
&= \left(\sum_{p=0}^{r-1}\sum_{j=1}^{L}\alpha_{j,p}\int_0^1 K_{\lambda_j}^{(p)}(\theta)e^{-2\pi i(mN+\ell)\theta}d\theta\right)_{m\in\mathbb{Z}}\\
&= \left(\sum_{p=0}^{r-1}\sum_{j=1}^{L}\alpha_{j,p}\,\widehat{K_{\lambda_j}^{(p)}}(mN+\ell)\right)_{m\in\mathbb{Z}} = \sum_{p=0}^{r-1}\sum_{j=1}^{L}\alpha_{j,p}\mathcal{T}K_{\lambda_j}^{(p)}(\ell,.) = \mathcal{A}^*_{K,\ell}\alpha,
\end{aligned}$$

where for $\ell \in \mathbb{Z}_N$, $\mathcal{A}_{K,\ell}$ is the analysis operator associated with the system $\{\mathcal{T}K_{\lambda_j}^{(p)}(\ell,.) : 1 \le j \le L, 0 \le p \le r-1\}$. This gives $\mathcal{A}^*_{K,\ell} = \mathcal{A}^*_{\Phi,\ell}\overline{\mathfrak{Z}_{\ell,\Lambda}}$ for all $\ell \in \mathbb{Z}_N$. Then note that for $c \in J(\ell)$, we get

$$\begin{aligned}
\left\langle\overline{\mathfrak{Z}_{\ell,\Lambda}}\,\mathfrak{Z}_{\ell,\Lambda}^t\,\mathcal{A}_{\Phi,\ell}c,\ \mathcal{A}_{\Phi,\ell}c\right\rangle &= \left\langle\mathcal{A}^*_{\Phi,\ell}\,\overline{\mathfrak{Z}_{\ell,\Lambda}}\,\mathfrak{Z}_{\ell,\Lambda}^t\,\mathcal{A}_{\Phi,\ell}c,\ c\right\rangle\\
&= \left\langle\mathcal{A}^*_{K,\ell}\,\mathcal{A}_{K,\ell}c,\ c\right\rangle = \langle\mathcal{A}_{K,\ell}c,\ \mathcal{A}_{K,\ell}c\rangle\\
&= \left\langle\left\{\left\langle c,\ \mathcal{T}K_{\lambda_j}^{(p)}(\ell,.)\right\rangle\right\}_{\substack{1\le j\le L\\ 0\le p\le r-1}},\ \left\{\left\langle c,\ \mathcal{T}K_{\lambda_j}^{(p)}(\ell,.)\right\rangle\right\}_{\substack{1\le j\le L\\ 0\le p\le r-1}}\right\rangle\\
&= \sum_{j=1}^{L}\sum_{p=0}^{r-1}\left|\left\langle c,\ \mathcal{T}K_{\lambda_j}^{(p)}(\ell,.)\right\rangle\right|^2,
\end{aligned}$$

which proves the equivalence (iii) $\Longleftrightarrow$ (iv). This completes the proof. $\square$

The following result provides a reformulation of the sampling inequalities (4.11). It provides a sufficient condition, expressed in terms of the matrix $\mathcal{B}_{\ell,\Lambda}$, under which the Zak transform based inequalities (4.14) and consequently, all equivalent conditions in Proposition 4.7 are satisfied.

Proposition 4.8. *Let $\mathcal{B}_{\ell,\Lambda} \in \mathbb{C}^{M\times L}$ be the matrix defined in (4.13). Suppose that for every $\ell \in \mathbb{Z}_N$, there exist constants $A, B > 0$ (independent of ℓ) such that*

$$A\|c\|^2 \leq \left\langle \overline{\mathcal{B}_{\ell,\Lambda}}\, \mathcal{B}^t_{\ell,\Lambda} c,\ c \right\rangle \leq B\|c\|^2 \tag{4.20}$$

for all $c \in \mathbb{C}^M$. Then, for every $\ell \in \mathbb{Z}_N$ and all $a \in \mathbb{C}^{rM}$, we have

$$A\|a\|^2 \leq \left\langle \overline{\mathfrak{Z}_{\ell,\Lambda}}\, \mathfrak{Z}^t_{\ell,\Lambda} a,\ a \right\rangle \leq B\|a\|^2.$$

Proof. Notice that any $a \in \mathbb{C}^{rM}$ can be written in the form:

$$a = \begin{bmatrix} a_0^t & a_1^t & \cdots & a_{r-1}^t \end{bmatrix}^t,$$

where $a_p \in \mathbb{C}^M$ for $0 \leq p \leq r-1$. Then for any $\ell \in \mathbb{Z}_N$ and $a \in \mathbb{C}^{rM}$, we have

$$\left\langle \overline{\mathfrak{Z}_{\ell,\Lambda}}\, \mathfrak{Z}^t_{\ell,\Lambda} a,\ a \right\rangle = \sum_{p=0}^{r-1} \left\langle \overline{\mathcal{B}_{\ell,\Lambda}}\, \mathcal{B}^t_{\ell,\Lambda} a_p,\ a_p \right\rangle.$$

Applying the assumption (4.20) to each term in the sum yields

$$A\|a_p\|^2 \leq \left\langle \overline{\mathcal{B}_{\ell,\Lambda}}\, \mathcal{B}^t_{\ell,\Lambda} a_p,\ a_p \right\rangle \leq B\|a_p\|^2,$$

for all $0 \leq p \leq r-1$. Summing these inequalities over p gives

$$A\sum_{p=0}^{r-1} \|a_p\|^2 \leq \sum_{p=0}^{r-1} \left\langle \overline{\mathcal{B}_{\ell,\Lambda}}\, \mathcal{B}^t_{\ell,\Lambda} a_p,\ a_p \right\rangle \leq B\sum_{p=0}^{r-1} \|a_p\|^2.$$

Noting $\|a\|^2 = \sum\limits_{p=0}^{r-1} \|a_p\|^2$, we conclude that

$$A\|a\|^2 \leq \left\langle \overline{\mathfrak{Z}_{\ell,\Lambda}}\, \mathfrak{Z}^t_{\ell,\Lambda} a,\ a \right\rangle \leq B\|a\|^2, \quad \text{for all } a \in \mathbb{C}^{rM}.$$

This completes the proof. $\square$

Observe that if the inequalities (4.20) hold for all $\ell \in \mathbb{Z}_N$ and all $a \in \mathbb{C}^M$, with constants A and B denoting the lower and upper bounds, respectively, then Proposition 4.8, Proposition 4.7 and the fact that $\mathcal{A}_{\Phi,\ell}(J(\ell)) \subset \mathbb{C}^{rM}$ ensure that the following sampling inequalities

$$AN\|f\|^2 \leq \sum_{p=0}^{r-1}\sum_{j=1}^{L}\sum_{\ell\in\mathbb{Z}_N}\left|f^{(p)}\left(\lambda_j+\frac{\ell}{N}\right)\right|^2 \leq BN\|f\|^2$$

hold for all $f \in V_N(\Phi)$ with scaled lower and upper bounds scaled by N, i.e., NA and NB, respectively.

In view of this, define the event $\mathcal{E}$ by

$$\mathcal{E} := \left\{\forall\, \ell \in \mathbb{Z}_N,\ \forall\, a \in \mathbb{C}^M,\ \|a\| = 1:\ L(1-\alpha) \leq \left\langle \overline{\mathcal{B}_{\ell,\Lambda}}\,\mathcal{B}_{\ell,\Lambda}^t a,\ a\right\rangle \leq L(1+\alpha)\right\}, \tag{4.21}$$

and let $\mathcal{G}$ denote the event that the sampling inequalities (4.11) hold for all $f \in V_N(\Phi)$. Then, by the above discussion and standard probabilistic arguments, we obtain

$$\mathbb{P}(\mathcal{E}) \leq \mathbb{P}(\mathcal{G}).$$

Consequently, if $\mathbb{P}(\mathcal{E}) \geq 1-\epsilon$, it necessarily follows that $\mathbb{P}(\mathcal{G}) \geq 1-\epsilon$ as well. Therefore, to establish Theorem 4.2, it suffices to show that the event $\mathcal{E}$ occurs with probability at least $1-\epsilon$.

The following lemma is required in the sequel.

Lemma 4.9. *Let $U_{\ell,\lambda_j} \in \mathbb{C}^{M\times M}$ be the random matrix defined by*

$$(U_{\ell,\lambda_j})_{i,i'} := N\,\overline{\mathcal{Z}\phi_i(\ell,\lambda_j)}\,\mathcal{Z}\phi_{i'}(\ell,\lambda_j), \quad i, i' \in \{1,\dots,M\}, \tag{4.22}$$

for each $j = 1,\dots,L$ and $\ell \in \mathbb{Z}_N$. If we assume that the condition (4.9) is satisfied, then for any $\ell \in \mathbb{Z}_N$ and $j = 1,\dots,L$, the following properties hold:

(i) $\mathbb{E}\left[U_{\ell,\lambda_j}\right] = I_M$.

(ii) $\left\|U_{\ell,\lambda_j} - I_M\right\| \leq NMC + 1$.

(iii) $\sigma^2 := \left\|\sum_{j=1}^{L}\mathbb{E}\left[\left(U_{\ell,\lambda_j} - I_M\right)^2\right]\right\| \leq L\,(NMC-1)$.

Proof. We first prove (i). By using Remark 4.4, the Fourier series of ϕ_i converges absolutely for $i = 1,2,\dots,M$. Consequently, the Zak transform of ϕ_i admits the following representation:

$$\begin{aligned}\mathcal{Z}\phi_i(\ell,\omega) &= \frac{1}{\sqrt{N}} \sum_{k\in\mathbb{Z}_N}\sum_{n\in\mathbb{Z}} \widehat{\phi}_i(n)\, e^{2\pi i n(\omega+k/N)}\, e^{-2\pi i k\ell/N} \\ &= \frac{1}{\sqrt{N}} \sum_{n\in\mathbb{Z}} \widehat{\phi}_i(n)\, e^{2\pi i n\omega} \sum_{k\in\mathbb{Z}_N} e^{2\pi i k(n-\ell)/N} \\ &= \sqrt{N} \sum_{m\in\mathbb{Z}} \widehat{\phi}_i(mN+\ell)\, e^{2\pi i(mN+\ell)\omega},\end{aligned}$$

for $\ell \in \mathbb{Z}_N$, $\omega \in \mathbb{T}$, and $i = 1, 2, \ldots, M$. We now compute

$$\begin{aligned}&\mathbb{E}[U_{\ell,\lambda_j}]_{i,i'} = N \int_0^1 \overline{\mathcal{Z}\phi_i(\ell,\theta)}\, \mathcal{Z}\phi_{i'}(\ell,\theta)\, d\theta \\ &= N^2 \sum_{m,n\in\mathbb{Z}} \overline{\widehat{\phi}_i(mN+\ell)}\, \widehat{\phi_{i'}}(nN+\ell) \int_0^1 e^{2\pi i(n-m)N\theta} d\theta \\ &= N \sum_{m\in\mathbb{Z}} \overline{\widehat{\phi}_i(mN+\ell)}\, \widehat{\phi_{i'}}(mN+\ell) = N\langle \mathcal{T}\phi_{i'}(\ell,.), \mathcal{T}\phi_i(\ell,.)\rangle.\end{aligned}$$

It is a routine exercise to show that the orthonormality of the system $E_N(\Phi)$ implies that for each $\ell \in \mathbb{Z}_N$, the set $\{\mathcal{T}\phi_i(\ell,.) : i = 1, 2, \ldots, M\}$ is an orthogonal basis for the subspace $J(\ell) \subset \ell^2(\mathbb{Z})$ and satisfies

$$\langle \mathcal{T}\phi_i(\ell,.), \mathcal{T}\phi_j(\ell,.)\rangle = \frac{1}{N}\delta_{i,j}, \quad i, j = 1, 2, \ldots, M. \tag{4.23}$$

As a consequence of (4.23), we get

$$\mathbb{E}[U_{\ell,\lambda_j}] = I_M, \quad \text{for } \ell \in \mathbb{Z}_N \text{ and } j = 1, 2, \ldots, L.$$

Now we prove (ii). Using (4.22) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\|U_{\ell,\lambda_j}\| = \sup\left\{\|U_{\ell,\lambda_j}a\|_2 : a \in \mathbb{C}^M,\ \|a\|_2 = 1\right\} \le \|U_{\ell,\lambda_j}\|_2 \\ &= N\left(\sum_{i=1}^{M}\sum_{i'=1}^{M} |\overline{\mathcal{Z}\phi_i(\ell,\lambda_r)}\mathcal{Z}\phi_{i'}(\ell,\lambda_j)|^2\right)^{1/2} = N\sum_{i=1}^{M} |\mathcal{Z}\phi_i(\ell,\lambda_r)|^2 \\ &= \sum_{i=1}^{M}\left|\sum_{k\in\mathbb{Z}_N} \phi_i\left(\lambda_j + \frac{k}{N}\right) e^{-2\pi i k\ell/N}\right|^2 \le N\sum_{i=1}^{M}\sum_{k\in\mathbb{Z}_N} \left|\phi_i\left(\lambda_j + \frac{k}{N}\right)\right|^2 \le NMC.\end{aligned} \tag{4.24}$$

This gives,

$$\|U_{\ell,\lambda_j} - I_M\| \le \|U_{\ell,\lambda_r}\| + \|I_M\| \le NMC + 1, \quad \ell \in \mathbb{Z}_N,\ j = 1, 2, \ldots, L.$$

Finally, we prove (iii). For each $\ell \in \mathbb{Z}_N$ and $j = 1, 2, \ldots, L$, observe that $U(\ell, \lambda_j)$ is a rank-one Hermitian matrix of the form

$$U_{\ell,\lambda_j} = Nvv^*, \quad \text{where } v = \begin{bmatrix} \overline{\mathcal{Z}\phi_1(\ell, \lambda_j)} & \overline{\mathcal{Z}\phi_2(\ell, \lambda_j)} & \cdots & \overline{\mathcal{Z}\phi_M(\ell, \lambda_j)} \end{bmatrix}^t \in \mathbb{C}^M.$$

This representation yields $x^* U_{\ell,\lambda_j} x = N|x^*v|^2 \geq 0$ for all $x \in \mathbb{C}^M$, and hence $U(\ell, \lambda_j)$ is positive semi-definite. Since $\|U_{\ell,\lambda_j}\|$ denotes the largest eigenvalue of U_{ℓ,λ_j}, the positive semi-definiteness of U_{ℓ,λ_j} implies

$$U^2_{\ell,\lambda_j} \leq \|U_{\ell,\lambda_j}\| \, U_{\ell,\lambda_j} \leq NMC \, U_{\ell,\lambda_j},$$

where the final inequality follows from (4.24). Using (i) and the linearity of the expectation operator, we obtain

$$NMC \, I_M - \mathbb{E}(U^2_{\ell,\lambda_j}) = \mathbb{E}(NMC \, U_{\ell,\lambda_j}) - \mathbb{E}(U^2_{\ell,\lambda_j}) = \mathbb{E}(NMC \, U_{\ell,\lambda_j} - U^2_{\ell,\lambda_j}) \geq 0,$$

where we have used the fact that the expectation of a positive semi-definite random matrix is again positive semi-definite. As a consequence, we compute

$$\begin{aligned} \mathbb{E}\left[(U_{\ell,\lambda_j} - I_M)^2\right] &= \mathbb{E}[U^2_{\ell,\lambda_j} - U_{\ell,\lambda_j} - U_{\ell,\lambda_j} + I_M] \\ &= \mathbb{E}[U^2_{\ell,\lambda_j}] - 2I_M + I_M = \mathbb{E}[U^2_{\ell,\lambda_j}] - I_M \\ &\leq (NMC - 1) \, I_M. \end{aligned}$$

Finally, summing over $j = 1, \ldots, L$ and applying the triangle inequality for the operator norm, we obtain

$$\left\| \sum_{j=1}^{L} \mathbb{E}\left[\left(U_{\ell,\lambda_j} - I_M\right)^2\right] \right\| \leq L(NMC - 1), \quad \ell \in \mathbb{Z}_N.$$

This completes the proof. □

We are now ready to provide the proof of our main result.

4.3.2 Proof of the main result

We will develop the proof of our main result in this section. The basic idea is similar as in the paper [14] by Antezana et. al. However, there are also significant differences due to the compact setting, which in turn leads to a simpler and less technical proof.

Let $\mathcal{E}$ be the success event (4.21). In view of the equations $\min_{\|a\|=1} \langle \overline{\mathcal{B}_{\ell,\Lambda}} \, \mathcal{B}^t_{\ell,\Lambda} a, a \rangle = \lambda_{\min}(\overline{\mathcal{B}_{\ell,\Lambda}} \, \mathcal{B}^t_{\ell,\Lambda})$ and $\max_{\|a\|=1} \langle \overline{\mathcal{B}_{\ell,\Lambda}} \, \mathcal{B}^t_{\ell,\Lambda} a, a \rangle = \lambda_{\max}(\overline{\mathcal{B}_{\ell,\Lambda}} \, \mathcal{B}^t_{\ell,\Lambda})$, the failure event $\mathcal{E}^c$ can be be expressed in terms of the extreme eigenvalues as

$$\mathcal{E}^c = \left\{ \min_{\ell\in\mathbb{Z}_N} \lambda_{\min}(\overline{\mathcal{B}_{\ell,\Lambda}}\mathcal{B}^t_{\ell,\Lambda}) < L(1-\alpha) \right\} \cup \left\{ \max_{\ell\in\mathbb{Z}_N} \lambda_{\max}(\overline{\mathcal{B}_{\ell,\Lambda}}\mathcal{B}^t_{\ell,\Lambda}) > L(1+\alpha) \right\}.$$

Therefore, we obtain the following upper bound on the failure probability:

$$\mathbb{P}(\mathcal{E}^c) \le \mathbb{P}\left(\min_{\ell\in\mathbb{Z}_N} \lambda_{\min}(\overline{\mathcal{B}_{\ell,\Lambda}}\mathcal{B}^t_{\ell,\Lambda}) < L(1-\alpha) \right) + \mathbb{P}\left(\max_{\ell\in\mathbb{Z}_N} \lambda_{\max}(\overline{\mathcal{B}_{\ell,\Lambda}}\mathcal{B}^t_{\ell,\Lambda}) > L(1+\alpha) \right). \tag{4.25}$$

To simplify (4.25), let $U_{\ell,\lambda_j} \in \mathbb{C}^{M\times M}$ be the random matrix as defined in (4.22) for $j = 1, \ldots, L$ and $\ell \in \mathbb{Z}^N$. It can be observed that

$$\overline{\mathcal{B}_{\ell,\Lambda}}\mathcal{B}^t_{\ell,\Lambda} = \sum_{j=1}^{L} U_{\ell,\lambda_j}.$$

As its consequence, we obtain the following implications:

$$\lambda_{\min}(\overline{\mathcal{B}_{\ell,\Lambda}}\mathcal{B}^t_{\ell,\Lambda}) < L\,(1-\alpha) \quad\Longleftrightarrow\quad \lambda_{\min}\left(\sum_{r=1}^{L} (U_{\ell,\lambda_r} - I_M) \right) < -L\alpha, \tag{4.26}$$

and

$$\lambda_{\max}(\overline{\mathcal{B}_{\ell,\Lambda}}\mathcal{B}^t_{\ell,\Lambda}) > L\,(1+\alpha) \quad\Longleftrightarrow\quad \lambda_{\max}\left(\sum_{r=1}^{L} (U_{\ell,\lambda_r} - I_M) \right) > L\alpha. \tag{4.27}$$

By (4.25), (4.26) and (4.27), we get

$$\begin{aligned} \mathbb{P}(\mathcal{E}^c) \le{}& \mathbb{P}\left(\min_{\ell\in\mathbb{Z}_N} \lambda_{\min}\left(\sum_{r=1}^{L} (U_{\ell,\lambda_r} - I_M) \right) < -L\alpha \right) \\ &+ \mathbb{P}\left(\max_{\ell\in\mathbb{Z}_N} \lambda_{\max}\left(\sum_{r=1}^{L} (U_{\ell,\lambda_r} - I_M) \right) > L\alpha \right). \end{aligned} \tag{4.28}$$

Define $X_j := U_{\ell,\lambda_j} - I_M$ for $j = 1, 2, \ldots, L$. From Lemma 4.9, we have $\mathbb{E}[X_j] = 0$ for all j. Applying Lemma 4.1 to the sequence $\{X_j\}_{j=1}^{L}$, with $B = NMC + 1$, $\sigma^2 \le L(NMC - 1)$, and $m = L\alpha$, we obtain the following bounds on the lower and upper tail probabilities stated in (4.28).

For the lower tail in (4.28), we have

$$\begin{aligned}
\mathbb{P}&\left(\min_{\ell\in\mathbb{Z}_N}\lambda_{\min}\Big(\sum_{r=1}^{L}(U_{\ell,\lambda_r}-I_M)\Big)<-L\alpha\right)\\
&\le\sum_{\ell\in\mathbb{Z}_N}\mathbb{P}\left(\lambda_{\min}\Big(\sum_{r=1}^{L}(U_{\ell,\lambda_r}-I_M)\Big)<-L\alpha\right)\\
&=\sum_{\ell\in\mathbb{Z}_N}\mathbb{P}\left(\lambda_{\max}\Big(\sum_{r=1}^{L}(-U_{\ell,\lambda_r}+I_M)\Big)>L\alpha\right)\\
&\le NM\cdot\exp\left(-\frac{L^2\alpha^2/2}{L(NMC-1)+(NMC+1)L\alpha/3}\right)\\
&\le NM\cdot\exp\left(-\frac{L\alpha^2}{2(NMC+1)(1+\alpha/3)}\right)\\
&\le NM\cdot\exp\left(-\frac{L\alpha^2}{3(NMC+1)}\right).
\end{aligned}\tag{4.29}$$

Similarly, for the upper tail in (4.28), an identical estimate yields:

$$\mathbb{P}\left(\max_{\ell\in\mathbb{Z}_N}\lambda_{\max}\left(\sum_{r=1}^{L}(U_{\ell,\lambda_r}-I_M)\right)>-L\alpha\right)\le NM\cdot\exp\left(-\frac{L\alpha^2}{3(NMC+1)}\right).\tag{4.30}$$

Substituting the bounds (4.29) and (4.30) into (4.28), we get the following success probability estimate:

$$\mathbb{P}(\mathcal{E})=1-\mathbb{P}(\mathcal{E}^c)\ge1-2NM\cdot\exp\left(-\frac{L\alpha^2}{3(NMC+1)}\right).$$

To ensure $\mathbb{P}(\mathcal{E})\ge1-\epsilon$, it suffices to impose

$$L\ge\frac{3\,(NMC+1)}{\alpha^2}\cdot\log\left(\frac{2NM}{\epsilon}\right),$$

under which the sampling inequalities

$$LNV(1-\alpha)\|f\|^2\le\sum_{p=0}^{r-1}\sum_{j=1}^{L}\sum_{\ell\in\mathbb{Z}_N}\left|f^{(p)}\left(\lambda_j+\frac{\ell}{N}\right)\right|^2\le LNU(1+\alpha)\|f\|^2,$$

hold for every $f\in V_N(\Phi)$ with probability at least $1-\epsilon$. □

4.4 Applications

In this section, we illustrate the applicability of our main result for various cases including the space of trigonometric polynomials. The reconstruction of trigonometric polynomials from random samples has been investigated in [17, 123], where the focus was on recovery using random point evaluations. In the present setting, our result directly applies to this problem, allowing stable reconstruction from random derivative samples.

Similarly, the result extends to periodic multi-band signals with well-separated frequency bands, providing a stable recovery framework under random derivative sampling. We further demonstrate its applicability to wavelet subspaces and periodic shift-invariant spaces generated by scaling functions associated with multi-resolution analyses.

4.4.1 Random sampling in the space of trigonometric polynomials

As an application of Theorem 4.2, we consider the periodic shift-invariant space generated by the Dirichlet kernel, which corresponds to the space of trigonometric polynomials of bounded degree. Let $R \in \mathbb{N}$, and define

$$\mathcal{P}_R := \left\{ f \in L^2(\mathbb{T}) : f(\theta) = \sum_{n=-R}^{R} c_n e^{2\pi i n\theta},\ c_n \in \mathbb{C} \right\},$$

the space of trigonometric polynomials of degree at most R. A natural generator for $\mathcal{P}_R$ is the *Dirichlet kernel*

$$D_R(\theta) = \sum_{n=-R}^{R} e^{2\pi i n\theta} = \frac{\sin(\pi(2R+1)\theta)}{\sin(\pi\theta)}, \qquad \theta \in \mathbb{T}. \tag{4.31}$$

Clearly, $D_R \in \mathcal{P}_R$.

Let $\phi = \frac{1}{\sqrt{2R+1}} D_R$. Then the collection $E_{2R+1}(\phi) := \{L_{k/(2R+1)}\phi : k = 0, 1, \ldots, 2R\}$ forms an orthonormal basis for $\mathcal{P}_R$. Indeed, for $k, \ell \in \{0, \ldots, 2R\}$,

$$\begin{aligned}
\left\langle \phi\left(\cdot - \tfrac{k}{2R+1}\right), \phi\left(\cdot - \tfrac{\ell}{2R+1}\right) \right\rangle &= \int_{\mathbb{T}} \phi\left(\theta - \tfrac{k}{2R+1}\right) \overline{\phi\left(\theta - \tfrac{\ell}{2R+1}\right)}\, d\theta \\
&= \sum_{n=-R}^{R} \sum_{m=-R}^{R} \tfrac{1}{2R+1} e^{-2\pi i n \frac{k}{2R+1}} e^{2\pi i m \frac{\ell}{2R+1}} \int_{\mathbb{T}} e^{2\pi i (n-m)\theta}\, d\theta \\
&= \sum_{n=-R}^{R} \tfrac{1}{2R+1} e^{2\pi i n \frac{\ell - k}{2R+1}} = \delta_{k,\ell}.
\end{aligned}$$

Hence, $E_{2R+1}(\phi)$ is an orthonormal basis for $\mathcal{P}_R$.

Next, we verify the assumptions of Theorem 4.2. Since $\phi \in \mathcal{P}_R$ is infinitely differentiable, the bounded frequency periodization condition (4.9) is trivially satisfied. To compute the constant C in (4.10), we have

$$\begin{aligned}\sum_{\ell=0}^{2R} \left|\phi\big(\theta + \tfrac{\ell}{2R+1}\big)\right|^2 &= \sum_{\ell=0}^{2R} \left|\tfrac{1}{\sqrt{2R+1}} \sum_{n=-R}^{R} e^{2\pi i n(\theta+\frac{\ell}{2R+1})}\right|^2 \\ &= \sum_{\ell=0}^{2R} \left|\tfrac{1}{\sqrt{2R+1}} \sum_{n=-R}^{R} e^{2\pi i n\theta} e^{2\pi i n\ell/(2R+1)}\right|^2 \\ &= \sum_{n=-R}^{R} \left|e^{2\pi i n\theta}\right|^2 = 2R+1, \qquad \theta \in \mathbb{T},\end{aligned}$$

where the last equality follows from Parseval's identity. Thus, $C = 2R + 1$.

Although the exact values of U and V are not computed here, the system $\mathcal{G}_\ell$ is finite-dimensional and spans $J(\ell)$, so the frame bounds exist and are strictly positive and finite. Consequently, all the hypotheses of Theorem 4.2 are satisfied, and the random sampling inequalities (4.11) hold in the space $\mathcal{P}_R$.

4.4.2 Random sampling in the space of periodic multi-band signals

As a further application of Theorem 4.2, we consider the case of periodic multi-band signals.

Let $\mathcal{W} \subset \mathbb{Z}$ be a finite union of s disjoint frequency bands of equal width, defined by

$$\mathcal{W} = \bigcup_{i=1}^{s} \big([-R+\delta_i,\ R+\delta_i] \cap \mathbb{Z}\big),$$

where $R \in \mathbb{N}$ and the shifts $\{\delta_i\}_{i=1}^{s} \subset \mathbb{Z}$ satisfy the separation condition

$$|\delta_i - \delta_j| > 2R, \quad i \neq j.$$

The corresponding periodic shift-invariant (PSI) space is

$$\mathcal{B}_{\mathcal{W}} := \left\{ f \in L^2(\mathbb{T}) : \operatorname{supp}(\widehat{f}) \subseteq \mathcal{W} \right\}.$$

A natural generating set for $\mathcal{B}_{\mathcal{W}}$ is given by

$$\Phi = \left\{ \phi_i \in L^2(\mathbb{T}) : \phi_i(\theta) = \frac{1}{\sqrt{2R+1}} D_R(\theta)\, e^{2\pi i \delta_i \theta}, \quad i = 1, \ldots, s \right\},$$

where D_R denotes the Dirichlet kernel defined in (4.31). For $k = 0, 1, \dots, 2R$ and each i, we have

$$\phi_i\left(\theta - \frac{k}{2R+1}\right) = \sum_{n=-R+\delta_i}^{R+\delta_i} \frac{1}{\sqrt{2R+1}}\, e^{2\pi i n\theta}\, e^{-2\pi i n k/(2R+1)}. \tag{4.32}$$

Since the bands in $\mathcal{W}$ are disjoint, the frequency supports of ϕ_i and ϕ_j are non-overlapping for $i \neq j$. Therefore, the orthogonality of the exponential system $\{e^{2\pi i n \cdot}\}_{n\in\mathbb{Z}}$ yields

$$\left\langle \phi_i\left(\cdot - \tfrac{k}{2R+1}\right), \phi_j\left(\cdot - \tfrac{\ell}{2R+1}\right) \right\rangle = 0, \quad 0 \le k, \ell \le 2R,\ i \neq j.$$

For $i = j$, following the computation in the trigonometric polynomial case, we get

$$\left\langle \phi_i\left(\cdot - \tfrac{k}{2R+1}\right), \phi_i\left(\cdot - \tfrac{\ell}{2R+1}\right) \right\rangle = \delta_{k,\ell}.$$

Combining both cases, the system

$$E_{2R+1}(\Phi) := \left\{ L_{k/(2R+1)}\phi_i : 0 \le k \le 2R,\ 1 \le i \le s \right\}$$

forms an orthonormal basis for $\mathcal{B}_{\mathcal{W}}$, which therefore has dimension $s(2R+1)$.

Next, we verify the assumptions of Theorem 4.2. By Parseval's identity and (4.32), for each i and $\theta \in \mathbb{T}$,

$$\sum_{\ell=0}^{2R} \left| \phi_i\left(\theta + \tfrac{\ell}{2R+1}\right) \right|^2 = \sum_{n=-R+\delta_i}^{R+\delta_i} \left| e^{2\pi i n\theta} \right|^2 = 2R + 1.$$

Summing over $i = 1, \dots, s$, we obtain

$$\sum_{i=1}^{s} \sum_{\ell=0}^{2R} \left| \phi_i\left(\theta + \tfrac{\ell}{2R+1}\right) \right|^2 = s(2R+1), \qquad \theta \in \mathbb{T}.$$

Hence, the constant C in (4.10) is

$$C = s(2R+1).$$

Finally, as in the trigonometric polynomial case, the derivative fiber system associated with Φ is finite-dimensional and spans the corresponding range spaces. Thus, all the hypotheses of Theorem 4.2 are satisfied, and the random sampling inequalities (4.11) hold in the space of periodic multi-band signals $\mathcal{B}_{\mathcal{W}}$.

4.4.3 Random sampling in wavelet subspaces

Let ψ be a 1-periodic real-valued function such that it is continuously differentiable and having the Fourier series decomposition

$$\psi(\theta) = \sum_{m\in\mathbb{Z}} \hat{\psi}(m)e^{2\pi i m\theta}, \quad \theta \in \mathbb{T},$$

where the coefficients satisfy $\hat{\psi}(m) > 0$ for all $m \in \mathbb{Z}$. Any such ψ will be called a *periodic basis function.* Periodic basis functions are special cases of a class of the positive definite spherical functions and are useful for solving scattered data interpolation (or least-squares) problems on the circle [127].

For a positive integer N, consider the shift-invariant systems $E_N(\psi)$ and $E_{2N}(\psi)$. Clearly, $E_N(\psi) \subset E_{2N}(\psi)$. Let U_N be the orthogonal complement of $E_N(\psi)$ in $E_{2N}(\psi)$, i.e.,

$$U_N = E_{2N}(\psi) \ominus E_N(\psi).$$

In the context of multiresolution theory, the space U_N serves as a detail space that captures the components of $E_{2N}(\psi)$ not contained in $E_N(\psi)$. The inclusion $E_N(\psi) \subset E_{2N}(\psi)$ reflects a refinement in resolution, and U_N represents the new information introduced at the finer scale.

The following result provides a *wavelet function* in U_N, i.e., a function $f \in U_N$ whose set of translates

$$\left\{ f\left(\cdot - \frac{k}{N}\right) : k \in \mathbb{Z}_N \right\}$$

forms an orthonormal basis for the space U_N.

Theorem 4.10. *[111] Let N be a positive integer and let ψ be a periodic basis function. For any $\ell \in \mathbb{Z}$ and $\theta \in \mathbb{T}$, define*

$$\psi_{\ell,N}(\theta) := \sum_{m\in\mathbb{Z}} \hat{\psi}(mN+\ell)e^{2\pi i m\theta}.$$

Further, let $\phi \in L^2(\mathbb{T})$ be defined as

$$\phi(\theta) = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{2N-1} e^{ik\pi/N} \sqrt{\frac{\|\psi_{\ell+N,2N}\|^2}{\|\psi_{\ell,2N}\|^2 \|\psi_{\ell,N}\|^2}}\ \psi_{\ell,2N}(\theta). \tag{4.33}$$

Then ϕ is real-valued and has the same smoothness as ψ, and the set

$$\left\{\phi\left(\cdot - \tfrac{\ell}{N}\right)\right\}_{\ell\in\mathbb{Z}_N}$$

is an orthonormal basis for U_N.

In addition, ϕ has the absolutely convergent Fourier series decomposition

$$\phi(\theta) = \sum_{m\in\mathbb{Z}} \hat{\phi}(m) e^{2\pi i m\theta},$$

where for $\ell \in \mathbb{Z}_{2N}$, the Fourier coefficients are given by

$$\hat{\phi}(2mN+\ell) = e^{ik\pi/N} \sqrt{\frac{\|\psi_{\ell+N,2N}\|^2}{\|\psi_{\ell,2N}\|^2 \|\psi_{\ell,N}\|^2}}\ \hat{\psi}(2mN+\ell).$$

The above result implies that ϕ is at least continuously differentiable, since ϕ inherits the smoothness of ψ and ψ is continuously differentiable. In particular, ϕ admits an absolutely convergent Fourier series. Therefore for $r = 2$, Theorem 4.2 applies to the space $V_N(\phi) = U_N$ by choosing $\Phi = \{\phi\}$, where ϕ is defined in (4.33).

4.4.4 Random sampling in Haar-type PSI spaces

We now consider periodic shift-invariant spaces generated by Haar-type scaling functions. Let $N = 8$, and define a trigonometric polynomial $\phi \in L^2(\mathbb{T})$ by

$$\begin{aligned}\phi(\theta) = \frac{1}{\sqrt{8}}\Big[&1 + 2\cos(2\pi\theta) + 2\cos(6\pi\theta) + 2\cos\big(4\pi\theta - \tfrac{\pi}{8}\big)\cos\big(\tfrac{\pi}{8}\big) \\ &+ 2\cos\big(12\pi\theta - \tfrac{3\pi}{8}\big)\cos\big(\tfrac{3\pi}{8}\big)\Big] + \frac{i}{2}\Big[\sin\big(8\pi\theta - \tfrac{3\pi}{8}\big)\cos\big(\tfrac{\pi}{8}\big) - \sin\big(24\pi\theta - \tfrac{\pi}{8}\big)\cos\big(\tfrac{3\pi}{8}\big)\Big].\end{aligned}$$

The function ϕ serves as a scaling function of an MRA of order $N = 8$, associated with the low-pass filter $u \in \ell^2(\mathbb{Z}_8)$ defined by

$$u(n) = e^{-\pi i n/8} \cos\left(\frac{n\pi}{8}\right), \quad n \in \mathbb{Z}_8.$$

The filter u is induced by the discrete Haar mask $c \in \ell(\mathbb{Z}_N)$, given by

$$c(0) = c(1) = \frac{1}{2}, \qquad c(n) = 0 \quad \text{for } n \neq 0, 1,$$

and satisfies the standard two-scale relation

$$\widehat{\phi}(2k) = u(k)\,\widehat{\phi}(k), \qquad k \in \mathbb{Z}.$$

As shown in [90], the Fourier coefficients of ϕ are real and supported on the set $[-12, 12] \cap \mathbb{Z}$. Consequently, ϕ is a real-valued trigonometric polynomial of degree at most 12. Since its Fourier expansion contains only finitely many nonzero terms, the series converges absolutely, allowing the frequency periodization condition (4.9) to be relaxed.

Furthermore, by [90, Theorem 3.5], the scaling function ϕ satisfies the orthonormality condition, ensuring that $E_N(\phi)$ forms an orthonormal system. Hence, all the assumptions of Theorem 4.2 are met, and the random sampling inequalities (4.11) hold for the periodic shift-invariant space $V_8(\phi)$ for any derivative order $r \geq 0$.

To determine the constant C in (4.10), we evaluate numerically the function

$$\theta \mapsto \sum_{\ell \in \mathbb{Z}_N} \left|\phi\left(\theta + \tfrac{\ell}{N}\right)\right|^2,$$

whose maximum value corresponds to C. As shown in Figure 4.1, the peak of this function is approximately 9.9059, giving $C = 9.9059$.

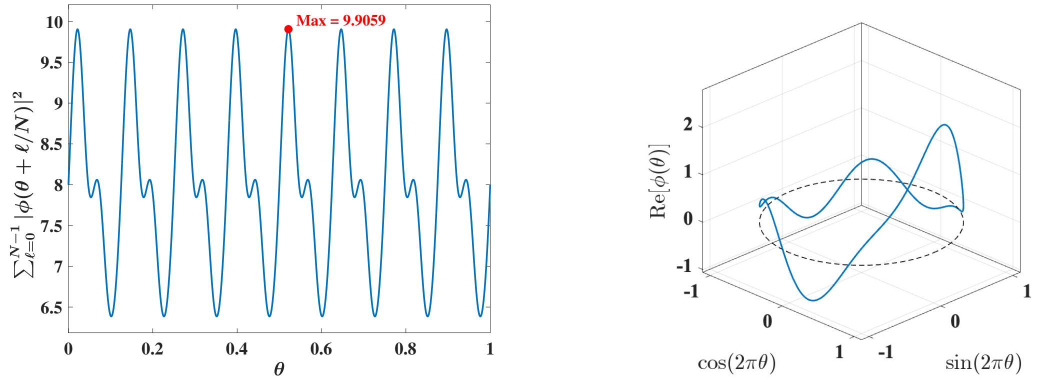


Figure 4.1: Left: The function $\theta \mapsto \sum_{\ell \in \mathbb{Z}_N} \left|\phi\left(\theta + \frac{\ell}{N}\right)\right|^2$ indicating the C-value. Right: The scaling function ϕ over unit circle.

Chapter 5

Multi-channel stable sampling for translation-invariant spaces

In this chapter, we study multi-channel stable sampling for translation-invariant (TI) spaces over locally compact groups, not necessarily abelian. The available literature on multi-channel sampling utilizes the abelian property of the underlying group. To develop the theory of multi-channel sampling for TI spaces over locally compact groups, we first develop the theory for multiplication-invariant (MI) spaces in the vector valued space $L^2(X, \mathcal{H})$, where X is a finite measure space and $\mathcal{H}$ is a separable Hilbert space. The idea is motivated by the work of Iverson [88], who studied the frame properties of the TI systems over locally compact groups using the generalized Zak transform. Utizing the tools developed by Bownik and Iverson [26], we characterize the stable sampling for MI spaces and develop necessary density conditions. Finally, employing the Zak transform, the results are transferred to TI spaces over locally compact groups.

5.1 Background on SI/TI spaces and Zak transform

Let $\mathscr{G}$ be a second countable locally compact group (need not to be abelian) and H be a closed abelian subgroup of $\mathscr{G}$. A closed subspace V in $L^2(\mathscr{G})$ is said to be *H-translation-invariant (H-TI)* if $L_\xi f \in V$ for all $f \in V$ and $\xi \in H$, where for each $\eta \in \mathscr{G}$ the *left translation* L_η on $L^2(\mathscr{G})$ is defined by

$$(L_\eta f)(\gamma) = f(\eta^{-1}\gamma), \ \gamma \in \mathscr{G} \text{ and } \ f \in L^2(\mathscr{G}).$$

For a countable family of functions $\mathscr{A} \subseteq L^2(\mathscr{G})$, let us consider a H-TI system $\mathcal{E}^H(\mathscr{A})$ and its associated H-TI space $\mathcal{S}^H(\mathscr{A})$ generated by $\mathscr{A}$, i.e.,

$$\mathcal{E}^H(\mathscr{A}) := \{L_\xi \varphi : \varphi \in \mathscr{A}, \xi \in H\} \text{ and } \mathcal{S}^H(\mathscr{A}) := \overline{\operatorname{span}}\, \mathcal{E}^H(\mathscr{A}),$$

The results presented in this chapter are based on our work:

Kalra, S. and Shukla, N. K. 2025. Multi-channel stable sampling for translation invariant spaces over locally compact groups, *Under preparation.*

respectively.

For $x \in \mathscr{G}$, a right coset of H in $\mathscr{G}$ with respect to x is denoted by Hx. For a function $f : \mathscr{G} \to \mathbb{C}$, we define a complex valued function f^{Hx} on H by $f^{Hx}(\xi) = f(\xi\, \Xi(Hx))$, $\xi \in H$, where the space of orbits $H\backslash\mathscr{G} = \{\Gamma x : x \in \mathscr{G}\}$ is the set of all right cosets of H in $\mathscr{G}$, and $\Xi : H\backslash\mathscr{G} \to \mathscr{G}$ is a *Borel section* for the quotient space $H\backslash\mathscr{G}$. Then the Fourier transform of $f^{Hx} \in L^1(H)$ is given by $\widehat{f^{Hx}}(\alpha) = \int_H f^{Hx}(\xi)\alpha(\xi^{-1})\, d\mu(\xi)$ for $\alpha \in \widehat{H}$, which can be extended to $L^2(H)$. The *Zak transformation* $\mathcal{Z} : L^2(\mathscr{G}) \to L^2(\widehat{H}, L^2(H\backslash\mathscr{G}))$ of $f \in L^2(\mathscr{G})$ for the pair $(\mathscr{G}, H)$ is defined by

$$(\mathcal{Z}f)(\alpha)(Hx) = \widehat{f^{Hx}}(\alpha) \text{ a.e. } \alpha \in \widehat{H} \text{ and } Hx \in H\backslash\mathscr{G}, \tag{5.1}$$

where $\hat{H}$ denotes the dual of H [88]. It is a unitary is closely associated with the fiberization map $\mathscr{F}$ when $\mathscr{G}$ becomes abelian. For a second countable locally compact abelian (LCA) group $\mathcal{G}$ and its closed subgroup Λ, the *fiberization* $\mathscr{F}$ is a unitary map from $L^2(\mathcal{G})$ to $L^2(\widehat{\mathcal{G}}/\Lambda^\perp, L^2(\Lambda^\perp))$ given by

$$(\mathscr{F}f)(\beta\Lambda^\perp)(x) = \widehat{f}(x\,\zeta(\beta\Lambda^\perp)),\ x \in \Lambda^\perp,\ \beta \in \widehat{\mathcal{G}}, \tag{5.2}$$

for $f \in L^2(\mathcal{G})$, where $\Lambda^\perp := \{\beta \in \widehat{\mathcal{G}} : \beta(\lambda) = 1 \text{ for all } \lambda \in \Lambda\}$, $\Lambda^\perp\backslash\widehat{\mathcal{G}} = \widehat{\mathcal{G}}/\Lambda^\perp$ and $\zeta : \widehat{\mathcal{G}}/\Lambda^\perp \to \widehat{\mathcal{G}}$ is a Borel section which maps compact sets to pre-compact sets. The Zak transform and the fiberization map on the Euclidean space $\mathbb{R}^n$ by the action of integers $\mathbb{Z}^n$ are

$$(\mathcal{Z}f)(\xi,\eta) = \sum_{k\in\mathbb{Z}^n} f(\xi+k)e^{-2\pi ik\eta}, \text{ and } (\mathscr{F}f)(\xi)(k) = \widehat{f}(\xi+k)$$

for $k \in \mathbb{Z}^n$, $\xi, \eta \in \mathbb{T}^n$ and $f \in L^1(\mathbb{R}^n)\bigcap L^2(\mathbb{R}^n)$. The fiberization map for the case $\mathcal{G} = \mathbb{T}$ and $\Lambda = \frac{1}{N}\mathbb{Z}_N$ is defined in (4.3).

The Zak transform provides a framework for handling TI spaces over locally compact groups, analogous to the way the fiberization map treats SI spaces in the abelian setting. Both transforms allow TI spaces to be studied via MI spaces by converting translation operators L_ξ into multiplication operators M_{ϕ_ξ} for suitable functions $\phi_\xi \in L^\infty(X)$. Specifically, for $f \in L^2(\mathscr{G})$, we have

$$(\mathcal{Z}L_\xi f)(\alpha) = (M_{\phi_\xi}\mathcal{Z}f)(\alpha), \quad \text{for a.e. } \alpha \in \widehat{H} \text{ and } \xi \in H,$$

where M_{ϕ_ξ} denote the multiplication operator on $L^2(\widehat{H}, L^2(H\backslash\mathscr{G}))$, defined by

$$(M_{\phi_\xi}F)(\eta) = \phi_\xi(\eta)F(\eta), \quad \text{for } F \in L^2(\widehat{H}, L^2(H\backslash\mathscr{G})),\ \phi_\xi \in L^\infty(\hat{H}) \text{ and } \eta \in \hat{H}.$$

Therefore our goal of studying sampling theory for SI/TI spaces can be established by converting the problem of H-TI space $\mathcal{S}^H(\mathscr{A})$ into the MI spaces on $L^2(X,\mathcal{H})$ with the help of the Zak transform, where $X=\widehat{H}$ and $\mathcal{H}=L^2(H\backslash\mathscr{G})$.

