

Signal Analysis using Ramanujan and Affine Ramanujan Sums

M.Sc. Thesis

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DEPARTMENT OF MATHEMATICS
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**Signal Analysis using Ramanujan and Affine Ramanujan
Sums**

A THESIS

*Submitted in partial fulfillment of the requirements for the award of
the degree of*

Master of Science

by

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Under the guidance of

Dr. Niraj K. Shukla



**DEPARTMENT OF MATHEMATICS
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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **Signal Analysis using Ramanujan and Affine Ramanujan Sums** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY INDORE**, is an authentic record of my own work carried out during the time period from July 2025 to May 2026 under the supervision of **Dr. Niraj K. Shukla**, Professor, Department of Mathematics, IIT Indore. The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.



Signature of the student with date

21/05/2026

(Vishal Kumar)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.



Signature of Thesis Supervisor with date

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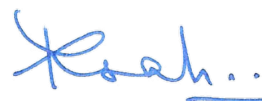
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Signature of Supervisor of M.Sc. Thesis

Date: **21-05-2026**



Signature of Convener, DPGC

Date: 21/05/2026

Dedicated to my
Mother and Father

“ Those times when you get up early and you work hard, those times when you stay up late and you work hard, those times when you don’t feel like working, you’re too tired, you don’t want to push yourself, but you do it anyway. That is actually the dream. That’s the dream. It’s not the destination, it’s the journey. And if you guys can understand that, then what you’ll see happen is you won’t accomplish your dreams, your dreams won’t come true; something greater will.”

– Kobe Bryant

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Abstract

This thesis studies the mathematical foundations and signal processing applications of Ramanujan sums, Ramanujan subspaces, and Ramanujan filter banks, together with their affine generalizations for non-stationary signal analysis. Ramanujan sums provide an efficient framework for detecting and representing periodic structures in discrete-time signals due to their orthogonality and arithmetic properties. Using these properties, finite-dimensional Ramanujan subspaces and finite Ramanujan representations are developed for signal decomposition and periodicity analysis.

The thesis further investigates Ramanujan filter banks constructed from Ramanujan sums and analyzes their frame-theoretic properties. In particular, sufficient conditions for frame and tight frame structures are established, ensuring stable signal representation and perfect reconstruction. These results demonstrate the usefulness of Ramanujan systems in signal denoising, reconstruction, and periodic component extraction.

To address the limitations of classical Ramanujan systems for non-stationary signals, affine Ramanujan sums and affine Ramanujan Fourier transforms are in-

troduced. Their orthogonality properties, recursive structures, and finite-dimensional representations are studied in detail. Affine Ramanujan subspaces and finite affine Ramanujan transforms are constructed for chirp and time-varying signal analysis. The thesis also discusses the potential development of affine Ramanujan filter banks for improved non-stationary signal processing applications.

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CHAPTER 1

Introduction

Signal processing is one of the most important areas of modern applied mathematics and engineering. It has wide applications in communication systems, radar technology, biomedical signal analysis, image processing, speech processing, and many other scientific fields. One of the central objectives in signal processing is to analyze and represent signals in terms of their frequency and periodic structures. Classical transforms such as the Fourier transform and wavelet transform play a fundamental role in this direction because they provide efficient tools for frequency analysis and signal decomposition.

For a discrete-time signal $x(n)$, the discrete Fourier transform (DFT) [9][5] is given by

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}},$$

which provides a representation of the signal in the frequency domain. Periodic signals naturally arise in many practical situations, and the identification of hidden periodic structures is essential in understanding signal behavior. In this

context, Ramanujan sums provide a remarkable mathematical framework for the study of periodicity in discrete-time signals. Ramanujan sums were introduced by the Indian mathematician Srinivasa Ramanujan in 1918 in connection with problems in analytic number theory. Due to their arithmetic and orthogonality properties, they later attracted considerable attention in harmonic analysis and signal processing.

The Ramanujan sum corresponding to a positive integer q [14][1] is defined by

$$C_q(n) = \sum_{\substack{1 \leq k \leq q \\ (k,q)=1}} \exp\left(\frac{2\pi i kn}{q}\right),$$

where the summation extends over all integers relatively prime to q . Unlike the conventional Fourier basis, Ramanujan sums are closely related to the divisibility structure of integers, which makes them highly effective in detecting periodic components in signals. Their orthogonality and recursive properties have led to the development of Ramanujan subspaces and Ramanujan Fourier representations for finite-dimensional signal spaces.

A Ramanujan subspace corresponding to a positive integer q is defined as

$$\mathcal{S}_q = \text{span} \{C_q(n), C_q(n-1), \dots, C_q(n - \varphi(q) + 1)\},$$

where $\varphi(q)$ denotes Euler's totient function. These subspaces consist of signals having exact period q and are mutually orthogonal for distinct values of q . Using shifted Ramanujan sums as basis elements, finite-duration signals can be decomposed into sums of orthogonal periodic components. Such representations are useful in periodicity estimation, denoising, reconstruction, and compression of signals.

Based on these ideas, Ramanujan filter banks were introduced as collections of filters generated from Ramanujan sums. Each filter extracts components associated with a specific period, this allows signals to be separated into different components based on their underlying periodic behavior [10]. A Ramanujan filter

corresponding to period q may be represented as

$$y_q(n) = (x * C_q)(n),$$

where $*$ denotes convolution. An important aspect of these filter banks is their frame-theoretic behavior[11]. In particular, the tight frame property guarantees stable signal representation and perfect reconstruction[3]. A collection $\{f_k\}_{k \in I}$ forms a tight frame for a Hilbert space $\mathcal{H}$ if

$$\sum_{k \in I} |\langle x, f_k \rangle|^2 = A \|x\|^2, \quad x \in \mathcal{H},$$

for some positive constant A .

However, classical Ramanujan sums are mainly suitable for stationary periodic signals and are not sufficiently effective for non-stationary signals whose spectral characteristics vary with time. Many practical signals, including radar signals[21, 15], biomedical signals, chirp signals, and communication signals[20, 26], exhibit non-stationary behavior. Therefore, there is a need to generalize Ramanujan-based methods for time-varying frequency analysis.

Affine Fourier analysis provides a natural framework for studying such signals. The affine Fourier transform generalizes the classical Fourier transform by incorporating chirp-modulated structures and affine parameters. The affine Fourier kernel may be written as

$$F_k(m, n) = \frac{1}{\sqrt{k}} \exp \left(j \frac{a}{2b} (m \Delta t)^2 - j \frac{2\pi mn}{k} + j \frac{d}{2b} (n \Delta u)^2 \right),$$

where

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1.$$

Combining affine Fourier analysis with Ramanujan sums leads to affine Ramanujan sums and affine Ramanujan Fourier transforms, which are more suitable for analyzing chirp-like and non-stationary signals.

The affine Ramanujan sum is defined by

$$\Psi_k^\alpha(m) = \sum_{\substack{n=1 \\ (n,k)=1}}^k F_k(m, n).$$

Affine Ramanujan sums retain several important properties analogous to those of classical Ramanujan sums, including orthogonality and recursive structures. Using these properties, affine Ramanujan subspaces and transform matrices can be constructed for finite-dimensional signal spaces. These developments provide a new multiresolution framework for chirp period estimation and non-stationary signal analysis.

The main objective of this thesis is to study the mathematical structure and signal processing applications of Ramanujan sums, Ramanujan subspaces, Ramanujan filter banks, and affine Ramanujan sums. Special emphasis is given to the frame-theoretic analysis of Ramanujan filter banks and their possible extensions toward affine Ramanujan filter banks for non-stationary signal analysis.

The thesis begins with a detailed study of Ramanujan sums and their mathematical properties. Fundamental results such as orthogonality relations, recursive formulas, autocorrelation properties, and discrete Fourier transform representations are established. The concept of Ramanujan subspaces is then introduced together with their basis structure, dimensionality, and orthogonality properties [29].

The next part of the thesis focuses on signal decomposition using Ramanujan sums. Two finite Ramanujan representations for finite-duration signals are developed. The first representation expresses a signal as a finite linear combination of Ramanujan sums, while the second representation utilizes Ramanujan subspaces and shifted Ramanujan sums to obtain structured orthogonal decompositions [19].

After establishing the decomposition framework, the thesis studies Ramanujan filter banks and their applications [16]. Ramanujan filters are analyzed through their frequency-selective properties and frame-theoretic structures. Sufficient conditions under which the associated filter bank forms a frame or a tight frame are established. Tight frame are useful in applications such as compressed sensing, denoising, and data compression [2][4][25][3]. The final part of the the-

sis introduces affine Ramanujan sums and affine Ramanujan Fourier transforms. Orthogonality relations, recursive structures, affine Ramanujan subspaces, and finite affine Ramanujan transforms are studied in detail. Two finite affine Ramanujan representations, namely ARFT-I and ARFT-II, are developed together with their matrix formulations and reconstruction properties.

Finally, the thesis concludes with future research directions related to affine Ramanujan filter banks and their frame properties. Since affine Ramanujan systems are capable of handling chirp-like and non-stationary signals, they may provide efficient tools for denoising, reconstruction, and time-varying signal analysis.

Overall, this thesis aims to build a bridge between number theory, harmonic analysis, frame theory, and signal processing by developing both theoretical and application-oriented aspects of Ramanujan and affine Ramanujan systems.

CHAPTER 2

Ramanujan Sums and its Properties

2.1 Ramanujan sums

Ramanujan sums is trigonometric summation which was introduced by Indian mathematician Srinivasa Ramanujan in 1918.

Definition 2.1. [14] For any positive integers $q \geq 1$ and any integer n , the Ramanujan sum $c_q(n)$ is defined by

$$c_q(n) = \sum_{\substack{1 \leq k \leq q \\ (k,q)=1}} \exp\left(\frac{2\pi i k n}{q}\right),$$

where the summation is taken over all integers k such that $1 \leq k \leq q$ and $\gcd(k, q) = 1$.

It can also be expressed in terms of the Möbius function μ and the divisor function as [27]

$$c_q(n) = \frac{\mu\left(\frac{q}{\gcd(q,n)}\right) \varphi(q)}{\varphi\left(\frac{q}{\gcd(q,n)}\right)},$$

where φ is Euler's totient function.

Equation (2.1) also shows that the DFT of $c_q(n)$ is

$$C_q[k] = \begin{cases} q & \text{if } (k, q) = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

For example, if $q = 10$, then $(k, q) = 1$ for $k \in \{1, 3, 7, 9\}$, thus

$$c_{10}(n) = e^{j2\pi n/10} + e^{j6\pi n/10} + e^{j14\pi n/10} + e^{j18\pi n/10}, \quad n \geq 1.$$

$$c_{10} = \{4, 1, -1, 1, -1, -4, -1, 1, -1, 1\}.$$

2.2 Properties of Ramanujan sums

One of the key properties of Ramanujan sum is that it is real valued

Theorem 2.1. [27] *The Ramanujan sum $c_q(n)$ is real valued sequence. Moreover, it is symmetric modulo q .*

Proof. If $(k, q) = 1$, then $(q - k, q) = 1$ as well. Further,

$$W_q^{-(q-k)n} = W_q^{kn} = \overline{W_q^{-kn}},$$

so the terms corresponding to k and $q - k$ occur as complex conjugate pairs.

Hence, their sum is real, and therefore the Ramanujan sum is real-valued. By

combining each conjugate pair, we obtain

$$W_q^{kn} + W_q^{-kn} = 2 \cos\left(\frac{2\pi kn}{q}\right),$$

which gives the cosine representation above.

