RAMANUJAN'S FIVE ENTRIES, WEIGHTED PARTITION IDENTITIES AND DIVISOR GENERATING q-SERIES WITH APPLICATIONS TO PROBABILITY THEORY AND RANDOM GRAPHS

Ph.D. Thesis

By

ARCHIT AGARWAL

DEPARTMENT OF MATHEMATICS

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **RAMANUJAN'S FIVE ENTRIES, WEIGHTED PARTITION IDENTITIES AND DIVISOR GENERATING *q*-SERIES WITH APPLICATIONS TO PROBABILITY THEORY AND RANDOM GRAPHS** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DEPARTMENT OF MATHEMATICS**, **Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from Decemeber 2020 to December 2025 under the supervision of **Dr. Bibekananda Maji**, Associate Professor, Department of Mathematics, IIT Indore.

The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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LIST OF PUBLICATIONS

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ABSTRACT

Ramanujan recorded many q-series identities in his notebooks and lost notebook. At the end of his second notebook, he mentioned five q-series identities. In 2021, Dixit and Maji obtained a q-series identity that enabled them to derive three of five Ramanujan's q-series identities. Later, a unified generalization of these five q-series identities was obtained by Bhoria, Eyyunni and Maji and a finite analogue of this generalization was subsequently established by Dixit and Patel, yielding finite analogues of all five identities of Ramanujan. One of the primary objectives of this thesis is to develop a one variable generalization of the aforementioned identity of Bhoria et. al., together with its finite analogue, thereby extending the work of Dixit and Patel. As a consequence, we derive one variable generalizations of each of Ramanujan's five identities along with their corresponding finite analogues.

Further, we study a divisor generating q-series identity of Uchimura (1981), which has applications in probability theory. Mainly, the following identity of Uchimura

$$\sum_{n=1}^{\infty} nq^n (q^{n+1})_\infty = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\frac{n(n+1)}{2}}}{(1-q^n)(q)_n} = \sum_{n=1}^{\infty} \frac{q^n}{1-q^n},$$

has inspired numerous generalizations. Uchimura himself extended it in 1987, followed by further refinements due to Dilcher (1995), Andrews–Crippa–Simon (1997) and Gupta–Kumar (2021). Any generalization of the right most expression of the above identity, we name as *divisor-type* sum, whereas a generalization of the middle expression we say *Ramanujan-type* sum and any generalization of the left most expression we refer it as *Uchimura-type* sum. Parallel to these developments, in 1984, Bressoud and Subbarao extracted a weighted partition identity from Uchimura's identity that connect partition function and divisor function. They not only proved their result using a purely combinatorial argument but also gave a more general result for a generalized divisor function. Quite remarkably, Simon, Crippa and Collenberg (1993) discovered that the same divisor generating function also arises in the study of random acyclic digraphs, offering a surprising bridge between number theory and random graphs. In this thesis, we revisit Bressoud and Subbarao's combinatorial argument and gave a more general weighted partition identity. Further, by employing a fractional differential operator, we establish several new Bressoud–Subbarao type identities arising from an identity of Andrews, Garvan and Liang. We also obtain a one variable generalization of an identity of Uchimura related to Bell polynomials and study its application to probability theory. We then extend the results of Andrews–Crippa–Simon by establishing explicit formulas for the limit of the t-th cumulant in terms of generalized divisor functions. This produces new limit forms for identities of Uchimura and Dilcher and offers an enriched probabilistic interpretation of divisor-type generating series. In doing so, the thesis highlights a surprising and elegant interplay between number theory and probabilistic combinatorics. These results collectively provide a fourth side to the *Uchimura–Ramanujan–divisor-type* identity.

Keywords: partitions, q-series, partition identities, generalized divisor function, Bressoud–Subbarao's identity, weighted partition identities, basic hypergeometric series, finite analogues, probability distributions, random graphs.

Contents

Chapter 1

INTRODUCTION

1.1 Partitions

Over time, number theory has fascinated mathematicians for its aesthetic beauty and deep structural richness. Among its many branches, partition theory and q-series play a central role, forming a vibrant intersection of combinatorics, analysis and arithmetic and offering a rich framework for understanding the additive structure of integers. At its core, partition theory concerns the decomposition of a positive integer into sums of positive integers, where the order of summands is disregarded. Classical results in this domain, prominently including those of Euler and Ramanujan provide elegant identities that enumerate partitions adhering to specific combinatorial restrictions, such as partitions into distinct parts, odd parts, or partitions satisfying modular conditions. These classical partition identities have not only shaped the development of combinatorics but have also found profound connections to areas such as modular forms, q-series, algebra, and mathematical physics.

A partition of a positive integer n is defined as a finite non-increasing sequence of positive integers whose sum is n. Each individual number in the sequence is referred as a part of the partition. For example, the number 4 can be partitioned in the following five ways:

$$\begin{aligned}
&4,\\
&3+1,\\
&2+2,\\
&2+1+1,\\
&1+1+1+1.
\end{aligned}$$

Thus, the number of partitions of 4 is 5. The partition function $p(n)$ denotes the number of partitions of the integer n, so $p(4) = 5$. The study of integer partitions dates back to the works of Euler, who laid the foundations by introducing generating functions to encode partition numbers. Euler showed that the generating function for the number of unrestricted partitions $p(n)$ is given by the infinite product:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$

where q be a complex number with $|q| < 1$. This infinite product not only encodes the combinatorics of partitions but also reveals deep connections with infinite series, modular forms, and asymptotic analysis. This is also a prototype of what is now known as a q-series, a power series in the indeterminate q, which serves as a discrete analogue of power series in analysis. These series often serve as generating functions that encode enumeration problems in number theory and combinatorics. The works of mathematicians such as Jacobi and, more notably, Ramanujan brought renewed focus to q-series and their surprising identities. Ramanujan's contributions include the discovery of mock theta functions, partition congruences, and intricate q-hypergeometric identities, many of which remain active areas of research today. In 1919, Ramanujan [37], in particular, discovered congruences for the partition function, such as:

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11},$$

which spurred a surge in the analytical and algebraic exploration of partition functions. He [37] proved first two congruences by giving the following q-series identities

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)^6},$$
$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\prod_{n=1}^{\infty} \frac{(1-q^{7n})^3}{(1-q^n)^4} + 49q^7\prod_{n=1}^{\infty} \frac{(1-q^{7n})^7}{(1-q^n)^8},$$

and later in 1920, announced in a short note [38] that he had also found a proof of third congruence. In the same paper, he also remark that "It appears that there are no equally simple properties for any moduli involving primes other than these three." In [37], Ramanujan gave a more general conjecture for partition congruences, namely,

$$p(n\delta+\lambda) \equiv 0 \pmod{\delta},$$

where δ is any natural number of the form $5^a7^b11^c$ and λ is a solution of $24\lambda \equiv 1 \pmod{\delta}$. Unfortunately, there was an error in the above conjecture which was later corrected by Chowla [20] in 1934.

Beyond their intrinsic interest, partitions and q-series have wide-ranging applications. They appear in the representation theory of symmetric groups, statistical mechanics, and the study of modular and automorphic forms. Their analytic structure allows the use of tools from complex analysis, such as the circle method developed by Hardy and Ramanujan, to estimate the growth of partition functions and to uncover asymptotic behavior. Modern development in partition theory include refined concept such as weighted partitions, partition statistics (like rank and crank), and connections with modular and mock modular forms. The exploration of partition identities between generating functions corresponding to different partition classes continues to be an active area of study. Many of these identities are proved or generalized using q-series techniques, Bailey pairs, the Rogers–Ramanujan identities, and their generalizations. Partition theory has evolved into a rich connections to combinatorics. The combinatorial methods has significantly advanced our understanding of weighted partition identities, such as those studied by Alladi [5, 6, 7], Berkovich and Uncu [15], Bressoud and Subbarao [18], Garvan [28], and has opened pathways to new identities. It provides an elegant framework for understanding additive structures in integers and yields numerous identities.

While these classical partition identities count partitions in a uniform manner, assigning equal weight to each partition, a more nuanced perspective arises through the introduction of weighted partition identities. In this generalized framework, each partition is assigned a numerical weight, often determined by statistics such as the number of parts, parity conditions, or other combinatorial parameters intrinsic to the partition's structure. The sum of these weights over all partitions of an integer produces generating functions that encode richer arithmetic and combinatorial information than ordinary counts. Such weighted identities frequently reveal deeper symmetries, modularity properties, and combinatorial bijections.

Over the past century, the study of weighted partition identities has become an active and fruitful area of research. Inspired by Ramanujan's extensive work, mathematicians have rigorously developed and extended these identities, uncovering connections to mock modular forms, q-hypergeometric series, and representation theory. These weighted sums not only generalize classical enumerations but also provide refined invariants that bridge discrete combinatorics with analytic structures. Consequently, weighted partition identities represent a powerful tool for capturing subtle phenomena in the arithmetic and combinatorial behavior of partitions, significantly enriching the theoretical landscape beyond classical partition theory.

1.2 Ramanujan's five q-series identities

The study of q-series identities has long been a central theme in partition theory, modular forms, and combinatorial number theory. Among the many contributions of Ramanujan, a particularly intriguing set of results lies at the end of his Second Notebook [39, pp. 354–355], [40, pp. 302–303], where he recorded five striking q-series identities. Unlike the more structured chapters in his notebooks, this list of five identities appears with minimal commentary and without proofs. For decades, these identities attracted comparatively less attention compared to his more famous formulas, such as the Rogers–Ramanujan identities or mock theta functions. These five identities were first rigorously analyzed and proved by Berndt [16, pp. 262–265], who systematically annotated Ramanujan's notebooks in a series of volumes. Before stating these five identities, we introduce an important notation in the theory of q-series, namely q-Pochhammer symbol. For complex numbers A and q, with $|q| < 1$, the q-Pochhammer symbol is defined as

$$(A)_0 := (A;q)_0 = 1, \quad (A)_n := (A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}), \quad n \geq 1,$$
$$(A)_\infty := (A;q)_\infty := \lim_{n\to\infty}(A;q)_n.$$

We are now ready to state five identities of Ramanujan.

Entry 1: Let $n \in \mathbb{N}$, $a \neq 0, b \neq q^{-n}$. Then

$$\frac{(-aq)_\infty}{(bq)_\infty} = \sum_{n=0}^{\infty} \frac{(-b/a)_n a^n q^{n(n+1)/2}}{(q)_n (bq)_n}. \tag{1.2.1}$$

One can also find (1.2.1) in the Lost Notebook [41, p. 370].

Entry 2: Let $1 - aq^n \neq 0$ for $n \geq 1$. Then

$$(aq)_\infty \sum_{n=1}^{\infty} \frac{na^n q^{n^2}}{(q)_n (aq)_n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} a^n q^{n(n+1)/2}}{1-q^n}. \tag{1.2.2}$$

Entry 3: For $a \neq 0$, and $1 - bq^n \neq 0, n \geq 0$, one has

$$\sum_{n=1}^{\infty} \frac{(b/a)_n a^n}{\left(1-q^n\right)(b)_n} = \sum_{n=1}^{\infty} \frac{a^n - b^n}{1-q^n}. \tag{1.2.3}$$

Letting $a \to 0$, and replacing b by aq gives the next identity.

Entry 4: For $1 - aq^n \neq 0, n \geq 1$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} a^n q^{\frac{n(n+1)}{2}}}{\left(1-q^n\right)(aq)_n} = \sum_{n=1}^{\infty} \frac{a^n q^n}{1-q^n}. \tag{1.2.4}$$

The last identity is the following:

Entry 5: For $1 - aq^n \neq 0, n \geq 0$, we have

$$\sum_{n=1}^{\infty} \frac{a^n (q)_{n-1}}{\left(1-q^n\right)(a)_n} = \sum_{n=1}^{\infty} \frac{na^n}{1-q^n}. \tag{1.2.5}$$

Berndt was the one, who presented these identities as Entries 1 through 5. The identity (1.2.4) was rediscovered by Uchimura [43, Equation (3)] and later by Garvan

[29]. Kluyver [35] discovered a particular case $a = 1$ of (1.2.4):

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\frac{n(n+1)}{2}}}{\left(1-q^{n}\right)(q)_n} = \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}. \tag{1.2.6}$$

This identity was also rediscovered by Fine [26, p. 14, Equation (12.4), (12.42)], Uchimura [43, Theorem 2] and Zudilin [46, p. 4]. Ando [8] gave a combinatorial proof of the above identity. Moreover, Uchimura not only recovered Kluyver's result but also gave a new expression to it. He make use of a sequence of polynomials $U_1(x) = x$, $U_n(x) = nx^n + (1-x^n)U_{n-1}(x)$ to get the new expression. These polynomials are also related to the analysis of the data structure called "heap". He showed that

$$\sum_{n=1}^{\infty} nq^n (q^{n+1})_\infty = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\frac{n(n+1)}{2}}}{\left(1-q^{n}\right)(q)_n} = \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}. \tag{1.2.7}$$

Bressoud and Subbarao [18] established a combinatorial interpretation of the above identity. They proved that

$$\sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)-1} s(\pi) = d(n), \tag{1.2.8}$$

where $\mathcal{D}(n)$ denotes the set of all distinct partitions of n, $\#(\pi)$ is the number of parts of the partition π, $s(\pi)$ is its smallest part, and $d(n)$ denotes the number of divisors of n. We will discuss this identity in detail in the upcoming sections.

Uchimura's identity (1.2.7) has been generalized by several mathematicians, and it also admits a natural partition-theoretic interpretation. Before presenting these generalizations, we introduce some terminology that will be used frequently throughout. Any generalization of the leftmost expression of the identity (1.2.7) will be referred to as an *Uchimura-type sum*. Similarly, any generalization of the middle expression will be called a *Ramanujan-type sum*, while any generalization of the rightmost expression will be termed a *divisor-type sum*, since it serves as a generating function for the well-known divisor function $d(n)$.

Recently in 2020, Dixit and Maji [24, Theorem 2.1] derived a one variable generalization of Ramanujan's Entry 3 (1.2.3). Specifically, for complex numbers a, b, and c satisfying $|a| < 1$ and $|cq| < 1$, they proved the following identity:

$$\sum_{n=1}^{\infty} \frac{(b/a)_n a^n}{(1-cq^n)(b)n} = \sum_{n=0}^{\infty} \frac{(b/c)_n c^n}{(b)_n} \left(\frac{aq^n}{1-aq^n} - \frac{bq^n}{1-bq^n} \right). \tag{1.2.9}$$

Furthermore, in the same work, they also derived a generalization of Entry 4 (1.2.4) by utilizing (1.2.9). For $|cq| < 1$, they showed that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n q^{\frac{n(n+1)}{2}}}{(1-cq^n)(zq)_n} = \frac{z}{c} \sum_{n=1}^{\infty} \frac{(zq/c)_{n-1}}{(zq)_n} (cq)^n. \tag{1.2.10}$$

This identity serves as a one-variable generalization of a result due to Andrews, Garvan, and Liang [14, Theorem 3.5]. By manipulating this identity, Dixit and Maji also provide a natural proof of the following identity of Garvan [29, Equation (1.3)], valid for $|z| \leq 1$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n q^{n^2}}{(zq;q^2)n(1-zq^{2n})} = \sum_{n=1}^{\infty} \frac{(q;q)_{n-1} z^n q^{\frac{n(n+1)}{2}}}{(zq;q)_n}. \tag{1.2.11}$$

In a recent work, Bhoria, Eyyunni, and Maji [17, Theorem 2.1] established a one-variable generalization of the identity (1.2.9) of Dixit and Maji. For complex numbers $a,\ b,\ c$, and d with $|ad| < 1$ and $|cq| < 1$, they proved that

$$\sum_{n=1}^{\infty} \frac{(b/a)_n (c/d)_n (ad)^n}{(b)_n (cq)_n} = \frac{(a-b)(d-c)}{(ad-b)} \sum_{n=0}^{\infty} \frac{(a)_n (bd/c)_n c^n}{(b)_n (ad)_n} \left(\frac{adq^n}{1-adq^n} - \frac{bq^n}{1-bq^n} \right). \tag{1.2.12}$$

Furthermore, they obtained a one-variable generalization of (1.2.10). Specially, for $|cq| < 1$, they proved that

$$\sum_{n=1}^{\infty} \frac{(-z)^n (c/d)_n d^n q^{\frac{n(n+1)}{2}}}{(cq)_n (zq)_n} = \frac{z(c-d)}{c} \sum_{n=1}^{\infty} \frac{(zdq/c)_{n-1}}{(zq)_n} (cq)^n.$$

By employing these general identities, Bhoria, Eyyunni, and Maji [17] successfully derived all five of Ramanujan's identities (1.2.1)–(1.2.5). In addition, they also recovered a well-known q-series identity due to Andrews [9, p. 24, Corollary 2.2], namely,

$$\sum_{n=1}^{\infty} \frac{z^n c^n q^{n^2}}{(zq)_n (cq)_n} = z \sum_{n=1}^{\infty} \frac{(cq)^n}{(zq)_n}. \tag{1.2.13}$$

In [1], we introduced a one-variable generalization of the identity (1.2.12) due to Bhoria, Eyyunni, and Maji. This generalization extends their result in a natural way by incorporating an additional parameter, which allows us to study a wider family of q-series. One of the main advantages of this extension is that it unifies several classical identities and provides a systematic approach to deriving their one-variable analogues. Using this generalization, we were able to obtain individual one-variable extensions of each of Ramanujan's five entries (1.2.1)–(1.2.5). These new identities not only refine the classical forms given by Ramanujan but also reveal deeper structural connections among them.

The detailed proofs, along with a discussion of the underlying ideas, applications, and consequences of these one-variable generalizations, will be presented in Chapter 2. There, we also highlight how these generalizations relate to other known results in the literature and how they fit into the broader framework of q-series identities.

The next section is devoted to exploring finite analogues of the q-series identities discussed in the preceding sections, providing a deeper understanding of how these classical infinite identities behave in a finite setting.

1.3 Finite analogue of Ramanujan's five identities

Finite analogues, or truncated versions, of q-series identities have received significant attention in recent years. They serve as polynomial generalizations of classical infinite identities and often reveal combinatorial and congruence properties not apparent in the original forms. Such finite analogues also appear naturally in the study of restricted partition functions. For example, van Hamme [34] obtained the following finite form:

$$\sum_{n=1}^{N} \frac{q^n}{1-q^n} = \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^{n-1} q^{n(n+1)/2}}{1-q^n}, \tag{1.3.1}$$

where

$$\begin{bmatrix} N \\ n \end{bmatrix} = \begin{bmatrix} N \\ n \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_N}{(q;q)_n (q;q)_{N-n}}, & \text{if } 0 \leq n \leq N, \\ 0, & \text{otherwise}, \end{cases}$$

is the q-binomial coefficient. It is immediate that, upon letting $N \to \infty$ in (1.3.1), one recovers the identity of Kluyver (1.2.6). In 2015, Guo and Zeng [31, Equation (3.8)] obtained a finite analogue of the first and third expressions in Uchimura's identity (1.2.7), namely,

$$\sum_{n=1}^{N} \frac{q^n}{1-q^n} = \sum_{n=1}^{N} n q^n (q^{n+1})_{N-1} - \sum_{n=1}^{\infty} n q^{n+N} (q^{n+1})_{N-1}.$$

They further provided a partition theoretic interpretation of the above identity, which gives finite analogue of Bressoud and Subbarao's identity (1.2.8). Let $d(n, N)$ denote the number of divisors of n which are less than or equal to N. Then

$$d(n, N) = t(n, N) - t(n - N, N),$$

where

$$t(n, N) := \sum_{\pi \in \mathcal{D}(n,N)} (-1)^{\#(\pi)-1} s(\pi),$$

and $\mathcal{D}(n, N)$ is the collection of partitions of n into distinct parts such that $\ell(\pi) - s(\pi) \leq N - 1$.

In 2021, Dixit, Eyyunni, Maji and Sood [23, Theorem 1.1] established a finite analogue of the identity (1.2.9). For any natural number N, complex numbers $a,\ b,\ c$ satisfying $a, b, c \neq q^{-n}, 1 \leq n \leq N-1, c \neq q^{-N}$, and $a, b \neq 1$, they proved that

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(a)_{N-n} (b/a)_n (q)_n a^n}{(a)_N (1 - cq^n)(b)_n}$$

$$= \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(cq)_{N-n}(b/c)_{n-1}(q)_n c^{n-1}}{(cq)_N (b)_{n-1}} \left(\frac{aq^{n-1}}{1-aq^{n-1}} - \frac{bq^{n-1}}{1-bq^{n-1}} \right). \tag{1.3.2}$$

Further, for complex numbers z and c satisfying $z, c \neq q^{-n}, 1 \leq n \leq N$, they obtained a finite analogue of (1.2.10), given by

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^{n-1} z^n q^{\frac{n(n+1)}{2}} (q)_n}{(1-cq^n)(zq)_n} = \frac{z}{c} \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(cq)_{N-n}(zq/c)_{n-1}(q)_n (cq)^n}{(cq)_N (zq)_{n-1}}. \tag{1.3.3}$$

This identity, apart from its combinatorial significance, also serves as a useful tool for deriving finite analogues of several other known q-series identities. In particular, from (1.3.3), Dixit et. al. [23, Theorem 1.3] further obtained a finite analogue of Garvan's identity (1.2.11). Specifically, for $z \neq q^{-n}, 1 \leq n \leq 4N-1$, they showed that

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} \frac{(-1)^{n-1}(q^2;q^2)_n z^n q^{n^2}}{(zq;q^2)_n (1-zq^{2n})}$$

$$= \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} \left(\frac{(q)_{2n-2} z^{2n-1} q^{n(2n-1)}}{(zq)_{2n-1}} + \frac{(q)_{2n-1} z^{2n} q^{n(2n+1)}}{(zq)_{2n}} \right) \frac{(q^2;q^2)_n}{(zq^{2N+1};q^2)_n}. \tag{1.3.4}$$

Building upon these results, Dixit and Patel [25, Theorem 2.1] introduced a one-variable generalization of the above finite analogue identities. They derived a finite analogue of the identity (1.2.12) of Bhoria et al., which itself generalizes (1.3.2). They proved that

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_n (b/a)_n (c/d)_n (ad)_{N-n} (ad)^n}{(b)_n (cq)_n (ad)_N} = \frac{(a-b)(d-c)}{(ad-b)}$$

$$\times \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_n (cq)_{N-n} (a)_{n-1} (bd/c)_{n-1} c^{n-1}}{(b)_{n-1} (ad)_{n-1} (cq)_N} \left(\frac{adq^{n-1}}{1-adq^{n-1}} - \frac{bq^{n-1}}{1-bq^{n-1}} \right). \tag{1.3.5}$$

Dixit and Patel also obtained the following one variable generalization of (1.3.3):

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_n (c/d)_n (-zd)^n q^{n(n+1)/2}}{(zq)_n (cq)_n} = \frac{z(c-d)}{c} \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_n (cq)_{N-n} (zdq/c)_{n-1} (cq)^n}{(zq)_n (cq)_N}. \tag{1.3.6}$$

In addition, Dixit and Patel derived finite analogues of all five entries (1.2.1)–(1.2.5) of Ramanujan as direct consequences of (1.3.5) and (1.3.6). These results elegantly demonstrate how Ramanujan's q-series identities can be interpreted and extended in the context of finite sums, thereby connecting classical theory with modern developments in basic hypergeometric transformations. Now we present finite analogues of five entries of Ramanujan obtained by Dixit and Patel [25]. The finite analogue of

Entry 1 (1.2.1) is given by

$$\sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{\left(\frac{-b}{a}\right)_n a^n q^{\frac{n(n+1)}{2}}}{(bq)_n} = \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{\left(\frac{-a}{b}\right)_n (bq)_{N-n} (bq)^n}{(bq)_N}. \tag{1.3.7}$$

A finite analogue of Entry 2 (1.2.2) is

$$(aq)_N \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{na^n q^{n^2}}{(aq)_n} = \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_n (-1)^{n-1} a^n q^{\frac{n(n+1)}{2}}}{1-q^n}. \tag{1.3.8}$$

A finite analogue of Entry 3 (1.2.3) takes the following form:

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_n \left(\frac{b}{a}\right)_n (a)_{N-n} a^n}{(b)_n (1-q^n)(a)_N} = \sum_{m=1}^{\infty} \frac{(a^m - b^m)(1-q^{mN})}{1-q^m}. \tag{1.3.9}$$

Similarly, a finite analogue of Entry 4 (1.2.4) is expressed as

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^{n-1} a^n q^{\frac{n(n+1)}{2}} (q)_n}{(1-q^n)(aq)_n} = \sum_{n=1}^{N} \frac{aq^n}{1-aq^n}. \tag{1.3.10}$$

Finally, a finite analogue of Entry 5 (1.2.5) can be written as

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_n (q)_{n-1} (a)_{N-n} a^n}{(a)_n (1-q^n)(a)_N} = \sum_{n=1}^{N} \frac{aq^{n-1}}{(1-aq^{n-1})^2}. \tag{1.3.11}$$

Letting $N \to \infty$ in the above finite analogues, one can easily recover five entries of Ramanujan. Furthermore, Dixit and Patel obtained a finite analogue identity involving basic hypergeometric series ${}_2\phi_1$, given by

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{n(-1)^{n-1}(c/d)_n d^n q^{n(n+1)/2}}{(cq)_n} + \frac{(c/d)_\infty (dq)_\infty}{(cq)_\infty (dq^{N+1})_\infty} \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{d^n q^{n(n+1)}}{(dq)_n (1-q^n)}$$
$$\times {}_2\phi_1 \left(\begin{matrix} dq, & dq^{N+1} \\ & dq^{n+1} \end{matrix} ; \frac{cq^{n+1}}{d} \right) = \frac{c}{c-d} \left(1 - \frac{(dq)_N}{(cq)_N}\right) + \frac{1}{(cq)_N} \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(cq/d)_n (dq)_{N-n} (dq)^n}{(1-q^n)}. \tag{1.3.12}$$

The identity (1.3.12) plays a crucial role in connecting finite q-series with partition-theoretic generating functions. As an application, Dixit and Patel utilized it to give a new proof of Andrews' celebrated identity for the smallest parts function [12, Theorem 3]. They also derived several elegant q-series identities as corollaries, demonstrating the depth and versatility of the finite analogue framework. A detailed discussion of these applications can be found in [25, Section 5].

In [1], we established a one variable generalization of the identity (1.3.5) due to Dixit and Patel. This generalization allowed to obtain a one variable extension of the finite analogues (1.3.7)–(1.3.11) of Ramanujan's five entries. In addition to this, we derived a one-variable generalization of a finite analogue (1.3.4) of Garvan's identity (1.2.11). Our work also produced an identity involving the basic hypergeometric series ${}_3\phi_2$ and ${}_2\phi_1$, which may be viewed as a one variable extension of the identity

(1.3.12) obtained earlier by Dixit and Patel. All these results, along with their proofs, implications, and connections to the theory of basic hypergeometric series, will be discussed in detail in Chapter 2 of this thesis.

In the next section, we discuss the weighted partition identity of Bressoud and Subbarao that arises from Uchimura's identity (1.2.7), and we also describe how this identity has been further generalized in subsequent work.

1.4 Bressoud–Subbarao type weighted partition identities

In 1984, by comparing the coefficients of q^n from the identity (1.2.7) of Uchimura, Bressoud and Subbarao [18] extracted a beautiful partition identity (1.2.8) that connect divisor function with the partition function. Their identity linked the divisor function to weighted counts over partitions, opening a rich interplay between additive combinatorics and arithmetic functions. These identities not only extended the combinatorial reach of partition theory but also hinted at deeper algebraic and analytic structures underlying the relationships between partition function and divisor function. Before proceeding further, we record a few notations that will be used frequently in what follows.

- π: an integer partition,
- $p(n)$:= the number of integer partitions of n,
- $p^{(t)}(n)$:= the number of partitions of n into exactly t distinct part sizes,
- $s(\pi)$:= the smallest part of π,
- $\ell(\pi)$:= the largest part of π,
- $\#(\pi)$:= the number of parts of π,
- $\nu_d(\pi)$:= the number of parts of π without multiplicity,
- $\mathcal{P}(n)$:= collection of all integer partitions of n,
- $\mathcal{D}(n)$:= collection of all partitions of n into distinct parts,

Bressoud and Subbarao's identity (1.2.8) was later rediscovered by Fokkink, Fokkink and Wang [27]. In his famous paper on the spt-function, Andrews [12] proved (1.2.8) by differentiating the q-analogue of Gauss's theorem [10, p. 20, Corollary 2.4]. Bressoud and Subbarao proved (1.2.8) using a purely combinatorial argument and further gave a more general result involving generalized divisor function. Mainly, they proved for any non-negative integer z,

$$\sum_{\pi\in\mathcal{D}(n)} (-1)^{\#(\pi)-1} \sum_{j=1}^{s(\pi)} (\ell(\pi) - s(\pi) + j)^z = \sigma_z(n), \tag{1.4.1}$$

where $\sigma_z(n) = \sum_{d|n} d^z$. However, they did not derive a corresponding q-series identity from which (1.4.1) could be obtained. Recently, Bhoria, Eyyunni and Maji [17, Remark 4] discovered a generating function identity that encapsulates (1.4.1). Furthermore, they [17, Theorem 2.4] also derived a generalization of (1.4.1). For any natural number n, integer z and a complex number c, they proved that

$$\sum_{\pi\in\mathcal{D}(n)} (-1)^{\#(\pi)-1} \sum_{j=1}^{s(\pi)} (\ell(\pi) - s(\pi) + j)^z \, c^{\ell(\pi)-s(\pi)+j} = \sigma_{z,c}(n), \tag{1.4.2}$$

where

$$\sigma_{z,c}(n) = \sum_{d|n} d^z c^d. \tag{1.4.3}$$

Substituting $c = 1$ in the above identity, one can see (1.4.1) is actually valid for *any* integer z. Their approach utilizes a combination of differential and integral operators applied successively to the partition-theoretic interpretation of Ramanujan's Entry 4 (1.2.4). In a related study, Andrews, Garvan, and Liang [14] examined (1.2.8), denoting its left-hand side by $\mathrm{FFW}(1, n)$ and subsequently introducing the generalized form $\mathrm{FFW}(c, n)$, defined by

$$\mathrm{FFW}(c, n) := \sum_{\pi\in\mathcal{D}(n)} (-1)^{\#(\pi)-1} \left(1 + c + \cdots + c^{s(\pi)-1}\right).$$

They [14, Theorem 3.5] further recovered a result of Yan and Fu [45, p. 117], valid for $|cq| < 1$ and $c \neq 1$,

$$\sum_{n=1}^{\infty} \mathrm{FFW}(c, n) q^n = \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(1 - cq^n)(q)_n} = \frac{1}{1-c}\left(1 - \frac{(q)_\infty}{(cq)_\infty}\right).$$

Another generalization of (1.2.8) was established by Patkowski [36, Corollary 2.4], derived via a 'sum of tails' identity.

In [2], along with Bhoria, Eyyunni, and Maji, we extended the domain of validity for the variable z in the identity (1.4.2) from integers to the entire complex plane. This extension allowed us to view the identity (1.4.2) in a broader analytic setting. In addition to this extension, we also established several further generalizations of (1.2.8), including new Bressoud–Subbarao type weighted partition identities. These identities refine and expand the classical results by introducing suitable weights on the parts of a partition, thereby revealing deeper combinatorial and analytic features. The details of these generalizations, along with their proofs and consequences, will be presented in Chapter 3 of this thesis.

In the next section, we look at how Uchimura's identity (1.2.7) has been further extended over the years, starting from Uchimura's original work and continuing with the contributions of Dilcher, Andrews–Crippa–Simon and many other mathematicians who have added new ideas and generalizations.

1.5 Work of Uchimura, Dilcher, Andrews and many other mathematicians

Uchimura [44] not only gave a new term but also provided a one variable generalization of his identity (1.2.7). To this end, he defined, for each non-negative integer m,

$$M_m := M_m(q) = \sum_{n=1}^{\infty} n^m q^n (q^{n+1})_\infty \text{ and } K_{m+1} := K_{m+1}(q) = \sum_{n=1}^{\infty} \sigma_m(n) q^n, \quad (1.5.1)$$

where $\sigma_m(n) = \sum_{d|n} d^m$. Then he proved the below result.