One of the benefits of the Zak transform for the pair $(\mathscr{G},H)$ is that the various previously inaccessible examples pairs, like $(\mathbb{R}^n,\mathbb{R}^m)$, $(\mathbb{R}^n,\mathbb{Z}^m)$, $(\mathbb{Q}_p,\mathbb{Z}_p)$, $(\mathcal{G},\Lambda)$, etc., can be accessed through it where $n\geq m$, Λ (not necessarily co-compact, i.e., $\mathcal{G}/\Lambda$-compact, or uniform lattice) is a closed subgroup of the second countable LCA group $\mathcal{G}$, and $\mathbb{Z}_p$ is the p-adic integer in the p-adic number field $\mathbb{Q}_p$.

5.2 Stable sampling for MI spaces

For studying stable sampling techniques for MI spaces, we fix (X,μ) to be a finite measure space such that $L^2(X)$ is separable. We also assume that the set $\mathfrak{B}:=\{g_m\}_{m\in\mathcal{M}}\subset L^\infty(X)$ is such that it forms an orthonormal basis for $L^2(X)$, where the index set $\mathcal{M}$ is countable and $\mu_{\mathcal{M}}$ is a counting measure.

We briefly discuss the MI systems and the associated spaces in $L^2(X,\mathcal{H})$, where

$$L^2(X;\mathcal{H})=\left\{\varphi \mid \varphi: X\to\mathcal{H} \text{ is measurable such that } \int_X \|\varphi(x)\|^2\, d\mu(x)<\infty\right\}.$$

Now we define the multiplication operator on $L^2(X,\mathcal{H})$.

Definition 5.1. Let $\phi\in L^\infty(X)$. The operator $\mathfrak{M}_\phi$ on $L^2(X,\mathcal{H})$ defined by

$$(\mathfrak{M}_\phi f)(x)=\phi(x)\,f(x),\quad \text{for a.e. } x\in X,\ f\in L^2(X,\mathcal{H}),$$

is called the *multiplication operator.*

The operator $\mathfrak{M}_\phi$ is a bounded linear operator on $L^2(X;\mathcal{H})$ satisfying $\|\mathfrak{M}_\phi\|=\|\phi\|_{L^\infty}$ provided X is a σ-finite measure space. If X is not a σ-finite measure space, then $\|\mathfrak{M}_\phi\|$ need not be same as $\|\phi\|_{L^\infty}$ [44, Theorem 1.5].

Definition 5.2. [88, Definition 2.1] Let V be a closed subspace of $L^2(X,\mathcal{H})$. We say V is a *multiplication invariant (MI) space* if

$$\mathfrak{M}_\phi f\in V,\ \text{for all } \phi\in L^\infty(X) \text{ and } f\in V.$$

Let us consider a finite collection of generators $\mathcal{A}=\{\Phi_1,\Phi_2,\ldots,\Phi_r\}\subset L^2(X,\mathcal{H})$. We define the *multiplication invariant (MI)* system $E_{\mathcal{A},\mathfrak{B}}$ and its associated MI space $W_{\mathcal{A},\mathfrak{B}}$

as follows:

$$E_{\mathcal{A},\mathfrak{B}} := \{\mathfrak{M}_{g_t}\Phi_k : t \in \mathcal{M},\ k \in \mathcal{I}_r\} \text{ and } W_{\mathcal{A},\mathfrak{B}} := \overline{\operatorname{span}}\, E_{\mathcal{A},\mathfrak{B}}, \tag{5.3}$$

respectively.

5.2.1 Local setting

We introduce essential definitions and fundamental concepts necessary for formulating localized versions of our results.

Definition 5.3. [26] Let $\mathcal{W} \subseteq L^2(X,\mathcal{H})$ and $\mathcal{W}' \subseteq L^2(X,\mathcal{H}')$ be two multiplication-invariant spaces. Then, a *multiplication-invariant operator* between them is a bounded linear operator $T : \mathcal{W} \to \mathcal{W}'$ such that

$$T\mathfrak{M}_\varphi = \mathfrak{M}'_\varphi T \quad \text{for all } \varphi \in L^\infty(X).$$

Definition 5.4. [26] An $\mathcal{H}$-valued *range function* on X is a mapping

$$J : X \to \{\text{closed subspaces of } \mathcal{H}\}.$$

In other words, it is a choice of an orthogonal projection $P_{J(x)} \in B(\mathcal{H})$ for every $x \in X$, where $B(\mathcal{H})$ denotes the set of bounded operators on $\mathcal{H}$ and $P_{J(x)}$ projects onto $J(x)$. A range function J is measurable if P_J is weakly measurable i.e., for any $u, v \in \mathcal{H}$, the mapping

$$x \mapsto \langle P_{J(x)}u, v\rangle$$

is measurable on X. With every range function $J : X \to \{\text{closed subspaces of } H\}$, we associate a closed subspace

$$V_J := \{\varphi \in L^2(X,\mathcal{H}) : \varphi(x) \in J(x) \text{ for a.e. } x \in X\}.$$

It is easy to see that V_J is multiplication-invariant.

The following result characterizes MI spaces using range functions. It is provided in [88] for determining sets [88, see Definition 2.1] in a σ-finite measure space. Here, we consider the case where $\mathfrak{B}$ is an orthonormal basis of $L^2(X)$ with functions in $L^\infty(X)$, which serves as a determining set for $L^1(X)$ assuming $\mu(X) < \infty$.

Theorem 5.5. *[26] Let $\mathcal{A} = \{\Phi_1, \Phi_2, \ldots, \Phi_r\} \subset L^2(X,\mathcal{H})$ be a finite collection of generators. Then, the following statements are equivalent:*

(i) $W_{\mathcal{A},\mathfrak{B}}$ *is an MI space.*

(ii) For every $g \in \mathfrak{B}$, $\mathfrak{M}_g W_{\mathcal{A},\mathfrak{B}} \subset W_{\mathcal{A},\mathfrak{B}}$.

(iii) There exists a measurable range function J *such that* $W_{\mathcal{A},\mathfrak{B}} = V_J$.

Moreover, the associated range function J *satisfies*

$$J(x) = \overline{\operatorname{span}}\{\Phi_i(x) : i \in \mathcal{I}_r\}, \quad \textit{for a.e. } x \in X.$$

Additionally, the mapping $J \mapsto V_J$ *establishes a one-to-one correspondence between the collection of measurable* $\mathcal{H}$*-valued range functions on* X *(considered up to equality a.e.) and the set of MI subspaces of* $L^2(X, \mathcal{H})$.

Assumptions: *Throughout this chapter*, we assume that $\mathcal{A} = \{\Phi_1, \Phi_2, \dots, \Phi_r\} \subset L^2(X, \mathcal{H})$ is chosen in such a way that the MI system $E_{\mathcal{A},\mathfrak{B}} = \{\mathfrak{M}_{g_t}\Phi_k : t \in \mathcal{M}, k \in \mathcal{I}_r\}$ is a *Riesz sequence* in $L^2(X, \mathcal{H})$, i.e., there exist constants $0 < A \le B < \infty$ such that for every finitely supported family of scalars $\{c_{k,t}\}_{k\in\mathcal{I}_r,\, t\in\mathcal{M}}$ the following inequality holds:

$$A \sum_{k\in\mathcal{I}_r} \sum_{t\in\mathcal{M}} |c_{k,t}|^2 \;\le\; \Big\| \sum_{k\in\mathcal{I}_r} \sum_{t\in\mathcal{M}} c_{k,t}\, \mathfrak{M}_{g_t} \Phi_k \Big\|^2_{L^2(X,\mathcal{H})} \;\le\; B \sum_{k\in\mathcal{I}_r} \sum_{t\in\mathcal{M}} |c_{k,t}|^2.$$

Under these assumptions, the space $W_{\mathcal{A},\mathfrak{B}}$ has the following form:

$$W_{\mathcal{A},\mathfrak{B}} = \left\{ \sum_{k\in\mathcal{I}_r} \sum_{m\in\mathcal{M}} \alpha_k(m) \mathfrak{M}_{g_m} \Phi_k \,\Big|\, (\alpha_k(m))_{m\in\mathcal{M}} \in \ell^2(\mathcal{M}), g_m \in \mathfrak{B},\ k \in \mathcal{I}_r \right\}. \tag{5.4}$$

We also assume that J is the range function associated with MI space $W_{\mathcal{A},\mathfrak{B}}$.

We now provide a few examples of Riesz sequences in the MI setting.

(i) Let $X = (0, 1)$ with the Lebesgue measure, $\mathcal{H} = \mathbb{C}$, and $\mathcal{M} = \mathbb{Z}$. For $\mathcal{A} = \{\chi_{[-1/2,\,1/2]}(\cdot)\}$, consider the system

$$E_{\mathcal{A},\mathfrak{B}} = \{e^{2\pi i n(\cdot)} \chi_{[-1/2,\,1/2]}(\cdot)\}_{n\in\mathbb{Z}},$$

where $\mathfrak{B} = \{e^{-2\pi i m(\cdot)}\}_{m\in\mathbb{Z}}$. Then $E_{\mathcal{A},\mathfrak{B}}$ forms a Riesz sequence in $L^2(0, 1)$. Indeed, it is well known that the system $\{\operatorname{sinc}(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $L^2(\mathbb{R})$ [42]. Since the inverse Fourier transform $\mathcal{F}^{-1}$ is unitary on $L^2(\mathbb{R})$, and

$$\mathcal{F}^{-1}(\operatorname{sinc}(\cdot - n))(y) = e^{2\pi i n y} \chi_{[-1/2,\,1/2]}(y), \quad y \in \mathbb{R},$$

the claim follows.

(ii) Let $\mathcal{G}$ be an LCA group with Haar measure, and let $\mathcal{M} = \Lambda$ be its uniform lattice. For $\psi \in L^2(\mathcal{G})$, consider the system

$$E_{\mathcal{A},\mathfrak{B}} = \{\chi_\xi(\cdot)\, \mathcal{Z}\psi\}_{\xi\in\Lambda},$$

where $X = \widehat{\Lambda}$ is the dual group of Λ, $\mathcal{H} = L^2(\Lambda\backslash\mathcal{G})$, $\mathfrak{B} = \{\chi_\xi\}_{\xi\in\Lambda}$, and $\mathcal{A} = \{\mathcal{Z}\psi\}$. Then $E_{\mathcal{A},\mathfrak{B}}$ forms a Riesz sequence in $L^2(X,\mathcal{H})$ if and only if there exist constants $0 < A \leq B < \infty$ such that

$$A \leq \|\mathcal{Z}\psi(\alpha)\|^2 \leq B \quad \text{for a.e. } \alpha \in \widehat{H}.$$

(iii) Let $X = [0,1) \times [0,1)$ with Lebesgue measure, $\mathcal{H} = \mathbb{C}$, and $\mathcal{M} = \mathbb{Z}\times\mathbb{Z}$. Then the family $\mathfrak{B} = \{e^{2\pi imx}e^{-2\pi iny}\}_{m,n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2([0,1)\times[0,1))$.

For $\Phi \in L^2([0,1)\times[0,1))$, the system

$$\{e^{2\pi imx}e^{-2\pi iny}\,\Phi(x,y)\}_{m,n\in\mathbb{Z}}$$

is a Riesz basis for $L^2([0,1)\times[0,1))$ with bounds $A, B > 0$ if and only if

$$A \leq |\Phi(x,y)|^2 \leq B \quad \text{for a.e. } (x,y) \in [0,1)\times[0,1).$$

(See [42].) Such systems are closely related to the Gabor system

$$\{M_{mb}T_{na}f\}_{m,n\in\mathbb{Z}} = \{e^{2\pi imbx}f(x-na)\}_{m,n\in\mathbb{Z}}, \quad \text{for } ab = 1,$$

via the transform $Z_a : L^2(\mathbb{R}) \to L^2([0,1)\times[0,1))$, defined by

$$(Z_a h)(x,y) = \sqrt{a}\sum_{n\in\mathbb{Z}} h(a(x-n))e^{2\pi iny}, \quad x,y\in\mathbb{R}$$

as this transform satisfies the identity $Z_a M_{mb}T_{na}f = e^{2\pi imx}e^{-2\pi iny}Z_a f$.

5.2.1.1 *The map $T_{\mathcal{A}}$*

Define the linear map $T_{\mathcal{A}}$ by

$$T_{\mathcal{A}} : L^2(X,\mathbb{C}^r) \to W_{\mathcal{A},\mathfrak{B}}, \quad \sum_{k\in\mathcal{I}_r}\sum_{m\in\mathcal{M}} \alpha_k(m)g_m(x)e_k \longmapsto \sum_{k\in\mathcal{I}_r}\sum_{m\in\mathcal{M}} \alpha_k(m)\mathfrak{M}_{g_m}\Phi_k(x), \tag{5.5}$$

for $(\alpha_k(m))_{m\in\mathcal{M}} \in \ell^2(\mathcal{M})$ and $x \in X$, where the standard basis vector e_k is given by

$$e_k = (0,\ldots,0, \underbrace{1}_{k\text{-th place}}, 0,\ldots,0).$$

It follows that $T_{\mathcal{A}}$ is linear, bounded, and invertible, since the system $\{\mathfrak{M}_{g_m}\Phi_k\}_{m\in\mathcal{M}}^{k\in\mathcal{I}_r}$ forms a Riesz basis for $W_{\mathcal{A},\mathfrak{B}}$.

To understand the structure of the operator $T_{\mathcal{A}}$, it is useful to examine its pointwise action. For each $x \in X$, the operator $T_{\mathcal{A}}$ can be represented through a local map

$$R(x) : \mathbb{C}^r \to \operatorname{span}\{\Phi_k(x)\}_{k \in \mathcal{I}_r},$$

defined by

$$R(x)u = \sum_{k \in \mathcal{I}_r} u_k \Phi_k(x), \quad u = (u_1, u_2, \dots, u_r) \in \mathbb{C}^r. \tag{5.6}$$

We observe that this pointwise operator $R(x)$ captures how the global synthesis map $T_{\mathcal{A}}$ acts locally on the fiber over each x. We first present a result characterizing the uniform boundedness of the operators $R(x)$ for a.e. $x \in X$.

Proposition 5.6. *For a.e. $x \in X$, the operator $R(x) : \mathbb{C}^r \longrightarrow \operatorname{span}\{\Phi_k(x)\}_{k \in \mathcal{I}_r}$ defined in (5.6) is uniformly bounded i.e., there exists a constant $C > 0$ such that $\|R(x)u\| \leq C\|u\|$ for any $u \in \mathbb{C}^r$ and a.e. $x \in X$ if and only if $\max_{k \in \mathcal{I}_r} \text{ess-sup}_{x \in X} \|\Phi_k(x)\|^2 < \infty$.*

Proof. By the Cauchy-Schwarz inequality, we have the following inequality for a.e. $x \in X$ and $u \in \mathbb{C}^r$:

$$\|R(x)u\|^2 = \sum_{k \in \mathcal{I}_r} \sum_{j \in \mathcal{I}_r} u_k \overline{u_j} \langle \Phi_k(x), \Phi_j(x) \rangle \leq \sum_{k \in \mathcal{I}_r} \sum_{j \in \mathcal{I}_r} |u_k||\overline{u_j}| \|\Phi_k(x)\| \|\Phi_j(x)\| \leq C(x)\|u\|^2, \tag{5.7}$$

where $C(x) = C_2^2 C_1^2(x)$, $C_1(x) := \max_{k \in \mathcal{I}_r} \|\Phi_k(x)\|$, and $\left(\sum_{k \in \mathcal{I}_r} |u_k|\right) \leq C_2\|u\|$ for some $C_2 > 0$.

If $\max_{k \in \mathcal{I}_r} \text{ess-sup}_{x \in X} \|\Phi_k(x)\|^2 < \infty$, then by using (5.7), $R(x)$ is uniformly bounded for a.e. $x \in X$ since $\text{ess-sup}_{x \in X} C(x) = C_2^2 \max_{k \in \mathcal{I}_r} \text{ess-sup}_{x \in X} \|\Phi_k(x)\|^2$. Conversely, assume $\max_{k \in \mathcal{I}_r} \text{ess-sup}_{x \in X} \|\Phi_k(x)\|^2$ is not finite. Then there exist $\ell \in \mathcal{I}_r$ and a set V of positive measure such that $\sup_{x \in V} \|\Phi_\ell(x)\|^2$ is not finite. Therefore for any $M > 0$, there exists a measurable set $W \subseteq V$ with $\mu(W) > 0$ such that $\|\Phi_\ell(x)\|^2 > M$ for $x \in W$. Now for $u = (0, , \dots, u_\ell, \dots, 0) \in \mathbb{C}^r$ with $\|u\| = 1$, we have $\|R(x)u\|^2 = \|\Phi_\ell(x)\|^2 > M$, which is a contradiction. Hence the claim follows. □

The next proposition provides a precise pointwise characterization of the synthesis operator $T_{\mathcal{A}}$. It shows that $T_{\mathcal{A}}$ can be represented as a direct integral of the local operators $R(x)$ over X, linking the global action of $T_{\mathcal{A}}$ to its behavior on each fiber. Moreover, under the uniform boundedness condition of $R(x)$ (as in Proposition 5.6), the operator $T_{\mathcal{A}}$ satisfies a natural intertwining relation with multiplication operators.

Proposition 5.7. *The mapping $T_{\mathcal{A}} : L^2(X, \mathbb{C}^r) \to W_{\mathcal{A},\mathfrak{B}}$ given by (5.5) has the representation of the form $T_{\mathcal{A}} = \int_X^{\oplus} R(x) d\mu(x)$, satisfying*

$$(T_{\mathcal{A}}F)(y) = \left(\int_X^{\oplus} R(x) d\mu(x) F \right)(y) = R(y)\big[F(y)\big], \;\; F \in L^2(X, \mathbb{C}^r), y \in X,$$

provided for a.e. $x \in X$, $R(x)$ is uniformly bounded (in the sense of Proposition 5.6). Moreover, it satisfies the intertwining relation

$$T_{\mathcal{A}} M_g = \mathfrak{M}_g T_{\mathcal{A}} \text{ for each } g \in \mathfrak{B}, \tag{5.8}$$

where for $g \in L^\infty(X)$, M_g denotes the multiplication operator on $L^2(X, \mathbb{C}^r)$.

Proof. Since $\Phi_k : X \to H$ is measurable, thus by Petti's Measurability Theorem it follows that the map $x \mapsto \langle R(x)u, v \rangle$ is measurable for $u \in \mathbb{C}^r, v \in \mathcal{H}$ and a.e. $x \in X$. Also for $F = \sum\limits_{k \in \mathcal{I}_r} \sum\limits_{m \in \mathcal{M}} \alpha_k(m) g_m e_k \in L^2(X, \mathbb{C}^r)$, $(\alpha_k(m))_{m \in \mathcal{M}} \in \ell^2(\mathcal{M})$, we have $R(x)F(x) = (T_{\mathcal{A}}F)(x)$ for a.e. $x \in X$ which follows by

$$\begin{aligned} R(x)F(x) &= R(x)\Big(\sum_{m \in \mathcal{M}} \alpha_1(m) g_m(x), \sum_{m \in \mathcal{M}} \alpha_2(m) g_m(x), \dots, \sum_{m \in \mathcal{M}} \alpha_r(m) g_m(x) \Big) \\ &= \sum_{k \in \mathcal{I}_r} \sum_{m \in \mathcal{M}} \alpha_k(m) g_m(x) \Phi_k(x). \end{aligned}$$

Employing the theory of direct integrals from [55], we can represent $T_{\mathcal{A}}$ in the form of direct integral of $R(x)$, i.e, $T_{\mathcal{A}} = \int_X^{\oplus} R(x) d\mu(x)$, provided $R(x)$ is *uniformly bounded* for a.e. $x \in X$. Consequently, $T_{\mathcal{A}} M_{g_m} = \mathfrak{M}_{g_m} T_{\mathcal{A}}$ for each $m \in \mathcal{M}$ [26, Theorem 3.7]. ☐

$$\begin{tikzcd} L^2(X, \mathbb{C}^r) \arrow[r, "T_{\mathcal{A}}"] \arrow[d, "M_g"'] & W_{\mathcal{A},\mathfrak{B}} \arrow[d, "\mathfrak{M}_g"] \\ L^2(X, \mathbb{C}^r) \arrow[r, "T_{\mathcal{A}}"] & W_{\mathcal{A},\mathfrak{B}} \end{tikzcd}$$

Figure 5.1: Commutative diagram representing the intertwining relation of $T_{\mathcal{A}}$.

Remark 5.8. It is important to observe that by applying (5.8) and invoking [26, Theorem 3.7] for $\mathfrak{B}$, it follows that $T_{\mathcal{A}}$ is an MI operator. Furthermore, an application of [26, Theorem 4.1] establishes that the adjoint operator $T_{\mathcal{A}}^*$ is also an MI operator, satisfying $(T_{\mathcal{A}}^* h)(x) = R^*(x)h(x)$, for a.e. $x \in X$ and $h \in W_{\mathcal{A},\mathfrak{B}}$, where $R^*(x) = (R(x))^*$. Moreover, it satisfies,

$$T_{\mathcal{A}}^* \mathfrak{M}_{g_m} = M_{g_m} T_{\mathcal{A}}^*, \quad m \in \mathcal{M}. \tag{5.9}$$

For the remainder of this chapter, we use the notation M_ϕ to denote the multiplication operator on $L^2(X, \mathbb{C}^r)$ for $\phi \in L^\infty(X)$.

5.2.2 Stability of the sampling set

Let us consider a collection of functions $\mathcal{N} = \{\eta_i\}_{i \in \mathcal{I}_s} \subset W_{\mathcal{A},\mathfrak{B}}$, where $s \geq r$ (over-sampling case), such that the system

$$E_{\mathcal{N},\mathfrak{B}} := \{\mathfrak{M}_{g_m} \eta_i\}_{m \in \mathcal{M}}^{i \in \mathcal{I}_s}$$

forms a Bessel system. That is, there exists a constant $B > 0$ satisfying

$$\sum_{i \in \mathcal{I}_s} \sum_{m \in \mathcal{M}} |\langle f, \mathfrak{M}_{g_m} \eta_i \rangle|^2 \leq B \|f\|^2, \quad \forall f \in W_{\mathcal{A},\mathfrak{B}}.$$

The *analysis operator* $S_\mathcal{N}$ associated with the Bessel system $E_{\mathcal{N},\mathfrak{B}}$ is given by

$$S_\mathcal{N} : W_{\mathcal{A},\mathfrak{B}} \to \ell^2(\mathcal{M} \times \mathcal{I}_s), \quad f \mapsto S_\mathcal{N} f, \tag{5.10}$$

where $S_\mathcal{N} f$ is defined component-wise as

$$(S_\mathcal{N} f)(m, i) = \langle f, \mathfrak{M}_{g_m} \eta_i \rangle_{L^2(X,\mathcal{H})}, \quad (m, i) \in \mathcal{M} \times \mathcal{I}_s. \tag{5.11}$$

The values $S_\mathcal{N} f(m, i)$ are the *generalized samples* of a signal $f \in W_{\mathcal{A},\mathfrak{B}}$, where the set $\mathcal{M} \times \mathcal{I}_s$ is called the *sampling set.*

Definition 5.9. A set $\mathcal{M} \times \mathcal{I}_s$ is said to be *a stable set of sampling* for $W_{\mathcal{A},\mathfrak{B}}$ if the collection $E_{\mathcal{N},\mathfrak{B}}$ is a frame for $W_{\mathcal{A},\mathfrak{B}}$, i.e., there exist constants $0 < a \leq b < \infty$ such that

$$a\|f\|_{L^2(X,\mathcal{H})} \leq \left(\sum_{i \in \mathcal{I}_s} \sum_{m \in \mathcal{M}} |S_\mathcal{N} f(m, i)|^2 \right)^{1/2} \leq b\|f\|_{L^2(X,\mathcal{H})} \quad \text{for every } f \in W_{\mathcal{A},\mathfrak{B}}. \tag{5.12}$$

Our main aim is to reconstruct any signal $f \in W_{\mathcal{A},\mathfrak{B}}$ from its samples $S_\mathcal{N} f(m, i)$. We begin with defining the *Fourier transform* $\mathcal{F}$ of $F \in L^1(X)$ associated with $\mathfrak{B}$ as follows:

$$(\mathcal{F}F)(m) = \int_X F(x)\overline{g_m(x)} d\mu(x), m \in \mathcal{M}. \tag{5.13}$$

The map $\mathcal{F}$ is surjective and satisfies the *Plancharel formula* $\|\mathcal{F}F\|_{\ell^2(\mathcal{M})} = \|F\|_{L^2(X)}$. Therefore, $\mathcal{F}$ is an isomorphism and its inverse Fourier transform $\mathcal{F}^{-1} : \ell^2(\mathcal{M}) \longrightarrow L^2(X)$ is defined by

$$(\mathcal{F}^{-1}\alpha)(x) = \sum_{m \in \mathcal{M}} \alpha(m) g_m(x), \text{ for } \alpha = (\alpha(m))_{m \in \mathcal{M}} \in \ell^2(\mathcal{M}), \ x \in X, \tag{5.14}$$

where the above sum is interpreted as its limit in $L^2(X)$ [126, Lemma 3.9]. The map $\mathcal{F}$ can be naturally extended to define the *k-valued Fourier transform* on $L^2(X, \mathbb{C}^k)$, for $k \in \mathbb{N}$, given by

$$\mathcal{F}^{(k)} : L^2(X, \mathbb{C}^k) \longrightarrow \ell^2(\mathcal{M}, \mathbb{C}^k), \quad (\mathcal{F}^{(k)}F)(m) = \big(\mathcal{F}F_1(m), \mathcal{F}F_2(m), \ldots, \mathcal{F}F_k(m)\big), \tag{5.15}$$

where $F = (F_1, F_2, \ldots, F_k) \in L^2(X, \mathbb{C}^k)$ and $m \in \mathcal{M}$.

The *k-valued Fourier transform* establishes a connection between the analysis operator $S_{\mathcal{N}} : W_{\mathcal{A},\mathfrak{B}} \to \ell^2(\mathcal{M} \times \mathcal{I}_s)$ and its pointwise representation, as stated in the following proposition. The following result is re-statement of [26, Theorem 5.7].

Proposition 5.10. *The analysis operator $S_{\mathcal{N}} : W_{\mathcal{A},\mathfrak{B}} \to \ell^2(\mathcal{M} \times \mathcal{I}_s)$ associated with $E_{\mathcal{N},\mathfrak{B}}$ maps into the range of the s-valued Fourier transform $\mathcal{F}^{(s)} : L^2(X, \mathbb{C}^s) \to \ell^2(\mathcal{M}, \mathbb{C}^s)$ via*

$$S_{\mathcal{N}} = \mathcal{F}^{(s)} \int_X^{\oplus} \tilde{S}_{\mathcal{N}}(x)\, d\mu(x), \tag{5.16}$$

where, for a.e. $x \in X$, the map $\tilde{S}_{\mathcal{N}}(x) : J(x) \to \mathbb{C}^s$ is given by

$$\tilde{S}_{\mathcal{N}}(x)(u) = \big(\tilde{S}_{\mathcal{N}}(x)(u,1), \tilde{S}_{\mathcal{N}}(x)(u,2), \ldots, \tilde{S}_{\mathcal{N}}(x)(u,s)\big), \quad u \in J(x) = \operatorname{span}\{\Phi_k(x)\}_{k \in \mathcal{I}_r},$$

where each component is defined as

$$\tilde{S}_{\mathcal{N}}(x)(u,i) = \langle u, \eta_i(x) \rangle, \quad i \in \mathcal{I}_s.$$

5.2.2.1 Matrix representation of samples:

The samples expression (5.11) can be represented in the matrix form. For that purpose, simplifying (5.11) further for

$$f = \sum_{k \in \mathcal{I}_r} \sum_{m \in \mathcal{M}} \alpha_k(m)\, \mathfrak{M}_{g_m} \Phi_k \in W_{\mathcal{A},\mathfrak{B}},$$

we get

$$S_{\mathcal{N}} f(m,i) = \left\langle \sum_{k \in \mathcal{I}_r} \sum_{t \in \mathcal{M}} \alpha_k(t) \mathfrak{M}_{g_t} \Phi_k, \mathfrak{M}_{g_m} \eta_i \right\rangle = \sum_{k \in \mathcal{I}_r} \sum_{t \in \mathcal{M}} \alpha_k(t) \langle \mathfrak{M}_{g_t} \Phi_k, \mathfrak{M}_{g_m} \eta_i \rangle, \tag{5.17}$$

for each $m \in \mathcal{M}, i \in \mathcal{I}_s$. We now define the block operator $\mathcal{L}_{S_{\mathcal{N}}}$, which acts on elements of $\ell^2(\mathcal{M}, \mathbb{C}^r)$ and maps them to $\ell^2(\mathcal{M}, \mathbb{C}^s)$, as

$$\mathcal{L}_{S_{\mathcal{N}}} := \big[\mathcal{L}_{k,i} \,:\, k \in \mathcal{I}_r, i \in \mathcal{I}_s\big].$$

where each operator $\mathcal{L}_{k,i}$ is defined by

$$(\mathcal{L}_{k,i}\alpha_k)(m) = \sum_{t\in\mathcal{M}} \alpha_k(t)\langle \mathfrak{M}_{g_t}\Phi_k, \mathfrak{M}_{g_m}\eta_i\rangle, \quad m \in \mathcal{M}.$$

Then for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)^T \in \ell^2(\mathcal{M}, \mathbb{C}^r)$, the multiplication $\mathcal{L}_{S_\mathcal{N}}\alpha$ is computed as follows:

$$(\mathcal{L}_{S_\mathcal{N}}\alpha)(m) = \begin{pmatrix} \sum\limits_{k\in\mathcal{I}_r}\sum\limits_{t\in\mathcal{M}} \alpha_k(t)\langle \mathfrak{M}_{g_t}\Phi_k, \mathfrak{M}_{g_m}\eta_1\rangle \\ \sum\limits_{k\in\mathcal{I}_r}\sum\limits_{t\in\mathcal{M}} \alpha_k(t)\langle \mathfrak{M}_{g_t}\Phi_k, \mathfrak{M}_{g_m}\eta_2\rangle \\ \vdots \\ \sum\limits_{k\in\mathcal{I}_r}\sum\limits_{t\in\mathcal{M}} \alpha_k(t)\langle \mathfrak{M}_{g_t}\Phi_k, \mathfrak{M}_{g_m}\eta_s\rangle \end{pmatrix}, \quad m \in \mathcal{M}. \tag{5.18}$$

Comparing with the definition of $S_\mathcal{N} f$, we obtain

$$(\mathcal{L}_{S_\mathcal{N}}\alpha)(m) = \begin{pmatrix} S_\mathcal{N} f(m,1) \\ S_\mathcal{N} f(m,2) \\ \vdots \\ S_\mathcal{N} f(m,s) \end{pmatrix}, \quad m \in \mathcal{M}.$$

For a.e. $x \in X$ and $u \in J(x)$, we write $u = \sum\limits_{k\in\mathcal{I}_r} \gamma_k \Phi_k(x)$, $\gamma_k \in \mathbb{C}$, which leads to

$$\langle u, \eta_i(x)\rangle = \sum_{k\in\mathcal{I}_r} \gamma_k \langle \Phi_k(x), \eta_i(x)\rangle.$$

Analogous to (5.18), we define the pointwise operator of order $s \times r$ as follows:

$$\tilde{\mathcal{L}}_{\tilde{S}_\mathcal{N}}(x) = \left[\langle \Phi_k(x), \eta_i(x)\rangle\right]_{k\in\mathcal{I}_r}^{i\in\mathcal{I}_s}.$$

Then, the multiplication $\tilde{\mathcal{L}}_{\tilde{S}_\mathcal{N}}(x)\gamma$ with an $r \times 1$ column vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_r)^T \in \mathbb{C}^r$, as follows:

$$\tilde{\mathcal{L}}_{\tilde{S}_\mathcal{N}}(x)\gamma = \begin{pmatrix} \sum\limits_{k\in\mathcal{I}_r} \gamma_k\langle \Phi_k(x), \eta_1(x)\rangle \\ \sum\limits_{k\in\mathcal{I}_r} \gamma_k\langle \Phi_k(x), \eta_2(x)\rangle \\ \vdots \\ \sum\limits_{k\in\mathcal{I}_r} \gamma_k\langle \Phi_k(x), \eta_s(x)\rangle \end{pmatrix} = \begin{pmatrix} \tilde{S}_\mathcal{N}(x)(u,1) \\ \tilde{S}_\mathcal{N}(x)(u,2) \\ \vdots \\ \tilde{S}_\mathcal{N}(x)(u,s) \end{pmatrix}. \tag{5.19}$$

We have the following result, which is an abstract version of the existing results on stability of the sampling set [10, 78, 129]. Additionally, it offers equivalent a.e. local conditions, which is a novel aspect of our findings.

Theorem 5.11. *Let $\mathcal{A} = \{\Phi_1, \Phi_2, \dots, \Phi_r\} \subset L^2(X, \mathcal{H})$ be such that $W_{\mathcal{A},\mathfrak{B}}$ has the form (5.4) with corresponding range function $J_{W_{\mathcal{A},\mathfrak{B}}} := J$. Further, let $\mathcal{N} = \{\eta_i\}_{i\in\mathcal{I}_s} \subset W_{\mathcal{A},\mathfrak{B}}$ be such that the system $E_{\mathcal{N},\mathfrak{B}} = \{\mathfrak{M}_{g_m}\eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is Bessel. Then the following statements are equivalent:*

(i) The set $\mathcal{M} \times \mathcal{I}_s$ is a stable set of sampling for $W_{\mathcal{A},\mathfrak{B}}$. That is, there exist constants $0 < a \leq b < \infty$ such that

$$a\|f\| \leq \left(\sum_{i\in\mathcal{I}_s}\sum_{m\in\mathcal{M}} |(S_{\mathcal{N}}f)(m,i)|^2\right)^{1/2} \leq b\|f\| \textit{ for all } f \in W_{\mathcal{A},\mathfrak{B}}.$$

(ii) For a.e. $x \in X$ and every $u \in J(x)$, there exist constants $0 < a \leq b < \infty$ such that

$$a\|u\|^2 \leq \sum_{i\in\mathcal{I}_s} |(\tilde{S}_{\mathcal{N}}(x)(u,i))|^2 \leq b\|u\|^2.$$

(iii) For each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r) \in \ell^2(\mathcal{M}, \mathbb{C}^r)$, there exist constants $0 < c \leq d < \infty$ such that

$$c\sum_{k\in\mathcal{I}_r} \|\alpha_k\|_{\ell^2(\mathcal{M})} \leq \|\mathcal{L}_{S_{\mathcal{N}}}\alpha\| \leq d\sum_{k\in\mathcal{I}_r} \|\alpha_k\|_{\ell^2(\mathcal{M})}.$$

(iv) For a.e. $x \in X$ and every $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_r) \in \mathbb{C}^r$, there exist constants $0 < c \leq d < \infty$ such that

$$c\sum_{k\in\mathcal{I}_r} |\gamma_k|^2 \leq \|\tilde{\mathcal{L}}_{\tilde{S}_{\mathcal{N}}}(x)\gamma\|^2 \leq d\sum_{k\in\mathcal{I}_r} |\gamma_k|^2.$$

Proof. (i) $\iff$ (ii). Since the stability of the sampling set $\mathcal{M} \times \mathcal{I}_s$ implies that $E_{\mathcal{N},\mathfrak{B}}$ forms a frame for $W_{\mathcal{A},\mathfrak{B}}$, the equivalence follows directly from [88, Theorem 2.10].

(i) $\iff$ (iii). Since the norm of $\mathcal{L}_{S_{\mathcal{N}}}\alpha$, defined in (5.18), is

$$\|\mathcal{L}_{S_{\mathcal{N}}}\alpha\| = (\sum_{i\in\mathcal{I}_s}\sum_{m\in\mathcal{M}} |S_{\mathcal{N}}f(m,i)|^2)^{1/2},$$

the result follows by utilizing the Riesz basis property of $E_{\mathcal{A},\mathfrak{B}}$.

(ii) $\iff$ (iv). Since the norm of $\tilde{\mathcal{L}}_{\tilde{S}_{\mathcal{N}}}(x)\gamma$ defined in (5.19) for a.e. $x \in X$ is $\|\tilde{\mathcal{L}}_{\tilde{S}_{\mathcal{N}}}(x)\gamma\|^2 = \sum_{i\in\mathcal{I}_s} |\tilde{S}_{\mathcal{N}}(x)(u,i)|^2$, the result follows by utilizing the Riesz basis property of $E_{\mathcal{A},\mathfrak{B}}$. □

5.2.3 Sampling formula: global and local form

In this subsection, we characterize the stability of the sampling set $\mathcal{M} \times \mathcal{I}_s$ through both the global and local formulations of a generalized sampling formula. In particular, the approach of range functions enable the sampling problem to be analyzed within finite-dimensional fiber spaces determined by the corresponding range function for a.e. $x \in X$.

Proposition 5.12. *Let $\mathcal{A} = \{\Phi_1, \Phi_2, \dots, \Phi_r\} \subset L^2(X, \mathcal{H})$ be such that $W_{\mathcal{A},\mathfrak{B}}$ has the form (5.4). Further, let $\mathcal{N} = \{\eta_i\}_{i \in \mathcal{I}_s} \subset W_{\mathcal{A},\mathfrak{B}}$ be such that the system $E_{\mathcal{N},\mathfrak{B}} = \{\mathfrak{M}_{g_m}\eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is Bessel. Then the following statements hold true under the standing assumptions of Proposition 5.7:*

(i) For each $f \in W_{\mathcal{A},\mathfrak{B}}$, the samples $S_{\mathcal{N}}f$ defined in (5.11) can be represented in the following form

$$S_{\mathcal{N}}f(m,i) = \langle F, M_{g_m} T_{\mathcal{A}}^* \eta_i \rangle, \ m \in \mathcal{M} \text{ and } i \in \mathcal{I}_s, \tag{5.20}$$

where $F \in L^2(X, \mathbb{C}^r)$ is such that $T_{\mathcal{A}}F = f$ and $T_{\mathcal{A}}^*$ is the adjoint of $T_{\mathcal{A}}$.

(ii) For any $F = (F_1, F_2, \dots, F_r) \in L^2(X, \mathbb{C}^r)$, the pointwise-multiplication $T_{\mathcal{A}}^*\eta_i.F^T$ (and hence $F.(T_{\mathcal{A}}^*\eta_i)^T$) is a member of $L^2(X)$, where

$$(T_{\mathcal{A}}^*\eta_i \cdot F^T)(x) := \langle T_{\mathcal{A}}^*\eta_i(x), F(x)\rangle_{\mathbb{C}^r}, \quad x \in X.$$

(iii) The collection $\{M_{g_m}T_{\mathcal{A}}^*\eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ associated with the samples $S_{\mathcal{N}}f$ is Bessel if and only if

$$\operatorname*{ess\text{-}sup}_{x\in X} \lambda_{max}[\mathbb{U}(x)] < \infty,$$

where $\mathbb{U}(x) := \mathbb{S}^*(x)\mathbb{S}(x)$, and $\mathbb{S}(x) := \Big(\varphi_1^T(x)\ \varphi_2^T(x) \cdots \varphi_s^T(x)\Big)^T$ is the $s \times r$ matrix for $\varphi_i(x) = \overline{T_{\mathcal{A}}^*\eta_i(x)} \in L^2(X, \mathbb{C}^r)$ for a.e. $x \in X$ and $i \in \mathcal{I}_s$.

(iv) Any $f \in W_{\mathcal{A},\mathfrak{B}}$ can be recovered from the samples $S_{\mathcal{N}}f$ if and only if the collection $\{M_{g_m}T_{\mathcal{A}}^*\eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is Bessel and

$$\operatorname*{ess\text{-}inf}_{x\in X} \lambda_{min}[\mathbb{U}(x)] > 0.$$

Remark 5.13. Referring to (5.20), it is evident that the samples $S_{\mathcal{N}}f$ will recover the function $f \in W_{\mathcal{A},\mathfrak{B}}$ uniquely if and only if the system $\{M_{g_m}T_{\mathcal{A}}^*\eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is complete. Therefore, the oversampling condition $s \geq r$ is necessary to ensure that the system $\{M_{g_m}T_{\mathcal{A}}^*\eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is complete in $L^2(X, \mathbb{C}^r)$. Indeed, it can be shown that the system

$\{M_{g_m}T_{\mathcal{A}}^*\eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is complete if and only if rank $\mathbb{S}(x) = r$, a.e. $x \in X$. However, we won't be presenting the proof here. Its proof for shift-invariant spaces and unitary invariant subspaces in the LCA group set-up can be found in [59, 60].

We have the following main result of this section:

Theorem 5.14. *Let $\mathcal{A} = \{\Phi_1, \Phi_2, \dots, \Phi_r\} \subset L^2(X, \mathcal{H})$ be such that $W_{\mathcal{A},\mathfrak{B}}$ has the form (5.4) with corresponding range function $J_{W_{\mathcal{A},\mathfrak{B}}} := J$. Further, let $\mathcal{N} = \{\eta_i\}_{i\in\mathcal{I}_s} \subset W_{\mathcal{A},\mathfrak{B}}$ be such that the system $E_{\mathcal{N},\mathfrak{B}} = \{\mathfrak{M}_{g_m}\eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is Bessel. Then the following statements are equivalent under the standing assumptions of Proposition 5.7:*

(i) The set $\mathcal{M} \times \mathcal{I}_s$ is a stable set of sampling for $W_{\mathcal{A},\mathfrak{B}}$.

(ii) There exist $h_i \in W_{\mathcal{A},\mathfrak{B}}$, $i \in \mathcal{I}_s$ such that the collection $\{\mathfrak{M}_{g_m}h_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is a frame for $W_{\mathcal{A},\mathfrak{B}}$ and the sampling formula (global form) for any $f \in W_{\mathcal{A},\mathfrak{B}}$ is

$$f = \sum_{i\in\mathcal{I}_s}\sum_{m\in\mathcal{M}} S_{\mathcal{N}}f(m,i)\,\mathfrak{M}_{g_m}h_i. \tag{5.21}$$

(iii) There exist $h_i \in W_{\mathcal{A},\mathfrak{B}}$, $i \in \mathcal{I}_s$ such that for a.e. $x \in X$, $\{h_i(x)\}_{i\in\mathcal{I}_s}$ is a frame for $J(x)$ and the sampling formula (local form) is

$$u = \sum_{i\in\mathcal{I}_s} \tilde{S}_{\mathcal{N}}(x)(u,i)h_i(x),\ \ u \in J(x) = \operatorname{span}\{\Phi_k(x)\}_{k\in\mathcal{I}_r}. \tag{5.22}$$

(iv) The collection $\{M_{g_m}T_{\mathcal{A}}^*\eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is a frame for $L^2(X,\mathbb{C}^r)$. That is, there exist $0 < a \le b < \infty$ such that

$$a\|F\|^2_{L^2(X,\mathbb{C}^r)} \le \sum_{i\in\mathcal{I}_s}\sum_{m\in\mathcal{M}}\left|\int_X \langle F(x), g_m(x)T_{\mathcal{A}}^*\eta_i(x)\rangle_{\mathbb{C}^r}d\mu(x)\right|^2 \le b\|F\|^2_{L^2(X,\mathbb{C}^r)},\ \ F \in L^2(X,\mathbb{C}^r).$$

(v) For a.e. $x \in X$ and every $u \in \mathbb{C}^r$, there exist $0 < a \le b < \infty$ such that

$$a\|u\|^2 \le \sum_{i\in\mathcal{I}_s} |\langle u, R^*(x)\eta_i(x)\rangle_{\mathbb{C}^r}|^2 \le b\|u\|^2 \ \text{ holds,}$$

where $R^*(x)$ is the adjoint of the operator $R(x)$ (defined in (5.6)) for a.e. $x \in X$.

(vi) There exists an $r \times s$ matrix $\Psi(x)$ with each column in $L^\infty(X,\mathbb{C}^r)$ satisfying

$$\Psi(x)\mathbb{S}(x) = I_r \text{ for a.e. } x \in X,$$

where $\mathbb{S}(x)$ is defined in Proposition 5.12 and I_r is the $r \times r$ identity matrix.

Example 5.15. Let $X = (0,1)$, $\mathcal{H} = \mathbb{C}$, $\mathcal{M} = \mathbb{Z}$, $\mathfrak{B} = \{e^{2\pi i m \cdot}\}_{m\in\mathbb{Z}}$ and $\mathcal{A} = \{\Phi\}$, where Φ is the Gaussian function $\Phi = e^{-x^2/2}, 0 < x < 1$. Then the collection

$$E_{\mathcal{A},\mathfrak{B}} := \{e^{2\pi i m x} e^{-x^2/2}\}_{m\in\mathbb{Z}}$$

is a Riesz basis for $L^2(0,1)$ since the Gaussian generator Φ satisfies

$$0 < \operatorname*{ess-inf}_{x\in(0,1)} e^{-x^2/2} < \operatorname*{ess-sup}_{x\in(0,1)} e^{-x^2/2} < \infty \tag{5.23}$$

[65, Theorem 2]. For $s = 1$, consider the cubic B-spline function defined by

$$\eta_1(x) := N_3(x) = \chi_{[0,1]} * \chi_{[0,1]} * \chi_{[0,1]} * \chi_{[0,1]}(x), \quad x \in (0,1),$$

where '$*$' denotes the convolution in $L^1(\mathbb{R})$. This also satisfies,

$$0 < \operatorname*{ess-inf}_{x\in(0,1)} |N_3(x)| < \operatorname*{ess-sup}_{x\in(0,1)} |N_3(x)| < \infty, \tag{5.24}$$

and therefore the system

$$E_{\mathcal{N},\mathfrak{B}} = \{e^{2\pi i m \cdot} N_3\}_{m\in\mathbb{Z}}$$

is a frame for $L^2(0,1)$ [65, Theorem 2]. This implies that $\mathbb{Z}$ is a stable set of sampling for $\overline{\operatorname{span}}\{e^{2\pi i m x} e^{-\pi x^2}\}_{m\in\mathbb{Z}}$. Then by application of Theorem 5.14, there exists $\Psi \in \overline{\operatorname{span}}\{e^{2\pi i m x} e^{-\pi x^2}\}_{m\in\mathbb{Z}}$ such that the following sampling formula holds for any $f \in L^2(0,1)$

$$f(x) = \sum_{m\in\mathbb{Z}} \langle f, e^{2\pi i m \cdot} N_3\rangle e^{2\pi i m x}\Psi(x) = \sum_{m\in\mathbb{Z}} \langle f, e^{2\pi i m \cdot}\Psi\rangle e^{2\pi i m x} N_3(x), \tag{5.25}$$

where the convergence of the series is pointwise due to the unconditional convergence of the frame expansion, which is uniform as a consequence of (5.24).