Since

$$\cos\left(\frac{2\pi kn}{q}\right) = \cos\left(-\frac{2\pi kn}{q}\right),$$

it follows that $c_q(n) = c_q(-n)$. Finally, using the periodicity of Ramanujan sums modulo q , we obtain $c_q(-n) = c_q(q - n)$, and hence $c_q(n) = c_q(q - n)$. $\square$

Next, we present two significant aggregate characteristics of Ramanujan sums over a complete period modulo q . The first identity shows that the sum of all the values of $c_q(n)$ over a full period is zero. This is because Ramanujan sums are oscillatory and balanced in nature. The second identity evaluates the sum of the squares of these values, yielding $q\phi(q)$, which represents the energy of the sequence over one period. These relationships are essential in the harmonic analysis of arithmetic functions and play an important role in applications related to orthogonality and discrete Fourier transforms.

Theorem 2.2. [27] *Let $q > 1$ be a positive integer and let $c_q(n)$ denote the Ramanujan sum. Then the following identities hold:*

$$\sum_{n=0}^{q-1} c_q(n) = 0, \quad (2.2)$$

and

$$\sum_{n=0}^{q-1} c_q^2(n) = q\phi(q), \quad (2.3)$$

where $\phi(q)$ is Euler's totient function.

Proof. Recall that the Ramanujan sum is given by

$$c_q(n) = \sum_{\substack{1 \leq k \leq q \\ (k,q)=1}} e^{\frac{2\pi i kn}{q}}.$$

First, consider the sum

$$\sum_{n=0}^{q-1} c_q(n).$$

Substituting the definition of $c_q(n)$, we obtain

$$\sum_{n=0}^{q-1} c_q(n) = \sum_{n=0}^{q-1} \sum_{\substack{1 \leq k \leq q \\ (k,q)=1}} e^{\frac{2\pi i kn}{q}}.$$

Interchanging the order of summation gives

$$\sum_{\substack{1 \leq k \leq q \\ (k, q) = 1}} \sum_{n=0}^{q-1} e^{\frac{2\pi i k n}{q}}.$$

For each k with $(k, q) = 1$, the inner sum is a geometric progression with ratio

$$e^{\frac{2\pi i k}{q}} \neq 1,$$

since $q > 1$. Hence,

$$\sum_{n=0}^{q-1} e^{\frac{2\pi i k n}{q}} = 0.$$

Therefore,

$$\sum_{n=0}^{q-1} c_q(n) = 0.$$

Next, we evaluate the sum of squares. By Parseval's identity for the discrete Fourier transform,

$$\sum_{n=0}^{q-1} c_q^2(n) = \frac{1}{q} \sum_{k=0}^{q-1} |C_q(k)|^2,$$

where $C_q(k)$ denotes the discrete Fourier transform of $\{c_q(n)\}$.

For Ramanujan sums, the transform satisfies

$$|C_q(k)| = \begin{cases} q, & \text{if } (k, q) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence exactly $\phi(q)$ values of k contribute, and so

$$\sum_{k=0}^{q-1} |C_q(k)|^2 = \phi(q) q^2.$$

Substituting into Parseval's identity, we obtain

$$\sum_{n=0}^{q-1} c_q^2(n) = \frac{1}{q} \phi(q) q^2 = q\phi(q).$$

This completes the proof. □

Ramanujan sums form an important arithmetic functions that exhibit orthogonality properties comparable to those of exponential functions in Fourier analysis. This orthogonality plays a crucial role in number theory, especially in the study of multiplicative functions and expansions of arithmetic functions.

Theorem 2.3. [27][Orthogonality of Ramanujan sums] Let $q_1, q_2 \in \mathbb{N}$ and let

$$m = \text{lcm}(q_1, q_2).$$

Then the Ramanujan sums $c_{q_1}(n)$ and $c_{q_2}(n)$ satisfy the orthogonality relation

$$\sum_{n=0}^{m-1} c_{q_1}(n) c_{q_2}(n) = \begin{cases} q_1 \phi(q_1), & \text{if } q_1 = q_2, \\ 0, & \text{if } q_1 \neq q_2, \end{cases}$$

where ϕ denotes Euler's totient function. More generally, for any integer ℓ , we have

$$\sum_{n=0}^{m-1} c_{q_1}(n) c_{q_2}(n - \ell) = 0, \quad \text{whenever } q_1 \neq q_2.$$

Proof. Recall that the Ramanujan sum admits the exponential representation

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^q e^{2\pi i k n / q}.$$

Let us write

$$W_q := e^{2\pi i / q}.$$

Then

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^q W_q^{k n}.$$

Consider

$$S := \sum_{n=0}^{m-1} c_{q_1}(n) c_{q_2}(n - \ell).$$

Substituting the exponential form,

$$S = \sum_{n=0}^{m-1} \left(\sum_{\substack{k_1=1 \\ (k_1, q_1)=1}}^{q_1} W_{q_1}^{k_1 n} \right) \left(\sum_{\substack{k_2=1 \\ (k_2, q_2)=1}}^{q_2} W_{q_2}^{k_2 (n - \ell)} \right).$$

Rearranging sums,

$$S = \sum_{\substack{k_1=1 \\ (k_1, q_1)=1}}^{q_1} \sum_{\substack{k_2=1 \\ (k_2, q_2)=1}}^{q_2} W_{q_2}^{-k_2 \ell} \sum_{n=0}^{m-1} W_{q_1}^{k_1 n} W_{q_2}^{k_2 n}.$$

Now, since $m = \text{lcm}(q_1, q_2)$, there exist integers ℓ_1, ℓ_2 such that

$$m = q_1 \ell_1 = q_2 \ell_2.$$

Thus,

$$W_{q_1}^{k_1 n} = W_m^{k_1 \ell_1 n}, \quad W_{q_2}^{k_2 n} = W_m^{k_2 \ell_2 n}.$$

Hence the inner sum becomes

$$\sum_{n=0}^{m-1} W_m^{(k_2 \ell_2 - k_1 \ell_1) n}.$$

Using the standard identity for roots of unity,

$$\sum_{n=0}^{m-1} W_m^{rn} = \begin{cases} m, & \text{if } r \equiv 0 \pmod{m}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the sum is zero unless

$$k_2 \ell_2 - k_1 \ell_1 \equiv 0 \pmod{m}.$$

But since $1 \leq k_i < q_i$, we have

$$|k_2 \ell_2 - k_1 \ell_1| < m,$$

so the only possibility is

$$k_2 \ell_2 - k_1 \ell_1 = 0,$$

which implies

$$\frac{k_2}{q_2} = \frac{k_1}{q_1}.$$

Since $(k_1, q_1) = 1$ and $(k_2, q_2) = 1$, this equality can hold only if

$$q_1 = q_2 \quad \text{and} \quad k_1 = k_2.$$

Therefore, if $q_1 \neq q_2$, all inner sums vanish, and hence

$$S = 0.$$

This proves the orthogonality relation

$$\sum_{n=0}^{m-1} c_{q_1}(n) c_{q_2}(n - \ell) = 0 \quad (q_1 \neq q_2).$$

Finally, when $q_1 = q_2 = q$, only the diagonal terms contribute, giving

$$\sum_{n=0}^{q-1} c_q(n)^2 = q\phi(q),$$

which completes the proof. □

Next we discuss some additional properties that will be used in later parts of the

thesis.

Theorem 2.4. [\[27\]](#) *[Recursive Formula for Ramanujan Sums] The Ramanujan sum $c_q(n)$ can be expressed recursively in terms of Ramanujan sums of lower order divisors of q as*

$$c_q(n) = q \delta((n)_q) - \sum_{\substack{q_k | q \\ q_k < q}} c_{q_k}(n), \quad (2.4)$$

where $q_k | q$ denotes that q_k runs over all positive divisors of q less than q , and $\delta((n)_q)$ is defined as

$$\delta((n)_q) = \begin{cases} 1, & q | n, \\ 0, & q \nmid n. \end{cases}$$

Proof. A fundamental identity satisfied by Ramanujan sums is

$$\sum_{d|q} c_d(n) = q \delta((n)_q), \quad (2.5)$$

where the summation is taken over all positive divisors d of q . Separating the term corresponding to $d = q$ from the above sum, we obtain

$$c_q(n) + \sum_{\substack{d|q \\ d < q}} c_d(n) = q \delta((n)_q).$$

Rearranging the terms gives

$$c_q(n) = q \delta((n)_q) - \sum_{\substack{d|q \\ d < q}} c_d(n),$$

which proves the recursive relation [\(2.4\)](#). □

The recursive formula is particularly useful for computational purposes, since it allows $c_q(n)$ to be determined successively from lower-order Ramanujan sums, beginning with the initial value

$$c_1(n) = 1.$$

For example,

$$c_{10}(n) = 10 \delta((n)_{10}) - c_1(n) - c_2(n) - c_5(n),$$

and

$$c_5(n) = 5 \delta((n)_5) - c_1(n).$$

Thus, only simple addition and subtraction operations are required.

2.3 Ramanujan subspace S_q

In this section, we show that starting from the Ramanujan sum $c_q(n)$, one can generate an entire subspace S_q of period- q signals with dimension $\phi(q)$, which we call the *Ramanujan subspace*. Such a subspace captures a much broader class of arithmetic structures representable via $c_q(n)$.

The purpose of this section is to define this subspace and develop some of its fundamental mathematical properties. Starting from $c_q(n)$, we first define the $q \times q$ integer circulant matrix as follows:

$$\mathbf{B}_q = \begin{bmatrix} c_q(0) & c_q(q-1) & c_q(q-2) & \cdots & c_q(1) \\ c_q(1) & c_q(0) & c_q(q-1) & \cdots & c_q(2) \\ c_q(2) & c_q(1) & c_q(0) & \cdots & c_q(3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_q(q-2) & c_q(q-3) & c_q(q-4) & \cdots & c_q(q-1) \\ c_q(q-1) & c_q(q-2) & c_q(q-3) & \cdots & c_q(0) \end{bmatrix} \quad (2.6)$$

The matrix $\mathbf{B}_q$ is circulant, meaning that each row is a cyclic shift of the previous one. This structure plays a key role in analyzing the Ramanujan subspace S_q and its algebraic properties. There are several characteristics of the matrix:

- The 0th row is the time-reversed Ramanujan sum sequence $c_q(-n)$.
- Since $c_q(-n) = c_q(n)$, the matrix B_q is symmetric.
- The matrix is Toeplitz because it is circulant, meaning that all entries along any line parallel to the principal diagonal are identical.

The Ramanujan space S_q is defined as the column space of B_q . It is obvious that the sequence $c_q(n)$ and all of its circularly shifted versions belong to the space S_q . The space $S_q \subset \mathbb{C}^q$ has dimension $\phi(q)$, since the matrix B_q has rank $\phi(q)$.

Any circulant matrix is diagonalized by the discrete Fourier transform (DFT) matrix. Thus,

$$B_q = W^{-1}A_qW, \quad (2.7)$$

where W is the $q \times q$ DFT matrix, and

$$A_q = \begin{bmatrix} C_q[0] & 0 & 0 & \cdots & 0 \\ 0 & C_q[1] & 0 & \cdots & 0 \\ 0 & 0 & C_q[2] & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C_q[q-1] \end{bmatrix}. \quad (2.8)$$

Here, $C_q[k]$ denotes the DFT coefficients of the Ramanujan sum sequence $c_q(n)$, given by

$$C_q[k] = \sum_{n=0}^{q-1} c_q(n) e^{-j2\pi nk/q} = \begin{cases} q, & \text{if } (k, q) = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (2.9)$$

Since $B_q = W^{-1}A_qW$, it follows that

$$B_qW = WA_q.$$

Because $W/\sqrt{q}$ is unitary, we also have $W^{-1} = W^H/q$, where W^H denotes the Hermitian transpose of W . Since both $c_q(n)$ and $C_q[k]$ are real-valued, one may equivalently write

$$B_qW = WA_q.$$

Hence, the eigenvectors of B_q are represented by the columns of W , and the associated eigenvalues are the diagonal entries $C_q[k]$. Since exactly $\phi(q)$ of the entries $C_q[k]$ are nonzero, the rank of the circulant matrix B_q , and consequently the dimension of the Ramanujan subspace, is

$$\dim(S_q) = \phi(q).$$

The circulant matrix B_q can therefore be factorized as

$$B_q = VV^*, \quad (2.10)$$

where V is a $q \times \phi(q)$ submatrix of the DFT matrix W , obtained by retaining only those columns indexed by k such that $(k, q) = 1$. Thus, the column space of B_q is precisely the column space of V , and is spanned by the $\phi(q)$ columns.