Theorem 1.5.1. *We have the following properties:*

(1) $\exp\left(\sum_{m=1}^{\infty} K_m t^m/m!\right) = 1 + \sum_{m=1}^{\infty} M_m t^m/m!.$

(2) *Let Y_m be the Bell polynomial defined by*

$$Y_m(u_1, u_2, \ldots, u_m) = \sum_{\Pi(m)} \frac{m!}{k_1! \ldots k_m!} \left(\frac{u_1}{1!}\right)^{k_1} \cdots \left(\frac{u_m}{m!}\right)^{k_m}, \quad (1.5.2)$$

where $\Pi(m)$ denotes a partition of m with

$$k_1 + 2k_2 + \cdots + mk_m = m.$$

Then for any $m \geq 1$,

$$M_m = Y_m(K_1, \ldots, K_m). \quad (1.5.3)$$

It is worth noting that Uchimura did not investigate the partition-theoretic implications of the above theorem, in particular, he did not formulate the corresponding generalization of (1.2.8). Subsequently, Dilcher [22, Corollary 1] recorded several combinatorial identities arising from Theorem 1.5.1. In particular, for $m \leq 4$, he expressed the coefficients $C_m(n)$ of q^n in M_m in terms of the classical divisor functions as follows:

$$C_1(n) = d(n),$$

$$C_2(n) = \sigma_1(n) + \sum_{j=1}^{n-1} d(j)d(n-j), \quad (1.5.4)$$

$$C_3(n) = \sigma_2(n) + 3\sum_{j=1}^{n-1} d(j)\sigma_1(n-j) + \sum_{\substack{j+k+\ell=n\\ j,k,\ell\geq 1}} d(j)d(k)d(\ell),$$

$$C_4(n) = \sigma_3(n) + 3\sum_{j=1}^{n-1} \sigma_1(j)\sigma_1(n-j) + 4\sum_{j=1}^{n-1} d(j)\sigma_2(n-j)$$
$$+ 6\sum_{\substack{j+k+\ell=n\\ j,k,\ell\geq 1}} d(j)d(k)\sigma_1(\ell) + \sum_{\substack{j_1+\cdots+j_4=n\\ j_1,\ldots,j_4\geq 1}} d(j_1)\cdots d(j_4).$$

More significantly, Dilcher demonstrated that the coefficients $C_m(n)$ admit a natural partition-theoretic interpretation, namely,

$$C_m(n) = \sum_{\pi\in\mathcal{D}(n)} (-1)^{\#(\pi)-1} s(\pi)^m. \tag{1.5.5}$$

In 1995, Dilcher [22, Corollary 2] obtained a significant generalization of Uchimura's identity (1.2.7). He proved for any natural number k, the following identity is true:

$$\sum_{n=k}^{\infty}\binom{n}{k} q^n (q^{n+1})_\infty = q^{-\binom{k}{2}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{\binom{n+k}{2}}}{(1-q^n)^k(q)_n} = \sum_{j_1=1}^{\infty}\frac{q^{j_1}}{1-q^{j_1}}\sum_{j_2=1}^{j_1}\frac{q^{j_2}}{1-q^{j_2}}\cdots\sum_{j_k=1}^{j_{k-1}}\frac{q^{j_k}}{1-q^{j_k}}. \tag{1.5.6}$$

The case $k=1$ in the above identity immediately gives (1.2.7). Furthermore, using (1.5.6), Dilcher [22, p. 87, Section 4] attempted to establish the following identity of Andrews, Crippa and Simon [13, Theorem 2.1], which also generalizes (1.2.6). Andrews, Crippa and Simon proved that, for a fixed $k\in\mathbb{N}$, there exist a polynomial $P_k(x_1, x_2, \cdots, x_k)$ with rational coefficients such that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{\binom{n+1}{2}}}{(1-q^n)^k(q)_n} = P_k\left(S_0(q), S_1(q), \cdots, S_{k-1}(q)\right), \tag{1.5.7}$$

where

$$S_m(q) = \sum_{n=1}^{\infty}\frac{n^m q^n}{1-q^n} \tag{1.5.8}$$

is the same as the generating function $K_{m-1}(q)$, defined in (1.5.1), of generalized divisor function $\sigma_m(n)$. It should be noted that a minor error appears in Dilcher's proof, which we rectified in [3] and we shall discuss this with more details in Chapter 4 of this thesis. Substituting $k=1$ in (1.5.7) gives Kluyver's identity (1.2.6). This naturally raises the question: what is the Uchimura-type expression corresponding to the identity (1.5.7)? The answer to this question was provided by Andrews, Crippa and Simon in [13, Lemma 2.2]. Mainly, they proved that

$$(q)_\infty\sum_{n=0}^{\infty}\frac{q^n}{(q)_n}\binom{k+n-1}{k} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{\binom{n+1}{2}}}{(1-q^n)^k(q)_n}. \tag{1.5.9}$$

Inspire from the identities (1.5.9) and (1.2.9), Gupta and Kumar [33, Theorem 1.1] recently established a one variable generalization of (1.5.9). Specifically, they obtained for $|a| < 1$ and $k \in \mathbb{N}$,

$$\frac{(q)_\infty}{(a)_\infty} \sum_{n=1}^{\infty} \frac{(a/q)_n q^n}{(q)_n} \binom{k+n-1}{k} = -\sum_{n=1}^{\infty} \frac{(q/a)_n a^n}{(1-q^n)^k (q)_n}. \tag{1.5.10}$$

Letting $a \to 0$ in the above identity recovers (1.5.9). In the same paper, they also obtained a divisor-type sum for (1.5.10). For that, they defined

$$\mathfrak{S}_{m,a}(q) := S_m(q) - R_{m,a}(q),$$

where $S_m(q)$ is the same as in (1.5.8) and

$$R_{m,a}(q) := \mathrm{Li}_{-m}(a) + \sum_{n=1}^{\infty} \frac{n^m a^n q^n}{1-q^n},$$

where $\mathrm{Li}_{-m}(a)$ denotes the well-known polylogarithm function. Further, Gupta and Kumar proved that, for $|a| < 1$ and $k \in \mathbb{N}$, there exists a polynomial $P_k(x_1, x_2, \cdots, x_k)$ with rational coefficients such that

$$-\sum_{n=1}^{\infty} \frac{(q/a)_n a^n}{(1-q^n)^k (q)_n} = P_k\big(\mathfrak{S}_{0,a}(q), \mathfrak{S}_{1,a}(q), \cdots, \mathfrak{S}_{k-1,a}(q)\big) := P_k(a,q). \tag{1.5.11}$$

Letting $a \to 0$ in (1.5.11) leads to (1.5.7).

In [2], in addition to establishing several weighted partition identities, we also obtained a generalization of Uchimura's identity (1.5.3). The details of this generalization, together with its consequences, will be presented in Chapter 3. Further, in [3], jointly with Bhoria, Eyyunni and Maji, we developed a one-variable extension of the identities (1.5.10) and (1.5.11) of Gupta and Kumar. As an implications of our results, we also obtained a different generalization of Andrews, Crippa, and Simon's identities (1.5.7) and (1.5.9). In addition, we established a Ramanujan-type summation formula for a generalized form of (1.5.3), which consequently yielded a Ramanujan-type sum for the original identity itself. These results form an important connection between divisor generating functions, q-series identities, and the structure of random variables that arise in probability theory and random graph models. All of these developments will be discussed in detail in Chapter 4 of this thesis.

The next section is devoted to examining how q-series techniques have been applied to problems in probability theory and the study of random graphs, highlighting the contributions of Uchimura as well as the developments by Simon–Crippa–Collenberg and Andrews–Crippa–Simon.

1.6 Applications to probability theory and random graphs

The works of Uchimura [44], Simon–Crippa–Collenberg [42] and Andrews–Crippa–Simon [13] demonstrate that identities involving divisor generating functions also arise in the study of probability theory and random graphs. These connections provide a striking perspective: analytic properties of q-series may encode structural information about random discrete systems.

Uchimura [44, p. 76] considered a random variable X with the probability mass function

$$\Pr(X = n) = q^n (q^{n+1})_\infty, \quad n \geq 0, \; 0 < q < 1,$$

and corresponding probability generating function

$$G(x, q) = \sum_{n=0}^{\infty} x^n \Pr(X = n).$$

He proved that, for any natural number m, the m-th moment of X is given by

$$\mathbb{E}(X^m) = M_m = Y_m(K_1, K_2, \ldots, K_m), \tag{1.6.1}$$

where M_m and K_m are defined in (1.5.1) and Y_m denotes the Bell polynomial defined in (1.5.2). Moreover, Uchimura showed that the m-th cumulant h_m of X is exactly the divisor generating function K_m, that is,

$$h_m = K_m, \quad \forall\, m \in \mathbb{N}. \tag{1.6.2}$$

This follows from the generating function

$$G(e^t, q) = \exp\left(\sum_{m=1}^{\infty} h_m \frac{t^m}{m!} \right).$$

The above identity (1.6.2) proves that the divisor generating function K_m is nothing but the m-th cumulant with respect to the random variable X. As we know that the second cumulant h_2 is same as $\mathrm{Var}(X)$, so we have

$$\mathrm{Var}(X) = \sum_{n=1}^{\infty} \sigma(n) q^n, \tag{1.6.3}$$

where $\sigma(n) = \sum_{d|n} d$. Uchimura also provided a combinatorial interpretation of the probability generating function $G(x, q)$ in the analysis of data structures known as heaps, specifically, in estimating the average number of exchanges required when inserting an element. For readers interested in a detailed exposition, further discussion can be seen in [44, Section 3].

A related probabilistic interpretation was later studied by Simon, Crippa, and Collenberg [42] in the context of random acyclic digraphs. They defined the $G_{n,p}$–model as a random acyclic digraph with a vertex set $V = \{1, 2, \ldots, n\}$ and directed edges appearing between vertices (i, j), for $1 \leq i < j \leq n$, with probability $p \in (0, 1)$. Let $\gamma_n^*(1)$ be a random variable denoting the number of vertices reachable from vertex 1 by directed path, including vertex 1 itself. They first established a probability function, for $1 \leq h \leq n$,

$$\Pr(\gamma_n^*(1) = h) = q^{n-h} \prod_{j=1}^{h-1} (1 - q^{n-j}), \tag{1.6.4}$$

where $q = 1 - p$. Furthermore, by interpreting the random variable corresponding to the size of the transitive closure as a discrete–time pure–birth process, they were able to derive explicit expressions for its distribution, mean, and variance. These computations were shown to be intimately connected to the divisor generating function. In particular, they proved that

$$\lim_{n \to \infty} (n - \mathbb{E}(\gamma_n^*(1)) = \sum_{n=1}^{\infty} d(n) q^n. \tag{1.6.5}$$

In 1997, Andrews, Crippa and Simon [13] further studied the same random variable and proved that

$$\lim_{n \to \infty} \mathrm{Var}(\gamma_n^*(1)) = \sum_{n=1}^{\infty} \sigma(n) q^n. \tag{1.6.6}$$

Comparing (1.6.3) and (1.6.6), we observe that

$$\mathrm{Var}(X) = \lim_{n \to \infty} \mathrm{Var}(\gamma_n^*(1)) = \sum_{n=1}^{\infty} \sigma(n) q^n = K_2,$$

where X is the same random variable studied by Uchimura.

More generally, as Uchimura showed in (1.6.2), the m-th cumulant h_m with respect to random variable X equals K_m, the generating function for the generalized divisor function $\sigma_{m-1}(n)$. This observation led the authors of [3, p. 31, Problem 2] to ask whether the generalized divisor generating function K_m, defined in (1.5.1), may also be interpreted as the m-th cumulant of the random variable $\gamma_n^*(1)$. In [4, Theorem 2.6], along with Bhoria, Eyyunni, Maji and Wakhare, we provide a complete affirmative answer to this question. We will discuss these results in Chapter 5 of this thesis.

In a further refinement, Andrews, Crippa and Simon [13] obtained a more general form of (1.6.5) through an application of q-series technique. To achieve this, they first introduced a sequence $\{t_n(q)\}$ of polynomials in q, defined recursively for $n \geq 1$ by,

$$t_n(q) = f(n) + (1 - q^{n-1}) t_{n-1}(q), \quad t_0(q) = 0, \tag{1.6.7}$$

where $f(n) = \sum_{k \geq 0} c_k n^k$ is a non-zero polynomial in n with rational coefficients.

Then they showed that there exist rational numbers h_j, such that

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n} f(i) - t_n(q)\right) = \sum_{j\geq 1} h_j P_j(q), \tag{1.6.8}$$

where $P_j(q) := P_j(K_1(q), K_2(q), \ldots, K_j(q))$ is some polynomial in j variables with rational coefficients, and the coefficients h_j are expressible in the following way:

$$h_1 = c_0, \quad h_j = \sum_{i\geq j-1} (-1)^{i-j+1} \binom{i-1}{j-2} i! \sum_{k\geq i} c_k \bar{s}(k,i), \tag{1.6.9}$$

where $\bar{s}(k,i)$ is the Stirling number of the second kind. In addition, they observed that the expected value of $\gamma_n^*(1)$, denoted as $\mathbb{E}(\gamma_n^*(1))$, satisfies the recurrence relation given in (1.6.7) with $f(n) = 1$ for all $n \in \mathbb{N}$. Substituting this choice of $f(n)$ in (1.6.8) yields (1.6.5). Further, as an immediate consequences of (1.6.8), they also derived (1.6.6). In the same paper, Andrews, Crippa and Simon conjectured that, if $f(n) = (-1)^n$, then

$$\lim_{n\to\infty}\left(\sum_{1\leq i\leq n} f(i) - t_n(q)\right) = \sum_{n=1}^{\infty} (-1)^n q^{n^2}, \tag{1.6.10}$$

where $\{t_n(q)\}$ be a sequence of polynomial satisfying (1.6.7). This conjecture was later proved by Bringmann and Jennings–Shaffer [19]. In fact, they proved a more general result for the sequence $f(n) = b^n$, where $b \neq 1$.

Theorem 1.6.1. *Let $b \in \mathbb{C}\backslash\{1\}$ and $f(n) = b^n$. Let $\{t_n(q)\}$ be a sequence of polynomials defined as in* (1.6.7)*. Then for $|q| < \min\left(1, |b|^{-1}\right)$, we have*

$$\lim_{n\to\infty}\left(\sum_{1\leq i\leq n} b^i - t_n(q)\right) = \frac{b}{1-b} - \frac{b(q)_\infty}{(b)_\infty}.$$

By choosing $b = -1$ in the above theorem, one immediately recovers the conjecture in (1.6.10). Bringmann and Jennings-Shaffer further considered the case when $f(n)$ is a periodic function with period N. Let $\zeta_N = e^{\frac{2\pi i}{N}}$ and $t_n(q)$ again be the sequence of polynomials defined in (1.6.7). They proved that

$$\lim_{n\to\infty}\left(\sum_{1\leq i\leq n} f(i) - t_n(q)\right) = c_0 S_0(q) - (q)_\infty \sum_{k=1}^{N-1} \frac{c_k}{(\zeta_N^k)_\infty} + \sum_{k=1}^{N-1} \frac{c_k}{1-\zeta_N^k}, \tag{1.6.11}$$

where

$$c_k = \frac{1}{N}\sum_{j=1}^{N} f(j)\zeta_N^{(1-j)k}.$$

More recently, Gupta and Kumar [33, Theorem 1.11] attempted to generalize the above result (1.6.11), but their argument contained a minor error. In Chapter 5, we correct it and provide a complete and valid extension of (1.6.11).

In [4], along with Bhoria, Eyyunni, Maji and Wakhare, we first established a new q-series identity and using that identity, we obtained a new Uchimura-type expression for the identity (1.6.8) of Andrews, Crippa, and Simon. As a corollary, we further derived limit forms for Uchimura's identity (1.5.3) and for Dilcher's identity (1.5.6). In addition to these results, we computed the third, fourth and fifth cumulants associated with the random variable $\gamma_n^*(1)$ introduced by Simon, Crippa, and Collenberg. Building on these computations, we subsequently established a general formula describing the higher order cumulants of this random variable. All these results will be discussed in detail in Chapter 5 of this thesis.

Chapter 2

ONE VARIABLE GENERALIZATIONS OF FIVE ENTRIES OF RAMANUJAN AND THEIR FINITE ANALOGUES

This chapter is based on our paper [1]. We begin by introducing one variable generalizations of two important identities, the identity (1.2.12) of Bhoria–Eyyunni–Maji and the identity (1.3.5) due to Dixit–Patel. These generalizations naturally extend the original results by incorporating an additional complex parameter, thereby allowing us to study a richer family of q-series identities. One of the key advantages of these extensions is that they unify several classical results and provide a systematic framework for deriving their one-variable analogues. Using these generalized identities, we obtain individual one variable extensions of each of Ramanujan's five entries (1.2.1)–(1.2.5), as well as their finite analogues (1.3.7)–(1.3.11). These new results not only refine the classical forms recorded by Ramanujan but also reveal deeper structural relationships among the five entries, highlighting the unifying role played by the additional parameter. Furthermore, we derive a one-variable generalization of the finite analogue (1.3.4) of Garvan's identity (1.2.11), thereby providing a unified extension that connects the classical and finite versions of this identity. In addition to these developments, we establish an identity involving the basic hypergeometric series ${}_3\phi_2$ and ${}_2\phi_1$, which is a one variable analogue of the identity (1.3.12) previously obtained by Dixit and Patel. This new identity further enriches the landscape of finite q-series transformations and opens up new avenues for exploring their analytic and combinatorial properties.

All these results are presented systematically in the next section. This is followed by a collection of preliminary lemmas and auxiliary identities that are essential for the proofs. The chapter concludes with detailed proofs of all the results described above.

2.1 Main results

We start with a one variable generalization of the identity (1.2.12) of Bhoria, Eyyunni and Maji [17, Theorem 2.1].

Theorem 2.1.1. *Let a, b, c, d, e be complex numbers such that $\left|ade\right| < 1$ and $|ceq| < 1$. Then*

$$\sum_{n=1}^{\infty} \frac{(b/a)_n (c/d)_n (q/e)_{n-1} (ad)^n e^{n-1}}{(b)_n (cq)_n (q)_{n-1}}$$
$$= \frac{(a-b)(d-c)}{(ad-b)} \frac{(ceq)_\infty (ad)_\infty}{(cq)_\infty (ade)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n (bd/c)_n (q/e)_n (ce)^n}{(b)_n (ad)_n (q)_n} \left(\frac{adq^n}{1-adq^n} - \frac{bq^n}{1-bq^n} \right).$$

By substituting $e = 1$ in the above theorem, one can easily obtain (1.2.12). Also, letting $a \to 0$ and then replacing b by zq in Theorem 2.1.1, we get the below result.

Corollary 2.1.2. *For $|ceq| < 1$, we have*

$$\sum_{n=1}^{\infty} \frac{(c/d)_n (q/e)_{n-1} (-dz)^n e^{n-1} q^{n(n+1)/2}}{(zq)_n (cq)_n (q)_{n-1}}$$
$$= (c-d) \frac{z}{c} \frac{(ceq)_\infty}{(cq)_\infty} \sum_{n=1}^{\infty} \frac{(zqd/c)_{n-1} (q/e)_{n-1} (cq)^n e^{n-1}}{(zq)_n (q)_{n-1}}.$$

Letting $e = 1$ in the above identity gives the following identity of Bhoria et. al. [17, Theorem 2.2]:

$$\sum_{n=1}^{\infty} \frac{(-z)^n (c/d)_n d^n q^{\frac{n(n+1)}{2}}}{(cq)_n (zq)_n} = \frac{z(c-d)}{c} \sum_{n=1}^{\infty} \frac{(zdq/c)_{n-1}}{(zq)_n} (cq)^n.$$

Further, setting $d \to 0$ in Corollary 2.1.2 gives a generalization of (1.2.13).

Corollary 2.1.3. *For $|ceq| < 1$, we have*

$$\sum_{n=1}^{\infty} \frac{(q/e)_{n-1} e^{n-1} (cz)^n q^{n^2}}{(zq)_n (cq)_n (q)_{n-1}} = \frac{z(ceq)_\infty}{(cq)_\infty} \sum_{n=1}^{\infty} \frac{(q/e)_{n-1} e^{n-1} (cq)^n}{(zq)_n (q)_{n-1}}. \tag{2.1.1}$$

Substituting $d=1$ in Theorem 2.1.1 yields an interesting new generalization of (1.2.9).

Corollary 2.1.4. *For $|ae|<1$ and $|ceq|<1$, we have*

$$\sum_{n=1}^{\infty}\frac{(b/a)_n(q/e)_{n-1}a^n e^{n-1}}{(1-cq^n)(b)_n(q)_{n-1}}=\frac{(ceq)_\infty(a)_\infty}{(cq)_\infty(ae)_\infty}\sum_{n=0}^{\infty}\frac{(b/c)_n(q/e)_n(ce)^n}{(b)_n(q)_n}\left(\frac{aq^n}{1-aq^n}-\frac{bq^n}{1-bq^n}\right).$$

Here we remark that the above generalization is different from the generalization (1.2.12) obtained by Bhoria, Eyyunni and Maji.

2.1.1 Finite analogue of Theorem 2.1.1

We now state a finite analogue of Theorem 2.1.1 which essentially generalizes the identity (1.3.5) of Dixit and Patel.

Theorem 2.1.5. *For any natural number N, we have*

$$\sum_{n=1}^{N}\begin{bmatrix}N\\n\end{bmatrix}\frac{(q)_n(b/a)_n(c/d)_n(q/e)_{n-1}(ade)_{N-n}(ad)^n e^{n-1}}{(b)_n(cq)_n(q)_{n-1}}=\frac{(a-b)(d-c)}{(ad-b)}\frac{(ad)_N}{(cq)_N}$$
$$\times\sum_{n=1}^{N}\begin{bmatrix}N\\n\end{bmatrix}\frac{(q)_n(ceq)_{N-n}(a)_{n-1}(bd/c)_{n-1}(q/e)_{n-1}(ce)^{n-1}}{(b)_{n-1}(ad)_{n-1}(q)_{n-1}}\left(\frac{adq^{n-1}}{1-adq^{n-1}}-\frac{bq^{n-1}}{1-bq^{n-1}}\right).\tag{2.1.2}$$

Setting $e=1$ in the above identity gives (1.3.5). Moreover, letting $a\to 0$ and then substituting $b=zq$ in Theorem 2.1.5, we get a finite analogue of (2.1.2).

Corollary 2.1.6. *We have*

$$\sum_{n=1}^{N}\begin{bmatrix}N\\n\end{bmatrix}\frac{(q)_n(c/d)_n(q/e)_{n-1}(-zd)^n q^{n(n+1)/2}e^{n-1}}{(zq)_n(cq)_n(q)_{n-1}}=\frac{z(c-d)}{c(cq)_N}$$
$$\times\sum_{n=1}^{N}\begin{bmatrix}N\\n\end{bmatrix}\frac{(q)_n(ceq)_{N-n}(zdq/c)_{n-1}(q/e)_{n-1}(cq)^n e^{n-1}}{(zq)_n(q)_{n-1}}.$$

Putting $e=1$ in the above identity gives the below identity of Dixit and Patel [25, Corollary 2.2],

$$\sum_{n=1}^{N}\begin{bmatrix}N\\n\end{bmatrix}\frac{(q)_n(c/d)_n(-zd)^n q^{n(n+1)/2}}{(zq)_n(cq)_n}=\frac{z(c-d)}{c}\sum_{n=1}^{N}\begin{bmatrix}N\\n\end{bmatrix}\frac{(q)_n(cq)_{N-n}(zdq/c)_{n-1}(cq)^n}{(zq)_n(cq)_N}.$$

Further, by allowing $d\to 0$ in Corollary 2.1.6, we arrive at the following finite analogue of the identity (2.1.1):

Corollary 2.1.7. *We have*

$$\sum_{n=1}^{N}\begin{bmatrix}N\\n\end{bmatrix}\frac{(q)_n(q/e)_{n-1}(zc)^n e^{n-1}q^{n^2}}{(zq)_n(cq)_n(q)_{n-1}}=\frac{z}{(cq)_N}\sum_{n=1}^{N}\begin{bmatrix}N\\n\end{bmatrix}\frac{(q)_n(ceq)_{N-n}(q/e)_{n-1}(cq)^n e^{n-1}}{(zq)_n(q)_{n-1}}.$$

Again, substituting $d = 1$ in Theorem 2.1.5, we get a finite analogue of (2.1.4).

Corollary 2.1.8. *We have*

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_n (b/a)_n (q/e)_{n-1} (ae)_{N-n} (a)^n e^{n-1}}{(1-cq^n)(b)_n (q)_{n-1}}$$
$$= \frac{(a)_N}{(cq)_N} \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_n (ceq)_{N-n} (b/c)_{n-1} (q/e)_{n-1} (ce)^{n-1}}{(b)_{n-1}(q)_{n-1}} \left(\frac{aq^{n-1}}{1-aq^{n-1}} - \frac{bq^{n-1}}{1-bq^{n-1}} \right).$$

Next, we state an identity that generalizes a finite analogue of Garvan's identity (1.3.4) due to Dixit, Eyyunnim Maji and Sood.

Theorem 2.1.9. *We have*

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} \frac{(-1)^n (q^2;q^2)_n (z/d;q^2)_n z^n d^n q^{n^2}}{(z-d)(zq)_{2n}}$$
$$= \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} \left(\frac{(dq)_{2n-2} z^{2n-1} q^{n(2n-1)}}{(zq)_{2n-1}} + \frac{(dq)_{2n-1} z^{2n} q^{n(2n+1)}}{(zq)_{2n}} \right) \frac{(q^2;q^2)_n}{(dzq^{2N+1};q^2)_n}.$$

Letting $N \to \infty$ in the above theorem gives a one variable generalization of an identity of Garvan (1.2.11). Namely, we obtain the following identity.

Theorem 2.1.10. *For $|dz| < 1$ and $|q| < 1$,*

$$\sum_{n=1}^{\infty} \frac{(-1)^n (z/d;q^2)_n z^n d^n q^{n^2}}{(z-d)(zq)_{2n}} = \sum_{n=1}^{\infty} \frac{(dq)_{n-1} z^n q^{n(n+1)/2}}{(zq)_n}. \tag{2.1.3}$$

Upon substituting $d = 1$ in (2.1.3), one can easily get Garvan's identity (1.2.11).

2.2 Preliminary results

For $|z| < 1$, the q-binomial theorem [30, p. 8, Equation (1.32)] is given by

$$\sum_{n=0}^{\infty} \frac{(A)_n z^n}{(q)_n} = \frac{(Az)_\infty}{(z)_\infty}. \tag{2.2.1}$$

The basic hypergeometric series ${}_{r+1}\phi_r$ is defined as,

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, & a_2, & \cdots, & a_{r+1} \\ b_1, & b_2, & \cdots, & b_r \end{matrix} ; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{r+1})_n}{(b_1)_n (b_2)_n \cdots (b_r)_n} \frac{z^n}{(q)_n}.$$

We recall Heine's transformation [30, p. 359, (III.1)]

$${}_2\phi_1 \left[\begin{matrix} A, & B \\ & C \end{matrix} ; q, z \right] = \frac{(Az)_\infty (B)_\infty}{(z)_\infty (C)_\infty} {}_2\phi_1 \left[\begin{matrix} \frac{C}{B}, & z \\ & Az \end{matrix} ; q, B \right], \tag{2.2.2}$$

In particular, when $z=\frac{C}{AB}$, it is known as the q-Gauss sum [30, p. 354, (II.8)]:

$$ {}_2\phi_1\left[\begin{matrix} A, & B \\ & C \end{matrix}; q, \frac{C}{AB}\right] = \frac{\left(\frac{C}{A}\right)_\infty \left(\frac{C}{B}\right)_\infty}{(C)_\infty \left(\frac{C}{AB}\right)_\infty}. \tag{2.2.3} $$

We shall recall two ${}_3\phi_2$ transformation formulas in [30, p. 359, Equation (III.9), p. 360, Equation (III.12)]:

$$ {}_3\phi_2\left[\begin{matrix} A, & B, & C \\ & D, & E \end{matrix}; q, \frac{DE}{ABC}\right] = \frac{\left(\frac{E}{A}\right)_\infty \left(\frac{DE}{BC}\right)_\infty}{(E)_\infty \left(\frac{DE}{ABC}\right)_\infty} {}_3\phi_2\left[\begin{matrix} A, & \frac{D}{B}, & \frac{D}{C} \\ & D, & \frac{DE}{BC} \end{matrix}; q, \frac{E}{A}\right], \tag{2.2.4} $$

$$ {}_3\phi_2\left[\begin{matrix} q^{-N}, & B, & C \\ & D, & E \end{matrix}; q, q\right] = \frac{\left(\frac{E}{C}\right)_N}{(E)_N} c^N {}_3\phi_2\left[\begin{matrix} q^{-N}, & C, & \frac{D}{B} \\ & D, & \frac{Cq^{1-N}}{E} \end{matrix}; q, q\right]. \tag{2.2.5} $$

A finite Heine transformation given by Andrews [11, Theorem 2] is

$$ {}_3\phi_2\left[\begin{matrix} q^{-N}, & \alpha, & \beta \\ & \gamma, & \frac{q^{1-N}}{\tau} \end{matrix}; q, q\right] = \frac{(\beta)_N (\alpha\tau)_N}{(\gamma)_N (\tau)_N} {}_3\phi_2\left[\begin{matrix} q^{-N}, & \frac{\gamma}{\beta}, & \tau \\ & \alpha\tau, & \frac{q^{1-N}}{\beta} \end{matrix}; q, q\right]. \tag{2.2.6} $$

We also need a corollary of (2.2.6), given below [11, Corollary 3, Equation (2.7)]

$$ {}_3\phi_2\left[\begin{matrix} q^{-N}, & \alpha, & \beta \\ & \gamma, & \frac{q^{1-N}}{\tau} \end{matrix}; q, q\right] = \frac{\left(\frac{\gamma}{\beta}\right)_N (\beta\tau)_N}{(\gamma)_N (\tau)_N} {}_3\phi_2\left[\begin{matrix} q^{-N}, & \frac{\alpha\beta\tau}{\gamma}, & \beta \\ & \beta\tau, & \frac{\beta q^{1-N}}{\gamma} \end{matrix}; q, q\right]. \tag{2.2.7} $$

Some basic formula from [23, Equation (4.1), (4.2)] stated below.

$$ \left(\frac{q^{-N}}{x}\right)_n = \frac{(-1)^n \left(xq^{N-n+1}\right)_n}{x^n q^{Nn}} q^{\frac{n(n-1)}{2}}, \tag{2.2.8} $$

$$ \frac{(q^{-N})_n}{\left(\frac{q^{-N}}{x}\right)_n} = \frac{(q^{N-n+1})_n x^n}{(xq^{N-n+1})_n} = \frac{(q)_N (xq)_{N-n}}{(q)_{N-n}(xq)_N} x^n. \tag{2.2.9} $$

We also state the following result which can be obtained from the q-Chu-Vandermonde sum [30, p. 354, Equation (II.7)]

$$ \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1/a)_n (ac)^n q^{n(n+1)/2}}{(cq)_n} = \frac{(-acq)_N}{(cq)_N}. \tag{2.2.10} $$

We shall also use a finite analogue of the Rogers-Fine identity [23, Lemma 9.2]:

$$ \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(\alpha q)_n (\tau)_{N-n} (q)_n \tau^n}{(\beta q)_n (\tau)_N} $$

$$ = (1-\tau q^N) \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(\alpha q)_n (q)_n \left(\frac{\alpha\tau q}{\beta}\right)_n (\alpha\tau q^2)_{N-1} q^{n^2} (1-\alpha\tau q^{2n+1})}{(\beta\tau)_n (\tau)_{n+1} (\alpha\tau q^2)_{N+n}}. \tag{2.2.11} $$

We also use [30, p. 351, Equation (I.11)]

$$ \frac{(b;q^2)_N (a;q^2)_{N-n}}{(a;q^2)_N (b;q^2)_{N-n}} \left(\frac{a}{b}\right)^n = \frac{(q^{2-2N}/b;q^2)_n}{(q^{2-2N}/a;q^2)_n}. \tag{2.2.12} $$

We also state Sears' transformation [30, p. 70, Equation (3.2.1)]

$$
{}_4\phi_3\left[\begin{matrix} q^{-N}, & A, & B, & C \\ & D, & E, & \frac{ABCq^{1-N}}{DE} \end{matrix} ; q, q\right]
$$

$$
= \frac{\left(\frac{E}{A}\right)_N \left(\frac{DE}{BC}\right)_N}{(E)_N \left(\frac{DE}{ABC}\right)_N} {}_4\phi_3\left[\begin{matrix} q^{-N}, & A, & \frac{D}{B}, & \frac{D}{C} \\ & D, & \frac{DE}{BC}, & \frac{Aq^{1-N}}{E} \end{matrix} ; q, q\right]. \tag{2.2.13}
$$

2.3 Proof of main results

Proof of Theorem 2.1.1. Let $A = \frac{q}{e}$, $B = \frac{bq}{a}$, $C = \frac{cq}{d}$, $D = bq$, and $E = cq^2$ in (2.2.4) so that

$$
\sum_{n=0}^{\infty} \frac{(q/e)_n (bq/a)_n (cq/d)_n (ade)^n}{(bq)_n (cq^2)_n (q)_n} = \frac{(ceq)_\infty (adq)_\infty}{(cq^2)_\infty (ade)_\infty} \sum_{n=0}^{\infty} \frac{(q/e)_n (a)_n (bd/c)_n (ceq)^n}{(bq)_n (adq)_n (q)_n}.
$$

Re-indexing the summation on the left hand side and then multiplying both sides by $\frac{(1-b/a)(1-c/d)ad}{(1-b)(1-cq)}$, we get

$$
\sum_{n=1}^{\infty} \frac{(b/a)_n (c/d)_n (q/e)_{n-1} (ad)^n e^{n-1}}{(b)_n (cq)_n (q)_{n-1}}
$$

$$
= \frac{(a-b)(d-c)}{(1-b)(1-ad)} \frac{(ceq)_\infty (ad)_\infty}{(cq)_\infty (ade)_\infty} \sum_{n=0}^{\infty} \frac{(q/e)_n (a)_n (bd/c)_n (ceq)^n}{(bq)_n (adq)_n (q)_n}
$$

$$
= \frac{(a-b)(d-c)}{(ad-b)} \frac{(ceq)_\infty (ad)_\infty}{(cq)_\infty (ade)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n (bd/c)_n (q/e)_n (ce)^n}{(b)_n (ad)_n (q)_n} \left(\frac{(ad-b)q^n}{(1-bq^n)(1-adq^n)} \right)
$$

$$
= \frac{(a-b)(d-c)}{(ad-b)} \frac{(ceq)_\infty (ad)_\infty}{(cq)_\infty (ade)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n (bd/c)_n (q/e)_n (ce)^n}{(b)_n (ad)_n (q)_n} \left(\frac{adq^n}{1-adq^n} - \frac{bq^n}{1-bq^n} \right).
$$

This proves the result. □

Proof of Theorem 2.1.5. Substituting $A = \frac{q}{e}$, $B = \frac{bq}{a}$, $C = \frac{cq}{d}$, $D = bq$, and $E = cq^2$ in (2.2.13), we get

$$
\sum_{n=0}^{N} \frac{\left(q^{-N}\right)_n (q/e)_n (bq/a)_n (cq/d)_n}{(bq)_n (cq^2)_n (q^{1-N}/ade)_n (q)_n} q^n = \frac{(ceq)_N (adq)_N}{(cq^2)_N (ade)_N} \sum_{n=0}^{N} \frac{\left(q^{-N}\right)_n (q/e)_n (a)_n (bd/c)_n}{(bq)_n (adq)_n (q^{-N}/ce)_n (q)_n} q^n.
$$

Now we use (2.2.9) with $x = ade/q$ for the left hand side and with $x = ce$ for the right hand side of the aforementioned expression to see that

$$
\sum_{n=0}^{N} \frac{(q)_N (ade)_{N-n} (q/e)_n (bq/a)_n (cq/d)_n (ade)^n}{(q)_{N-n} (bq)_n (cq^2)_n (q)_n}
$$

$$
= \frac{(adq)_N}{(cq^2)_N} \sum_{n=0}^{N} \frac{(q)_N (ceq)_{N-n} (q/e)_n (a)_n (bd/c)_n (ceq)^n}{(q)_{N-n} (bq)_n (adq)_n (q)_n}.
$$

Re-indexing the sums of both sides by replacing n by $n-1$ and then multiplying the resultant by $\frac{(1-q^N)(1-b/a)(1-c/d)ad}{(1-b)(1-cq)}$ yields

$$\sum_{n=1}^{N+1} \begin{bmatrix} N+1 \\ n \end{bmatrix} \frac{(q)_n(ade)_{N-n+1}(q/e)_{n-1}(b/a)_n(c/d)_n(ad)^n e^{n-1}}{(b)_n(cq)_n(q)_{n-1}}$$
$$= (a-b)(d-c)\frac{(adq)_N}{(cq)_{N+1}} \sum_{n=1}^{N+1} \begin{bmatrix} N+1 \\ n \end{bmatrix} \frac{(q)_n(ceq)_{N-n+1}(q/e)_{n-1}(a)_{n-1}(bd/c)_{n-1}(ceq)^{n-1}}{(b)_n(adq)_{n-1}(q)_{n-1}}$$
$$= \frac{(a-b)(d-c)}{(ad-b)} \frac{(ad)_{N+1}}{(cq)_{N+1}} \sum_{n=1}^{N+1} \begin{bmatrix} N+1 \\ n \end{bmatrix} \frac{(q)_n(a)_{n-1}(bd/c)_{n-1}(ceq)_{N-n+1}(q/e)_{n-1}(ce)^{n-1}}{(b)_{n-1}(ad)_{n-1}(q)_{n-1}}$$
$$\times \left(\frac{adq^{n-1}}{1-adq^{n-1}} - \frac{bq^{n-1}}{1-bq^{n-1}} \right).$$

Now replace N by $N-1$ to get the desired result. □

2.3.1 Generalization of five entries of Ramanujan and their finite analogue

In this section, our aim is to derive a one variable generalizations of each of the five entries of Ramanujan and their corresponding finite analogues. We will start with a finite analogue of Entry 1 (1.2.1).