Further, for any $u \in \operatorname{span}\{e^{-\pi x^2}\}$, the local form of (5.25) gives:

$$u = \langle u, N_3(x)\rangle \Psi(x) = \langle u, \Psi(x)\rangle N_3(x), \quad \text{for a.e. } x \in (0,1).$$

We are now ready to provide the proof of Proposition 5.12.

Proof of Proposition 5.12. Note that (i) holds using (5.9) for $f \in W_{\mathcal{A},\mathfrak{B}}$ with $T_{\mathcal{A}} f = F$, we have

$$S_{\mathcal{N}} f(m,i) = \langle T_{\mathcal{A}} F, \mathfrak{M}_{g_m} \eta_i\rangle = \langle F, T_{\mathcal{A}}^* \mathfrak{M}_{g_m} \eta_i\rangle = \langle F, M_{g_m} T_{\mathcal{A}}^* \eta_i\rangle.$$

For (ii), fix $i \in \mathcal{I}_r$. Then by the application of the Cauchy–Schwarz inequality, we get

$$
\begin{aligned}
\|T_{\mathcal{A}}^*\eta_i \cdot F^T\|^2 &= \int_X \left|T_{\mathcal{A}}^*\eta_i(x) \cdot F^T(x)\right|^2 d\mu(x) = \int_X \left|\langle \overline{T_{\mathcal{A}}^*\eta_i(x)}, F(x)\rangle\right|^2 d\mu(x) \\
&\le \int_X \|T_{\mathcal{A}}^*\eta_i(x)\|^2 \|F(x)\|^2 d\mu(x) = \int_X \|R^*(x)\eta_i(x)\|^2 \|F(x)\|^2 d\mu(x).
\end{aligned}
$$

Since $R(x)$ is uniformly bounded for a.e. $x \in X$, then $\|R^*(x)\eta_i(x)\|^2 \le C_1\|\eta_i(x)\|^2, \quad \text{a.e. } x \in X$ for some $C_1 > 0$. Furthermore, the functions $\eta_i(x) \in J(x)$, for a.e. $x \in X$, can be expressed as $\eta_i(x) = \sum_{k\in\mathcal{I}_r} \gamma_k \Phi_k(x)$, for some coefficients $\{\gamma_k\}_{k\in\mathcal{I}_r} \subset \mathbb{C}^r$. By Proposition 5.6, we obtain the following bound for some constant $C_2 > 0$:

$$
\begin{aligned}
\operatorname*{ess\,sup}_{x\in X} \|\eta_i(x)\|^2 &\le \sum_{k\in\mathcal{I}_r}\sum_{j\in\mathcal{I}_r} |\gamma_k||\overline{\gamma_j}| \operatorname*{ess\,sup}_{x\in X} \left(\|\Phi_k(x)\|\|\Phi_j(x)\|\right) \\
&\le C_2 \max_{k\in\mathcal{I}_r} \operatorname*{ess\,sup}_{x\in X} \|\Phi_k(x)\|^2 \|\gamma\|^2 < \infty.
\end{aligned}
\tag{5.26}
$$

By combining the above estimates, we conclude that

$$
\|T_{\mathcal{A}}^*\eta_i \cdot F^T\|^2 \le C_1 C_2 \max_{k\in\mathcal{I}_r} \operatorname*{ess\,sup}_{x\in X} \|\Phi_k(x)\|^2 \|\gamma\|^2 \|F\|^2 < \infty.
$$

Now we prove (iii). Let the collection $\{M_{g_m} T_{\mathcal{A}}^*\eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ be Bessel with bound B. Using (5.20), we have the inequality $\sum_{i\in\mathcal{I}_s}\sum_{m\in\mathcal{M}} |S_{\mathcal{N}} f(m,i)|^2 \le B\|F\|^2$ for $f = T_A F$. The samples $S_{\mathcal{N}} f$ can be further simplified for $F = (F_1, F_2, \ldots, F_r)$ and $\varphi_i = \overline{T_{\mathcal{A}}^*\eta_i} = (\varphi_{i,1}, \varphi_{i,2}, \ldots, \varphi_{i,r})$ in $L^2(X, \mathbb{C}^r)$ as follows:

$$
\begin{aligned}
S_{\mathcal{N}} f(m,i) = \langle F, M_{g_m}\overline{\varphi}_i\rangle &= \int_X \sum_{k\in\mathcal{I}_r} F_k(x)\varphi_{i,k}(x)\overline{g_m(x)} d\mu(x) \\
&= \int_X (\varphi_i.F^T)(x)\overline{g_m(x)} d\mu(x) = \mathcal{F}(\varphi_i.F^T)(m),
\end{aligned}
\tag{5.27}
$$

for each $m \in \mathcal{M}$ and $i \in \mathcal{I}_s$ since $\varphi_i.F^T \in L^2(X)$ using (ii). Then, by using Plancherel formula and using (5.27), we have

$$
\sum_{i\in\mathcal{I}_s}\sum_{m\in\mathcal{M}} |S_{\mathcal{N}} f(m,i)|^2 = \sum_{i\in\mathcal{I}_s} \|\mathcal{F}(\varphi_i.F^T)\|_{\ell^2(\mathcal{M})}^2 = \sum_{i\in\mathcal{I}_s} \|\varphi_i.F^T\|_{L^2(X)}^2 = \|\mathbb{S}F^T\|_{L^2(X,\mathbb{C}^s)}^2,
\tag{5.28}
$$

where for a.e. $x \in X$, $\mathbb{S}(x)F^T(x) = \left[\varphi_1(x)F^T(x), \varphi_2(x)F^T(x), \ldots, \varphi_s(x)F^T(x)\right]^T$. Since

$$
\begin{aligned}
\|\mathbb{S}F^T\|^2 &= \int_X \|\mathbb{S}(x)F^T(x)\|^2 d\mu(x) = \int_X \langle \mathbb{S}(x)F^T(x), \mathbb{S}(x)F^T(x)\rangle d\mu(x) \\
&= \int_X \left(\mathbb{S}(x)F^T(x)\right)^* \mathbb{S}(x)F^T(x) d\mu(x) = \int_X \overline{F(x)}\mathbb{U}(x)F^T(x) d\mu(x) \\
&\le \operatorname*{ess\text{-}sup}_{x\in X} \lambda_{max}[\mathbb{U}(x)] \int_X \|F(x)\|^2 d\mu(x),
\end{aligned}
\tag{5.29}
$$

therefore by using (5.28), we get $\sum_{i\in\mathcal{I}_s}\sum_{m\in\mathcal{M}}|S_{\mathcal{N}}f(m,i)|^2 \leq L\|F\|^2$ for each $F \in L^2(X,\mathbb{C}^r)$, where $L := \text{ess-sup}_{x\in X}\lambda_{max}[\mathbb{U}(x)]$. We now claim that L is the smallest constant satisfying the above inequality. For that, choose $L^{'} < L$. Then by the definition of essential supremum, there exists a set $V \subset X$ such that $\mu(V) > 0$ and $\lambda_{max}[\mathbb{U}(x)] > L^{'}$ for a.e. $x \in V$. Choose $G \in L^2(X,\mathbb{C}^r)$ such that

$$G(x) = \begin{cases} 0, & \text{if } x \in X\setminus V, \\ \gamma(x), & \text{if } x \in V, \end{cases}$$

where $\gamma(x)$ is the eigenvector corresponding to the maximum eigenvalue of $\mathbb{U}(x)$. Then by (5.28), we get

$$\sum_{i\in\mathcal{I}_s}\sum_{m\in\mathcal{M}}|S_{\mathcal{N}}g(m,i)|^2 = \|\mathbb{S}G^T\|^2 = \int_X \overline{G(x)}\mathbb{U}(x)G^T(x)d\mu(x) \geq L^{'}\|G\|^2,$$

where $g = T_{\mathcal{A}}G$. This gives $L \leq B < \infty$ and proves the claim. Converse part follows from (i), (5.28) and (5.29).

For (iv), it is clear that any $f \in W_{\mathcal{A},\mathfrak{B}}$ can be stably recovered from the samples $\{S_{\mathcal{N}}f(m,i)\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ if and only if $\{M_{g_m}T_{\mathcal{A}}^*\eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is a frame for $L^2(X,\mathbb{C}^r)$. Assume that the system $\{M_{g_m}T_{\mathcal{A}}^*\eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is a frame for $L^2(X,\mathbb{C}^r)$ with frame bounds A, B. Consider $M := \text{ess-inf}_{x\in X}\lambda_{min}[\mathbb{U}(x)]$. Then by (5.28),

$$\begin{aligned}\sum_{i\in\mathcal{I}_s}\sum_{m\in\mathcal{M}}|S_{\mathcal{N}}f(m,i)|^2 = \|\mathbb{S}F^T\|^2 &= \int_X \overline{F(x)}\mathbb{U}(x)F^T(x)d\mu(x) \\ &\geq \operatorname*{ess-inf}_{x\in X}\lambda_{min}[\mathbb{U}(x)]\|F\|^2 \geq M\|F\|^2,\end{aligned} \tag{5.30}$$

where $f = T_{\mathcal{A}}F$. Similar to the proof of (iii), it can be observed that M is the largest constant satisfying this inequality. Therefore $0 < A \leq M$. The converse part holds using the above inequality and (iii). □

Finally, we provide the proof of Theorem 5.14.

Proof of Theorem 5.14. (i)$\Longleftrightarrow$(iv) follows from 5.12 (i) and the Riesz basis property of $E_{\mathcal{A},\mathfrak{B}}$.

(iv)$\Longleftrightarrow$(v). Suppose (v) holds. For $F \in L^2(X,\mathbb{C}^r)$, there exist constants $0 < a \leq b < \infty$ such that we have the following for a.e. $x \in X$:

$$a\|F(x)\|^2 \leq \sum_{i\in\mathcal{I}_s}|\langle F(x), R^*(x)\eta_i(x)\rangle_{\mathbb{C}^r}|^2 \leq b\|F(x)\|^2, \ F(x) \in \mathbb{C}^r. \tag{5.31}$$

Integrating (5.31), we get

$$a\|F\|^2 \leq \sum_{i\in\mathcal{I}_s} \int_X |\langle F(x), (T_{\mathcal{A}}^*\eta_i)(x)\rangle|^2 d\mu(x) \leq b\|F\|^2.$$

Then by using the Plancherel formula, we get

$$\begin{aligned}\sum_{i\in\mathcal{I}_s} \int_X |\langle F(x), (T_{\mathcal{A}}^*\eta_i)(x)\rangle|^2 d\mu(x) &= \sum_{i\in\mathcal{I}_s}\sum_{m\in\mathcal{M}} \left| \int_X \langle F(x), (T_{\mathcal{A}}^*\eta_i)(x)\rangle \overline{g_m(x)} d\mu(x)\right|^2 \\ = \sum_{i\in\mathcal{I}_s}\sum_{m\in\mathcal{M}} \left| \int_X \langle F(x), g_m(x) T_{\mathcal{A}}^*\eta_i(x)\rangle d\mu(x)\right|^2 &\end{aligned}$$

and hence (iv) holds with bounds as in (v).

For the converse part (iv) $\implies$ (v), we choose a countable dense subset $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}^r$. We use the same technique as used in proving Theorem 5.11. If (v) is not true, then either $\mu\big(\{x \in X : \sum_{i\in\mathcal{I}_s} |\langle z_n, R^*(x)\eta_i(x)\rangle|^2 - b\|z_n\|^2 > 0\}\big) > 0$ or $\mu\big(\{x \in X : \sum_{i\in\mathcal{I}_s} |\langle z_n, R^*(x)\eta_i(x)\rangle|^2 - a\|z_n\|^2 < 0\}\big) > 0$ for some $n \in \mathbb{N}$. Equivalently,

$$\mu\Bigg(\bigcup_{v=1}^{\infty}\Bigg\{x \in X : \sum_{i\in\mathcal{I}_s} |\langle z_n, R^*(x)\eta_i(x)\rangle|^2 - (b+1/v)\|z_n\|^2 > 0\Bigg\}\Bigg) > 0 \quad \text{or}$$

$$\mu\Bigg(\bigcup_{v=1}^{\infty}\Bigg\{x \in X : \sum_{i\in\mathcal{I}_s} |\langle z_n, R^*(x)\eta_i(x)\rangle|^2 - (a-1/v)\|z_n\|^2 < 0\Bigg\}\Bigg) > 0,$$

Similar to the proof of Theorem 5.11, we can observe that both the above conditions lead to a contradiction to (iv). Hence (iv) $\iff$ (v) holds.

(vi) $\implies$ (ii). Assume (vi) holds. Observe that by using Proposition 5.12 (ii) and the Fourier transform $\mathcal{F}$, we have the following equality for $\varphi_i := \overline{T_{\mathcal{A}}^*\eta_i}$, $i \in \mathcal{I}_s$, and for any $F \in L^2(X, \mathbb{C}^r)$:

$$\begin{aligned}\sum_{m\in\mathcal{M}} \langle F, M_{g_m} T_{\mathcal{A}}^*\eta_i\rangle g_m(x) &= \sum_{m\in\mathcal{M}} \left(\int_X (F.\varphi_i^T)(y)\overline{g_m(y)} d\mu(y)\right) g_m(x) \\ = \sum_{m\in\mathcal{M}} \mathcal{F}(F.\varphi_i^T)(m) g_m(x) &= \mathcal{F}^{-1}\mathcal{F}(F.\varphi_i^T)(x) = F(x).\varphi_i^T(x) = \varphi_i(x).F^T(x), \quad x \in X.\end{aligned} \tag{5.32}$$

Consequently, we have the following equality for a.e. $x \in X$:

$$\begin{aligned}\mathbb{S}(x)F^T(x) &= \left[\varphi_1(x)F^T(x), \varphi_2(x)F^T(x), \ldots, \varphi_s(x)F^T(x)\right]^T \\ &= \left(\sum_{m\in\mathcal{M}} \langle F, M_{g_m} T_{\mathcal{A}}^*\eta_1\rangle g_m(x), \sum_{m\in\mathcal{M}} \langle F, M_{g_m} T_{\mathcal{A}}^*\eta_2\rangle g_m(x), \ldots, \sum_{m\in\mathcal{M}} \langle F, M_{g_m} T_{\mathcal{A}}^*\eta_s\rangle g_m(x)\right)^T,\end{aligned}$$

Let $\Psi(x) = (\psi_1^T(x), \psi_2^T(x), \ldots, \psi_s^T(x))$ be the $r \times s$ matrix satisfying $\Psi(x)\mathbb{S}(x) = I_r$ for a.e. $x \in X$ with $\psi_i \in L^\infty(X, \mathbb{C}^r), i \in \mathcal{I}_s$. By left multiplication of $\Psi(x)$ with $\mathbb{S}(x)F^T(x)$, we get $F^T(x) = \sum_{i\in\mathcal{I}_s}\sum_{m\in\mathcal{M}}\langle F, M_{g_m}T_{\mathcal{A}}^*\eta_i\rangle\psi_i^T(x)g_m(x)$ and hence

$$F(x) = \sum_{i\in\mathcal{I}_s}\sum_{m\in\mathcal{M}}\langle F, M_{g_m}T_{\mathcal{A}}^*\eta_i\rangle\psi_i(x)g_m(x) \quad \text{for a.e. } x \in X. \tag{5.33}$$

Since the functions $\psi_i \in L^\infty(X, \mathbb{C}^r)$ for $i \in \mathcal{I}_s$, therefore $F.\psi^T \in L^2(X)$ and we get

$$\begin{aligned}\sum_{i\in\mathcal{I}_s}\sum_{m\in\mathcal{M}} |\langle F, M_{g_m}\psi_i\rangle|^2 &= \sum_{i\in\mathcal{I}_s}\sum_{m\in\mathcal{M}} \left|\mathcal{F}(F.\psi_i^T)(m)\right|^2 \\ &= \sum_{i\in\mathcal{I}_s}\|F.\psi_i^T\|^2 \le \left(\sum_{i\in\mathcal{I}_s}\|\psi_i\|_{L^\infty(X,\mathbb{C}^r)}\right)\|F\|^2.\end{aligned}$$

As its consequence, $\{M_{g_m}\psi_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is a Bessel sequence in $L^2(X, \mathbb{C}^r)$. Further, $E_{\mathcal{N},\mathfrak{B}}$ is Bessel and $T_{\mathcal{A}}$ is a bounded operator implies that the collection $\{T_{\mathcal{A}}^*(M_{g_m}\eta_i)\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s} = \{M_{g_m}T_{\mathcal{A}}^*\eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is Bessel. Then the Bessel sequences $\{M_{g_m}T_{\mathcal{A}}^*\eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ and $\{M_{g_m}\psi_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ satisfy (5.33) and according to [42, Lemma 6.3.2], they are dual frames for $L^2(X, \mathbb{C}^r)$. Applying the map $T_{\mathcal{A}}$ on both sides of the equation (5.33) and using Proposition 5.7 along with equation (5.20), we get

$$f(x) = \sum_{i\in\mathcal{I}_s}\sum_{m\in\mathcal{M}} S_{\mathcal{N}}f(m,i)T_{\mathcal{A}}(M_{g_m}\psi_i(.))(x) = \sum_{i\in\mathcal{I}_s}\sum_{m\in\mathcal{M}} S_{\mathcal{N}}f(m,i)\mathfrak{M}_{g_m}T_{\mathcal{A}}(\psi_i)(x), \tag{5.34}$$

for a.e. $x \in X$. Finally by [42, Corollary 5.3.2], $\{\mathfrak{M}_{g_m}T_{\mathcal{A}}(\psi_i)\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is a frame for $L^2(X, \mathbb{C}^r)$. Hence (ii) holds, where $h_i := T_{\mathcal{A}}(\psi_i) \in W_{\mathcal{A},\mathfrak{B}}$.

(iv) $\Longrightarrow$ (vi). Assume (iv) holds. Then, by using Proposition 5.12, we have ess-inf$_{x\in X}\,\lambda_{min}[\mathbb{S}^*(x)\mathbb{S}(x)] > 0$. Consequently, the Moore-Penrose pseudo inverse matrix is defined and is given by $\mathbb{S}^\dagger(x) = [\mathbb{S}^*(x)\mathbb{S}(x)]^{-1}\mathbb{S}^*(x)$ for a.e. $x \in X$ and it satisfies $\mathbb{S}^\dagger(x)\mathbb{S}(x) = I_r$. Since $\lambda_{\min}[\mathbb{S}^*(x)\mathbb{S}(x)] \ge c > 0$ for some c and a.e. $x \in X$, then the matrix, $\mathbb{S}^*(x)\mathbb{S}(x)$ is Hermitian positive semi-definite and therefore

$$\|[\mathbb{S}^*(x)\mathbb{S}(x)]^{-1}\| = \frac{1}{\lambda_{\min}(\mathbb{S}^*(x)\mathbb{S}(x))} \le c^{-1} \quad a.e.\ x \in X.$$

This shows that the entries of the matrix $\mathbb{S}^*(x)\mathbb{S}(x)$ are essentially bounded. Further, the entries of $\mathbb{S}(x)$ are also bounded a.e. since

$$\operatorname*{ess\text{-}sup}_{x\in X}\|T_{\mathcal{A}}^*\eta_i(x)\| = \operatorname*{ess\text{-}sup}_{x\in X}\|R(x)^*\eta_i(x)\| \le \infty,$$

where the last inequality follows since $R(x)$ (hence $R^*(x)$) is *uniformly bounded* for a.e. $x \in X$ and using (5.26). Consequently, the entries and hence the columns of $\mathbb{S}^\dagger(x)$ are essentially bounded. Hence (vi) holds.

(ii) $\Longrightarrow$ (iv). Assume (ii) holds. Applying $T_{\mathcal{A}}^{-1}$ on both sides of (5.21), we get

$$F = \sum_{i \in \mathcal{I}_s} \sum_{m \in \mathcal{M}} \langle F, M_{g_m} T_{\mathcal{A}}^* \eta_i \rangle T_{\mathcal{A}}^{-1}(\mathfrak{M}_{g_m} r_i).$$

Since $T_{\mathcal{A}}^{-1}$ is bounded (by open mapping theorem) and surjective, then again by [42, Corollary 5.3.2], the collection $\{T_{\mathcal{A}}^{-1}(\mathfrak{M}_{g_m} r_i)\}_{m \in \mathcal{M}}^{i \in \mathcal{I}_s}$ is a frame for $L^2(X, \mathbb{C}^r)$. Finally by [42, Lemma 6.3.2], the collections $\{M_{g_m} T_{\mathcal{A}}^* \eta_i\}_{m \in \mathcal{M}}^{i \in \mathcal{I}_s}$ and $\{T_{\mathcal{A}}^{-1}(\mathfrak{M}_{g_m} r_i)\}_{m \in \mathcal{M}}^{i \in \mathcal{I}_s}$ are dual frames for $L^2(X, \mathbb{C}^r)$ and hence (iv) holds.

(ii)$\Longleftrightarrow$(iii). Assume (iii) holds. For any $f \in W_{\mathcal{A},\mathfrak{B}}$, there exist $h_i \in W_{\mathcal{A},\mathfrak{B}}, i \in \mathcal{I}_s$ such that for a.e. $x \in X$, $f(x) = \sum_{i \in \mathcal{I}_s} \tilde{S}_{\mathcal{N}}(x)(f(x), i) h_i(x)$ and the collection $\{h_i(x)\}_{i \in \mathcal{I}_s}$ is a frame for $J(x)$ for a.e. $x \in X$. Thus the collection $\{\mathfrak{M}_{g_m} h_i\}_{m \in \mathcal{M}}^{i \in \mathcal{I}_s}$ is a frame for $W_{\mathcal{A},\mathfrak{B}}$. Further note that by using [26, Theorem 5.5], the collection $\{\eta_i(x)\}_{i \in \mathcal{I}_s}$ is Bessel with same bound as that of $E_{\mathcal{N},\mathfrak{B}}$ (say, B). Then, we have

$$\sum_{i \in \mathcal{I}_s} \int_X \left|\langle f(x), \eta_i(x)\rangle\right|^2 d\mu(x) = \int_X \sum_{i \in \mathcal{I}_s} \left|\langle f(x), \eta_i(x)\rangle\right|^2 d\mu(x) \leq B \int_X \|f(x)\|^2 d\mu(x) < \infty.$$

This implies that the map $x \mapsto \tilde{S}_{\mathcal{N}}(x)(f(x), i) = \langle f(x), \eta_i(x)\rangle$ is in $L^2(X)$ for each i. Then by using the property $\mathcal{F}^{-1}\mathcal{F} = I$ and Proposition (5.10) to the above expression of $f(x)$, we have

$$\begin{aligned} f(x) &= \sum_{i \in \mathcal{I}_s} \sum_{m \in \mathcal{M}} \left(\int_X \tilde{S}_{\mathcal{N}}(y)(f(y), i) \overline{g_m(y)} d\mu(y) \right) g_m(x) h_i(x) \\ &= \sum_{i \in \mathcal{I}_s} \sum_{m \in \mathcal{M}} \left(\mathcal{F}^{(s)} \int_X^{\oplus} \tilde{S}_{\mathcal{N}}(x) d\mu(x) f \right)(m, i) g_m(x) h_i(x) \qquad (5.35) \\ &= \sum_{i \in \mathcal{I}_s} \sum_{m \in \mathcal{M}} S_{\mathcal{N}} f(m, i) (\mathfrak{M}_{g_m} h_i)(x), \text{ a.e. } x \in X. \end{aligned}$$

Hence (ii) holds true. Conversely, assume (iii) is not true. Then for any set $\{h_i\}_{i \in \mathcal{I}_s}$ of functions in $W_{\mathcal{A},\mathfrak{B}}$, there exists a set $V \subset X$ of positive measure such that for any $x \in V$, either the collection $\{h_i(x)\}_{i \in \mathcal{I}_s}$ is not a frame for $J(x)$ or it is a frame for a.e. $x \in X$ but (5.22) does not hold for some $u \in J(x)$ and $x \in V$. If the first case holds, then the collection $\{\mathfrak{M}_{g_m} h_i\}_{m \in \mathcal{M}}^{i \in \mathcal{I}_s}$ is not a frame for $W_{\mathcal{A},\mathfrak{B}}$, which is a contradiction to (ii).

If the second case holds, then the R.H.S. of (5.22), given by

$$\sum_{i\in\mathcal{I}_s}\tilde{S}_{\mathcal{N}}(x)(u,i)h_i(x)=\sum_{i\in\mathcal{I}_s}\langle u,\eta_i(x)\rangle h_i(x),\ u\in J(x)=\operatorname{span}\{\Phi_k(x)\}_{k\in\mathcal{I}_r},$$

is not equal to u for $x\in V$. This is only possible when $\{\eta_i(x)\}_{i\in\mathcal{I}_s}$ is not a dual of the frame $\{h_i(x)\}_{i\in\mathcal{I}_s}$ for $x\in V$. Therefore, $\{\mathfrak{M}_{g_m}h_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is not a dual of the frame $\{\mathfrak{M}_{g_m}\eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ [26, Corollary 5.13], which contradicts (ii) using (5.11) and (5.21). Hence the claim follows. □

5.2.4 Stability of the multi-channel filter bank

We begin with assuming H and Γ to be uniform lattices of $\mathbb{R}^d$ such that $H\subset\Gamma$ and Γ/H is finite, say q. The filter bank in our case is designed to process signals in $\ell^2(\Gamma)$ with the help of low-pass filters $a_k\in\ell^1(\Gamma)$ for $k\in\mathcal{I}_s$. The analysis stage of the filter bank involves the convolution of the input signal with the low-pass filters where the convolution of two sequences $x,y\in\ell^1(\Gamma)$ is defined by

$$(x*y)(n)=\sum_{\gamma\in\Gamma}x(\gamma)y(n-\gamma),\quad n\in\Gamma.$$

The above series converges absolutely for any $n\in\Gamma$. The condition $a_k\in\ell^1(\Gamma)$ for $k\in\mathcal{I}_s$ on low-pass filters guarantees that $x*a_k\in\ell^2(\Gamma)$ and $\|x*a_k\|_2\leq\|x\|_2\|a_k\|_1$. The output from the low-pass filters is downsampled using the *H-fold downsampling* operator $D_H:\ell^2(\Gamma)\to\ell^2(H)$ defined by $D_Hx(\xi)=x(\xi)$, $\xi\in H$.

Fig. 5.2 is the schematic diagram of the analysis phase of a p-channel filter bank, where $\downarrow H$ represent the *H-fold downsampling* operator D_H.

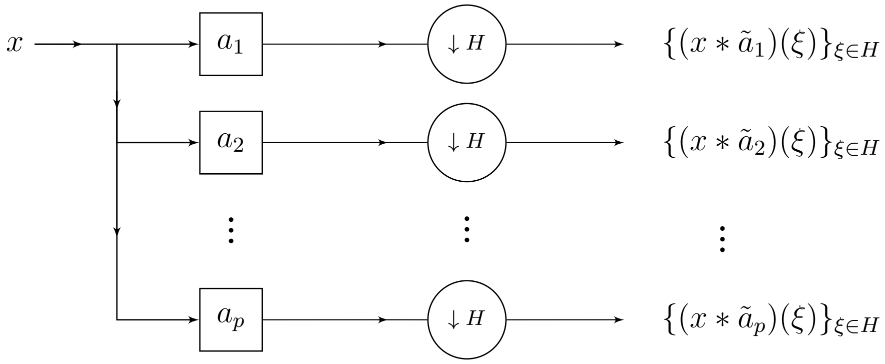


Figure 5.2: Analysis phase of a p-channel filter bank

The stability of filter bank is governed by the *analysis operator* $\mathbb{H} : \ell^2(\Gamma) \to \oplus_{k\in\mathcal{I}_p}\ell^2(H)$ associated with the filter bank and is defined as follows for each $x \in \ell^2(\Gamma)$:

$$\mathbb{H} : x \to \{\mathbb{H}x(\xi, k)\}_{\xi\in H}^{k\in\mathcal{I}_p}, \ \mathbb{H}x(\xi, k) = D_H(x * \tilde{a}_k)(\xi) = x * \tilde{a}_k(\xi), \tag{5.36}$$

where the *involution* $\tilde{a}$ for any sequence $a \in \ell^2(\Gamma)$ is defined by $\tilde{a}(\gamma) = \overline{a(-\gamma)}, \gamma \in \Gamma$. One can notice that,

$$\mathbb{H}x(\xi, k) = \sum_{\gamma\in\Gamma} x(\gamma)\tilde{a}_k(\xi - \gamma) = \sum_{\gamma\in\Gamma} x(\gamma)\overline{a_k(\gamma - \xi)} = \sum_{\gamma\in\Gamma} x(\gamma)\overline{L_\xi a_k(\gamma)} = \langle x, L_\xi a_k\rangle \tag{5.37}$$

where $(L_\gamma a)(n) = a(n-\gamma), a \in \ell^2(\Gamma)$ and $\gamma \in \Gamma$. For $x \in \ell^2(\Gamma)$, the collection $\{\mathbb{H}x(\xi, k)|\xi \in H, k \in \mathcal{I}_p\}$ is the *output of the analysis phase of the filter bank* as shown in Fig. 5.2.

The filter bank given by the low-pass filters $a_1, a_2, \ldots, a_p$ is said to be *stable* if there exist constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{k\in\mathcal{I}_p}\sum_{\xi\in H} |\mathbb{H}x(\xi, k)|^2 \leq B\|x\|^2, \quad \text{for } x \in \ell^2(\Gamma).$$

In view of (5.37) and the above condition, the underlying filter bank is stable if and only if the collection $\{L_\xi a_k\}_{\xi\in H}^{k\in\mathcal{I}_p}$ is a frame for $\ell^2(\Gamma)$.

Connection with a finitely generated Fourier-like system:

Let $\hat{H}$ denote the dual group of H. We assume that the measure on $\hat{H}$ is normalized. Then the collection $\{\chi_\xi \mid \chi_\xi(\omega) = \omega(\xi),\ \omega \in \hat{H}\}_{\xi\in H}$ of characters on $\hat{H}$ forms an orthonormal basis for $L^2(\hat{H})$ [55]. To characterize the stability of the filter bank, we utilize the Zak transform for the setting of discrete abelian groups. It is the map $\mathfrak{Z} : \ell^2(\Gamma) \to L^2(\hat{H}, \mathbb{C}^q)$ defined by

$$(\mathfrak{Z}x)(\eta, \ell) = \sum_{\xi\in H} x(\xi + \ell)e^{-2\pi i\xi\eta}, \quad \text{for } \eta \in \hat{H}, \ell \in \mathbb{C}^q,$$

and satisfies $\langle \mathfrak{Z}x, \mathfrak{Z}y\rangle = \sum_{\ell\in\mathcal{I}_q}\langle \mathfrak{Z}x(., \ell), \mathfrak{Z}y(., \ell)\rangle_{L^2(\hat{H})} = \sum_{\ell\in\mathcal{I}_q}\sum_{\xi\in H} x(\xi + \ell)\overline{y(\xi + \ell)} = \langle x, y\rangle$. Moreover, it satisfies the intertwining relation $\mathfrak{Z}(L_\xi x) = \chi_\xi \mathfrak{Z}x$ for $x \in \ell^2(\Gamma)$ and $\xi \in H$.

Fix a generator $\mathfrak{g} \in L^2(\hat{H}, \mathbb{C}^q)$ such that the system $E_{\mathfrak{g},\mathcal{B}} := \{\chi_\xi\mathfrak{g}\}_{\xi\in H}$ is a Riesz basis for $W_{\mathfrak{g},\mathcal{B}} := \overline{\operatorname{span}}\ E_{\mathfrak{g},\mathcal{B}}$. To set-up the connection, we fix $\mathcal{A} = \{\mathfrak{g}\}, \mathfrak{B} = \mathcal{B}, X = \hat{H}, \mathcal{M} = H$ and $\mathcal{H} = \mathbb{C}^q$.

The subsequent theorem asserts that the stability of a filter with appropriate choice of low-pass filters bank is equivalent to the stability of the sampling set associated to $W_{\mathfrak{g},\mathcal{B}}$.

Theorem 5.16. *Assume* $W_{\mathfrak{g},\mathcal{B}} := \overline{\operatorname{span}}\{e^{-2\pi i\xi\cdot}\mathfrak{g}\}_{\xi\in H} \subset L^2(\hat{H},\mathbb{C}^q)$ *for* $\mathfrak{g} \in L^2(\hat{H},\mathbb{C}^q)$ *and* $\mathcal{B} = \{\chi_\xi\}_{\xi\in H}$. *Let the filter bank be given by the low-pass filters* $a_1, a_2, \cdots, a_p$ *in* $\ell^1(\Gamma)$ *such that*

$$a_i = \mathfrak{Z}^{-1}\eta_i, \quad i \in \mathcal{I}_s.$$

Then the following statements are equivalent:

- (i) *The set* $H \times \mathcal{I}_p$ *is a stable set of sampling for* $W_{\mathfrak{g},\mathcal{B}}$.
- (ii) *The filter bank given by the low-pass filters* $a_1, a_2, \ldots, a_p$ *is stable.*

Proof. The proof is a consequence of the following equality:

$$S_{\mathcal{N}}f(\xi,i) = \langle f, \chi_\xi\eta_i\rangle = \langle x, \mathfrak{Z}^{-1}(\chi_\xi\eta_i)\rangle = \langle x, L_\xi\mathfrak{Z}^{-1}\eta_i\rangle = \langle x, L_\xi a_i\rangle, \quad \xi \in H, i \in \mathcal{I}_p, \tag{5.38}$$

where $f = \mathfrak{Z}x \in W_{\mathfrak{g},\mathcal{B}}$. □

Now we define a $\mathcal{I}_p \times \mathcal{I}_q$ matrix $A(\omega)$ for a.e. $\omega \in \hat{H}$ associated with the low-pass filters $a_1, a_2, \ldots, a_p$ as follows:

$$A(\omega) = \begin{pmatrix} \overline{(\mathfrak{Z}a_1)(\omega,1)} & \overline{(\mathfrak{Z}a_1)(\omega,2)} & \cdots & \overline{(\mathfrak{Z}a_1)(\omega,q)} \\ \overline{(\mathfrak{Z}a_2)(\omega,1)} & \overline{(\mathfrak{Z}a_2)(\omega,2)} & \cdots & \overline{(\mathfrak{Z}a_2)(\omega,q)} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{(\mathfrak{Z}a_p)(\omega,1)} & \overline{(\mathfrak{Z}a_p)(\omega,2)} & \cdots & \overline{(\mathfrak{Z}a_p)(\omega,q)} \end{pmatrix}.$$

The following result is a consequence of Proposition 5.12.

Theorem 5.17. *With the assumptions of Theorem 5.16, the following statements are equivalent:*

- (i) *The filter bank given by the low-pass filters* $a_1, a_2, \ldots, a_p$ *is stable.*
- (ii) *The collection* $\{\chi_\xi T_{\mathfrak{g}}^*\mathfrak{Z}a_i\}_{\xi\in H}^{k\in\mathcal{I}_p}$ *is a frame for* $L^2(\hat{H})$, *where* $T_{\mathfrak{g}}^*$ *is the adjoint of* $T_{\mathfrak{g}} : L^2(\hat{H}) \to L^2(\hat{H},\mathbb{C}^q)$ *mapping the orthonormal basis* $\{\chi_\xi\}_{\xi\in H}$ *to the Riesz basis* $E_{\mathfrak{g},\mathcal{B}}$.
- (iii) $\operatorname{ess-inf}_{\omega\in\hat{H}}(\lambda_{min}[A^*(\omega)A(\omega)]) > 0$.

Proof. (i)$\Longleftrightarrow$(ii) is a consequence of Theorem 5.16 and Proposition 5.12.

(i)$\Longleftrightarrow$(iii). Assume (i) holds. Let the given collection be frame with lower frame bound K. Then by the isometry property of the Zak transform, we get the following equality for any $x \in \ell^2(\Gamma)$:

$$\begin{aligned}\sum_{k\in\mathcal{I}_p}\sum_{\xi\in H}|\langle x, L_\xi a_i\rangle|^2 &= \sum_{k\in\mathcal{I}_p}\sum_{\xi\in H}|\langle \mathfrak{Z}x, \chi_\xi \mathfrak{Z}a_i\rangle|^2 = \sum_{k\in\mathcal{I}_p}\sum_{\xi\in H}|\mathcal{F}(\sum_{\ell\in\mathcal{I}_q}\mathfrak{Z}x(.,\ell)\overline{\mathfrak{Z}a_k(.,\ell)})(\xi)|^2\\ &= \sum_{k\in\mathcal{I}_p}\|\mathcal{F}(\sum_{\ell\in\mathcal{I}_q}\mathfrak{Z}x(.,\ell)\overline{\mathfrak{Z}a_k(.,\ell)})\|^2 = \sum_{k\in\mathcal{I}_p}\|\sum_{\ell\in\mathcal{I}_q}\mathfrak{Z}x(.,\ell)\overline{\mathfrak{Z}a_k(.,\ell)}\|^2_{L^2(\hat{H})}\\ &= \|AX\|^2_{L^2(\Omega,\mathbb{C}^s)},\end{aligned} \tag{5.39}$$

where $X = (\mathfrak{Z}x(.,1), \mathfrak{Z}x(.,2), \ldots, \mathfrak{Z}x(.,q))^T$ is a $q \times 1$ column vector. Finally by (5.39) and the equality $\|X\|^2_{L^2(\hat{H},\mathbb{C}^q)} = \|\mathfrak{Z}x\|^2_{L^2(\hat{H},\mathbb{C}^q)} = \|x\|^2$, we have

$$\begin{aligned}\sum_{k\in\mathcal{I}_p}\sum_{\xi\in H}|\langle x, L_\xi a_i\rangle|^2 = \|AX\|^2 &= \int_{\hat{H}} X^*(\omega)A^*(\omega)A(\omega)X(\omega)d\omega\\ &\geq \operatorname*{ess-inf}_{\omega\in\hat{H}}(\lambda_{min}[A^*(\omega)A(\omega)])\|x\|^2.\end{aligned} \tag{5.40}$$

Now consider $\mathcal{U} := \text{ess-inf}_{\omega\in\Omega}(\lambda_{min}[A^*(\omega)A(\omega)])$. Then similar to the reasoning provided in the proof of Proposition 5.12, it can be shown that $\mathcal{U}$ is the largest constant satisfying (5.40). Now K being the lower frame bound satisfies (5.40), therefore $0 < K \leq \mathcal{U}$. Hence, (iii) holds. Conversely, if (iii) holds, then by (5.40), it is clear that $\mathcal{U}$ is the lower frame bound of the collection $\{L_\xi a_k\}_{\xi\in H}^{k\in\mathcal{I}_p}$. Moreover, by using (5.36) and (5.37), we get

$$\begin{aligned}\sum_{k\in\mathcal{I}_p}\sum_{\xi\in H}|\langle x, L_\xi a_k\rangle|^2 = \sum_{k\in\mathcal{I}_p}\sum_{\xi\in H}|x * \bar{a}_k(\xi)|^2 &\leq \sum_{k\in\mathcal{I}_p}\|x * \bar{a}_k\|^2\\ &\leq \sum_{k\in\mathcal{I}_p}\|x\|^2\|a_k\|_1^2 = B\|x\|^2,\end{aligned} \tag{5.41}$$

where $B = \sum_{k\in\mathcal{I}_p}\|a_k\|_1^2$. Hence (i) holds. This proves the result. $\square$

The following result is an application of Theorem 5.14 for the set up of filter banks.

Theorem 5.18. *Under the assumptions of Theorem 5.16, the following statements are equivalent:*

(i) The filter bank given by the low-pass filters $a_1, a_2, \ldots, a_p$ is stable.

(ii) There exist $d_i \in \ell^2(\Gamma)$ for $i \in \mathcal{I}_p$ such that the sequence $\{L_\xi d_i\}_{\xi\in H}^{i\in\mathcal{I}_p}$ forms a frame for $\ell^2(\Gamma)$, and the sampling formula for any $x \in \ell^2(\Gamma)$ is given by

$$x = \sum_{i\in\mathcal{I}_p}\sum_{\xi\in H}\mathbb{H}x(\xi, i)L_\xi d_i. \tag{5.42}$$

(iii) The set $H \times \mathcal{I}_p$ is a stable sampling set for $W_{\mathfrak{g},\mathcal{B}}$.

(iv) There exist $r_i \in \ell^2(\Gamma)$ for $i \in \mathcal{I}_p$ such that the sequence $\{\chi_\xi \mathfrak{Z} r_i\}_{\xi \in H}^{i \in \mathcal{I}_p}$ forms a frame for $W_{\mathfrak{g},\mathcal{B}}$, and the sampling formula for any $f \in W_{\mathfrak{g},\mathcal{B}}$ is given by

$$f = \sum_{i \in \mathcal{I}_p} \sum_{\xi \in H} S_{\mathcal{N}} f(\xi, i) \chi_\xi \mathfrak{Z} r_i. \tag{5.43}$$

(v) The collection $\{\chi_\xi T_{\mathfrak{g}}^* \mathfrak{Z} a_i\}_{\xi \in H}^{i \in \mathcal{I}_p}$ forms a frame for $L^2(\hat{H})$.

(vi) There exists a $q \times p$ matrix $\Psi(\omega)$ with each column in $L^\infty(\hat{H}, \mathbb{C}^q)$ satisfying

$$\Psi(\omega) A(\omega) = I_q \quad \text{for a.e. } \omega \in \hat{H},$$

where I_q is the $q \times q$ identity matrix.

Proof. (i)$\iff$(iii) follows from Theorem 5.16. (i)$\iff$(v) follows from Theorem 5.17. The proofs of (iv)$\iff$(v), (v) $\implies$ (vi) and (vi) $\implies$ (iv) follow from the reasoning provided in the proof of Theorem 5.14 by taking $X = \hat{H}, r = q$, $\mathcal{A} = \{\mathfrak{g}\}$, $\mathfrak{B} = \mathcal{B}$ and $\mathbb{S}(\omega) = A(\omega)$ for $\omega \in \hat{H}$ and using Theorem 5.17. Finally (ii)$\iff$(iv) follows from the unitary property of the Zak transform and using [42, Corollary 5.3.4]. Hence, the result holds. $\square$

Remark 5.19. Unlike Theorem 5.14, the uniform boundedness condition is not required to establish the above result. This is because, in Theorem 5.14, uniform boundedness was specifically used to prove that $T_{\mathcal{A}}$ satisfies the intertwining relation. However, in the context of filter banks, the operator $\mathcal{T}_{\mathfrak{g}}$ inherently satisfies the intertwining relation $T_{\mathfrak{g}} \chi_\xi = \chi_\xi T_{\mathfrak{g}}$ for $\xi \in H$ by definition, as demonstrated in the following computation:

$$T_{\mathfrak{g}}\Bigg(\sum_{\xi \in H} \alpha(\xi)\, \chi_{\xi+\beta}\Bigg) = T_{\mathfrak{g}}\Bigg(\sum_{\xi \in H} \alpha(\xi - \beta)\, \chi_\xi\Bigg) = \sum_{\xi \in H} \alpha(\xi - \beta)\, \chi_\xi \mathfrak{g} = \sum_{\xi \in H} \alpha(\xi)\, \chi_{\xi+\beta} \mathfrak{g},$$

for $(\alpha(\xi))_{\xi \in H} \in \ell^2(H)$ and $\beta \in H$. Furthermore, the Bessel condition required in the hypothesis of Theorem 5.14 is satisfied by virtue of (5.41).

5.3 Density conditions for stable sampling and interpolation

In this section, we establish the necessary density conditions on the sampling set $\mathcal{M} \times \mathcal{I}_s$ for stable sampling and stable interpolation. Our work is inspired by the contributions of Führ et al. [57], where necessary density conditions for stable sampling and interpolation were derived for reproducing kernel Hilbert spaces. Additionally, we draw motivation from Mitkovski et al. [110], who obtained density conditions for sampling and interpolation by comparing two continuous frames (not necessarily discrete) in a Hilbert space. Utilizing the unifying framework developed in [57], we extend these ideas to derive density conditions for stable sampling and interpolation in $L^2(X,\mathcal{H})$.

We begin with characterizing the stable interpolation condition for $W_{\mathcal{A},\mathfrak{B}}$.

5.3.1 Stable interpolation

A set $\mathcal{M}\times\mathcal{I}_s$ is called a *stable set of interpolation* for $W_{\mathcal{A},\mathfrak{B}}$ if for any sequence $(c_{m,i})_{(m,i)\in\mathcal{M}\times\mathcal{I}_s}\in\ell^2(\mathcal{M}\times\mathcal{I}_s)$, the interpolation problem for the analysis operator $S_{\mathcal{N}}$ (defined in (5.11)),

$$S_{\mathcal{N}}f(m,i)=c_{m,i}$$

has a solution $f\in W_{\mathcal{A},\mathfrak{B}}$.

We characterize the stable interpolation in $W_{\mathcal{A},\mathfrak{B}}$ with respect to the analysis operator $S_{\mathcal{N}}$ in terms of a Riesz basis in $L^2(X,\mathbb{C}^r)$ utilizing the concept of the moment sequences. A *moment sequence* of any $F\in L^2(X,\mathbb{C}^r)$ with respect to a fixed sequence of vectors $\{G_\ell\}_{\ell\in\mathcal{L}}\subset L^2(X,\mathbb{C}^r)$, for any countable set $\mathcal{L}$, is the collection $\{\langle F,G_\ell\rangle\}_{\ell\in\mathcal{L}}$. The collection of all such sequences for all $F\in L^2(X,\mathbb{C}^r)$ is called the *moment space* of $\{G_\ell\}_{\ell\in\mathcal{L}}\subset L^2(X,\mathbb{C}^r)$. Further, the stable interpolation of the set $\mathcal{M}\times\mathcal{I}_s$ is also characterized in terms of local condition involving $\tilde{S}_{\mathcal{N}}(x)$ (defined in Proposition 5.10) for a.e. $x\in X$. Our result is an abstract version of [154, Theorem 9], [13, Proposition 2.7] and [59, Lemma 2] in the measure-theoretic set-up.