Let $W = e^{-2\pi i/q}$ denote the primitive q th root of unity. Consider the following set of column vectors:

$$\begin{bmatrix} 1 \\ W^{k_1} \\ W^{2k_1} \\ \vdots \\ W^{(q-1)k_1} \end{bmatrix} \quad \begin{bmatrix} 1 \\ W^{k_2} \\ W^{2k_2} \\ \vdots \\ W^{(q-1)k_2} \end{bmatrix} \quad \dots \quad \begin{bmatrix} 1 \\ W^{k_{\phi(q)}} \\ W^{2k_{\phi(q)}} \\ \vdots \\ W^{(q-1)k_{\phi(q)}} \end{bmatrix},$$

where $(k_i, q) = 1$ for each i . These vectors correspond to selected columns of the $q \times q$ Discrete Fourier Transform (DFT) matrix. Therefore, the connection between the DFT matrix and the Ramanujan subspace S_q can be summarized as follows:

The Ramanujan subspace $S_q \subset \mathbb{C}^q$ is identical to the subspace spanned by those $\phi(q)$ columns of the $q \times q$ DFT matrix whose column indices k are coprime to q .

Thus, the Ramanujan subspace S_q is given by the span

$$S_q = \text{span} \{ W_q^{kn} = e^{-2\pi i kn/q} : 1 \leq k \leq q, (k, q) = 1 \},$$

where (k, q) denotes the greatest common divisor of k and q .

Observe that W_q^k is a primitive q th root of unity if and only if $(k, q) = 1$. In this case,

$$W_q^{kq} = 1 \quad \text{only when } n \text{ is a multiple of } q.$$

Equivalently, if $(k, q) = 1$, then the sequence

$$x(n) = W_q^{kn},$$

has fundamental period q .

Theorem 2.5. [27] *Let M be a submatrix of the circulant matrix $\mathbf{B}_q$ obtained by retaining any consecutive $\phi(q)$ columns, i.e., $r, r+1, \dots, r+\phi(q)-1$ for some r . Then submatrix M has rank $\phi(q)$.*

Proof. Based on (2.10), the matrix $\mathbf{B}_q$ can be expressed as

$$\mathbf{B}_q = VV^*,$$

where V is a $q \times \phi(q)$ matrix and V^* denotes its conjugate transpose. Consequently, the submatrix mentioned in the theorem can be represented as

$$M = VV_1,$$

where V_1 is a $\phi(q) \times \phi(q)$ matrix. Now, let us consider the j th row of the V_1 matrix, which can be written as

$$\begin{bmatrix} a_j^r \\ a_j^{r+1} \\ \vdots \\ a_j^{r+\phi(q)-1} \end{bmatrix} = a_j^r \begin{bmatrix} 1 \\ a_j \\ \vdots \\ a_j^{\phi(q)-1} \end{bmatrix} \quad (2.11)$$

where

$$a_j = e^{-2\pi i k_j / q}, \quad (k_j, q) = 1.$$

Considering (2.11), the matrix V_1 can be expressed as the product of a diagonal matrix and a Vandermonde matrix. This is possible because the values of a_j are distinct for different rows. Specifically, the matrix V_1 takes the form

$$V_1 = \Lambda V_2,$$

where Λ represents a diagonal matrix with non-zero diagonal elements, and V_2 denotes a $\phi(q) \times \phi(q)$ row-wise Vandermonde matrix.

Consequently, the matrix V_1 is non-singular due to the presence of the non-zero diagonal elements in Λ and the Vandermonde structure of V_2 . Additionally, considering the definition of V , which is a column-wise Vandermonde matrix with

$\phi(q)$ distinct columns, it can be inferred that V has rank $\phi(q)$. As a result, the product matrix $M = VV_1$ also has rank $\phi(q)$. $\square$

Theorem 2.6. [27] *The Ramanujan sum $c_q(n)$ and its $\phi(q)-1$ consecutive shifts, that is, the first $\phi(q)$ columns of the circular matrix $\mathbf{B}_q$, constitute an integer basis for the Ramanujan space S_q .*

Proof. According to the Theorem (2.5) mentioned, it can be established that the first $\phi(q)$ columns of the circular matrix $\mathbf{B}_q$ exhibit linear independence. This implies that these columns form a linearly independent set. Additionally, the dimension of the Ramanujan space S_q is precisely $\phi(q)$. As a consequence, the Ramanujan sum $c_q(n)$, along with its $\phi(q)$ consecutive shifts, serves as an integer basis for the Ramanujan space S_q . In other words, they form a set of integers that spans the entirety of S_q and are linearly independent, allowing them to represent any element within S_q uniquely. $\square$

In simple terms, the integer basis for the Ramanujan space S_q can be represented by the following set:

$$\{c_q(n), c_q(n-1), \dots, c_q(n-\phi(q)+1)\}.$$

This set of $\phi(q)$ circularly shifted Ramanujan sequences forms an integer basis for S_q . Consequently, any signal $x(n)$ belonging to S_q can be expressed in either of the following equivalent forms:

$$x(n) = \sum_{i=0}^{\phi(q)-1} b_i c_q(n-i) = \sum_{\substack{k=1 \\ (k,q)=1}}^q a_k W_q^{kn},$$

where b_i represents the coefficients in the integer basis and a_k denotes the coefficients in the complex basis $\{W_q^{kn} : 1 \leq k \leq q, (k, q) = 1\}$. Both representations hold true for expressing signals within the Ramanujan space S_q .

One of the key properties of the Ramanujan space S_q is that any signal belonging to this space has a period of q . This property can be stated as follows:

Theorem 2.7. [8] Any signal $x(n)$ in the Ramanujan space S_q has period q unless $x(n) = 0$ for all n .

Proof. Any signal $x(n) \in S_q$ can be expressed as

$$x(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^q a_k e^{-2\pi i k n / q}.$$

Now, let us consider $x(n+q)$:

$$x(n+q) = \sum_{\substack{k=1 \\ (k,q)=1}}^q a_k e^{-2\pi i k (n+q) / q} = \sum_{\substack{k=1 \\ (k,q)=1}}^q a_k e^{-2\pi i k n / q} e^{-2\pi i k} = \sum_{\substack{k=1 \\ (k,q)=1}}^q a_k e^{-2\pi i k n / q} = x(n).$$

Since we have shown that $x(n) = x(n+q)$, it remains to prove that the period is not a proper divisor of q . We will use proof by contradiction. Suppose P is the period of $x(n)$, so that $P \mid q$. Assume that $P < q$, implying $q = Pr$ for some integer $r > 1$. Since $x(n) = x(n+P)$, we can express $x(n)$ uniquely in terms of the P -point DFT as follows:

$$x(n) = \sum_{\ell=0}^{P-1} b_\ell e^{-2\pi i \ell n / P}. \quad (2.12)$$

Since $q = Pr$, (2.12) can be rewritten as:

$$x(n) = \sum_{\ell=0}^{P-1} b_\ell e^{-2\pi i \ell r n / q}. \quad (2.13)$$

Both (2.12) and (2.13) can be considered as q -point DFT representations of $x(n)$ for $0 \leq n \leq q-1$. Therefore, the terms of the summation must match. This implies that there exists a pair k and ℓ such that $k = r\ell$. Hence, $(k, q) = r > 1$, which contradicts the condition $(k, q) = 1$. Therefore, it can be concluded that $x(n) \in S_q$ has a period of q unless $x(n) = 0$ for all n . $\square$

The second property of the Ramanujan space is its orthogonality for different values of q_i and q_j . Now, we present a theorem that establishes the orthogonality within the Ramanujan space.

Theorem 2.8. [27] The spaces S_{q_i} and S_{q_j} are orthogonal for $q_i \neq q_j$.

Proof. Suppose we consider m , which is the least common multiple (lcm) of (q_i, q_j) , where q_i and q_j are distinct elements. We can express m as

$$m = q_i \ell_i = q_j \ell_j$$

for some integers ℓ_i and ℓ_j . By rearranging terms, we obtain:

$$\sum_{n=0}^{m-1} W_{q_i}^{k_i n} W_{q_j}^{-k_j n} = \sum_{n=0}^{m-1} W_m^{(k_i \ell_i - k_j \ell_j) n}.$$

This sum evaluates to zero unless $k_i \ell_i - k_j \ell_j = mk$ for some $k \in \mathbb{N}$. However, since $1 \leq k_\ell \leq q_\ell$, it follows that $k_\ell \ell_\ell < m$. As a result,

$$|k_i \ell_i - k_j \ell_j| < m,$$

implying that the sum is zero unless $k_i \ell_i - k_j \ell_j = 0$. In other words,

$$\frac{k_i}{q_i} = \frac{k_j}{q_j} \quad \text{or equivalently} \quad \frac{k_i}{\ell_i} = \frac{k_j}{\ell_j}.$$

Nevertheless, this scenario is not feasible since both k_i and k_j are coprime with q_i and q_j , respectively, denoted by $(k_i, q_i) = (k_j, q_j) = 1$. Thus, the assumption $\frac{k_i}{q_i} = \frac{k_j}{q_j}$ cannot hold, leading to a contradiction. $\square$

The elements in the space S_q exhibit the following exquisite autocorrelation properties:

Theorem 2.9. [27] *The circular autocorrelation of the Ramanujan sum $c_q(n)$ is expressed as follows:*

$$r_{qq}(l) = \sum_{n=0}^{q-1} c_q(n) c_q(n-l) = q c_q(l),$$

where the term $(n-l)$ is considered modulo q due to the periodic nature of $c_q(n)$ with period q .

Proof. Given $r_{qq}(l) = \sum_{n=0}^{q-1} c_q(n) c_q(n-l)$. So by (2.1) the DFT of $r_{qq}(l)$ is given by

$$R_{qq}[k] = C_q[k] C_q^*[k] = \begin{cases} q^2 & \text{if } (k, q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $R_{qq}[k] = q C_q[k]$. So $r_{qq}(l) = q c_q(l)$ indeed. $\square$

Building upon these mathematical properties, the next chapter will bridge its properties and digital signal processing. We will investigate Signal Decomposition using Ramanujan Sums, focusing specifically on time-invariant systems through the lens of the First and Second Ramanujan Finite Impulse Response (FIR) representations. This mathematical framework will serve as the core engine for the signal analysis applications developed throughout the remainder of this thesis.

CHAPTER 3

Signal Decomposition using Ramanujan Sums

Ramanujan showed that several standard arithmetic functions in number theory can be expressed as a linear combination of the form

$$x(n) = \sum_{q=1}^{\infty} \alpha_q c_q(n), \quad n \geq 1. \quad (3.1)$$

This is called the *Ramanujan sum expansion* of $x(n)$.

The purpose is to develop Ramanujan sum expansions for finite duration (FIR) signals based on these Ramanujan spaces, and thereby to develop techniques to identify periodic components in signals. It turns out that (3.1) is not useful for this purpose. Instead, we develop two different approaches to expand FIR signals in terms of Ramanujan sums. The second approach is particularly useful for identifying periodic components.