Theorem 2.3.1 (Finite analogue of Entry 1)**.** *For $N \in \mathbb{N}$, we have*

$$\frac{(-aq)_N(be)_N}{(bq)_N} = \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^n(-ae)_{N-n}(-b/a)_n(q/e)_n(ae)^n}{(bq)_n}. \tag{2.3.1}$$

Proof. Letting $a \to 0$, then substituting $d = 1/q$ and $c = -a/q$ in Theorem 2.1.5, we get

$$\sum_{n=1}^{N} \frac{(q)_N(q/e)_{n-1}(-be)^{n-1}q^{(n-1)(n-2)/2}}{(q)_{N-n}(b)_n(q)_{n-1}} = \sum_{n=1}^{N} \frac{(q)_N(-ae)_{N-n}(-b/a)_{n-1}(q/e)_{n-1}(-ae)^{n-1}}{(q)_{N-n}(-aq)_{N-1}(b)_n(q)_{n-1}}.$$

Re-indexing both sums above and then replacing N by $N+1$ yields

$$\sum_{n=0}^{N} \frac{(q)_{N+1}(q/e)_n(-be)^n q^{n(n-1)/2}}{(q)_{N-n}(b)_{n+1}(q)_n} = \sum_{n=0}^{N} \frac{(q)_{N+1}(-ae)_{N-n}(-b/a)_n(q/e)_n(-ae)^n}{(q)_{N-n}(-aq)_N(b)_{n+1}(q)_n}.$$

Multiply both sides by $\frac{1-b}{1-q^{N+1}}$ in the above equation and then further simplifying, one gets

$$(-aq)_N \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q/e)_n(-be)^n q^{n(n-1)/2}}{(bq)_n} = \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-ae)_{N-n}(-b/a)_n(q/e)_n(-ae)^n}{(bq)_n}.$$

Now the result follows by using (2.2.10) with $a = -e/q,\ c = b$ in the above identity. □

Remark 1. *We point out that the identity* (2.3.1) *can also be derived from the* q*-Pfaff-Saalschütz sum [30, p. 355, Equation (II.12)].*

To obtain the identity (1.3.7) of Dixit and Patel, let $e \to 0$ in (2.3.1) to get

$$\frac{(-aq)_N}{(bq)_N} = \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-b/a)_n a^n q^{n(n+1)/2}}{(bq)_n}. \tag{2.3.2}$$

By doing a simple calculation, one can observe that

$$\frac{(-aq)_N}{(bq)_N} = \frac{\left(q^{-N}/a\right)_N}{\left(q^{-N}/b\right)_N} \left(\frac{-a}{b}\right)^N. \tag{2.3.3}$$

Apply the q-Chu-Vandermonde sum [30, p. 354, Equation (II.6)] in the right side of the above expression to get

$$\begin{aligned} \frac{\left(q^{-N}/a\right)_N}{\left(q^{-N}/b\right)_N} \left(\frac{-a}{b}\right)^N &= {}_2\phi_1 \left[\begin{matrix} -a/b, & q^{-N} \\ & q^{-N}/b \end{matrix}; q, z \right] \\ &= \sum_{n=0}^{N} \frac{(q^{-N})_n(-a/b)_n}{(q^{-N}/b)_n(q)_n} q^n. \end{aligned}$$

Apply (2.2.9) with $x = b$ in the above expression to obtain

$$\begin{aligned} \frac{\left(q^{-N}/a\right)_N}{\left(q^{-N}/b\right)_N} \left(\frac{-a}{b}\right)^N &= \sum_{n=0}^{N} \frac{(q)_N(bq)_{N-n}(-a/b)_n}{(q)_{N-n}(bq)_N(q)_n} (bq)^n \\ &= \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-a/b)_n(bq)_{N-n}(bq)^n}{(bq)_N}. \end{aligned} \tag{2.3.4}$$

Employing (2.3.3) and (2.3.4) in (2.3.2) gives required identity (1.3.7) of Entry 1 due to Dixit and Patel.

Letting $N \to \infty$ in Theorem 2.3.1 gives a one variable generalization of (1.2.1).

Corollary 2.3.2 (One variable generalization of Entry 1)**.** *For* $|ae| < 1$*, we have*

$$\frac{(-aq)_\infty(be)_\infty}{(-ae)_\infty(bq)_\infty} = \sum_{n=0}^{\infty} \frac{(-1)^n(-b/a)_n(q/e)_n(ae)^n}{(q)_n(bq)_n}. \tag{2.3.5}$$

It can be easily observed that letting $e \to 0$ in Corollary 2.3.2 gives Ramanujan's Entry 1 (1.2.1). The identity (2.3.5) can also be obtained from the q-Gauss sum (2.2.3).

The following identity is a finite analogue of Entry 2 (1.2.2).

Theorem 2.3.3 (Finite analogue of Entry 2)**.** *For* $N \in \mathbb{N}$*, we have*

$$(aq)_N \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^{n-1} n(q/e)_n(ae)^n q^{n(n-1)/2}}{(aq)_n} = \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(ae)_{N-n}(q)_n(q/e)_n(ae)^n}{1-q^n}. \tag{2.3.6}$$

Proof. Letting $d \to 0$ in Theorem 2.1.5, we get

$$\sum_{n=1}^{N} \frac{(q)_N (b/a)_n (q/e)_{n-1} (-ace)^{n-1} q^{n(n-1)/2}}{(q)_{N-n} (b)_n (cq)_n (q)_{n-1}}$$

$$= \frac{(a-b)}{a(cq)_N} \sum_{n=1}^{N} \frac{(q)_N (ceq)_{N-n} (a)_{n-1} (q/e)_{n-1} (ceq)^{n-1}}{(q)_{N-n} (b)_n (q)_{n-1}}.$$

Now substitute $b = 0$ in the above identity and then re-index the sum of both sides to obtain

$$\sum_{n=0}^{N-1} \frac{(q)_N (q/e)_n (-ace)^n q^{n(n+1)/2}}{(q)_{N-n-1} (cq)_{n+1} (q)_n} = \frac{1}{(cq)_N} \sum_{n=0}^{N-1} \frac{(q)_N (ceq)_{N-n-1} (a)_n (q/e)_n (ceq)^n}{(q)_{N-n-1} (q)_n}.$$

Replacing N by $N+1$ and then multiplying both sides by $\frac{(cq)_{N+1}}{1-q^{N+1}}$ in the resulting expression yields

$$(cq^2)_N \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^n (q/e)_n (ace)^n q^{n(n+1)/2}}{(cq^2)_n} = \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} (ceq)_{N-n} (a)_n (q/e)_n (ceq)^n.$$

Now differentiate this identity with respect to the variable a to get

$$(cq^2)_N \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^n (q/e)_n n a^{n-1} (ce)^n q^{n(n+1)/2}}{(cq^2)_n} = \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} (ceq)_{N-n} (a)_n (q/e)_n (ceq)^n \times \sum_{k=0}^{n-1} \frac{-q^k}{1-aq^k}.$$

Finally, let $a \to 1$ and then substitute $c = a/q$ in the above identity and use the fact that

$$\lim_{a \to 1} (a)_n \sum_{k=0}^{n-1} \frac{q^k}{1-aq^k} = \lim_{a \to 1} \left[\frac{(a)_n}{1-a} + (a)_n \sum_{k=1}^{n-1} \frac{q^k}{1-aq^k} \right] = (q)_{n-1},$$

to prove (2.3.6). □

In Theorem 2.3.3, letting $e \to 0$ leads us to an identity (1.3.8) of Dixit and Patel which essentially serves as a finite analogue of Ramanujan's Entry 2 (1.2.2). Letting $N \to \infty$ in Theorem 2.3.3 allows us to obtain the following one variable generalization of Entry 2 (1.2.2).

Corollary 2.3.4 (One variable generalization of Entry 2)**.** *For* $|ae| < 1$*, we have*

$$\frac{(aq)_\infty}{(ae)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n n (q/e)_n (ae)^n q^{n(n-1)/2}}{(aq)_n (q)_n} = -\sum_{n=1}^{\infty} \frac{(q/e)_n (ae)^n}{1-q^n}.$$

One can easily check that, (1.2.2) follows from Corollary 2.3.4 by letting $e \to 0$.

A finite analogue of Entry 3 (1.2.3) is given below.

Theorem 2.3.5 (Finite analogue of Entry 3)**.** *For a natural number* N*, we have*

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^n (q)_{n-1} (b/a)_n a^n q^{n(n+1)/2}}{(b)_n (q/e)_n e^n} = \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_{n-1} (b)_{N-n} (aq/be)_n b^n}{(b)_N (q/e)_n}$$

$$-\sum_{n=1}^{N}\frac{bq^{n-1}}{1-bq^{n-1}}. \qquad (2.3.7)$$

Proof. Differentiating both sides of Corollary 2.1.6 with respect to e yields

$$\sum_{n=2}^{N}\begin{bmatrix}N\\n\end{bmatrix}\frac{(q)_n(c/d)_n(-zd)^n q^{n(n+1)/2}}{(zq)_n(cq)_n(q)_{n-1}}\frac{\mathrm{d}}{\mathrm{de}}\left(e^{n-1}(q/e)_{n-1}\right)=\frac{zq(1-q^N)(c-d)}{(1-zq)(cq)_N}\frac{\mathrm{d}}{\mathrm{de}}(ceq)_{N-1}$$
$$+\frac{zq(c-d)}{(cq)_N}\sum_{n=2}^{N}\begin{bmatrix}N\\n\end{bmatrix}\frac{(q)_n(zdq/c)_{n-1}(cq)^{n-1}}{(zq)_n(q)_{n-1}}\frac{\mathrm{d}}{\mathrm{de}}\left(e^{n-1}(q/e)_{n-1}(ceq)_{N-n}\right).$$

Letting $e\to q$ in the above expression, and using the fact that

$$\lim_{e\to q}\left[\frac{\mathrm{d}}{\mathrm{de}}\left(e^{n-1}(q/e)_{n-1}\right)\right]=q^{n-2}(q)_{n-2},$$

and

$$\lim_{e\to q}\frac{d}{de}(ceq)_{N-1}=(cq^2)_{N-1}\sum_{n=1}^{N-1}\frac{-cq^n}{1-cq^{n+1}},$$

gives

$$\sum_{n=2}^{N}\begin{bmatrix}N\\n\end{bmatrix}\frac{(-1)^n(q)_n(c/d)_n(zd)^n q^{(n-1)(n+4)/2}}{(zq)_n(cq)_n(1-q^{n-1})}=\frac{z(1-q^N)(c-d)}{(1-zq)(1-cq)}\sum_{k=1}^{N-1}\frac{-cq^{k+1}}{1-cq^{k+1}}$$
$$+\frac{z(c-d)}{(cq)_N}\sum_{n=2}^{N}\begin{bmatrix}N\\n\end{bmatrix}\frac{(q)_n(cq^2)_{N-n}(zdq/c)_{n-1}c^{n-1}q^{2n-2}}{(zq)_n(1-q^{n-1})}.$$

Replace N by $N+1$ and then re-index the first and the last sum to get

$$\sum_{n=1}^{N}\frac{(-1)^{n+1}(q)_{N+1}(c/d)_{n+1}(zd)^{n+1}q^{n(n+5)/2}}{(q)_{N-n}(zq)_{n+1}(cq)_{n+1}(1-q^n)}=\frac{z(1-q^{N+1})(c-d)}{(1-zq)(1-cq)}\sum_{n=1}^{N}\frac{-cq^{n+1}}{1-cq^{n+1}}$$
$$+\frac{z(c-d)}{(cq)_{N+1}}\sum_{n=1}^{N}\frac{(q)_{N+1}(cq^2)_{N-n}(zdq/c)_n c^n q^{2n}}{(q)_{N-n}(zq)_{n+1}(1-q^n)}.$$

Multiplying both sides of the above identity by $\frac{(1-zq)(1-cq)}{z(1-q^{N+1})(c-d)}$, we arrive

$$\sum_{n=1}^{N}\begin{bmatrix}N\\n\end{bmatrix}\frac{(-1)^n(q)_{n-1}(cq/d)_n(zd)^n q^{n(n+5)/2}}{(zq^2)_n(cq^2)_n}=\sum_{n=1}^{N}\begin{bmatrix}N\\n\end{bmatrix}\frac{(q)_{n-1}(cq^2)_{N-n}(zdq/c)_n c^n q^{2n}}{(cq^2)_N(zq^2)_n}$$
$$-\sum_{n=1}^{N}\frac{cq^{n+1}}{1-cq^{n+1}}.$$

Now replace c by b/q^2, d by a/q and z by $1/eq$ to get

$$\sum_{n=1}^{N}\begin{bmatrix}N\\n\end{bmatrix}\frac{(-1)^n(q)_{n-1}(b/a)_n a^n q^{n(n+1)/2}}{(b)_n(q/e)_n e^n}=\sum_{n=1}^{N}\begin{bmatrix}N\\n\end{bmatrix}\frac{(q)_{n-1}(b)_{N-n}(aq/be)_n b^n}{(b)_N(q/e)_n}-\sum_{n=1}^{N}\frac{bq^{n-1}}{1-bq^{n-1}}.$$

This proves (2.3.7). □

One can check that $e\to 0$ in the above theorem gives a new finite analogue of (1.2.3) which is different than Dixit and Patel's result (1.3.9). Also, letting $N\to\infty$

in (2.3.7) gives a one variable generalization to Entry 3 (1.2.3).

Corollary 2.3.6 (One variable generalization of Entry 3)**.** *For* $|b| < 1$*, we have*

$$\sum_{n=1}^{\infty} \frac{(-1)^n (b/a)_n a^n q^{n(n+1)/2}}{(1-q^n)(b)_n (q/e)_n e^n} = \sum_{n=1}^{\infty} \frac{(aq/be)_n b^n}{(1-q^n)(q/e)_n} - \sum_{n=1}^{\infty} \frac{b^n}{1-q^n}. \tag{2.3.8}$$

Remark 2. *Letting* $e \to 0$ *in* (2.3.8) *and using the fact that*

$$\lim_{e\to 0} \frac{(aq/be)_n}{(q/e)_n} = \left(\frac{a}{b}\right)^n$$

serves Entry 3 (1.2.3)*. We can also recover Entry 1* (1.2.1) *from* (2.3.8) *by multiplying both sides by* $1 - \frac{q}{e}$ *and then taking* $e \to q$ *in the resultant identity gives*

$$\sum_{n=1}^{\infty} \frac{(-1)^n (b/a)_n a^n q^{n(n-1)/2}}{(b)_n (q)_n} = \sum_{n=1}^{\infty} \frac{(a/b)_n b^n}{(q)_n}.$$

Now use the q-binomial theorem (2.2.1) *and then replace* a *by* $-aq$ *and* b *by* bq *to get* (1.2.1)*.*

Allowing $a \to 0$ and then replacing b by zq in Theorem 2.3.5 gives a finite analogue of Entry 4 (1.2.4).

Theorem 2.3.7 (Finite analogue of Entry 4)**.** *Let* $N \in \mathbb{N}$*. Then we have*

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_{n-1} z^n q^{n(n+1)}}{(zq)_n (q/e)_n e^n} = \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_{n-1} (zq)_{N-n} (aq)^n}{(zq)_N (q/e)_n} - \sum_{n=1}^{N} \frac{zq^n}{1-zq^n}. \tag{2.3.9}$$

Letting $e \to 0$ allow us to recover a finite analogue of Entry 4 (1.2.4) given by Dixit and Patel (1.3.10). Further, letting $N \to \infty$ in (2.3.9) gives a one variable generalization of Entry 4 (1.2.4).

Corollary 2.3.8 (One variable generalization of Entry 4)**.** *For* $|zq| < 1$*, we have*

$$\sum_{n=1}^{\infty} \frac{z^n q^{n(n+1)}}{(1-q^n)(zq)_n (q/e)_n e^n} = \sum_{n=1}^{\infty} \frac{(zq)^n}{(1-q^n)} \left(\frac{1}{(q/e)_n} - 1 \right).$$

A finite analogue of Entry 5 (1.2.5) is given below.

Theorem 2.3.9 (Finite analogue of Entry 5)**.** *For a natural number* N*, the following identity is true:*

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^n (q)_{n-1}^2 a^n q^{n(n+1)/2}}{(a)_n (q/e)_n e^n} = -\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_{n-1} (a)_{N-n} a^n}{(a)_N} \left(\sum_{k=1}^{n} \frac{q^k}{e - q^k} \right).$$

Proof. Utilizing (1.3.2) with $b = 0$ and $c = 1$ in Theorem 2.3.5, we get

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^n (q)_{n-1} (b/a)_n a^n q^{n(n+1)/2}}{(b)_n (q/e)_n e^n} = \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_{n-1} (b)_{N-n} b^n}{(b)_N} \left(\frac{(aq/be)_n}{(q/e)_n} - 1 \right).$$

Now divide both sides by $1 - b/a$ in the above expression and then let $b \to a$ to get the result. ∎

Letting $N \to \infty$ in the above theorem gives a one variable generalization of Entry 5 (1.2.5).

Corollary 2.3.10 (One variable generalization of Entry 5)**.** *For $|a| < 1$, one can see*

$$\sum_{n=1}^{\infty} \frac{(-1)^n (q)_{n-1} a^n q^{n(n+1)/2}}{(1-q^n)(a)_n (q/e)_n e^n} = -\sum_{n=1}^{\infty} \frac{a^n}{1-q^n} \sum_{k=1}^{n} \frac{q^k}{e-q^k}. \tag{2.3.10}$$

Letting $e \to 0$ in (2.3.10) gives Entry 5 (1.2.5).

2.3.2 Proof of Theorem 2.1.9 and some of its corollaries

In this section, our goal is to prove Theorem 2.1.9. However, before proving this theorem, it is essential to introduce a few key results. These lemmas play a vital role in the proof of our theorem.

Lemma 2.3.11. *For $N \in \mathbb{N}$, we have*

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} \frac{(q^2;q^2)_n (dq^2;q^2)_{n-1} (zq;q^2)_{N-n} (zq)^n}{(zq^2;q^2)_n (zq;q^2)_N}$$
$$= \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} \frac{(q^2;q^2)_n (dq;q^2)_{n-1} (zq^2;q^2)_{N-n} z^n q^{2n-1}}{(zq;q^2)_n (zq^2;q^2)_N}.$$

Proof. Let us define the left hand side expression as $S_1(z,d,q,N)$, so we have

$$\begin{aligned} S_1(z,d,q,N) &= \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} \frac{(q^2;q^2)_n (dq^2;q^2)_{n-1} (zq;q^2)_{N-n} (zq)^n}{(zq^2;q^2)_n (zq;q^2)_N} \\ &= \sum_{n=0}^{N-1} \frac{(q^2;q^2)_N (dq^2;q^2)_n (zq;q^2)_{N-n-1} (zq)^{n+1}}{(q^2;q^2)_{N-n-1} (zq^2;q^2)_{n+1} (zq;q^2)_N} \\ &= \frac{zq(1-q^{2N})}{(1-zq^{2N-1})(1-zq^2)} \sum_{n=0}^{N-1} \frac{(q^2;q^2)_{N-1} (zq;q^2)_{N-n-1} (dq^2;q^2)_n (zq)^n}{(zq;q^2)_{N-1} (q^2;q^2)_{N-n-1} (zq^4;q^2)_n}. \end{aligned}$$

Now use (2.2.12) with $a = zq$ and $b = q^2$ and N replaced by $N-1$. We have

$$\begin{aligned} S_1(z,d,q,N) &= \frac{zq(1-q^{2N})}{(1-zq^{2N-1})(1-zq^2)} \sum_{n=0}^{N-1} \frac{(q^{2-2N};q^2)_n (dq^2;q^2)_n}{(q^{3-2N}/z;q^2)_n (zq^4;q^2)_n} q^{2n} \\ &= \frac{zq(1-q^{2N})}{(1-zq^{2N-1})(1-zq^2)} {}_3\phi_2 \left[\begin{matrix} q^{2-2N}, & dq^2, & q^2 \\ & zq^4, & \frac{q^{3-2N}}{z} \end{matrix} ; q^2, q^2 \right]. \end{aligned} \tag{2.3.11}$$

At this point, make use of (2.2.7) with N and q replaced by $N-1$ and q^2 respectively, and taking $\alpha = dq^2$, $\beta = q^2$, $\gamma = zq^4$ and $\tau = zq$ in (2.3.11) yields

$$S_1(z,d,q,N) = \frac{zq(1-q^{2N})}{(1-zq^{2N-1})(1-zq^2)} \frac{(zq^2;q^2)_{N-1} (zq^3;q^2)_{N-1}}{(zq^4;q^2)_{N-1} (zq;q^2)_{N-1}}$$

$$\times {}_3\phi_2\left[\begin{matrix} q^{2-2N}, & dq, & q^2 \\ & zq^3, & \frac{q^{2-2N}}{z} \end{matrix}; q^2, q^2\right]$$

$$= \frac{zq(1-q^{2N})}{(1-zq)(1-zq^{2N})}\sum_{n=0}^{N-1}\frac{(q^{2-2N};q^2)_n(dq;q^2)_n}{(q^{2-2N}/z)_n(zq^3;q^2)_n}q^{2n}.$$

Here again we use (2.2.12) with N replaced by $N-1$, $a = zq^2$ and $b = q^2$, we get

$$S_1(z,d,q,N) = \frac{zq(1-q^{2N})}{(1-zq)(1-zq^{2N})}\sum_{n=0}^{N-1}\frac{(q^2;q^2)_{N-1}(zq^2;q^2)_{N-n-1}(dq;q^2)_n}{(zq^2;q^2)_{N-1}(q^2;q^2)_{N-n-1}(zq^3;q^2)_n}\left(zq^2\right)^n$$

$$= \sum_{n=1}^{N}\begin{bmatrix} N \\ n \end{bmatrix}_{q^2}\frac{(q^2;q^2)_n(zq^2;q^2)_{N-n}(dq;q^2)_{n-1}}{(zq^2;q^2)_N(zq;q^2)_n}z^nq^{2n-1}.$$

This proves the result. □

Lemma 2.3.12. *For $N \in \mathbb{N}$, we have*

$$S_1(z,d,q,N) = \sum_{n=1}^{N}\begin{bmatrix} N \\ n \end{bmatrix}_{q^2}\left(\frac{(dq)_{2n-2}z^{2n-1}q^{n(2n-1)}}{(zq)_{2n-1}} + \frac{(dq)_{2n-1}z^{2n}q^{n(2n+1)}}{(zq)_{2n}}\right)\frac{(q^2;q^2)_n}{(dzq^{2N+1};q^2)_n}.$$

Proof. Replace q by q^2 and let $\alpha = \frac{d}{q}$, $\beta = zq$ and $\tau = zq^2$ in (2.2.11) and then multiplying both sides of the resulting identity by $\frac{(1-dzq^{2N+3})}{(1-zq)(1-zq^{2N+2})}$, we get

$$(1-dzq^{2N+3})\sum_{n=0}^{N}\frac{(q^2;q^2)_N(dq;q^2)_n(zq^2;q^2)_{N-n}z^nq^{2n}}{(q^2;q^2)_{N-n}(zq;q^2)_{n+1}(zq^2;q^2)_{N+1}}$$

$$= \sum_{n=0}^{N}\frac{(q^2;q^2)_N(dq;q^2)_n(dq^2;q^2)_n(dzq^5;q^2)_Nz^{2n}q^{3n+2n^2}(1-dzq^{4n+3})}{(q^2;q^2)_{N-n}(zq;q^2)_{n+1}(zq^2;q^2)_{n+1}(dzq^5;q^2)_{N+n}}$$

$$= \sum_{n=0}^{N}\frac{(dq;q)_{2n}z^{2n}q^{n(2n+3)}(1-dzq^{4n+3})}{(zq;q)_{2n+2}}\frac{(q^2;q^2)_N(dzq^5;q^2)_N}{(q^2;q^2)_{N-n}(dzq^5;q^2)_{N+n}}$$

$$= \sum_{n=0}^{N}\left[\frac{(dq;q)_{2n}z^{2n}q^{n(2n+3)}}{(zq;q)_{2n+1}} + \frac{(dq;q)_{2n+1}z^{2n+1}q^{(n+2)(2n+1)}}{(zq;q)_{2n+2}}\right]\frac{(q^2;q^2)_N}{(q^2;q^2)_{N-n}(dzq^{2N+5};q^2)_n}.$$

Now re-indexing both sums and then multiplying both sides by $zq(1-q^{2N+2})$ and further replace N by $N-1$, we get

$$\sum_{n=1}^{N}\frac{(q^2;q^2)_N(dq;q^2)_{n-1}(zq^2;q^2)_{N-n}z^nq^{2n-1}}{(q^2;q^2)_{N-n}(zq;q^2)_n(zq^2;q^2)_N}$$

$$= \sum_{n=1}^{N}\left[\frac{(dq;q)_{2n-2}z^{2n-2}q^{(n)(2n+1)}}{(zq;q)_{2n-1}} + \frac{(dq;q)_{2n-1}z^{2n}q^{(n)(2n+1)}}{(zq;q)_{2n}}\right]\frac{(q^2;q^2)_N}{(q^2;q^2)_{N-n}(dzq^{2N+3};q^2)_n}.$$

The result follows from the above equation and Lemma 2.3.11. □

Now we are ready to prove Theorem 2.1.9.

Proof of Theorem 2.1.9. Take $e = 1$, replace q by q^2 then z by z/q and put $c = z$ in Corollary 2.1.6. This yields

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} \frac{(-1)^n (q^2;q^2)_n (z/d;q^2)_n (zd)^n q^{n^2}}{(z-d)(zq;q^2)_n (zq^2;q^2)_n}$$
$$= \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} \frac{(q^2;q^2)_n (zq^2;q^2)_{N-n} (dq;q^2)_{n-1} z^n q^{2n-1}}{(zq^2;q^2)_N (zq;q^2)_n}.$$

Now the result follows from the above equation and Lemmas 2.3.11 and 2.3.12. □

Here are some immediate implications of Theorem 2.1.9.

Corollary 2.3.13. *For $N \in \mathbb{N}$, we have*

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} \frac{(-1)^{n-1} (q^2/d;q^2)_{n-1} d^{n-1} q^{n^2}}{(q;q^2)_n} = \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} \frac{(dq^2;q^2)_{n-1} (q;q^2)_{N-n} q^n}{(q;q^2)_N}$$
$$= \sum_{n=1}^{N} \frac{(dq;q^2)_{n-1} q^{2n-1}}{(q;q^2)_n} = \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} \frac{(q^2;q^2)_{n-1} (dq)_{2n-1} (1-dq^{4n-1}) q^{n(2n-1)}}{(dq^{2N+1};q^2)_n (q)_{2n-1} (1-dq^{2n-1})}.$$

Proof. Substituting $z = 1$ in Theorem 2.1.9, Lemmas 2.3.11 and 2.3.12, then combining the result together, we get the required results. □

Substituting $d = 1$ in Corollary 2.3.13 gives [23, Corollary 9.5]. Also, letting $N \to \infty$ in Corollary 2.3.13 gives a one variable generalization to [24, p. 345, Remark 3], which is stated below.

Corollary 2.3.14. *We have*

$$\sum_{n=1}^{\infty} \frac{(-1)^n (1/d;q^2)_n d^n q^{n^2}}{(1-d)(q;q)_{2n}} = \sum_{n=1}^{\infty} \frac{(dq^2;q^2)_{n-1} q^n}{(q^2;q^2)_n} = \sum_{n=1}^{\infty} \frac{(dq;q^2)_{n-1} q^{2n-1}}{(q;q^2)_n}$$
$$= \sum_{n=1}^{\infty} \frac{(dq)_{2n-1} (1-dq^{4n-1}) q^{n(2n-1)}}{(q)_{2n-1} (1-q^{2n})(1-dq^{2n-1})}.$$

A further implication of Theorem 2.1.9 is given below.

Corollary 2.3.15. *For $N \in \mathbb{N}$, we have*

$$\frac{1}{(d-q)} \left(1 - \frac{(dq^2;q^2)_N}{(q^3;q^2)_N} \right) = \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} \frac{(-1)^{n-1} (q/d;q^2)_n d^n q^{n(n+1)}}{(d-q)(q^3;q^2)_n}$$
$$= \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} \frac{(dq;q^2)_{n-1} (q^3;q^2)_{N-n} q^{3n-1}}{(q^3;q^2)_N} = \sum_{n=1}^{N} \frac{(dq^2;q^2)_{n-1}}{(q^3;q^2)_n} q^{2n}$$
$$= (1-q) \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} \frac{(q^2;q^2)_{n-1} (dq)_{2n-1} (1-dq^{4n}) q^{2n^2+n-1}}{(dq^{2N+2};q^2)_n (q)_{2n-1} (1-dq^{2n-1})(1-q^{2n+1})}.$$

Proof. In [25, Lemma 4.2], replace q by q^2 and then put $c = q$. We get

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} \frac{(-1)^{n-1}(q/d;q^2)_n d^n q^{n(n+1)}}{(q^3;q^2)_n} = 1 - \frac{(dq^2;q^2)_N}{(q^3;q^2)_N}.$$

Now, substitute $z = q$ in Theorem 2.1.9, Lemma 2.3.11, and 2.3.12. Combining all the resulting identities and use the above identity to get the desired result. □

Note that, substituting $d = 1$ in Corollary 2.3.15 gives [23, Corollary 9.7].