Theorem 5.20. *Let $\mathcal{A}=\{\Phi_1,\Phi_2,\dots,\Phi_r\}\subset L^2(X,\mathcal{H})$ be such that $W_{\mathcal{A},\mathfrak{B}}$ has the form (5.4). Further, let $\mathcal{N}=\{\eta_i\}_{i\in\mathcal{I}_s}\subset W_{\mathcal{A},\mathfrak{B}}$ is such that the system $E_{\mathcal{N},\mathfrak{B}}=\{\mathfrak{M}_{g_m}\eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is Bessel and complete in $W_{\mathcal{A},\mathfrak{B}}$, i.e., $\overline{\operatorname{span}}\,E_{\mathcal{N},\mathfrak{B}}=W_{\mathcal{A},\mathfrak{B}}$. Then the following statements are equivalent under the assumptions of Proposition 5.7:*

- *(i) The set $\mathcal{M}\times\mathcal{I}_s$ is a stable set of interpolation for $W_{\mathcal{A},\mathfrak{B}}$.*
- *(ii) The system $\{M_{g_m}T_{\mathcal{A}}^*\eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is a Riesz sequence in $L^2(X,\mathbb{C}^r)$ or equivalently it is a Riesz basis for a closed subspace of $L^2(X,\mathbb{C}^r)$.*
- *(iii) For any sequence $(c_{m,i})_{(m,i)\in\mathcal{M}\times\mathcal{I}_s}\in\ell^2(\mathcal{M}\times\mathcal{I}_s)$, the problem*

$$\mathcal{F}\big[\tilde{S}_{\mathcal{N}}(.)(f(.),i)\big](m)=c_{m,i}$$

has a solution $f\in W_{\mathcal{A},\mathfrak{B}}$, where $\mathcal{F}$ is the Fourier transform defined in (5.13).

Proof. (i)$\Longleftrightarrow$(iii). For each $f \in W_{\mathcal{A},\mathfrak{B}}$ and $i \in \mathcal{I}_s$, the following equation holds using (5.16)

$$S_{\mathcal{N}} f(m,i) = \left(\mathcal{F}^{(s)} \int_X^{\oplus} \tilde{S}_{\mathcal{N}}(x) d\mu(x) f\right)(m,i) = \int_X (\tilde{S}_{\mathcal{N}}(x)(f(x),i)\overline{g_m(x)} d\mu(x). \tag{5.44}$$

Since the mapping $x \mapsto (\tilde{S}_{\mathcal{N}}(x)(f(x),i)) = \langle f(x), \eta_i(x)\rangle$ is in $L^2(X)$ for each $i \in \mathcal{I}_s$, then the equation (5.44) can be written as

$$S_{\mathcal{N}} f(m,i) = \mathcal{F}\big[\tilde{S}_{\mathcal{N}}(.)(f(.),i)\big](m), \quad f \in W_{\mathcal{A},\mathfrak{B}}, \text{ and } i \in \mathcal{I}_s.$$

Consequently, the equivalence holds. (i)$\Longleftrightarrow$(ii). Note that by using (5.20), (i) is equivalent to (i'): The moment space of $\{M_{g_m} T^*_{\mathcal{A}} \eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$ is equal to $\ell^2(\mathcal{M}\times\mathcal{I}_s)$. Then (i')$\Longleftrightarrow$(ii) is a consequence of [154, Theorem 8, Sec. 4]. □

Critically sampled filter banks: The filter bank given by the low-pass filters $a_1, a_2, \ldots, a_s$ in $\ell^1(\Gamma)$ is *critically sampled* if the collection $\{L_\xi a_k\}_{\xi\in H}^{k\in\mathcal{I}_s}$, given in (5.36), governing the output of the analysis phase of the filter bank is a Riesz basis for $\ell^2(\Gamma)$. The following result is an application of Theorem 5.20 for the setting of filter banks. It provides a characterization of critically sampled filter bank given by the low-pass filters $a_1, a_2, \ldots, a_s$ in terms of stable interpolation condition on the set $H \times \mathcal{I}_s$ in $W_{\mathfrak{g},\mathcal{B}} := \overline{\operatorname{span}}\{\chi_\xi \mathfrak{g}\}_{\xi\in H} \subset L^2(\hat{H}, \mathbb{C}^q)$.

Theorem 5.21. *In addition to the standing assumptions of Theorem 5.16, if the collection* $\{L_\xi a_k\}_{\xi\in H}^{k\in\mathcal{I}_s}$ *is complete in* $\ell^2(\Gamma)$*, then the following statements are equivalent:*

(i) The set $H \times \mathcal{I}_s$ *is a stable set of interpolation for* $W_{\mathfrak{g},\mathcal{B}}$.

(ii) The system $\{\chi_\xi T^*_{\mathfrak{g}} \mathfrak{Z} a_i\}_{\xi\in H}^{i\in\mathcal{I}_s}$ *is a Riesz basis for* $L^2(\hat{H})$.

(iii) The filter bank given by the low-pass filters $a_1, a_2, \ldots, a_s$ *is critically sampled.*

Proof. (i)$\Longleftrightarrow$(ii) follows from Theorem 5.20 by taking $X = \hat{H}$, $\mathcal{A} = \{\mathfrak{g}\}$, $\mathfrak{B} = \mathcal{B}$ and using Theorem 5.17. For the proof of (ii)$\Longleftrightarrow$(iii), note that $(T^*_{\mathfrak{g}})^{-1}$ satisfies $(T^*_{\mathfrak{g}})^{-1}\chi_\xi = \chi_\xi (T^*_{\mathfrak{g}})^{-1}$ for $\xi \in H$. Then the bounded invertible operator $U := \mathfrak{Z}^{-1}(T^*_{\mathfrak{g}})^{-1}$ maps the system $\{\chi_\xi T^*_{\mathfrak{g}} \mathfrak{Z} a_i\}_{\xi\in H}^{i\in\mathcal{I}_s}$ to the system $\{L_\xi a_i\}_{\xi\in H}^{i\in\mathcal{I}_s}$ and preserves Riesz basis, proving the equivalence. □

5.3.2 Density conditions for stable sampling and stable interpolation

The results presented in Fuhr et al. [57] establish fundamental density conditions for stable sampling and interpolation in a reproducing kernel Hilbert space $\mathcal{H}$. In their framework, sampling is based on function evaluations at discrete points, i.e., $\{f(x) : x \in \Lambda\}$, where $\mathcal{X}$ is a metric measure space with some geometrical conditions (see [57] for more details) and $\Lambda \subset \mathcal{X}$ is relatively separated.

This way, if $\{K_x : x \in X\}$ is the set of reproducing kernels, the stability of the sampling set is characterized by the system $\{K_\gamma\}_{\gamma\in\Lambda}$ forming a discrete frame for $\mathcal{H}$ and the stability of the interpolation is characterized by the system $\{K_\gamma\}_{\gamma\in\Lambda}$ forming a Riesz sequence for $\mathcal{H}$.

To extend the density conditions established in [57] to the more general sampling model of the current work, we replace the reproducing kernel system $\{K_x\}_{x\in X}$ in Fuhr et al.'s framework with the continuous system

$$\{M_{g_y} T_{\mathcal{A}}^* \eta_i\}_{y\in Y, i\in\mathcal{I}_s}, \tag{5.45}$$

where Y is a metric measure space chosen appropriately such that $\mathcal{M} \subset Y$. The discrete system $\{K_\gamma\}_{\gamma\in\Lambda}$ is then replaced by the corresponding discrete system

$$\{M_{g_m} T_{\mathcal{A}}^* \eta_i\}_{m\in\mathcal{M}, i\in\mathcal{I}_s} \tag{5.46}$$

involved in our sample expression (5.20) This adaptation requires a careful analysis of localization and approximation properties to ensure that the MI system satisfies analogous frame conditions to those considered in Führ et al. To this end, we begin with the following assumptions:

Assumptions: In addition to the standing assumptions of the previous section,

(i) We assume that (Y, d, ν) is a metric measure space with Borel measure ν and metric d. The metric d is chosen in such a way that for all $r > 0$ and $a \in Y$, $\nu(B) < \infty$, where $B := B(a, r)$, an open ball centered at a with radius r. We fix this notation unless some other center or radius is specified. In addition to this, we assume that there exists $r > 0$ such that $\inf_{a\in Y} \nu(B) > 0$ and for every $t_1, t_2 \geq 0$,

$$\lim_{r\to\infty} \sup_{y\in Y} \frac{\nu(B(y, r + t_1)\backslash B(y, r - t_2))}{\nu(B(y, r))} = 0. \tag{5.47}$$

The condition (5.47) is known as the *annular decay property* of the measure ν.

(ii) We also assume that the set $\mathcal{M} \subset Y$ is uniformly discrete i.e., there exists $\rho > 0$ such that $d(t, m) > \rho$ for every $t, m \in \mathcal{M}$ with $t \neq m \in \mathcal{M}$ and counting measure $\#_{\mathcal{M}}$, where $\#_{\mathcal{M}}(S) = \#(S \cap \mathcal{M})$ for every Borel set $S \subset Y$.

(*iii*) We fix a continuous family $\{g_y\}_{y\in Y} \subset L^\infty(X)$ such that its sub-family $\{g_m\}_{m\in\mathcal{M}}$ corresponds to $\mathfrak{B}$.

We denote $D^-(\mathcal{M})$ and $D^+(\mathcal{M})$ as the lower density and upper density of the set $\mathcal{M} \subset Y$ with respect to ν and are defined by

$$D^-(\mathcal{M}) = \liminf_{r\to\infty} \inf_{y\in Y} \frac{\#_{\mathcal{M}}(B(y,r))}{\nu(B(y,r))}, \quad D^+(\mathcal{M}) = \limsup_{r\to\infty} \sup_{y\in Y} \frac{\#_{\mathcal{M}}(B(y,r))}{\nu(B(y,r))}.$$

An essential requirement for establishing density conditions on $\mathcal{M}$ is that the system $\mathcal{Q} := \{M_{g_y} T^*_{\mathcal{A}} \eta_i\}_{y\in Y}^{i\in\mathcal{I}_s}$ forms a *continuous frame* for $L^2(X,\mathbb{C}^r)$, i.e., the map $(y,i) \mapsto \langle f, M_{g_y} T^*_{\mathcal{A}} \eta_i\rangle$ is a measurable function on $Y \times \mathcal{I}_s$ and there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{i\in\mathcal{I}_s} \int_Y |\langle f, M_{g_y} T^*_{\mathcal{A}} \eta_i\rangle|^2 d\nu(y) \le B\|f\|^2 \text{ for all } f \in L^2(X,\mathbb{R}).$$

The continuous frame $\{M_{g_y} T^*_{\mathcal{A}} \eta_i\}_{y\in Y}^{i\in\mathcal{I}_s}$ is *Parseval continuous frame* if $A = B = 1$ and it is called *normalized continuous frame* if $\|M_{g_y} T^*_{\mathcal{A}} \eta_i\| = 1$ for $y \in Y, i \in \mathcal{I}_s$.

Associated with $\mathcal{Q}$, we define the quantities:

$$\mathcal{T}^-(\mathcal{Q}) := \liminf_{r\to\infty} \inf_{a\in Y} \frac{1}{\nu(B(a,r))} \left| \frac{1}{s} \sum_{i\in\mathcal{I}_s} \int_{B(a,r)} \langle M_{g_y} T^*_{\mathcal{A}} \eta_i, \widetilde{M_{g_y} T^*_{\mathcal{A}}} \eta_i\rangle d\nu(y) \right|,$$

$$\mathcal{T}^+(\mathcal{Q}) := \limsup_{r\to\infty} \sup_{a\in Y} \frac{1}{\nu(B(a,r))} \left| \frac{1}{s} \sum_{i\in\mathcal{I}_s} \int_{B(a,r)} \langle M_{g_y} T^*_{\mathcal{A}} \eta_i, \widetilde{M_{g_y} T^*_{\mathcal{A}}} \eta_i\rangle d\nu(y) \right|,$$

where $\widetilde{Q} := \{\widetilde{M_{g_y} T^*_{\mathcal{A}}} \eta_i\}_{y\in Y}^{i\in\mathcal{I}_s}$ denotes the canonical dual of the frame $\mathcal{Q}$.

For the remainder of this section, we use the notation $\mathcal{Q} = \{M_{g_y} T^*_{\mathcal{A}} \eta_i\}_{y\in Y, i\in\mathcal{I}_s}$ for the continuous system and $\mathcal{E} = \{M_{g_m} T^*_{\mathcal{A}} \eta_i\}_{m\in\mathcal{M}, i\in\mathcal{I}_s}$ for the discrete system. Additionally, if $\mathcal{E}$ forms a frame for $L^2(X,\mathbb{C}^r)$, we denote its canonical dual as $\widetilde{\mathcal{E}} = \{\widetilde{M_{g_m} T^*_{\mathcal{A}}} \eta_i\}_{m\in\mathcal{M}, i\in\mathcal{I}_s}$.

The following main results of this subsection establish necessary density conditions on $\mathcal{M}$ for stable sampling and interpolation in $W_{\mathcal{A},\mathfrak{B}}$.

Theorem 5.22. *In addition to the standing assumptions of Theorem 5.14, assume that the system $\mathcal{Q}$ forms a continuous frame for $L^2(X,\mathbb{C}^r)$. Moreover, suppose that for every $\delta > 0$, there exists $R > 0$ such that for all $i \in \mathcal{I}_s$, the following conditions hold:*

(i) (Weak localization property)

$$\sup_{x\in Y} \sum_{j\in\mathcal{I}_s} \int_{Y\setminus B(x,R)} |\langle M_{g_x} T^*_{\mathcal{A}} \eta_i, M_{g_y} T^*_{\mathcal{A}} \eta_j\rangle|^2 d\nu(y) < \delta^2,$$

(ii) (Homogeneous approximation property)

$$\sup_{y\in Y}\sum_{j\in\mathcal{I}_s}\sum_{m\in\mathcal{M}\cap Y\setminus B(y,R)}|\langle M_{g_y}T^*_{\mathcal{A}}\eta_i, M_{g_m}T^*_{\mathcal{A}}\eta_j\rangle|^2<\delta^2.$$

Then the following results hold:

1. *If $\mathcal{M}\times\mathcal{I}_s$ is a stable set of sampling for $W_{\mathcal{A},\mathfrak{B}}$, then*

$$D^-(\mathcal{M})\geq\mathcal{T}^-(\mathcal{Q}),\ \ \textit{and}\ D^+(\mathcal{M})\geq\mathcal{T}^+(\mathcal{Q}).$$

2. *If $\mathcal{M}\times\mathcal{I}_s$ is a stable set of interpolation for $W_{\mathcal{A},\mathfrak{B}}$, then*

$$D^-(\mathcal{M})\leq\mathcal{T}^-(\mathcal{Q}),\ \ \textit{and}\ D^+(\mathcal{M})\leq\mathcal{T}^+(\mathcal{Q}).$$

Theorem 5.23. *In addition to the standing assumptions of Theorem 5.22, if $\mathcal{Q}$ is a normalized continuous Parseval frame for $L^2(X,\mathbb{C}^r)$, then the following results hold:*

1. *If $\mathcal{M}\times\mathcal{I}_s$ is a stable set of sampling for $W_{\mathcal{A},\mathfrak{B}}$, then $D^-(\mathcal{M})\geq 1$.*
2. *If $\mathcal{M}\times\mathcal{I}_s$ is a stable set of interpolation for $W_{\mathcal{A},\mathfrak{B}}$, then $D^+(\mathcal{M})\leq 1$.*

Before proceeding with the proofs of the theorems, we introduce a key result inspired by Mitkovski and Ramirez's work [110, Theorem 3.2]. Their result provides a density comparison of the measures m_1 and m_2 associated with continuous frames $\{f_x\}_{x\in\mathcal{U}}\subset\mathcal{F}$ and $\{g_x\}_{x\in\mathcal{U}}\subset\mathcal{G}$, with indexing spaces $(\mathcal{U},r,m_1)$ and $(\mathcal{U},r,m_2)$, respectively, where $\mathcal{F}$ and $\mathcal{G}$ are two closed subspaces of a Hilbert space $\mathcal{H}$, r is the metric on $\mathcal{U}$, and m_1 and m_2 are Borel measures on $\mathcal{U}$.

Our goal is to apply this framework to compare a continuous frame, $\mathcal{Q}$, with its discrete counterpart, $\mathcal{E}$. We now present the following result:

Theorem 5.24. *Let $\mathcal{Q}$ be a continuous frame for $L^2(X,\mathbb{C}^r)$ and let $\mathcal{E}$ be a frame for $L^2(X,\mathbb{C}^r)$. Further, suppose that for any $\delta>0$, there exists $R>0$ such that for all $r\geq R$ and $B=B(a,r)$ with $a\in Y$, we have*

$$\Bigg|\sum_{i\in\mathcal{I}_s}\sum_{j\in\mathcal{I}_s}\sum_{m\in\mathcal{M}\cap(Y\setminus B)}\int_B\langle M_{g_m}T^*_{\mathcal{A}}\eta_i, M_{g_y}T^*_{\mathcal{A}}\eta_i\rangle\langle\widetilde{M_{g_y}T^*_{\mathcal{A}}}\eta_i,\widetilde{M_{g_m}T^*_{\mathcal{A}}}\eta_i\rangle d\nu(y)$$
$$-\sum_{i\in\mathcal{I}_s}\sum_{j\in\mathcal{I}_s}\int_{Y\setminus B}\sum_{m\in\mathcal{M}\cap B}\langle M_{g_m}T^*_{\mathcal{A}}\eta_i, M_{g_y}T^*_{\mathcal{A}}\eta_i\rangle\langle\widetilde{M_{g_y}T^*_{\mathcal{A}}}\eta_i,\widetilde{M_{g_m}T^*_{\mathcal{A}}}\eta_i\rangle d\nu(y)\Bigg|<\delta\big(\nu(B)+\#_{\mathcal{M}}(B)\big), \tag{5.48}$$

then the following conditions hold:

(i) $D^-(\mathcal{M})\geq\mathcal{T}^-(\mathcal{Q})$, *and* $D^+(\mathcal{M})\geq\mathcal{T}^+(\mathcal{Q})$.

(*ii*) *Moreover, if* $\langle M_{g_m} T^*_{\mathcal{A}} \eta_i, \widetilde{M_{g_m} T^*_{\mathcal{A}} \eta_i} \rangle = 1$ *for all* $m \in \mathcal{M}$ *and* $i \in \mathcal{I}_s$*, then*

$$D^-(\mathcal{M}) \leq \mathcal{T}^-(\mathcal{Q}), \quad \textit{and} \quad D^+(\mathcal{M}) \leq \mathcal{T}^+(\mathcal{Q}).$$

Proof. Note that by using [57, Lemma 3.9], we have $|\langle M_{g_m} T^*_{\mathcal{A}} \eta_i, \widetilde{M_{g_m} T^*_{\mathcal{A}} \eta_i} \rangle| \leq 1$. If $\#_{\mathcal{I}_s}$ denotes the counting measure on $\mathcal{I}$, then by setting one of the appropriate continuous frame to be $\mathcal{Q}$ with indexing space $(Y \times \mathcal{I}_s, d, \nu \times \#_{\mathcal{I}_s})$ and other continuous frame to be $\mathcal{E}$ with indexing space $(Y \times \mathcal{I}_s, d, \#_{\mathcal{M}} \times \#_{\mathcal{I}_s})$ in [110, Theorem 3.2(i)], (i) holds.

Similarly, (ii) is an application of [110, Theorem 3.2(ii)]. □

From the above theorem, it is certain that the condition (5.48) is the key condition to obtain the required density conditions. Now we provide a result which gives the conditions under which (5.48) holds true. For this purpose, we define

($\mathcal{P}_1$): For all $j \in \mathcal{I}_s$,

$$\lim_{r\to\infty} \sup_{y \in Y} \sum_{i \in \mathcal{I}_s} \sum_{m \in \mathcal{M} \cap Y \setminus B(y,r)} |\langle M_{g_y} T^*_{\mathcal{A}} \eta_j, M_{g_m} T^*_{\mathcal{A}} \eta_i \rangle|^2 = 0, \text{ and}$$

$$\lim_{r\to\infty} \sup_{m \in \mathcal{M}} \sum_{i \in \mathcal{I}_s} \int_{Y \setminus B(m,r)} |\langle M_{g_y} T^*_{\mathcal{A}} \eta_i, M_{g_m} T^*_{\mathcal{A}} \eta_j \rangle|^2 d\nu(y) = 0.$$

($\mathcal{P}_2$): For every $\omega > 0$, $\sup_{a \in Y} \#_{\mathcal{M}}\big(B(a, r+\omega) \setminus B(a,r)\big) = O\big(\nu(B(a,r))\big)$, as $r \to \infty$.

The first condition of $\mathcal{P}_1$ is homogeneous approximation property and second condition of $\mathcal{P}_1$ is a modified version of weak localization property. Further, the condition $\mathcal{P}_2$ is a slight modification of the annular decay property of the measure $\#_{\mathcal{M}}$ [57, 110].

The following proposition provides the necessary conditions for the validity of (5.48), adapting [110, Theorem 3.7] to our setting. For brevity, we omit its proof.

Proposition 5.25. *Let* $\mathcal{Q}$ *be a continuous frame for* $L^2(X, \mathbb{C}^r)$*, and let* $\mathcal{E}$ *be a frame for* $L^2(X, \mathbb{C}^r)$*. Suppose there exists a constant* $L > 0$ *such that* $\|M_{g_y} T^*_{\mathcal{A}} \eta_i\| \leq L$ *for a.e.* $y \in Y$ *and all* $i \in \mathcal{I}_s$*. Furthermore, if the conditions* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *hold, then* (5.48) *is satisfied.*

Now we are ready to provide the proof of Theorem 5.22. The idea of its proof is adopted from [110, Theorem 4.1].

Proof of Theorem 5.22. To apply Theorem 5.24 in proving the result, we first verify that (5.48) holds by utilizing Proposition 5.25. The first equality in $\mathcal{P}_1$ follows from the homogeneous approximation property, while the second equality follows from the weak localization property. It remains to establish that $\mathcal{P}_2$ holds.

Choose $\delta = 1$. Then by applying the first condition of the hypothesis, there exists $R > 0$ such that for all $x \in Y$ and $i \in \mathcal{I}_s$, we have

$$\sum_{j\in\mathcal{I}_s} \int_{Y\setminus B(x,R)} |\langle M_{g_x} T_{\mathcal{A}}^* \eta_i, M_{g_y} T_{\mathcal{A}}^* \eta_j\rangle|^2 d\nu(y) < 1.$$

Therefore,

$$1 \le \sum_{j\in\mathcal{I}_s} \int_{B(x,R)} |\langle M_{g_x} T_{\mathcal{A}}^* \eta_i, M_{g_y} T_{\mathcal{A}}^* \eta_j\rangle|^2 d\nu(y)$$

for all $x \in Y$ and $i \in \mathcal{I}_s$. This implies for any $\omega > 0, r > R$ and $i \in \mathcal{I}_s$, we have

$$\mathcal{M} \cap \big(B(a, r+\omega)\setminus B(a,r)\big) \le \sum_{j\in\mathcal{I}_s} \sum_{m\in\mathcal{M}\cap B(a,r+\omega)\setminus B(a,r)} \int_{B(m,R)} |\langle M_{g_m} T_{\mathcal{A}}^* \eta_i, M_{g_y} T_{\mathcal{A}}^* \eta_j\rangle|^2 d\nu(y). \tag{5.49}$$

Since $B(m,R) \subset B(a, r+\omega+R)\setminus B(a, r-R)$ for $r > R, \omega > 0$ and $m \in B(a, r+\omega)\setminus B(a,r)$, therefore,

$$\begin{aligned}
&\sum_{j\in\mathcal{I}_s} \sum_{m\in\mathcal{M}\cap B(a,r+\omega)\setminus B(a,r)} \int_{B(m,R)} |\langle M_{g_m} T_{\mathcal{A}}^* \eta_i, M_{g_y} T_{\mathcal{A}}^* \eta_j\rangle|^2 d\nu(y) \\
&\quad \le \sum_{j\in\mathcal{I}_s} \sum_{i\in\mathcal{I}_s} \sum_{m\in\mathcal{M}} \int_{B(a,r+\omega+R)\setminus B(a,r-R)} |\langle M_{g_m} T_{\mathcal{A}}^* \eta_i, M_{g_y} T_{\mathcal{A}}^* \eta_j\rangle|^2 d\nu(y) \qquad (5.50) \\
&\quad \le s\, C\, L \nu\big(B(a, r+\omega+R)\setminus B(a, r-R)\big) \le s\, C\, L\nu(B(a,r)),
\end{aligned}$$

using the annular decay property of ν in the last inequality, where C is the Bessel bound of $\{M_{g_m} T_{\mathcal{A}}^* \eta_i\}_{m\in\mathcal{M}}^{i\in\mathcal{I}_s}$. Combining (5.49) and (5.50), $\mathcal{P}_2$ is satisfied and hence (5.48) holds true using Proposition 5.25. Now we are ready to use Theorem 5.24.

If $\mathcal{M} \times \mathcal{I}_s$ is a stable set of sampling for $W_{\mathcal{A},\mathfrak{B}}$, then by Theorem 5.14, the system $\mathcal{E}$ is a frame for $L^2(X, \mathbb{C}^r)$. Consequently, it follows from [57, Lemma 3.9] that

$$|\langle M_{g_m} T_{\mathcal{A}}^* \eta_i, \widetilde{M_{g_m} T_{\mathcal{A}}^* \eta_i}\rangle| \le 1, \quad \forall m \in \mathcal{M}, i \in \mathcal{I}_s.$$

Thus, (1) holds by using Proposition 5.24.

Similarly, if $\mathcal{M} \times \mathcal{I}_s$ is a stable set of interpolation for $W_{\mathcal{A},\mathfrak{B}}$, then by Theorem 5.20, the system $\{M_{g_m} T_{\mathcal{A}}^* \eta_i\}_{m\in\mathcal{M}, i\in\mathcal{I}_s}$ is a Riesz sequence in $L^2(X, \mathbb{C}^r)$. In this case, it follows from [57, Lemma 3.9]

$$\langle M_{g_m} T_{\mathcal{A}}^* \eta_i, \widetilde{M_{g_m} T_{\mathcal{A}}^* \eta_i}\rangle = 1, \quad \forall m \in \mathcal{M}, i \in \mathcal{I}_s.$$

Thus, (2) follows by Proposition 5.24, completing the proof. □

Proof of Theorem 5.23. From Theorem 5.22, we already have

$$D^-(\mathcal{M}) \geq \mathcal{T}^-(\mathcal{Q}), \quad \text{and} \quad D^+(\mathcal{M}) \leq \mathcal{T}^+(\mathcal{Q}),$$

under the assumptions that $\mathcal{M} \times \mathcal{I}_s$ is a stable set of sampling for $W_{\mathcal{A},\mathfrak{B}}$ and $\mathcal{M} \times \mathcal{I}_s$ is a stable set of interpolation for $W_{\mathcal{A},\mathfrak{B}}$, respectively.

If $\mathcal{Q}$ is a normalized continuous Parseval frame, then its canonical dual coincides with $\mathcal{Q}$ itself. Consequently, we obtain

$$\langle M_{g_y} T_{\mathcal{A}}^* \eta_i, \widetilde{M_{g_y} T_{\mathcal{A}}^* \eta_i} \rangle = \langle M_{g_y} T_{\mathcal{A}}^* \eta_i, M_{g_y} T_{\mathcal{A}}^* \eta_i \rangle = \|M_{g_y} T_{\mathcal{A}}^* \eta_i\|^2 = 1.$$

Thus, we conclude that $\mathcal{T}^-(\mathcal{Q}) = \mathcal{T}^+(\mathcal{Q}) = 1$. This completes the proof. □

5.3.3 Applications of density results

In this subsection, we discuss applications of our density results.

5.3.3.1 Paley-Wiener Spaces

For a measurable set $\Omega \subset \mathbb{R}^d$ with finite Lebesgue measure μ, the *Paley-Wiener* space

$$PW_\Omega = \{f \in L^2(\mathbb{R}^d) \mid \operatorname{supp} \hat{f} \subseteq \Omega\}$$

consists of band-limited functions whose spectrum is contained in Ω.

The density result established in Theorem 5.22 provides the necessary density conditions for the stability of sampling and interpolation sets in the space

$$\widehat{PW}_\Omega := \{\hat{f} \mid f \in PW_\Omega\} \subset L^2(\Omega),$$

where $\hat{f}$ denotes the Fourier transform of f on $\mathbb{R}^d$.

Let $\Lambda \subset \mathbb{R}^d$ be a discrete subgroup such that the set $\{e^{-2\pi i\lambda\cdot}\}_{\lambda\in\Lambda}$ forms an orthonormal basis for $L^2(\Omega)$. We set $X = \Omega$, $\mathcal{H} = \mathbb{C}$, and $\mathfrak{B} = \{e^{-2\pi i\lambda\cdot}\}_{\lambda\in\Lambda}$. Additionally, we take $Y = \mathbb{R}^d$ equipped with the Lebesgue measure ν, and choose $s = 1$ along with η_1 such that $T_{\mathcal{A}}^*\eta_1 = 1_\Omega$, where $T_{\mathcal{A}} : L^2(\Omega) \to \widehat{PW}_\Omega$ maps the orthonormal basis $\mathfrak{B}$ onto a Riesz basis for $\widehat{PW}_\Omega$.

Let $f \in PW_\Omega$. The samples in this setting involves the collection:

$$\{\langle \hat{f}, e^{2\pi i\lambda\cdot} 1_\Omega \rangle : \lambda \in \Lambda\} = \{f(\lambda) : \lambda \in \Lambda\}. \tag{5.51}$$

It is straightforward to verify that the collection $\mathcal{Q} := \{e^{-2\pi i\xi\cdot} 1_\Omega\}_{\xi\in\mathbb{R}^d}$ forms a continuous Parseval frame for $L^2(\Omega)$. The quantities $\mathcal{T}^-(\mathcal{Q})$ and $\mathcal{T}^+(\mathcal{Q})$ both equal $\mu(\Omega)$.

Specifically, we have

$$\begin{aligned}\mathcal{T}^-(\mathcal{Q}) &= \liminf_{r\to\infty} \inf_{a\in\mathbb{R}^d} \frac{1}{\nu(B(a,r))}\left| \int_{B(a,r)} \langle e^{-2\pi i\xi\cdot}, e^{-2\pi i\xi\cdot}\rangle_{L^2(\Omega)} d\nu(\xi)\right| \\ &= \liminf_{r\to\infty} \inf_{a\in X} \frac{1}{\mu(B(a,r))}\left| \int_{B(a,r)} \mu(\Omega)\ d\mu(x)\right| = \mu(\Omega).\end{aligned}$$

Similarly, we obtain $\mathcal{T}^+(\mathcal{Q}) = \mu(\Omega)$.

Since $\mathbb{R}^d$ with the Euclidean metric and Lebesgue measure satisfies the assumptions of Subsection 5.3.2, we verify the weak localization and homogeneous approximation properties for this case. The central quantity of the weak localization property is given by

$$\langle e^{-2\pi i y\cdot}, e^{-2\pi i x\cdot}\rangle_{L^2(\Omega)} = \widehat{1_\Omega}(y-x) = K(x,y),$$

where $K(x,y)$ denotes the reproducing kernel of the Paley-Wiener space PW_Ω. Then the verification of these properties can be adopted from [57, Subsection 5.1].

Consequently, we are in a position to apply Theorem 5.22. This yields the following density conditions:

- If Λ is a stable sampling set for $\widehat{PW}_\Omega$, then $D^-(\Lambda) \geq \mu(\Omega)$.
- If Λ is a stable interpolation set for $\widehat{PW}_\Omega$, then $D^+(\Lambda) \leq \mu(\Omega)$.

Thus, considering (5.51) along with the aforementioned density conditions, Theorem 5.22 yields Landau's fundamental density theorem for band-limited functions.

5.3.3.2 For multi-channel filter banks.

For the setting of the fiter bank considered in Subsection 5.2.4, Theorem 5.22 establishes necessary density conditions on H that ensure the stability or critical sampling of a multi-channel filter bank composed of low-pass filters $a_1, a_2, \ldots, a_s$ in $\ell^1(\Gamma)$.

Specifically, by setting $Y = \mathbb{R}^d$ with Lebesgue measure and Euclidean metric, $X = \hat{H}$, $\mathcal{A} = \{\mathfrak{g}\}$, $\mathfrak{B} = \mathcal{B}$, $\mathcal{M} = H$ and defining $\mathcal{Q} = \{e^{-2\pi i\xi\cdot} T_{\mathfrak{g}}^* \mathfrak{Z} a_i\}_{\xi\in\mathbb{R}^d}^{i\in\mathcal{I}_s}$, Theorem 5.22, along with Theorem 5.18 and Theorem 5.21, provides the following conditions:

(i) The filter bank given by the low-pass filters $a_1, a_2, \ldots, a_s$ is stable, then

$$D^+(H) \leq \mathcal{T}^+(\mathcal{Q}).$$

(ii) The filter bank given by the low-pass filters $a_1, a_2, \ldots, a_s$ is critically sampled, then

$$D^-(H) \geq \mathcal{T}^-(\mathcal{Q}).$$

5.4 Applications for shift-invariant spaces

In this section, we provide an interpretation of the measure-theoretic results from the previous sections, emphasizing their application to SI spaces. To achieve this, we utilize the Zak transform introduced in [88], which extends the classical Zak transform on $\mathbb{R}$ in a non-trivial way and converts translations into multiplications. This transformation allows us to translate our earlier findings into the setting of SI spaces invariant under translations by an abelian subgroup. We begin by outlining a few key assumptions.

5.4.1 Stable sampling associated to translation-invariant systems

Throughout the remainder of this section, we assume that $\mathscr{G}$ is a second-countable, locally compact group (not necessarily abelian) and that H is a discrete abelian subgroup of $\mathscr{G}$. As usual, we denote the Pontryagin dual group of H by $\hat{H}$. Let ρ and σ represent the Haar measures on $\mathscr{G}$ and $\hat{H}$, respectively. Our objective is to interpret the results from the previous sections within the specific framework: $X = \hat{H}$, $\mathcal{H} = L^2(H\backslash\mathscr{G})$, $\mu = \sigma$, $Y = \mathscr{G}$, $\nu = \rho$, $\mathcal{M} = H$, and $\mathfrak{B} = \{\chi_\xi\}_{\xi\in H}$, where $\chi_\xi : \hat{H} \to \mathbb{T}$ is defined by $\chi_\xi(\gamma) = \gamma(\xi)$, $\gamma \in \hat{H}$. Furthermore, assuming that the Haar measure on $\hat{H}$ is normalized, Pontryagin duality implies that $\mathfrak{B}$ forms a set of characters on $\hat{H}$ and thus constitutes an orthonormal basis for $L^2(\hat{H})$ [55].

Let us consider $\mathscr{A} = \{\phi_1, \phi_2, \dots, \phi_r\} \subset L^2(\mathscr{G})$ such that the TI system

$$\mathcal{E}^H(\mathscr{A}) := \{L_\xi \phi_k : \xi \in H,\ k \in \mathcal{I}_r\}$$

is a Riesz basis for $\mathcal{S}^H(\mathscr{A})$. Then $\mathcal{S}^H(\mathscr{A})$ has the representation of the form:

$$\mathcal{S}^H(\mathscr{A}) = \left\{ \sum_{k\in\mathcal{I}_r} \sum_{\xi\in H} \alpha_k(\xi) L_\xi \phi_k \,\middle|\, (\alpha_k(\xi))_{\xi\in H} \in \ell^2(H), \phi_k \in \mathscr{A}, k \in \mathcal{I}_r \right\}, \tag{5.52}$$

In this setup, the range function corresponding to the H-TI space $\mathcal{S}^H(\mathscr{A})$ in $L^2(\mathscr{G})$ is the map $J_{\mathscr{A}} : \widehat{H} \to \{\text{closed subspaces of } L^2(H\backslash\mathscr{G})\}$, where for a.e. $\alpha \in \widehat{H}$, $J_{\mathscr{A}}(\alpha)$ is defined by

$$J_{\mathscr{A}}(\alpha) = \overline{\operatorname{span}}\{(\mathcal{Z}\phi_i)(\alpha) : k \in \mathcal{I}_r\} \tag{5.53}$$

(see [88]).

By applying the Zak transform $\mathcal{Z}$ (see (5.1)), the TI space $\mathcal{S}^H(\mathscr{A})$ can be expressed as an MI space of the form (5.4), defined by

$$W_{\mathcal{Z}\mathscr{A},\mathfrak{B}} := \mathcal{Z}\mathcal{S}^H(\mathscr{A}) = \overline{\operatorname{span}}\{\mathfrak{M}_{\chi_\xi}\mathcal{Z}\phi_k\}_{\xi\in H}^{k\in\mathcal{I}_r}.$$

Let $\mathcal{N} = \{d_i\}_{i\in\mathcal{I}_s} \subset \mathcal{S}^H(\mathscr{A})$ be such that $\mathcal{E}_{\mathcal{N}} := \{L_\xi d_i\}_{i\in\mathcal{I}_s}$ is Bessel. Then the analysis operator associated with $\mathcal{E}_{\mathcal{N}}$ is the map $\mathfrak{S}_{\mathcal{N}} : \mathcal{S}^H(\mathscr{A}) \longrightarrow \ell^2(H\times\mathcal{I}_s)$ defined as

$$\mathfrak{S}_{\mathcal{N}} f(\xi, i) = \langle f, L_\xi d_i\rangle_{L^2(G)}.$$

The set $H\times\mathcal{I}_s$ is said to be *stable set of sampling* for $\mathcal{S}^H(\mathscr{A})$ if the collection $\mathcal{E}_{\mathcal{N}}$ is a frame for $\mathcal{S}^H(\mathscr{A})$.

Proposition 5.26. *Let the collection $\mathcal{E}_{\mathcal{N}}$ be Bessel. Then the following statements are equivalent:*

- (*i*) *The set $H\times\mathcal{I}_s$ is a stable set of sampling for $\mathcal{S}^H(\mathscr{A})$.*
- (*ii*) *The set $H\times\mathcal{I}_s$ is a stable set of sampling for $W_{\mathcal{Z}\mathscr{A},\mathfrak{B}}$.*

Proof. Observe that the system $E_{\mathcal{Z}\mathcal{N},\mathfrak{B}} := \{\mathfrak{M}_{\chi_\xi}\mathcal{Z}d_i\}_{\xi\in H}^{i\in\mathcal{I}_s}$ is a Bessel system due to the Bessel property of $\mathcal{E}_{\mathcal{N}}$ and the unitary property of $\mathcal{Z}$. The following equality:

$$\mathfrak{S}_{\mathcal{N}} f(\xi, i) = \langle f, L_\xi d_i\rangle = \langle \mathcal{Z}f, \mathfrak{M}_{\chi_\xi}\mathcal{Z}d_i\rangle = S_{\mathcal{Z}\mathcal{N}}(\mathcal{Z}f)(\xi, i), \quad \xi\in H,\, i\in\mathcal{I}_s, \tag{5.54}$$

implies that the samples of $f\in\mathcal{S}^H(\mathscr{A})$ with respect to the Bessel system $\mathcal{E}_{\mathcal{N}}$ are identical to the samples of $\mathcal{Z}f\in W_{\mathcal{Z}\mathscr{A},\mathfrak{B}}$ with respect to the Bessel system $E_{\mathcal{Z}\mathcal{N},\mathfrak{B}}$, where $\mathcal{Z}\mathcal{N} = \{\mathcal{Z}d_1, \mathcal{Z}d_2, \ldots, \mathcal{Z}d_s\}$. Thus (5.54) establishes that the frame property of $\mathcal{E}_{\mathcal{N}}$ and $E_{\mathcal{Z}\mathcal{N},\mathfrak{B}}$ are mutually implied, proving the claim. □

For local analysis, we define the pointwise operator $\tilde{\mathfrak{S}}_{\mathcal{Z}\mathcal{N}}(\eta) : J_{\mathscr{A}}(\eta) \to \mathbb{C}^s$ for a.e. $\eta\in\hat{H}$, associated with $\mathcal{Z}\mathcal{N}$ and is given by

$$\tilde{\mathfrak{S}}_{\mathcal{Z}\mathcal{N}}(\eta)(u) = \big(\tilde{\mathfrak{S}}_{\mathcal{Z}\mathcal{N}}(\eta)(u,1), \tilde{\mathfrak{S}}_{\mathcal{Z}\mathcal{N}}(\eta)(u,2), \ldots, \tilde{\mathfrak{S}}_{\mathcal{Z}\mathcal{N}}(\eta)(u,s)\big),$$

for $u\in J_{\mathscr{A}}(\eta) = \operatorname{span}\{\mathcal{Z}\varphi_k(\eta) \mid k\in\mathcal{I}_r\}$. The components of the operator are defined as

$$\tilde{\mathfrak{S}}_{\mathcal{Z}\mathcal{N}}(\eta)(u,i) = \langle u, \mathcal{Z}d_i(\eta)\rangle, \quad \text{for a.e. } \eta\in\hat{H},\, i\in\mathcal{I}_s.$$

The following theorem is an application of Theorem 5.11 and can be proved using it by taking $X = \hat{H}$, $\mathcal{H} = L^2(H\backslash\mathscr{G}), \mathfrak{B} = \{\chi_\xi\}_{\xi\in H}$ and utilizing Proposition 5.26. It generalizes the results of [10, Theorem 4.2], [129, Theorem 2.1] and [9, Proposition 1] for the TI subpaces of $L^2(\mathscr{G})$ in the group setting:

Theorem 5.27. *Let* $\mathscr{A} = \{\phi_1, \phi_2, \dots, \phi_r\} \subset L^2(\mathscr{G})$ *be such that* $\mathcal{S}^H(\mathscr{A})$ *has the form* (5.52) *with corresponding range function* $J_{\mathscr{A}}$. *Further, let* $\mathcal{N} = \{d_i\}_{i\in\mathcal{I}_s} \subset \mathcal{S}^H(\mathscr{A})$ *be such that the system* $\mathcal{E}_{\mathcal{N}} = \{L_\xi d_i\}_{\xi\in H}^{i\in\mathcal{I}_s}$ *is Bessel. Then the following statements are equivalent:*

(i) The set $H \times \mathcal{I}_s$ *is a stable set of sampling for* $\mathcal{S}^H(\mathscr{A})$. *That is, there exist constants* $0 < a \le b < \infty$ *such that*

$$a\|f\| \le \left(\sum_{i\in\mathcal{I}_s}\sum_{\xi\in H} |(\mathfrak{S}_{\mathcal{N}} f)(\xi, i)|^2\right)^{1/2} \le b\|f\|, \quad \textit{for all } f \in \mathcal{S}^H(\mathscr{A}).$$

(ii) For a.e. $\eta \in \hat{H}$ *and every* $u \in J_{\mathscr{A}}(\eta)$, *there exist constants* $0 < a \le b < \infty$ *such that*

$$a\|u\|^2 \le \sum_{i\in\mathcal{I}_s} |(\tilde{\mathfrak{S}}_{\mathcal{Z}\mathcal{N}}(\eta)(u, i))|^2 \le b\|u\|^2.$$

(iii) For each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r) \in \ell^2(H, \mathbb{C}^r)$, *there exist constants* $0 < c \le d < \infty$ *such that*

$$c\sum_{k\in\mathcal{I}_r} \|\alpha_k\|_{\ell^2(H)} \le \|\mathcal{L}_{\mathfrak{S}_{\mathcal{N}}}\alpha\| \le d\sum_{k\in\mathcal{I}_r} \|\alpha_k\|_{\ell^2(H)},$$

where $\mathcal{L}_{\mathfrak{S}_{\mathcal{N}}}$ *is defined analogously to* (5.18) *for* $\mathfrak{S}_{\mathcal{N}}$.

(iv) For a.e. $\eta \in \hat{H}$ *and every* $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_r) \in \mathbb{C}^r$, *there exist constants* $0 < c \le d < \infty$ *such that*

$$c\sum_{k\in\mathcal{I}_r} |\gamma_k|^2 \le \|\tilde{\mathcal{L}}_{\tilde{\mathfrak{S}}_{\mathcal{Z}\mathcal{N}}}(\eta)\gamma\|^2 \le d\sum_{k\in\mathcal{I}_r} |\gamma_k|^2,$$

where $\tilde{\mathcal{L}}_{\tilde{\mathfrak{S}}_{\mathcal{Z}\mathcal{N}}}$ *is defined analogously to* (5.19) *for* $\tilde{\mathfrak{S}}_{\mathcal{Z}\mathcal{N}}$.

Next, we characterize the stability of the sampling set $H \times \mathcal{I}_s$ in terms of a sampling formula for $f \in \mathcal{S}^H(\mathscr{A})$ along with its associated local conditions. The next result is an application of Theorem 5.14 for locally compact groups and generalizes the results of [60, Theorem 1], [59, Theorem 2.1] and [64, Theorem 1] and [95, Theorem 3].

Theorem 5.28. *In addition to the standing assumptions of Theorem 5.27, assume*

$$\max_{k\in\mathcal{I}_r} \operatorname*{ess\text{-}sup}_{\xi\in\hat{H}} \|(\mathcal{Z}\phi_k)(\xi)\|^2 < \infty.$$

Then the following statements are equivalent:

(i) *The set $H \times \mathcal{I}_s$ is a stable set of sampling for $\mathcal{S}^H(\mathscr{A})$.*

(ii) *There exist $h_i \in \mathcal{S}^H(\mathscr{A})$, $i \in \mathcal{I}_s$ such that the collection $\{L_\xi h_i\}_{h\in H}^{i\in\mathcal{I}_s}$ is a frame for $\mathcal{S}^H(\mathscr{A})$ and the sampling formula (global form) for any $f \in \mathcal{S}^H(\mathscr{A})$ is*

$$f = \sum_{i\in\mathcal{I}_s}\sum_{\xi\in H} \mathfrak{S}_{\mathcal{N}} f(\xi, i)\, L_\xi h_i.$$

(iii) *There exist $h_i \in \mathcal{S}^H(\mathscr{A})$, $i \in \mathcal{I}_s$ such that for a.e. $\eta \in \hat{H}$, $\{\mathcal{Z}h_i(\eta)\}_{i\in\mathcal{I}_s}$ is a frame for $J_{\mathscr{A}}(\eta)$ and the sampling formula (local form) is*

$$u = \sum_{i\in\mathcal{I}_s} \tilde{\mathfrak{S}}_{\mathcal{Z}\mathcal{N}}(\eta)(u,i)(\mathcal{Z}h_i)(\eta),\ \ u \in J_{\mathscr{A}}(\eta) = \operatorname{span}\{\mathcal{Z}\phi_k(\eta)\}_{k\in\mathcal{I}_r}. \tag{5.55}$$

(iv) *The collection $\{M_{\chi_\xi} T^*_{\mathcal{Z}\mathscr{A}} \mathcal{Z}d_i\}_{\xi\in H}^{i\in\mathcal{I}_s}$ is a frame for $L^2(\hat{H}, \mathbb{C}^r)$.*

(v) *For a.e. $\eta \in \hat{H}$ and every $u = (u_1, u_2, \ldots, u_r) \in \mathbb{C}^r$, there exist $0 < a \leq b < \infty$ such that*

$$a\|u\|^2 \leq \sum_{i\in\mathcal{I}_s} |\sum_{k\in\mathcal{I}_r} u_k \langle(\mathcal{Z}\phi_k)(\eta), (\mathcal{Z}d_i)(\eta)\rangle_{\mathbb{C}^r}|^2 \leq b\|u\|^2 \ \ \textit{holds.}$$

(vi) *There exists a $r\times s$ matrix $\Psi(\eta)$ with each column in $L^\infty(\hat{H}, \mathbb{C}^r)$ satisfying $\Psi(\eta)\mathbb{S}(\eta) = I_r$ for a.e. $\eta \in \hat{H}$, where $\mathbb{S}(\eta) := \Big(\varphi_1^T(\eta)\ \varphi_2^T(\eta)\ \cdots\ \varphi_s^T(\eta)\Big)^T$ is the $s \times r$ matrix for $1 \times r$ matrix $\varphi_i(\eta) = \overline{T^*_{\mathcal{Z}\mathscr{A}}\mathcal{Z}d_i(\eta)}$ for a.e. $\eta \in \hat{H}$, $i \in \mathcal{I}_s$ and I_r is the $r \times r$ identity matrix.*

Proof. Notice that the equivalences (i)$\Longleftrightarrow$(iii) $\Longleftrightarrow$(iv)$\Longleftrightarrow$(v)$\Longleftrightarrow$(vi) follow directly from Theorem 5.14 by taking $X = \hat{H}$, $\mathcal{H} = L^2(H\backslash\mathscr{G}), \mathfrak{B} = \{\chi_\xi\}_{\xi\in H}$ and Proposition 5.26.