3.1 First Ramanujan FIR representation

We consider the finite Ramanujan sum expansion of the form

$$x(n) = \sum_{q=1}^N a_q c_q(n), \quad 0 \leq n \leq N-1. \quad (3.2)$$

This can be written in matrix-vector form as

$$\underbrace{\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}}_{\mathbf{x}} = \mathbf{A}_N \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}}_{\mathbf{a}}. \quad (3.3)$$

The q th column of $\mathbf{A}_N$ is formed by the sequence $c_q(n)$, repeated periodically with period q until N samples are obtained, i.e.,

$$\mathbf{A}_N(:, q) = \begin{bmatrix} c_q(0) \\ c_q(1) \\ \vdots \\ c_q(N-1) \end{bmatrix}, \quad 1 \leq q \leq N. \quad (3.4)$$

Theorem 3.1. [28] *The matrix $\mathbf{A}_N$ has full rank N .*

Proof. The recursion relation shows that each column $\mathbf{c}_m$ can be written in the form

$$\mathbf{c}_m = \mathbf{e}_m - \mathbf{c}_{m_1} - \mathbf{c}_{m_2} - \cdots, \quad (3.5)$$

where $\mathbf{e}_m$ is a vector having the value m at positions $0, m, 2m, \dots$, and zeros elsewhere. Here, $\mathbf{c}_{m_i}$ are columns of the matrix $\mathbf{A}_N$ corresponding to indices $m_i < m$ that are factors of m .

This representation allows us to perform elementary column operations on $\mathbf{A}_N$ to reduce it to a simpler form. In particular, an elementary column operation consists of replacing a column $\mathbf{c}_m$ by $\mathbf{c}_m + \alpha \mathbf{c}_l$, where $\alpha \neq 0$ and $l \neq m$. This is equivalent to multiplying $\mathbf{A}_N$ on the right by a matrix of the form $(\mathbf{I} + \alpha \mathbf{J})$, where $\mathbf{J}$ is a matrix with all zero entries except for a 1 in the (l, m) -th position. Such

elementary column operations do not change the determinant, and hence do not change the rank of the matrix. Starting from the N th column, we can successively apply these operations to eliminate contributions from columns corresponding to its proper divisors. Repeating this process for all columns, the matrix $\mathbf{A}_N$ can be reduced to a diagonal form:

$$\mathbf{A}_N \sim \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \end{bmatrix}. \quad (3.6)$$

Since the resulting matrix is diagonal with all nonzero entries on the diagonal, its determinant is nonzero. Therefore, $\mathbf{A}_N$ has full rank N . $\square$

Example 3.1. Consider the signal

$$x(n) = c_{24}(n), \quad 0 \leq n \leq 255,$$

where $c_{24}(n)$ denotes the Ramanujan sum of period 24. The corresponding coefficient magnitudes are shown in (3.1). In this case, a single dominant peak occurs at $k = 24$, confirming that the Ramanujan representation accurately captures the exact periodic structure of the signal.

Now consider a circularly shifted version of the Ramanujan sum,

$$x(n) = c_{24}(n - 3),$$

with signal length $N = 256$. Since the signal has period $M = 24$, the corresponding coefficient magnitudes are displayed in (3.2). Unlike the previous case, several coefficients become significant, and the energy spreads across multiple indices. This demonstrates that the first finite Ramanujan representation is sensitive to circular shifts of the signal. Consequently, shifted periodic signals may not always be represented by a single dominant Ramanujan coefficient.

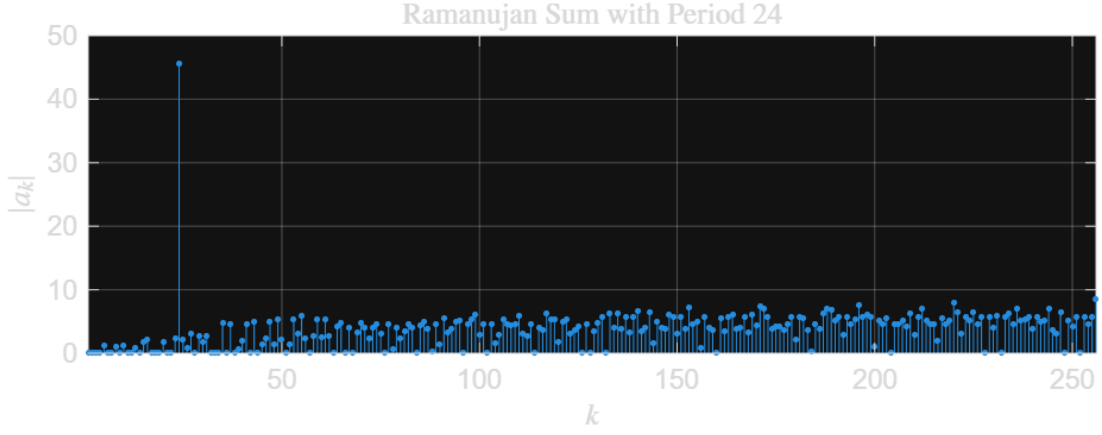


Figure 3.1: $x(n) = c_{24}(n)$

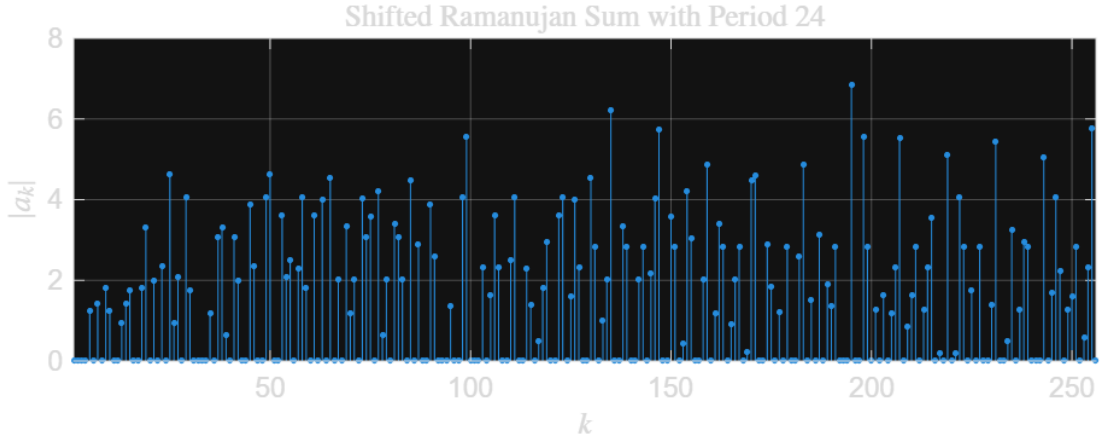


Figure 3.2: $x(n) = c_{24}(n - 3)$.

From the above figure, it is evident that the coefficient distribution is not concentrated at a single value of k . Hence, we cannot directly conclude that $x(n)$ consists of only one periodic component.

3.2 Second Ramanujan FIR representation

As discussed in the earlier section, the representation using Ramanujan sums does not, in general, possess orthogonality properties.

we know that each Ramanujan sum $c_q(n)$ generates an entire subspace S_q of

dimension $\phi(q)$. By combining such subspaces corresponding to different values of q , we can construct representations for finite-duration (FIR) sequences.

In (3.1), we represented $x(n)$ as

$$\mathbf{x} = [\mathbf{c}_{q_1} \quad \mathbf{c}_{q_2} \quad \cdots \quad \mathbf{c}_{q_K}] \mathbf{d}, \quad (3.7)$$

where q_i are the K divisors of N , with $K < N$. However, each $c_q(n)$ corresponds to only a single vector in the Ramanujan subspace S_q , whose dimension is $\phi(q)$. To fully exploit the structure of S_q , we replace each $\mathbf{c}_{q_i}$ by a basis for the corresponding subspace:

$$\mathbf{G}_{q_i} = [\mathbf{c}_{q_i} \quad \mathbf{c}_{q_i}^{(1)} \quad \cdots \quad \mathbf{c}_{q_i}^{(\phi(q_i)-1)}], \quad (3.8)$$

where $\mathbf{c}_{q_i}^{(k)}$ denotes a circular downward shift of $\mathbf{c}_{q_i}$ by k samples. Thus, $\mathbf{G}_{q_i}$ is an $N \times \phi(q_i)$ matrix whose columns form a basis for the subspace S_{q_i} . now we replace $\mathbf{c}_{q_i}$ by $\mathbf{G}_{q_i}$ for each q_i then we get

$$\mathbf{F}_N = [\mathbf{G}_{q_1} \quad \mathbf{G}_{q_2} \quad \cdots \quad \mathbf{G}_{q_K}].$$

This has N rows and $\sum_{q_i|N} \phi(q_i)$ columns. But it is well-known that

$$\sum_{q_i|N} \phi(q_i) = N.$$

We observe two crucial properties of $\mathbf{F}_N$

Theorem 3.2. [28][Orthogonality] *Let $\mathbf{F}_N$ be the transform matrix and let $\mathbf{G}_{q_i}$ and $\mathbf{G}_{q_k}$ be its submatrices corresponding to q_i and q_k , respectively. If $i \neq k$, then the column spaces of $\mathbf{G}_{q_i}$ and $\mathbf{G}_{q_k}$ span orthogonal subspaces of $\mathbb{C}^N$.*

Proof. Since $q_i \mid N$, each column in (3.8) has an integer number of periods of $c_{q_i}(n)$. Therefore, in view of the orthogonality property given in Theorem (2.2), it follows that the column space of $\mathbf{G}_{q_i}$ is orthogonal to that of $\mathbf{G}_{q_k}$ for $i \neq k$. $\square$

Theorem 3.3. [28] *The matrix $\mathbf{F}_N$ has full rank N , so that any $N \times 1$ vector $\mathbf{x}$ can be represented in the form*

$$\mathbf{x} = \mathbf{F}_N \mathbf{b}. \quad (3.9)$$

Thus $\mathbf{F}_N$ defines an integer basis for $\mathbb{C}^N$.

Proof. The $\phi(q_i)$ columns of each $\mathbf{C}_{q_i}$ are independent, and form a basis for the $\phi(q_i)$ -dimensional space S_{q_i} , as shown in theorem 5 of [27]. Since each $\mathbf{G}_{q_i}$ has rank $\phi(q_i)$ and the column spaces of $\mathbf{G}_{q_i}$ are orthogonal for different q_i , it follows that $\mathbf{F}_N$ has rank

$$\sum_{q_i|N} \phi(q_i) = N.$$

□

Now let us examine the implications of this Ramanujan representation. From (3.8) of $\mathbf{F}_N$, we have

$$\mathbf{x} = \begin{bmatrix} \mathbf{G}_{q_1} & \mathbf{G}_{q_2} & \cdots & \mathbf{G}_{q_K} \end{bmatrix} \mathbf{b} = \sum_{i=1}^K \mathbf{G}_{q_i} \mathbf{b}_{q_i}. \quad (3.10)$$

Let $\mathbf{x}_{q_i} = \mathbf{G}_{q_i} \mathbf{b}_{q_i}$. Then,

$$\mathbf{x} = \sum_{i=1}^K \mathbf{x}_{q_i}. \quad (3.11)$$

Since the columns of $\mathbf{G}_{q_i}$ are circularly shifted versions of $c_{q_i}(n)$, the components of $\mathbf{x}_{q_i}$ are given by

$$x_{q_i}(n) = \sum_{l=0}^{\phi(q_i)-1} \beta_{i,l} c_{q_i}(n-l). \quad (3.12)$$

In addition to the first representation based on Ramanujan subspaces, an alternative formulation expresses a finite-length sequence directly as a linear combination of shifted Ramanujan sums. This provides a constructive and structured way to decompose signals in terms of arithmetic basis functions associated with the divisors of the sequence length.

Definition 3.1. [28]/[Second Ramanujan FIR Representation] Any sequence $x(n)$ of length N can be represented as

$$x(n) = \sum_{q_i|N} \sum_{l=0}^{\phi(q_i)-1} \beta_{il} c_{q_i}(n-l), \quad (3.13)$$

where q_i are the positive divisors of N , $\phi(\cdot)$ denotes Euler's totient function, β_{il} are expansion coefficients, and $c_{q_i}(n)$ is the Ramanujan sum of order q_i .