2.3.3 Identity involving finite sum of ${}_3\phi_2$ and ${}_2\phi_1$

In this section, our objective is to establish an identity that involves a finite sum of the basic hypergeometric series ${}_3\phi_2$ and ${}_2\phi_1$. This identity is significant as it effectively serves as a generalization of an identity (1.3.12) of Dixit and Patel. Before going into the proof of this identity, we will begin by introducing and proving a few important lemmas. These lemmas will be essential to prove our theorem.

Lemma 2.3.16. *We have*

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^{n-1}(c/d)_n(q/e)_{n-1}e^{n-1}d^n q^{n(n+1)/2}}{(cq)_n(q)_{n-1}} \left(\sum_{k=1}^{n} \frac{q^k}{1-q^k}\right)$$
$$= \frac{(c/d)_\infty(deq)_\infty(q/e)_\infty}{(cq)_\infty(deq^{N+1})_\infty(q)_\infty} \sum_{k=1}^{N} \begin{bmatrix} N \\ k \end{bmatrix} \frac{e^{k-1}d^k q^{k(k+1)}}{(deq)_k(1-q^k)} \sum_{m=0}^{\infty} \frac{(dq)_m(deq^{N+1})_m}{(deq^{k+1})_m(q)_m} \left(\frac{cq^k}{d}\right)^m$$
$$\times {}_2\phi_1 \left[\begin{matrix} e, & deq^{N+m+1} \\ & deq^{k+m+1} \end{matrix} ; \ q, \ \frac{q^k}{e} \right]. \tag{2.3.12}$$

Proof. Using van Hamme's identity [34]

$$\sum_{k=1}^{n} \frac{q^k}{1-q^k} = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^{k-1}q^{k(k+1)/2}}{1-q^k},$$

the left hand side expression of (2.3.12) becomes

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^{n-1}(c/d)_n(q/e)_{n-1}e^{n-1}d^n q^{n(n+1)/2}}{(cq)_n(q)_{n-1}} \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^{k-1}q^{k(k+1)/2}}{1-q^k}$$
$$= \sum_{k=1}^{N}\sum_{n=k}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^{n-1}(c/d)_n(q/e)_{n-1}e^{n-1}d^n q^{n(n+1)/2}}{(cq)_n(q)_{n-1}} \frac{(-1)^{k-1}q^{k(k+1)/2}}{1-q^k}.$$

Now re-indexing the second sum of the above expression by replacing n by $n+k$ yields

$$(q)_N \sum_{k=1}^{N} \frac{(-1)^{k-1}q^{k(k+1)/2}}{(q)_k(1-q^k)} \sum_{n=0}^{N-k} \frac{(-1)^{n+k-1}(c/d)_{n+k}(q/e)_{n+k-1}e^{n+k-1}d^{n+k}q^{(n+k)(n+k+1)/2}}{(cq)_{n+k}(q)_n(q)_{N-n-k}(q)_{n+k-1}}$$

$$= (q)_N \sum_{k=1}^{N} \frac{(c/d)_k (q/e)_{k-1} q^{k(k+1)} e^{k-1} d^k}{(cq)_k (q)_k (q)_{k-1} (1-q^k)} \sum_{n=0}^{N-k} \frac{(-1)^n (cq^k/d)_n (q^k/e)_n (de)^n q^{\frac{n(n+1)}{2}+nk}}{(cq^{k+1})_n (q)_n (q)_{N-n-k} (q^k)_n}.$$

Taking $x = 1$ and replacing N by $N-k$ in (2.2.8) and making use of it in the second sum of the above expression to obtain

$$(q)_N \sum_{k=1}^{N} \frac{(c/d)_k (q/e)_{k-1} q^{k(k+1)} e^{k-1} d^k}{(cq)_k (q)_k (q)_{k-1} (1-q^k)} \sum_{n=0}^{N-k} \frac{\left(q^{-(N-k)}\right)_n (cq^k/d)_n (q^k/e)_n}{(cq^{k+1})_n (q)_n (q)_{N-k} (q^k)_n} \left(deq^{(N+1)}\right)^n$$

$$= \sum_{k=1}^{N} \begin{bmatrix} N \\ k \end{bmatrix} \frac{(c/d)_k (q/e)_{k-1} q^{k(k+1)} e^{k-1} d^k}{(cq)_k (q)_{k-1} (1-q^k)} \frac{(cq^k/d)_\infty}{(cq^{k+1})_\infty} \sum_{n=0}^{N-k} \frac{\left(q^{-(N-k)}\right)_n (q^k/e)_n}{(q)_n (q^k)_n} \left(deq^{(N+1)}\right)^n \times \frac{(cq^{k+n+1})_\infty}{(cq^{k+n}/d)_\infty}.$$

Now make use of the q-binomial theorem (2.2.1) and get

$$\frac{(c/d)_\infty}{(cq)_\infty} \sum_{k=1}^{N} \begin{bmatrix} N \\ k \end{bmatrix} \frac{(q/e)_{k-1} q^{k(k+1)} e^{k-1} d^k}{(q)_{k-1} (1-q^k)} \sum_{n=0}^{N-k} \frac{\left(q^{-(N-k)}\right)_n (q^k/e)_n}{(q)_n (q^k)_n} \left(deq^{(N+1)}\right)^n$$

$$\times \sum_{m=0}^{\infty} \frac{(dq)_m}{(q)_m} \left(\frac{cq^{k+n}}{d}\right)^m = \frac{(c/d)_\infty}{(cq)_\infty} \sum_{k=1}^{N} \begin{bmatrix} N \\ k \end{bmatrix} \frac{(q/e)_{k-1} q^{k(k+1)} e^{k-1} d^k}{(q)_{k-1} (1-q^k)} \sum_{m=0}^{\infty} \frac{(dq)_m}{(q)_m} \left(\frac{cq^k}{d}\right)^m \times {}_2\phi_1 \left[\begin{matrix} q^{-(N-k)}, & q^k/e \\ & q^k \end{matrix} ;\ q,\ deq^{N+m+1} \right].$$

Now apply Heine's transformation (2.2.2) with $a = q^{-(N-k)},\ b = q^k/e,\ c = q^k$ and $z = deq^{N+m+1}$ and get the required right hand side of (2.3.12) upon simplification.

□

Lemma 2.3.17. *We have*

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^{n-1} (c/d)_n (q/e)_{n-1} e^{n-1} d^n q^{n(n+1)/2}}{(cq)_n (q)_{n-1}} = \frac{(deq)_N}{(cq)_N} \sum_{n=1}^{N} \frac{(q^{-N})_n (d/c)_n (eq)_{n-1}}{(deq)_n (q)_n (q)_{n-1}} \left(cq^{N+1}\right)^n.$$

Proof. Putting $z = 1$ in Corollary 2.1.6, we have

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^n (c/d)_n (q/e)_{n-1} q^{n(n+1)/2} d^n e^{n-1}}{(cq)_n (q)_{n-1}}$$

$$= \frac{1}{(cq)_N} \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(ceq)_{N-n} (d/c)_n (q/e)_{n-1} (cq)^n e^{n-1}}{(q)_{n-1}}$$

$$= \frac{(ceq)_N}{e(cq)_N} \sum_{n=1}^{N} \frac{(q)_N (ceq)_{N-n} (d/c)_n (q/e)_{n-1} (ceq)^n}{(q)_{N-n} (ceq)_N (q)_n (q)_{n-1}}.$$

Utilizing (2.2.9) with $x = ce$ in right hand side of the above expression, we get

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^n (c/d)_n (q/e)_{n-1} q^{n(n+1)/2} d^n e^{n-1}}{(cq)_n (q)_{n-1}} = \frac{(ceq)_N}{e(cq)_N} \sum_{n=1}^{N} \frac{\left(q^{-N}\right)_n (d/c)_n (q/e)_{n-1} (q)^n}{\left(q^{-N}/ce\right)_n (q)_n (q)_{n-1}}$$

$$= \frac{q}{e} \frac{(ceq)_N}{(cq)_N} \frac{\left(1-q^{-N}\right)(1-d/c)}{\left(1-q^{-N}/ce\right)(1-q)} {}_3\phi_2 \left[\begin{matrix} q^{-(N-1)}, & q/e, & dq/c \\ q^2 & q^{-(N-1)}/ce & \end{matrix} ; \ q, \ q \right].$$

Applying ${}_3\phi_2$ transformation with N replaced by $N-1$ and substituting $B = q/e,\ C = dq/c,\ D = q^2$ and $E = q^{-(N-1)}/ce$ in (2.2.5), we get

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^n (c/d)_n (q/e)_{n-1} q^{n(n+1)/2} d^n e^{n-1}}{(cq)_n (q)_{n-1}}$$

$$= \frac{q}{e} \frac{(ceq)_N}{(cq)_N} \frac{\left(1-q^{-N}\right)(1-d/c)}{\left(1-q^{-N}/ce\right)(1-q)} \frac{\left(q^{-N}/de\right)_{N-1}}{\left(q^{-(N-1)}/ce\right)_{N-1}} \left(\frac{dq}{c}\right)^{N-1}$$

$$\times {}_3\phi_2 \left[\begin{matrix} q^{-(N-1)}, & dq/c, & eq \\ q^2 & deq^2 & \end{matrix} ; \ q, \ cq^{N+1} \right]$$

$$= \frac{q}{e} \frac{(ceq)_N}{(cq)_N} \frac{\left(1-q^{-N}\right)(1-d/c)}{\left(1-q^{-N}/ce\right)(1-q)} \frac{(deq^2)_{N-1}}{(ceq)_{N-1}} \sum_{n=0}^{N-1} \frac{\left(q^{-(N-1)}\right)_n (dq/c)_n (eq)_n}{(q^2)_n (deq^2)_n (q)_n} \left(cq^{N+1}\right)^n$$

$$= \frac{(deq)_N}{(cq)_N} \sum_{n=0}^{N-1} \frac{\left(q^{-N}\right)_{n+1} (d/c)_{n+1} (eq)_n}{(q)_{n+1} (deq)_{n+1} (q)_n} \left(cq^{N+1}\right)^{n+1}.$$

This completes the proof of Lemma 2.3.17 by re-indexing the above sum by replacing n by $n-1$. □

We now ready to prove our main identity of this section, which is a one variable generalization of an identity (1.3.12) of Dixit and Patel.

Theorem 2.3.18. *We have*

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{n(-1)^{n-1} (c/d)_n (q/e)_{n-1} e^{n-1} d^n q^{n(n+1)/2}}{(cq)_n (q)_{n-1}} + \frac{(c/d)_\infty (deq)_\infty (q/e)_\infty}{(cq)_\infty (deq^{N+1})_\infty (q)_\infty}$$

$$\times \sum_{k=1}^{N} \begin{bmatrix} N \\ k \end{bmatrix} \frac{e^{k-1} d^k q^{k(k+1)}}{(deq)_k (1-q^k)} \sum_{m=0}^{\infty} \frac{(dq)_m (deq^{N+1})_m}{(deq^{k+1})_m (q)_m} \left(\frac{cq^k}{d}\right)^m {}_2\phi_1 \left[\begin{matrix} e, & deq^{N+m+1} \\ & deq^{k+m+1} \end{matrix} ; \ q, \ \frac{q^k}{e} \right]$$

$$= \frac{c}{c-d} \frac{(deq)_N}{(cq)_N} \sum_{n=1}^{N} \frac{(q^{-N})_n (d/c)_n (eq)_{n-1}}{(deq)_n (q)_n (q)_{n-1}} \left(cq^{N+1}\right)^n + \frac{(q/e)_{N-1}}{(cq)_N (q)_{N-1}}$$

$$\times \sum_{k=1}^{N} \begin{bmatrix} N \\ k \end{bmatrix} \frac{(cq/d)_k (deq)_{N-k} (dq)^k e^{k-1}}{(1-q^k)} {}_3\phi_2 \left[\begin{matrix} q^{-(N-k)}, & e, & ceq \\ deq & eq^{1-N} & \end{matrix} ; \ q, \ q \right]. \qquad (2.3.13)$$

Remark 3. *It can be easily observed that, $e = 1$ in the above identity gives* (1.3.12).

Proof of Theorem 2.3.18. Differentiating both sides of Corollary 2.1.6 with respect to z and then put $z = 1$ to get

$$\sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^{n-1}(c/d)_n(q/e)_{n-1}e^{n-1}d^n q^{n(n+1)/2}}{(cq)_n(q)_{n-1}} \left(n + \sum_{k=1}^{n} \frac{q^k}{1-q^k} \right) =: T_1 + T_2, \tag{2.3.14}$$

where

$$T_1 = \frac{(d-c)}{c(cq)_N} \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(ceq)_{N-n}(dq/c)_{n-1}(q/e)_{n-1}(cq)^n e^{n-1}}{(q)_{n-1}}, \tag{2.3.15}$$

$$T_2 = \frac{(d-c)}{c(cq)_N} \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(ceq)_{N-n}(dq/c)_{n-1}(q/e)_{n-1}(cq)^n e^{n-1}}{(q)_{n-1}} \left(\sum_{k=1}^{n} \frac{q^k}{1-q^k} - \sum_{k=1}^{n-1} \frac{q^k d/c}{1-q^k d/c} \right). \tag{2.3.16}$$

Using Corollary 2.1.6 with $z = 1$ and Lemma 2.3.17 in (2.3.15), we get

$$T_1 = \frac{(deq)_N}{(cq)_N} \sum_{n=1}^{N} \frac{(q^{-N})_n(d/c)_n(eq)_{n-1}}{(deq)_n(q)_n(q)_{n-1}} \left(cq^{N+1} \right)^n. \tag{2.3.17}$$

Guo and Zhang [32, Corollary 3.1] proved the following identity for $n \geq 0$ and $0 \leq m \leq n$:

$$\sum_{\substack{k=0 \\ k \neq m}}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(q/x)_k(x)_{n-k}}{1-q^{k-m}} x^k = (-1)^m q^{\frac{m(m+1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix} \left(xq^{-m} \right)_n \left(\sum_{k=0}^{n-1} \frac{xq^{k-m}}{1-xq^{k-m}} - \sum_{\substack{k=0 \\ k \neq m}}^{n} \frac{q^{k-m}}{1-q^{k-m}} \right).$$

Substituting $m = 0$ and $x = d/c$ in the above identity gives

$$\sum_{k=1}^{n} \frac{q^k}{1-q^k} - \sum_{k=1}^{n-1} \frac{q^k d/c}{1-q^k d/c} = \frac{d}{c-d} - \frac{1}{(d/c)_n} \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(cq/d)_k(d/c)_{n-k}}{1-q^k} \left(\frac{d}{c} \right)^k.$$

Use the above expression and employing (2.3.15) in (2.3.16) yields

$$T_2 = \frac{d}{c-d} T_1 + T_3, \tag{2.3.18}$$

where

$$\begin{aligned}
T_3 &= \frac{1}{(cq)_N} \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(ceq)_{N-n}(q/e)_{n-1}(cq)^n e^{n-1}}{(q)_{n-1}} \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(cq/d)_n(d/c)_{n-k}(d/c)^k}{1-q^k} \\
&= \frac{(q)_N}{(cq)_N} \sum_{n=1}^{N} \frac{(ceq)_{N-n}(q/e)_{n-1}(cq)^n e^{n-1}}{(q)_{N-n}(q)_{n-1}} \sum_{k=1}^{n} \frac{(cq/d)_k(d/c)_{n-k}(d/c)^k}{(q)_k(q)_{n-k}(1-q^k)} \\
&= \frac{(q)_N}{(cq)_N} \sum_{k=1}^{N} \frac{(cq/d)_k(d/c)^k}{(q)_k(1-q^k)} \sum_{n=k}^{N} \frac{(ceq)_{N-n}(d/c)_{n-k}(q/e)_{n-1}(cq)^n e^{n-1}}{(q)_{n-k}(q)_{N-n}(q)_{n-1}} \\
&= \frac{(q)_N}{(cq)_N} \sum_{k=1}^{N} \frac{(cq/d)_k(d/c)^k}{(q)_k(1-q^k)} \sum_{n=0}^{N-k} \frac{(ceq)_{N-n-k}(d/c)_n(q/e)_{n+k-1}(cq)^{n+k} e^{n+k-1}}{(q)_n(q)_{N-n-k}(q)_{n+k-1}}
\end{aligned}$$

$$= \frac{(q)_N}{(cq)_N} \sum_{k=1}^{N} \frac{(cq/d)_k (q/e)_{k-1} (dq)^k e^{k-1}}{(q)_k (q)_{k-1} (1-q^k)} \sum_{n=0}^{N-k} \frac{(ceq)_{N-n-k} (d/c)_n (q^k/e)_n (ceq)^n}{(q)_{N-n-k} (q)_n (q^k)_n}$$

$$= \frac{(q)_N}{(cq)_N} \sum_{k=1}^{N} \frac{(cq/d)_k (ceq)_{N-k} (q/e)_{k-1} (dq)^k e^{k-1}}{(q)_k (q)_{N-k} (q)_{k-1} (1-q^k)} \sum_{n=0}^{N-k} \frac{\left(q^{-(N-k)}\right)_n (d/c)_n (q^k/e)_n q^n}{(q^k)_n \left(q^{-(N-k)}/ce\right)_n (q)_n},$$

where in the last step, we used (2.2.9) with N replaced by $N-k$ and $x = ce$. Now use (2.2.6) with N replaced by $N-k$, $\alpha = d/c$, $\beta = q^k/e$, $\gamma = q^k$ and $\tau = ceq$ to see that

$$\sum_{n=0}^{N-k} \frac{\left(q^{-(N-k)}\right)_n (d/c)_n (q^k/e)_n q^n}{(q^k)_n \left(q^{-(N-k)}/ce\right)_n (q)_n} = \frac{(q^k/e)_{N-k} (deq)_{N-k}}{(q^k)_{N-k} (ceq)_{N-k}} {}_3\phi_2 \left[\begin{matrix} q^{-(N-k)}, & e, & ceq \\ deq, & eq^{1-N} & \end{matrix} ; q, q \right].$$

Using the above identity in the final expression of T_3, we get

$$T_3 = \frac{(q/e)_{N-1}}{(cq)_N (q)_{N-1}} \sum_{k=1}^{N} \begin{bmatrix} N \\ k \end{bmatrix} \frac{(cq/d)_k (deq)_{N-k} (dq)^k e^{k-1}}{(1-q^k)} {}_3\phi_2 \left[\begin{matrix} q^{-(N-k)}, & e, & ceq \\ deq, & eq^{1-N} & \end{matrix} ; q, q \right]. \tag{2.3.19}$$

Now utilize (2.3.17) and (2.3.19) in (2.3.18) to deduce

$$T_2 = \frac{d}{c-d} \frac{(deq)_N}{(cq)_N} \sum_{n=1}^{N} \frac{(q^{-N})_n (d/c)_n (eq)_{n-1}}{(deq)_n (q)_n (q)_{n-1}} \left(cq^{N+1}\right)^n + \frac{(q/e)_{N-1}}{(cq)_N (q)_{N-1}}$$

$$\times \sum_{k=1}^{N} \begin{bmatrix} N \\ k \end{bmatrix} \frac{(cq/d)_k (deq)_{N-k} (dq)^k e^{k-1}}{(1-q^k)} {}_3\phi_2 \left[\begin{matrix} q^{-(N-k)}, & e, & ceq \\ deq, & eq^{1-N} & \end{matrix} ; q, q \right]. \tag{2.3.20}$$

Now combine (2.3.17), (2.3.20) and Lemma 2.3.16 in (2.3.14) to get the desired identity (2.3.13). □

Chapter 3

BRESSOUD–SUBBARAO TYPE WEIGHTED PARTITION IDENTITIES FOR A GENERALIZED DIVISOR FUNCTION

All the results presented in this chapter are taken from our joint work with Bhoria, Eyyunni and Maji [2]. We begin this chapter by presenting an extension of the identity (1.4.2) due to Bhoria, Eyyunni, and Maji. This extension enlarges the domain of the parameter z appearing in the weighted partition identity of Bressoud and Subbarao (1.4.1), allowing z to vary over the entire complex plane rather than being restricted to non-negative integers. We then establish a one variable generalization of Uchimura's Theorem 1.5.1, thereby broadening the scope of his identity within the framework of weighted partition theory. Following this, we discuss several new weighted partition identities that further generalize the classical Bressoud and Subbarao's identity (1.2.8). These results collectively deepen the understanding of weighted partitions and their associated q-series structures, and they form the central focus of this chapter. In the following section, we collect all the results, while the proofs of these results are provided in the section that follows.

3.1 Main results

We start with an extension of an identity (1.4.2) of Bhoria, Eyyunni and Maji.

Theorem 3.1.1. *Let c and z be two complex numbers. Then, for any $n \in \mathbb{N}$, we have*

$$\sum_{\pi\in\mathcal{D}(n)} (-1)^{\#(\pi)-1} \sum_{j=1}^{s(\pi)} (\ell(\pi) - s(\pi) + j)^z c^{\ell(\pi)-s(\pi)+j} = \sigma_{z,c}(n), \tag{3.1.1}$$

where $\sigma_{z,c}(n)$ is the generalized divisor function defined in (1.4.3).

The next theorem is a one-variable generalization of Uchimura's result i.e. Theorem 1.5.1.

Theorem 3.1.2. *For a non-negative integer m and a complex number c with $|cq| < 1$, we define*

$$\begin{aligned} M_{m,c} &:= M_{m,c}(q) = \sum_{n=1}^{\infty} n^m c^n q^n (q^{n+1})_\infty, \\ K_{m+1,c} &:= K_{m+1,c}(q) = \sum_{n=1}^{\infty} \sigma_{m,c}(n) q^n. \end{aligned} \tag{3.1.2}$$

Let Y_m be the Bell polynomial defined by

$$Y_m(u_1, u_2, \cdots, u_m) := \sum_{\Pi(m)} \frac{m!}{k_1! \cdots k_m!} \left(\frac{u_1}{1!}\right)^{k_1} \cdots \left(\frac{u_m}{m!}\right)^{k_m},$$

where $\Pi(m)$ denotes a partition of m with

$$k_1 + 2k_2 + \cdots + mk_m = m.$$

Then the exponential generating functions of $M_{m,c}$ and $K_{m,c}$ are related by

$$\frac{(q)_\infty}{(cq)_\infty} \exp\left(\sum_{m=1}^{\infty} K_{m,c} \frac{t^m}{m!}\right) = \frac{(q)_\infty}{(cq)_\infty} + \sum_{m=1}^{\infty} M_{m,c} \frac{t^m}{m!},$$

and for any $m \geq 1$, we have

$$M_{m,c} = \frac{(q)_\infty}{(cq)_\infty} Y_m(K_{1,c}, \cdots, K_{m,c}). \tag{3.1.3}$$

Next, we mention a few Bressoud-Subbarao type weighted partition identities.

Theorem 3.1.3. *Let k and c be two complex numbers. We have the following identity:*

$$\sum_{\pi\in\mathcal{D}(n)} (-1)^{\#(\pi)-1} s(\pi)^k c^{s(\pi)} = \sum_{\pi\in\mathcal{P}(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} (\ell(\pi) - j)^k c^{\ell(\pi)-j}. \tag{3.1.4}$$

Note that the left hand side above arises precisely from the partition-theoretic interpretation of $M_{m,c}$ in Theorem 3.1.2. Setting $c = 1$ in (3.1.4) naturally leads to the partition-theoretic explanation of the coefficient of q^n in the series M_m in (1.5.1).

Corollary 3.1.4. *Suppose that k is a complex number. Then we have*

$$\sum_{\pi\in\mathcal{D}(n)} (-1)^{\#(\pi)-1} s(\pi)^k = \sum_{\pi\in\mathcal{P}(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} (\ell(\pi) - j)^k. \tag{3.1.5}$$

The above result is also a one variable generalization of Bressoud and Subbarao's identity (1.2.8). We explain why this is so. The left hand side readily reduces to the left side of (1.2.8) for $k = 1$. The right hand side takes the form $\sum_{\pi\in\mathcal{P}(n)}\sum_{j=0}^{\nu_d(\pi)}(-1)^j\binom{\nu_d(\pi)}{j}(\ell(\pi)-j)$. This can be written as

$$\begin{aligned}
&\sum_{\pi\in\mathcal{P}(n)}\ell(\pi)\sum_{j=0}^{\nu_d(\pi)}(-1)^j\binom{\nu_d(\pi)}{j}-\sum_{\pi\in\mathcal{P}(n)}\sum_{j=0}^{\nu_d(\pi)}(-1)^j\binom{\nu_d(\pi)}{j}j\\
&=\sum_{\pi\in\mathcal{P}(n)}\ell(\pi)(1-1)^{\nu_d(\pi)}-\sum_{\pi\in\mathcal{P}(n)}\sum_{j=0}^{\nu_d(\pi)}(-1)^j\binom{\nu_d(\pi)}{j}j\\
&=-\sum_{\pi\in\mathcal{P}(n)}\sum_{j=0}^{\nu_d(\pi)}(-1)^j\binom{\nu_d(\pi)}{j}j. \qquad (3.1.6)
\end{aligned}$$

Now, for a complex number α and a partition π of a positive integer, consider the identity

$$(1-\alpha)^{\nu_d(\pi)}=\sum_{j=0}^{\nu_d(\pi)}(-\alpha)^j\binom{\nu_d(\pi)}{j}.$$

Differentiating with respect to α, we get

$$-\nu_d(\pi)(1-\alpha)^{\nu_d(\pi)-1}=\sum_{j=0}^{\nu_d(\pi)}(-1)^j j\alpha^{j-1}\binom{\nu_d(\pi)}{j}. \qquad (3.1.7)$$

We let α approach 1, and see that the right side above reduces to the inner sum in (3.1.6). The behaviour of the left side above depends on the value of $\nu_d(\pi)$. Note that, as $\alpha\to 1$, $-\nu_d(\pi)(1-\alpha)^{\nu_d(\pi)-1}$ tends to 0 if $\nu_d(\pi)>1$ and to $-\nu_d(\pi)=-1$ when $\nu_d(\pi)=1$. Hence, putting this information in (3.1.6), we can write

$$\begin{aligned}
\sum_{\pi\in\mathcal{P}(n)}\sum_{j=0}^{\nu_d(\pi)}(-1)^j\binom{\nu_d(\pi)}{j}(\ell(\pi)-j)&=-\sum_{\pi\in\mathcal{P}(n)}\sum_{j=0}^{\nu_d(\pi)}(-1)^j\binom{\nu_d(\pi)}{j}j\\
&=-\sum_{\substack{\pi\in\mathcal{P}(n)\\ \nu_d(\pi)=1}}-1=\sum_{\substack{\pi\in\mathcal{P}(n)\\ \nu_d(\pi)=1}}1, \qquad (3.1.8)
\end{aligned}$$

the number of partitions π of n with $\nu_d(\pi)=1$. But the partitions of n with only one distinct part are of the form $m+m+\cdots+m$, where m is a positive divisor of n, and conversely, for each positive divisor m of n, we do have the partition $m+\cdots+m$ with n/m summands, which is a partition with one distinct part. Thus, the last sum in (3.1.8) equals $d(n)$, and we have showed that (3.1.5) is indeed a generalization of (1.2.8).

Recall that $p^{(2)}(n)$ denotes the number of partitions of n with exactly two distinct parts sizes. From the identities (1.5.4) and (1.5.5) of Dilcher, one can get,

$$\sum_{\pi\in\mathcal{D}(n)}(-1)^{\#(\pi)-1}s(\pi)^2=\sigma(n)+\sum_{j=1}^{n-1}d(j)d(n-j), \tag{3.1.9}$$

and the special case $k=2$ of (3.1.5), we obtain a representation for $p^{(2)}(n)$, the proof of which will be given in the next section.

Corollary 3.1.5. *For each positive integer n, we have*

$$p^{(2)}(n)=\frac{1}{2}\left\{\sum_{j=1}^{n-1}d(j)d(n-j)+d(n)-\sigma(n)\right\}. \tag{3.1.10}$$

Again, we mention a Bressoud-Subbarao type weighted partition identity for the generalized divisor function $\sigma_{k,c}(n)$.

Theorem 3.1.6. *Let $n\in\mathbb{N}$ and $k,c\in\mathbb{C}$. Then we get*

$$\sum_{\pi\in\mathcal{D}(n)}(-1)^{\#(\pi)-1}\sum_{j=1}^{s(\pi)}j^kc^j=\sum_{\substack{\pi\in\mathcal{P}(n)\\ \nu_d(\pi)\geq 2}}\sum_{j=0}^{\nu_d(\pi)-1}(-1)^j\binom{\nu_d(\pi)-1}{j}(\ell(\pi)-j)^kc^{\ell(\pi)-j}+\sigma_{k,c}(n). \tag{3.1.11}$$

Remark 4. *Corresponding to $k=0$ and $c=1$, the above identity reduces to* (1.2.8). *Note that $d(n)=p^{(1)}(n)$, where $p^{(1)}(n)$ denotes the number of partitions of n into exactly* 1 *part size. Hence, identity* (1.2.8) *can be rewritten as*

$$p^{(1)}(n)=\sum_{\pi\in\mathcal{D}(n)}(-1)^{\#(\pi)-1}s(\pi). \tag{3.1.12}$$

Letting $k=1$ and $c=1$ in Theorem 3.1.6, we obtain an interesting analog of (3.1.12).

Corollary 3.1.7. *Let n be a positive integer. Then we have*

$$p^{(2)}(n)=\sum_{\pi\in\mathcal{D}(n)}(-1)^{\#(\pi)}s(\pi)\left(\ell(\pi)-s(\pi)\right).$$

In the next section, we demonstrate proofs of all the main results stated above.

3.2 Proof of the main results

Proof of Theorem 3.1.1. The main idea of the proof of this theorem is due to Bressoud and Subbarao. Here we explain the details while simultaneously extending their result to complex number z. We first define, for each positive integer N, the set $\mathcal{C}(N)$ of partitions π into distinct parts satisfying the following inequalities:

$$\ell(\pi)\geq N>\ell(\pi)-s(\pi).$$

Let $\pi \in \mathcal{C}(N)$ be any partition into distinct parts. The definition of $\mathcal{C}(N)$ implies that the only possibilities for N are $\ell(\pi) - s(\pi) + j$, with $1 \leq j \leq s(\pi)$. Therefore for any partition π into distinct parts, there are exactly $s(\pi)$ many integers N such that $\pi \in \mathcal{C}(N)$. With the help of this fact the left hand side of (3.1.1) can be written as

$$\begin{aligned}
\sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)-1} \sum_{j=1}^{s(\pi)} (\ell(\pi) - s(\pi) + j)^z c^{\ell(\pi)-s(\pi)+j} \\
= \sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)-1} \sum_{\substack{N=1 \\ \pi \in \mathcal{C}(N)}}^{n} N^z c^N \\
= \sum_{N=1}^{n} N^z c^N \sum_{\substack{\pi \in \mathcal{D}(n) \\ \pi \in \mathcal{C}(N)}} (-1)^{\#(\pi)-1}. \qquad (3.2.1)
\end{aligned}$$

Now our goal is to show that

$$\sum_{\substack{\pi \in \mathcal{D}(n) \\ \pi \in \mathcal{C}(N)}} (-1)^{\#(\pi)-1} = \begin{cases} 1, & \text{if } N \mid n, \\ 0, & \text{if } N \nmid n. \end{cases} \qquad (3.2.2)$$

To prove (3.2.2), we shall try to pair the distinct parts partitions of n in $\mathcal{C}(N)$ that have opposite parity in their respective number of parts. The contribution of such pairs will be a $+1$ and a -1 to the sum in the left side of (3.2.2), since the weight is $(-1)^{\#(\pi)-1}$. Thus, the sum of the contributions of each such pair will vanish. When $N \mid n$, the only distinct parts partition that will remain is n itself. And when $N \nmid n$, all the distinct parts partitions of n will be paired.

Now we shall explain the algorithm to create such pairs. We consider two different cases.

Case 1: Let $\pi \in \mathcal{D}(n) \cap \mathcal{C}(N)$. If π contains a part which is divisible by N and if π has at least one other part, (note that the other part cannot be a multiple of N since $\ell(\pi) - s(\pi) < N$) then remove the part which is divisible by N and add N to the smallest remaining part. Continue this way to create new partitions by adding N to the smallest part in the previous partition until we get a distinct parts partition of n.

A natural question arises here as to what the guarantee is that this algorithm gives a new distinct parts partition of n that also lies in $\mathcal{C}(N)$. In the next paragraph, we shall explain by taking a general distinct parts partition of n.

Let $\pi \in \mathcal{D}(n) \cap \mathcal{C}(N)$ with parts $a_1 < a_2 < \cdots < a_k$ and suppose N divides a_i for some $1 \leq i \leq k$, say $a_i = j \cdot N$ for some $j \geq 1$. First, we remove a_i and add N to the smallest remaining part. We continue adding N to the smallest part in the previous partition to create a new partition until we again have a partition of n.