The equivalence (i)$\Longleftrightarrow$(ii) is a consequence of Proposition 5.26 and the properties of the Zak transform. This proves the claim. □

Next, we characterize the stable interpolation for TI spaces. The sampling set $H \times \mathcal{I}_s$ is called a *stable set of interpolation* for $\mathcal{S}^H(\mathscr{A})$ if for any $(c_{\xi,i})_{(\xi,i)\in H\times\mathcal{I}_s} \in \ell^2(H \times \mathcal{I}_s)$, the interpolation problem $\mathfrak{S}_{\mathcal{N}} f(\xi, i) = c_{\xi,i}$ has a solution $f \in \mathcal{S}^H(\mathscr{A})$. The following result is an application of Theorem 5.20 for TI spaces and generalizes the results of [59, Lemma 2], [154, Theorem 9] and [13, Proposition 2.7].

Theorem 5.29. *In addition to the standing assumptions of Theorem 5.28, assume $\mathcal{E}_{\mathcal{N}} = \{L_\xi d_i\}_{\xi\in H}^{i\in\mathcal{I}_s}$ is complete in $\mathcal{S}^H(\mathscr{A})$, i.e., $\overline{\operatorname{span}}\,\mathcal{E}_{\mathcal{N}} = \mathcal{S}^H(\mathscr{A})$. Then the following statements are equivalent:*

(*i*) *The set* $H \times \mathcal{I}_s$ *is a stable set of interpolation for* $\mathcal{S}^H(\mathscr{A})$.

(*ii*) *The system* $\{M_{\chi_\xi} T^*_{\mathcal{Z}\mathscr{A}} \mathcal{Z} d_i\}_{\xi \in H}^{i \in \mathcal{I}_s}$ *is a Riesz basis for* $L^2(\hat{H}, \mathbb{C}^r)$.

(*iii*) *For any* $c = \{c_{\xi,i}\} \in \ell^2(H \times \mathcal{I}_s)$, *the problem* $\mathcal{F}\big(\tilde{\mathfrak{G}}_{\mathcal{Z}\mathcal{N}}(.)\mathcal{Z}f(.)\big)_i(\xi) = c_{\xi,i}$ *has a solution* $f \in \mathcal{S}^H(\mathscr{A})$.

Proof. First, note that by using (5.54), statement (i) is equivalent to the following reformulation:

(i') The set $H \times \mathcal{I}_s$ is a stable set of interpolation for $W_{\mathcal{Z}\mathcal{A},\mathfrak{B}}$.

The equivalences (i') $\iff$ (iii) and (i') $\iff$ (ii) follow from Theorem 5.20. by taking $X = \hat{H}$, $\mathcal{H} = L^2(H\backslash\mathscr{G})$, and $\mathfrak{B} = \{\chi_\xi\}_{\xi \in H}$. □

Finally, we establish the necessary density conditions on the subgroup H for stable sampling and stable interpolation by applying the results from Subsection 5.3.2 within the framework of locally compact groups.

We make the following additional assumptions.

(*i*) The subgroup $H \subset \mathscr{G}$ is uniformly discrete, meaning, there exists an open set U such that the sets ξU are pairwise disjoint for all $\xi \in H$. It is equipped with the counting measure $\#_H$.

(*ii*) The group $\mathscr{G}$ is compactly generated, meaning, there exists an open set N containing the identity element with the property $N = N^{-1}$ and with compact closure such that $\mathscr{G} = \bigcup_{i=0}^{\infty} N^i$. The corresponding metric on $\mathscr{G}$ is the word metric, which is defined as

$$d(\alpha, \beta) := \min\{i \in (0, \infty) : \alpha^{-1}\beta \in N^i\}, \quad \alpha, \beta \in \mathscr{G}.$$

(*iii*) The group $\mathscr{G}$ has polynomial growth, meaning, there exist positive constants A, B such that

$$\rho(N^i) \leq A i^B, \quad i \in \mathbb{N}.$$

Under these assumptions and [141, Corollary 10], the metric measure space $(\mathscr{G}, d, \rho)$ satisfies the assumptions made in the beginning of Subsection 5.3.2. See, [30, 57] for more details.

The next two results are application of Theorems 5.22 and 5.23 for the group setting. They generalize the results of [75, Theorem 1], [3, Theorem 3.4] and [4, Theorem 2.18].

Theorem 5.30. *In addition to the standing assumptions of Theorem 5.27, assume that the system*

$$\mathcal{Q} = \{M_{\chi_y} T^*_{\mathcal{Z}\mathscr{A}} \mathcal{Z} d_i\}_{y\in\mathscr{G}}^{i\in\mathcal{I}_s}$$

forms a continuous frame for $L^2(\hat{H}, \mathbb{C}^r)$, where $\{\chi_y\}_{y\in\mathscr{G}}$ is the set of continuous characters on $\hat{\mathscr{G}}$, given by $\chi_y : \hat{\mathscr{G}} \to \mathbb{T}, \quad \chi_y(\omega) = \omega(y), \quad \omega \in \hat{\mathscr{G}}$.

If for every $\epsilon > 0$, there exists $R > 0$ such that for all $i \in \mathcal{I}_s$, the following conditions hold:

(*i*) *(Weak localization property)*

$$\sup_{x\in\mathscr{G}} \sum_{j\in\mathcal{I}_s} \int_{\mathscr{G}\setminus B(x,R)} \left|\langle M_{\chi_x} T^*_{\mathcal{Z}\mathscr{A}} \mathcal{Z} d_i, M_{\chi_y} T^*_{\mathcal{Z}\mathscr{A}} \mathcal{Z} d_j\rangle\right|^2 d\rho(y) < \epsilon^2,$$

(*ii*) *(Homogeneous approximation property)*

$$\sup_{y\in\mathscr{G}} \sum_{j\in\mathcal{I}_s} \sum_{\xi\in H\cap\mathscr{G}\setminus B(y,R)} \left|\langle M_{\chi_y} T^*_{\mathcal{Z}\mathscr{A}} \mathcal{Z} d_i, M_{\chi_\xi} T^*_{\mathcal{Z}\mathscr{A}} \mathcal{Z} d_j\rangle\right|^2 < \epsilon^2,$$

then the following results hold:

1. *If $H \times \mathcal{I}_s$ is a stable set of sampling for $\mathcal{S}^H(\mathscr{A})$, then*

$$D^-(H) \geq \mathcal{T}^-(\mathcal{Q}) \quad \text{and} \quad D^+(H) \geq \mathcal{T}^+(\mathcal{Q}).$$

2. *If $H \times \mathcal{I}_s$ is a stable set of interpolation for $\mathcal{S}^H(\mathscr{A})$, then*

$$D^-(H) \leq \mathcal{T}^-(\mathcal{Q}) \quad \text{and} \quad D^+(H) \leq \mathcal{T}^+(\mathcal{Q}).$$

Here, $D^-(H)$ and $D^+(H)$ denote the lower and upper densities of the set H, respectively, and are defined analogously as in Section 5.3.

Theorem 5.31. *In addition to the standing assumptions of Theorem 5.30, if $\mathcal{Q}$ is a normalized continuous Parseval frame for $L^2(\hat{H}, \mathbb{C}^r)$, then the following results hold:*

1. *If $H \times \mathcal{I}_s$ is stable set of sampling for $\mathcal{S}^H(\mathscr{A})$, then $D^-(H) \geq 1$.*
2. *If $H \times \mathcal{I}_s$ is stable set of interpolation for $\mathcal{S}^H(\mathscr{A})$, then $D^+(H) \leq 1$.*

The proofs of Theorems 5.30 and 5.31 follow directly from Theorems 5.22 and 5.23, respectively, upon noting that condition (1) (or (2)) in both theorems ensures that H is a stable set of sampling (or interpolation) for $W_{\mathcal{Z}\mathscr{A},\mathfrak{B}}$.

For locally compact abelian groups. At the end, we provide a discussion for the case when $\mathscr{G}$ is abelian. In the abelian setting, the Zak transform has a sibling, known as the fiberization map. For a second countable locally compact abelian group $\mathcal{G}$ and its discrete subgroup H, we can utilize the fiberization map as defined in (5.2). It is worth to note that $\hat{G}/H^{\perp} = H^{\perp}\backslash\hat{G}$ in the abelian setting. In this case, the set $\mathfrak{B}$ for $L^2(\hat{G}/H^{\perp})$ is given by $\mathfrak{B} = \{\chi_\xi \in L^{\infty}(\hat{G}/H^{\perp}) : \xi \in H, \chi_\xi(\eta H^{\perp}) = \overline{\eta(\xi)} \text{ for } \eta H^{\perp} \in \hat{G}/H^{\perp}\}$. By replacing the Zak transform with the fiberization map, the Theorems 5.27, 5.28, 5.29, 5.30 and 5.31 can be provided for this case with the corresponding assumptions.

The p-adic group $\mathbb{Q}_p$ by the action of its discrete subgroup. For a prime number p, consider the group of p-adic numbers $\mathbb{Q}_p$. The group $\mathbb{Q}_p$ possess a *quasi-metric* $|.|_p$ such that $|a+b|_p \leq \max\{|a|_p, |b|_p\}$ for all $a, b \in \mathbb{Q}_p$. The p-adic group $\mathbb{Q}_p$ is an LCA group and all its proper subgroups (say K, one of them) are compact and open, and hence $\mathbb{Q}_p/K$ is not compact. Therefore $\mathbb{Q}_p$ does not have uniform lattice while the maximum literature for sampling on LCA groups is mainly based on the action of uniform lattices [3, 60]. Note that in our situation, we only require a discrete subgroup. For that purpose, let $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ be the compact open subgroup of $\mathbb{Q}_p$. Then we can consider the TI space generated by a finite family $\mathcal{A} \subset L^2(\mathbb{Q}_p)$ where the translations are given by the action of its discrete subgroup $\mathbb{Q}_p/\mathbb{Z}_p$. To define the Zak transform, let $\mathfrak{R}$ be a fundamental domain for $\mathbb{Z}_p^{\perp} = \{\chi_\xi : \xi \in \mathbb{Z}_p\}$, where for each $\xi \in \mathbb{Q}_p$, χ_ξ defines a character on $\mathbb{Q}_p$ given by $\chi_\xi(x) = e^{2\pi i x\xi}$. Then for $f \in L^1(\mathbb{Q}_p) \cap L^2(\mathbb{Q}_p)$, the Zak transform is defined by $\mathcal{Z}f(x,y) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} f(y+\xi)e^{-2\pi i x\xi} d\mu_{\mathbb{Q}_p/\mathbb{Z}_p}(\xi)$ for $x, y \in \mathfrak{R}$. Thus, we can derive the characterization results for the stability of the sampling and interpolation sets in this set-up using $\mathcal{Z}$ and the results of this section.

In the next chapter, we study the supremum cosine angle between TI spaces and establish its relation to the injectivity of the sampling operator associated with a TI space defined as a union of TI subspaces. Following a similar approach as in this chapter, we first develop the theory for MI spaces and subsequently employ the Zak transform $\mathcal{Z}$ to transfer the results to the setting of TI spaces.

Chapter 6

Supremum cosine angle and sampling injectivity

In this chapter, we derive equivalent conditions for the injectivity of the sampling operator associated with a translation-invariant (TI) system defined on a union of TI spaces. Motivated by [107], where sampling on unions of finitely generated shift-invariant (SI) spaces was analyzed to model signals lying in unions of subspaces, we extend this framework to the TI setting. Following the approach in Chapter 5, the problem is first studied for multiplication-invariant (MI) spaces by considering a union of MI spaces. The supremum cosine angle between any two MI spaces in the union yields sufficient conditions for injectivity through its link with the closedness of their sum. These results are finally extended to TI spaces over locally compact groups using the Zak transform.

6.1 Supremum cosine angle between MI spaces

Let E and F be two closed subspaces of a separable Hilbert space $\mathcal{H}$, then the *supremum cosine angle* between E and F is defined by

$$\vartheta(E,F) := \sup\left\{\frac{\|P_F u\|}{\|u\|} : u \in E\backslash\{0\}\right\} = \|P_F|_E\|, \tag{6.1}$$

where P_F is the orthogonal projection of $\mathcal{H}$ onto F and $P_F|_E$ is its restriction on E. It is clear from the definition that $\vartheta(E,F) = 0$ if either $E = \{0\}$ or $F = \{0\}$. In general, $\vartheta(E,F) = \vartheta(F,E)$. The application of the supremum cosine angle has many practical uses in the field of sampling theory. The closedness of the sum of two closed subspaces of a Hilbert space has been shown to be connected with the supremum cosine angle in several studies [6, 25, 98, 99, 101, 137]. Kim et al. [97] established conditions for the closedness of

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The results presented in this chapter are based on our published work [92]:

Kalra, S., Sarkar, S. and Shukla, N. K. 2025. An application of the supremum cosine angle between multiplication invariant spaces in $L^2(X,\mathcal{H})$, *Proc. Indian Acad. Sci. Math. Sci.* **134** (1), Paper No. 15, 16.

the sum of two singly generated shift-invariant subspaces of $L^2(\mathbb{R}^d)$. Later, in [100], Kim et al. extended the result to shift-invariant subspaces of $L^2(\mathbb{R}^d)$ with multiple generators.

To investigate the injectivity of the sampling operator, we consider a broader class of MI spaces in $L^2(X, \mathcal{H})$ defined through a Parseval determining set, in contrast to the orthonormal basis $\mathfrak{B}$ employed in Chapter 5. Here, (X, μ) denotes a σ-finite, complete measure space such that $L^2(X)$ is separable, and $\mathcal{H}$ is a separable Hilbert space. The notion of a Parseval determining set was introduced by Iverson [88] for $L^1(X)$ and generalizes the role of characters to a measure-theoretic setting on LCA groups.

Definition 6.1. Let $(\mathcal{M}, \mu_{\mathcal{M}})$ be a σ-finite measure space. A set $\mathfrak{B} = \{g_s \in L^\infty(X) : s \in \mathcal{M}\}$ is said to be *Parseval determining set* for $L^1(X)$ if for each $f \in L^1(X)$, $s \mapsto \int_X f(x)\overline{g_s(x)}\, d\mu(x)$ is measurable on $\mathcal{M}$ and

$$\int_{\mathcal{M}} \left| \int_X f(x)\overline{g_s(x)}\, d\mu(x) \right|^2 d\mu_{\mathcal{M}}(s) = \int_X |f(x)|^2\, d\mu(x).$$

Remark 6.2. Note that if $\mathfrak{B} = \{g_t\}_{t\in\mathcal{M}}$ is an orthonormal basis for $L^2(X)$, where $\mathcal{M}$ is a countable set and $\mu_{\mathcal{M}}$ is the counting measure, then for any $f \in L^2(X)$, Parseval's identity gives

$$\sum_{t\in\mathcal{M}} \left| \int_X f(x)\overline{g_t(x)}\, d\mu(x) \right|^2 = \int_X |f(x)|^2\, d\mu(x).$$

For each $f \in L^1(X)$, the coefficient mapping

$$t \mapsto \int_X f(x)\overline{g_t(x)}\, d\mu(x)$$

is measurable on $\mathcal{M}$, since it is a function from a countable set to $\mathbb{C}$ and every such function is measurable. Moreover, the basis functions satisfy $g_t \in L^\infty(X) \subseteq L^2(X)$, so these integrals are defined for all $f \in L^1(X)$. Consequently, an orthonormal basis $\mathfrak{B}$ and the measure space $(\mathcal{M}, \mu_{\mathcal{M}})$ satisfy the requirements of a Parseval determining set for $L^1(X)$ as in Definition 6.1.

Thus, the MI spaces considered in this chapter form a more general setting than those studied in Chapter 5.

Let us consider a countable set of generators $\mathcal{A} = \{\varphi_i : i \in I\} \subset L^2(X, \mathcal{H})$ and a Parseval determining set $\mathfrak{B} = \{g_t\}_{t\in\mathcal{M}}$ such that the the MI system $E_{\mathcal{A},\mathfrak{B}}$ is Bessel, where I is an index set and $\mathcal{M}$ is a σ-finite measure space. Then for a.e. $x \in X$, the system $\{\varphi_i(x) : i \in I\}$ is Bessel [88, Theorem 2.10]. Let J denote the range function associated with the MI space $W_{\mathcal{A},\mathfrak{B}} = \overline{\operatorname{span}}\, E_{\mathcal{A},\mathfrak{B}}$. The *analysis operator* $\mathcal{T}_{\mathcal{A}}(x) : J(x) \to \ell^2(I)$ associated with $\{\varphi_i(x) : i \in I\}$ is given by

$$\mathcal{T}_{\mathcal{A}}(x)(h) = \{\langle h, \varphi_i(x)\rangle\}_{i\in I} \text{ for every } h \in J(x).$$

The *synthesis operator* of $\{\varphi_i(x) : i \in I\}$ is the adjoint of analysis operator $\mathcal{T}_{\mathcal{A}}(x)$ and is given by

$$\mathcal{T}^*_{\mathcal{A}}(x) : \ell^2(I) \to J(x),\ \mathcal{T}^*_{\mathcal{A}}(x)(c) = \sum_{i\in I} c_i\varphi_i(x)$$

for $c = \{c_i\}_{i\in I}$ having finitely many non-zero terms. The operator $G_{\mathcal{A}}(x) := \mathcal{T}_{\mathcal{A}}(x)\mathcal{T}^*_{\mathcal{A}}(x)$ on $\ell^2(I)$ is the *Gramian* of $\{\varphi_i(x) : i \in I\}$. Let $\mathcal{A}' = \{\psi_i\}_{i\in I} \subset L^2(X, \mathcal{H})$ be another countable collection such that $E_{\mathcal{A}',\mathfrak{B}}$ is also Bessel, then the operator $G_{\mathcal{A},\mathcal{A}'}(x) := \mathcal{T}_{\mathcal{A}}(x)\mathcal{T}^*_{\mathcal{A}'}(x)$ is the *mixed Gramian* of $\{\varphi_i(x)\}_{i\in I}$ and $\{\psi_i(x)\}_{i\in I}$ [42]. The matrix representations of Gramian and mixed Gramian operator corresponding to the collections $\mathcal{A} = \{\varphi_i : i \in I\}$ and $\mathcal{A}' = \{\psi_i : i \in I\}$ is

$$G_{\mathcal{A}}(x) = [\langle\varphi_j(x), \varphi_i(x)\rangle]_{i,j\in I} \text{ and } G_{\mathcal{A},\mathcal{A}'}(x) = [\langle\varphi_j(x), \psi_i(x)\rangle]_{i,j\in I}\,.$$

Our next result explains the supremum cosine angle between two multiplication invariant spaces by utilizing range functions and connecting it with the supremum cosine angle between the corresponding fibers pointwise. This theorem also establishes a link between the supremum cosine angle between two multiplication invariant spaces and the Gramians and mixed Gramians connected with the respective generating sets, all within a measure-theoretic context. This result is essentially a measure-theoretic version of [100, Lemma 3.6].

For an MI space V in $L^2(X, \mathcal{H})$, we denote $\sigma(V) = \{x \in X : J(x) \neq 0\}$.

Theorem 6.3. *Let $\mathcal{A} = \{\varphi_i : i \in I\}$ and $\mathcal{A}' = \{\psi_i : i \in I\}$ be two countable collections of functions in $L^2(X, \mathcal{H})$. Define $\Omega := \sigma(W_{\mathcal{A},\mathfrak{B}}) \cap \sigma(W_{\mathcal{A}',\mathfrak{B}})$. Then the supremum cosine angle between $W_{\mathcal{A},\mathfrak{B}}$ and $W_{\mathcal{A}',\mathfrak{B}}$ is given by*

$$\vartheta(W_{\mathcal{A},\mathfrak{B}}, W_{\mathcal{A}',\mathfrak{B}}) = \operatorname*{ess\text{-}sup}_{x\in\Omega} \big\{\vartheta\big(J(x), J'(x)\big)\big\},$$

where J' denote the range function associated with $W_{\mathcal{A}',\mathfrak{B}}$. In addition, if the multiplication generated systems $E_{\mathcal{A},\mathfrak{B}}$ and $E_{\mathcal{A}',\mathfrak{B}}$ are frames for $W_{\mathcal{A},\mathfrak{B}}$ and $W_{\mathcal{A}',\mathfrak{B}}$, respectively, then

$$\vartheta(W_{\mathcal{A},\mathfrak{B}}, W_{\mathcal{A}',\mathfrak{B}}) = \operatorname*{ess\text{-}sup}_{x\in\Omega} \left\|(G_{\mathcal{A}'}(x)^\dagger)^{\frac{1}{2}} G_{\mathcal{A},\mathcal{A}'}(x)(G_{\mathcal{A}}(x)^\dagger)^{\frac{1}{2}}\right\|,$$

where † denotes the pseudoinverse.

Our second main result, which is presented in Theorem 6.4, provides a condition that determines when the sum of two MI spaces will be closed in terms of the supremum cosine angle. Furthermore, we demonstrate that this condition is preserved by the corresponding fiber spaces almost everywhere. Specifically, we show that the sum of two MI spaces will be closed if and only if the supremum cosine angle between the corresponding fiber spaces is less than 1 almost everywhere. This result generalizes [100, Lemma 3.6] for the MI spaces, providing a set-theoretic abstraction of the aforementioned lemma.

Theorem 6.4. *In addition to the standing assumptions as in Theorem 6.3, let*

$$\Omega' := \Omega \backslash \sigma\big(W_{\mathcal{A},\mathfrak{B}} \cap W_{\mathcal{A}',\mathfrak{B}}\big),$$

and $W_{\mathcal{A},\mathfrak{B}}|_{\Omega'} := \{f \in W_{\mathcal{A},\mathfrak{B}} : f(x) = 0 \text{ a.e. on } X\backslash\Omega'\}$. *Then the following are equivalent:*

(i) $W_{\mathcal{A},\mathfrak{B}}|_{\Omega'} + W_{\mathcal{A}',\mathfrak{B}}|_{\Omega'}$ *is closed.*

(ii) $\vartheta(W_{\mathcal{A},\mathfrak{B}}|_{\Omega'}, W_{\mathcal{A}',\mathfrak{B}}|_{\Omega'}) < 1$.

(iii) $J(x) + J'(x)$ *is closed for a.e.* $x \in \Omega'$.

(iv) $\vartheta(J(x), J'(x)) < 1$ *for a.e.* $x \in \Omega'$.

Before proving the above results, we first provide a couple of propositions which will be required in the sequel. The following Proposition 6.5 is a set-theoretic version of [100, Lemmas 3.3 and 3.4]. The technique of the proof follows on the same line.

From now on, we frequently use $V(x) := \{f(x) : f \in V\}$ to denote the fiber of the MI space V at $x \in X$.

Proposition 6.5. *Let* $\mathcal{A} = \{\varphi_i : i \in I\}$ *and* $\mathcal{A}' = \{\psi_i : i \in I\}$ *be two countable collections of functions in* $L^2(X, \mathcal{H})$. *For the MI spaces* $W_{\mathcal{A},\mathfrak{B}}$ *and* $W_{\mathcal{A}',\mathfrak{B}}$ *in* $L^2(X, \mathcal{H})$, *the following hold true:*

(i) If $W_{\mathcal{A},\mathfrak{B}} \cap W_{\mathcal{A}',\mathfrak{B}} = \{0\}$, *then* $J(x) \cap J'(x) = \{0\}$ *for a.e.* $x \in X$.

(ii) $\sigma\big(W_{\mathcal{A},\mathfrak{B}} \cap W_{\mathcal{A}',\mathfrak{B}}\big) = \{x \in X : J(x) \cap J'(x) \neq \{0\}\} \subseteq \sigma(W_{\mathcal{A},\mathfrak{B}}) \cap \sigma(W_{\mathcal{A}',\mathfrak{B}})$.

Proof. (i) Firstly, it is easy to observe that $W_{\mathcal{A},\mathfrak{B}} \cap W_{\mathcal{A}',\mathfrak{B}}$ is again an MI space. Also,

$$W_{\mathcal{A},\mathfrak{B}} \cap W_{\mathcal{A}',\mathfrak{B}} = \{f \in L^2(X, \mathcal{H}) : f(x) \in J(x) \cap J'(x) \text{ for a.e. } x \in X\},$$

by the definitions of $W_{\mathcal{A},\mathfrak{B}}$ and $W_{\mathcal{A}',\mathfrak{B}}$. Now, by the application of Theorem [28, Theorem 2.4], the corresponding range function is $J \cap J'$. Now, we begin the proof of (i). Let us assume $J(x) \cap J'(x) \neq \{0\}$ for a.e. $x \in X$, then there exists a measurable set $\Delta \subset X$ such that $\mu(\Delta) > 0$ and for all $x \in \Delta$, $J(x) \cap J'(x) \neq \{0\}$. Let $\{d_1, d_2, \dots, d_n, \dots\}$ be a countable dense subset of $\mathcal{H}$ and $U(x) = J(x) \cap J'(x)$. Since $U(x) \neq \{0\}$ on Δ, then for each $x \in \Delta$, there exists d_n such that $P_{U(x)}d_n \neq 0$, where $P_{U(x)}$ projects onto $U(x)$. Let $i(x)$ be the least positive integer for which $P_{U(x)}d_{i(x)} \neq 0$. Consider the set $\Delta_n = \{x \in \Delta : i(x) = n\}$, then it is clear that $\Delta = \oplus_{n\in\mathbb{N}}\Delta_n$. Since $\mu(\Delta) > 0$, there exists $N \in \mathbb{N}$ such that Δ_N has positive measure. Define a function $f : X \to \mathcal{H}$ such that :

$$f(x) = \begin{cases} P_{U(x)}d_N \text{ when } x \in \Delta_N, \\ 0, \text{ otherwise.} \end{cases}$$

Since $f(x) \in J(x) \cap J'(x)$, then by [28, Theorem 2.4], $0 \neq f \in W_{\mathcal{A},\mathfrak{B}} \cap W_{\mathcal{A}',\mathfrak{B}}$, which is a contradiction.

(ii) We prove that $\sigma\big(W_{\mathcal{A},\mathfrak{B}} \cap W_{\mathcal{A}',\mathfrak{B}}\big) = \{x \in X : J(x) \cap J'(x) \neq \{0\}\}$ as the later containment of sets is quite obvious. Let $\mathscr{W} := W_{\mathcal{A},\mathfrak{B}} \cap W_{\mathcal{A}',\mathfrak{B}}$, $E' := W_{\mathcal{A},\mathfrak{B}} \ominus \mathscr{W}$ and $F' := W_{\mathcal{A}',\mathfrak{B}} \ominus \mathscr{W}$. Since $\mathscr{W}$ is the intersection of two MI spaces, therefore $\mathscr{W}$ is again an MI space. Further from definition (2.2), it is clear that if $g_s \in \mathcal{D}$, then $\overline{g}_s \in \mathcal{D}$ for any $s \in \mathcal{M}$ and hence $\mathcal{D}$ is closed under conjugation. Then by [44, Ch. II, Proposition 3.7], $\mathscr{W}^{\perp}$ is also an MI space. Consequently, E' and F' are MI spaces. Then $W_{\mathcal{A},\mathfrak{B}} = E' \oplus \mathscr{W}$ and $W_{\mathcal{A}',\mathfrak{B}} = F' \oplus \mathscr{W}$ implies $J(x) = E'(x) \oplus \mathscr{W}(x)$ and $J'(x) = F'(x) \oplus \mathscr{W}(x)$, by the pointwise projection property of $\mathcal{D}$-multiplication invariant spaces. Now if $x \in \sigma(W_{\mathcal{A},\mathfrak{B}} \cap W_{\mathcal{A}',\mathfrak{B}}) = \sigma(\mathscr{W})$, then $\{0\} \neq \mathscr{W}(x) \subseteq J(x) \cap J'(x)$. Conversly, let $x \in X$ is such that $J(x) \cap J'(x) \neq \{0\}$. Then, there exists $0 \neq h \in J(x) \cap J'(x)$. Also there exists $h_1 \in E'(x), h_2 \in F'(x)$ and $w_1, w_2 \in \mathscr{W}(x)$ such that $h = h_1 + w_1 = h_2 + w_2$. Since $h_1 - h_2 \in \mathscr{W}(x)^{\perp}$ and $w_1 - w_2 \in \mathscr{W}(x)$ implies $h_1 = h_2$ and $w_1 = w_2$. Therefore, $h = h_1 + w_1$ with $h_1 \in E'(x) \cap F'(x)$. Also it is clear that, $E' \cap F' = \{0\}$ which further by Proposition 6.5 implies $E'(x) \cap F'(x) = \{0\}$. Hence, $h_1 = 0$ and $0 \neq h \in \mathscr{W}(x) = \big(W_{\mathcal{A},\mathfrak{B}} \cap W_{\mathcal{A}',\mathfrak{B}}\big)(x)$. □

The next proposition proves that the restriction of a multiplication invariant space to a measurable subset is again multiplication invariant. The following result is a set-theoretic abstraction of [100, Lemma 2.5].

Proposition 6.6. *For a countable collection of functions $\mathcal{A}$ in $L^2(X,\mathcal{H})$, consider the MI space $S_{\mathfrak{D}}(\mathcal{A})$ and its restriction to a measurable subset Λ of X defined by,*

$$W_{\mathcal{A},\mathfrak{B}}|_{\Lambda} := \{f \in W_{\mathcal{A},\mathfrak{B}} : f(x) = 0 \ a.e. \ on \ X\backslash\Lambda\}.$$

Then $W_{\mathcal{A},\mathfrak{B}}|_{\Lambda}$ is also an MI space.

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $W_{\mathcal{A},\mathfrak{B}}|_{\Lambda}$ such that $u_n \longrightarrow u$ in $L^2(X,\mathcal{H})$ as $n \longrightarrow \infty$ for some $u \in L^2(X,\mathcal{H})$. Since $W_{\mathcal{A},\mathfrak{B}}$ is closed and $W_{\mathcal{A},\mathfrak{B}}|_{\Lambda} \subseteq W_{\mathcal{A},\mathfrak{B}}$, we have $u \in W_{\mathcal{A},\mathfrak{B}}$. Also note that, $\operatorname{supp}(u_n) = \Lambda$ implies $\operatorname{supp}(u) = \Lambda$. Hence, $u \in W_{\mathcal{A},\mathfrak{B}}|_{\Lambda}$. Therefore $W_{\mathcal{A},\mathfrak{B}}|_{\Lambda}$ is closed and an MI space. $\square$

We are now ready to prove our main results Theorems (6.3) and (6.4).

Proof of Theorem 6.3 .

Let $\{d_n : n \in \mathbb{N}\}$ be an orthonormal basis of $L^2(X,\mathcal{H})$. For each $n \in \mathbb{N}$, define a function $s_n : X \to \mathbb{R}, x \mapsto s_n(x) := \langle P_{J(x)}|_{J'(x)} d_n, d_n\rangle$. By definition of range function, the sequence $(s_n)_{n\in\mathbb{N}}$ is measurable. Denote $s := \sup_{n\in\mathbb{N}} s_n$ defined by, $x \mapsto \mathfrak{S}(J(x), J'(x)) = \sup_{n\in\mathbb{N}}\langle P_{J(x)}|_{J'(x)} d_n, d_n\rangle = \left\| P_{J'(x)}|_{J(x)} \right\|$. Then the map s is weakly measurable, where the last equality holds because projection maps are self adjoint. Let $c = \vartheta(W_{\mathcal{A},\mathfrak{B}}, W_{\mathcal{A}',\mathfrak{B}}) = \|P_{W_{\mathcal{A}',\mathfrak{B}}}|_{W_{\mathcal{A},\mathfrak{B}}}\|$ and $\tilde{c} = \text{ess-sup}_{x\in\Omega}\left\{\vartheta\big(J(x), J'(x)\big)\right\}$. For $u \in W_{\mathcal{A},\mathfrak{B}}$, we have $\|P_{J'(x)}u(x)\| \le \vartheta(J(x), J'(x))\|u(x)\|$ for a.e. $x \in X$ and hence

$$\begin{aligned}\|P_{W_{\mathcal{A}',\mathfrak{B}}}(u)\|^2 &= \int_X \|P_{W_{\mathcal{A}',\mathfrak{B}}}(u)(x)\|^2 \, d\mu(x) = \int_X \|P_{J'(x)}u(x)\|^2 \, d\mu(x) \\ &= \int_\Omega \|P_{J'(x)}u(x)\|^2 \, d\mu(x) \le \tilde{c}^2 \int_X \|u(x)\|^2 \, d\mu(x) \le \tilde{c}^2\|u\|^2,\end{aligned}$$

by observing $P_{W_{\mathcal{A}',\mathfrak{B}}}(u)(x) = P_{J'(x)}u(x)$ given in [28, Proposition 2.2]. Taking the supremum over all $u \in W_{\mathcal{A},\mathfrak{B}}$ we have $c \le \tilde{c}$.

By choosing $c < \tilde{c}$, there exist $\epsilon > 0$ and a measurable subset $\Delta \subseteq \Omega$, with $\mu(\Delta) > 0$ such that $u + \epsilon < \vartheta(J(x), J'(x))$ for a.e. $x \in X$. Construct $\Delta_i = \{x \in W_{\mathcal{A},\mathfrak{B}} : 0 < (c+\epsilon)\|P_{J(x)}d_i\| \le \|P_{J'(x)}(P_{J(x)}d_i)\|\}$. Then $X = \bigcup_{i=1}^{\infty}\Delta_i$. Hence there exists i_0 such that $\mu(\Delta_{i_0}) > 0$. Define a function h by

$$h(x) = \begin{cases} P_{J(x)}d_i \text{ if } x \in \Delta_{i_0}, \\ 0, \text{ otherwise.} \end{cases}$$

Then, $h \in W_{\mathcal{A},\mathfrak{B}}$ and hence the following calculation

$$\begin{aligned} \|P_{W_{\mathcal{A}',\mathfrak{B}}}h\|^2 &= \int_{\Omega} \|P_{J'(x)}h(x)\|^2 \, d\mu(x) = \int_{\Delta_{i_0}} \|P_{J'(x)}P_{J(x)}d_{i_0}\|^2 \, d\mu(x) \\ &\geq (c+\epsilon)^2 \int_{\Delta_{i_0}} \|P_{J(x)}d_{i_0}\|^2 \, d\mu(x) = (c+\epsilon)^2 \int_{\Delta_{i_0}} \|h(x)\|^2 \, d\mu(x) \\ &= (c+\epsilon)^2 \|h\|^2 > 0, \end{aligned}$$

implies that $c = \vartheta(W_{\mathcal{A},\mathfrak{B}}, W_{\mathcal{A}',\mathfrak{B}}) \geq \frac{\|P_{W_{\mathcal{A}',\mathfrak{B}}}h\|}{\|h\|} \geq c+\epsilon$, a contradiction. So $c \geq \tilde{c}$. Hence $c = \tilde{c}$.

For the remaining part, first observe that the fibers $\{\varphi_i(x) : i \in I\}$ and $\{\psi_i(x) : i \in I\}$ are frames for $J_{\mathcal{A}}(x)$ and $J'_{\mathcal{A}}(x)$ for a.e. $x \in X$, since $E_{\mathcal{A},\mathfrak{B}}$ and $E_{\mathcal{A}',\mathfrak{B}}$ are frames for $W_{\mathcal{A},\mathfrak{B}}$ and $W_{\mathcal{A}',\mathfrak{B}}$ by [88, Theorem 2.10]. The hypothesis imply that

$$\mathcal{T}_{\mathcal{A}}(x),\ T_{\mathcal{A}'}(x),\ T^*_{\mathcal{A}}(x),\ \mathcal{T}^*_{\mathcal{A}'}(x),\ G_{\mathcal{A}}(x), \text{ and } G_{\mathcal{A}'}(x)$$

have closed ranges and their pseudo-inverses exist. Now, the result follows using [100, Theorem 2.1]. □

Next, we provide a proof of Theorem 6.4 by observing that for two closed subspaces E and F of $\mathcal{H}$, the sum $E+F$ is closed and $E \cap F = \{0\}$ if and only if $\vartheta(E,F) < 1$ [137, Theorem 2.9].

Proof of Theorem 6.4 . (i) $\Leftrightarrow$ (ii) By Proposition 6.6, $W_{\mathcal{A},\mathfrak{B}}|_{\Omega'}$ and $W_{\mathcal{A}',\mathfrak{B}}|_{\Omega'}$ are multiplication invariant spaces. Now applying Proposition 6.5, $\sigma(W_{\mathcal{A},\mathfrak{B}}|_{\Omega'} \cap W_{\mathcal{A}',\mathfrak{B}}|_{\Omega'}) = \phi$ therefore $W_{\mathcal{A},\mathfrak{B}}|_{\Omega'} \cap W_{\mathcal{A}',\mathfrak{B}}|_{\Omega'} = \{0\}$. Applying [137, Theorem 2.9], (i) and (ii) are equivalent. The statements (iii) and (iv) are also equivalent by following the same arguments since for a.e. $x \in \Omega'$, $J(x) \cap J'(x) = \{0\}$.

The statements (ii) and (iv) are equivalent by noting

$$\vartheta(W_{\mathcal{A},\mathfrak{B}}|_{\Omega'}, W_{\mathcal{A}',\mathfrak{B}}|_{\Omega'}) = \operatorname*{ess\text{-}sup}_{x\in\Omega'} \left\{\vartheta\big(J(x), J'(x)\big)\right\}$$

using Theorem 6.3 and $\vartheta(W_{\mathcal{A},\mathfrak{B}}|_{\Omega'}, W_{\mathcal{A}',\mathfrak{B}}|_{\Omega'}) < 1$ if and only if $\vartheta(J(x), J'(x)) < 1$ for a.e. $x \in \Omega'$. □

In the next section, we will provide an application of the supremum cosine angle for the sampling problem.

6.2 Injectivity of the sampling operator

In this section, we discuss the application part of the of supremum cosine angle. In sampling theory, it is desirable for the sampling operator to be one-to-one for the unique representation of a function. We find out the conditions under which the sampling operator associated with a Bessel MI system in $L^2(X, \mathcal{H})$ is one-to-one. We also connect our result with the injectivity of the pointwise sampling operators defined on fiber spaces for each $x \in X$. The results presented in this section are a generalization of the previously established results for the closed subspaces of a separable Hilbert space or, more generally, shift-invariant spaces [12, 107]. We begin with some definitions.

Definition 6.7. Let $\mathfrak{B} = \{g_t\}_{t\in\mathcal{M}}$ be a Parseval determining set and $\mathcal{A} = \{\varphi_i\}_{i\in I} \subset L^2(X, \mathcal{H})$ be a countable set of generators such that the MI system $E_{\mathcal{A},\mathfrak{B}}$ is Bessel. Then the *sampling operator* associated with the system $E_{\mathcal{A},\mathfrak{B}}$ is defined by

$$T : L^2(X, \mathcal{H}) \to L^2(\mathcal{M} \times I),\ Tf := \{\langle f, M_{g_t}\varphi_i\rangle\}_{t\in\mathcal{M},i\in I}. \tag{6.2}$$

With every sampling operator T, we can associate pointwise sampling operator $\tilde{T}(x)$ of Bessel family $\{\varphi_i(x)\}_{i\in I}$ defined on the fiber space $J(x)$ for a.e. $x \in X$ and is given by

$$\tilde{T}(x) : J(x) \to \ell^2(I),\ \tilde{T}(x)(g) = \{\langle g, \varphi_i(x)\rangle\}_{i\in I}.$$

The sampling operator T associated with the Bessel system $E_{\mathcal{A},\mathfrak{B}}$ is the analysis operator $\mathcal{T}_{\mathcal{A}}$ of the Bessel system $E_{\mathcal{A},\mathfrak{B}}$. The requirement is to obtain conditions for the sampling operator to be one-to-one to uniquely obtain a signal from its samples. The following result is an abstract version of [12, Theorem 5.12] for the setting of MI spaces. Additionally, it provides the equivalent conditions for the sampling operator T to be one-to-one in terms of injectivity of the pointwise sampling operators $\tilde{T}(x)$ for a.e. $x \in X$.

Proposition 6.8. *Let $\mathcal{A} = \{\varphi_i\}_{i\in I} \subset L^2(X, \mathcal{H})$ be a countable family of functions and $\mathfrak{B} = \{g_t\}_{t\in\mathcal{M}}$ be a Parseval determining set such that $E_{\mathcal{A},\mathfrak{B}}$ is Bessel in $L^2(X, \mathcal{H})$ and T be the associated sampling operator. Then for a finite set $\mathcal{A}' = \{\psi_1, \psi_2, ..., \psi_r\} \subset L^2(X, \mathcal{H})$, the following are equivalent:*

- *(i) The sampling operator T is one-to-one on $W_{\mathcal{A},\mathfrak{B}}$.*
- *(ii) $E_{\mathcal{A},\mathfrak{B}}$ is complete in $W_{\mathcal{A}',\mathfrak{B}}$, i.e., $W_{\mathcal{A}',\mathfrak{B}} = \overline{\operatorname{span}}\, E_{\mathcal{A},\mathfrak{B}}$.*

(iii) $\tilde{T}(x)$ *is one-to-one on* $J'(x)$ *for a.e.* $x \in X$.

(iv) For a.e. $x \in X$, *the system* $\{\varphi_i(x) : i \in I\}$ *is complete in* $J(x)$.

(v) $\ker(G_{\mathcal{A}',\mathcal{A}})(x) = \ker(\mathcal{T}^*_{\mathcal{A}'}(x))$ *for a.e.* $x \in X$.

(vi) $\dim(\text{range}(G_{\mathcal{A}',\mathcal{A}})(x)) = \dim J'(x)$ *for a.e.* $x \in X$.

Proof. The sampling operator T is one-to-one on $W_{\mathcal{A}',\mathfrak{B}}$ if and only if for all $t \in \mathcal{M}$ and $i \in I$, $\langle f, M_{g_t}\varphi_i\rangle = 0$ will imply $f = 0$ for any $f \in W_{\mathcal{A}',\mathfrak{B}}$ which is equivalent to (ii). Therefore statements (i) and (ii) are equivalent. Similarly, statements (iii) and (iv) are equivalent. Also, if (ii) is true, then by [26, Theorem 3.5], the corresponding range function J' satisfies

$$J'(x) = \overline{\text{span}}\{\varphi_i(x) : i \in I\}.$$

Since $J'(x)$ is closed, then by [12, Proposition 3.2], $\tilde{T}(x)$ is one-to-one on $J'(x)$ for a.e. $x \in X$ and hence the statements (ii) and (iii) are equivalent. The equivalence of the statements (i), (v), and (vi) follows from the fact that the fiber space $J'(x)$ is finite dimensional and the range of the synthesis operator $\mathcal{T}^*_{\mathcal{A}'}(x)$ is $J'(x)$. □

In classical sampling theory, the signals to be sampled come from a single space. Practically, signals for sampling should be considered from the union of subspaces. This approach has applications in sparse signal representation, compressed sampling, nonuniform splines, and more. The extension of the sampling problem to a union of subspaces was introduced by Lu and Do [107] and extended further by M. Anastasio and C. Cabrelli [12]. In our setup, we consider a union of the finitely generated multiplication invariant subspaces of $L^2(X, \mathcal{H})$ and discuss the injectivity of sampling operator T on the union.

Question: *Corresponding to an arbitrary index set* Δ, *let* $\{\mathcal{N}_\delta\}_{\delta\in\Delta}$ *be the finitely generated MI spaces in* $L^2(X, \mathcal{H})$ *and*

$$\mathcal{X} := \bigcup_{\delta\in\Delta} \mathcal{N}_\delta. \tag{6.3}$$

When can each element $f \in \mathcal{X}$ *be uniquely represented by its samples?*

The discussion of injectivity of the sampling operator for a union of MI spaces in the collection $\{\mathcal{N}_\delta\}_{\delta\in\Delta}$ comes with a problem. The definition of an injective operator (say A) says that for any x_1, x_2 in the domain of the operator, $Ax_1 = Ax_2$ implies $x_1 = x_2$. The issue while considering the union of subspaces is that the points x_1 and x_2 can be from different subspaces of the union and exploits the linearity. To deal with the issue, we consider the union of a sum of MI subspaces of $L^2(X, \mathcal{H})$. The idea was originated in [107] and supported further by [12] where the author considered the collection $\{\mathcal{S}_\delta\}_{\delta\in\Delta}$ of shift-invariant spaces and took the union of the sum of each shift-invariant spaces in the collection. Similarly, we consider the union

$$\mathcal{X} := \bigcup_{\delta\in\Delta} \mathcal{N}_{\delta,\theta} \tag{6.4}$$

of the sum of the finitely generated MI spaces in the collection $\{\mathcal{N}_\delta\}_{\delta\in\Delta}$. Each space $\mathcal{N}_{\delta,\theta}$ in the union has the following form

$$\mathcal{N}_{\delta,\theta} := \mathcal{N}_\delta + \mathcal{N}_\theta = \{u + v : u \in \mathcal{N}_\delta, v \in \mathcal{N}_\theta\}, \tag{6.5}$$

and every vector in the union is a secant vector which has importance in dimensionality reduction [31].