This representation shows that any finite-length discrete-time signal can be decomposed into a superposition of shifted Ramanujan sums corresponding to the divisors of N . The total number of basis functions involved is

$$\sum_{q_i|N} \phi(q_i) = N,$$

which ensures completeness of the representation. The coefficients β_{il} determine the contribution of each shifted Ramanujan component to the overall signal.

Theorem 3.4. [28] *The $N \times N$ matrix $\mathbf{F}_N$ is an orthogonal matrix (i.e., its columns are pairwise orthogonal) if and only if N is a power of two, i.e., $N = 2^m$.*

Proof. When $N = 2^m$, all its divisors q_i are powers of two. From Lemma 2 in [27], each of the matrices $\mathbf{C}_{q_i}$ is orthogonal, i.e., $\mathbf{C}_{q_i}^T \mathbf{C}_{q_i}$ is diagonal, so the same holds for $\mathbf{G}_{q_i}$. Furthermore, the columns of $\mathbf{G}_{q_i}$ and $\mathbf{G}_{q_k}$ are orthogonal for $i \neq k$ (Theorem (2.2)). So the columns of $\mathbf{F}_N$ are pairwise orthogonal.

For the converse, assume N is not a power of two, so it has a prime factor $q_i > 2$. We know from Lemma 2 of [27] that $\mathbf{C}_{q_i}$ (hence $\mathbf{G}_{q_i}$) is not orthogonal, so $\mathbf{F}_N$ is not orthogonal. $\square$

The first and second finite Ramanujan representations are both useful for identifying hidden periodic patterns in finite-length signals. However, their performance depends on the nature of the signal as well as the relationship between the signal's true period and the data length. The first Ramanujan representation expresses a signal directly in terms of Ramanujan sums. In many situations, this method identifies the hidden period very effectively by producing a strong peak at the correct period value. The representation clearly detects the correct period. This shows that the first representation can successfully identify periodicities even when the signal period is not a divisor of the data length. On the other hand, the second Ramanujan representation is based on projections onto Ramanujan subspaces. This approach works particularly well when the hidden period divides the signal length. An important advantage of this method is that

it is less sensitive to circular shifts of the signal because shifted versions of Ramanujan sums are embedded within the subspace framework. As a result, shifted periodic signals that may not be detected clearly by the first representation can often be identified accurately by the second one.

Despite this advantage, the second representation also has limitations. When the signal period is not a divisor of the data length, the projection energy spreads over several subspaces instead of concentrating at a single period. In such cases, identifying the true hidden period becomes difficult.

CHAPTER 4

Ramanujan Filter Banks

Ramanujan sums and Ramanujan subspaces have gained attention in signal processing due to their ability to extract periodic components from discrete-time signals. The primary objective of our study is to analyze the perfect reconstruction properties of the filter bank associated with Ramanujan sums, also known as Ramanujan filter bank.

4.1 Ramanujan filters and filter banks

Definition 4.1. [23] *If the Ramanujan sum $c_q(n)$ is regarded as the impulse response of a digital filter, then from (2.1) its frequency response is given by*

$$C_q(e^{j\omega}) = 2\pi \sum_{\substack{1 \leq k \leq q \\ (k,q)=1}} \delta\left(\omega - \frac{2\pi k}{q}\right), \quad 0 \leq \omega < 2\pi, \quad (4.1)$$

and is periodically extended with period 2π outside this interval. Thus, $C_q(e^{j\omega})$ is zero everywhere except at the coprime frequencies $2\pi k/q$, where k is coprime to

q .

A collection of such filters $\{C_1(e^{j\omega}), C_2(e^{j\omega}), \dots, C_Q(e^{j\omega})\}$ forms a *Ramanujan filter bank*. Each filter extracts the signal components corresponding to period q , thereby enabling decomposition of a signal into exactly periodic components of different periods.

Lemma 4.1. [23] *The Ramanujan filter $C_q(e^{j\omega})$ can produce a nonzero output in response to a period- P signal $x(n)$ only if q is a divisor of P .*

Proof. Let $x(n)$ be a discrete-time signal with period P . Since $x(n)$ is periodic, it can be represented by its discrete Fourier series as

$$x(n) = \sum_{k=0}^{P-1} X_k e^{j\frac{2\pi k}{P}n},$$

where X_k are the Fourier series coefficients. The Ramanujan filter $C_q(e^{j\omega})$ is designed to pass only those frequency components corresponding to the Ramanujan subspace of period q . These frequencies are of the form

$$\omega_l = \frac{2\pi l}{q}, \quad l \in \{1, 2, \dots, q\},$$

where $\gcd(l, q) = 1$. For the filter output to be nonzero, at least one frequency component of $x(n)$ must coincide with one of the frequencies passed by $C_q(e^{j\omega})$. Therefore, there must exist integers k and l such that $\frac{2\pi k}{P} = \frac{2\pi l}{q}$. Equivalently, $\frac{k}{P} = \frac{l}{q}$, which implies $kq = lP$.

Since k and l are integers, this equality can hold only when q divides P , i.e., $q \mid P$. Hence, the Ramanujan filter $C_q(e^{j\omega})$ produces a nonzero output for a period- P signal only if q is a divisor of P . Therefore, the lemma is proved. $\square$

Theorem 4.2. [23][24][Ramanujan filter banks] *Consider a Ramanujan analysis filter bank $\{c_q(n)\}$ with $1 \leq q \leq N$, and let $x(n)$ be a period- P input signal with $1 \leq P \leq N$. Then non-zero outputs can only be produced by those filters*

$c_q(n)$ such that the filter index q is a divisor of P . Furthermore, assuming non-zero outputs are produced by the subset of filters $c_{q_1}(n)$ with periods $q_1, q_2, \dots, q_K$, the input period P is exactly equal to $\text{lcm}(q_1, q_2, \dots, q_K)$.

Proof. We already know that nonzero outputs can only be produced by Ramanujan filters with periods $q \mid P$. So clearly P is a common multiple of q_i . Suppose it is not the lcm. Let $L = \text{lcm}\{q_1, q_2, \dots, q_K\}$ so that $P = LM$ for some integer $M > 1$.

Now, $x(n)$ has the frequency components $2\pi l/P$ for $0 \leq l \leq P-1$ where $1 \leq P \leq N$. The filter $c_q(n)$ on the other hand passes the components $2\pi k/q$ where $1 \leq k \leq q$ with $(k, q) = 1$. But the set of rationals l/P where $1 \leq l \leq P$ and $1 \leq P \leq N$ is the same as the set of rationals k/q satisfying

$$(k, q) = 1, \quad 1 \leq k \leq q, \quad 1 \leq q \leq N. \quad (4.2)$$

Thus, every nonzero frequency component in $x(n)$ will be seen by one or other of the filters, and will produce a nonzero output in that filter. So $x(n)$ can be represented as a linear combination of $W_{q_i}^{k_i n}$ where $(k_i, q_i) = 1$ and $1 \leq k_i \leq q_i$. Now, since $L = \text{lcm}\{q_1, q_2, \dots, q_K\}$ it follows that $L = q_i m_i$ for some integer m_i . So that $W_{q_i} = W_L^{m_i}$. This shows that $x(n)$ is a linear combination of the form

$$x(n) = \sum_{m=0}^{L-1} \beta_m W_L^{mn}. \quad (4.3)$$

That is, $x(n)$ has a period L (or a divisor of L) which is smaller than P . This is a contradiction. $\square$

4.2 Frame properties of Ramanujan filter bank

In signal processing, a fundamental requirement for perfect reconstruction in filter banks is that the underlying system forms a *frame* [12, 6, 7, 13]. Consider a K -channel filter bank with analysis filters $h_1, h_2, \dots, h_K$. The associated system is

defined as:

$$\mathcal{F}_{\mathfrak{B},K} := \{L_m h_1\}_{m \in \mathfrak{B}\mathbb{Z}_N^d} \cup \{L_m h_2\}_{m \in \mathfrak{B}\mathbb{Z}_N^d} \cup \cdots \cup \{L_m h_K\}_{m \in \mathfrak{B}\mathbb{Z}_N^d},$$

where L_m denotes the translation operator.

Definition 4.2 (Frame). *The system $\mathcal{F}_{\mathfrak{B},K}$ is a frame for $\ell^2(\mathbb{Z}_N^d)$ if there exist constants $A, C > 0$ (called frame bounds) such that:*

$$A\|x\|^2 \leq \sum_{i=1}^K \sum_{m \in \mathfrak{B}\mathbb{Z}_N^d} |\langle x, L_m h_i \rangle|^2 \leq C\|x\|^2, \quad \text{for all } x \in \ell^2(\mathbb{Z}_N^d).$$

If $A = C$, the frame is called a tight frame with bound A .

In the context of Ramanujan filter banks, we consider a K -channel system where the i -th analysis channel employs the q_i -th Ramanujan sum c_{q_i} . The corresponding system is:

$$\mathcal{Q}_N := \{L_k c_{q_1}\}_{k=0}^{N-1} \cup \{L_k c_{q_2}\}_{k=0}^{N-1} \cup \cdots \cup \{L_k c_{q_K}\}_{k=0}^{N-1}.$$

A key structural result ensures that any signal $x \in \ell^2(\mathbb{Z}_N)$ admits an orthogonal decomposition:

$$x(n) = \sum_{i=1}^K x_{q_i}(n),$$

where each component x_{q_i} lies in a distinct subspace spanned by shifts of c_{q_i} , and $\langle x_{q_i}, x_{q_j} \rangle = 0$ for $i \neq j$.

Sahil Kalra and Niraj K. Shukla [17, 18] provided a key breakthrough in this area by introducing this foundational theorem within the context of Ramanujan filter banks. This result is important as it connects the frame properties of Ramanujan filter banks with signal recovery under noise and incomplete data and effectively expanding the tight frame property to dictionaries constructed from Ramanujan sums. By doing so, they proved an important result which provides the conditions under which the collection $\mathcal{Q}_N$ forms a tight frame for $\ell^2(\mathbb{Z}_N)$ for different downsampling factors.

Theorem 4.3. [18]/[Tight Frame Property] The collection $\mathcal{Q}_N$ forms a tight frame for $\ell^2(\mathbb{Z}_N)$ with frame bound N^2 .

Proof. By using above equation for $x \in \ell^2(\mathbb{Z}_N)$, the central quantity in def (4.2) with respect to $\mathcal{Q}_N$ simplifies to:

$$\begin{aligned}
\sum_{i=1}^K \sum_{k=0}^{N-1} |\langle x, L_k c_{q_i} \rangle|^2 &= \sum_{i=1}^K \sum_{k=0}^{N-1} \left| \left\langle \sum_{j=1}^K \sum_{\ell=0}^{\phi(q_j)-1} \alpha_{\ell,i} L_{\ell} c_{q_j}, L_k c_{q_i} \right\rangle \right|^2 \\
&= \sum_{i=1}^K \sum_{k=0}^{N-1} \left| \sum_{j=1}^K \sum_{\ell=0}^{\phi(q_j)-1} \alpha_{\ell,i} \langle L_{\ell} c_{q_j}, L_k c_{q_i} \rangle \right|^2 \\
&= \sum_{i=1}^K \sum_{k=0}^{N-1} \left| \sum_{j=1}^K \sum_{\ell=0}^{\phi(q_j)-1} \alpha_{\ell,i} \sum_{n=0}^{N-1} c_{q_j}(n) c_{q_i}(n - (k - \ell)) \right|^2 \\
&= \sum_{i=1}^K \sum_{k=0}^{N-1} \left| \sum_{\ell=0}^{\phi(q_i)-1} \alpha_{\ell,i} \sum_{n=0}^{N-1} c_{q_i}(n) c_{q_i}(n - (k - \ell)) \right|^2 \\
&= \sum_{i=1}^K \sum_{k=0}^{N-1} \left| \sum_{\ell=0}^{\phi(q_i)-1} \alpha_{\ell,i} N c_{q_i}(k - \ell) \right|^2 \\
&= N^2 \sum_{i=1}^K \sum_{k=0}^{N-1} \left| \sum_{\ell=0}^{\phi(q_i)-1} \alpha_{\ell,i} c_{q_i}(k - \ell) \right|^2 \\
&= N^2 \sum_{i=1}^K \sum_{k=0}^{N-1} |x_{q_i}(k)|^2 = N^2 \|x\|^2.
\end{aligned}$$

This confirms the tight frame condition with bound N^2 . □

The Ramanujan filter bank system in [17] is generalized as:

$$\mathcal{R}_{p,N} = \bigcup_{i=1}^K \{L_{pk} c_{q_i}\}_{k=0}^{d-1},$$

where $N = pd$. Known results include:

- For $p = 1$: $\mathcal{R}_{1,N}$ is a tight frame for $\ell^2(\mathbb{Z}_N)$ with bound N^2 .
- For $p = 2$ with M odd: $\mathcal{R}_{2,N}$ is a tight frame for $\ell^2(\mathbb{Z}_N)$ with bound $2M^2$.