Since $a_i = j \cdot N$, we have to do this process j times. Let π'_j denote the new partition of n.

Here we look at various cases depending on the value of j. If $j < k-1$, then π'_j equals

$$\begin{cases} a_{j+1} + \cdots + a_{i-1} + a_{i+1} + \cdots + a_k + (a_1 + N) + \cdots + (a_j + N), & \text{if } 1 \leq j < i-1, \\ a_{i+1} + \cdots + a_k + (a_1 + N) + \cdots + (a_{i-1} + N), & \text{if } j = i-1, \\ a_{j+2} + \cdots + a_k + (a_1 + N) + \cdots + (a_{i-1} + N) + (a_{i+1} + N) + \cdots + (a_{j+1} + N), & \text{if } i-1 < j < k-1. \end{cases}$$

One can easily check that the new partition π'_j is a distinct parts partition of n and lies in $\mathcal{C}(N)$. Now, if $j = k-1$, then we simply add N to each part and get a new distinct parts partition of n, namely, $\pi'_j = (a_1 + N) + \cdots + (a_{i-1} + N) + (a_{i+1} + N) + \cdots + (a_k + N)$, which also belongs to $\mathcal{C}(N)$. Note that π and π'_j have opposite parity in their number of parts. Again, if $j > k-1$, then with the help of division algorithm and utilizing previous cases, one can construct $\pi'_j \in \mathcal{D}(n) \cap \mathcal{C}(N)$ such that π and π'_j will have opposite parity in their number of parts.

Case 2: Let $\pi \in \mathcal{D}(n) \cap \mathcal{C}(N)$ with parts $a_1 < a_2 < \cdots < a_k$ such that $N \nmid a_i$ for all $1 \leq i \leq k$. In this case, we must reverse the procedure, that is, we subtract N from the largest part in the previous partition until we reach a unique partition π' for which

$$\ell(\pi') - N < \text{ the total amount subtracted} < s(\pi') + N. \tag{3.2.3}$$

Finally, this total amount subtracted is then inserted as a new part. The above condition (3.2.3) may look artificial, but soon we will explain why this condition comes in naturally.

First, we subtract N from the largest part a_k. Then we have $a_k - N < a_1$ since $N > \ell(\pi) - s(\pi) = a_k - a_1$. Let $\pi_1 := (a_k - N) + a_1 + \cdots + a_{k-1}$. Note that π_1 is not a partition of n, so we need to add N to π_1 to get a distinct parts partition of n. This brings up three possibilities.

Sub case 1: Firstly, suppose that $N < a_k - N$. Let us define $\pi'_1 := N + (a_k - N) + a_1 + \cdots + a_{k-1}$. See that $\pi'_1 \in \mathcal{D}(n)$. Moreover, $\pi'_1 \in \mathcal{C}(N)$ if and only if $\ell(\pi'_1) = a_{k-1} \geq N > \ell(\pi'_1) - s(\pi'_1) = \ell(\pi'_1) - N$. In this sub case, the left-most inequality of (3.2.3) will be satisfied naturally since we assumed that $N < a_k - N$, but the right-most inequality may or may not be true. The right-most inequality will be true if $N > \ell(\pi'_1) - N$(the amount subtracted), that is, N(the amount subtracted) $> \ell(\pi'_1) - N$.

Sub case 2: Next, suppose we have $a_k - N < N < a_{k-1}$. In this sub case, let us define $\pi'_1 := (a_k - N) + \cdots + N + \cdots + a_{k-1}$, which is a distinct parts partition of n. Now $\pi'_1 \in \mathcal{C}(N)$ if and only if $\ell(\pi'_1) = a_{k-1} \geq N > \ell(\pi'_1) - s(\pi'_1) = a_{k-1} - (a_k - N)$. In this sub case, one can readily see that both of the inequalities in (3.2.3) are true.

Sub case 3: Finally, suppose $a_{k-1} < N$ is true. Now we define $\pi'_1 := (a_k - N) + a_1 + \cdots + a_{k-1} + N$, which is a distinct parts partition of n. Here N is the largest part.

Now $\pi_1' \in \mathcal{C}(N)$ if and only if $\ell(\pi_1') \geq N > \ell(\pi_1') - s(\pi_1') = N - s(\pi_1')$. The left-most inequality is true since $\ell(\pi_1') = N$, but the right-most inequality will be true if $s(\pi_1') + N > N$(the amount subtracted). Combining all three sub cases, we can clearly see that $\pi_1' \in \mathcal{C}(N)$ if and only if

$$\ell(\pi_1') - N < N \,(\text{the amount subtracted}) < s(\pi_1') + N. \tag{3.2.4}$$

Thus, the above condition justifies why we need (3.2.3). If the chain of inequalities in (3.2.4) does not hold at this stage, then again we subtract N from the largest part a_{k-1} in π_1. And we define $\pi_2 := (a_{k-1} - N) + (a_k - N) + a_1 + \cdots + a_{k-2}$. Now one can observe that we have subtracted $2N$ from the original partition π of n, so we have to add $2N$ to obtain a distinct parts partition of n. Again, we face three sub cases. Firstly, if $2N < (a_{k-1} - N)$, then we write $\pi_2' := 2N + (a_{k-1} - N) + \cdots + a_{k-2}$. Secondly, if $(a_{k-1} - N) < 2N < a_{k-2}$, then put $\pi_2' := (a_{k-1} - N) + \cdots + 2N + \cdots + a_{k-2}$. And thirdly, if $a_{k-2} < 2N$, then set $\pi_2' := (a_{k-1} - N) + \cdots + a_{k-2} + 2N$. Along the same lines, we can show that $\pi_2' \in \mathcal{C}(N)$ if and only if

$$\ell(\pi_2') - N < 2N \,(\text{the amount subtracted}) < s(\pi_2') + N.$$

If the partition π_2' does not satisfy the above conditions, then we apply the algorithm once again. More generally, suppose that we have subtracted the integer N 'j' many times. We then have to add $j \cdot N$ as a new part. Note that this part may be the smallest part, the largest part or neither of them, and so we have to consider three subcases. If $j < k$, then our new distinct parts partition of n, again denoted by π_j', will be

$$\begin{cases} j \cdot N + (a_{k-j+1} - N) + \cdots + a_k + \cdots + a_{k-j}, & \text{if } j \cdot N \text{ is the smallest part,} \\ (a_{k-j+1} - N) + \cdots + j \cdot N + \cdots + a_{k-j}, & \text{if } j \cdot N \text{ is neither the smallest nor the largest part,} \\ (a_{k-j+1} - N) + \cdots + a_k + \cdots + a_{k-j} + j \cdot N, & \text{if } j \cdot N \text{is the largest part.} \end{cases}$$

In a similar vein, one can show that $\pi_j' \in \mathcal{C}(N)$ if and only if

$$\ell(\pi_j') - N < j \cdot N \,(\text{the total amount subtracted}) < s(\pi_j') + N.$$

This justifies why condition (3.2.3) is necessary. If $j \geq k$, then we can use division algorithm and give similar arguments. Hence, summarizing Case 2, we started with a distinct parts partition π with no part divisible by N and then constructed another distinct parts partition π_j' which has only one part divisible by N, namely, $j \cdot N$ for some suitable positive integer j.

The above algorithm explains that if $N \nmid n$, then any partition $\pi \in \mathcal{D}(n) \cap \mathcal{C}(N)$ can be paired with another partition $\pi_j' \in \mathcal{D}(n) \cap \mathcal{C}(N)$ such that they have opposite parity in their number of parts. And if $N | n$, the only partition of n which will remain unpaired is the partition n itself since the partition n neither belongs to Case 1 nor to Case 2. This completes the proof of (3.2.2). Finally, combining (3.2.1) and (3.2.2), one can obtain (3.1.1). □

Remark 5. *In [17, Section 5], the authors showed that Theorem 3.1.1 is valid for any integer z, by applying the differential operator $D[f(c)] := c\frac{d}{dc}\{f(c)\}$, and the integral operator $I[f(c)] := \int_0^c \frac{f(t)}{t}dt$, on the partition theoretic interpretation of* (1.2.4). *Here we have stated that Theorem 3.1.1 is in fact true for any complex number z. Moreover, for positive real numbers z and j, one can define the fractional derivative*

$$\frac{d^z}{dc^z}(c^j) := \frac{\Gamma(j+1)}{\Gamma(j-z+1)}c^{j-z}.$$

This definition matches with the usual definition of the derivative when z is any positive integer. Therefore, using this fractional derivative, we have $D^z(c^j) = j^z c^j$ for any $z > 0$. Hence applying this fractional operator on the partition theoretic interpretation [17, Lemma 5.1] of (1.2.4)*, one can first show that, Theorem 3.1.1 is true for any positive real number z and then using analytic continuation, we can prove that the identity* (3.1.1) *is in fact valid for any complex z.*

Proof of Theorem 3.1.2. Let us define a function

$$A(x,q) := \frac{(q)_\infty}{(xq)_\infty}. \tag{3.2.5}$$

We next invoke Euler's identity, which is a special case of the q-binomial theorem. For $|q|, |t| < 1$, we have

$$\sum_{n=0}^{\infty} \frac{t^n}{(q)_n} = \frac{1}{(t)_\infty}. \tag{3.2.6}$$

Replacing t by xq in (3.2.6) and utilizing in (3.2.5), we get, for $|xq| < 1$,

$$A(x,q) = (q)_\infty \sum_{n=0}^{\infty} \frac{x^n q^n}{(q)_n} = \sum_{n=0}^{\infty} x^n q^n (q^{n+1})_\infty.$$

Putting $x = ce^t$, for $m \geq 1$, one can see that

$$\frac{\partial^m}{\partial t^m} A(ce^t, q)\bigg|_{t=0} = \sum_{n=1}^{\infty} n^m c^n q^n (q^{n+1})_\infty = M_{m,c}.$$

Thus, the power series for $A(ce^t, q)$ in the variable t takes the shape as

$$A(ce^t, q) = A(c,q) + \sum_{m=1}^{\infty} M_{m,c} \frac{t^m}{m!}. \tag{3.2.7}$$

Suppose we write

$$\log(A(x,q)) = \sum_{n=0}^{\infty} h_n(c) \frac{t^n}{n!}. \tag{3.2.8}$$

From (3.2.5), we get

$$\begin{aligned}\log(A(x,q)) = \log((q)_\infty) - \log((xq)_\infty) &= \log((q)_\infty) - \sum_{n=1}^{\infty}\log(1-xq^n)\\ &= \log((q)_\infty) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(xq^n)^m}{m}\\ &= \log((q)_\infty) + \sum_{m=1}^{\infty}\frac{x^m}{m}\frac{q^m}{1-q^m}.\end{aligned}$$

Putting $x = ce^t$, we find that

$$\begin{aligned}\log(A(ce^t,q)) &= \log((q)_\infty) + \sum_{m=1}^{\infty}\frac{q^m}{1-q^m}\frac{c^m e^{tm}}{m}\\ &= \log((q)_\infty) + \sum_{m=1}^{\infty}\frac{q^m}{1-q^m}\frac{c^m}{m}\sum_{n=0}^{\infty}\frac{(tm)^n}{n!}\\ &= \log((q)_\infty) + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{c^m m^{n-1} q^m}{1-q^m}\frac{t^n}{n!}\\ &= \log((q)_\infty) + \sum_{m=1}^{\infty}\frac{c^m}{m}\frac{q^m}{1-q^m} + \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{c^m m^{n-1} q^m}{1-q^m}\right)\frac{t^n}{n!}\\ &= \log((q)_\infty) - \log((cq)_\infty) + \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{c^m m^{n-1} q^m}{1-q^m}\right)\frac{t^n}{n!}\\ &= \log\left(\frac{(q)_\infty}{(cq)_\infty}\right) + \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{c^m m^{n-1} q^m}{1-q^m}\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing coefficients in (3.2.8), we get

$$h_0(c) = \log\left(\frac{(q)_\infty}{(cq)_\infty}\right), \tag{3.2.9}$$

and for any $n \geq 0$, we have

$$h_{n+1}(c) = \sum_{m=1}^{\infty}\frac{c^m m^n q^m}{1-q^m} = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}c^m m^n q^{mk} = \sum_{\ell=1}^{\infty}\sum_{d|\ell}c^d d^n q^\ell = K_{n+1,c}. \tag{3.2.10}$$

Now in view of (3.2.8), (3.2.9), and (3.2.10), one can see that

$$A(ce^t,q) = \frac{(q)_\infty}{(cq)_\infty}\exp\left(\sum_{n=1}^{\infty}K_{n,c}\frac{t^n}{n!}\right),$$

and finally using (3.2.7), we arrive at

$$\frac{(q)_\infty}{(cq)_\infty}\exp\left(\sum_{n=1}^{\infty}K_{n,c}\frac{t^n}{n!}\right) = \frac{(q)_\infty}{(cq)_\infty} + \sum_{n=1}^{\infty}M_{n,c}\frac{t^n}{n!}.$$

Finally comparing the coefficients of $\frac{t^n}{n!}$, for $n \geq 1$, and using the definition of the Bell polynomial, one can obtain (3.1.3). □

Proof of Theorem 3.1.3. We start with the partition-theoretic interpretation of an identity due to Andrews, Garvan and Liang [14], first noted down by Dixit and Maji [24, Equation (2.6), Corollary 2.4]

$$\sum_{\pi\in\mathcal{D}(n)}(-1)^{\#(\pi)-1}\left(1+c+\cdots+c^{s(\pi)-1}\right)=\sum_{\pi\in\mathcal{P}(n)}c^{\ell(\pi)-\nu_d(\pi)}(c-1)^{\nu_d(\pi)-1}. \quad (3.2.11)$$

Multiplying throughout by $(c-1)$, we get an identity with which we are going to work.

$$\sum_{\pi\in\mathcal{D}(n)}(-1)^{\#(\pi)-1}\left(c^{s(\pi)}-1\right)=\sum_{\pi\in\mathcal{P}(n)}c^{\ell(\pi)-\nu_d(\pi)}(c-1)^{\nu_d(\pi)}. \quad (3.2.12)$$

The left side is clearly a polynomial in c. The same is true of the right side as well, as $\ell(\pi)\geq\nu_d(\pi)$ for any partition π. To see why, observe that if $\ell(\pi)$ is the largest part in a partition, then the possible parts in the partition come from the set $\{1,2,\dots,\ell(\pi)\}$. This means that $\nu_d(\pi)$, the number of distinct parts appearing in the partition π is at most $\ell(\pi)$, with equality occuring if and only if each of the integers from 1 to $\ell(\pi)$ appears in the partition. We now apply the fractional differential operator $D^k:=\left(c\frac{d}{dc}\right)^k$, with $k>0$, to both sides of (3.2.12). The left side then transforms into

$$\sum_{\pi\in\mathcal{D}(n)}(-1)^{\#(\pi)-1}s(\pi)^k\, c^{s(\pi)}. \quad (3.2.13)$$

Before applying D^k to the right side of (3.2.12), we expand it using the binomial theorem to get

$$\sum_{\pi\in\mathcal{P}(n)}c^{\ell(\pi)-\nu_d(\pi)}\sum_{j=0}^{\nu_d(\pi)}\binom{\nu_d(\pi)}{j}(-1)^j c^{\nu_d(\pi)-j}=\sum_{\pi\in\mathcal{P}(n)}\sum_{j=0}^{\nu_d(\pi)}c^{\ell(\pi)-j}(-1)^j\binom{\nu_d(\pi)}{j}.$$

Operating by D^k, we obtain

$$\sum_{\pi\in\mathcal{P}(n)}\sum_{j=0}^{\nu_d(\pi)}(\ell(\pi)-j)^k c^{\ell(\pi)-j}(-1)^j\binom{\nu_d(\pi)}{j}. \quad (3.2.14)$$

Equating the differentiated expressions in (3.2.13) and (3.2.14), we get (3.1.4) for any $k>0$. Finally, making use of analytic continuation on the variable k, the proof of Theorem 3.1.3 is over. □

Proof of Corollary 3.1.5. The idea of this proof is to consider the case $k=2$ of (3.1.5) in Corollary 3.1.4, and compare it with (3.1.9) due to Dilcher. Begin by setting $k=2$ in (3.1.5) to see that

$$\begin{aligned}\sum_{\pi\in\mathcal{D}(n)}(-1)^{\#(\pi)-1}s(\pi)^2&=\sum_{\pi\in\mathcal{P}(n)}\sum_{j=0}^{\nu_d(\pi)}(-1)^j\binom{\nu_d(\pi)}{j}(\ell(\pi)-j)^2\\&=\sum_{\pi\in\mathcal{P}(n)}\sum_{j=0}^{\nu_d(\pi)}(-1)^j\binom{\nu_d(\pi)}{j}\left(\ell(\pi)^2-2\ell(\pi)j+j^2\right)\end{aligned}$$

$$= \sum_{\pi \in \mathcal{P}(n)} \ell(\pi)^2 (1-1)^{\nu_d(\pi)} - 2 \sum_{\pi \in \mathcal{P}(n)} \ell(\pi) \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} j$$

$$+ \sum_{\pi \in \mathcal{P}(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} j^2$$

$$= -2 \sum_{\pi \in \mathcal{P}(n)} \ell(\pi) \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} j + \sum_{\pi \in \mathcal{P}(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} j^2. \quad (3.2.15)$$

In the first sum in (3.2.15), we have already seen in (3.1.6) that the inner sum equals -1 if $\nu_d(\pi) = 1$, and 0 if $\nu_d(\pi) > 1$. So the first sum in (3.2.15) reduces to $2 \sum_{\substack{\pi \in \mathcal{P}(n) \\ \nu_d(\pi)=1}} \ell(\pi)$. We have previously observed that partitions of a positive integer n with one distinct part correspond to divisors of n. This means that if $m + \cdots + m$ is a partition of n, corresponding to a divisor m of n, then the largest part is also m. Thus,

$$2 \sum_{\substack{\pi \in \mathcal{P}(n) \\ \nu_d(\pi)=1}} \ell(\pi) = 2 \sum_{d \mid n} d.$$

Putting this in (3.2.15) brings us to

$$\sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)-1} s(\pi)^2 = 2 \sum_{d \mid n} d + \sum_{\pi \in \mathcal{P}(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} j^2. \quad (3.2.16)$$

Recall the identity (3.1.7) and multiply it with c to get

$$-\nu_d(\pi) \cdot c \cdot (1-c)^{\nu_d(\pi)-1} = \sum_{j=0}^{\nu_d(\pi)} (-1)^j j c^j \binom{\nu_d(\pi)}{j}.$$

Now differentiate this with respect to c to derive the identity

$$-\nu_d(\pi) \left[(1-c)^{\nu_d(\pi)-1} - c(\nu_d(\pi)-1)(1-c)^{\nu_d(\pi)-2} \right] = \sum_{j=0}^{\nu_d(\pi)} (-1)^j \cdot j^2 \cdot c^{j-1} \cdot \binom{\nu_d(\pi)}{j}. \quad (3.2.17)$$

Letting $c \to 1$ in (3.2.17), we see that

$$\sum_{j=0}^{\nu_d(\pi)} (-1)^j \cdot j^2 \cdot \binom{\nu_d(\pi)}{j} = F(\nu_d(\pi)), \quad (3.2.18)$$

where

$$F(\nu_d(\pi)) = \lim_{c \to 1} -\nu_d(\pi) \left[(1-c)^{\nu_d(\pi)-1} - c(\nu_d(\pi)-1)(1-c)^{\nu_d(\pi)-2} \right].$$

If $\nu_d(\pi) > 2$, we can see that $F(\nu_d(\pi))$ approaches 0. When $\nu_d(\pi) = 2$, the first term in the parenthetical sum in $F(\nu_d(\pi))$ vanishes and the second term nears the value -1, so that in effect, $F(2) = 2$. Finally, when $\nu_d(\pi) = 1$, the second term in

$F(\nu_d(\pi))$ goes to 0 and the first term to 1, thereby giving us the value of $F(1)$ as -1. In summary,

$$F(\nu_d(\pi)) = \begin{cases} -1, & \text{if } \nu_d(\pi) = 1, \\ 2, & \text{if } \nu_d(\pi) = 2, \\ 0, & \text{if } \nu_d(\pi) > 2. \end{cases}$$

Using this, via (3.2.18), in (3.2.16), we arrive at

$$\sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)-1} s(\pi)^2 = 2\sum_{d \mid n} d - \sum_{\substack{\pi \in \mathcal{P}(n) \\ \nu_d(\pi)=1}} 1 + \sum_{\substack{\pi \in \mathcal{P}(n) \\ \nu_d(\pi)=2}} 2 = 2\sigma(n) - d(n) + 2p_d^{(2)}(n). \tag{3.2.19}$$

Comparing (3.2.19) with (3.1.9), we see that

$$\sigma(n) + \sum_{j=1}^{n-1} d(j)d(n-j) = 2\sigma(n) - d(n) + 2p_d^{(2)}(n),$$

which on rearrangement yields (3.1.10) and the proof is complete. □

Proof of Theorem 3.1.6. We start with (3.2.11) and multiplying throughout by c, we get

$$\sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)-1} \sum_{j=1}^{s(\pi)} c^j = \sum_{\pi \in \mathcal{P}(n)} c^{\ell(\pi)-\nu_d(\pi)+1}(c-1)^{\nu_d(\pi)-1}. \tag{3.2.20}$$

Now apply the fractional differential operator $D^k := \left(c\frac{d}{dc}\right)^k$, with $k > 0$, to both sides of (3.2.20). The left side transforms into

$$\sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)-1} \sum_{j=1}^{s(\pi)} j^k c^j. \tag{3.2.21}$$

Before applying the fractional differential operator to the right hand side of (3.2.20), we use the binomial theorem to expand

$$\sum_{\pi \in \mathcal{P}(n)} c^{\ell(\pi)-\nu_d(\pi)+1}(c-1)^{\nu_d(\pi)-1} = \sum_{\pi \in \mathcal{P}(n)} \sum_{j=0}^{\nu_d(\pi)-1} (-1)^j \binom{\nu_d(\pi)-1}{j} c^{\ell(\pi)-j}.$$

Now applying the fractional differential operator D^k, we obtain

$$\begin{aligned} &\sum_{\pi \in \mathcal{P}(n)} \sum_{j=0}^{\nu_d(\pi)-1} (-1)^j \binom{\nu_d(\pi)-1}{j} (\ell(\pi)-j)^k c^{\ell(\pi)-j} \\ &= \sum_{\substack{\pi \in \mathcal{P}(n) \\ \nu_d(\pi)\geq 2}} \sum_{j=0}^{\nu_d(\pi)-1} (-1)^j \binom{\nu_d(\pi)-1}{j} (\ell(\pi)-j)^k c^{\ell(\pi)-j} + \sum_{\substack{\pi \in \mathcal{P}(n) \\ \nu_d(\pi)=1}} \ell(\pi)^k c^{\ell(\pi)}. \end{aligned} \tag{3.2.22}$$

We have previously observed that partitions of a positive integer n with one distinct part correspond to divisors of n. This means that if $m + \cdots + m$ is a partition of n, corresponding to a divisor m of n, then the largest part is also m. Thus,

$$\sum_{\substack{\pi\in\mathcal{P}(n)\\ \nu_d(\pi)=1}} \ell(\pi)^k c^{\ell(\pi)} = \sum_{d|n} d^k c^d$$

Putting this in (3.2.22), we get

$$\sum_{\substack{\pi\in\mathcal{P}(n)\\ \nu_d(\pi)\geq 2}} \sum_{j=0}^{\nu_d(\pi)-1} (-1)^j \binom{\nu_d(\pi)-1}{j} (\ell(\pi)-j)^k c^{\ell(\pi)-j} + \sum_{d|n} d^k c^d \tag{3.2.23}$$

Equating the differentiated expressions in (3.2.21) and (3.2.23), we can obtain (3.1.11) for $k > 0$. Finally, using analytic continuation on k, one can see that Theorem 3.1.6 is valid for any complex k. □

Proof of Corollary 3.1.7. Putting $k = 1$ and $c = 1$ in Theorem 3.1.6, the identity becomes

$$\sum_{\pi\in\mathcal{D}(n)} (-1)^{\#(\pi)-1} \sum_{j=1}^{s(\pi)} j = \sum_{\substack{\pi\in\mathcal{P}(n)\\ \nu_d(\pi)\geq 2}} \sum_{j=0}^{\nu_d(\pi)-1} (-1)^j \binom{\nu_d(\pi)-1}{j} (\ell(\pi)-j) + \sum_{d|n} d. \tag{3.2.24}$$

Letting $z = c = 1$ in Theorem 3.1.1, we have

$$\sum_{\pi\in\mathcal{D}(n)} (-1)^{\#(\pi)-1} \sum_{j=1}^{s(\pi)} (\ell(\pi)-s(\pi)+j) = \sum_{d|n} d. \tag{3.2.25}$$

From (3.2.24) and (3.2.25), one can see that

$$\sum_{\pi\in\mathcal{D}(n)} (-1)^{\#(\pi)} s(\pi)\left(\ell(\pi)-s(\pi)\right) = \sum_{\substack{\pi\in\mathcal{P}(n)\\ \nu_d(\pi)\geq 2}} \sum_{j=0}^{\nu_d(\pi)-1} (-1)^j \binom{\nu_d(\pi)-1}{j} (\ell(\pi)-j).$$

Now we will divide the right hand sum into two parts corresponding to $\nu_d(\pi) = 2$ and $\nu_d(\pi) \geq 3$. Thus, we have

$$\sum_{\substack{\pi\in\mathcal{P}(n)\\ \nu_d(\pi)=2}} \sum_{j=0}^{1} (-1)^j \binom{1}{j} (\ell(\pi)-j) + \sum_{\substack{\pi\in\mathcal{P}(n)\\ \nu_d(\pi)\geq 3}} \sum_{j=0}^{\nu_d(\pi)-1} (-1)^j \binom{\nu_d(\pi)-1}{j} (\ell(\pi)-j). \tag{3.2.26}$$

One can easily see that the first sum reduces to $p^{(2)}(n)$. To show that the second sum vanishes, we mention the following expressions:

$$(1-x)^m = \sum_{j=0}^{m} (-1)^j \binom{m}{j} x^j,$$

$$\implies -m(1-x)^{m-1} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} j x^{j-1}. \tag{3.2.27}$$

Now letting $x = 1$ and $m = \nu_d(\pi) - 1$ in (3.2.27) and using them, one can see that the second sum in (3.2.26) vanishes. □

Chapter 4

A DIVISOR GENERATING q-SERIES AND ITS APPLICATIONS TO PROBABILITY THEORY AND RANDOM GRAPHS

All the results presented in this chapter are taken from our work [3]. We begin by establishing a q-series identity involving an analytic function $f(z)$ with power series expansion $\sum_{j=0}^{\infty} \lambda_j z^j$. This identity provides an extended form of Gupta and Kumar's identity (1.5.10) and yields several new families of q-series identities as special cases. Furthermore, we correct an identity of Dilcher [22, Theorem 3] and derive a generalization of his result. We also extend the divisor-type identity (1.5.11) of Gupta and Kumar, thereby broadening its range of applications. A Ramanujan-type summation formula is then established for the identity (3.1.3), which yields, as immediate corollaries, a Ramanujan-type expression for Uchimura's identity (1.5.3) and a Uchimura-type representation for Ramanujan's Entry 4 (1.2.4), which was not known before. The chapter further explores applications of these results to probability theory and random graphs. In particular, we provide a corrected version of an identity of Gupta and Kumar [33, Theorem 1.11], thereby resolving an issue in the original formulation. We gather all the results discussed above in the next section. The subsequent section is then devoted to presenting detailed proofs.

4.1 Main results

Theorem 4.1.1. *Let $a,\ b,\ c,\ d$ be complex numbers with $|a|<1,\ |c|<1$ and $f(z)$ be an analytic function with power series $\sum_{j=0}^{\infty}\lambda_j z^j$ under the condition $|z|<1$. Then we have*

$$\sum_{n=0}^{\infty}\frac{(b/a)_n a^n}{(d)_n}f(cq^n)=\sum_{j=0}^{\infty}\lambda_j c^j\sum_{n=0}^{\infty}\frac{(ad/b)_n(-b)^n q^{\binom{n}{2}+jn}}{(d)_n(aq^j)_{n+1}}.$$

By letting $f(z)=(1-z)^{-\alpha}$, $\alpha\in\mathbb{C}$, an extended version of the Gupta-Kumar's identity (1.5.10) is obtained as an immediate corollary of the above theorem. This corollary is significant enough to be presented as a separate theorem as it will be frequently availed throughout the chapter. Prior to revealing this identity, it is essential to define the binomial coefficient for complex numbers as the quotient of gamma functions. For a complex number α such that $-\alpha\in\mathbb{C}\backslash\mathbb{N}$, and a natural number n, we define the binomial coefficient as follows:

$$\binom{\alpha}{n}:=\frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)}. \tag{4.1.1}$$

We now move on to explore several interesting results stemming from Theorem 4.1.1.

Theorem 4.1.2. *Let $a,\ b,\ c,\ d,\ \alpha$ be complex numbers with $|a|<1,\ |c|<1$. Then*

$$\sum_{n=0}^{\infty}\frac{(b/a)_n a^n}{(1-cq^n)^{\alpha}(d)_n}=\sum_{j=0}^{\infty}c^j\binom{\alpha+j-1}{j}\sum_{n=0}^{\infty}\frac{(ad/b)_n(-b)^n q^{\binom{n}{2}+jn}}{(d)_n(aq^j)_{n+1}}.$$

By substituting $d=q$ into the above identity, we arrive at a two variable generalization to Gupta and Kumar's identity (1.5.10). Mainly, we obtain the following identity.

Corollary 4.1.3. *Let $a,\ b,\ c,\ \alpha$ be complex numbers with $|c|<1$, then we have*

$$\sum_{n=0}^{\infty}\frac{(b/a)_n a^n}{(1-cq^n)^{\alpha}(q)_n}=\sum_{n=0}^{\infty}c^n\binom{\alpha+n-1}{n}\frac{(bq^n)_\infty}{(aq^n)_\infty}.$$

Letting $b=q^2$ and $c=q$ in Corollary 4.1.3 and upon simplification, we get the following immediate extended form of (1.5.10).

Corollary 4.1.4. *For complex numbers a and α with $|a|<1$, we have*

$$\sum_{n=1}^{\infty}\frac{(q/a)_n a^n}{(1-q^n)^{\alpha}(q)_n}=-\frac{(q)_\infty}{(a)_\infty}\sum_{n=1}^{\infty}q^n\binom{\alpha+n-1}{n-1}\frac{(a/q)_n}{(q)_n}.$$

Gupta and Kumar [33, Theorem 1.1] originally derived the above result under the assumption that α is a natural number. However, it is important to note that their identity is in fact valid for any complex number α as well.

Further, as an application of Theorem 4.1.2, we obtain an elegant two variable generalization of Uchimura's identity (1.2.7).

Corollary 4.1.5. *For a complex number α and a natural number r, we have*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n}{2}+nr}}{(1-q^n)^{\alpha}(q)_n} = \sum_{j=r}^{\infty} \binom{\alpha+j-r}{j-r} q^j (q^{j+1})_{\infty}. \tag{4.1.2}$$

In particular, when $\alpha = r = 1$, it yields the first equality of Uchimura's identity (1.2.7). Quite surprisingly, letting $\alpha = r$, we recover the first equality of Dilcher's identity (1.5.6). Thus, the above identity is a unified generalization of Uchimura's identity (1.2.7) as well as Dilcher's identity (1.5.6).

4.1.1 A generalization of Dilcher's identity

Before proceeding to the next theorem, we would like to introduce an identity formulated by Dilcher [22, Theorem 3]. For $k \in \mathbb{N}$, the identity is the following:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n+1}{2}}}{(1-q^n)^k (q)_n} = \sum_{t=1}^{k} \left\{ \sum_{j=0}^{k-t} \binom{k-1}{j+t-1} \frac{s(j+t,t)}{(j+t)!} \right\} U_t(q), \tag{4.1.3}$$

where $s(m,n)$ represents the Stirling numbers of the first kind, and

$$U_t(q) := \sum_{n=1}^{\infty} n^t q^n (q^{n+1})_{\infty}.$$

However, it is important to note that there is an error in the above identity (4.1.3). Instead of $U_t(q)$ on the right side expression of (4.1.3), it should be the tail $U_{t,j+t}(q)$ of $U_t(q)$, defined in (4.1.6). We not only correct the above identity (4.1.3) but also give a one-variable generalization of it. Here is the generalized identity.

Theorem 4.1.6. *For natural numbers r and k, we have*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n}{2}+nr}}{(1-q^n)^k (q)_n} &= \sum_{t=1}^{k} \sum_{j=0}^{k-t} \binom{k-1}{j+t-1} A(j+t,r,t) U_{t,r+j+t-1}(q) \\ &\quad + \sum_{j=1}^{k} \binom{k-1}{j-1} A(j,r,0) U_{0,r+j-1}(q),\end{aligned} \tag{4.1.4}$$

where

$$A(j,r,t) := \sum_{l=t}^{j} \frac{s(j,l)}{j!} \binom{l}{t} (1-r)^{l-t}, \tag{4.1.5}$$

and

$$U_{m,i}(q) := \sum_{n=i}^{\infty} n^m q^n (q^{n+1})_{\infty} \tag{4.1.6}$$

is the tail of Uchimura's sum $M_m(q)$ defined in (1.5.1).