The following proposition implies that the closedness of the sum of the two multiplication invariant spaces results in another multiplication invariant space, and this fact will be necessary in the subsequent discussion.

Proposition 6.9. *For* $\mathcal{A}, \mathcal{A}' \subset L^2(X, \mathcal{H})$, $W_{\mathcal{A}\cup\mathcal{A}',\mathfrak{B}} = \overline{W_{\mathcal{A},\mathfrak{B}} + W_{\mathcal{A}',\mathfrak{B}}}$.

Proof. The proof follows by the linearity of multiplication operator M_ϕ and observing

$$\begin{aligned} \overline{W_{\mathcal{A},\mathfrak{B}} + W_{\mathcal{A}',\mathfrak{B}}} &= \overline{\overline{\operatorname{span}}\{M_\phi\varphi : \varphi \in \mathcal{A}, \phi \in \mathcal{D}\} + \overline{\operatorname{span}}\{M_\phi\varphi : \varphi \in \mathcal{A}', \phi \in \mathcal{D}\}} \\ &= \overline{\operatorname{span}}\{M_\phi(\varphi + \varphi') : \varphi \in \mathcal{A}, \varphi' \in \mathcal{A}', \phi \in \mathcal{D}\}. \end{aligned}$$

□

Now, we deal with the injectivity of sampling operator T on $\mathcal{X}$ given in (6.4). The conditions for the injectivity of the sampling operator for the case of the union of closed subspaces have been passed to those of single space [107, Proposition 1]. But here the problem is sum of two MI spaces $\mathcal{N}_{\delta,\theta} := \mathcal{N}_\delta + \mathcal{N}_\theta = \{u + v : u \in \mathcal{N}_\delta, v \in \mathcal{N}_\theta\}$ need not to be closed and therefore $\mathcal{N}_{\delta,\theta}$ need not to be an MI space. This directly shifts the problem into the field of closedness of subspaces where we will take the help of Section 6.1. The following Proposition suggests that the closure of the sum of two MI spaces is again an MI space generated by the union of the generators of two MI spaces and thus for the space $\mathcal{N}_{\delta,\theta}$ to be an MI space, it is enough for it to be closed. A similar result for the case of shift-invariant spaces can be found in [12].

The following result is a measure-theoretic abstraction of [12, Theorem 6.5].

Theorem 6.10. *Let $\mathcal{A} = \{\psi_i\}_{i\in I}$ be a sequence in $L^2(X, \mathcal{H})$ such that $E_{\mathcal{A},\mathfrak{B}}$ is Bessel and the supremum cosine angle*

$$\vartheta(\mathcal{N}_\delta, \mathcal{N}_\theta) < 1 \text{ for every } \delta, \theta \in \Delta.$$

If $\mathcal{A}'_{\delta,\theta}$ is a finite set of generators for $\mathcal{N}_{\delta,\theta}$, defined in (6.5), then the sampling operator associated with $E_{\mathcal{A},\mathfrak{B}}$ is one-to-one for $\bigcup_{\delta,\theta\in\Delta} \mathcal{N}_{\delta,\theta}$ if and only if $\dim(\operatorname{range}(G_{\mathcal{A}'_{\delta,\theta},\mathcal{A}})(x)) = \dim J_{\mathcal{A}'_{\delta,\theta}}(x)$ for a.e. $x \in X$, and $\delta, \theta \in \Delta$.

Proof. Using Theorem 6.3, $\mathcal{N}_{\delta,\theta}$ is closed for each $\delta, \theta \in \Delta$, and then by applying Proposition 6.9, it is clear that for each $\delta, \theta \in \Delta$ the space $\mathcal{N}_{\delta,\theta}$ is an MI space. Also, the sampling operator T is one-to-one on $\bigcup_{\delta,\theta\in\Delta} \mathcal{N}_{\delta,\theta}$ if and only if T is one-to-one on $\mathcal{N}_{\delta,\theta}$ for each $\delta, \theta \in \Delta$ [107, Proposition 1]. Finally by applying Proposition 6.8, we get the desired result. □

6.3 Application to locally compact groups

To transfer our results to TI spaces in the setting of locally compact groups, we adopt the terminology introduced in Section 5.4 and employ the Zak transform defined therein. Since $\mathcal{Z} : L^2(\mathscr{G}) \to L^2(\widehat{H}, H\backslash\mathscr{G})$ is an unitary operator, we have

$$\begin{aligned} \vartheta(\mathcal{S}^H(\mathscr{A}), \mathcal{S}^H(\mathscr{A}')) &= \sup\left\{ \frac{|\langle u, v\rangle|}{\|u\|\|v\|} : u \in \mathcal{S}^H(\mathscr{A})\backslash\{0\}, v \in \mathcal{S}^H(\mathscr{A}')\backslash\{0\} \right\} \\ &= \sup\left\{ \frac{|\langle \mathcal{Z}u, \mathcal{Z}v\rangle|}{\|\mathcal{Z}u\|\|\mathcal{Z}v\|} : \mathcal{Z}u \in \mathcal{Z}\mathcal{S}^H(\mathscr{A})\backslash\{0\}, \mathcal{Z}v \in \mathcal{Z}\mathcal{S}^H(\mathscr{A}')\backslash\{0\} \right\} \\ &= \vartheta(\mathcal{Z}\mathcal{S}^H(\mathscr{A}), \mathcal{Z}\mathcal{S}^H(\mathscr{A}')). \end{aligned} \tag{6.6}$$

The relation (6.6) of the angle between two MI spaces and the corresponding TI spaces will allow us to have results similar to Theorem 6.3 and Theorem 6.4 for the set up of TI spaces. The next theorem is an application of the Theorem 6.3 to TI spaces. A similar result has been proved for shift-invariant spaces [100, Theorem 3.6].

Following the terminology of Chapter 5, we denote by $J_{\mathcal{A}}$ the range function associated with the TI space $\mathcal{S}^H(\mathscr{A})$, corresponding to the generating set $\mathscr{A} \in L^2(X, \mathcal{H})$.

Theorem 6.11. *Let $\mathscr{A} = \{\varphi_i : i \in I\}$ and $\mathscr{A}' = \{\psi_i : i \in I\}$ be two countable collections of functions in $L^2(\mathscr{G})$. Define $\sigma(\mathcal{S}^H(\mathscr{A})) = \{\alpha \in \widehat{H} : J_{\mathscr{A}}(\alpha) \neq \{0\}\}$ and $\Omega := \sigma(\mathcal{S}^H(\mathscr{A}))\cap$*

$\sigma(\mathcal{S}^H(\mathscr{A}'))$. Then the supremum cosine angle between $\mathcal{S}^H(\mathscr{A})$ and $\mathcal{S}^H(\mathscr{A}')$ is given by

$$\vartheta(\mathcal{S}^H(\mathscr{A}), \mathcal{S}^H(\mathscr{A}')) = \operatorname*{ess-sup}_{\alpha \in \Omega}\{\vartheta(J_{\mathscr{A}}(\alpha), J_{\mathscr{A}'}(\alpha))\}.$$

In addition, if the translation-invariant systems $\mathcal{E}^H(\mathscr{A})$ and $\mathcal{E}^H(\mathscr{A}')$ are frames for $\mathcal{S}^H(\mathscr{A})$ and $\mathcal{S}^H(\mathscr{A}')$, respectively, then

$$\vartheta(\mathcal{S}^H(\mathscr{A}), \mathcal{S}^H(\mathscr{A}')) = \text{ess-sup}_{\alpha \in \Omega} \left\| (G_{\mathscr{A}'}(\alpha)^{\dagger})^{\frac{1}{2}} G_{\mathscr{A},\mathscr{A}'}(\alpha) (G_{\mathscr{A}}(\alpha)^{\dagger})^{\frac{1}{2}} \right\|,$$

where $\dagger$ denotes the pseudoinverse.

Similarly, the next result is an application of Theorem 6.4 to TI spaces.

Theorem 6.12. *In addition to the standing assumptions as in Theorem 6.11, let*

$$\Omega' := \{\alpha \in \Omega : J_{\mathscr{A}}(\alpha) \cap J_{\mathscr{A}'}(\alpha) = \{0\}\} = \Omega \backslash \sigma\left(\mathcal{S}^H(\mathscr{A}) \cap \mathcal{S}^H(\mathscr{A}')\right).$$

Then the following are equivalent:

(i) $\mathcal{S}^H(\mathscr{A})|_{\Omega'} + \mathcal{S}^H(\mathscr{A}')|_{\Omega'}$ is closed.

(ii) $\vartheta(\mathcal{S}^H(\mathscr{A})|_{\Omega'}, \mathcal{S}^H(\mathscr{A}')|_{\Omega'}) < 1$.

(iii) $J_{\mathscr{A}}(\alpha) + J_{\mathscr{A}'}(\alpha)$ is closed for a.e. $\alpha \in \Omega'$.

(iv) $\vartheta(J_{\mathscr{A}}(\alpha), J_{\mathscr{A}'}(\alpha)) < 1$ for a.e. $\alpha \in \Omega'$.

Remark 6.13. In the case of LCA group $\mathcal{G}$ and its closed abelian subgroup Γ, we can state the above theorems using the fiberization map (5.2). For this case, the equation (6.6) will be modified as follows

$$\vartheta(\mathcal{S}^{\Gamma}(\mathscr{A}), \mathcal{S}^{\Gamma}(\mathscr{A}')) = \vartheta(\mathscr{F}\mathcal{S}^{\Gamma}(\mathscr{A}), \mathscr{F}\mathcal{S}^{\Gamma}(\mathscr{A}')).$$

Similar to the case of MI spaces, here also we can talk about the injectivity of the sampling operator associated to TI space $\mathcal{S}^H(\mathscr{A})$. Let us first give some definitions.

Definition 6.14. Let $\mathscr{A} = \{\varphi_i\}_{i \in I} \subset L^2(\mathscr{G})$ be a countable collection such that the H-TI system $\mathcal{E}^H(\mathscr{A})$ is Bessel. Then the *sampling operator* assocated with the system $\mathcal{E}^H(\mathscr{A})$ is defined by

$$T : L^2(\mathscr{G}) \to L^2(H \times I),\ Tf := \{\langle f, L_\xi \varphi_i \rangle\}_{\xi \in H, i \in I}. \tag{6.7}$$

Corresponding to a arbitrary index set Δ, let $\{\mathcal{S}_\delta\}_{\delta\in\Delta}$ be a collection of finitely generated TI spaces in $L^2(\mathscr{G})$. Let

$$\mathcal{X} := \bigcup_{\delta\in\Delta} \mathcal{S}_{\delta,\theta},$$

where $\mathcal{S}_{\delta,\theta} := \mathcal{S}_\delta + \mathcal{S}_\theta = \{u+v : u \in \mathcal{S}_\delta, v \in \mathcal{S}_\theta\}$. The next result is an application of Theorem 6.10 for translation-invariant spaces.

Theorem 6.15. *Let $\mathscr{A} = \{\varphi_i\}_{i\in I}$ be a countable collection in $L^2(\mathscr{G})$ such that $\mathcal{E}^H(\mathscr{A})$ is Bessel and the supremum cosine angle*

$$\vartheta(\mathcal{S}_\delta, \mathcal{S}_\theta) < 1 \textit{ for every } \delta, \theta \in \Delta.$$

If $\mathscr{A}'_{\delta,\theta}$ is a finite set of generators for $\mathcal{S}_{\delta,\theta}$, then the sampling operator associated with $\mathcal{E}^H(\mathscr{A})$ is one-to-one for $\bigcup_{\delta\in\Delta} \mathcal{S}_{\delta,\theta}$ if and only if $\dim(\mathrm{range}(G_{\mathscr{A}'_{\delta,\theta},\mathscr{A}})(\alpha)) = \dim J_{\mathcal{A}'_{\delta,\theta}}(\alpha)$ *for a.e. $\alpha \in \widehat{H}$, and $\delta, \theta \in \Delta$.*

Example 6.16. For a prime number p, let $\mathbb{Q}_p$ be the group of p-adic numbers and $\mathbb{Z}_p$ be its closed subgroup. It is a locally compact abelian group. All its subgroups are compact and open. Let the fundamental domain for $\mathbb{Z}_p$ is Δ which is discrete. The Zak transform for $(\mathbb{Q}_p, \mathbb{Z}_p)$ is $\tilde{\mathcal{Z}} f(x,y) = \int_{\mathbb{Z}_p} f(y+\xi) e^{-2\pi i x\xi}\, d\mu_{\mathbb{Z}_p}(\xi)$ for $x, y \in \Delta$. We can find the supremum cosine angle between two $\mathbb{Z}_p$-invariant subspaces by the Theorem 6.3.

Example 6.17. Let $\mathcal{G} = \mathbb{R}^n$ and $\Lambda = \mathbb{Z}^n$. Then, $\widehat{\mathcal{G}} = \mathbb{R}^n$, $\Lambda^\perp = \mathbb{Z}^n$ and the fundamental domain for $\mathbb{Z}^n$ is $\widehat{\mathcal{G}}\backslash\Lambda^\perp = \mathbb{T}^n$. Then, the fiberization map $\mathscr{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{T}^n; \ell^2(\mathbb{Z}^n))$ is defined by $\mathscr{F} f(\xi) = \{\widehat{f}(\xi + k)\}_{k\in\mathbb{Z}^n}, \xi \in \mathbb{T}^n$. Therefore by considering countable families $\mathscr{A}$ and $\mathscr{A}'$ in $L^2(\mathbb{R}^n)$, we can find supremum cosine angle between $\mathcal{S}^\Lambda(\mathscr{A})$ and $\mathcal{S}^\Lambda(\mathscr{A}')$ and discuss their closedness using Theorem 6.11.

Chapter 7

Stable sampling for generalized twisted shift-invariant spaces

In this chapter, we extend the results of Chapter 5 on multi-channel stable sampling for translation-invariant (TI) spaces to the setting of generalized twisted shift-invariant (GTSI) spaces on the Heisenberg group. Twisted shift-invariant spaces were introduced by Radha and Saswata [116], extending shift-invariant spaces to the non-abelian setting of the Heisenberg group. Using the Weyl–Zak transform [121], we establish necessary and sufficient conditions for a system of generalized twisted translates to form a Riesz sequence or a frame. As a consequence, we obtain reconstruction formulas for functions in GTSI spaces from generalized samples. As a special case, by selecting appropriate filters, we derive a sampling formula for GTSI spaces involving functional values as samples. Finally, we demonstrate the results for the GTSI space generated by a twisted B-spline.

7.1 History and background

A simple and natural example of a non-abelian group is the famous Heisenberg group $\mathbb{H}^n$. It is a nilpotent Lie group whose underlying manifold is $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the group operation defined by $(x, y, t)(u, v, s) = (x + u, y + v, t + s + \frac{1}{2}(u \cdot y - v \cdot x))$ and the Haar measure is the Lebesgue measure $dx\, dy\, dt$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. By using the Schrödinger representation π_λ, $\lambda \in \mathbb{R}^*$, given by

$$\pi_\lambda(x, y, t)\phi(\xi) = e^{2\pi i\lambda t} e^{2\pi i\lambda(x\xi + \frac{1}{2}xy)}\phi(\xi + y),\ \ \phi \in L^2(\mathbb{R}),$$

we define the group Fourier transform of $f \in L^1(\mathbb{H}^n)$ as

$$\widehat{f}(\lambda) = \int_{\mathbb{H}} f(x, y, t)\pi_\lambda(x, y, t)\, dxdydt,\ \text{ where } \lambda \in \mathbb{R}^*,$$

The results presented in this chapter are based on our work:

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which is a bounded operator on $L^2(\mathbb{R}^n)$. It can be extended from $L^1 \cap L^2(\mathbb{H}^n)$ to $L^2(\mathbb{H}^n)$ by using the density argument. Let $f \in L^2(\mathbb{H}^n)$, then we define a function $f^\lambda \in L^1(\mathbb{R}^{2n})$ by $f^\lambda(x,y) = \int_{\mathbb{R}} f(x,y,t)e^{2\pi i\lambda t}\, dt$, which is the inverse Fourier transform of f in the t variable. An essential approach to study the problems on the Heisenberg group is to take the partial Fourier transform in the t variable and reduce the problem to the case of $\mathbb{R}^{2n}$. In fact, for $f, g \in L^1(\mathbb{H}^n)$, the group convolution of f and g is defined by

$$f * g(x,y,t) = \int_{\mathbb{H}^n} f((x,y,t)(u,v,s)^{-1})g(u,v,s)\, dudvds.$$

It is easy to show that $(f * g)^\lambda(x,y) = \int_{\mathbb{R}^{2n}} f^\lambda(x-u, y-v)g^\lambda(u,v)e^{\pi i\lambda(uy-vx)}\, dudv$. This led us to define a new convolution (non-standard convolution) operation on $L^1(\mathbb{R}^{2n})$ in the following way. Let $F, G \in L^1(\mathbb{R}^{2n})$, then we define

$$F *_\lambda G(x,y) = \int_{\mathbb{R}^{2n}} e^{\pi i\lambda(uy-vx)}F(x-u, y-v)G(u,v)\, dudv.$$

These are usually referred as the λ-twisted convolution. When $\lambda = 1$, it is called the twisted convolution. With respect to the twisted convolution, $L^1(\mathbb{R}^{2n})$ turns out to be a non-commutative Banach algebra. For a detailed study of analysis on the Heisenberg group we refer to [142]. One can notice that the twisted convolution involves the non-standard translation (λ-twisted translation) on $\mathbb{R}^{2n}$.

7.1.1 Generalized Weyl-Zak transform

To define generalized Weyl-Zak transform, we first study Weyl-transform. Let $g \in L^1(\mathbb{R}^2)$. Then, the Weyl transform of g, denoted by $W(g)$, is defined by

$$W(g)\phi(\xi) := \int_{\mathbb{R}^2} g(x,y)e^{2\pi i(x\xi + \frac{1}{2}xy)}\phi(\xi + y)\, dxdy,$$

where $\phi \in L^2(\mathbb{R})$. The Weyl transform is an integral operator with kernel

$$K_g(\xi, \eta) = \int_{\mathbb{R}} g(x, \eta - \xi)e^{\pi ix(\xi+\eta)}dx.$$

If $g \in L^1 \cap L^2(\mathbb{R}^2)$, then $K_g \in L^2(\mathbb{R}^2)$, which implies that $W(g)$ is a Hilbert-Schmidt operator whose norm is given by $\|W(g)\|_{\mathcal{B}_2}^2 = \|K_g\|_{L^2(\mathbb{R}^2)}^2$, where $\mathcal{B}_2$ is the Hilbert space of Hilbert-Schmidt operators on $L^2(\mathbb{R})$ with inner product $(T, S) = \operatorname{tr}(TS^*)$. The Plancherel formula for the Weyl transform is given by $\|W(g)\|_{\mathcal{B}_2} = \|g\|_{L^2(\mathbb{R})}$. For the study of the Weyl transform, we refer to [54, 142].

Now, we introduce the generalized Weyl-Zak transform which will be used to study a GTSI space. It is denoted by Z_W^m. When $m = 1$, we shall denote Z_W^1 by Z_W. It was introduced and referred as the Weyl-Zak transform in [121]. The map $Z_W^m : L^2(\mathbb{R}^2) \to L^2([0, \frac{1}{m}) \times [0, m) \times \mathbb{R})$ is defined by

$$Z_W^m \phi(\xi, \xi', \eta) = \sum_{p \in \mathbb{Z}} \frac{1}{\sqrt{m}} K_\phi(\xi + \frac{p}{m}, \eta) e^{-2\pi i \frac{p}{m} \xi'}.$$

Let $\phi \in L^2(\mathbb{R}^2)$. Then by using Fubini's theorem, we have

$$\begin{aligned}
\|Z_W^m \phi\|^2_{L^2([0,\frac{1}{m}) \times [0,m) \times \mathbb{R})} &= \frac{1}{m} \int_0^{\frac{1}{m}} \int_{\mathbb{R}} \int_0^m \left| \sum_{p \in \mathbb{Z}} K_\phi(\xi + \frac{p}{m}, \eta) e^{-2\pi i \frac{p}{m} \xi'} \right|^2 d\xi' d\eta d\xi \\
&= \frac{1}{m} \int_0^{\frac{1}{m}} \int_{\mathbb{R}} \int_0^m \sum_{p \in \mathbb{Z}} K_\phi(\xi + \frac{p}{m}, \eta) e^{-2\pi i \frac{p}{m} \xi'} \overline{\sum_{p' \in \mathbb{Z}} K_\phi(\xi + \frac{p'}{m}, \eta) e^{-2\pi i \frac{p'}{m} \xi'}} \, d\xi' d\eta d\xi \\
&= \frac{1}{m} \int_0^{\frac{1}{m}} \int_{\mathbb{R}} \sum_{p, p' \in \mathbb{Z}} K_\phi(\xi + \frac{p}{m}, \eta) \overline{K_\phi(\xi + \frac{p'}{m}, \eta)} \int_0^m e^{-2\pi i \frac{(p - p')}{m} \xi'} \, d\xi' d\eta d\xi \\
&= \int_0^{\frac{1}{m}} \int_{\mathbb{R}} \sum_{p \in \mathbb{Z}} K_\phi(\xi + \frac{p}{m}, \eta) \overline{K_\phi(\xi + \frac{p}{m}, \eta)} \, d\eta d\xi \\
&= \int_0^{\frac{1}{m}} \int_{\mathbb{R}} \sum_{p \in \mathbb{Z}} |K_\phi(\xi + \frac{p}{m}, \eta)|^2 \, d\eta d\xi \\
&= \|K_\phi\|^2_{L^2(\mathbb{R}^2)} = \|\phi\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Now, we aim to show that Z_W^m is surjective. Let $F \in L^2([0, \frac{1}{m}) \times [0, m) \times \mathbb{R})$. Then the Fourier series of F with respect to the ξ' variable is given by

$$F(\xi, \xi', \eta) = \sum_{p \in \mathbb{Z}} a^{\xi, \eta}_{\frac{p}{m}} e^{2\pi i \frac{p}{m} \xi'}, \text{ for some } \left\{ a^{\xi, \eta}_{\frac{p}{m}} \right\}_{p \in \mathbb{Z}} \in \ell^2 \left(\frac{\mathbb{Z}}{m} \right).$$

For each $\xi \in \mathbb{R}$, there exist unique elements $p \in \mathbb{Z}$ and $\gamma \in [0, \frac{1}{m})$ such that $\xi = \gamma + \frac{p}{m}$. Using this representation, we define a function K on $\mathbb{R}^2$ by $K(\xi, \eta) = a^{\gamma, \eta}_{\frac{p}{m}}$. Now, by making use of the Plancherel theorem for the Fourier series, we have

$$\begin{aligned}
\int_{\mathbb{R}} \int_{\mathbb{R}} |K(\xi, \eta)|^2 \, d\xi \, d\eta &= \int_{\mathbb{R}} \int_0^{\frac{1}{m}} \sum_{p \in \mathbb{Z}} |K(\gamma + \frac{p}{m}, \eta)|^2 \, d\gamma \, d\eta \\
&= \int_{\mathbb{R}} \int_0^{\frac{1}{m}} \sum_{p \in \mathbb{Z}} |a^{\gamma, \eta}_{\frac{p}{m}}|^2 \, d\gamma \, d\eta \\
&= \int_{\mathbb{R}} \int_0^{\frac{1}{m}} \int_0^m |F(\gamma, \xi', \eta)|^2 \, d\xi' \, d\gamma \, d\eta < \infty.
\end{aligned}$$

Since the Weyl transform W is onto, there exists $f \in L^2(\mathbb{R}^2)$ such that $W(f) = K$, which is a Hilbert Schmidt operator with the kernel $K \in L^2(\mathbb{R}^2)$. Moreover, we can show that $Z_W^m(f) = F$. Therefore Z_W^m is an isometric isomorphism between $L^2(\mathbb{R}^2)$ and $L^2([0, \frac{1}{m}) \times [0, m) \times \mathbb{R})$.

Given two functions $\phi, \psi \in L^2(\mathbb{R}^2)$, we define the bracket $[\phi, \psi]_m$ by

$$[\phi, \psi]_m (\xi, \xi^{'}) = \int_{\mathbb{R}} Z_W^m \phi(\xi, \xi^{'}, \eta) \overline{Z_W^m \psi(\xi, \xi^{'}, \eta)} \, d\eta \quad \text{for a.e. } (\xi, \xi^{'}) \in [0, \frac{1}{m}) \times [0, m).$$

In fact, by using Cauchy-Schwartz inequality, we get

$$|\, [\phi, \psi]_m (\xi, \xi^{'})| \leq \left(\int_{\mathbb{R}} |Z_W^m \phi(\xi, \xi^{'}, \eta)|^2 d\eta \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |Z_W^m \psi(\xi, \xi^{'}, \eta)|^2 d\eta \right)^{\frac{1}{2}} < \infty, \tag{7.1}$$

Further,

$$[\phi, \phi]_m (\xi, \xi^{'}) = \int_{\mathbb{R}} |Z_W^m \phi(\xi, \xi^{'}, \eta)|^2 \, d\eta. \tag{7.2}$$

Now, consider

$$\begin{aligned}
\|\, [\phi, \psi]_m \|_{L^1([0, \frac{1}{m}) \times [0, m))} &\leq \int_0^{\frac{1}{m}} \int_0^m \left(\int_{\mathbb{R}} |Z_W^m \phi(\xi, \xi^{'}, \eta)|^2 d\eta \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |Z_W^m \psi(\xi, \xi^{'}, \eta)|^2 d\eta \right)^{\frac{1}{2}} d\xi^{'} d\xi \\
&= \left(\int_0^{\frac{1}{m}} \int_0^m \int_{\mathbb{R}} |Z_W^m \phi(\xi, \xi^{'}, \eta)|^2 \, d\eta d\xi^{'} d\xi \right)^{\frac{1}{2}} \left(\int_0^{\frac{1}{m}} \int_0^m \int_{\mathbb{R}} |Z_W^m \psi(\xi, \xi^{'}, \eta)|^2 \, d\eta d\xi^{'} d\xi \right)^{\frac{1}{2}} \\
&= \|Z_W^m \phi\|_{L^2(W_m \times \mathbb{R})} \|Z_W^m \psi\|_{L^2(W_m \times \mathbb{R})} \\
&= \|\phi\|_{L^2(\mathbb{R}^2)} \|\psi\|_{L^2(\mathbb{R}^2)},
\end{aligned}$$

by applying Cauchy-Schwarz inequality and using (7.1).

Hereafter, we shall denote $[\phi, \phi]_m$ by ω_ϕ^m and $[0, \frac{1}{m}) \times [0, m)$ by W_m. Recall that $E_{k,l}^m(\xi, \xi^{'}) := e^{2\pi i k m \xi} e^{2\pi i \frac{l}{m} \xi^{'}}$, for $(\xi, \xi^{'}) \in W_m$.

7.2 Generalized twisted shift-invariant spaces

We begin by recalling the definition of twisted shift-invariant spaces. A closed subspace V of $L^2(\mathbb{R}^{2n})$ is said to be a *twisted shift-invariant* (TSI) space if $f \in V$, then $T_{(k,l)}^t f \in V$ for $k, \ell \in \mathbb{Z}^n$. Here $T_{(k,l)}^t$ denotes the twisted translation given by

$$T_{(k,l)}^t f(x, y) = e^{\pi i (x \cdot l - y \cdot k)} f(x - k, y - l)$$

for $x, y \in \mathbb{R}^n,\ k, l \in \mathbb{Z}^n$.

In this chapter, we assume that functions belong to the GTSI space of the form

$$V_m^t(\phi) := \overline{\operatorname{span}}\{T^t_{(mk,\frac{l}{m})}\phi : k,l \in \mathbb{Z}\},$$

for some $\phi \in L^2(\mathbb{R}^{2n})$ and $m \in \mathbb{N}$. The main objective of this chapter is to identify the conditions under which any function $f \in V_m^t(\phi)$ can be recovered from the samples $\{\langle f, T^t_{(mk,\frac{l}{m})}\eta_j\rangle : (k,l) \in \mathbb{Z}^2, j = 1,\ldots,s\}$, for some suitable collection of filters $\{\eta_j\}_{j=1}^s \subset V_m^t(\phi)$, through a sampling formula. This falls within the framework of generalized sampling, where the aim is to reconstruct a function using a sampling formula based on the samples $\{\langle f, T^t_{(mk,\frac{l}{m})}\eta_j\rangle : (k,l) \in \mathbb{Z}^2, j = 1,2,\ldots,s\}$. This idea traces back to the work of Papoulis [113], which presents a sampling formula for reconstructing a bandlimited function $\{\langle f, T^t_{(mk,\frac{l}{m})}\eta_j\rangle : (k,l) \in \mathbb{Z}^2, j = 1,2,\ldots,s\}$. Papoulis' result was later extended to general shift-invariant spaces using filter bank techniques (see, e.g., [48, 59, 63, 118, 144]).

We now formally define GTSI spaces.

Definition 7.1. [120] Let $\phi \in L^2(\mathbb{R}^2)$. For $(k,l) \in \mathbb{Z}^2$ and $m \in \mathbb{N}$, the *generalized twisted translation* $T^t_{(mk,\frac{l}{m})}\phi$ of ϕ, is defined by

$$T^t_{(mk,\frac{l}{m})}\phi(x,y) := e^{\pi i(x\frac{l}{m} - ymk)}\phi(x-k, y-l), \quad (x,y) \in \mathbb{R}^2.$$

Using the definition of generalized twisted translation, we have

$$T^t_{(mk_1,\frac{l_1}{m})}T^t_{(mk_2,\frac{l_2}{m})} = e^{-\pi i(k_1 l_2 - l_1 k_2)}T^t_{(m(k_1+k_2),\frac{l_1+l_2}{m})}, \qquad \forall\ (k_1,l_1),(k_2,l_2) \in \mathbb{Z}^2. \tag{7.3}$$

A closed subspace $V \subset L^2(\mathbb{R}^2)$ is said to be a *generalized twisted shift-invariant space* (GTSI) if $f \in V$, then $T^t_{(mk,\frac{l}{m})}f \in V$, for any $k,l \in \mathbb{Z}$, where $T^t_{(mk,\frac{l}{m})}f$ denotes the generalized twisted translation of f.

We begin with some auxiliary results which are required in the sequel.

Theorem 7.2. *[116] Let $\phi \in L^2(\mathbb{R}^2)$. Suppose $\{T^t_{(mk,\frac{l}{m})}\phi : k,l \in \mathbb{Z}\}$ is a frame for $V_m^t(\phi)$ with frame operator S. Then $S^{-1}T^t_{(mk,\frac{l}{m})} = T^t_{(mk,\frac{l}{m})}S^{-1}$.*

Remark 7.3. In this chapter, we consider the orthonormal basis $\{e^{2\pi i(mk\xi + \frac{l}{m}\xi')}e^{\pi ikl} : k,l \in \mathbb{Z}\}$ in $L^2([0,\frac{1}{m}) \times [0,m))$. With respect to this system, the Plancherel theorem is given by

$$\int_0^{\frac{1}{m}}\int_0^m |f(x,y)|^2\,dxdy = \sum_{k,l\in\mathbb{Z}} |\langle f, E^m_{k,l}\rangle|^2, \quad f \in L^2([0,\frac{1}{m}) \times [0,m)),$$

where $E^m_{k,l}(\xi,\xi^{'}) = e^{2\pi i(mk\xi+\frac{l}{m}\xi^{'})}e^{\pi ikl}$.

Lemma 7.4. *Let* $\{c_{mk,\frac{l}{m}}\}_{k,l\in\mathbb{Z}}$, $\{d_{mk,\frac{l}{m}}\}_{k,l\in\mathbb{Z}}$ *be the Fourier coefficients of* $f,g \in L^2([0,\frac{1}{m})\times[0,m))$ *with respect to the orthonormal system* $\{E^m_{k,l}e^{\pi ikl} : k,l \in \mathbb{Z}\}$. *Then*

$$\int_0^{\frac{1}{m}}\int_0^m |f(\xi,\xi^{'})g(\xi,\xi^{'})|^2\,d\xi d\xi^{'} = \sum_{k,l\in\mathbb{Z}}\left|\sum_{p,q\in\mathbb{Z}} c_{mp,\frac{q}{m}}d_{m(k-p),\frac{(l-q)}{m}}\,e^{\pi i(pl-kq)}e^{\pi ikl}\right|^2. \tag{7.4}$$

Proof. The Fourier series of f,g with respect to the system $\{E^m_{k,l}e^{\pi ikl} : k,l \in \mathbb{Z}\}$ is given by

$$f = \sum_{k,l\in\mathbb{Z}} c_{mk,\frac{l}{m}}e^{\pi ikl}E^m_{k,l} \quad \text{and} \quad g = \sum_{k,l\in\mathbb{Z}} d_{mk,\frac{l}{m}}e^{\pi ikl}E^m_{k,l}.$$

Let us define $c = \{c_{mk,\frac{l}{m}}e^{\pi ikl}\}_{k,l\in\mathbb{Z}}$ and $d = \{d_{mk,\frac{l}{m}}e^{\pi ikl}\}_{k,l\in\mathbb{Z}}$. Then, for $k,l \in \mathbb{Z}$, we have $(c*d)(mk,\frac{l}{m}) = \sum_{p,q\in\mathbb{Z}} c_{mp,\frac{q}{m}}e^{\pi ipq}d_{m(k-p),\frac{(l-q)}{m}}e^{\pi i(k-p)(l-q)}$, , where $*$ denotes the usual convolution on $L^2(\mathbb{R}^2)$. Assume L.H.S of (7.4) or R.H.S of (7.4) is finite. By making use of the Parseval's identity with respect to the orthonormal system $\{E^m_{k,l} : k,l \in \mathbb{Z}\}$, we obtain our required result. □

Remark 7.5. Let $f \in L^2(W_m)$, which can be extended to a function on $\mathbb{R}^2$ and $\tau_{(u,v)}$ be a translation operator defined by $\tau_{(u,v)}f(x,y) = f(x-u,y-v)$ for $u,v \in \mathbb{R}$. Then the $(mk,\frac{l}{m})^{th}$ Fourier coefficient of $\tau_{(u,v)}f$ can be computed as

$$\widehat{\tau_{(u,v)}f}(mk,\frac{l}{m}) = e^{2\pi i(umk+\frac{vl}{m})}\widehat{f}(mk,\frac{l}{m}).$$

Proposition 7.6. *Let* $\phi \in L^2(\mathbb{R}^2)$. *Then the generalized Weyl-Zak transform of* $T^t_{(mk,\frac{l}{m})}\phi$ *is given by*

$$Z^m_W T^t_{(mk,\frac{l}{m})}\phi(\xi,\xi^{'},\eta) = E^m_{k,l}(\xi,\xi^{'})e^{\pi ikl}Z^m_W\phi(\xi,\xi^{'},\eta), \quad k,l\in\mathbb{Z},\ (\xi,\xi^{'})\in W_m,\ \eta\in\mathbb{R}. \tag{7.5}$$

Proof. Now, we shall compute the kernel of the Weyl transform of $T^t_{(mk,\frac{l}{m})}\phi$. Consider

$$\begin{aligned} K_{T^t_{(mk,\frac{l}{m})}\phi}(\xi,\xi^{'}) &= \int_{\mathbb{R}} T^t_{(mk,\frac{l}{m})}\phi(x,\eta-\xi)e^{\pi ix(\eta+\xi)}\,dx \\ &= \int_{\mathbb{R}} e^{\pi i(\frac{xl}{m}-(\eta-\xi)mk)}\phi(x-mk,\eta-\xi-\frac{l}{m})e^{\pi ix(\eta+\xi)}\,dx \\ &= e^{\pi ikl}e^{2\pi imk\xi}\int_{\mathbb{R}}\phi(x,\eta-(\xi+\frac{l}{m}))e^{\pi ix(\eta+\xi+\frac{l}{m})}\,dx \\ &= e^{\pi ikl}e^{2\pi imk\xi}K_\phi(\xi+\frac{l}{m},\eta) \end{aligned} \tag{7.6}$$

By applying (7.6) in the definition of Z_W^n transform, we obtain

$$\begin{aligned} Z_W^m T^t_{(mk,\frac{l}{m})}\phi(\xi,\xi^{'},\eta) &= \sum_{p\in\mathbb{Z}} \frac{1}{\sqrt{m}} K_{T^t_{(mk,\frac{l}{m})}\phi}(\xi+\frac{p}{m},\eta)e^{-2\pi i\frac{p}{m}\xi^{'}} \\ &= \sum_{p\in\mathbb{Z}} \frac{1}{\sqrt{m}} e^{\pi ikl}e^{2\pi imk(\xi+\frac{p}{m})}K_\phi(\xi+\frac{l}{m}+\frac{p}{m},\eta)e^{-2\pi i\frac{p}{m}\xi^{'}} \\ &= e^{\pi ikl}e^{2\pi imk\xi}e^{2\pi i\frac{l}{m}\xi^{'}}\sum_{p\in\mathbb{Z}} \frac{1}{\sqrt{m}} K_\phi(\xi+\frac{p}{m},\eta)e^{-2\pi i\frac{p}{m}\xi^{'}} \\ &= e^{\pi ikl}E^m_{k,l}(\xi,\xi^{'})Z_W^m\phi(\xi,\xi^{'},\eta), \end{aligned}$$

proving our assertion. □

The following proposition characterizes the elements of GTSI spaces in terms of their generalized Weyl–Zak transform. Specifically, any $f \in V_m^t(\phi)$ can be expressed through a multiplier r_f acting on $Z_W^m\phi$, with an explicit formula for r_f in terms of the expansion coefficients of f.

Proposition 7.7. *Let $\phi \in L^2(\mathbb{R}^2)$. Then $f \in V_m^t(\phi)$ if and only if*

$$Z_W^m f(\xi,\xi^{'},\eta) = r_f(\xi,\xi^{'})Z_W^m\phi(\xi,\xi^{'},\eta), \ \textit{for a.e.}\ (\xi,\xi^{'}) \in W_m, \eta\in\mathbb{R},$$

for some $r_f \in L^2(W_m;\omega_\phi^m)$. The explicit form of r_f is given by

$$r_f = \sum_{k,l\in\mathbb{Z}} c_{mk,\frac{l}{m}}E^m_{k,l}e^{\pi ikl}, \quad \textit{for}\ f = \sum_{k,l\in\mathbb{Z}} c_{mk,\frac{l}{m}}T^t_{mk,\frac{l}{m}}\phi.$$

Proof. Let $f \in \operatorname{span}\{T^t_{(mk,\frac{l}{m})}\phi : k,l\in\mathbb{Z}\}$, and let $c_{0,0}(\mathbb{Z}^2)$ denote the space of all sequences in $\ell^2(\mathbb{Z}^2)$ that are eventually zero. Then, for some $\{a_{mk,\frac{l}{m}}\}_{k,l\in\mathbb{Z}} \in c_{0,0}(\mathbb{Z}^2)$, we have

$$f = \sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}}T^t_{(mk,\frac{l}{m})}\phi.$$

By applying Proposition 7.6 in the definition of Z_W^m, we get

$$\begin{aligned} Z_W^m f(\xi,\xi^{'},\eta) &= \sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} Z_W^m T^t_{(mk,\frac{l}{m})}\phi(\xi,\xi^{'},\eta) \\ &= \sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} e^{2\pi i(mk\xi+\frac{l}{m}\xi^{'})}e^{\pi ikl}Z_W^m\phi(\xi,\xi^{'},\eta) \\ &= \left(\sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} e^{2\pi i(mk\xi+\frac{l}{m}\xi^{'})}e^{\pi ikl}\right)Z_W^m\phi(\xi,\xi^{'},\eta) \\ &= r_f(\xi,\xi^{'})Z_W^m\phi(\xi,\xi^{'},\eta). \end{aligned} \tag{7.7}$$

Then by the isometry of Z_W^m, we have

$$\begin{aligned}\|f\|_{L^2(\mathbb{R}^2)}^2 &= \|Z_W^m f\|_{L^2(W_m\times\mathbb{R})}^2\\ &= \int_{W_m} |r_f(\xi,\xi^{'})|^2 \int_{\mathbb{R}} |Z_W^m\phi(\xi,\xi^{'},\eta)|^2\, d\eta\, d\xi^{'}\, d\xi\\ &= \int_{W_m} |r_f(\xi,\xi^{'})|^2\, [\phi,\phi]_m\,(\xi,\xi^{'})\, d\xi^{'}\, d\xi\\ &= \|r_f\|_{L^2(W_m;\omega_\phi^m)}^2. \qquad (7.8)\end{aligned}$$

Thus the map $f\mapsto r_f$ is an isometric isomorphism of $\operatorname{span}\{T^t_{(mk,\frac{l}{m})}\phi : k,l\in\mathbb{Z}\}$ into the collection of all trigonometric polynomials in $L^2(W_m;\omega_\phi^m)$. The surjectivity follows from retracing the steps as in (7.7). Moreover, the map $f\mapsto r_f$ can be extended from $V_m^t(\phi)$ onto $L^2(W_m;\omega_\phi^m)$, which gives an isometric isomorphism between them. □

The following result characterizes when the system of twisted translates $\{\mathcal{T}^t_{k,l}\phi : k,l\in\mathbb{Z}\}$ forms a frame sequence for $V_m^t(\phi)$ in terms of the essential bounds of the associated weight function ω_ϕ^m on its support.

Theorem 7.8. *Let $\phi\in L^2(\mathbb{R}^2)$. Then $\{T^t_{(mk,\frac{l}{m})}\phi : k,l\in\mathbb{Z}\}$ is a frame sequence for $V_m^t(\phi)$ if and only if*

$$0<A\leq \omega_\phi^m(\xi,\xi^{'})\leq B<\infty \ \ a.e\ (\xi,\xi^{'})\in\Omega_\phi^m, \qquad (7.9)$$

where $\Omega_\phi^m=\{(\xi,\xi^{'})\in W_m : \omega_\phi^m(\xi,\xi^{'})\neq 0\}$.

Proof. Let $f\in V_m^t(\phi)$. Now, by applying Proposition 7.6 and Proposition 7.7, we obtain

$$\begin{aligned}\langle f, T^t_{(mk,\frac{l}{m})}\phi\rangle_{L^2(\mathbb{R}^2)} &= \langle Z_W^m f, Z_W^m T^t_{(mk,\frac{l}{m})}\phi\rangle_{L^2(W_n\times\mathbb{R})}\\ &= \int_{W_m}\int_{\mathbb{R}} r_f(\xi,\xi^{'}) Z_W^m\phi(\xi,\xi^{'},\eta) e^{-2\pi i(mk\xi+\frac{l}{m}\xi^{'})} e^{-\pi ikl}\overline{Z_W^m\phi(\xi,\xi^{'},\eta)}\, d\eta d\xi^{'} d\xi.\end{aligned}$$

Thus

$$\langle f, T^t_{(mk,\frac{l}{m})}\phi\rangle_{L^2(\mathbb{R}^2)} = \int_{W_m} r_f(\xi,\xi^{'})\omega_\phi^m(\xi,\xi^{'}) E^m_{-k,-l}(\xi,\xi^{'}) e^{-\pi ikl}\, d\xi^{'} d\xi.$$

Now, by using Remark 7.3, we have

$$\begin{aligned}\sum_{k,l\in\mathbb{Z}} |\langle f, T^t_{(mk,\frac{l}{m})}\phi\rangle_{L^2(\mathbb{R}^2)}|^2 &= \sum_{k,l\in\mathbb{Z}}\left|\int_{W_m} r_f(\xi,\xi^{'})\omega_\phi^m(\xi,\xi^{'}) E^m_{-k,-l}(\xi,\xi^{'}) e^{-\pi ikl}\, d\xi^{'} d\xi\right|^2\\ &= \|r_f\omega_\phi^m\|_{L^2(W_m)}^2\\ &= \int_{W_m} |r_f(\xi,\xi^{'})\omega_\phi^m(\xi,\xi^{'})|^2\, d\xi^{'} d\xi. \qquad (7.10)\end{aligned}$$

Assume that the system $\{T^t_{(mk,\frac{l}{m})}\phi : k, l \in \mathbb{Z}\}$ is a frame sequence for $V^t_m(\phi)$, then there exist $A, B > 0$ such that

$$A\|f\|^2_{L^2(\mathbb{R}^2)} \leq \sum_{k,l\in\mathbb{Z}} |\langle f, T^t_{(mk,\frac{l}{m})}\phi\rangle_{L^2(\mathbb{R}^2)}|^2 \leq B\|f\|^2_{L^2(\mathbb{R}^2)}, \quad \forall\ f \in V^t_m(\phi). \tag{7.11}$$

Substituting (7.8) and (7.10) in (7.11), we get

$$A\|r_f\|^2_{L^2(W_m;\omega^m_\phi)} \leq \int_{W_m} |r_f(\xi,\xi^{'})\omega^m_\phi(\xi,\xi^{'})|^2\, d\xi^{'} d\xi, \tag{7.12}$$

and

$$\int_{W_m} |r_f(\xi,\xi^{'})\omega^m_\phi(\xi,\xi^{'})|^2\, d\xi^{'} d\xi \leq B\|r_f\|^2_{L^2(W_m;\omega^m_\phi)}. \tag{7.13}$$

Let $N := \{(\xi,\xi^{'}) \in \Omega^m_\phi : \omega^m_\phi(\xi,\xi^{'}) - B > 0\}$ be a measurable subset of W_m. By choosing $f \in V^t_m(\phi)$ such that $r_f = \chi_N$ in (7.13), we obtain

$$\iint\limits_N \omega^m_\phi(\xi,\xi^{'})(\omega^m_\phi(\xi,\xi^{'}) - B)\, d\xi^{'} d\xi = \iint\limits_{\Omega^m_\phi} |r_f(\xi,\xi^{'})|^2\omega^m_\phi(\xi,\xi^{'})(\omega^m_\phi(\xi,\xi^{'}) - B)\, d\xi^{'} d\xi \leq 0,$$

which implies that $\mu(N) = 0$, where μ is the Lebesgue measure $d\xi^{'}\, d\xi$ on W_m. Similarly, we can show that $A \leq \omega^m_\phi(\xi,\xi^{'})$ for a.e $(\xi,\xi^{'}) \in \Omega^m_\phi$, which proves our assertion. Conversely, assume that (7.9) holds. Then we have

$$\iint\limits_{\Omega^m_\phi} |r_f(\xi,\xi^{'})|^2\omega^m_\phi(\xi,\xi^{'})(A - \omega^m_\phi(\xi,\xi^{'}))\, d\xi^{'}\, d\xi \leq 0,$$

and

$$\iint\limits_{\Omega^m_\phi} |r_f(\xi,\xi^{'})|^2\omega^m_\phi(\xi,\xi^{'})(\omega^m_\phi(\xi,\xi^{'}) - B)\, d\xi^{'}\, d\xi \leq 0.$$

Now retracing the steps back, we get $\{T^t_{(mk,\frac{l}{m})}\phi : k, l \in \mathbb{Z}\}$ is a frame sequence with bounds A and B. □

Remark 7.9. The above results (Proposition 7.6, Proposition 7.7, Theorem 7.8) were proved for the case $m = 1$ in [121].