- For $p > 2$: The system is not a frame for $\ell^2(\mathbb{Z}_N)$.

In their analysis, author established that the collection $\mathcal{R}_{p,N}$, for $p > 2$, does not span the entire Hilbert space $\ell^2(\mathbb{Z}_N)$, and consequently does not form a frame for $\ell^2(\mathbb{Z}_N)$. The main reason is that, for $p > 2$, certain individual subcollections $\{L_{pk}c_{q_i}\}_{k \in \mathcal{J}_d}$ within $\mathcal{R}_{p,N}$ fail to span their corresponding Ramanujan subspaces of $\ell^2(\mathbb{Z}_N)$. As a result, the union of these subcollections is insufficient to provide a complete representation of the ambient signal space.

CHAPTER 5

Affine Ramanujan Sums

The Ramanujan Fourier transform is one of the efficient multiresolution analysis tools, but it only works for stationary or periodically characterized signals. The Affine Fourier transform, a general form of the Fourier transform, is suitable for nonstationary signal analysis but with fixed resolution. In this study, Ramanujan sums are combined with the affine Fourier transform to provide affine multiresolution analysis tools for nonstationary signals.

5.1 Affine Ramanujan sums and its properties

Definition 5.1. [\[22\]](#) *[Discrete Affine Fourier Transform] For a sampled signal*

$$x = [x(\Delta_t), \dots, x(k\Delta_t)]^T,$$

the discrete affine Fourier transform (DAFT) is defined as

$$X = Fx,$$

where F is the kernel matrix with elements

$$F_k(m, n) = \sqrt{\frac{1}{k}} \exp \left(i \frac{a}{2b} (m\Delta_t)^2 - i \frac{2\pi mn}{k} + i \frac{d}{2b} (n\Delta_u)^2 \right).$$

The parameters satisfy

$$\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{with } ad - bc = 1,$$

and the sampling condition is given by $\Delta_t \Delta_u = \frac{2\pi|b|}{q}$.

Theorem 5.1. [22][Chirp Periodicity] The function $\Psi_k^\alpha(m)$ satisfies the chirp periodicity property

$$e^{-j\frac{\beta}{2\delta}(m+k)\Delta t^2} \Psi_k^\alpha(m+k) = e^{-j\frac{\beta}{2\delta}m\Delta t^2} \Psi_k^\alpha(m), \quad (5.1)$$

where k is the chirp period.

Proof. It can be verified that the kernel function satisfies

$$e^{-j\frac{\beta}{2\delta}(m+k)\Delta t^2} F_k(m+k, n) = e^{-j\frac{\beta}{2\delta}m\Delta t^2} F_k(m, n). \quad (5.2)$$

From this relation, the chirp periodicity of $\Psi_k^\alpha(m)$ follows directly. $\square$

Theorem 5.2. [22][Orthogonality] The functions $\Psi_k^\alpha(m)$ satisfy the orthogonality property

$$\sum_{m=1}^{k_1 k_2} \Psi_{k_1}^\alpha(m) (\Psi_{k_2}^\alpha(m))^* = \begin{cases} 0, & k_1 \neq k_2, \\ k_1 \varphi(k_1), & k_1 = k_2. \end{cases} \quad (5.3)$$

Proof. Substituting the expression of $\Psi_k^\alpha(m)$ into the left-hand side of (5.3), we obtain

$$\begin{aligned} \sum_{m=1}^{k_1 k_2} \Psi_{k_1}^\alpha(m) (\Psi_{k_2}^\alpha(m))^* &= \sqrt{\frac{1}{k_1 k_2}} \sum_{m=1}^{k_1 k_2} \sum_{\substack{n_1=1 \\ \gcd(n_1, k_1)=1}}^{k_1} \\ &\quad \times \sum_{\substack{n_2=1 \\ \gcd(n_2, k_2)=1}}^{k_2} e^{-j2\pi m \left(\frac{n_1}{k_1} - \frac{n_2}{k_2} \right)} e^{j\frac{\beta}{2\delta}(n_1^2 - n_2^2)(\Delta t)^2}. \end{aligned} \quad (5.4)$$

The value of the above expression depends on the relationship between k_1 and k_2 .

Case (i): $k_1 \neq k_2$. In this case, $\frac{n_1}{k_1} \neq \frac{n_2}{k_2}$, and hence

$$\sum_{m=1}^{k_1 k_2} e^{-j2\pi m \left(\frac{n_1}{k_1} - \frac{n_2}{k_2} \right)} = 0, \quad (5.5)$$

which implies

$$\sum_{m=1}^{k_1 k_2} \Psi_{k_1}^\alpha(m) (\Psi_{k_2}^\alpha(m))^* = 0. \quad (5.6)$$

Case (ii): $k_1 = k_2 = k$. Then the summation becomes

$$\sum_{m=1}^{k^2} e^{-j2\pi m \frac{(n_1 - n_2)}{k}}. \quad (5.7)$$

If $k \nmid (n_1 - n_2)$, the sum evaluates to zero. If $k \mid (n_1 - n_2)$, i.e., $n_1 = n_2$, then

$$\sum_{m=1}^{k^2} 1 = k^2. \quad (5.8)$$

Thus, only terms with $n_1 = n_2$ contribute, and the expression reduces to

$$\sum_{m=1}^{k^2} \Psi_k^\alpha(m) (\Psi_k^\alpha(m))^* = k \varphi(k). \quad (5.9)$$

Hence, the orthogonality property is established. $\square$

Based on affine Ramanujan sums, the ARFT for infinite sequences is defined as follows.

Definition 5.2. *Let $x(m)$ be a infinite discrete-time sequence. The Affine Ramanujan Fourier Transform (ARFT) of $x(m)$ is defined by*

$$X^\alpha(k) = \frac{1}{\varphi(k)} \left(\lim_{M \rightarrow \infty} \frac{1}{M} \sum_{m=1}^M x(m) \overline{\Psi_k^\alpha(m)} \right), \quad (5.10)$$

where $\varphi(k)$ denotes Euler's totient function and $\Psi_k^\alpha(m)$ represents the affine Ramanujan basis function. The corresponding inverse Affine Ramanujan Fourier Transform is given by

$$x(m) = \sum_{k=1}^{\infty} X^\alpha(k) \Psi_k^{\alpha^{-1}}(m). \quad (5.11)$$

We introduce a canonical circular time-shift operator $\hat{\mathcal{F}}_n$. Specifically, the canonical time-shift operator can be represented as

$$\mathcal{F}_n(x)(m) = x(m - n) e^{j \frac{a}{2b} ((m-n)\Delta_t)^2} e^{-j \frac{a}{2b} (m\Delta_t)^2}. \quad (5.12)$$

Replacing the time-shift operator with circular time-shift operator in (5.12), the canonical circular time-shift operator is defined as follows

$$\hat{\mathcal{F}}_n(x)(m) = x(m - n) e^{j \frac{a}{2b} ((m-n)k\Delta_t)^2} e^{-j \frac{a}{2b} (m\Delta_t)^2}. \quad (5.13)$$

5.2 The ARFT-I for finite sequences

A natural idea to calculate the ARFT for finite sequences is assigning M in (5.10) as a finite number, such as assigning M as the length of sequence $x(m)$, $m = 1, \dots, M$. In this way, the first type of ARFT (ARFT-I) for finite sequences is obtained as follows.

Definition 5.3. [22] *The ARFT-I for a finite sequence $x(m)$, $m = 1, \dots, M$, is defined as*

$$X^\alpha(k) = \frac{1}{\phi(k)M} \sum_{m=1}^M x(m) \Psi_k^\alpha(m), \quad k = 1, \dots, M. \quad (5.14)$$

Accordingly, its matrix form is

$$\mathbf{X}^\alpha = \frac{1}{M} \mathbf{\Psi}^\alpha \mathbf{x}, \quad (5.15)$$

where

$$\mathbf{X}^\alpha = [X^\alpha(1), \dots, X^\alpha(M)]^T, \quad \mathbf{x} = [x(1), \dots, x(M)]^T,$$

and $\mathbf{\Psi}^\alpha$ is an $M \times M$ square transform matrix expressed as

$$\mathbf{\Psi}^\alpha = \begin{bmatrix} \frac{1}{\phi(1)} \Psi_1^\alpha(1) & \dots & \frac{1}{\phi(1)} \Psi_1^\alpha(M) \\ \vdots & \ddots & \vdots \\ \frac{1}{\phi(M)} \Psi_M^\alpha(1) & \dots & \frac{1}{\phi(M)} \Psi_M^\alpha(M) \end{bmatrix}. \quad (5.16)$$

If matrix $\mathbf{\Psi}^\alpha$ is reversible, $\mathbf{x}$ can be obtained from $\mathbf{X}^\alpha$, which is

$$\mathbf{x} = M (\mathbf{\Psi}^\alpha)^{-1} \mathbf{X}^\alpha. \quad (5.17)$$

Theorem 5.3. [22]/[Recursive Representation and Reversibility of ARFT-I] Let $F_k(m, n)$ be the function introduced in Theorem , and let m be fixed. Then the affine Ramanujan sums satisfy the relation

$$\sum_{n=1}^k F_k(m, n) = \sum_{q|k} \sum_{\substack{n=1 \\ (n,q)=1}}^q F_q(m, n) = \sum_{q|k} \Psi_q^\alpha(m). \quad (5.18)$$

Consequently, the affine Ramanujan sum $\Psi_k^\alpha(m)$ admits the recursive representation

$$\Psi_k^\alpha(m) = \sum_{n=1}^k F_k(m, n) - \sum_{\substack{q|k \\ q < k}} \Psi_q^\alpha(m). \quad (5.19)$$

Furthermore, the transform matrix Ψ^α defined in (5.16) can be reduced by elementary row operations to the matrix $\mathbf{Q}^\alpha$ given by

$$\mathbf{Q}^\alpha = \begin{bmatrix} \frac{1}{\phi(1)} F_1(1, 1) & \cdots & \frac{1}{\phi(1)} F_1(M, 1) \\ \vdots & \ddots & \vdots \\ \frac{1}{\phi(M)} \sum_{n=1}^M F_M(1, n) & \cdots & \frac{1}{\phi(M)} \sum_{n=1}^M F_M(M, n) \end{bmatrix}. \quad (5.20)$$

Moreover,

$$|\Psi^\alpha| = |\mathbf{Q}^\alpha|. \quad (5.21)$$

Hence, the reversibility of Ψ^α is equivalent to the reversibility of $\mathbf{Q}^\alpha$.