Letting $r = 1$ in the left-hand side expression of (4.1.4), it transforms into the form presented in (4.1.3). However, substituting $r = 1$ on the right-hand side of (4.1.4), we see that the second finite sum vanishes as $A(j, 1, 0) = 0$ because $s(j, 0) = 0$ for any $j \geq 1$. Then the first sum becomes

$$\sum_{t=1}^{k}\sum_{j=0}^{k-t}\binom{k-1}{j+t-1}A(j+t,1,t)U_{t,j+t}(q). \tag{4.1.7}$$

Now from the definition (4.1.5) of $A(j+t,1,t)$, it follows that $A(j+t,1,t) = \frac{s(j+t,t)}{(j+t)!}$. Thus, (4.1.7) becomes

$$\sum_{t=1}^{k}\sum_{j=0}^{k-t}\binom{k-1}{j+t-1}\frac{s(j+t,t)}{(j+t)!}U_{t,j+t}(q),$$

which is slightly different from the right hand side expression of Dilcher's identity (4.1.3). The only difference is that the function $U_t(q)$ in (4.1.3) is being replaced by $U_{t,j+t}(q)$.

Motivated by the identity (4.1.3), Dilcher [22, p. 89] defined an interesting sequence of rational numbers involving the Stirling numbers of the first kind. Mainly, for $k \in \mathbb{N}$ and $1 \leq t \leq k$, he defined

$$a(k,t) = \sum_{j=0}^{k-t}\binom{k-1}{j+t-1}\frac{s(j+t,t)}{(j+t)!}.$$

He further proved the following recursive relation for $a(k,t)$ with the help of the recursion formula for the Stirling numbers of the first kind,

$$a(1,1) = 1, \quad a(k+1,t+1) = \frac{k}{k+1}a(k,t+1) + \frac{1}{k+1}a(k,t). \tag{4.1.8}$$

Utilizing the above recursion formula, Dilcher [22, Equation (4.10)] obtained an interesting identity, namely,

$$\sum_{t=1}^{k} a(k,t) = 1. \tag{4.1.9}$$

Now inspired by the generalization of Dilcher's identity i.e., Theorem 4.1.6, we define a new sequence of numbers. First, we rewrite Theorem 4.1.6 as follows:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{\binom{n}{2}+nr}}{(1-q^n)^k(q)_n} = \sum_{t=1}^{k} C(k,r,t)U_{t,r+j+t-1}(q) + C(k,r,0)U_{0,r+j-1}(q)$$

where the coefficients $C(k,r,t)$ and $C(k,r,0)$ defined as

$$\begin{aligned} C(k,r,t) &:= \sum_{j=0}^{k-t}\binom{k-1}{j+t-1}A(j+t,r,t), \\ C(k,r,0) &:= \sum_{j=1}^{k}\binom{k-1}{j-1}A(j,r,0), \end{aligned} \tag{4.1.10}$$

where $A(j,r,t)$ is defined in (4.1.5). Letting $r=1$, one can easily verify that $A(j,1,t)=\frac{s(j,t)}{(j)!}$ and $C(k,1,t)$ is nothing but Dilcher's function $a(k,t)$. Analysing these coefficients, we prove the following beautiful recursion formulae:

Theorem 4.1.7. *Let $r,j,k\in\mathbb{N}$. For $1\le t\le j$, we have*

$$A(1,r,1)=1,\quad A(j,r,t)=(j+1)A(j+1,r,t+1)+(r+j-1)A(j,r,t+1), \tag{4.1.11}$$

and

$$C(1,r,1)=1,\quad C(k+1,r,t+1)=\frac{k+1-r}{k+1}C(k,r,t+1)+\frac{1}{k+1}C(k,r,t). \tag{4.1.12}$$

Remark 6. *When $r=1$, the recursion* (4.1.11) *reduces to the well-known recursion [21, p. 214] of the Stirling numbers of the first kind, that is,*

$$s(j,t)=s(j+1,t+1)+js(j,t+1),$$

and the recursion (4.1.12) *reduces to the recursion* (4.1.8) *for $a(k,t)$.*

Using the definitions (4.1.5), (4.1.10) of $A(j,r,t)$ and $C(k,r,t)$, one can also verify that, for any $r,k\in\mathbb{N}$,

$$C(k,r,k)=A(k,r,k)=\frac{1}{k!}.$$

Further, we prove the following interesting properties related to $A(j,r,t)$ and $C(k,r,t)$.

Theorem 4.1.8. *For any natural number j and k, we have*

$$\sum_{t=1}^{j}A(j,r,t)=\frac{(-1)^{j-1}}{(j-1)!}(r-1)r(r+1)\cdots(r+j-3), \tag{4.1.13}$$

$$\sum_{t=1}^{k}C(k,r,t)=\frac{(-1)^{k-1}}{(k-1)!}(r-2)(r-3)(r-4)\cdots(r-k). \tag{4.1.14}$$

Substituting $r=1$, we obtain a trivial identity related to Stirling numbers of the first kind and also recover Dilcher's identity (4.1.9).

Corollary 4.1.9. *For any $j,k\in\mathbb{N}$, we have*

$$\sum_{t=1}^{j}s(j,t)=0,\quad \sum_{t=1}^{k}a(k,t)=1.$$

4.1.2 A generalization of identities of Gupta–Kumar and Andrews–Crippa–Simon

Now we state a one variable generalization of Gupta-Kumar's identity (1.5.11). As a result, we obtain a two variable generalization of the identity (1.5.7) of Andrews, Crippa and Simon.

Theorem 4.1.10. *Let $m \in \mathbb{N} \cup \{0\}$. Let $a, c \in \mathbb{C}$ such that $|ac| < 1, |cq| < 1$, we define $\mathfrak{S}_{m,a,c}(q)$ by*

$$\mathfrak{S}_{m,a,c}(q) := S_{m,c}(q) - R_{m,ac}(q), \tag{4.1.15}$$

where

$$S_{m,c}(q) = \sum_{n=1}^{\infty} \frac{n^m c^n q^n}{1-q^n} = \sum_{n=1}^{\infty} \sigma_{m,c}(n) q^n, \tag{4.1.16}$$

$$R_{m,ac}(q) = \mathrm{Li}_{-m}(ac) + \sum_{n=1}^{\infty} \frac{n^m a^n c^n q^n}{1-q^n} = \mathrm{Li}_{-m}(ac) + \sum_{n=1}^{\infty} \sigma_{m,ac}(n) q^n. \tag{4.1.17}$$

For $k \in \mathbb{N}$, there exists a polynomial $P_k(x_1, x_2, \dots, x_k)$ with rational coefficients, such that

$$\sum_{n=1}^{\infty} \frac{(q/a)_n a^n}{(1-cq^n)^{k+1}(q)_{n-1}} = -\frac{1}{c} \frac{(q)_\infty (ac)_\infty}{(cq)_\infty (a)_\infty} P_k(\mathfrak{S}_{0,a,c}(q), \mathfrak{S}_{1,a,c}(q), \dots, \mathfrak{S}_{k-1,a,c}(q)). \tag{4.1.18}$$

Substituting $c = 1$ in the above theorem and upon simplification, one can immediately recover the identity (1.5.11) of Gupta and Kumar. Furthermore, letting $a \to 0$ in (4.1.18) gives us a new one variable generalization of the identity (1.5.7) of Andrews, Crippa and Simon.

Corollary 4.1.11. *For $|c| < 1$ and $k \in \mathbb{N}$, there exists a polynomial $P_k(x_1, x_2, \dots, x_k)$ with rational coefficients, such that*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n+1}{2}}}{(1-cq^n)^{k+1}(q)_{n-1}} = \frac{1}{c} \frac{(q)_\infty}{(cq)_\infty} P_k(S_{0,c}(q), S_{1,c}(q), \dots, S_{k-1,c}(q)).$$

In particular when $c = 1$, one can easily verify that the above identity reduces to the identity (1.5.7).

Now we state a *Ramanujan-type* expression for the identity (3.1.3). Let $K_{m,c}(q)$ be the generalized divisor generating function defined in (3.1.2) and Y_n be the Bell polynomial defined in (1.5.2). Let $A_n(x)$ be the Eulerian polynomial of degree n defined as

$$\sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!} = \frac{(1-x)e^t}{e^{xt} - xe^t}.$$

Then we have the following result.

Theorem 4.1.12. *For any non-negative integer m and a complex number c, we have*

$$\sum_{n=1}^{\infty} n^m c^n q^n (q^{n+1})_\infty = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n}{2}} (cq^n) A_m(cq^n)}{(1-cq^n)^{m+1}(q)_{n-1}} = \frac{(q)_\infty}{(cq)_\infty} Y_m(K_{1,c}(q), K_{2,c}(q), \cdots, K_{m,c}(q)).$$

Interestingly, letting $m = 1$ in the above theorem gives a *Uchimura-type* expression for the Ramanujan's identity (1.2.4).

Corollary 4.1.13. *For any complex number c, we found that*

$$\frac{(cq)_\infty}{(q)_\infty}\sum_{n=1}^{\infty} nc^n q^n (q^{n+1})_\infty = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}c^n q^{\frac{n(n+1)}{2}}}{(1-q^n)(cq)_n} = \sum_{n=1}^{\infty}\frac{c^n q^n}{1-q^n}. \tag{4.1.19}$$

Substituting $c=1$ in (4.1.19) reduces to Uchimura's identity (1.2.7). Further, putting $c=1$ in Theorem 4.1.12 produces a *Ramanujan-type* expression for the Uchimura's identity (1.5.3) which seems to be a new expression to the best of our knowledge.

Corollary 4.1.14. *For any non-negative integer m, we have the following expression*

$$\sum_{n=1}^{\infty} n^m q^n (q^{n+1})_\infty = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{\binom{n}{2}+n}A_m(q^n)}{(1-q^n)^m (q)_n} = Y_m(K_1(q), K_2(q), \ldots, K_m(q)).$$

Letting $m=1$ in the above corollary and upon simplification, it yields Uchimura's identity (1.2.7).

Before stating next result, we first present a one variable generalization of the identity (1.6.8) of Andrews, Crippa and Simon. For that, consider the sequence of polynomials $t_n(a,q)$ depending on a and q, defined as

$$t_n(a,q) = \frac{f(n)-af(n+1)}{1-aq^n} + \left(\frac{1-q^{n-1}}{1-aq^n}\right) t_{n-1}(a,q),\ n\geq 1 \text{ and } t_0(a,q)=0, \tag{4.1.20}$$

where $f(n)=\sum_{k\geq 0} c_k n^k \in \mathbb{Q}[n]$ be a polynomial in n with $af(1)=0$.

Theorem 4.1.15. *Let $t_n(a,q)$ be the sequence defined in* (4.1.20)*. Then we have*

$$\lim_{\ell\to\infty}\left(\sum_{n=1}^{\ell} f(n) - \frac{a}{1-a}f(\ell+1) - \left(\frac{1-aq^\ell}{1-a}\right)t_\ell(a,q)\right) = \sum_{j=1}^{\infty} h_j P_j(a,q),$$

where the constant h_j is same as in (1.6.9) *and $P_j(a,q)$ is a polynomial in j variables defined in* (1.5.11)*.*

Remark 7. *Letting $a=0$ in Theorem* 4.1.15*, one can easily recover the identity* (1.6.8) *of Andrews, Crippa and Simon. Note that the above result is the corrected version of Theorem* 4.2 *in [33].*

Now we state a corrected version of another identity of Gupta and Kumar [33, Theorem 1.11].

Theorem 4.1.16. *Let $f(n)$ be a periodic sequence with period N and $af(1)=0$. Let*

$t_n(a,q)$ *be the sequence defined in* (4.1.20). *Then for* $|a|<1$ *and* $|q|<1$, *we have*

$$\lim_{\ell\to\infty}\left(\sum_{n=1}^{\ell} f(n) - \frac{a}{1-a} f(\ell+1) - \left(\frac{1-aq^{\ell}}{1-a}\right) t_\ell(a,q)\right)$$
$$= c_0\mathfrak{S}_{0,a}(q) + \sum_{k=1}^{N-1} \frac{c_k}{1-\zeta_N^k} - \frac{(q)_\infty}{(a)_\infty}\sum_{k=1}^{N-1} c_k \frac{(a\zeta_N^k)_\infty}{(\zeta_N^k)\infty},$$

where

$$c_k = \frac{1}{N}\sum_{j=1}^{N} f(j)\zeta_N^{(1-j)k}.$$

Corollary 4.1.17. *Let* $f(n)$ *and* $t_n(a,q)$ *be same as in Theorem 4.1.16. Then for* $|q|<1$, *we have*

$$\lim_{\ell\to\infty}\left(\sum_{n=1}^{\ell} f(n) - \frac{a}{1-a} f(\ell+1) - \left(\frac{1-aq^{\ell}}{1-a}\right) t_\ell(a,q)\right) = \frac{(q)_\infty}{(a)_\infty}\sum_{n=0}^{\infty}\frac{(a/q)_n q^n}{(q)_n}\sum_{j=1}^{N} f(j)\left\lceil\frac{n+1-j}{N}\right\rceil.$$

This is the corrected version of Corollary 4.6 in [33]. Letting $a=0$ and $f(n) = (-1)^n$ in Corollary 4.1.17, one can recover the identity (1.6.10) of Bringmann and Jennings-Shaffer [19].

4.2 Proof of main results

Proof of Theorem 4.1.1. Using the power series representation of $f(z)$ and interchanging the summations, we can write

$$\sum_{n=0}^{\infty}\frac{(b/a)_n a^n}{(d)_n} f(cq^n) = \sum_{j=0}^{\infty}\lambda_j c^j \sum_{n=0}^{\infty}\frac{(b/a)_n}{(d)_n}(aq^j)^n.$$

The right side of the above expression can be re-written as

$$\sum_{j=0}^{\infty}\lambda_j c^j \sum_{n=0}^{\infty}\frac{(b/a)_n}{(d)_n}(aq^j)^n = \sum_{j=0}^{\infty}\lambda_j c^j \lim_{\gamma\to 0} {}_3\phi_2\left(\begin{matrix} q, & b/a, & \gamma d \\ & d, & \gamma b q^{j+1} \end{matrix}; q; aq^j\right).$$

Now we make use of ${}_3\phi_2$ transformation formula (2.2.4) to simplify further. Thus, we obtain

$$\sum_{j=0}^{\infty}\lambda_j c^j \lim_{\gamma\to 0}\frac{(\gamma b q^j)_\infty (aq^{j+1})_\infty}{(\gamma b q^{j+1})_\infty (aq^j)_\infty}\sum_{n=0}^{\infty}\frac{(ad/b)_n (1/\gamma)_n}{(d)_n (aq^{j+1})_n}(\gamma b q^j)^n = \sum_{j=0}^{\infty}\lambda_j c^j \sum_{n=0}^{\infty}\frac{(ad/b)_n (-b)^n q^{\binom{n}{2}+jn}}{(d)_n (aq^j)_{n+1}}.$$

This completes the proof of Theorem 4.1.1. □

Proof of Theorem 4.1.2. For any $\alpha\in\mathbb{C}$ and $|z|<1$, by considering the function

$$f(z) = 1/(1-z)^\alpha = \sum_{j=0}^{\infty}\binom{\alpha+j-1}{j} z^j,$$

in Theorem 4.1.1, where the binomial coefficient is defined as in (4.1.1), we can immediately obtain this result. □

Proof of Corollary 4.1.3. Replace d by q in Theorem 4.1.2 to see that

$$\sum_{n=0}^{\infty}\frac{(b/a)_n a^n}{(1-cq^n)^{\alpha}(q)_n}=\sum_{j=0}^{\infty}c^j\binom{\alpha+j-1}{j}\sum_{n=0}^{\infty}\frac{(aq/b)_n(-b)^n q^{\binom{n}{2}+jn}}{(q)_n(aq^j)_{n+1}}. \tag{4.2.1}$$

The right hand side of (4.2.1) can be written as

$$\sum_{j=0}^{\infty}\frac{c^j}{1-aq^j}\binom{\alpha+j-1}{j}\lim_{\gamma\to 0}\sum_{n=0}^{\infty}\frac{\left(\frac{aq}{b}\right)_n\left(\frac{b}{\gamma}\right)_n}{(aq^{j+1})_n}\frac{(\gamma q^j)^n}{(q)_n}.$$

Now applying the q-Gauss sum (2.2.3) in the inner summation and upon simplification, we get

$$\sum_{j=0}^{\infty}\frac{c^j}{1-aq^j}\binom{\alpha+j-1}{j}\frac{(bq^j)_\infty}{(aq^{j+1})_\infty}=\sum_{j=0}^{\infty}c^j\binom{\alpha+j-1}{j}\frac{(bq^j)_\infty}{(aq^j)_\infty}.$$

This finishes the proof. □

Proof of Corollary 4.1.4. Using the substitution $b=q^2,\ c=q$ in Corollary 4.1.3, we get

$$\sum_{n=0}^{\infty}\frac{(q^2/a)_n a^n}{(1-q^{n+1})^{\alpha}(q)_n}=\sum_{n=0}^{\infty}q^n\binom{\alpha+n-1}{n}\frac{(q^{n+2})_\infty}{(aq^n)_\infty}.$$

Multiply both sides by $a-q$ and replace α by $\alpha+1$ to derive

$$\sum_{n=0}^{\infty}\frac{(q/a)_{n+1}a^{n+1}}{(1-q^{n+1})^{\alpha}(q)_{n+1}}=-\frac{(q)_\infty}{(a)_\infty}\sum_{n=0}^{\infty}q^{n+1}\binom{\alpha+n}{n}\frac{\left(\frac{a}{q}\right)_{n+1}}{(q)_{n+1}}.$$

Finally, replacing the index n of the sum by $n-1$ on both sides yields the desired identity. □

Proof of Corollary 4.1.5. Letting $a\to 0$ in Theorem 4.1.2, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n b^n q^{\binom{n}{2}}}{(1-cq^n)^{\alpha}(d)_n}&=\sum_{j=0}^{\infty}c^j\binom{\alpha+j-1}{j}\sum_{n=0}^{\infty}\frac{(-b)^n q^{\binom{n}{2}+jn}}{(d)_n},\\&=\sum_{j=0}^{\infty}c^j\binom{\alpha+j-1}{j}\lim_{\gamma\to 0}{}_2\phi_1\left(\begin{matrix}b/\gamma, & q\\ & d\end{matrix};q;\gamma q^j\right)\\&=\sum_{j=0}^{\infty}c^j\binom{\alpha+j-1}{j}\lim_{\gamma\to 0}\frac{(bq^j)_\infty(q)_\infty}{(\gamma q^j)_\infty(d)_\infty}{}_2\phi_1\left(\begin{matrix}d/q & \gamma q^j\\ & bq^j\end{matrix};q;q\right)\\&=\frac{(q)_\infty}{(d)_\infty}\sum_{j=0}^{\infty}c^j\binom{\alpha+j-1}{j}(bq^j)_\infty\sum_{n=0}^{\infty}\frac{(d/q)_n q^n}{(bq^j)_n(q)_n}.\end{aligned}$$

Note that we have used Heine's transformation formula (2.2.2) in the penultimate step. Now replace α by $\alpha+1$, d by c and put $b=q^{r+1}$, for some non-negative integer

r, then we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n+1}{2}+nr}}{(1-cq^n)^{\alpha}(c)_{n+1}}=\frac{(q)_\infty}{(c)_\infty}\sum_{j=0}^{\infty}c^j\binom{\alpha+j}{j}(q^{j+r+1})_\infty\sum_{n=0}^{\infty}\frac{(c/q)_n q^n}{(q^{j+r+1})_n(q)_n}.$$

Now changing the index by replacing n by $n-1$ in the left side and j by $j-r$ in the right side, we get

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1} q^{\binom{n}{2}+(n-1)r}c^r}{(1-cq^{n-1})^{\alpha}(c)_{n}}=\frac{(q)_\infty}{(c)_\infty}\sum_{j=r}^{\infty}c^j\binom{\alpha+j-r}{j-r}(q^{j+1})_\infty\sum_{n=0}^{\infty}\frac{(c/q)_n q^n}{(q^{j+1})_n(q)_n}.$$

Finaly, replacing c by q in the above identity, we will get the identity (4.1.2) as the inner sum of the above summation shall only survive if $n=0$. □

Proof of Theorem 4.1.6. For $k,r\in\mathbb{N}$, let us define

$$V_{k,r}(q):=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{\binom{n}{2}+nr}}{(1-q^n)^k(q)_n}. \tag{4.2.2}$$

With the help of the binomial theorem, one can verify that

$$x^{r-1}\sum_{j=1}^{k}\binom{k-1}{j-1}\frac{x^j}{(1-x)^j}=\frac{x^r}{(1-x)^k}.$$

Substituting $x=q^n$, we have

$$q^{n(r-1)}\sum_{j=1}^{k}\binom{k-1}{j-1}\frac{q^{nj}}{(1-q^n)^j}=\frac{q^{nr}}{(1-q^n)^k}. \tag{4.2.3}$$

Utilize (4.2.3) in (4.2.2) to see that

$$V_{k,r}(q)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{\binom{n}{2}}}{(q)_n}\sum_{j=1}^{k}\binom{k-1}{j-1}\frac{q^{n(j+r-1)}}{(1-q^n)^j}.$$

Now we interchange the summation and use the definition (4.2.2) to get

$$V_{k,r}(q)=\sum_{j=1}^{k}\binom{k-1}{j-1}V_{j,r+j-1}(q). \tag{4.2.4}$$

Employing Corollary 4.1.5, it can be seen that

$$V_{j,r+j-1}(q)=\sum_{n=r+j-1}^{\infty}\binom{n-r+1}{j}q^n(q^{n+1})_\infty. \tag{4.2.5}$$

Substitute (4.2.5) in (4.2.4) to see that

$$V_{k,r}(q)=\sum_{j=1}^{k}\binom{k-1}{j-1}\sum_{n=r+j-1}^{\infty}\binom{n-r+1}{j}q^n(q^{n+1})_\infty. \tag{4.2.6}$$

From the definition of Stirling numbers of the first kind, one knows that

$$\binom{n}{m}=\sum_{l=1}^{m}\frac{s(m,l)}{m!}n^l. \tag{4.2.7}$$

We utilize (4.2.7) to write

$$\binom{n-r+1}{j} = \sum_{l=1}^{j} \frac{s(j,l)}{j!}(n-r+1)^l = \sum_{l=1}^{j} \frac{s(j,l)}{j!} \sum_{t=0}^{l} \binom{l}{t}(1-r)^{l-t} n^t$$

$$= \sum_{t=0}^{j} A(j,r,t) n^t, \tag{4.2.8}$$

where $A(j,r,t)$ is defined by

$$A(j,r,t) = \sum_{l=t}^{j} \frac{s(j,l)}{j!} \binom{l}{t} (1-r)^{l-t}.$$

Here, we have used the fact that $s(j,0) = 0$. Now employing (4.2.8) in (4.2.6), we get

$$V_{k,r}(q) = \sum_{j=1}^{k} \binom{k-1}{j-1} \sum_{t=0}^{j} A(j,r,t) U_{t,r+j-1}(q),$$

where $U_{t,r+j-1}(q)$ is the tail of the Uchimura's function defined by

$$U_{t,r+j-1}(q) = \sum_{n=j+r-1}^{\infty} n^t q^n (q^{n+1})_\infty.$$

Now we simplify $V_{k,r}(q)$ further by separating the term corresponding to $t = 0$. Doing this, we get

$$V_{k,r}(q) = \sum_{j=1}^{k} \binom{k-1}{j-1} \sum_{t=1}^{j} A(j,r,t) U_{t,r+j-1}(q) + \sum_{j=1}^{k} \binom{k-1}{j-1} A(j,r,0) U_{0,r+j-1}(q)$$

$$= \sum_{t=1}^{k} \sum_{j=t}^{k} \binom{k-1}{j-1} A(j,r,t) U_{t,r+j-1}(q) + \sum_{j=1}^{k} \binom{k-1}{j-1} A(j,r,0) U_{0,r+j-1}(q).$$

Finally, replacing j by $j+t$ in the first sum, one can complete the proof of Theorem 4.1.6. □

Proof of Theorem 4.1.7. First, we shall prove the recurrence (4.1.11). From [21, Equation (6C), p. 215], we know that

$$\sum_{l=0}^{j} s(j,l) x^l = (x)_j = x(x-1)(x-2)\cdots(x-j+1). \tag{4.2.9}$$

Successive differentiation of the above equation, t times, with respect to x yields

$$A(j,1-x,t) = \sum_{l=t}^{j} \frac{s(j,l)}{j!} \binom{l}{t} x^{l-t} = \frac{1}{j!\ t!} \frac{d^t}{dx^t} (x)_j \tag{4.2.10}$$

$$= \frac{1}{j!\ t!} \frac{d^t}{dx^t} \left((x)_{j-1} (x-j+1) \right). \tag{4.2.11}$$

Using Leibniz rule for differentiation in (4.2.11), we get

$$A(j,1-x,t) = \frac{1}{j!\ t!} \left((x-j+1) \frac{d^t}{dx^t} (x)_{j-1} + t \frac{d^{t-1}}{dx^{t-1}} (x)_{j-1} \right)$$

$$= \frac{(x-j+1)}{j} \frac{1}{(j-1)!\ t!} \frac{d^t}{dx^t}(x)_{j-1} + \frac{1}{j\ (j-1)!\ (t-1)!} \frac{d^{t-1}}{dx^{t-1}}(x)_{j-1}.$$

Now utilize (4.2.10) in the above equation to see that

$$A(j-1, 1-x, t-1) = jA(j, 1-x, t) + (j-x-1)A(j-1, 1-x, t).$$

The identity (4.1.11) follows by replacing $j,\ x,\ t$ by $j+1,\ 1-r,\ t+1$, respectively, in the above equation. To prove (4.1.12), we shall start with

$$\frac{k+1-r}{k+1} C(k, r, t+1) + \frac{1}{k+1} C(k, r, t)$$
$$= \frac{k+1-r}{k+1} \sum_{j=0}^{k-t-1} \binom{k-1}{j+t} A(j+t+1, r, t+1) + \frac{1}{k+1} \sum_{j=0}^{k-t} \binom{k-1}{j+t-1} A(j+t, r, t)$$
$$= \frac{k+1-r}{k+1} \sum_{j=0}^{k-t-1} \binom{k-1}{j+t} A(j+t+1, r, t+1)$$
$$+ \frac{1}{k+1} \sum_{j=0}^{k-t} \binom{k-1}{j+t-1} \left[(j+t+1)A(j+t+1, r, t+1) + (r+j+t-1)A(j+t, r, t+1)\right].$$

Note that to obtain the last step we used the recurrence (4.1.11). Using the fact that $\binom{k-1}{k} = 0$, the first sum is also valid for $j = k-t$. Using this and simplifying further, we obtain

$$\frac{k+1-r}{k+1} C(k, r, t+1) + \frac{1}{k+1} C(k, r, t)$$
$$= \frac{1}{k+1} \sum_{j=0}^{k-t} \left((k+1-r)\binom{k-1}{j+t} + (j+t+1)\binom{k-1}{j+t-1} \right) A(j+t+1, r, t+1)$$
$$+ \frac{1}{k+1} \sum_{j=1}^{k-t} \binom{k-1}{j+t-1} (r+j+t-1)A(j+t, r, t+1),$$

where in the last step we used the fact that $A(t, r, t+1) = 0$. By changing the variable j by $j+1$ in the second sum and simplifying further, we see that

$$\frac{k+1-r}{k+1} C(k, r, t+1) + \frac{1}{k+1} C(k, r, t)$$
$$= \frac{1}{k+1} \sum_{j=0}^{k-t} \left((k+j+t+1)\binom{k-1}{j+t} + (j+t+1)\binom{k-1}{j+t-1} \right) A(j+t+1, r, t+1)$$
$$= \sum_{j=0}^{k-t} \binom{k}{j+t} A(j+t+1, r, t+1) = C(k+1, r, t+1).$$

This completes the proof of (4.1.12). □

Proof of Theorem 4.1.8. From the definition (4.1.5) of $A(j, r, t)$, we can write

$$\sum_{t=1}^{j} A(j, r, t) = \sum_{t=1}^{j} \sum_{l=1}^{j} \frac{s(j, l)}{j!} \binom{l}{t} (1-r)^{l-t}$$

$$= \sum_{l=1}^{j} \frac{s(j,l)}{j!} \sum_{t=1}^{l} \binom{l}{t} (1-r)^{l-t}$$

$$= \sum_{l=1}^{j} \frac{s(j,l)}{j!} \left((2-r)^l - (1-r)^l\right)$$

$$= \frac{1}{j!} \left(\sum_{l=1}^{j} s(j,l)(2-r)^l - \sum_{l=1}^{j} s(j,l)(1-r)^l \right).$$

Now utilizing (4.2.9), one can rewrite the above equation as follows:

$$\sum_{t=1}^{j} A(j,r,t) = \frac{1}{j!} \left((2-r)_j - (1-r)_j\right)$$

$$= \frac{(-1)^{j-1}}{(j-1)!}(r-1)r(r+1)\cdots(r+j-3).$$

This completes the proof of (4.1.13). Now to prove (4.1.14), we will start with

$$\sum_{t=1}^{k} C(k,r,t) = \sum_{t=1}^{k} \sum_{j=0}^{k-t} \binom{k-1}{j+t-1} A(j+t,r,t)$$

$$= \sum_{t=1}^{k} \sum_{j=t}^{k} \binom{k-1}{j-1} A(j,r,t)$$

$$= \sum_{j=1}^{k} \binom{k-1}{j-1} \sum_{t=1}^{j} A(j,r,t)$$

$$= \sum_{j=1}^{k} \binom{k-1}{j-1} \frac{(-1)^{j-1}}{(j-1)!}(r-1)r(r+1)\cdots(r+j-3)$$

$$= \sum_{j=1}^{k} (-1)^{j-1} \binom{k-1}{j-1} \binom{r+j-3}{j-1}$$

$$= \sum_{j=0}^{k-1} (-1)^{j} \binom{k-1}{j} \binom{r+j-2}{j}$$

$$= (-1)^{k-1} \binom{r-2}{k-1},$$

where to obtain the last equality we used the following binomial identity, for $a, b \in \mathbb{N}$,

$$\sum_{j=0}^{a} (-1)^j \binom{a}{j} \binom{b+j}{j} = (-1)^a \binom{b}{a}.$$

This finishes the proof of (4.1.14). □

Before proving Theorem 4.1.10, we need the following results.