Corollary 7.10. *Let $\phi \in L^2(\mathbb{R}^2)$. Then $\{T^t_{(mk,\frac{l}{m})}\phi : k, l \in \mathbb{Z}\}$ is a Bessel sequence in $V^t_m(\phi)$ if and only if*

$$\omega^m_\phi(\xi,\xi^{'}) \leq B < \infty \ a.e\ (\xi,\xi^{'}) \in \Omega^m_\phi,$$

where $\Omega^m_\phi = \{(\xi,\xi^{'}) \in W_m : \omega^m_\phi(\xi,\xi^{'}) \neq 0\}$.

The following result characterizes the canonical dual of a twisted shift-invariant frame system.

Theorem 7.11. *Let* $\phi \in L^2(\mathbb{R}^2)$ *be such that the family* $\{T^t_{(mk,\frac{l}{m})}\phi : k,l \in \mathbb{Z}\}$ *is a frame for* $V^t_m(\phi)$*. Define* $\tilde{\phi} \in V^t_m(\phi)$ *by*

$$Z^m_W\tilde{\phi}(\xi,\xi^{'}\eta) = \begin{cases} \frac{1}{\omega^m_\phi(\xi,\xi^{'})} Z^m_W\phi(\xi,\xi^{'},\eta) & \textit{if } \ (\xi,\xi^{'}) \in \Omega^m_\phi \\ 0 & \textit{otherwise.} \end{cases} \tag{7.14}$$

Then $\{T^t_{(mk,\frac{l}{m})}\tilde{\phi} : k,l \in \mathbb{Z}\}$ *is the canonical dual frame for* $V^t_m(\phi)$*. Moreover, we have* $\Omega^m_{\tilde{\phi}} = \Omega^m_\phi$*.*

Proof. The canonical dual frame for the system $\{T^t_{(mk,\frac{l}{m})}\phi : k,l \in \mathbb{Z}\}$ is given by $\{T^t_{(mk,\frac{l}{m})}S^{-1}\phi : k,l \in \mathbb{Z}\}$ (see Theorem 7.2). It remains to show that $\tilde{\phi} = S^{-1}\phi$. Now, by using Proposition 7.6, we obtain

$$\begin{aligned}
\langle\tilde{\phi}, T^t_{(mk,\frac{l}{m})}\phi\rangle &= \langle Z^m_W\tilde{\phi}, Z^m_W T^t_{(mk,\frac{l}{m})}\psi\rangle \\
&= \int_{W_m}\int_{\mathbb{R}} Z^m_W\tilde{\phi}(\xi,\xi^{'},\eta)\overline{Z^m_W T^t_{(mk,\frac{l}{m})}\phi(\xi,\xi^{'},\eta)}\, d\eta d\xi d\xi^{'} \\
&= \iint_{\Omega^m_\phi}\int_{\mathbb{R}} \frac{1}{\omega^m_\phi(\xi,\xi^{'})} Z^m_W\phi(\xi,\xi^{'},\eta)\overline{Z^m_W\phi(\xi,\xi^{'},\eta)} e^{-2\pi i(mk\xi+\frac{l}{m}\xi^{'})}e^{-\pi ikl}\, d\eta d\xi d\xi^{'} \\
&= \iint_{\Omega^m_\phi}\left(\frac{1}{\omega^m_\phi(\xi,\xi^{'})}\int_{\mathbb{R}}|Z^m_W\phi(\xi,\xi^{'},\eta)|^2\, d\eta\right)E^m_{-k,-l}(\xi,\xi^{'})e^{-\pi ikl}d\xi d\xi^{'} \\
&= \int_{W_m}\chi_{\Omega^m_\phi}(\xi,\xi^{'})E^m_{-k,-l}(\xi,\xi^{'})e^{-\pi ikl}\, d\eta d\xi d\xi^{'} \\
&= \widehat{(\chi_{\Omega^m_\phi})}(k,l).
\end{aligned}$$

The frame operator corresponding to the system $\{T^t_{(mk,\frac{l}{m})}\phi : k,l \in \mathbb{Z}\}$ is given by

$$Sf = \sum_{k,l\in\mathbb{Z}}\langle f, T^t_{(mk,\frac{l}{m})}\phi\rangle T^t_{(mk,\frac{l}{m})}\phi, \text{ for } f \in V^t_m(\phi).$$

In particular, by choosing $f = \tilde{\phi}$, we have $S\tilde{\phi} = \sum_{k,l\in\mathbb{Z}}\langle\tilde{\phi}, T^t_{(mk,\frac{l}{m})}\phi\rangle T^t_{(mk,\frac{l}{m})}\phi$. Again, by making use of Proposition 7.6, we obtain

$$Z^m_W(S\tilde{\phi})(\xi,\xi^{'},\eta) = \left(\sum_{k,l\in\mathbb{Z}}\widehat{(\chi_{\Omega^m_\phi})}(mk,\frac{l}{m})E^m_{k,l}(\xi,\xi^{'})e^{\pi ikl}\right)Z^m_W\phi(\xi,\xi^{'},\eta), \quad \text{for a.e. } \ \xi,\xi^{'} \in W_m, \eta \in \mathbb{R},$$

which implies that

$$Z^m_W(S\tilde{\phi})(\xi,\xi^{'},\eta) = \chi_{\Omega^m_\phi}(\xi,\xi^{'})Z^m_W\phi(\xi,\xi^{'},\eta), \quad \text{for a.e. } \ \xi,\xi^{'} \in W_m, \eta \in \mathbb{R}.$$

However, it can be shown that $\chi_{\Omega_\phi^m}(\xi,\xi^{'})Z_W^m\phi(\xi,\xi^{'},\eta) = Z_W^m\phi(\xi,\xi^{'},\eta)$. This leads to the fact that $Z_W^m(S\tilde{\phi}) = Z_W^m\phi$, proving our assertion. $\square$

We now focus on Riesz sequences in twisted shift-invariant spaces. The following result gives equivalent conditions under which the system $\{T^t_{(mk,\frac{l}{m})}\phi : k,l \in \mathbb{Z}\}$ forms a Riesz sequence for $V_m^t(\phi)$.

Theorem 7.12. *Let $\phi \in L^2(\mathbb{R}^2)$. Then the following conditions hold:*

(i) The system $\{T^t_{(mk,\frac{l}{m})}\phi : k,l \in \mathbb{Z}\}$ is a Riesz sequence for $V_m^t(\phi)$.

(ii) $0 < A \leq \omega_\phi^m(\xi,\xi^{'}) \leq B < \infty$ a.e $(\xi,\xi^{'}) \in W_n$.

(iii) The system $\{e^{\pi ikl}E^m_{k,l} : k,l \in \mathbb{Z}\}$ is a Riesz sequence for $L^2(W_m;\omega_\phi^m)$.

Proof. (i) $\Longleftrightarrow$ (ii) Assume (i) holds. Let $\{a_{mk,\frac{l}{m}}\} \in \ell^2(m\mathbb{Z} \times \frac{\mathbb{Z}}{m})$. By using Proposition (7.6), we get

$$\begin{aligned}
\left\|\sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}}T^t_{(mk,\frac{l}{m})}\phi\right\|^2_{L^2(\mathbb{R}^2)} &= \left\|Z_W^m\left(\sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}}T^t_{(mk,\frac{l}{m})}\phi\right)\right\|^2_{L^2(W_m\times\mathbb{R})}\\
&= \left\|\sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}}Z_W^mT^t_{(mk,\frac{l}{m})}\phi\right\|^2_{L^2(W_m\times\mathbb{R})}\\
&= \int_{W_m}\int_{\mathbb{R}}\left|\sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}}e^{2\pi i(mk\xi+\frac{l}{m}\xi^{'})}e^{\pi ikl}Z_W^m\phi(\xi,\xi^{'},\eta)\right|^2 d\eta d\xi^{'} d\xi\\
&= \int_{W_m}\left|\sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}}E^m_{k,l}(\xi,\xi^{'})e^{\pi ikl}\right|^2\int_{\mathbb{R}}\left|Z_W^m\phi(\xi,\xi^{'},\eta)\right|^2 d\eta d\xi^{'} d\xi\\
&= \int_{W_m}\left|\sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}}E^m_{k,l}(\xi,\xi^{'})e^{\pi ikl}\right|^2\omega_\phi^m(\xi,\xi^{'})d\xi^{'}d\xi. \qquad (7.15)
\end{aligned}$$

Let $p \in L^2(W_m)$. Then the Fourier series representation of p is given by

$$p = \sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}}E^m_{k,l}e^{\pi ikl}.$$

It follows from (7.15) that

$$\left\|\sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}}T^t_{(mk,\frac{l}{m})}\phi\right\|^2_{L^2(\mathbb{R}^2)} = \int_{W_m}|p(\xi,\xi^{'})|^2\omega_\phi^m(\xi,\xi^{'})d\xi^{'}d\xi.$$

Moreover, by using Remark 7.3, we have $\|p\|_{L^2(W_m)} = \sum\limits_{k,l\in\mathbb{Z}}|a_{mk,\frac{l}{m}}|^2$ for any $p \in L^2(W_m)$. Assume that $\{T^t_{(mk,\frac{l}{m})}\phi : k,l \in \mathbb{Z}\}$ is a Riesz sequence. Then there exists $A, B > 0$ such that for all $\{a_{mk,\frac{l}{m}} : k,\ell \in \mathbb{Z}\} \in \ell^2(m\mathbb{Z} \times \frac{\mathbb{Z}}{m})$, we get

$$A \sum_{k,l\in\mathbb{Z}} |a_{mk,\frac{l}{m}}|^2 \leq \left\| \sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} T^t_{(mk,\frac{l}{m})}\phi \right\|^2_{L^2(\mathbb{R}^2)} \leq B \sum_{k,l\in\mathbb{Z}} |a_{mk,\frac{l}{m}}|^2, \tag{7.16}$$

Thus,

$$A \int_{W_m} |p(\xi,\xi^{'})|^2 \, d\xi^{'} d\xi \leq \int_{W_m} |p(\xi,\xi^{'})|^2 \omega^m_\phi(\xi,\xi^{'}) \, d\xi^{'} d\xi \leq B \int_{W_m} |p(\xi,\xi^{'})|^2 \, d\xi^{'} d\xi, \tag{7.17}$$

for any $p \in L^2(W_m)$. Let $M := \{(\xi,\xi^{'}) \in W_m : \omega^m_\phi(\xi,\xi^{'}) > B\}$ be a measurable subset of W_n. By taking $p = \chi_M$ in R.H.S of (7.17), we get

$$\iint\limits_M (\omega^m_\phi(\xi,\xi^{'}) - B) \, d\xi^{'} d\xi \leq 0,$$

which implies that $\mu(M) = 0$, where μ is the Lebesgue measure $d\xi^{'}\, d\xi$ on W_m. Similarly, we can show that $\mu\{(\xi,\xi^{'}) \in W_m : \omega^m_\phi(\xi,\xi^{'}) < A\} = 0$. Thus

$$0 < A \leq \omega^m_\phi(\xi,\xi^{'}) \leq B < \infty \text{ a.e } (\xi,\xi^{'}) \in W_m.$$

Now, assume (ii) holds. Let $\{a_{mk,\frac{l}{m}}\} \in c_{00}(m\mathbb{Z} \times \frac{\mathbb{Z}}{m})$. Then by using (7.15), we get

$$\begin{aligned} A \int_{W_m} \left| \sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} e^{2\pi i(mk\xi + \frac{l}{m}\xi^{'})} e^{\pi ikl} \right|^2 d\xi^{'} d\xi &\leq \left\| \sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} T^t_{(mk,\frac{l}{m})}\phi \right\|^2_{L^2(\mathbb{R}^2)} \\ &\leq B \int_{W_m} \left| \sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} e^{2\pi i(mk\xi + \frac{l}{m}\xi^{'})} e^{\pi ikl} \right|^2 d\xi^{'} d\xi. \end{aligned} \tag{7.18}$$

Then the result follows by using Remark 7.3.

(i) $\iff$ (iii). It follows from (7.15) that

$$\left\| \sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} T^t_{(mk,\frac{l}{m})}\phi \right\|^2_{L^2(\mathbb{R}^2)} = \left\| \sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} E^m_{k,l} e^{\pi ikl} \right\|^2_{L^2(W_m;\omega^m_\phi)}, \tag{7.19}$$

for any $\{a_{mk,\frac{l}{m}}\} \in \ell^2(m\mathbb{Z} \times \frac{\mathbb{Z}}{m})$. As an immediate consequence of (7.19), we can prove (iii) $\implies$ (i) directly. Assume (i) holds. The boundedness of ω^m_ϕ leads to the fact that

$$\overline{\text{span}}\{e^{\pi ikl} E^m_{k,l} : k,l \in \mathbb{Z}\} = L^2(W_m; \omega^m_\phi).$$

By making use of (7.19), we obtain our required result. □

Remark 7.13. The equivalence (i) $\iff$ (ii) in Theorem 7.12 was proved for the case $m = 1$ in [119].

The following lemma characterizes the continuity of the functions in $V_m^t(\phi)$ in terms of the continuity of the generator and its uniform boundedness.

Lemma 7.14. *Suppose $\phi \in L^2(\mathbb{R}^2)$, Then the following statements are equivalent:*

1. *For any $\{a_{mk,\frac{l}{m}}\}_{k,l\in\mathbb{Z}} \in \ell^2(m\mathbb{Z} \times \frac{\mathbb{Z}}{m})$,*

$$\sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} T^t_{(mk,\frac{l}{m})}\phi$$

converges pointwise to a continuous function.

2. *$\phi \in C(\mathbb{R}^2)$ and*

$$\sup_{x,y\in\mathbb{R}^2} \sum_{k,l\in\mathbb{Z}} |\phi(x - mk, y - \frac{l}{m})|^2 < \infty.$$

Proof. Assume (1) holds. By choosing $a_{0,0} = 1$ and $a_{mk,\frac{l}{m}} = 0$ for all $(k,l) \neq (0,0)$ in (1), we obtain $\phi \in C(\mathbb{R}^2)$. Since $\phi \in L^2(\mathbb{R}^2)$, we have

$$\begin{aligned}
\int_0^m \int_0^{\frac{1}{m}} \sum_{k,l\in\mathbb{Z}} \big|\phi\big(x - mk, y - \frac{l}{m}\big)\big|^2 \, dy \, dx &= \sum_{k,l\in\mathbb{Z}} \int_0^m \int_0^{\frac{1}{m}} |\phi(x - mk, y - \frac{l}{m})|^2 \, dy \, dx \\
&= \sum_{k,l\in\mathbb{Z}} \int_{mk}^{m(k+1)} \int_{\frac{l}{m}}^{\frac{l+1}{m}} |\phi(x,y)|^2 \, dy \, dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} |\phi(x,y)|^2 \, dx \, dy < \infty,
\end{aligned}$$

Hence $\sum\limits_{k,l\in\mathbb{Z}} \big|\phi\big(x - mk, y - \frac{l}{m}\big)\big|^2 < \infty$ for a.e. $x \in [0, m)$, $y \in [0, \frac{1}{m})$. Moreover, by using continuity of ϕ, we obtain $\sum\limits_{k,l\in\mathbb{Z}} \big|\phi\big(x - mk, y - \frac{l}{m}\big)\big|^2 < \infty$ for all $x, y \in \mathbb{R}$. For each $x, y \in \mathbb{R}$, define an operator $T_{(x,y)}$ on $\ell^2(m\mathbb{Z} \times \frac{\mathbb{Z}}{m})$ by

$$T_{(x,y)}(\{a_{mk,\frac{l}{m}}\}) = \sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} T^t_{(mk,\frac{l}{m})}\phi(x,y).$$

Consider

$$\begin{aligned}
|T_{(x,y)}(\{a_{mk,\frac{l}{m}}\})| &\leq \sum_{k,l\in\mathbb{Z}} |a_{mk,\frac{l}{m}}||T^t_{(mk,\frac{l}{m})}\phi(x,y)| \\
&= \sum_{k,l\in\mathbb{Z}} |a_{mk,\frac{l}{m}}||e^{\pi i(\frac{xl}{m}) - ymk}\phi(x - mk, y - \frac{l}{m})| \\
&\leq \Big(\sum_{k,l\in\mathbb{Z}} |a_{mk,\frac{l}{m}}|^2\Big)^{\frac{1}{2}} \Big(\sum_{k,l\in\mathbb{Z}} \big|\phi\big(x - mk, y - \frac{l}{m}\big)\big|^2\Big)^{\frac{1}{2}},
\end{aligned}$$

which in turn leads to $\|T_{x,y}\| \leq \Big(\sum\limits_{k,l\in\mathbb{Z}} \big|\phi\big(x-mk, y-\frac{l}{m}\big)\big|^2\Big)^{\frac{1}{2}}$ for all $x, y \in \mathbb{R}$. Furthermore, we can show that $T_{x,y}$ is a bounded operator with $\|T_{x,y}\| = \Big(\sum\limits_{k,l\in\mathbb{Z}} \big|\phi\big(x-mk, y-\frac{l}{m}\big)\big|^2\Big)^{\frac{1}{2}}$ by choosing $c_{mk,\frac{l}{m}} = \overline{\phi(x-mk, y-\frac{l}{m})}e^{-\pi i(\frac{xl}{m})-ymk}$. From our assumption (1), we have

$$\sup_{(x,y)\in[0,m)\times[0,\frac{1}{m})} \Big|\sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} T^t_{(mk,\frac{l}{m})}\phi(x,y)\Big| < \infty,$$

which implies that

$$\sup_{(x,y)\in[0,m)\times[0,\frac{1}{m})} |T_{x,y}(\{c_{mk,\frac{l}{m}}\})| < \infty.$$

By applying the uniform boundedness principle for the family $\{T_{x,y} : (x,y) \in [0,m) \times [0,\frac{1}{m})\}$, we obtain $\sup\limits_{(x,y)\in[0,m)\times[0,\frac{1}{m})} \|T_{x,y}\| < \infty$. That is, there exists $M > 0$ such that

$$\sum_{k,l\in\mathbb{Z}} \big|\phi\big(x-mk, y-\frac{l}{m}\big)\big|^2 \leq M^2 \quad \text{for all } (x,y) \in [0,m) \times [0,\frac{1}{m}).$$

Since the function $(x,y) \mapsto \sum\limits_{k,l\in\mathbb{Z}} \big|\phi\big(x-mk, y-\frac{l}{m}\big)\big|^2$ is $m \times \frac{1}{m}$ periodic on $\mathbb{R}^2$, we get $\sup\limits_{x,y\in\mathbb{R}} \sum\limits_{k,l\in\mathbb{Z}} |\phi(x-mk, y-\frac{l}{m})|^2 \leq M^2$, proving our assertion. Conversely assume that (2) holds. For all $x, y \in \mathbb{R}$,

$$\begin{aligned}
\sum_{k,l\in\mathbb{Z}} |a_{mk,\frac{l}{m}} T^t_{(mk,\frac{l}{m})}\phi(x,y)| &= \sum_{k,l\in\mathbb{Z}} |a_{mk,\frac{l}{m}}||e^{\pi i(\frac{xl}{m}-ymk)}\phi(x-mk, y-\frac{l}{m})| \\
&\leq \sum_{k,l\in\mathbb{Z}} |a_{mk,\frac{l}{m}}||\phi(x-mk, y-\frac{l}{m})| \\
&= \Big(\sum_{k,l\in\mathbb{Z}} |a_{mk,\frac{l}{m}}|^2\Big)^{\frac{1}{2}} \Big(\sum_{k,l\in\mathbb{Z}} |\phi(x-mk, y-\frac{l}{m})|^2\Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{k,l\in\mathbb{Z}} |a_{mk,\frac{l}{m}}|^2\Big)^{\frac{1}{2}} \sup_{x,y\in\mathbb{R}} \Big(\sum_{k,l\in\mathbb{Z}} |\phi(x-mk, y-\frac{l}{m})|^2\Big)^{\frac{1}{2}} < \infty,
\end{aligned}$$

which follows that the series $\sum\limits_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} T^t_{(mk,\frac{l}{m})}\phi$ converges uniformly on $\mathbb{R}^2$. Since ϕ is continuous, then the limit function is continuous. □

Remark 7.15. For $m = 1$, the above characterization lemma was proved in [119].

With the assumption on ϕ being continuous and uniformly bounded, the following lemma assures the continuous property of functions in $V^t_m(\psi)$ for $\psi \in V^t_m(\phi)$,

Lemma 7.16. *Let $\phi \in C(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ be a function such that the system $\{T^t_{(mk,\frac{l}{m})}\phi : k,l \in \mathbb{Z}\}$ is a frame for $V^t_m(\phi)$ and $\sum\limits_{k,l\in\mathbb{Z}} |\phi(x-mk, y-\frac{l}{m})|^2 < M$, for all $x, y \in \mathbb{R}$. Suppose, for $\psi \in V^t_m(\phi)$, the system $\{T^t_{(mk,\frac{l}{m})}\psi : k,l \in \mathbb{Z}\}$ is a frame for $V^t_m(\phi)$. Then there exists a constant $C > 0$ such that*

$$\sum_{k,l\in\mathbb{Z}} |\psi(x-mk, y-\frac{l}{m})|^2 \leq C \sum_{k,l\in\mathbb{Z}} |\phi(x-mk, y-\frac{l}{m})|^2.$$

Proof. The generalized Weyl-Zak transform of ψ is given by

$$Z^n_W\psi(\xi,\xi',\eta) = r_\psi(\xi,\xi')Z^n_W\phi(\xi,\xi',\eta), \quad \text{for a.e. } (\xi,\xi') \in W_n, \eta \in \mathbb{R},$$

(see Proposition 7.7). By applying Theorem 7.8, we can show that r_ψ is bounded a.e. on Ω^n_ϕ. Define $\tilde{r} \in L^2(W_m;\omega^m_\phi)$ by $\tilde{r} = r_\psi \chi_{\Omega^n_\phi}$. One can notice that $\tilde{r}$ is bounded a.e. on W_m. Thus the Fourier series representation of $\tilde{r}$ is given by $\tilde{r} = \sum\limits_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} E^m_{p,q} e^{\pi i p q}$. It follows from Remark 7.5 that the $(mk,\frac{l}{m})^{th}$ Fourier coefficient of $\tau_{(-\frac{y}{2},\frac{x}{2})}(\tilde{r})$ is given by

$$\widehat{\tau_{(-\frac{y}{2},\frac{x}{2})}(\tilde{r})}(mk,\frac{l}{m}) = e^{\pi i(\frac{xl}{m}-ymk)} a_{mk,\frac{l}{m}}, \quad \text{for all } x, y \in \mathbb{R}.$$

Now, for a.e. $(\xi,\xi') \in W_n$, $\eta \in \mathbb{R}$, we have $\tilde{r}(\xi,\xi')Z^n_W\phi(\xi,\xi',\eta) = r_\psi(\xi,\xi')Z^n_W\phi(\xi,\xi',\eta)$, which implies that $Z^n_W\psi(\xi,\xi',\eta) = \tilde{r}(\xi,\xi')Z^m_W\phi(\xi,\xi',\eta)$. As a consequence of Proposition 7.7 and Lemma 7.14, we obtain $\psi(x,y) = \sum\limits_{k,l} a_{mk,\frac{l}{m}} T^t_{(mk,\frac{l}{m})}\phi(x,y)$, $x, y \in \mathbb{R}$. Hence, by making use of Lemma 7.4, we get

$$\begin{aligned}
\sum_{k,l\in\mathbb{Z}} |\psi(x+mk, y+\frac{l}{m})|^2 &= \sum_{k,l\in\mathbb{Z}} \left| \sum_{p,q\in\mathbb{Z}} a_{mp,\frac{q}{m}} T^t_{(mp,\frac{q}{m})}\phi(x+mk, y+\frac{l}{m}) \right|^2 \\
&= \sum_{k,l\in\mathbb{Z}} \left| \sum_{p,q\in\mathbb{Z}} a_{mp,\frac{q}{m}} e^{\pi i((x+mk)\frac{q}{m}-(y+\frac{l}{m})mp)} \phi(x+m(k-p), y+\frac{(l-q)}{m}) \right|^2 \\
&= \sum_{k,l\in\mathbb{Z}} \left| e^{\pi ikl} \sum_{p,q\in\mathbb{Z}} a_{mp,\frac{q}{m}} e^{\pi i(\frac{xq}{m}-ymp)} e^{\pi i(pl-kq)} \phi(x+m(k-p), y+\frac{(l-q)}{m}) \right|^2 \\
&= \int_{W_m} \left| \tau_{(\frac{-y}{2},\frac{x}{2})}(\tilde{r})(\xi,\xi') \Big(\sum_{p,q\in\mathbb{Z}} \phi(x+mp, y+\frac{q}{m}) E^m_{p,q} e^{\pi ipq} \Big)(\xi,\xi') \right|^2 d\xi d\xi' \\
&\leq \|\tau_{(\frac{-y}{2},\frac{x}{2})}(\tilde{r})\|^2_\infty \int_{W_m} \left| \Big(\sum_{p,q\in\mathbb{Z}} \phi(x+mp, y+\frac{q}{m}) E^m_{p,q} e^{\pi ipq} \Big)(\xi,\xi') \right|^2 d\xi d\xi' \\
&\leq \|\tilde{r}\|^2_\infty \sum_{p,q\in\mathbb{Z}} |\phi(x+mp, y+\frac{q}{m})|^2,
\end{aligned}$$

which completes the proof. □

Remark 7.17. Note that if any one of the condition of the Lemma 7.14 is satisfied, then any $f \in V_m^t(\phi)$ is defined on $\mathbb{R}^2$ as the pointwise sum $f(x,y) = \sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} T^t_{(mk,\frac{l}{m})}\phi(x,y)$. Moreover, if $\{T^t_{(mk,\frac{l}{m})}\phi : k,l \in \mathbb{Z}\}$ is a Riesz basis for $V_m^t(\phi)$, i.e.,

$$A \sum_{k,l\in\mathbb{Z}} |a_{mk,\frac{l}{m}}|^2 \leq \Big\| \sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} T^t_{(mk,\frac{l}{m})}\phi \Big\|^2 \leq B \sum_{k,l\in\mathbb{Z}} |a_{mk,\frac{l}{m}}|^2, \tag{7.20}$$

then $V_m^t(\phi)$ is a RKHS since the *point evaluation functionals* $\mathcal{L}_{(x,y)}$ for any $(x,y) \in \mathbb{R}^2$, defined by the formula

$$\mathcal{L}_{(x,y)} f = f(x,y) = \sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} T^t_{(mk,\frac{l}{m})}\phi(x,y), \;\; f \in V_m^t(\phi)$$

are bounded in $V_m^t(\phi)$. Indeed, by applying Cauchy-Schwartz inequality in the above equation and using (7.20) for any $(x,y) \in \mathbb{R}^2$ and $f \in V_m^t(\phi)$, we have

$$\begin{aligned} |f(x,y)|^2 &\leq \frac{\|f\|^2}{A} \sum_{k,l\in\mathbb{Z}} |T^t_{(mk,\frac{l}{m})}\phi(x,y)|^2 \\ &\leq \frac{\|f\|^2}{A} \sum_{k,l\in\mathbb{Z}} \left| e^{\pi i(\frac{xl}{m} - ymk)} \phi(x - mk, y - \frac{l}{m}) \right|^2 \\ &\leq \sup_{x,y\in\mathbb{R}^2} \sum_{k,l\in\mathbb{Z}} |\phi(x-mk, y-\frac{l}{m})|^2 \frac{\|f\|^2}{A}. \end{aligned} \tag{7.21}$$

In the next section, we characterize the recovery of a function $f \in V_m^t(\phi)$ from its generalized samples, which are given by the inner products of the signal with elements of a Bessel family constructed using twisted translations of appropriately chosen functions. Additionally, we establish conditions under which these samples correspond to the function's pointwise values, leading to the classical Shannon sampling formula for $V_m^t(\phi)$.

7.3 Stable sampling for generalized twisted shift-invariant spaces

We choose $\{\eta_j\}_{j=1}^s \in V_m^t(\phi)$ such that the collection $\{T^t_{(mk,\frac{l}{m})}\eta_j\}_{k,l\in\mathbb{Z}}^{j=1,2,\ldots,s}$ is a Bessel sequence in $V_m^t(\phi)$. For $j = 1,2,\ldots,s$, the sampling operator $S^j : V_n^t(\phi) \to \ell^2(\mathbb{Z}^2)$ corresponding to the Bessel sequence $\{T^t_{(mk,\frac{l}{m})}\eta_j\}_{k,l\in\mathbb{Z}}^{j=1,2,\ldots,s}$ in $V_m^t(\phi)$ is defined by

$$S^j(f)(k,l) := \langle f, T^t_{(mk,\frac{l}{m})}\eta_j \rangle, \;\; j = 1,2\ldots,s. \tag{7.22}$$

We call the collection $\{S^j(f)(k,l)\}_{(k,l)\in\mathbb{Z}^2}^{j=1,2,\ldots,s}$ as the *generalized samples of f*. We ssume that the generator ϕ is continuous on $\mathbb{R}^2$ and $\sup\limits_{x,y\in\mathbb{R}^2} \sum\limits_{k,l\in\mathbb{Z}} |\phi(x-mk, y-\frac{l}{m})|^2 < \infty$. In addition, we also assume that the system $\{T^t_{(mk,\frac{l}{m})}\phi : k,l \in \mathbb{Z}\}$ is a Riesz sequence. Define $T^m_\phi : L^2(W_m;\omega^m_\phi) \mapsto V^t_m(\phi)$ by

$$T^m_\phi(F) = \sum_{k,l\in\mathbb{Z}} \langle F, E^m_{k,l}e^{\pi ikl}\rangle_{L^2(W_m;\omega^m_\phi)} T^t_{(mk,\frac{l}{m})}\phi.$$

Since the system $\{T^t_{(mk,\frac{l}{m})}\phi : k,l \in \mathbb{Z}\}$ is a Riesz sequence with bounds A and B, we have the following inequality:

$$A\sum_{k,l\in\mathbb{Z}} |\langle F, E^n_{k,l}e^{\pi ikl}\rangle|^2 \le \left\| \sum_{k,l\in\mathbb{Z}} \langle F, E^n_{k,l}e^{\pi ikl}\rangle T^t_{(mk,\frac{l}{m})}\phi \right\|^2 \le B\sum_{k,l\in\mathbb{Z}} |\langle F, E^n_{k,l}e^{\pi ikl}\rangle|^2, \tag{7.23}$$

for all $F \in L^2(W_m;\omega^m_\phi)$. Moreover, from Theorem 7.12, we have

$$A^{'}\|F\|_{L^2(W_m;\omega^m_\phi)} \le \|F\|_{L^2(W_m)} \le B^{'}\|F\|_{L^2(W_m;\omega^m_\phi)}, \quad \forall\ F \in L^2(W_m;\omega^m_\phi) \tag{7.24}$$

for some $A^{'}, B^{'} > 0$. Hence, the boundedness and injectivity of T^m_ϕ follows from (7.23) and (7.24). Now, we aim to show that T^m_ϕ is surjective. Let $f \in V^t_m(\phi)$. Then f can be represented as $f = \sum\limits_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} T^t_{(mk,\frac{l}{m})}\phi$, for some $\{a_{mk,\frac{l}{m}}\} \in \ell^2(m\mathbb{Z}\times\frac{\mathbb{Z}}{m})$. Define $F \in L^2(W_m)$ by $F = \sum\limits_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} E^n_{k,l}e^{\pi ikl}$. Now,

$$\begin{aligned} T^m_\phi\left(\frac{F}{\omega^m_\phi}\right) &= \sum_{k,l\in\mathbb{Z}} \left\langle \frac{F}{\omega^m_\phi}, E^m_{k,l}e^{\pi ikl}\right\rangle_{L^2(W_m;\omega^m_\phi)} T^t_{(mk,\frac{l}{m})}\phi \\ &= \sum_{k,l\in\mathbb{Z}} \left\langle \sum_{p,q\in\mathbb{Z}} a_{mk,\frac{l}{m}} E^m_{p,q}e^{\pi ipq}, E^m_{k,l}e^{\pi ikl}\right\rangle_{L^2(W_m)} T^t_{(mk,\frac{l}{m})}\phi \\ &= \sum_{k,l\in\mathbb{Z}}\sum_{p,q\in\mathbb{Z}} a_{mk,\frac{l}{m}} \langle E^m_{p,q}e^{\pi ipq}, E^m_{k,l}e^{\pi ikl}\rangle_{L^2(W_m)} T^t_{(mk,\frac{l}{m})}\phi \\ &= \sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} T^t_{(mk,\frac{l}{m})}\phi = f, \end{aligned} \tag{7.25}$$

proving our assertion. Thus the operator T^m_ϕ is bounded and invertible.

The adjoint of T^m_ϕ is denoted by $(T^m_\phi)^*$ and it is calculated as

$$(T^m_\phi)^*(f) = \sum_{k,l\in\mathbb{Z}} \langle f, T^t_{(mk,\frac{l}{m})}\phi\rangle_{L^2(\mathbb{R}^2)} E^m_{k,l}e^{\pi ikl},\ \forall\ f \in L^2(\mathbb{R}^2), \tag{7.26}$$

which satisfies $\langle T^m_\phi F, f\rangle_{L^2(W_m;\omega^m_\phi)} = \langle F, (T^m_\phi)^* f\rangle_{L^2(\mathbb{R}^2)}$.

Remark 7.18. Recall that $\Omega^m_\phi = \{(\xi,\xi^{'}) \in W_m : \omega^m_\phi(\xi,\xi^{'}) \neq 0\}$. Since the system $\{T^t_{(mk,\frac{l}{m})}\phi : k,l \in \mathbb{Z}\}$ is a Riesz sequence, by using Theorem 7.12, we have $\mu(W_m\backslash\Omega^m_\phi) = 0$, where μ is the Lebesgue measure on W_m.

Remark 7.19. Throughout this chapter, we assume that the system $\{T^t_{(mk,\frac{l}{m})}\phi : k, l \in \mathbb{Z}\}$ is a Riesz basis for $V^t_m(\phi)$. Then, by making use of Theorem 7.12, there exist $\alpha, \beta > 0$ such that $\alpha \leq \omega^m_\phi(\xi, \xi^{'}) \leq \beta$ for a.e. $(\xi, \xi^{'}) \in W_n$.

The following proposition provides an expression of the samples $\{S^j(f)(k,l)\}^{j=1,2,\dots,s}_{(k,l)\in\mathbb{Z}^2}$ in terms of inner product in $L^2(W_m; \omega^m_\phi)$.

Proposition 7.20. *The sample expression* (7.22) *has the equivalent form:*

$$S^j(f)(k,l) = \big\langle F, e^{\pi ikl} E^m_{k,l}\overline{g_{j,m}}\big\rangle_{L^2(W_m;\omega^m_\phi)}, \tag{7.27}$$

where $F \in L^2(W_m; \omega^m_\phi)$ *is such that* $T^m_\phi(F) = f$, $g_{j,m} := \sum\limits_{p,q\in\mathbb{Z}} S^j\phi(p,q)E^m_{p,q}e^{-\pi ipq}$.

Proof. Let $f \in V^t_m(\phi)$. Then there exists $F \in L^2(W_m; \omega^m_\phi)$ such that $T^m_\phi(F) = f$. By making use of (7.26) and (7.3), we get

$$\begin{aligned}
(T^m_\phi)^* T^t_{(mk,\frac{l}{m})}\eta_j &= \sum_{p,q\in\mathbb{Z}} \langle (T^t_{(mp,\frac{q}{m})})^* T^t_{(mk,\frac{l}{m})}\eta_j, \phi\rangle E^m_{p,q} e^{\pi ipq}\\
&= \sum_{p,q\in\mathbb{Z}} \langle T^t_{(-mp,-\frac{q}{m})} T^t_{(mk,\frac{l}{m})}\eta_j, \phi\rangle E^m_{p,q} e^{\pi ipq}\\
&= \sum_{p,q\in\mathbb{Z}} \langle e^{-\pi i(qk-pl)} T^t_{(m(k-p),\frac{(l-q)}{m})}\eta_j, \phi\rangle E^m_{p,q} e^{\pi ipq}\\
&= \sum_{p,q\in\mathbb{Z}} \langle T^t_{(mp,\frac{q}{m})}\eta_j, \phi\rangle E^m_{k-p',l-q'} e^{\pi ipq} e^{\pi ikl}\\
&= E^m_{k,l} e^{\pi ikl} \sum_{p,q\in\mathbb{Z}} \langle T^t_{(mp,\frac{q}{m})}\eta_j, \phi\rangle E^m_{-p,-q} e^{\pi ipq}\\
&= E^m_{k,l} e^{\pi ikl} \overline{\sum_{p,q\in\mathbb{Z}} S^j\phi(p,q) E^m_{p,q} e^{-\pi ipq}}.
\end{aligned} \tag{7.28}$$

From (7.28), we obtain

$$S^j f(k,l) = \big\langle T^m_\phi(F), T^t_{(mk,\frac{l}{m})}\eta_j\big\rangle_{L^2(\mathbb{R}^2)} = \big\langle F, (T^m_\phi)^* T^t_{(mk,\frac{l}{m})}\eta_j\big\rangle_{L^2(W_m;\omega^m_\phi)} = \big\langle F, e^{\pi ikl} E^m_{k,l}\overline{g_{j,m}}\big\rangle, \tag{7.29}$$

where $g_{j,m} := \sum\limits_{p,q\in\mathbb{Z}} S^j\phi(p,q)E^m_{p,q}e^{-\pi ipq}$ in $L^2(W_m; \omega^m_\phi)$, proving our assertion. □

From above Proposition, it is clear that the recovery of $f \in V^t_m(\phi)$ from the samples $\{S^j(f)(k,l)\}^{j=1,2,\dots,s}_{(k,l)\in\mathbb{Z}^2}$ reduces to the recovery of $F \in L^2(W_m; \omega^m_\phi)$ from the sequence $\big\{\big\langle F, e^{\pi ikl} E^m_{k,l}\overline{g_{j,m}}\big\rangle_{L^2(W_m;\omega^m_\phi)}\big\}^{j=1,2,\dots,s}_{(k,l)\in\mathbb{Z}^2}$. To study the frame properties of the system

$$\{e^{\pi ikl} E^m_{k,l}\overline{g_{j,m}}\}^{j=1,2,\dots,s}_{(k,l)\in\mathbb{Z}^2},$$

we now consider a $s \times 1$ matrix for each $(\xi, \xi') \in W_m$ as follows:

$$\mathcal{U}(\xi,\xi') = \sqrt{\omega_\phi^m(\xi,\xi')}\,\Big((g_{1,n}(\xi,\xi'), g_{2,n}(\xi,\xi'), \dots, g_{s,n}(\xi,\xi')\Big)^T, \ \xi \in W_m. \tag{7.30}$$

We shall denote the transpose conjugate of the matrix $\mathcal{U}(\xi,\xi')$ by $\mathcal{U}^*(\xi,\xi')$ and the function $\mathcal{U}^*(\xi,\xi')\mathcal{U}(\xi,\xi')$ by $\mathcal{E}(\xi,\xi')$.

We have the following result:

Theorem 7.21. *Let $\{\eta_j\}_{j=1}^s \subset V_m^t(\phi)$ be chosen such that the samples (7.22) are defined. For $g_{j,n} = \sum\limits_{p,q\in\mathbb{Z}} S^j\phi(p,q)E_{p,q}^m e^{-\pi ipq}$, consider the matrix $\mathcal{U}(\xi,\xi')$ defined in (7.30). Then the following statements are equivalent:*

(*i*) *The collection $\{T^t_{(mk,\frac{l}{m})}\eta_j\}_{j=1}^s$ is a Bessel sequence in $V_m^t(\phi)$.*

(*ii*) *The collection $\{e^{\pi ikl}E_{k,l}^m\overline{g_{j,m}}\}_{(k,l)\in\mathbb{Z}^2}^{j=1,2,\dots,s}$ is a Bessel sequence in $L^2(W_m;\omega_\phi^m)$.*

(*iii*) $\text{ess-sup}_{(\xi,\xi')\in W_m}\left(\sum\limits_{j=1}^s |g_{j,m}(\xi,\xi')|^2\right) < \infty.$

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) follows from (7.29) and boundedness of T_ϕ^m.

(ii) $\Longleftrightarrow$ (iii). Let $F \in L^2(W_m;\omega_\phi^m)$. Then, by using the Plancherel theorem with respect to an orthonormal basis $\{E_{k,l}^m e^{\pi ikl} : k,l\in\mathbb{Z}\}$, we have

$$\begin{aligned}\sum_{j=1}^s\sum_{k,\ell\in\mathbb{Z}}\left|\langle F, e^{\pi ikl}E_{k,l}^m\overline{g_{j,m}}\rangle_{L^2(W_m;\omega_\phi^m)}\right|^2 &= \sum_{j=1}^s\left\|Fg_{j,m}\omega_\phi^m\right\|^2_{L^2(W_m)}\\ &= \iint_{W_m}\left(\sum_{j=1}^s\left|\sqrt{\omega_\phi^m}(\xi,\xi')g_{j,m}(\xi,\xi')\right|^2\right)\left|F(\xi,\xi')\right|^2\omega_\phi^m(\xi,\xi')\,d\xi'\,d\xi\\ &= \iint_{W_m}\mathcal{E}(\xi,\xi')|F(\xi,\xi')|^2\omega_\phi^m(\xi,\xi')d\xi'd\xi. \end{aligned}\tag{7.31}$$

From equation (7.31), it is clear that

$$\sum_{j=1}^s\sum_{k,\ell\in\mathbb{Z}}\left|\langle F, e^{\pi ikl}E_{k,l}^m\overline{g_{j,m}}\rangle_{L^2(W_m;\omega_\phi^m)}\right|^2 \le \operatorname*{ess-sup}_{(\xi,\xi')\in W_m}\mathcal{E}(\xi,\xi')\|F\|^2_{L^2(W_m;\omega_\phi^m)}, \tag{7.32}$$

for every $F \in L^2(W_m;\omega_\phi^m)$. If (iii) holds, then (ii) holds directly from (7.32) and Remark 7.19. Conversely, assume (ii) holds. Let $B > 0$ be such that

$$\sum_{j=1}^s\sum_{k,\ell\in\mathbb{Z}}\left|\langle F, e^{\pi ikl}E_{k,l}^m\overline{g_{j,m}}\rangle_{L^2(W_m;\omega_\phi^m)}\right|^2 \le B\|F\|^2_{L^2(W_m;\omega_\phi^m)}, \quad \text{for all } F \in L^2(W_m;\omega_\phi^m). \tag{7.33}$$

In order to prove (iii), it is sufficient to show that $\text{ess-sup}_{(\xi,\xi')\in W_m}\mathcal{E}(\xi,\xi') < B$ (in view of Remark 7.19). Suppose not, then there would be a positive measurable set $W \subset W_m$ such that $\mu_{\omega_\phi^m}(W) > 0$ and $\mathcal{E}(\xi,\xi') > B$ for a.e. $(\xi,\xi') \in W$, where $\mu_{\omega_\phi^m}$ is the weighted Lebesgue measure with respect to the weight ω_ϕ^m on W_m. Define $H \in L^\infty(W_m;\omega_\phi^m)$ such that

$$H(\xi,\xi') = \begin{cases} 0 & \text{if } (\xi,\xi') \in ([0,\frac{1}{m}) \times [0,m)) \setminus W \\ 1 & \text{if } (\xi,\xi') \in W. \end{cases}.$$

Then, by using (7.31), we get

$$\sum_{j=1}^{s}\sum_{k,\ell\in\mathbb{Z}} \left|\langle H, e^{\pi ikl}\overline{E_{k,l}^m g_{j,m}}\rangle\right|^2 = \iint_{W_m} \mathcal{E}(\xi,\xi')|H(\xi,\xi')|^2\omega_\phi^m(\xi,\xi')d\xi'd\xi > B\|H\|^2_{L^2(W_m;\omega_\phi^m)}, \tag{7.34}$$

which is a contradiction to (7.33), proving our assertion. Now we prove (i). If the system $\{e^{\pi ikl}\overline{E_{k,l}^m g_{j,m}}\}_{(k,l)\in\mathbb{Z}^2}^{j=1,2,\ldots,s}$ is a Bessel sequence in $L^2(W_m;\omega_\phi^m)$ with Bessel bound $\mathcal{B}$. Then from our claim, it is clear that $B < \mathcal{B} < \infty$. Hence the claim follows. □

We have the following main result:

Theorem 7.22. *Let $\{\eta_j\}_{j=1}^s \subset V_m^t(\phi)$ be chosen so that the collection $\{T^t_{(mk,\frac{l}{m})}\eta_j\}_{j=1}^s$ is a Bessel sequence in $V_m^t(\phi)$. Assume $g_{j,m} \in L^\infty(W_m;\omega_\phi^m)$ for which the matrix $\mathcal{U}(\xi,\xi')$ is defined in (7.30). Then the following conditions are equivalent:*

- *(i) The set $m\mathbb{Z} \times \frac{\mathbb{Z}}{m}$ is a stable set of sampling for $V_m^t(\phi)$.*
- *(ii) There exist functions $\beta_j, j = 1,2,\ldots,s$ such that the collection $\{T^t_{(mk,\frac{l}{m})}\beta_j\}_{j=1}^s$ is a frame for $V_m^t(\phi)$ and the following sampling formula holds for any $f \in V_m^t(\phi)$:*

$$f(x,y) = \sum_{j=1}^{s}\sum_{k,\ell\in\mathbb{Z}} S^j f(k,l)T^t_{(mk,\frac{l}{m})}\beta_j(x,y), \ \ (x,y) \in \mathbb{R}^2. \tag{7.35}$$

  where the convergence of the series in (7.35) is in the $L^2(\mathbb{R}^2)$-sense and uniform on $\mathbb{R}^2$.
- *(iii) The system $\{e^{\pi ikl}\overline{E_{k,l}^m g_{j,m}}\}_{(k,l)\in\mathbb{Z}^2}^{j=1,2,\ldots,s}$ is a frame for $L^2(W_m;\omega_\phi^m)$. That is, there exist constants $0 < A \leq B < \infty$ such that the following inequality holds for any $F \in L^2(W_m;\omega_\phi^m)$:*

$$A\|F\|^2 \leq \sum_{j=1}^{s}\sum_{k,l\in\mathbb{Z}} |\langle F, e^{\pi ikl}\overline{E_{k,l}^m g_{j,m}}\rangle|^2 \leq B\|F\|^2.$$

(*iv*) *There exists a* $1 \times s$ *matrix* $\mathcal{W}$ *with entries in* $L^{\infty}(W_m; \omega_\phi^m)$ *for a.e.* $\xi \in W_m$ *satisfying*

$$\sqrt{\omega_\phi^m(\xi, \xi')}\mathcal{W}(\xi, \xi')\mathcal{U}(\xi, \xi') = 1 \ \ \textit{for a.e.} \ (\xi, \xi') \in W_m.$$

(*v*) $\text{ess-inf}_{(\xi,\xi')\in W_m} \left(\sum_{j=1}^{s} |g_{j,m}(\xi, \xi')|^2 \right) > 0.$

Proof. (i) $\iff$ (iii). Assume (i) holds. Then there exist two constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k,l\in\mathbb{Z}} |S^j f(k, l)|^2 \leq B\|f\|^2, \ \forall \ f \in V_m^t(\phi). \tag{7.36}$$

By applying Proposition 7.20 and bounded property of T_ϕ^m and $(T_\phi^m)^{-1}$ in (7.36), we get the following inequality for any $F \in L^2(W_m; \omega_\phi^m)$:

$$\frac{A}{\|(T_\phi^m)^{-1}\|^2}\|F\|^2 \leq \sum_{j=1}^{s} \sum_{k,l\in\mathbb{Z}} |\langle F, e^{\pi ikl} E_{k,l}^m \overline{g_{j,m}}\rangle|^2 \leq B\|T_\phi^m\|^2\|F\|^2,$$

proving our assertion. Conversely, assume (iii) holds. Then proceeding as before we obtain our required result.