Proof. Fix an integer m . By the definition of $F_k(m, n)$ introduced in Theorem 1, we obtain

$$\sum_{n=1}^k F_k(m, n) = \sum_{q|k} \sum_{\substack{n=1 \\ (n,q)=1}}^q F_q(m, n).$$

Recognizing that the inner summation corresponds exactly to the affine Ramanujan sum $\Psi_q^\alpha(m)$, we obtain (5.18). Rearranging the terms yields the recursive formula (5.19). Now, consider the matrix Ψ^α defined in (5.16). By performing elementary row operations while preserving the determinant, we successively transform Ψ^α . Specifically, adding $\phi(q)$ times the q th row to the last row, for

every divisor $q \mid M$ with $q < M$, transforms the last row into

$$\sum_{n=1}^M F_M(m, n) / \phi(M).$$

Similarly, adding the q th row to the $(M - 1)$ th row, for every $q \mid (M - 1)$ with $q < M - 1$, transforms the $(M - 1)$ th row into

$$\sum_{n=1}^{M-1} F_{M-1}(m, n) / \phi(M - 1).$$

Repeating this process successively up to the first row, the matrix Ψ^α is reduced to the matrix $\mathbf{Q}^\alpha$ given in (5.20). Since elementary row operations preserve the determinant up to non-zero constants, it follows that

$$|\Psi^\alpha| = |\mathbf{Q}^\alpha|.$$

Therefore, Ψ^α is reversible if and only if $\mathbf{Q}^\alpha$ is reversible. $\square$

Example 5.1. [ARFT-I Amplitude Illustration with Circular Shift] Fig. (5.1) illustrates the amplitudes of the ARFT-I of $\Psi_8^\alpha(m)$ and its one-step canonical circular time shift given by $\Psi_8^\alpha(m - 1) e^{j \frac{a}{2b} (m-1)^2 - m^2}$.

This example demonstrates how a circular shift in the time domain affects the amplitude distribution in the ARFT domain. While the magnitude structure remains preserved, the phase term introduces a modulation governed by the quadratic exponential factor. The significant strength located at the 8th point in Fig. (5.1) represents the chirp period 8, but the chirp period strength is distributed on almost all independent variables of ARFT-I in Fig. (5.2). In the second case, ARFT-I fails in estimating the chirp period, which explains that $X^\alpha(k)$ is sensitive to the canonical circular time shift.

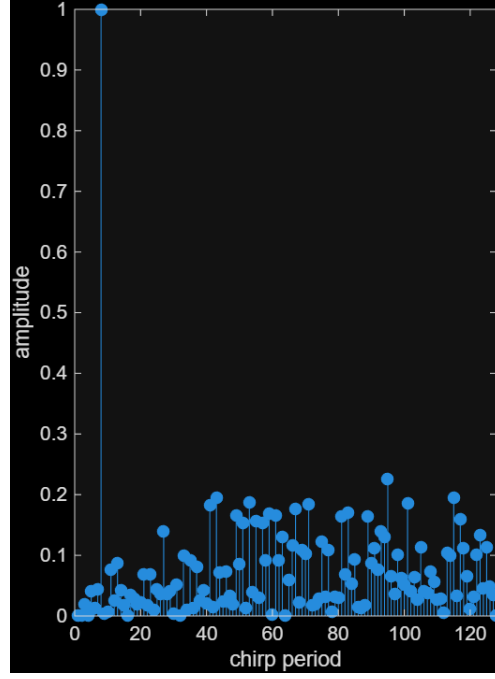


Figure 5.1: Amplitudes of ARFT-I of $\Psi_8^\alpha(m)$

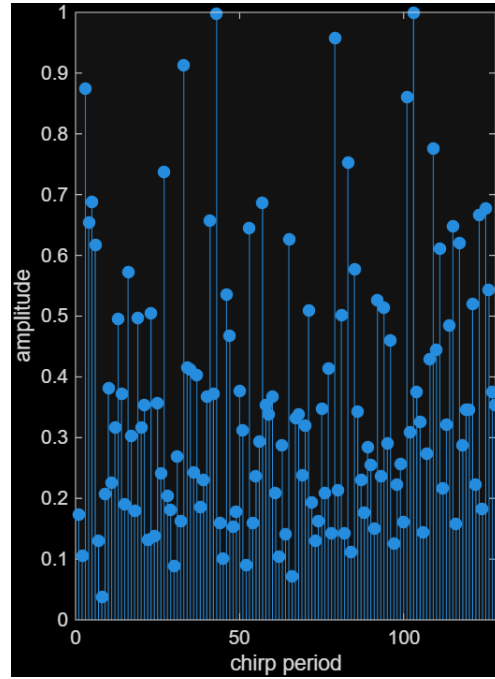


Figure 5.2: Amplitude of ARFT-I of one step canonical circular time shift of $\Psi_8^\alpha(m)$

: The affine Ramanujan basis matrix $\mathbb{B}_k^\alpha$ is given by

$$\mathbb{B}_k^\alpha = \begin{bmatrix} \psi_k^\alpha(0) & \psi_k^\alpha(k-1) e^{-\frac{1}{2\theta_0}((k-1)\Delta t)^2} & \dots & \psi_k^\alpha(1) e^{-\frac{1}{2\theta_0}(\Delta t)^2} \\ \psi_k^\alpha(1) & \psi_k^\alpha(0) e^{\frac{1}{2\theta_0}(\Delta t)^2} & \dots & \psi_k^\alpha(2) e^{\frac{1}{2\theta_0}(1-2^2)(\Delta t)^2} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_k^\alpha(k-1) & \psi_k^\alpha(k-2) e^{-\frac{1}{2\theta_0}((k-2)^2-(k-1)^2)(\Delta t)^2} & \dots & \psi_k^\alpha(0) e^{\frac{1}{2\theta_0}(k-1)^2(\Delta t)^2} \end{bmatrix} \quad (5.22)$$

Next we will discuss affine Ramanujan subspace and some of its properties

Theorem 5.4. [22] *Any consecutive $\varphi(k)$ columns of B_k^α are linearly independent.*

Proof. Denote the submatrix by selecting the l th to the $(l + \varphi(k) - 1)$ th column of B_k^α as

$$S_k^\alpha = B_k^\alpha A, \quad (5.23)$$

where $A = [0_{\varphi(k) \times (l-1)} \ I_{\varphi(k)} \ 0_{\varphi(k) \times (k-l+1-\varphi(k))}]^T$, $0_{\varphi(k) \times (l-1)}$ is a $\varphi(k) \times (l-1)$ matrix with all elements being 0, I is an identity matrix, and $1 \leq l \leq k - \varphi(k) \in \mathbb{Z}$. S_k^α becomes

$$S_k^\alpha = \text{sub}(F^{-1})F_1, \quad (5.24)$$

where $F_1 = \text{sub}(F_0^T)^T A$ is obtained by retaining the l th to $(l + \varphi(k) - 1)$ th column of $\text{sub}(F_0^T)^T$. As explained in Theorem 4 of [28], F_1 is full-rank. Due to the columns of a reversible matrix F^{-1} being linearly independent, the rank of $\text{sub}(F^{-1})$ is $\varphi(k)$. Further, S_k^α has rank $\varphi(k)$. $\square$

Without loss of generality, the first $\varphi(k)$ columns of $\mathbf{B}_k^\alpha$ are selected as $\mathbf{S}_k^\alpha$ in the rest part of this paper. The k th affine Ramanujan sum and its $\varphi(k) - 1$ canonical circular time shift, which are actually the columns of $\mathbf{S}_k^\alpha$, constitute a set of basis for a $\varphi(k)$ -dimensional affine Ramanujan subspace $\mathcal{S}_k$. A signal satisfies $x(m) \in \mathcal{S}_k$ if it can be decomposed as

$$\mathbf{x} = \mathbf{S}_k^\alpha [X(0), \dots, X(\varphi(k) - 1)]^T, \quad (5.25)$$

for some complex numbers $X(m) \in \mathbb{C}, m = 0, \dots, \varphi(k) - 1$, where $\mathbb{C}$ is a set of complex numbers. In detail, every element $x(m)$ is represented as

$$x(m) = \sum_{l=0}^{\varphi(k)-1} X(l) \Psi_k^\alpha(m-l)_k e^{-j \frac{a}{2b} ((m-l)_k \Delta_t)^2 + j \frac{a}{2b} (m \Delta_t)^2}. \quad (5.26)$$

As explained in (5.25), $\mathbf{S}_k^\alpha$ is a $k \times \varphi(k)$ matrix. The $M \times M$ transform matrix for $x(m), m = 1, \dots, M$ can be built by the following steps:

1. Determining all the divisors $k_1, \dots, k_p$ of M satisfying $\sum_{l=1}^p \varphi(k_l) = M$;
2. Chirp periodically extending every $\mathbf{S}_{k_l}^\alpha$ vertically by canonical time shifting it M/k_l times, that is,

$$\tilde{\mathbf{S}}_{k_l}^\alpha = \begin{bmatrix} \mathbf{S}_{k_l}^\alpha \\ \mathbf{D}_1 \mathbf{S}_{k_l}^\alpha \\ \vdots \\ \mathbf{D}_{M/k_l-1} \mathbf{S}_{k_l}^\alpha \end{bmatrix}, \quad (5.27)$$

where the size of $\tilde{\mathbf{S}}_{k_l}^\alpha$ is $M \times \varphi(k_l)$. $\mathbf{D}_m, m = 1, \dots, M/k_l - 1$ is a diagonal matrix and it is expressed as

$$\mathbf{D}_m = \text{diag} \left(e^{-j \frac{a}{2b} (0^2 - (0+mk)^2) (\Delta_t)^2}, e^{-j \frac{a}{2b} (1^2 - (1+mk)^2) (\Delta_t)^2}, \dots, e^{-j \frac{a}{2b} ((k-1)^2 - (k-1+mk)^2) (\Delta_t)^2} \right), m = 1, \dots, M.$$

3. Ranging the matrices $\tilde{\mathbf{S}}_{k_l}^\alpha$ horizontally, that is

$$\mathbf{G}^\alpha = [\tilde{\mathbf{S}}_{k_1}^\alpha, \dots, \tilde{\mathbf{S}}_{k_p}^\alpha]. \quad (5.28)$$

Theorem 5.5. [22]/[Block Orthogonality] The columns in $\tilde{\mathbf{S}}_{k_l}$ and $\tilde{\mathbf{S}}_{k_n}$ are orthogonal, such that:

$$\tilde{\mathbf{S}}_{k_l}^\dagger \tilde{\mathbf{S}}_{k_n} = 0, \quad \text{for } l \neq n, \quad (5.29)$$

where $\mathbf{S}^\dagger$ denotes the conjugate transpose of $\mathbf{S}$.

Proof. According to the orthogonality of the affine Ramanujan sum shown in (5.3), the orthogonality of the columns in two different matrices $\tilde{\mathbf{S}}_{k_l}$ and $\tilde{\mathbf{S}}_{k_n}$ is similarly obtained. $\square$

Theorem 5.6 (Full-rank property). *The rank of matrix $\mathbf{G}^\alpha$ is M .*

Proof. As explained in Theorem (5.4), columns in $\mathbf{S}_{k_l}$ are linearly independent. Thus, columns in $\tilde{\mathbf{S}}_{k_l}$ are linearly independent. Moreover, as explained in orthogonality, columns in different matrices are orthogonal. Thus, columns in $\mathbf{G}^\alpha$ are linearly independent, which means $\mathbf{G}^\alpha$ is full-rank. $\square$

Based on this theorem, the second type of ARFT for finite sequences $\mathbf{x} = [x(1), \dots, x(M)]^T$ is given as follows:

Definition 5.4. [22][ARFT-II] *The ARFT-II for a finite sequence $x(m), m = 1, \dots, M$ is defined as*

$$\mathbf{x} = \mathbf{G}^\alpha \mathbf{X}, \quad (5.30)$$

where $\mathbf{x} = [x(1), \dots, x(M)]^T$, and $\mathbf{X}$ is the coefficient vector that can be calculated as

$$\mathbf{X} = (\mathbf{G}^\alpha)^{-1} \mathbf{x}. \quad (5.31)$$

We want to show by example that if the periods are divisor of the length of signal, then which perform better and how it is able to do it .