Lemma 4.2.1. *For $r \in \mathbb{N}$, $|a| < 1$ and $|cq| < 1$, we have*

$$\frac{d^r}{dx^r}\left[\frac{(xac)_\infty}{(xcq)_\infty}\right]_{x=1} = r! \sum_{n=0}^{\infty} \binom{n}{r} \frac{(a/q)_n}{(q)_n} c^n q^n.$$

Proof. Using the q-binomial theorem (2.2.1), we have

$$\frac{(xac)_\infty}{(xcq)_\infty} = \sum_{n=0}^{\infty} \frac{(a/q)_n}{(q)_n} (xcq)^n.$$

Differentiating both sides with respect x successively r times, we have

$$\begin{aligned}\frac{d^r}{dx^r}\left[\frac{(xac)_\infty}{(xcq)_\infty}\right] &= \sum_{n=0}^{\infty} \frac{(a/q)_n}{(q)_n} n(n-1)\cdots(n-r+1)x^{n-r}c^n q^n \\ &= r! \sum_{n=0}^{\infty} \frac{(a/q)_n}{(q)_n} \binom{n}{r} x^{n-r} c^n q^n.\end{aligned}$$

Letting $x \to 1$ in the above equation, we get the result. □

Now let us consider the following function

$$T_{r,a,c} = T_{r,a,c}(x,q) := \sum_{n=1}^{\infty} \frac{c^r q^{nr}}{(1-xcq^n)^r} - \sum_{n=0}^{\infty} \frac{a^r c^r q^{nr}}{(1-xacq^n)^r}. \tag{4.2.12}$$

One can easily verify that

$$\frac{d}{dx} T_{r,a,c}(x,q) = r T_{r+1,a,c}(x,q). \tag{4.2.13}$$

Lemma 4.2.2. *For each $i \in \mathbb{N}$, $a, c \in \mathbb{C}$ and $|q| < 1$ there exists a polynomial $N_i(x_1, x_2, \ldots, x_i) \in \mathbb{Q}[x_1, x_2, \ldots, x_i]$ of degree i such that*

$$\frac{d^i}{dx^i}\left[\frac{(xac)_\infty}{(xcq)_\infty}\right] = \frac{(xac)_\infty}{(xcq)_\infty} N_i(T_{1,a,c}, T_{2,a,c}, \ldots, T_{i,a,c}).$$

Proof. We start with the following product representation

$$\frac{(xac)_\infty}{(xcq)_\infty} = \prod_{n=0}^{\infty} \frac{1-xacq^n}{1-xcq^{n+1}}.$$

Then differentiating both sides with respect to x, we see that

$$\begin{aligned}\frac{d}{dx}\left[\frac{(xac)_\infty}{(xcq)_\infty}\right] &= \frac{d}{dx}\left[\exp\left\{\log\left(\prod_{n=0}^{\infty} \frac{1-xacq^n}{1-xcq^{n+1}}\right)\right\}\right] \\ &= \prod_{n=0}^{\infty} \frac{1-xacq^n}{1-xcq^{n+1}} \frac{d}{dx}\left[\sum_{n=0}^{\infty} \log\left(\frac{1-xacq^n}{1-xcq^{n+1}}\right)\right] \\ &= \frac{(xac)_\infty}{(xcq)_\infty} T_{1,a,c}(x,q) \\ &= \frac{(xac)_\infty}{(xcq)_\infty} N_1(T_{1,a,c}),\end{aligned}$$

with $N_1(x_1) = x_1$ and $T_{1,a,c}$ is defined as in (4.2.12). Further, differentiating the above equality with respect to x, it reduces to

$$\frac{d^2}{dx^2}\left[\frac{(xac)_\infty}{(xcq)_\infty}\right] = \frac{(xac)_\infty}{(xcq)_\infty}T^2_{1,a,c}(x,q) + \frac{(xac)_\infty}{(xcq)_\infty}\frac{d}{dx}T_{1,a,c}(x,q).$$

Now utilizing (4.2.13), one sees that

$$\frac{d^2}{dx^2}\left[\frac{(xac)_\infty}{(xcq)_\infty}\right] = \frac{(xac)_\infty}{(xcq)_\infty}T^2_{1,a,c}(x,q) + \frac{(xac)_\infty}{(xcq)_\infty}T_{2,a,c}(x,q) = \frac{(xac)_\infty}{(xcq)_\infty}N_2(T_{1,a,c}, T_{2,a,c}),$$

with $N_2(x_1, x_2) = x_1^2 + x_2$. Continuing differentiating with respect to x and using the relation (4.2.13), one can complete the proof of Corollary 4.2.2. □

Lemma 4.2.3. *Let $\mathfrak{S}_{m,a,c}(q)$ and $T_{r,a,c}(x,q)$ be the functions defined as in* (4.1.15) *and* (4.2.12), *respectively. For every $r \in \mathbb{N}$, we have*

$$T_{r,a,c}(1,q) = \sum_{h=0}^{r-1} Q_{h,r}\mathfrak{S}_{h,a,c}(q),$$

where $Q_{h,r} \in \mathbb{Q}$.

Proof. From (4.2.12), we have

$$T_{r,a,c}(1,q) = \sum_{n=1}^{\infty}\frac{c^r q^{nr}}{(1-cq^n)^r} - \sum_{n=0}^{\infty}\frac{a^r c^r q^{nr}}{(1-acq^n)^r}. \tag{4.2.14}$$

Using binomial theorem twice, for $|cq| < 1$, one can show that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{c^r q^{nr}}{(1-cq^n)^r} &= \sum_{n=1}^{\infty}\frac{cq^n(1-(1-cq^n))^{r-1}}{(1-cq^n)^r} \\
&= \sum_{n=1}^{\infty}\frac{cq^n}{(1-cq^n)^r}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^j(1-cq^n)^j \\
&= \sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^j\sum_{n=1}^{\infty}\frac{cq^n}{(1-cq^n)^{r-j}} \\
&= \sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^j\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\binom{r-j+m-1}{m}(cq^n)^{m+1} \\
&= \sum_{j=0}^{r-1}\binom{r-1}{j}\frac{(-1)^j}{(r-j-1)!}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(cq^n)^m m(m+1)\cdots(m+r-j-2) \\
&= \sum_{j=0}^{r-1}\binom{r-1}{j}\frac{(-1)^{r-1}}{(r-j-1)!}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(cq^n)^m(-m)(-m-1)\cdots(-m-r+j+2).
\end{aligned} \tag{4.2.15}$$

Now using the definition of the Stirling numbers of the first kind, that is,

$$x(x-1)\cdots(x-j+1) = \sum_{h=0}^{j} s(j,h)x^h,$$

the above expression in (4.2.15) can be written as follows:

$$\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{(-1)^{r-1}}{(r-j-1)!}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(cq^n)^m\sum_{h=0}^{r-j-1}s(r-j-1,h)(-m)^h$$

$$=\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{(-1)^{r-1}}{(r-j-1)!}\sum_{h=0}^{r-j-1}(-1)^h s(r-j-1,h)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}m^h(cq^n)^m$$

$$=\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{(-1)^{r-1}}{(r-j-1)!}\sum_{h=0}^{r-j-1}(-1)^h s(r-j-1,h)S_{h,c}(q)$$

$$=\sum_{h=0}^{r-1}\sum_{j=0}^{r-h-1}\binom{r-1}{j}\frac{(-1)^{h+r-1}}{(r-j-1)!}s(r-j-1,h)S_{h,c}(q),$$

$$=\sum_{h=0}^{r-1}Q_{h,r}S_{h,c}(q),$$

where $S_{h,c}(q)$ is defined as in (4.1.16) and the rational numbers $Q_{h,r}$ are defined by

$$Q_{h,r}=\sum_{j=0}^{r-h-1}\frac{(-1)^{r+h-1}}{(r-j-1)!}\binom{r-1}{j}s(r-j-1,h).$$

In a similar way, we can prove that the second sum in (4.2.14) is

$$\sum_{n=0}^{\infty}\frac{a^rc^rq^{nr}}{(1-acq^n)^r}=\sum_{h=0}^{r-1}Q_{h,r}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}m^h(acq^n)^m. \tag{4.2.16}$$

Now one can show that

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}m^h(acq^n)^m=\mathrm{Li}_{-h}(ac)+S_{h,ac}(q). \tag{4.2.17}$$

Thus, substituting (4.2.17) in (4.2.16), we have

$$\sum_{n=0}^{\infty}\frac{a^rc^rq^{nr}}{(1-acq^n)^r}=\sum_{h=0}^{r-1}Q_{h,r}R_{h,ac}(q),$$

where $R_{h,ac}(q)$ is as defined in (4.1.17). Utilizing (5.3.5) and (5.3.6) in (4.2.14) and together with the definition (4.1.15) of $\mathfrak{S}_{h,a,c}(q)=S_{h,c}(q)-R_{h,ac}(q)$, we finish the proof of this lemma. □

Proof of Theorem 4.1.10. Substituting $b=q^2$ and taking α to be a natural number, say $k+1$, in Corollary 4.1.3, we have

$$\sum_{n=0}^{\infty}\frac{(q^2/a)_na^n}{(1-cq^n)^{k+1}(q)_n}=\sum_{n=0}^{\infty}c^n\binom{k+n}{n}\frac{(q^{n+2})_\infty}{(aq^n)_\infty}$$

$$=\frac{(q)_\infty}{(a/q)_\infty}\sum_{n=1}^{\infty}c^{n-1}\binom{k+n-1}{k}\frac{(a/q)_n}{(q)_n}.$$

Further, replace c by cq and use the Chu-Vandemonde identity

$$\binom{k+n-1}{k} = \sum_{r=1}^{k} \binom{n}{r}\binom{k-1}{k-r},$$

in the above equation to get

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(q^2/a)_n a^n}{(1-cq^{n+1})^{k+1}(q)_n} &= \frac{1}{cq}\frac{(q)_\infty}{(a/q)_\infty} \sum_{n=1}^{\infty}\sum_{r=1}^{k} \binom{n}{r}\binom{k-1}{k-r} c^n q^n \frac{(a/q)_n}{(q)_n} \\ &= \frac{1}{cq}\frac{(q)_\infty}{(a/q)_\infty} \sum_{r=1}^{k} \binom{k-1}{k-r} \sum_{n=1}^{\infty} \binom{n}{r} \frac{(a/q)_n}{(q)_n} c^n q^n.\end{aligned}$$

Now we apply Lemma 4.2.1 in the right side of the above expression to see that

$$\sum_{n=0}^{\infty} \frac{(q^2/a)_n a^n}{(1-cq^{n+1})^{k+1}(q)_n} = \frac{1}{cq}\frac{(q)_\infty}{(a/q)_\infty} \sum_{r=1}^{k} \binom{k-1}{k-r} \frac{1}{r!} \left[\frac{d^r}{dx^r}\frac{(xac)_\infty}{(xcq)_\infty}\right]_{x=1}.$$

At this moment, employing Lemma 4.2.2, we obtain

$$\sum_{n=0}^{\infty} \frac{(q^2/a)_n a^n}{(1-cq^{n+1})^{k+1}(q)_n} = \frac{1}{cq}\frac{(q)_\infty}{(a/q)_\infty} \sum_{r=1}^{k} \binom{k-1}{k-r} \frac{1}{r!} \left[\frac{(ac)_\infty}{(cq)_\infty} N_r(T_{1,a,c}, T_{2,a,c}, \cdots, T_{r,a,c})\right].$$

Now utilizing Lemma 4.2.3, we can find a polynomial $P_k[x_1, x_2, \ldots, x_k] \in \mathbb{Q}[x_1, x_2, \ldots, x_k]$ of degree k such that

$$\sum_{n=0}^{\infty} \frac{(q^2/a)_n a^n}{(1-cq^{n+1})^{k+1}(q)_n} = \frac{1}{cq}\frac{(q)_\infty (ac)_\infty}{(a/q)_\infty (cq)_\infty} P_k(\mathfrak{S}_{0,a,c}(q), \mathfrak{S}_{1,a,c}(q), \ldots, \mathfrak{S}_{k-1,a,c}(q)).$$

Finally, multiplying both sides by $(1-a/q)$ and upon simplification, we complete the proof of Theorem 4.1.10. □

Proof of Theorem 4.1.12. Letting $a \to 0$ and $\alpha = 1$ in Corollary 4.1.3 gives

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}} b^n}{(q)_n (1-cq^n)} = \sum_{n=0}^{\infty} c^n (bq^n)_\infty.$$

Further, replacing c by cq and b by q^2, the above identity reduces to

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}} q^n}{(q)_n (1-cq^{n+1})} = \sum_{n=0}^{\infty} c^n q^n (q^{n+2})_\infty.$$

Changing the variable n by $n-1$ on both sides and simplifying, we get

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\frac{n(n-1)}{2}}}{(q)_{n-1}} \frac{cq^n}{(1-cq^n)} = \sum_{n=1}^{\infty} c^n q^n (q^{n+1})_\infty.$$

Now apply the differential operator $D = c\frac{d}{dc}$, m-times on both sides of the above identity to obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n}{2}}}{(q)_{n-1}} D^m\left(\frac{cq^n}{1-cq^n}\right) = \sum_{n=1}^{\infty} n^m c^n q^n (q^{n+1})_\infty. \tag{4.2.18}$$

To simplify further, we shall make use of the following property of Eulerian polynomials:

$$\frac{A_m(x)}{(1-x)^{m+1}} = \sum_{j=0}^{\infty} x^j (j+1)^m.$$

Letting $x = cq^n$ in the above identity, one can easily verify that

$$\frac{cq^n A_m(cq^n)}{(1-cq^n)^{m+1}} = D^m \left(\frac{cq^n}{1-cq^n} \right). \tag{4.2.19}$$

Now substituting (4.2.19) in (4.2.18), we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n}{2}} cq^n A_m(cq^n)}{(1-cq^n)^{m+1}(q)_{n-1}} = \sum_{n=1}^{\infty} n^m c^n q^n (q^{n+1})_\infty.$$

This completes the proof of the first equality in Corollary 4.1.3. The second equality has already been proved by the present authors in [2, Theorem 2.2] (see (3.1.3)). □

Proof of Corollary 4.1.13. Letting $m = 1$ in Theorem 4.1.12, it simplifies to

$$\sum_{n=1}^{\infty} nc^n q^n (q^{n+1})_\infty = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n}{2}} (cq^n) A_1(cq^n)}{(1-cq^n)^2 (q)_{n-1}} = \frac{(q)_\infty}{(cq)_\infty} Y_1(K_{1,c}(q)).$$

Note that the Eulerian polynomial $A_1(x) = 1$ and the Bell polynomial $Y_1(x) = x$. Moreover, using the definition (1.5.1) of $K_{1,c}(q)$, the above identity reduces to

$$\sum_{n=1}^{\infty} nc^n q^n (q^{n+1})_\infty = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n}{2}+n} c}{(1-cq^n)^2 (q)_{n-1}} = \frac{(q)_\infty}{(cq)_\infty} \sum_{n=1}^{\infty} \sigma_{0,c}(n) q^n.$$

Further, multiplying by $\frac{(cq)_\infty}{(q)_\infty}$ throughout the identity and using the definition of $\sigma_{0,c}(n)$, we obtain

$$\frac{(cq)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} nc^n q^n (q^{n+1})_\infty = \frac{(cq)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} cq^{\frac{n(n+1)}{2}}}{(1-cq^n)^2 (q)_{n-1}} = \sum_{n=1}^{\infty} \frac{c^n q^n}{1-q^n}.$$

Now our main aim is to simplify the middle term of the above identity since the left most and the right most expressions are exactly in the same form as we wanted in (4.1.19). Changing the variable n by $n+1$, the middle expression can be rewritten in the following way:

$$\begin{aligned}
&\frac{(cq)_\infty}{(q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n cq^{\frac{(n+1)(n+2)}{2}}}{(1-cq^{n+1})^2 (q)_n} \\
&= cq \frac{(cq)_\infty}{(q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}} (cq^2)_{n-1}^2 q^n}{(q)_n (cq^2)_n^2} \\
&= \frac{(cq)_\infty}{(q)_\infty} \frac{cq}{(1-cq)^2} \lim_{\gamma \to 0} \sum_{n=0}^{\infty} \frac{(cq)_n (cq)_n (q/\gamma)_n (\gamma q)^n}{(cq^2)_n (cq^2)_n (q)_n}, \\
&= \frac{(cq)_\infty}{(q)_\infty} \frac{cq}{(1-cq)^2} \lim_{\gamma \to 0} {}_3\phi_2 \left[\begin{matrix} q/\gamma, & cq, & cq \\ & cq^2, & cq^2 \end{matrix} ; q, \gamma q \right].
\end{aligned}$$

Now applying ${}_3\phi_2$ transformation formula (2.2.4) to the above expression yields that

$$\frac{(cq)_\infty}{(q)_\infty}\frac{cq}{(1-cq)^2}\lim_{\gamma\to 0}\frac{(\gamma cq)_\infty (q^2)_\infty}{(cq^2)_\infty(\gamma q)_\infty}{}_3\phi_2\left[\begin{matrix} q/\gamma, & q, & q \\ & cq^2, & q^2\end{matrix}; q, \gamma cq\right].$$

Further, upon simplification, it reduces to

$$\begin{aligned}\frac{1}{(1-q)}\frac{cq}{(1-cq)}\sum_{n=0}^{\infty}\frac{(-1)^n (q)_n q^{\frac{n(n+1)}{2}}(cq)^n}{(cq^2)_n(q^2)_n} &= \sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n+1)}{2}}(cq)^{n+1}}{(cq)_{n+1}(1-q^{n+1})},\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1} q^{\frac{n(n-1)}{2}}(cq)^n}{(cq)_n(1-q^n)},\end{aligned}$$

which is exactly same as the middle expression in Corollary 4.1.13. This completes the proof. □

Proof of Corollary 4.1.14. The proof of this corollary immediately follows by substituting $c = 1$ in Theorem 4.1.12. □

Proof of Theorem 4.1.15. First, we define the following two generating functions for the sequence of polynomials $\{t_n(a,q)\}$ and $\{f(n)\}$ as

$$T(a,\alpha,q) := \sum_{n=1}^{\infty} t_n(a,q)\alpha^n,$$

$$F(\alpha) := \sum_{n=1}^{\infty} f(n)\alpha^n.$$

From (4.1.20), we see that

$$(1-aq^n)t_n(a,q) = (f(n) - af(n+1)) + (1-q^{n-1})t_{n-1}(a,q).$$

Multiplying both sides by α^n and summing over all the natural numbers n, we get

$$T(a,\alpha,q) = \frac{(1-a/\alpha)}{(1-\alpha)}F(\alpha) - \frac{(1-a/\alpha)\alpha}{(1-\alpha)}T(a,\alpha q,q).$$

The above recurrence relation leads us to

$$T(a,\alpha,q) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(a/\alpha q^{n-1})_n F(\alpha q^{n-1})\alpha^{n-1}q^{\frac{(n-1)(n-2)}{2}}}{(\alpha)_n}.$$

Substitute $\alpha = q$ to see that

$$\begin{aligned}T(a,q,q) = \sum_{n=1}^{\infty} t_n(a,q)q^n &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(a/q^n)_n F(q^n) q^{\frac{n(n-1)}{2}}}{(q)_n}\\ &= -\sum_{n=1}^{\infty}\frac{(q/a)_n F(q^n)(a/q)^n}{(q)_n}.\end{aligned} \tag{4.2.20}$$

Utilizing (4.1.20), it can be shown that

$$t_n(a,q) - t_{n-1}(a,q) = f(n) - af(n+1) - q^{n-1}t_{n-1}(a,q) + aq^n t_n(a,q).$$

Taking summation over n from 1 to ℓ on both sides and upon simplification, one can show that

$$\lim_{\ell\to\infty}\left(\sum_{n=1}^{\ell} f(n) - \frac{a}{1-a} f(\ell+1) - \left(\frac{1-aq^{\ell}}{1-a}\right) t_\ell(a,q)\right) = \sum_{n=1}^{\infty} t_n(a,q) q^n = T(a,q,q). \tag{4.2.21}$$

Now we make use of another form of generating function for $f(n)$ given in [13, p. 50], namely,

$$F(\alpha) = \sum_{m\geq 0}\sum_{k\geq m} c_k \tilde{s}(k,m) m! \frac{\alpha^m}{(1-\alpha)^{m+1}} - c_0,$$

where $\tilde{s}(k,m)$ denotes the Stirling numbers of the second kind. Substituting $\alpha = q^n$ and upon simplification, one can show that

$$F(q^n) = \sum_{m\geq 1} d_m \sum_{j=0}^{m-1} e_{m,j} \frac{q^n}{(1-q^n)^{m+1-j}} + c_0 \frac{q^n}{1-q^n}, \tag{4.2.22}$$

where

$$d_m = \sum_{k\geq m} c_k \tilde{s}(k,m) m!, \text{ for } m \geq 1, \; d_0 = c_0,$$
$$e_{m,j} = (-1)^j \binom{m-1}{j}.$$

Finally, substituting the expression (4.2.22) for $F(q^n)$ in (4.2.20) and then substituting $T(a,q,q)$ in (4.2.21), it follows that

$$\begin{aligned}
&\lim_{\ell\to\infty}\left(\sum_{n=1}^{\ell} f(n) - \frac{a}{1-a} f(\ell+1) - \left(\frac{1-aq^{\ell}}{1-a}\right) t_\ell(a,q)\right) \\
&= -\sum_{n=1}^{\infty} \frac{(q/a)_n F(q^n)(a/q)^n}{(q)_n} \\
&= -\sum_{m\geq 1} d_m \sum_{j=0}^{m-1} e_{m,j} \sum_{n=1}^{\infty} \frac{(q/a)_n a^n}{(1-q^n)^{m+1-j}(q)_n} - c_0 \sum_{n=1}^{\infty} \frac{(q/a)_n a^n}{(1-q^n)(q)_n}.
\end{aligned} \tag{4.2.23}$$

Now from the definition (1.6.9) of h_j and the polynomial $P_k(a,q)$ defined in (1.5.11), it is clear that

$$\begin{aligned}
&\lim_{\ell\to\infty}\left(\sum_{n=1}^{\ell} f(n) - \frac{a}{1-a} f(\ell+1) - \left(\frac{1-aq^{\ell}}{1-a}\right) t_\ell(a,q)\right) \\
&= \sum_{m\geq 1} d_m \sum_{j=0}^{m-1} e_{m,j} P_{m+1-j}(a,q) + c_0 P_1(a,q) \\
&= \sum_{j=1}^{\infty} h_j P_j(a,q).
\end{aligned}$$

This completes the proof of Theorem 4.1.15. □

Proof of Theorem 4.1.16. The main idea of the proof of this result goes along the same lines as in [19, 33]. However, for completeness we give necessary details of the proof. As $f(n)$ is a periodic function of period N, one can write

$$f(n) = \frac{1}{N}\sum_{j=1}^{N} f(j) \sum_{k=0}^{N-1} \zeta_N^{(n-j)k}, \tag{4.2.24}$$

where $\zeta_N = e^{\frac{2\pi i}{N}}$. Now using (4.2.24), we can show that the generating function for $f(n)$, for $|\alpha| < 1$, is

$$F(\alpha) = \sum_{n=1}^{\infty} f(n)\alpha^n = \sum_{k=0}^{N-1} \frac{c_k \alpha}{1-\zeta_N^k \alpha}, \tag{4.2.25}$$

where $c_k = \frac{1}{N}\sum_{j=1}^{N} f(j)\zeta_N^{(1-j)k}$. From equation (4.2.23), we have seen that

$$\lim_{\ell\to\infty}\left(\sum_{n=1}^{\ell} f(n) - \frac{a}{1-a} f(\ell+1) - \left(\frac{1-aq^\ell}{1-a}\right) t_\ell(a,q)\right) = -\sum_{n=1}^{\infty} \frac{(q/a)_n F(q^n)(a/q)^n}{(q)_n}.$$

Now substituting the expression (4.2.25) for $F(q^n)$, we see that

$$\begin{aligned}\lim_{\ell\to\infty}\left(\sum_{n=1}^{\ell} f(n) - \frac{a}{1-a} f(\ell+1) - \left(\frac{1-aq^\ell}{1-a}\right) t_\ell(a,q)\right) &= -\sum_{n=1}^{\infty}\sum_{k=0}^{N-1} c_k \frac{(q/a)_n a^n}{(1-\zeta_N^k q^n)(q)_n} \\ &= -c_0 \sum_{n=1}^{\infty} \frac{(q/a)_n a^n}{(1-q^n)(q)_n} - \sum_{k=1}^{N-1} c_k \sum_{n=1}^{\infty} \frac{(q/a)_n a^n}{(1-\zeta_N^k q^n)(q)_n} \\ &= c_0 P_1(\mathfrak{S}_{0,a}(q)) - \sum_{k=1}^{N-1} \frac{c_k}{(1-\zeta_N^k)} \left({}_2\phi_1 \left[\begin{matrix} q/a, & \zeta_N^k \\ & \zeta_N^k q \end{matrix}; q, a\right] - 1\right),\end{aligned}$$

where P_1 is defined in (1.5.11) and one can verify that $P_1(x) = x$. Finally, using the q-Gauss summation formula (2.2.3), we have

$$\begin{aligned}\lim_{\ell\to\infty}\left(\sum_{n=1}^{\ell} f(n) - \frac{a}{1-a} f(\ell+1) - \left(\frac{1-aq^\ell}{1-a}\right) t_\ell(a,q)\right) &= c_0 \mathfrak{S}_{0,a}(q) + \sum_{k=1}^{N-1} \frac{c_k}{(1-\zeta_N^k)} \\ &\quad - \frac{(q)_\infty}{(a)_\infty} \sum_{k=1}^{N-1} c_k \frac{(a\zeta_N^k)_\infty}{(\zeta_N^k)_\infty}.\end{aligned}$$

This finishes the proof of Theorem 4.1.16. □

Proof of Corollary 4.1.17. The right hand side expression of this corollary has already been derived in [33, Corollary 4.6], whereas the left side expression is obtained from Theorem 4.1.16. This finishes the proof of this corollary.

□

Chapter 5

A DIVISOR GENERATING *q*-SERIES AND CUMULANTS ARISING FROM RANDOM GRAPHS

All the results presented in this chapter are drawn from our joint work [4], carried out in collaboration with Bhoria, Eyyunni, Maji and Wakhare. We first established a new q-series identity which serves as an analogue of Corollary 4.1.14 stated in Chapter 4. This identity plays a central role in our analysis. Using it, we derived a new Uchimura-type expression for the identity (1.6.8) of Andrews, Crippa and Simon. As an immediate corollary, we also obtained new limit forms for Uchimura's identity (1.5.3) and for Dilcher's identity (1.5.6), thereby extending the scope of these classical results. In addition to these analytic contributions, we also investigated the random variable $\gamma_n^*(1)$ introduced by Simon, Crippa, and Collenberg in their study of random acyclic digraphs. For this random variable, we explicitly computed the third, fourth, and fifth cumulants, which had not been previously determined. Building upon these computations, we further established a general formula describing all higher-order cumulants associated with $\gamma_n^*(1)$.

The remainder of this chapter is organized as follows. In the next section, we present all the results described above. This is followed by a section collecting the necessary preliminary results and tools that will be instrumental in proving our main theorems. The final section contains detailed proofs of all main results presented in this chapter.

5.1 Main results

We first proved a result motivated from Corollary 4.1.14.

Theorem 5.1.1. *For any non-negative integer k, we have the following expression*

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n} m^k\right) q^n (q^{n+1})_\infty = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n+1}{2}} A_k(q^n)}{(1-q^n)^{k+1}(q)_n}.$$

Utilizing this result, we obtain a new expression for the identity (1.6.8) of Andrews, Crippa and Simon.

Theorem 5.1.2. *Let $a_n(q)$ be a polynomial in q defined by the recurrence relation*

$$a_0(q) = 1, \quad a_n(q) = f(n) + (1-q^{n-1})a_{n-1}(q), \quad \text{for} \quad n \geq 1,$$

where $f(n) = \sum_{k\geq 0} c_k n^k$ is a polynomial in n. Then

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n} f(i) - a_n(q)\right) = \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} f(i)\right) q^n (q^{n+1})_\infty.$$

We present two applications of this theorem. First, we give a limit form for Uchimura's function M_k, defined in (1.5.1) and the second application presents a limit form for Dilcher's identity (1.5.6).

Corollary 5.1.3. *Let k be a non-negative integer and $a_{n,k}(q)$ be a sequence of polynomials in q defined by the recurrence relation*

$$a_{0,k}(q) = 1, \quad a_{n,k}(q) = f_k(n) + (1-q^{n-1})a_{n-1,k}(q), \quad \text{for} \quad n \geq 1, \tag{5.1.1}$$

where

$$f_k(n) = \sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1} n^{k-j}.$$

Then

$$\lim_{n\to\infty}\left(n^k - a_{n,k}(q)\right) = \sum_{n=1}^{\infty} n^k q^n (q^{n+1})_\infty.$$

As an application of Theorem 5.1.2, we obtain a limit expression for Dilcher's identity (1.5.6).

Corollary 5.1.4. *Let k be a non-negative integer and $a_{n,k}(q)$ be a sequence of polynomials in q defined by the recurrence relation*

$$a_{0,k}(q) = 1, a_{n,k}(q) = f_k(n) + (1-q^{n-1})a_{n-1,k}(q), \quad \textit{for} \;\; n \geq 1, \tag{5.1.2}$$

where

$$f_k(n) = \binom{n-1}{k-1}.$$

Then we have

$$\lim_{n\to\infty}\left(\binom{n}{k}-a_{n,k}(q)\right)=\sum_{n=k}^{\infty}\binom{n}{k}q^n(q^{n+1})_\infty. \tag{5.1.3}$$

Before proceeding to the next result, let $\gamma_n^*(1)$ be the random variable, studied by Simon–Crippa–Collenberg, as defined just above equation (1.6.4) in the introduction. For a random variable X, it is well-known that the cumulant generating function is given by

$$\log\left(\mathbb{E}[e^{uX}]\right)=\sum_{t=1}^{\infty}\kappa_t(X)\frac{u^t}{t!}, \tag{5.1.4}$$

where $\kappa_t(X)$ is the t-th cumulant with respect to the random variable X.

Simon et al. [42, p. 7, Equation (18)] and later Andrews et al. [13, p. 52, Equation (36)] proved that

$$\lim_{n\to\infty}(n-\kappa_1(\gamma_n^*(1)))=\lim_{n\to\infty}(n-\mathbb{E}(\gamma_n^*(1))=\sum_{n=1}^{\infty}d(n)q^n=K_1(q),$$

$$\lim_{n\to\infty}\kappa_2(\gamma_n^*(1))=\lim_{n\to\infty}\mathrm{Var}(\gamma_n^*(1))=\sum_{n=1}^{\infty}\sigma(n)q^n=K_2(q),$$

where $K_{m+1}(q)=\sum_{n=1}^{\infty}\sigma_m(n)q^n$. In this thesis, we first calculate the limit of third, fourth and fifth cumulants and then state a general result for higher cumulants.

Theorem 5.1.5. *We have*

$$\lim_{n\to\infty}\kappa_3(\gamma_n^*(1))=-K_3(q), \tag{5.1.5}$$

$$\lim_{n\to\infty}\kappa_4(\gamma_n^*(1))=K_4(q), \tag{5.1.6}$$

$$\lim_{n\to\infty}\kappa_5(\gamma_n^*(1))=-K_5(q). \tag{5.1.7}$$

More generally, we have the following result.

Theorem 5.1.6. *For any natural number t with $t>1$, we have*

$$\lim_{n\to\infty}\kappa_t(\gamma_n^*(1))=(-1)^tK_t(q).$$

5.2 Preliminary results

The generating function for Bernoulli numbers is given by

$$\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}=\frac{x}{e^x-1},\quad |x|<2\pi.$$

The generating function for Eulerian polynomials [21, p. 244] is as follows

$$\sum_{n=0}^{\infty}A_n(t)\frac{x^n}{n!}=\frac{t-1}{t-e^{x(t-1)}}.$$

Eulerian polynomials satisfy the following recurrence relation:

$$A_0(t) = 1,\ A_k(t) = \sum_{j=0}^{k-1} \binom{k}{j} A_j(t)(t-1)^{k-1-j} \quad \text{for} \quad k \geq 1. \tag{5.2.1}$$

One can also show that

$$\sum_{n=1}^{\infty} n^k x^n = \frac{xA_k(x)}{(1-x)^{k+1}}. \tag{5.2.2}$$

Bernoulli showed that the sum of k-th powers of the first $n-1$ natural numbers can be explicitly written as

$$\sum_{m=1}^{n-1} m^k = \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j n^{k+1-j}. \tag{5.2.3}$$

Now we state and prove a lemma that will be crucial to prove Theorem 5.1.1. This result gives a relation between Bernoulli numbers and Eulerian polynomials.