(ii) $\Longrightarrow$ (iii). By applying $(T_\phi^m)^{-1}$ on both sides of (7.35) and using Proposition 7.20, we get the following expression for any $F \in L^2(W_m; \omega_\phi^m)$

$$F = \sum_{j=1}^{s} \sum_{k,l\in\mathbb{Z}} \langle F, e^{\pi ikl} E_{k,l}^m \overline{g_{j,m}}\rangle (T_\phi^m)^{-1}(T^t_{(mk, \frac{l}{m})}\beta_j). \tag{7.37}$$

Since $(T_\phi^m)^{-1}$ is bounded and surjective, then by [42, Corollary 5.3.2], the collection

$$\{(T_\phi^m)^{-1}(T^t_{(mk,\frac{l}{m})}\beta_j)\}_{(k,l)\in\mathbb{Z}^2}^{j=1,2,\ldots,s}$$

is a frame for $L^2(W_m; \omega_\phi^m)$. Further, by Lemma [42, Lemma 6.3.2], the collections $\{e^{\pi ikl} E_{k,l}^m \overline{g_{j,m}}\}_{(k,l)\in\mathbb{Z}^2}^{j=1,2,\ldots,s}$ and $\{(T_\phi^m)^{-1}(T^t_{(mk,\frac{l}{m})}\beta_j)\}_{(k,l)\in\mathbb{Z}^2}^{j=1,2,\ldots,s}$ are dual frames for $L^2(W_m; \omega_\phi^m)$ and hence (iii) holds.

(v) $\iff$ (iii). Assume (v) holds. Then by using hypothesis that $\{T^t_{(mk,\frac{l}{m})}\eta_j\}_{j=1}^s$ is a Bessel sequence in $V_m^t(\phi)$ and Theorem 7.21, we have $\text{ess-sup}_{(\xi,\xi')\in W_m} \left(\sum_{j=1}^{s} |g_{j,m}(\xi, \xi')|^2 \right) < \infty$. Consequently, by using Remark 7.19 and (7.31), we have

$$\underset{(\xi,\xi')\in W_m}{\text{ess-inf}}\ \mathcal{E}(\xi, \xi')\|F\|^2 \leq \sum_{j=1}^{s} \sum_{k,\ell\in\mathbb{Z}} \left|\langle F, e^{\pi ikl} E_{k,l}^m \overline{g_{j,m}}\rangle_{L^2(W_m;\omega_\phi^m)}\right|^2 \leq \underset{(\xi,\xi')\in W_m}{\text{ess-sup}}\ \mathcal{E}(\xi, \xi')\|F\|^2, \tag{7.38}$$

for any $F \in L^2(W_m;\omega_\phi^m)$. This leads to the fact that the collection $\{e^{\pi ikl}E_{k,l}^m\overline{g_{j,m}}\}_{(k,l)\in\mathbb{Z}^2}^{j=1,2,\ldots,s}$ is a frame for $L^2(W_m;\omega_\phi^m)$. Finally, by using (7.29) and bounded property of T_ϕ^m and $(T_\phi^m)^{-1}$ in (7.38), we get

$$\frac{\text{ess-inf}_{(\xi,\xi')\in W_m}\mathcal{E}(\xi,\xi')}{\|(T_\phi^m)\|^2}\|f\|^2 \le \sum_{j=1}^{s}\sum_{k,l\in\mathbb{Z}}\left|\langle f, T^t_{(mk,\frac{l}{m})}\eta_j\rangle_{L^2(\mathbb{R}^2)}\right|^2 \le \underset{(\xi,\xi')\in W_m}{\text{ess-sup}}\ \mathcal{E}(\xi,\xi')\|(T_\phi^m)^{-1}\|^2\|f\|^2,$$

Hence (iii) holds. Conversely, assume (iii) holds with lower frame bound $A > 0$. Following the proof of Theorem 7.21, it can be easily shown that $\text{ess-inf}_{(\xi,\xi')\in W_m}\mathcal{E}(\xi,\xi')$ is the largest constant satisfying

$$\underset{(\xi,\xi')\in W_m}{\text{ess-inf}}\ \mathcal{E}(\xi,\xi')\|F\|^2 \le \sum_{j=1}^{s}\sum_{k,\ell\in\mathbb{Z}}\left|\left\langle F, e^{\pi ikl}E_{k,l}^m\overline{g_{j,m}}\right\rangle_{L^2(W_m;\omega_\phi^m)}\right|^2 \quad \forall\ F\in L^2(W_m;\omega_\phi^m).$$

This gives $0 < A < \text{ess-inf}_{(\xi,\xi')\in W_m}\mathcal{E}(\xi,\xi')$ and hence (v) holds. This completes the proof.

(v) $\Longrightarrow$ (iv). Assume (v) holds. Then, from Remark 7.19, we have

$$\underset{(\xi,\xi')\in W_m}{\text{ess-inf}}\ \mathcal{U}^*(\xi,\xi')\mathcal{U}(\xi,\xi') > 0.$$

Thus the Moore-Penrose pseudo inverse matrix is given by

$$\mathcal{U}^\dagger(\xi,\xi') = [\mathcal{U}^*(\xi,\xi')\mathcal{U}(\xi,\xi')]^{-1}\mathcal{U}^*(\xi,\xi') \quad \text{for a.e.}(\xi,\xi')\in W_m, \tag{7.39}$$

and it satisfies $\mathcal{U}^\dagger(\xi,\xi')\mathcal{U}(\xi,\xi') = 1$. Moreover, the entries of $\mathcal{U}^\dagger$ are essentially bounded since the functions $g_{j,m}$, $j = 1,\ldots,s$, are essentially bounded and

$$\underset{(\xi,\xi')\in W_m}{\text{ess-inf}}\ \mathcal{U}^*(\xi,\xi')\mathcal{U}(\xi,\xi') > 0.$$

(iv) $\Longrightarrow$ (ii). Let $F \in L^2(W_m;\omega_\phi^m)$. From the assumption that there exists $1\times s$ matrix $\mathcal{W}$ such that $\sqrt{\omega_\phi^m(\xi,\xi')}\mathcal{W}(\xi,\xi')\mathcal{U}(\xi,\xi')F(\xi,\xi') = F(\xi,\xi')$ for a.e. $(\xi,\xi')\in W_m$. Let us define $\mathcal{W}(\xi,\xi') = (\Psi_1(\xi,\xi'),\ldots,\Psi_s(\xi,\xi'))$ for some $\Psi_1,\ldots,\Psi_s \in L^\infty(W_m,\omega_\phi^m)$. Then, we have

$$\begin{aligned}
F(\xi,\xi') &= \sqrt{\omega_\phi^m(\xi,\xi')}\mathcal{W}(\xi,\xi')\mathcal{U}(\xi,\xi')F(\xi,\xi') \\
&= \omega_\phi^m(\xi,\xi')\Big(\Psi_1(\xi,\xi'),\cdots,\Psi_s(\xi,\xi')\Big)\begin{pmatrix} g_{1,m}(\xi,\xi')F(\xi,\xi') \\ \vdots \\ g_{1,s}(\xi,\xi')F(\xi,\xi') \end{pmatrix} \\
&= \omega_\phi^m(\xi,\xi')\sum_{j=1}^{s} g_{j,m}(\xi,\xi')F(\xi,\xi')\Psi_j(\xi,\xi') \\
&= \big(\sum_{j=1}^{s} F g_{j,m}\omega_\phi^m \Psi_j\big)(\xi,\xi').
\end{aligned} \tag{7.40}$$

On the other hand, since the functions $g_{j,m}$, $j = 1, ..., s$ are essentially bounded, we have the following Fourier series representation

$$\sum_{k,l\in\mathbb{Z}} \langle F g_{j,m}\omega_\phi^m, E_{k,l}^m e^{\pi ikl}\rangle_{L^2(W_m)} E_{k,l}^m e^{\pi ikl} = F g_{j,m}\omega_\phi^m. \tag{7.41}$$

By substituting (7.41) in (7.40), we obtain

$$\begin{aligned}
&\sum_{j=1}^{s}\sum_{k,l\in\mathbb{Z}} \langle F, E_{k,l}^m e^{\pi ikl}\overline{g_{j,m}}\rangle_{L^2(W_m;\omega_\phi^m)} E_{k,l}^m e^{\pi ikl}\Psi_j \\
&\qquad = \sum_{j=1}^{s}\sum_{k,l\in\mathbb{Z}} \langle F g_{j,m}\omega_\phi^m, E_{k,l}^m e^{\pi ikl}\rangle_{L^2(W_m)} E_{k,l}^m e^{\pi ikl}\Psi_j = F.
\end{aligned} \tag{7.42}$$

By applying T_ϕ^m on both sides, we have the following equation:

$$f = \sum_{j=1}^{s}\sum_{k,l\in\mathbb{Z}} \langle F, E_{k,l}^m e^{\pi ikl}\overline{g_{j,m}}\rangle_{L^2(W_m;\omega_\phi^m)} T_\phi^m(E_{k,l}^m e^{\pi ikl}\Psi_j), \tag{7.43}$$

where $T_\phi^m F = f$. Now, by the definition of T_ϕ^m, we get

$$\begin{aligned}
T_\phi^m(E_{k,l}^m e^{\pi ikl}\Psi_j) &= \sum_{p,q\in\mathbb{Z}} \langle E_{k,l}^m e^{\pi ikl}\Psi_j, E_{p,q}^m e^{\pi ipq}\rangle T^t_{(mp,\frac{q}{m})}\phi \\
&= \sum_{p,q\in\mathbb{Z}} \langle \Psi_j, E_{p-k,q-l}^m e^{-\pi ikl} e^{\pi ipq}\rangle T^t_{(mp,\frac{q}{m})}\phi \\
&= \sum_{p,q\in\mathbb{Z}} \langle \Psi_j, E_{p,q}^m e^{\pi ipq} e^{-\pi i(pl+qk)}\rangle T^t_{(m(p+k),\frac{(q+l)}{m})}\phi.
\end{aligned}$$

By making use of (7.3), we get

$$T_\phi^m(E_{k,l}^m e^{\pi ikl}\Psi_j) = \sum_{p,q\in\mathbb{Z}} \langle \Psi_j, E_{p,q}^m e^{\pi ipq}\rangle T^t_{(mk,\frac{l}{m})} T^t_{(mp,\frac{q}{m})}\phi = T^t_{(mk,\frac{l}{m})}\beta_j, \tag{7.44}$$

where $\beta_j = \sum\limits_{p,q\in\mathbb{Z}} \langle \Psi_j, E^m_{p,q} e^{\pi ipq} \rangle T^t_{(mp,\frac{q}{m})}\phi$. By substituting (7.44) in (7.43), we obtain

$$f = \sum_{j=1}^{s} \sum_{k,l\in\mathbb{Z}} \langle F, E^m_{k,l} e^{\pi ikl} \overline{g_{j,m}} \rangle_{L^2(W_m;\omega^m_\phi)} T^t_{(mk,\frac{l}{m})}\beta_j.$$

Now, we aim to show that the system $\{T^t_{(mk,\frac{l}{m})}\beta_j\}^{j=1,2,\dots,s}_{(k,l)\in\mathbb{Z}^2}$ is a frame for $L^2(W_m;\omega^m_\phi)$. As a consequence of the Plancherel Theorem for the Fourier series, we have

$$\begin{aligned}
\sum_{j=1}^{s} \sum_{k,\ell\in\mathbb{Z}} |\langle F, E^m_{k,l} e^{\pi ikl} \Psi_j \rangle_{L^2(W_m;\omega^m_\phi)}|^2 &= \sum_{j=1}^{s} \sum_{k,l\in\mathbb{Z}} |\langle F\overline{\Psi_j}\omega^m_\phi, E^m_{k,l} e^{\pi ikl} \rangle_{L^2(W_m)}|^2 \\
&= \sum_{j=1}^{s} \|F\overline{\Psi}_j\omega^m_\phi\|^2_{L^2(W_m)} \\
&\leq \left(\sum_{j=1}^{s} \|\Psi_j\omega^m_\phi\|^2_{L^\infty(W_m;\omega^m_\phi)} \right) \|F\|^2_{L^2(W_m;\omega^m_\phi)}.
\end{aligned}$$

This shows that the system $\{E^m_{k,l} e^{\pi ikl} \Psi_j\}^{j=1,2,\dots,s}_{(k,l)\in\mathbb{Z}^2}$ is a Bessel sequence in $L^2(W_m;\omega^m_\phi)$. Moreover, the system $\{T^t_{(mk,\frac{l}{m})}\beta_j\}^{j=1,2,\dots,s}_{(k,l)\in\mathbb{Z}^2}$ is a Bessel sequence for $V^t_m(\phi)$ since T^m_ϕ is invertible.

Further, by Theorem [42, Lemma 6.3.2], the collections

$$\{T^t_{(mk,\frac{l}{m})}\eta_j\}^{j=1,2,\dots,s}_{(k,l)\in\mathbb{Z}^2} \quad \text{and} \quad \{T^t_{(mk,\frac{l}{m})}\beta_j\}^{j=1,2,\dots,s}_{(k,l)\in\mathbb{Z}^2}$$

are dual frames for $L^2(W_m;\omega^m_\phi)$ and hence (7.35) holds. The convergence of the series (7.35) in $L^2(\mathbb{R}^2)$ norm is due to the Bessel property of the collection $\{T^t_{(mk,\frac{l}{m})}\beta_j\}^{j=1,2,\dots,s}_{(k,l)\in\mathbb{Z}^2}$. For its uniform convergence, note that for given ϵ, there exists $N, M \in \mathbb{N}$ such that we have

$$\begin{aligned}
&\left| \sum_{j=1}^{s} \sum_{k=-N}^{N} \sum_{\ell=-M}^{M} S^j f(k,l) T^t_{(mk,\frac{l}{m})}\beta_j(x,y) - \sum_{j=1}^{s} \sum_{k,\ell\in\mathbb{Z}} S^j f(k,l) T^t_{(mk,\frac{l}{m})}\beta_j(x,y) \right| \\
&\leq \frac{1}{A} \sup_{x,y\in\mathbb{R}^2} \sum_{k,l\in\mathbb{Z}} |\phi(x-mk, y-\frac{l}{m})|^2 \|f_{N,M} - f\|^2 < \epsilon,
\end{aligned} \tag{7.45}$$

where $f_{N,M} := \sum_{j=1}^{s} \sum_{k=-N}^{N} \sum_{\ell=-M}^{M} S^j f(k,l) T^t_{(mk,\frac{l}{m})}\beta_j$, the first inequality is due to (7.21), the last inequality follows from the convergence of the series (7.35) in $L^2(\mathbb{R}^2)$-norm. This completes the proof. □

Remark 7.23. To establish the equivalences $(i) \iff (iii)$ and $(v) \iff (iii)$ as well as the implication $(ii) \implies (iii)$, we can relax the condition that g is in $L^\infty(W_n;\omega^n_\phi)$. In fact, the proof shows it is sufficient for g is in $L^2(W_n;\omega^n_\phi)$.

In the above theorem, we obtained a sampling formula for a function f in $V_m^t(\phi)$ with the generalized samples $\{\langle f, T^t_{(mk,\frac{l}{m})}\eta_j\rangle\}_{(k,l)\in\mathbb{Z}^2}^{j=1,2,\ldots,s}$ as the coefficients. Now, we wish to obtain a characterization theorem for the sampling formula with functional value samples $\{f(mk,\frac{l}{m})\}_{k,l\in\mathbb{Z}}$ as the coefficients. The following lemma will be used to prove such a characterization theorem. In fact, in Lemma 7.24, we shall express the generalized samples $\{\langle f, T^t_{(mk,\frac{l}{m})}\psi\rangle\}_{(k,l)\in\mathbb{Z}^2}$ in terms of the functional value samples $\{f(mk,\frac{l}{m})\}_{k,l\in\mathbb{Z}}$ for some $\psi\in V_m^t(\phi)$.

Lemma 7.24. *Let $\phi\in C(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ be a function such that*

$$0 < A \leq |\Phi(\xi,\xi^{'})| \leq B < \infty, \quad \textit{for a.e. } (\xi,\xi^{'})\in W_m. \tag{7.46}$$

Define $\psi\in V_m^t(\phi)$ by

$$Z_W^m\psi(\xi,\xi^{'},\eta) = \begin{cases} \frac{1}{\Phi(\xi,\xi^{'})} Z_W^m\phi(\xi,\xi^{'},\eta) & \textit{if } \ (\xi,\xi^{'})\in\Omega_\phi^m \\ 0 & \textit{otherwise.}\end{cases} \tag{7.47}$$

Then there exists a function $\tilde{\psi}\in L^2(\mathbb{R}^2)$ such that the system $\{T^t_{(mk,\frac{l}{m})}\tilde{\psi} : k,l\in\mathbb{Z}\}$ is the canonical dual frame for $\{T^t_{(mk,\frac{l}{m})}\psi : k,l\in\mathbb{Z}\}$ and satisfies

$$\langle f, T^t_{mk,\frac{l}{m}}\tilde{\psi}\rangle = f(mk,\frac{l}{m}), \quad \textit{for all } f\in V_m^t(\phi). \tag{7.48}$$

Proof. Let $f\in V_m^t(\phi)$. Then, f can be expressed as

$$f(x,y) = \sum_{k,l\in\mathbb{Z}} a_{mk,\frac{l}{m}} T^t_{(mk,\frac{l}{m})}\phi(x,y), \text{ for all } x,y\in\mathbb{R}. \tag{7.49}$$

One can notice that $\Omega_\phi^m = \Omega_\psi^m$. By using (7.46) and Remark 7.19 in Theorem 7.12, we can show that the system $\{T^t_{(mk,\frac{l}{m})}\psi : k,l\in\mathbb{Z}\}$ is a Riesz sequence for $V_m^t(\phi)$. Now, by making use of the Theorem 7.11, the canonical dual frame of the system $\{T^t_{(mk,\frac{l}{m})}\psi : k,l\in\mathbb{Z}\}$ is given by $\{T^t_{(mk,\frac{l}{m})}\tilde{\psi} : k,l\in\mathbb{Z}\}$, where

$$Z_W^m\tilde{\psi}(\xi,\xi^{'}\eta) = \begin{cases} \frac{1}{\omega_\psi^m(\xi,\xi^{'})} Z_W^m\psi(\xi,\xi^{'},\eta) & \text{if } \ (\xi,\xi^{'})\in\Omega_\phi^m \\ \quad 0 & \quad\text{otherwise.}\end{cases}$$

Hence every function $f\in V_m^t(\phi)$ can be written as

$$f = \sum_{k,l\in\mathbb{Z}} \langle f, T^t_{(mk,\frac{l}{m})}\tilde{\psi}\rangle T^t_{(mk,\frac{l}{m})}\psi \quad \text{in } L^2(\mathbb{R}^2).$$

By using (7.46) and (7.47), the generalized Weyl-Zak transform of η_1 can be rewritten as

$$Z_W^m\tilde{\psi}(\xi,\xi^{'}\eta)=\begin{cases}\frac{\overline{\Phi(\xi,\xi^{'})}}{\omega_\phi^m(\xi,\xi^{'})}Z_W^m\phi(\xi,\xi^{'},\eta) & \text{if} \;\; (\xi,\xi^{'})\in\Omega_\phi^m\\ \qquad 0 & \qquad \text{otherwise.}\end{cases} \tag{7.50}$$

Now, by making use of Proposition 7.6, Proposition 7.7 and (7.50), we get

$$\begin{aligned}\langle f,T^t_{(mk,\frac{l}{m})}\tilde{\psi}\rangle &= \langle Z_W^m f, Z_W^m T^t_{(mk,\frac{l}{m})}\tilde{\psi}\rangle\\ &=\iint_{W_m}\int_{\mathbb{R}} Z_W^m f(\xi,\xi^{'},\eta)\overline{Z_W^m T^t_{(mk,\frac{l}{m})}\tilde{\psi}(\xi,\xi^{'},\eta)}\,d\eta d\xi^{'}d\xi\\ &=\iint_{W_m}\int_{\mathbb{R}} Z_W^m f(\xi,\xi^{'},\eta)\overline{Z_W^m\tilde{\psi}(\xi,\xi^{'},\eta)}e^{-2\pi i(mk\xi+\frac{l}{m}\xi^{'})}e^{-\pi ikl}\,d\eta d\xi^{'}d\xi\\ &=\iint_{W_m}\int_{\mathbb{R}} r(\xi,\xi^{'})Z_W^m\phi(\xi,\xi^{'},\eta)\frac{\Phi(\xi,\xi^{'})}{\omega_\phi^m(\xi,\xi^{'})}\overline{Z_W^m\phi(\xi,\xi^{'},\eta)}e^{-2\pi i(mk\xi+\frac{l}{m}\xi^{'})}e^{-\pi ikl}\,d\eta d\xi^{'}d\xi\\ &=\iint_{W_m}\int_{\mathbb{R}} r(\xi,\xi^{'})Z_W^m\phi(\xi,\xi^{'},\eta)\frac{\Phi(\xi,\xi^{'})}{\omega_\phi^m(\xi,\xi^{'})}\overline{Z_W^m\phi(\xi,\xi^{'},\eta)}e^{-2\pi i(mk\xi+\frac{l}{m}\xi^{'})}e^{-\pi ikl}\,d\eta d\xi^{'}d\xi\\ &=\iint_{W_m} r(\xi,\xi^{'})\frac{\Phi(\xi,\xi^{'})}{\omega_\phi^m(\xi,\xi^{'})}\left(\int_{\mathbb{R}}|Z_W^m\phi(\xi,\xi^{'},\eta)|^2\,d\eta\right)e^{-2\pi i(mk\xi+\frac{l}{m}\xi^{'})}e^{-\pi ikl}\,d\xi^{'}d\xi\\ &=\iint_{W_m} r(\xi,\xi^{'})\Phi(\xi,\xi^{'})e^{-2\pi i(mk\xi+\frac{l}{m}\xi^{'})}e^{-\pi ikl}\,d\xi^{'}d\xi.\end{aligned} \tag{7.51}$$

Since $r\Phi\in L^2(W_m)$, then the $(mk,\frac{l}{m})^{th}$ Fourier coefficient of $r\Phi$ is given by

$$\sum_{p,q\in\mathbb{Z}} a_{mp,\frac{q}{m}}\phi(m(k-p),\frac{(l-q)}{m})e^{\pi i(pl-kq)}e^{\pi ikl}.$$

Thus,

$$\begin{aligned}\langle f,T^t_{(mk,\frac{l}{m})}\tilde{\psi}\rangle &= e^{-\pi ikl}\iint_{W_m} r(\xi,\xi^{'})\Phi(\xi,\xi^{'})e^{-2\pi i(mk\xi+\frac{l}{m}\xi^{'})}\,d\xi d\xi^{'}\\ &= e^{-\pi ikl}\sum_{p,q\in\mathbb{Z}} a_{mp,\frac{q}{m}}\phi(m(k-p),\frac{(l-q)}{m})e^{\pi i(pl-kq)}e^{\pi ikl}\\ &=\sum_{p,q\in\mathbb{Z}} a_{mp,\frac{q}{m}}e^{\pi i(pl-kq)}\phi(m(k-p),\frac{(l-q)}{m})\\ &=\sum_{p,q\in\mathbb{Z}} a_{mp,\frac{q}{m}}T^t_{(mp,\frac{q}{m})}\phi(mk,\frac{l}{m}).\end{aligned} \tag{7.52}$$

Then the result follows by substituting (7.49) in (7.52). □

Theorem 7.25. *Let ϕ be a function such that $\Phi \in L^\infty(W_m;\omega_\phi^m)$. Define $\eta_1 \in V_m^t(\phi)$ by*

$$Z_W^m\eta_1(\xi,\xi^{'}\,\eta) = \begin{cases} \frac{\overline{\Phi(\xi,\xi^{'})}}{\omega_\phi^m(\xi,\xi^{'})} Z_W^m\phi(\xi,\xi^{'},\eta) & \text{if } (\xi,\xi^{'}) \in \Omega_\phi^m \\ 0 & \text{otherwise.} \end{cases}$$

Then the following conditions are equivalent.

(i) *The set $m\mathbb{Z}\times\frac{\mathbb{Z}}{m}$ is a stable set of sampling for $V_m^t(\phi)$.*

(ii) *There exists a function $\psi \in V_m^t(\phi)$ such that the collection $\{T^t_{(mk,\frac{l}{m})}\psi : k,l \in \mathbb{Z}\}$ is a frame for $V_m^t(\phi)$ and the following sampling formula holds for any $f \in V_m^t(\phi)$:*

$$f = \sum_{k,l\in\mathbb{Z}} f(mk,\frac{l}{m})T^t_{(mk,\frac{l}{m})}\psi, \tag{7.53}$$

where the convergence of the series (7.53) is in the $L^2(\mathbb{R}^2)$-sense and uniform on $\mathbb{R}^2$.

(iii) *The system $\{e^{\pi ikl}E^m_{k,l}\Phi\}_{(k,l)\in\mathbb{Z}^2}$ is a frame for $L^2(W_m;\omega_\phi^m)$.*

(iv) *There exists a function $\mathcal{W}$ in $L^\infty(W_m;\omega_\phi^m)$ satisfying*

$$\sqrt{\omega_\phi^m(\xi,\xi^{'})}\mathcal{W}(\xi,\xi^{'})\mathcal{U}(\xi,\xi^{'}) = 1 \quad \text{for a.e. } (\xi,\xi^{'}) \in W_m,$$

where $\mathcal{U}(\xi,\xi^{'}) = \sqrt{\omega_\phi^m}(\xi,\xi^{'})\Phi(\xi,\xi^{'})$.

(v) *There exists a positive constant A such that*

$$|\Phi(\xi,\xi^{'})| \geq A, \ \text{for a.e. } (\xi,\xi^{'}) \in W_m.$$

Proof. $(ii) \iff (v)$ Assume (ii) holds. Then, by choosing $f=\phi$ in (7.53), we get

$$\phi = \sum_{k,l\in\mathbb{Z}} \phi(mk,\frac{l}{m})T^t_{(mk,\frac{l}{m})}\psi \quad \text{in } L^2(\mathbb{R}^2).$$

By applying Proposition 7.7, we have

$$Z_W^m\phi(\xi,\xi^{'},\eta) = \left(\sum_{k,l}\phi(mk,\frac{l}{m})e^{2\pi i(mk\xi+\frac{l}{m}\xi^{'})}e^{\pi ikl}\right)Z_W^m\psi(\xi,\xi^{'},\eta) \quad ,$$

for a.e. $(\xi,\xi^{'}) \in W_m, \eta\in\mathbb{R}$, which implies that

$$Z_W^m\phi(\xi,\xi^{'},\eta) = \Phi(\xi,\xi^{'})Z_W^m\psi(\xi,\xi^{'},\eta) \ \text{ for a.e. } (\xi,\xi^{'}) \in W_m, \eta\in\mathbb{R}.$$

Hence one can obtain $\omega_\phi^m(\xi,\xi^{'}) = \int_{\mathbb{R}} |Z_W^m\phi(\xi,\xi^{'},\eta)|^2\,d\eta = |\Phi(\xi,\xi^{'})|^2\omega_\psi^m(\xi,\xi^{'})$, which in turn leads to the fact that $\Omega_\phi^m \subset \Omega_\psi^m$. By applying Theorem 7.8 for the spaces $V_m^t(\phi)$ and $V_m^t(\psi)$, we can show that there exists a constant $A>0$ such that $|\Phi(\xi,\xi^{'})| \geq A > 0$, for a.e. $(\xi,\xi^{'}) \in W_m$. Conversely, assume (v) holds. From Theorem 7.22, there exists a function $\psi \in V_m^t(\phi)$ such that the system $\{T^t_{(mk,\frac{l}{m})}\psi : k,l \in \mathbb{Z}\}$ is the canonical dual frame for the system $\{T^t_{(mk,\frac{l}{m})}\eta_1 : k,l \in \mathbb{Z}\}$ and the following sampling formula holds:

$$f = \sum_{k,l\in\mathbb{Z}} \langle f, T^t_{(mk,\frac{l}{m})}\eta_1\rangle T^t_{(mk,\frac{l}{m})}\psi \quad \forall\ f \in V_m^t(\phi).$$

This leads to the fact that the system $\{T^t_{(mk,\frac{l}{m})}\psi : k,l \in \mathbb{Z}\}$ is a frame for $V_m^t(\phi)$. Then the result follows from Lemma 7.24. Now, from Remark 7.19 and Corollary 7.10, we have that the system $\{T^t_{(mk,\frac{l}{m})}\eta_1\}$ is a Bessel sequence in $V_m^t(\phi)$. Recall that $g_m = \sum\limits_{k,l\in\mathbb{Z}} \langle \phi, T^t_{(mk,\frac{l}{m})}\eta_1\rangle E^m_{k,l} e^{\pi ikl}$ and $\Phi = \sum\limits_{k,l\in\mathbb{Z}} \phi(mk,\frac{l}{m})E^m_{k,l}e^{\pi ikl}$. It follows from the similar steps as in (7.51), we obtain

$$\langle \phi, T^t_{(mk,\frac{l}{m})}\eta_1\rangle = \iint_{W_m} \Phi(\xi,\xi')e^{-2\pi i(mk\xi+\frac{l}{m}\xi')}e^{-\pi ikl}\, d\xi d\xi' = \phi(mk,\frac{l}{m}).$$

This shows that $g_m = \Phi$. Hence the proof of the remaining equivalence conditions follows directly from the Theorem 7.22. ☐

7.4 An application using twisted B-spline as a generator

Example 7.26. Define ϕ_1 by $\phi_1(x,y) = \chi_{[0,1)}(x)\chi_{[0,1)}(y)$. Let $\phi_2 = \phi_1 \times \phi_1$, where $\times$ is a twisted convolution. Then, the explicit form of ϕ_2 is given by

$$\phi_2(x,y) \begin{cases} \frac{2}{\pi^2 xy}(1-\cos(\pi xy)), & \text{if} \quad (x,y) \in (0,1]\times(0,1] \\ \frac{2}{\pi^2 xy}(\cos(\pi x(y-1))-\cos(\pi x)), & \text{if} \quad (x,y) \in (0,1]\times(1,2] \\ \frac{2}{\pi^2 xy}(\cos(\pi(x-1)y)-\cos(\pi y)), & \text{if} \quad (x,y) \in (1,2]\times(0,1] \\ \frac{2}{\pi^2 xy}(\cos(\pi(x-y))-\cos(\pi(x+y-xy))), & \text{if} \quad (x,y) \in (1,2)\times(1,2) \\ 0 & \text{otherwise.} \end{cases} \tag{7.54}$$

(see [46]) and is depicted in Figure 7.1.

Now, we aim to show that ϕ_2 is continuous on $\mathbb{R}^2$. Let E denote the boundary points of the square $[0,2]\times[0,2]$ The continuity of ϕ_2 on $\mathbb{R}^2 \setminus E$ follows directly from the definition of ϕ_2 since the polynomial functions with two variables and the cosine function are continuous. Let us split the set $E \setminus \{(0,0),(0,2),(2,2),(2,0)\}$ into eight segments, namely, $E_1,E_2,E_3,E_4,E_5,E_6,E_7,E_8$, where $E_1 = \{(a,0) : 0 < a \leq 1\}$, $E_2 = \{(a,0) : 1 < a < 2\}$, $E_3 = \{(2,b) : 0 < b \leq 1\}$, $E_4 = \{(2,b) : 1 < b < 2\}$, $E_5 = \{(a,2) : 1 < a < 2\}$,

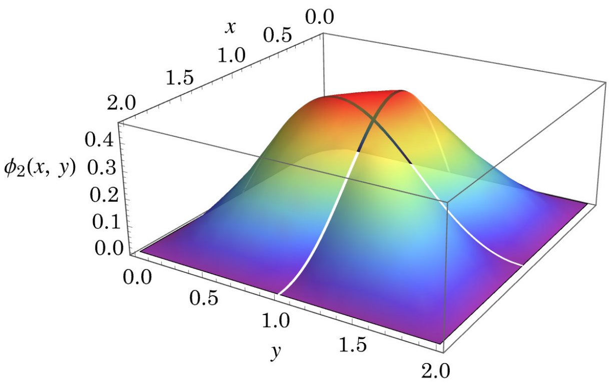

Figure 7.1: Twisted B-spline ϕ_2.

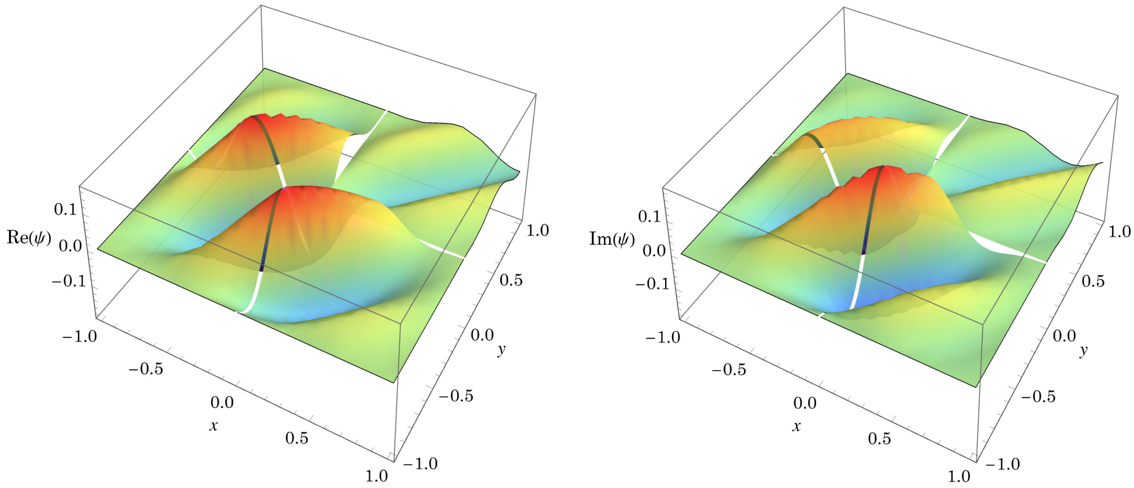

Figure 7.2: Real part of ψ (left) and imaginary part of ψ (right).

$E_6 = \{(a, 2) : 0 < a \leq 1\}$, $E_7 = \{(0, b) : 1 < b < 2\}$ and $E_8 = \{(0, b) : 0 < b \leq 1\}$. One can notice that $\phi_2(a, b) = 0$ for all $(a, b) \in E$. Hence, it is enough to show that

$$\lim_{(x,y)\to(a,b)} \phi_2(x, y) = 0 \quad \forall \quad (a, b) \in E.$$

Fix $(a, b) \in E_1 \cup E_8 \cup \{(0, 0)\}$. Now, by using Taylor series expansion for any $(x, y) \in (0, 1] \times (0, 1)$, we have

$$\frac{2}{\pi^2 xy}(1 - \cos(\pi xy)) = \frac{2}{\pi} \sum_{n\in\mathbb{N}} \frac{(-1)^n}{(2n)!} (\pi xy)^{2n-1}.$$

Thus $\phi_2(x, y) \to 0$ as $(x, y) \to (a, b)$ along any direction in a neighbourhood of (a, b), for all $(a, b) \in E_1 \cup E_2 \cup \{(0, 0)\}$.

Let $(a,0) \in E_2 \cup \{(2,0)\}$. Then, by applying the fact that $\cos(\pi xy - \pi y) - \cos(\pi y) = -2\sin(\frac{\pi xy}{2})\sin(\frac{\pi y(x-2)}{2})$ and $|\sin(\frac{\pi xy}{2})| \leq |\frac{\pi xy}{2}|$, we have

$$|\phi_2(x,y)| \leq \frac{2}{\pi}\Big|\sin(\frac{\pi y(x-2)}{2})\Big|, \quad \text{for all } x,y \in (1,2) \times (0,1).$$

Hence $\phi_2(x,y) \to 0$ as $(x,y) \to (a,0)$ along any direction in a neighbourhood of $(a,0)$, for $1 < a \leq 2$.

It is easy to show that the continuity of ϕ_2 on E_3. In fact, as (x,y) approaches $(2,b)$ along any direction in the neighbourhood $(1,2) \times (0,1)$, we have

$$\phi_2(x,y) \to \frac{2}{\pi^2 2b}(\cos(\pi b) - \cos(\pi b)) = 0.$$

Furthermore, the continuity of ϕ_2 on $E_4 \cup \{(2,2)\}$ follows in a similar way. Indeed, we have

$$\lim_{(x,y)\to(2,b)} \phi_2(x,y) = \frac{2}{\pi^2 2b}(\cos(\pi(2-b)) - \cos(\pi(2-b))) = 0. \tag{7.55}$$

Similary, we can show that the function ϕ_2 is continuous on $E_5 \cup E_6$.

Fix $(0,b) \in E_7 \cup \{(0,2)\}$. Since $\cos(\pi xy - \pi x) - \cos(\pi x) = -2\sin(\frac{\pi xy}{2})\sin(\frac{\pi x(y-2)}{2})$ and $|\sin(\frac{\pi xy}{2})| \leq |\frac{\pi xy}{2}|$, we have

$$|\phi_2(x,y)| \leq \frac{2}{\pi}\Big|\sin(\frac{\pi x(y-2)}{2})\Big|, \quad \text{for all } x,y \in (0,1) \times (1,2).$$

Thus $\phi_2(x,y) \to 0$ as $(x,y) \to (0,b)$ along any direction in a neighbourhood of $(0,b)$, for $1 < b \leq 2$. This completes the proof that the function ϕ_2 is continuous on $\mathbb{R}^2$.

In order to apply the Theorem 7.25, it is necessary to verify the uniform boundedness of ϕ_2 and Riesz property of the system $\{T^t_{(k,l)}\phi_2 : k,l \in \mathbb{Z}\}$. Since supp $\phi_2 = [0,2] \times [0,2]$, which in turn leads to the fact that

$$\sup_{x,y\in\mathbb{R}} \sum_{k,l\in\mathbb{Z}} |\phi_2(x-k, y-l)|^2 < \infty.$$

It has been proved that $\{T^t_{(k,l)}\phi_2 : k,l \in \mathbb{Z}\}$ is a Riesz sequence by using Gram matrix in [46]. Now, we aim to obtain a sampling formula for a function $f \in V^t(\phi_2)$ by using the Theorem 7.25. As $\phi_2(k,l) = 0$ for all $k,l \in \mathbb{Z} \setminus \{1\}$, we have

$$\Phi(\xi,\xi') = -\phi_2(1,1)e^{2\pi i(\xi+\xi')} = -\frac{4}{\pi^2}e^{2\pi i(\xi+\xi')}. \tag{7.56}$$

Moreover, there exist two positive constants A, B such that $A \leq |\Phi(\xi, \xi^{'})| \leq B$ for all $(\xi, \xi^{'}) \in [0, 1) \times [0, 1)$. It follows from the proof of the Theorem 7.25 that the relation between the functions ϕ_2 and ψ is given by

$$Z_W\phi_2(\xi, \xi^{'}, \eta) = \Phi(\xi, \xi^{'})Z_W\psi(\xi, \xi^{'}, \eta) \quad \text{for a.e. } (\xi, \xi^{'}) \in [0, 1) \times [0, 1), \eta \in \mathbb{R}. \tag{7.57}$$

By substituting (7.56) in (7.57), we obtain

$$Z_W\psi(\xi, \xi^{'}, \eta) = -\frac{4}{\pi^2}e^{2\pi i(-\xi-\xi^{'})}Z_W\phi_2(\xi, \xi^{'}, \eta) \quad \text{for a.e.} \, (\xi, \xi^{'}) \in W_1, \eta \in \mathbb{R}.$$

Now, by making use of the Proposition 7.7, we have

$$\psi = -\phi_2(1, 1)T^t_{(-1,-1)}\phi_2,$$

whose real and imaginary parts are shown in Fig. 7.2. Then, for any $f \in V^t(\phi_2)$, we have

$$f = \sum_{k,l\in\mathbb{Z}} f(k, l)T^t_{(k,l)}\psi,$$

where the convergence is in the $L^2(\mathbb{R}^2)$-sense and uniform on $\mathbb{R}^2$. In addition, the sampling formula can be rewritten as

$$\begin{aligned} f &= -\frac{4}{\pi^2}\sum_{k,l\in\mathbb{Z}} f(k, l)T^t_{(k,l)}T^t_{(-1,-1)}\phi_2 \\ &= -\frac{4}{\pi^2}\sum_{k,l\in\mathbb{Z}} e^{\pi i(k-l)}f(k, l)T^t_{(k-1,l-1)}\phi_2 \\ &= -\frac{4}{\pi^2}\sum_{k,l\in\mathbb{Z}} e^{\pi i(k-l)}f(k + 1, l + 1)T^t_{(k,l)}\phi_2 \end{aligned}$$

for any $f \in V^t(\phi_2)$.

Chapter 8

Summary and future directions

The first chapter of this thesis provides an overview of the research area, outlining the motivation, background, and organization of the work presented. **Chapter 2** investigates a Donoho–Stark type uncertainty principle for convolutional tight frames arising from finite-dimensional filter banks. The classical uncertainty bound between a signal and its Fourier transform is extended to a frame-theoretic setting by linking the support of frame coefficients with stable signal recovery under erasures. The results are further refined using restriction-type estimates for filter banks, with applications to signal recovery and sparse representation. **Chapter 3** explores perfect reconstruction properties of filter banks constructed from Ramanujan sums, analyzing their robustness to erasures in the finite-dimensional space $\ell^2(\mathbb{Z}_N)$. Non-uniform Ramanujan filter banks are developed to overcome the limitations of uniform ones, and an uncertainty principle associated with Ramanujan tight frames is established, providing sufficient conditions for perfect recovery in noisy or incomplete sampling scenarios. **Chapter 4** focuses on random periodic sampling involving derivatives in periodic shift-invariant (PSI) spaces over $L^2(\mathbb{T})$. Extending probabilistic sampling techniques, the chapter establishes sampling inequalities with high probability using tools such as the Zak transform and fiberization, and demonstrates applications to trigonometric polynomials, periodic multiband signals, and wavelet subspaces. **Chapter 5** develops a unified framework for multi-channel stable sampling in shift- and translation-invariant (SI/TI) spaces on locally compact, possibly non-abelian, groups. Through the theory of multiplication-invariant (MI) spaces in $L^2(X, \mathcal{H})$, and employing range functions and the generalized Zak transform, the chapter establishes stability and density conditions that generalize classical sampling results beyond abelian settings. **Chapter 6** addresses the injectivity of sampling operators associated with translation-invariant systems defined on unions of TI spaces. By analyzing the supremum cosine angle between constituent MI spaces, sufficient conditions for injectivity and closedness of the sum of spaces are derived and extended to TI spaces on locally compact groups through the generalized Zak transform. **Chapter 7** extends the framework of multi-channel stable sampling to generalized twisted shift-invariant (GTSI) spaces on the Heisenberg group. Using the generalized Weyl–Zak transform, the chapter provides necessary and sufficient conditions for twisted translates to form a Riesz sequence or a frame, yielding sampling and reconstruction formulas within a non-commutative setting. As a special case, a Whittaker–Shannon–Kotelnikov type sampling theorem is derived, and the theory is illustrated for the GTSI space generated by a twisted B-spline.

An exciting extension of the work done in Chapter 2 is to obtain sharp uncertainty principles for filter banks using the concept of additive energy. Recent work by K. Aldahleh *et al.* [5] employed combinatorial techniques to derive refined forms of the Donoho–Stark uncertainty principle in terms of additive energy. Motivated by their approach, one may pursue analogous results in the filter-bank setting, thereby establishing new quantitative bounds connecting sparsity and spectral concentration.

A natural continuation of the ideas developed in Chapter 3 involves the formulation of *Affine Ramanujan sums*, which generalize classical Ramanujan sums through the discrete affine Fourier transform (DAFT). These affine extensions preserve key number-theoretic and orthogonality properties while introducing an additional affine degree of freedom, enabling the analysis of non-stationary and chirp-periodic signals that classical Ramanujan sums cannot capture. Future research will focus on establishing frame-theoretic conditions for perfect reconstruction, exploring uncertainty relations between canonical and affine representations, and developing applications in denoising and reconstruction of non-stationary or modulated signals. This direction naturally bridges arithmetic harmonic analysis with affine signal models, opening avenues for robust, structure-aware signal recovery.

Another important direction is to extend the results of Chapter 5 to the *Heisenberg group* in particular. Such an extension would explore the interplay between non-commutative group structures and filter-bank or sampling-theoretic frameworks, potentially leading to new uncertainty principles, frame constructions, and reconstruction algorithms in the setting of the Heisenberg group. This direction also connects harmonic analysis on groups with applications in quantum signal processing and time-frequency analysis.

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