Lemma 5.2.1. *For any non-negative integer k and complex number t, we have*

$$S_k(t) := \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j (1-t)^j A_{k+1-j}(t) = tA_k(t).$$

Proof. Let us consider the generating function for $S_k(t)$,

$$\begin{aligned}
\sum_{k=0}^{\infty} S_k(t) \frac{x^{k+1}}{k!} &= \sum_{k=0}^{\infty} \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j (1-t)^j A_{k+1-j}(t) \frac{x^{k+1}}{k!} \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{B_j (1-t)^j}{j!} \frac{A_{k+1-j}(t)}{(k+1-j)!} x^{k+1} \\
&= \sum_{m=0}^{\infty} B_m \frac{(x(1-t))^m}{m!} \sum_{n=0}^{\infty} A_{n+1}(t) \frac{x^{n+1}}{(n+1)!} \\
&= \frac{x(1-t)}{e^{x(1-t)} - 1} \left(\frac{t-1}{t - e^{x(t-1)}} - 1 \right) \\
&= \frac{t-1}{t - e^{x(t-1)}} \times x e^{x(t-1)} \\
&= \sum_{n=0}^{\infty} A_n(t) \frac{x^n}{n!} \sum_{m=0}^{\infty} (t-1)^m \frac{x^{m+1}}{m!} \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} A_j(t)(t-1)^{k-j} \frac{x^{k+1}}{k!}.
\end{aligned}$$

Upon comparing the coefficients of $\frac{x^{k+1}}{k!}$, we get

$$S_k(t) = \sum_{j=0}^{k} \binom{k}{j} A_j(t)(t-1)^{k-j}$$

$$= \sum_{j=0}^{k-1} \binom{k}{j} A_j(t)(t-1)^{k-j} + A_k(t)$$

$$= (t-1)A_k(t) + A_k(t) = tA_k(t),$$

where in the penultimate step we used (5.2.1). This completes the proof. □

5.3 Proof of main results

Proof of Theorem 5.1.1. We will start with the left hand side of Theorem 5.1.1, that is,

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} m^k \right) q^n (q^{n+1})_\infty = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n-1} m^k \right) q^n (q^{n+1})_\infty + \sum_{n=1}^{\infty} n^k q^n (q^{n+1})_\infty$$

$$= \sum_{n=1}^{\infty} \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j n^{k+1-j} q^n (q^{n+1})_\infty$$

$$+ \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n+1}{2}} A_k(q^n)}{(1-q^n)^k (q)_n}, \tag{5.3.1}$$

where in the final step we used (5.2.3) to obtain the first sum and Corollary 4.1.14 to get the second sum. Now we shall try to simplify the first sum. Interchanging the sums and again making use of the Corollary 4.1.14, one can see that

$$\sum_{n=1}^{\infty} \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j n^{k+1-j} q^n (q^{n+1})_\infty$$

$$= \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n+1}{2}} A_{k+1-j}(q^n)}{(1-q^n)^{k+1-j} (q)_n}$$

$$= \frac{1}{k+1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n+1}{2}}}{(1-q^n)^{k+1} (q)_n} \sum_{j=0}^{k} \binom{k+1}{j} B_j (1-q^n)^j A_{k+1-j}(q^n)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n+1}{2}}}{(1-q^n)^{k+1} (q)_n} S_k(q^n) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n+1}{2}} q^n A_k(q^n)}{(1-q^n)^{k+1} (q)_n}, \tag{5.3.2}$$

where in the last step we employed Lemma 5.2.1. Now utilizing (5.3.2) in (5.3.1), we get

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} m^k \right) q^n (q^{n+1})_\infty = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n+1}{2}} q^n A_k(q^n)}{(1-q^n)^{k+1} (q)_n} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n+1}{2}} A_k(q^n)}{(1-q^n)^k (q)_n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n+1}{2}} A_k(q^n)}{(1-q^n)^k (q)_n} \left(\frac{q^n}{1-q^n} + 1 \right)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n+1}{2}} A_k(q^n)}{(1-q^n)^{k+1} (q)_n}.$$

This finishes the proof of Theorem 5.1.1. □

Proof of Theorem 5.1.2. Given that $f(n) = \sum_{k \geq 0} c_k n^k$ is a polynomial in n. Consider $F(x)$ to be the generating function for $f(n)$, that is,

$$F(x) := \sum_{n=1}^{\infty} f(n) x^n = \sum_{n=1}^{\infty} \sum_{k \geq 0} c_k n^k x^n = \sum_{k \geq 0} c_k \sum_{n=1}^{\infty} n^k x^n = \sum_{k \geq 0} c_k \frac{x A_k(x)}{(1-x)^{k+1}},$$

where in the last equality we used (5.2.2). Substituting $x = q^n$ in the above expression, we see that

$$F(q^n) = \sum_{k \geq 0} c_k \frac{q^n A_k(q^n)}{(1-q^n)^{k+1}}. \tag{5.3.3}$$

Now define the generating function for the sequence $a_n(q)$ as follows:

$$\begin{aligned} A(x) &:= \sum_{n=1}^{\infty} a_n(q) x^n \\ &= \sum_{n=1}^{\infty} \left(f(n) + (1 - q^{n-1}) a_{n-1}(q) \right) x^n \\ &= F(x) + x A(x) - x A(qx). \end{aligned}$$

Thus, we obtain

$$A(x) = \frac{F(x)}{1-x} - \frac{x}{1-x} A(qx).$$

This recurrence relation suggests that

$$A(x) = \sum_{n=0}^{\infty} \frac{(-1)^n F(q^n x) x^n q^{\binom{n}{2}}}{(x)_{n+1}}.$$

Put $x = q$ in the above expression and then use (5.3.3) to see that

$$\begin{aligned} A(q) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n}{2}}}{(q)_n} \sum_{k \geq 0} c_k \frac{q^n A_k(q^n)}{(1-q^n)^{k+1}} \\ &= \sum_{k \geq 0} c_k \sum_{n=1}^{\infty} \frac{(-1)^{n-1} A_k(q^n) q^{\binom{n+1}{2}}}{(1-q^n)^{k+1} (q)_n}. \end{aligned}$$

Now apply Theorem 5.1.1 to get

$$\begin{aligned} A(q) &= \sum_{k \geq 0} c_k \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} i^k \right) q^n (q^{n+1})_\infty \\ &= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} f(i) \right) q^n (q^{n+1})_\infty. \end{aligned} \tag{5.3.4}$$

From the recurrence relation of the sequence of polynomials $a_i(q)$, it is evident that

$$a_i(q) = f(i) + (1 - q^{i-1}) a_{i-1}(q)$$

$$\implies f(i) - a_i(q) + a_{i-1}(q) = q^{i-1} a_{i-1}(q).$$

Now taking the sum over i from 1 to n, then letting $n \to \infty$ and finally using (5.3.4), the result follows. □

Proof of Corollary 5.1.3. Observe that

$$\sum_{i=1}^{n} f_k(i) = \sum_{i=1}^{n}\sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} i^{k-j},$$
$$= \sum_{i=1}^{n} \left(i^k - (i-1)^k\right) = n^k.$$

Now apply Theorem 5.1.2 to complete the proof of the corollary. □

Proof of Corollary 5.1.4. As we have taken $f_k(n) = \binom{n-1}{k-1}$, we find that

$$\sum_{i=1}^{n} f_k(i) = \sum_{i=1}^{n} \binom{i-1}{k-1} = \binom{n}{k}.$$

Hence the proof of (5.1.3) immediately follows from Theorem 5.1.2. □

Proof of Theorem 5.1.5. For simplicity, throughout this proof, let us denote $\gamma_n^*(1)$ as X_n. We define $e_{n,k} := \mathbb{E}(X_n^k)$. Simon, Crippa and Collenberg [42] proved that $e_{n,1}$ and $e_{n,2}$ satisfy the following recurrence relations,

$$e_{n,1} = 1 + (1 - q^{n-1}) e_{n-1,1}, \tag{5.3.5}$$
$$e_{n,2} = 2n e_{n,1} - a_{n,2}, \tag{5.3.6}$$

where

$$a_{n,2} = \sum_{i=1}^{n} f_2(i) \prod_{j=i}^{n-1} (1 - q^j), \quad \text{and} \quad f_2(i) = (2i - 1).$$

More generally, for any fixed $k \geq 1$ and $n \geq 1$, we will show that $e_{n,k}$ satisfies the following relation,

$$e_{n,k} = \sum_{\ell=1}^{k} \binom{k}{\ell} (-1)^{\ell-1} n^{k-\ell} a_{n,\ell}, \tag{5.3.7}$$

where

$$a_{n,1} = e_{n,1}, \qquad a_{n,k} = \sum_{i=1}^{n} f_k(i) \prod_{j=i}^{n-1} (1 - q^j), \qquad \text{for } k \geq 2, \tag{5.3.8}$$

and

$$f_k(i) = \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} i^{k-j} = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^{k-j-1} i^{j}. \tag{5.3.9}$$

It is easy to see that the relation (5.3.7) is true for $k = 1$ as $e_{n,1} = a_{n,1}$ for all $n \geq 1$ by our assumption. Moreover, the relation (5.3.7) is also true for $k = 2$ as we know that (5.3.6) holds for all $n \geq 1$. Let us assume that the relation for $e_{m,j}$ is true for any $1 \leq m < \infty$ when $3 \leq j \leq k-1$ and $1 \leq m \leq n-1$ for $j = k$. Now we shall show that the relation holds for $e_{n,k}$. Since $e_{n,k} = \mathbb{E}(X_n^k)$, we can write

$$
\begin{aligned}
e_{n,k} &= \sum_{h=1}^{n} h^k P(X_n = h) \\
&= \sum_{h=1}^{n} h^k q^{n-h} \prod_{i=1}^{h-1} (1 - q^{n-i}) \\
&= q^{n-1} + (1 - q^{n-1}) \sum_{h=2}^{n} h^k q^{n-h} \prod_{i=2}^{h-1} (1 - q^{n-i}) \\
&= q^{n-1} + (1 - q^{n-1}) \sum_{h=1}^{n-1} (h+1)^k q^{n-h-1} \prod_{i=1}^{h-1} (1 - q^{n-i-1}) \\
&= q^{n-1} + (1 - q^{n-1}) \sum_{h=1}^{n-1} \left(1 + \sum_{j=1}^{k} \binom{k}{j} h^j \right) q^{n-h-1} \prod_{i=1}^{h-1} (1 - q^{n-i-1}) \\
&= q^{n-1} + (1 - q^{n-1}) \sum_{h=1}^{n-1} P(X_{n-1} = h) + (1 - q^{n-1}) \sum_{h=1}^{n-1} \sum_{j=1}^{k} \binom{k}{j} h^j P(X_{n-1} = h) \\
&= 1 + (1 - q^{n-1}) \sum_{j=1}^{k} \binom{k}{j} \sum_{h=1}^{n-1} h^j P(X_{n-1} = h) \\
&= 1 + (1 - q^{n-1}) \sum_{j=1}^{k} \binom{k}{j} e_{n-1,j} \\
&= 1 + (1 - q^{n-1}) \sum_{j=1}^{k} \binom{k}{j} \sum_{\ell=1}^{j} \binom{j}{\ell} (-1)^{\ell-1} (n-1)^{j-\ell} a_{n-1,\ell} \quad \text{(using inductive hypothesis)} \\
&= 1 + (1 - q^{n-1}) \sum_{\ell=1}^{k} (-1)^{\ell-1} a_{n-1,\ell} \sum_{j=\ell}^{k} \binom{k}{j} \binom{j}{\ell} (n-1)^{j-\ell} \\
&= 1 + (1 - q^{n-1}) \sum_{\ell=1}^{k} (-1)^{\ell-1} a_{n-1,\ell} \sum_{j=0}^{k-\ell} \binom{k}{j+\ell} \binom{j+\ell}{\ell} (n-1)^{j} \\
&= 1 + (1 - q^{n-1}) \sum_{\ell=1}^{k} \binom{k}{\ell} (-1)^{\ell-1} a_{n-1,\ell} \sum_{j=0}^{k-\ell} \binom{k-\ell}{j} (n-1)^{j} \\
&= 1 + (1 - q^{n-1}) \sum_{\ell=1}^{k} \binom{k}{\ell} (-1)^{\ell-1} a_{n-1,\ell}\, n^{k-\ell}. \qquad (5.3.10)
\end{aligned}
$$

From (5.3.8) and (5.3.9), one can easily observe that, for any $k \in \mathbb{N}$, the following recurrence relation for $a_{n,k}$ holds:

$$
a_{n,k} = f_k(n) + (1 - q^{n-1}) a_{n-1,k}, \quad \text{where} \quad \sum_{i=1}^{n} f_k(i) = n^k. \qquad (5.3.11)
$$

Use this recurrence relation in (5.3.10) to see that

$$\begin{aligned} e_{n,k} &= 1 + \sum_{\ell=1}^{k} \binom{k}{\ell} (-1)^{\ell-1} n^{k-\ell} \left(a_{n,\ell} - f_\ell(n)\right) \\ &= 1 + \sum_{\ell=1}^{k} \binom{k}{\ell} (-1)^{\ell-1} n^{k-\ell} a_{n,\ell} - \sum_{\ell=1}^{k} \binom{k}{\ell} (-1)^{\ell-1} n^{k-\ell} f_\ell(n). \end{aligned}$$

It becomes clear at this juncture that to prove (5.3.7) it is enough to show

$$\sum_{\ell=1}^{k} \binom{k}{\ell} (-1)^{\ell-1} n^{k-\ell} f_\ell(n) = 1.$$

Using the definition (5.3.9) of $f_\ell(n)$, we can see that

$$\begin{aligned} \sum_{\ell=1}^{k} \binom{k}{\ell} (-1)^{\ell-1} n^{k-\ell} f_\ell(n) &= \sum_{\ell=1}^{k} \sum_{j=1}^{\ell} \binom{k}{\ell} \binom{\ell}{j} (-1)^{\ell+j} n^{k-j} \\ &= \sum_{j=1}^{k} \sum_{\ell=j}^{k} \binom{k}{j} \binom{k-j}{\ell-j} (-1)^{\ell+j} n^{k-j} \\ &= \sum_{j=1}^{k} \binom{k}{j} n^{k-j} \sum_{\ell=0}^{k-j} \binom{k-j}{\ell} (-1)^{\ell} \\ &= 1 + \sum_{j=1}^{k-1} \binom{k}{j} n^{k-j} (1-1)^{k-j} \\ &= 1. \end{aligned}$$

This completes the proof of the relation (5.3.7) for $e_{n,k}$. As we know that the sequence $a_{n,k}$ satisfies the relation (5.3.11), so by applying Corollary 5.1.3, we have

$$\lim_{n\to\infty} \left(n^k - a_{n,k}(q)\right) = \sum_{n=1}^{\infty} n^k q^n (q^{n+1})_\infty.$$

Further, utilize Uchimura's identity (1.6.1) to see that

$$\lim_{n\to\infty} \left(n^k - a_{n,k}(q)\right) = Y_k(K_1, K_2, \ldots, K_m), \tag{5.3.12}$$

where Y_k denotes the Bell polynomial defined in (1.5.2). Now we are ready to calculate the limiting value of the third cumulant, that is,

$$\begin{aligned} \lim_{n\to\infty} (\kappa_3(X_n)) &= \lim_{n\to\infty} \mathbb{E}(X_n - \mathbb{E}(X_n))^3 \\ &= \lim_{n\to\infty} \left(\mathbb{E}(X_n^3) - 3\mathbb{E}(X_n)\mathbb{E}(X_n^2) + 2\mathbb{E}(X_n)^3\right) \\ &= \lim_{n\to\infty} \left(e_{n,3} - 3e_{n,1}e_{n,2} + 2e_{n,1}^3\right) \\ &= -\lim_{n\to\infty} \left((n^3 - a_{n,3}) - 3(n - a_{n,1})(n^2 - a_{n,2}) + 2(n - a_{n,1})^3\right) \\ &= -\left(Y_3 - 3Y_1Y_2 + 2Y_1^3\right) = -K_2, \end{aligned}$$

where in the ante-penultimate step we have used the recurrence relation (5.3.7), whereas in the penultimate step we used (5.3.12) and in the final step we used Dilcher's identity [22, p. 85, Equation (2.2)]. This proves (5.1.5).

Now we shall calculate the limiting value of the fourth cumulant, that is,

$$\begin{aligned}
\lim_{n\to\infty}(\kappa_4(X_n)) &= \lim_{n\to\infty}\left(\mathbb{E}(X_n-\mathbb{E}(X_n))^4 - 3\left(\mathbb{E}(X_n-\mathbb{E}(X_n))^2\right)^2\right)\\
&= \lim_{n\to\infty}\left(\mathbb{E}(X_n^4) - 4\mathbb{E}(X_n)\mathbb{E}(X_n^3) - 3\mathbb{E}(X_n^2)^2 + 12\mathbb{E}(X_n)^2\mathbb{E}(X_n^2) - 6\mathbb{E}(X_n)^4\right)\\
&= \lim_{n\to\infty}\left(e_{n,4} - 4e_{n,1}e_{n,3} - 3e_{n,2}^2 + 12e_{n,2}e_{n,1}^2 - 6e_{n,1}^4\right)\\
&= \lim_{n\to\infty}(n^4 - a_{n,4}) - 4(n-a_{n,1})(n^3-a_{n,3}) - 3(n^2-a_{n,2})^2 + 12(n^2-a_{n,2})(n-a_{n,1})^2\\
&\quad - 6(n-a_{n,1})^4)\\
&= (Y_4 - 4Y_1Y_3 - 3Y_2^2 + 12Y_2Y_1^2 - 6Y_1^4)\\
&= K_4.
\end{aligned}$$

Here again, we made use of (5.3.7), (5.3.12) and Dilcher's identity [22, p. 85, Equation (2.3)]. This completes the proof of (5.1.6). Finally, to prove (5.1.7), we evaluate the limiting value of the fifth cumulant in the following way:

$$\begin{aligned}
\lim_{n\to\infty}(\kappa_5(X_n)) &= \lim_{n\to\infty}\left(\mathbb{E}(X_n-\mathbb{E}(X_n))^5 - 10\mathbb{E}(X_n-\mathbb{E}(X_n))^3\mathbb{E}(X_n-\mathbb{E}(X_n))^2\right)\\
&= \lim_{n\to\infty}(\mathbb{E}(X_n^5) - 5\mathbb{E}(X_n^4)\mathbb{E}(X_n) + 20\mathbb{E}(X_n^3)\mathbb{E}(X_n)^2 - 40\mathbb{E}(X_n^2)\mathbb{E}(X_n)^3\\
&\qquad + 24\mathbb{E}(X_n)^5 - 10\mathbb{E}(X_n^3)\mathbb{E}(X_n^2) + 10\mathbb{E}(X_n^2)^2\mathbb{E}(X_n))\\
&= \lim_{n\to\infty}\left(e_{n,5} - 5e_{n,4}e_{n,1} + 20e_{n,3}e_{n,1}^2 - 40e_{n,2}e_{n,1}^3 + 24e_{n,1}^5 - 10e_{n,3}e_{n,2} + 10e_{n,2}^2e_{n,1}\right)\\
&= -\lim_{n\to\infty}((n^5-a_{n,5}) - 5(n^4-a_{n,4})(n-a_{n,1}) + 20(n^3-a_{n,3})(n-a_{n,1})^2\\
&\quad - 40(n^2-a_{n,2})(n-a_{n,1})^3 + 24(n-a_{n,1})^5 - 10(n^3-a_{n,3})(n^2-a_{n,2})\\
&\qquad + 10(n^2-a_{n,2})^2(n-a_{n,1}))\\
&= -(Y_5 - 5Y_4Y_1 + 20Y_3Y_1^2 - 40Y_2Y_1^3 + 24Y_1^5 - 10Y_3Y_2 + 10Y_2^2Y_1)\\
&= -K_5.
\end{aligned}$$

Here also to obtain the final step we employed Dilcher's identity [22, p. 85, Theorem 1] and in the penultimate step we invoked (5.3.12). □

So far we have not been able to extend the above technique for higher cumulants. However, we use another method to prove the general case of the t-th cumulant for any t.

5.3.1 Proof of Theorem 5.1.6

We use the notations as in the proof of Theorem 5.1.5. Let us define $Z_n := n - X_n$, where $X_n = \gamma_n^*(1)$. Let κ_t denote the t-th cumulant and $e_{n,k} = \mathbb{E}(X_n^k)$. For any random variable X and constant c, it is well-known that $\kappa_1(X+c) = \kappa_1(X) + c$, and for $t \geq 2$ one has $\kappa_t(X+c) = \kappa_t(X)$, and $\kappa_t(cX) = c^t \kappa_t(X)$ for any $t \geq 1$. Hence, for $t \geq 2$, one can see that

$$\lim_{n\to\infty} \kappa_t(Z_n) = \lim_{n\to\infty} (-1)^t \kappa_t(X_n).$$

Thus, for $t \geq 2$, Theorem 5.1.6 is equivalent to the fact that

$$\lim_{n\to\infty} \kappa_t(Z_n) = K_t(q), \tag{5.3.13}$$

where

$$K_{t+1}(q) = \sum_{n=1}^{\infty} \sigma_t(n) q^n = \sum_{n=1}^{\infty} \frac{n^t q^n}{1-q^n}$$

be the divisor generating function and for $t = 1$, we already know

$$\lim_{n\to\infty} \kappa_1(Z_n) = \lim_{n\to\infty} (n - \kappa_1(X_n)) = K_1(q).$$

We first prove a lemma that plays a vital role to prove (5.3.13).

Lemma 5.3.1. *We have*

$$\mathbb{E}(Z_n^k) = n^k - a_{n,k},$$

where $a_{n,k}$ is defined as in (5.3.8).

Proof. As $Z_n = n - X_n$, so using Binomial expansion

$$\begin{aligned}\mathbb{E}(Z_n^k) = \mathbb{E}\left((n - X_n)^k\right) &= \sum_{j=0}^{k} \binom{k}{j} n^{k-j} (-1)^j \mathbb{E}(X_n^j) \\ &= n^k + \sum_{j=1}^{k} \binom{k}{j} n^{k-j} (-1)^j e_{n,j}.\end{aligned}$$

Now we use (5.3.7) in the right side of the above expression to get

$$\begin{aligned}\mathbb{E}(Z_n^k) &= n^k + \sum_{j=1}^{k} \binom{k}{j} n^{k-j} (-1)^j \sum_{\ell=1}^{j} \binom{j}{\ell} (-1)^{\ell-1} n^{j-\ell} a_{n,\ell} \\ &= n^k + \sum_{\ell=1}^{k} (-1)^{\ell-1} n^{k-\ell} a_{n,\ell} \sum_{j=\ell}^{k} \binom{k}{j}\binom{j}{\ell} (-1)^j \\ &= n^k - \sum_{\ell=1}^{k} \binom{k}{\ell} n^{k-\ell} a_{n,\ell} \sum_{j=\ell}^{k} \binom{k-\ell}{j-\ell} (-1)^{j-\ell}.\end{aligned}$$

By using binomial theorem, the inner sum gives value 1 if $k = \ell$ and 0 otherwise. Therefore, we get

$$\mathbb{E}(n - X_n)^k = n^k - a_{n,k}.$$

□

Proof of Theorem 5.1.6. Now we are ready to prove Theorem 5.1.6, which is equivalent to proving identity (5.3.13). To prove (5.3.13), we further derive an equivalent statement by multiplying both sides of (5.3.13) by $\frac{z^t}{t!}$ and then summing over t to obtain

$$\lim_{n\to\infty}\sum_{t=1}^{\infty}\kappa_t(Z_n)\frac{z^t}{t!}=\sum_{t=1}^{\infty}K_t(q)\frac{z^t}{t!}. \tag{5.3.14}$$

Now we expand the sum on the right hand side to see

$$\begin{aligned}\sum_{t=1}^{\infty}\frac{z^t}{t!}K_t(q) &= \sum_{t=1}^{\infty}\frac{z^t}{t!}\sum_{k=1}^{\infty}\frac{k^{t-1}q^k}{1-q^k}\\ &= \sum_{k=1}^{\infty}\frac{q^k}{1-q^k}\frac{1}{k}\sum_{t=1}^{\infty}\frac{(zk)^t}{t!}\\ &= \sum_{k=1}^{\infty}\frac{q^k}{1-q^k}\frac{e^{kz}-1}{k}\\ &= \sum_{k=1}^{\infty}\frac{e^{kz}-1}{k}\sum_{\ell=1}^{\infty}q^{k\ell}\\ &= \sum_{\ell=1}^{\infty}\sum_{k=1}^{\infty}\frac{e^{kz}q^{k\ell}}{k}-\sum_{\ell=1}^{\infty}\sum_{k=1}^{\infty}\frac{q^{k\ell}}{k}\\ &= -\sum_{\ell=1}^{\infty}\log(1-e^zq^\ell)+\sum_{\ell=1}^{\infty}\log(1-q^\ell)\\ &= \log\prod_{\ell=1}^{\infty}\frac{(1-q^\ell)}{(1-e^zq^\ell)}=\log\frac{(q)_\infty}{(e^zq)_\infty}.\end{aligned} \tag{5.3.15}$$

Combining (5.3.14) and (5.3.15), we get

$$\lim_{n\to\infty}\sum_{t=1}^{\infty}\kappa_t(Z_n)\frac{z^t}{t!}=\log\frac{(q)_\infty}{(e^zq)_\infty}.$$

Hence, we only need to show the identity

$$\lim_{n\to\infty}\exp\left(\sum_{t=1}^{\infty}\kappa_t(Z_n)\frac{z^t}{t!}\right)=\frac{(q)_\infty}{(e^zq)_\infty}. \tag{5.3.16}$$

From the definition of the cumulant generating function (5.1.4), we know that

$$\lim_{n\to\infty}\mathbb{E}\left(e^{Z_nz}\right)=\lim_{n\to\infty}\exp\left(\sum_{t=1}^{\infty}\kappa_t(Z_n)\frac{z^t}{t!}\right).$$

Thus, to prove (5.3.16), we have to show that

$$\lim_{n\to\infty}\mathbb{E}\left(e^{Z_nz}\right)=\frac{(q)_\infty}{(e^zq)_\infty}.$$

Now we start with

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\left(e^{Z_nz}\right) &= \lim_{n\to\infty}\mathbb{E}\left(e^{(n-X_n)z}\right)\\ &= \lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{z^k}{k!}\mathbb{E}\left((n-X_n)^k\right)\end{aligned}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{z^k}{k!} \lim_{n\to\infty} (n^k - a_{n,k})$$

$$= 1 + \sum_{k=1}^{\infty} \frac{z^k}{k!} \sum_{\ell=1}^{\infty} \ell^k q^\ell (q^{\ell+1})_\infty, \tag{5.3.17}$$

where we used Lemma 5.3.1 in the penultimate step and Corollary 5.1.3 in the last step. Further, we use

$$\sum_{\ell=0}^{\infty} q^\ell (q^{\ell+1})_\infty = 1$$

in (5.3.17) to obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left(e^{Z_n z}\right) &= \sum_{\ell=0}^{\infty} q^\ell (q^{\ell+1})_\infty + \sum_{k=1}^{\infty} \frac{z^k}{k!} \sum_{\ell=1}^{\infty} \ell^k q^\ell (q^{\ell+1})_\infty \\
&= (q)_\infty + \sum_{\ell=1}^{\infty} q^\ell (q^{\ell+1})_\infty + \sum_{k=1}^{\infty} \frac{z^k}{k!} \sum_{\ell=1}^{\infty} \ell^k q^\ell (q^{\ell+1})_\infty \\
&= (q)_\infty + \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{\ell=1}^{\infty} \ell^k q^\ell (q^{\ell+1})_\infty \\
&= (q)_\infty + \sum_{\ell=1}^{\infty} q^\ell (q^{\ell+1})_\infty \sum_{k=0}^{\infty} \frac{z^k}{k!} \ell^k \\
&= (q)_\infty + \sum_{\ell=1}^{\infty} e^{\ell z} q^\ell (q^{\ell+1})_\infty \\
&= \sum_{\ell=0}^{\infty} e^{\ell z} q^\ell (q^{\ell+1})_\infty \\
&= \frac{(q)_\infty}{(e^z q)_\infty},
\end{aligned}$$

where in the last step, we used [44, Equation 2.3] with $x = e^z$. This completes the proof of Theorem 5.1.6.

□

Chapter 6

CONCLUDING REMARKS AND FUTURE DIRECTIONS

In Chapter 2, we introduced one variable generalizations of two key identities: Bhoria–Eyyunni–Maji's identity (1.2.12) and Dixit–Patel's identity (1.3.5) (see Theorem 2.1.1 and 2.1.5). By adding a new complex parameter, these results extend the original identities and allow us to study a broader class of q-series. This approach brought several classical results into a single framework and gave individual one variable extension of Ramanujan's five entries (1.2.1)–(1.2.5) and their finite analogues (1.3.7)–(1.3.11). These extensions refined Ramanujan's identities and reveal deeper connections among them. We also obtained a one variable generalization of the finite analogue (1.3.4) of Garvan's identity (1.2.11) (see Theorem 2.1.9), linking both the classical and finite forms. In addition, we proved an identity involving the basic hypergeometric series ${}_3\phi_2$ and ${}_2\phi_1$, which serves as a one-variable analogue of a result of Dixit and Patel. This further enriches the theory of finite q-series transformations.

In Chapter 3, we established a weighted partition identity (3.1.1) for the generalized divisor function $\sigma_{z,c}(n) = \sum_{d|n} d^z c^d$, following the combinatorial approach of Bressoud and Subbarao. We further observed that Theorem 3.1.1 can be recovered from the partition theoretic interpretation of (1.2.4) using a fractional differentiation operator. Applying this operator to an identity of Andrews, Garvan, and Liang yielded several additional Bressoud–Subbarao type weighted partition identities.

We also generalized Uchimura's identity, Theorem 1.5.1, by introducing a complex parameter c. In this setting, the generating functions K_n for $\sigma_{z,1}(n)$ naturally extend to those for $\sigma_{z,c}(n)$, for $z \in \mathbb{N}$. Since the generalization of the Bressoud–Subbarao identity in (1.4.2) holds for complex z, it is natural to ask whether an analogous extension of Theorem 1.5.1 exists for complex m, that is, whether one can obtain an analogue of Theorem 3.1.2 in the complex setting.

The identity (3.1.12) presented an alternative form of the Bressoud–Subbarao identity (1.2.8). We then found an interesting analogue of this form, namely Corollary 3.1.7, which provides a weighted partition identity for $p^{(2)}(n)$. It would be highly desirable to obtain a Bressoud–Subbarao type combinatorial proof for Corollary 3.1.7.

In Chapter 4, our main objective was to study various generalizations of Uchimura's identity (1.2.7). We classified the three sides of (1.2.7) as the *Uchimura-type* sum, *Ramanujan-type* sum and *divisor-type* sum and used this viewpoint to develop a unified framework for identities due to Uchimura (1.5.3), Dilcher (1.5.6), Andrews–Crippa–Simon (1.5.7), (1.5.9), Gupta–Kumar (1.5.10), (1.5.11) and our identity (3.1.3). We showed that all of these follow from our general identity, Theorem 4.1.1. For instance, although Uchimura's identity (1.2.7) and Dilcher's identity (1.5.6) appear unrelated, both arise from Corollary 4.1.5. We also obtained a one variable generalization of Gupta–Kumar's identity (1.5.11) (see Theorem 4.1.10), which simultaneously generalizes the identity (1.5.7) of Andrews, Crippa and Simon.

Further, we established a *Ramanujan-type* expression for the identity (3.1.3), yielding a Ramanujan-type sum for Uchimura's generalization (1.5.3) and a Uchimura-type sum for Ramanujan's identity (1.2.4).

We also corrected an identity of Dilcher (4.1.3) and, in doing so, introduced two sequences of rational numbers involving Stirling numbers of the first kind, namely $A(j, r, t)$ (4.1.5) and $C(k, r, t)$ (4.1.10), which satisfy interesting recurrence relations (see Theorem 4.1.7).

Applications to probability theory were discussed through connections between Uchimura's identities and the theory of random acyclic digraphs developed by Simon–Crippa–Collenberg and Andrews–Crippa–Simon. We gave a one-variable generalization of an identity of Andrews–Crippa–Simon (see Theorem 4.1.15) and a corrected version of an identity of Gupta and Kumar (see Theorem 4.1.16). We conclude with two open problems:

1. Corollary 4.1.5 simultaneously generalizes Uchimura's identity (1.2.7) and Dilcher's identity (1.5.6), but provides only the first equality. A natural question is: what is the divisor-type sum corresponding to Corollary 4.1.5?

2. As an application of Uchimura's generalization (1.5.3), Andrews–Crippa–Simon proved (1.6.8). Analogously, from Gupta–Kumar's identity (1.5.11) we obtained Theorem 4.1.15. Since Theorem 4.1.10 is a one-variable generalization of (1.5.11), it is natural to seek an application of Theorem 4.1.10 that simultaneously generalizes (1.6.8) and Theorem 4.1.15.

In Chapter 5, we discussed the work of Simon–Crippa–Collenberg [42], which provides an additional representation of Uchimura's identity (1.2.7):

$$\lim_{n\to\infty}\big(n-\mathbb{E}(\gamma_n^*(1))\big)=\sum_{n=1}^{\infty} nq^n(q^{n+1})_\infty=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{\frac{n(n+1)}{2}}}{(1-q^n)(q)_n}=\sum_{n=1}^{\infty} d(n)q^n.$$

Andrews, Crippa and Simon [13] further showed that

$$\lim_{n\to\infty}\mathrm{Var}(\gamma_n^*(1))=\sum_{n=1}^{\infty}\sigma(n)q^n=K_2(q).$$

Uchimura subsequently generalized his identity:

$$\sum_{n=1}^{\infty} n^k q^n(q^{n+1})_\infty=Y_k(K_1,K_2,\dots,K_k).$$

Combining Corollaries 4.1.14 and 5.1.3 yields

$$\begin{aligned}\lim_{n\to\infty}\Big(n^k-a_{n,k}(q)\Big)=\sum_{n=1}^{\infty} n^k q^n(q^{n+1})_\infty &=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{\binom{n+1}{2}}A_k(q^n)}{(1-q^n)^k(q)_n}\\ &=Y_k(K_1,K_2,\dots,K_k),\end{aligned}$$

where $a_{n,k}$ is defined in (5.1.1). Similarly, we obtained a limiting form of Dilcher's identity:

$$\begin{aligned}\lim_{n\to\infty}\left(\binom{n}{k}-a_{n,k}(q)\right)=\sum_{n=k}^{\infty}\binom{n}{k}q^n(q^{n+1})_\infty &=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{\binom{n+k}{2}-\binom{k}{2}}}{(1-q^n)^k(q)_n}\\ &=\sum_{j_1=1}^{\infty}\frac{q^{j_1}}{1-q^{j_1}}\cdots\sum_{j_k=1}^{j_{k-1}}\frac{q^{j_k}}{1-q^{j_k}},\end{aligned}$$

with $a_{n,k}$ defined in (5.1.2). These results provide limit expressions for Uchimura's identity (1.6.1) and Dilcher's identity (1.5.6), suggesting the broader problem of finding analogous limit forms for all generalizations of Uchimura-type identities within the Uchimura–Ramanujan–divisor framework studied in Chapter 3

We further generalized the work of Simon–Crippa–Collenberg [42] and Andrews–Crippa–Simon [13] by showing that for every $t\geq 1$, t-th cumulant is,

$$\lim_{n\to\infty}\kappa_t(Z_n)=K_t(q),\qquad Z_n=n-\gamma_n^*(1).$$

In Theorem 5.1.5, we explicitly evaluated the limits of the third, fourth and fifth cumulants using the limit forms (5.3.12) and Dilcher's identities [22, p. 85, Eq. (2.1)–(2.3)]. It would be interesting to extend this approach to obtain an alternative proof for the general t-th cumulant.

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