Example 5.2. The periodic signal $x(m)$ is composed of N complex periodic components with quadric polynomial initial phases, which is

$$x(m) = \sum_{n=1}^N e^{j\frac{2\pi}{k}\omega_n m - j\pi\omega_n^2}, \quad m = 1, \dots, M.$$

where N denotes the number of periodic components and ω_n represents the corresponding frequency indices.

For this experiment, the signal length and basic period are selected as $M = k = 128$. The number of periodic components is chosen as $N = 2$ with frequency indices $\omega_1 = 4, \omega_2 = 7$. The affine parameter vector used for both ARFT-I and ARFT-II is given by $\alpha = [0, 1, -1, 1]$

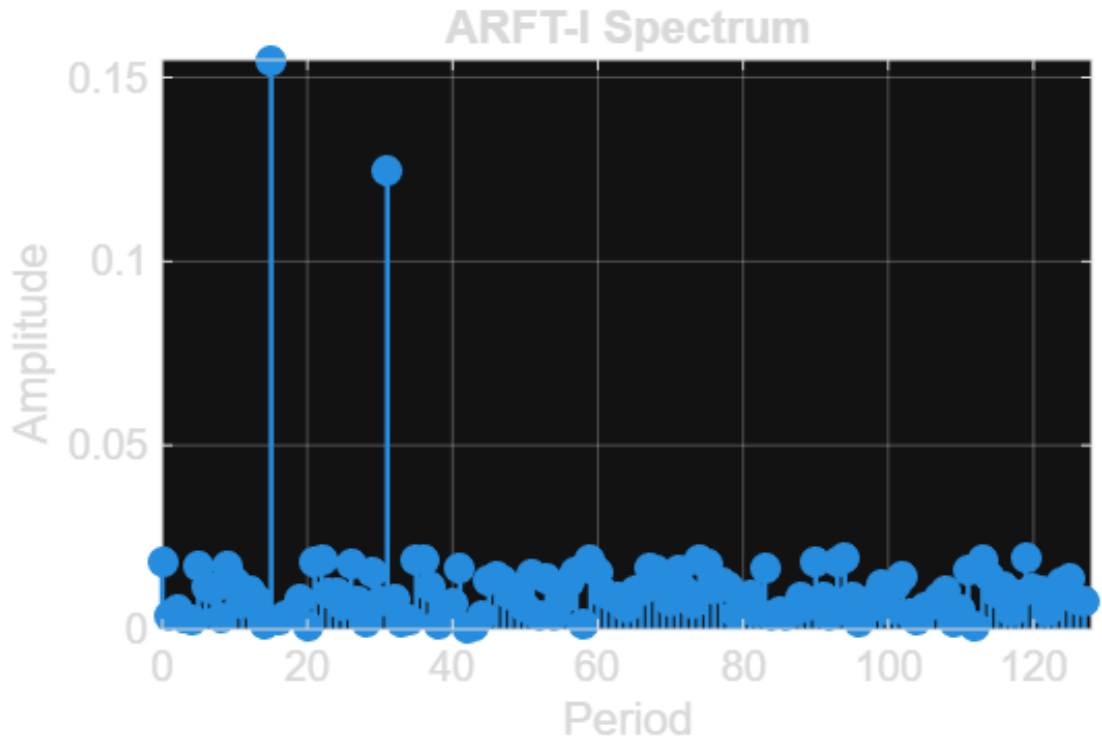


Figure 5.3: Amplitude of ARFT-I of $x(m)$

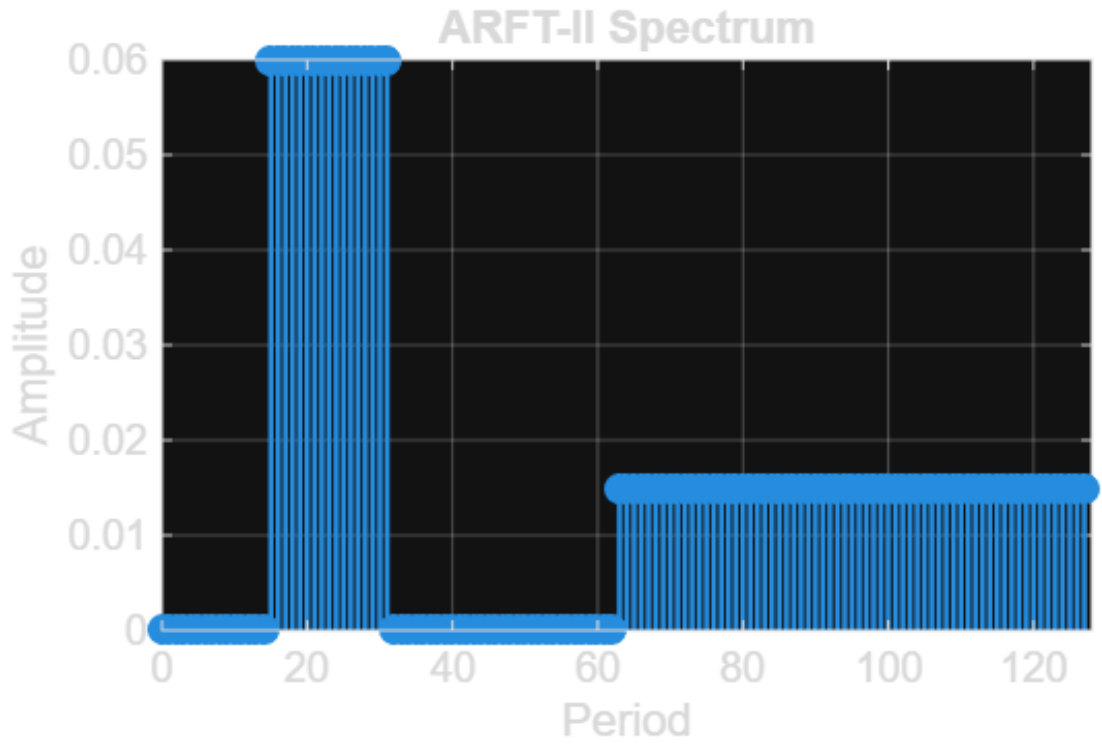


Figure 5.4: Amplitude of ARFT-II of $x(m)$

As show in Fig.(5.3) that ARFT-I can identify the hidden periodic components through concentrated spectral peaks. Since the affine frequencies

$\omega_1 = 2\pi \times \frac{4}{128}$ and $\omega_2 = 2\pi \times \frac{7}{128}$ are integer multiples of the frequency resolution interval $\frac{2\pi}{128}$. The hidden period corresponding to 32 can be estimated accurately. Furthermore, the second hidden period is determined using the greatest common divisor relation as $\frac{2\pi}{\gcd(4,7) \times \frac{2\pi}{128}} = 128$ which correctly estimates the second hidden period.

As shown in Fig.(5.4), ARFT-II also detects the existence of hidden periodicity; however, the transform coefficients are distributed over a range of neighboring indices instead of appearing as sharp peaks. The method estimates hidden periods based on nonzero projections over several subspaces generated by the divisors of the signal length. The divisors of 128 are 1, 2, 4, 8, 16, 32, 64, 128 and the corresponding Euler totient values are $\varphi(1) = 1$, $\varphi(2) = 1$, $\varphi(4) = 2$, $\varphi(8) = 4$, $\varphi(16) = 8$, $\varphi(32) = 16$, $\varphi(64) = 32$, $\varphi(128) = 64$ Therefore, corresponds to hidden period 32, the nonzero coefficient should start from 17th position and continue until 32th. and similarly period of 128 start from 65th position to 128th position. ARFT-II is superior for periodic estimation if it is divisor of M over ARFT-I.

CHAPTER 6

Future Work

In this thesis, we studied the tight frame property of Ramanujan filter banks and their applications in signal reconstruction and denoising. The obtained results show that Ramanujan filter banks are effective in recovering missing components of periodic and stationary signals, as well as in reducing noise contamination. However, a major limitation of classical Ramanujan sums is that they are primarily suited for stationary signals and are not sufficiently effective for analyzing non-stationary signals whose spectral characteristics vary with time.

In future work, we aim to extend this framework to the study of non-stationary signals by using affine Ramanujan sums. In particular, we will investigate the tight frame property of affine Ramanujan filter banks, which are defined by

$$\mathfrak{R}_{p,N}^\alpha = \bigcup_{i=1}^K \{L_{pk} \Psi_{q_i}^\alpha\}_{k=0}^{d-1}, \quad \alpha \in SL_2(\mathbb{Z}),$$

where $\Psi_{q_i}^\alpha$ denote the affine Ramanujan sums.

The proposed framework is expected to provide a more suitable representa-

tion for non-stationary signals, particularly signals exhibiting chirp-like behavior and time-varying frequency characteristics. We also plan to investigate whether affine Ramanujan filter banks preserve useful properties such as signal reconstruction, recovery of missing data, and denoising capability.

Establishing these properties would contribute toward the development of more efficient mathematical and signal processing tools for the analysis of non-stationary signals.

Bibliography

- [1] Tom M. Apostol. *Mathematical Analysis*. Addison-Wesley, Dedham, MA, USA, 1974.
- [2] J. J. Benedetto and M. Fickus. Finite normalized tight frames. *Advances in Computational Mathematics*, 18(2-4):357–385, 2003.
- [3] M. Bownik and Z. Rzesotnik. Construction and reconstruction of tight framelets and wavelets via matrix mask functions. *Journal of Functional Analysis*, 256(4):1065–1105, 2009.
- [4] J.-F. Cai, H. Ji, Z. Shen, and G.-B. Ye. Data-driven tight frame construction and image denoising. *Applied and Computational Harmonic Analysis*, 37(1):89–105, 2014.
- [5] N. Chen, Z. Chen, and Y. Wei. Multidimensional inverse lattice problem and a uniformly sampled arithmetic Fourier transform. *Physical Review E*, 55(1):R5–R8, 1997.
- [6] O. Christensen. *An Introduction to Frames and Riesz Bases*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, 2003.

- [7] O. Christensen. *An Introduction to Frames and Riesz Bases*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, 2003.
- [8] G. Kuldeep D. K. Yadav and S. D. Joshi. Orthogonal ramanujan sums, its properties, and applications in multiresolution analysis. *IEEE Transactions on Signal Processing*, 66(21), 2018.
- [9] Michael W. Frazier. *An Introduction to Wavelets Through Linear Algebra*. Springer Science & Business Media, 2006.
- [10] A. G. García, M. A. Hernández-Medina, and G. Pérez-Villalón. Filter banks on discrete abelian groups. *International Journal of Wavelets, Multiresolution and Information Processing*, 16(20):1850029, 2018.
- [11] A. G. García, M. A. Hernández-Medina, and G. Pérez-Villalón. Convolution systems on discrete abelian groups as a unifying strategy in sampling theory. *Results in Mathematics*, 75(1):Paper No. 40, 22, 2020.
- [12] B. Han. *Framelets and wavelets: Algorithms, analysis, and applications*. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, Cham, 2017.
- [13] D. Han and D. Larson. *Frames, bases and group representations*, volume 147. American Mathematical Society, 2000.
- [14] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*. Oxford University Press, 1975.
- [15] W. Jia, Y. Cao, S. Zhang, and W.-Q. Wang. Detecting high-speed maneuvering targets by exploiting range-doppler relationship for lfm radar. *IEEE Transactions on Vehicular Technology*, 70(3):2209–2218, 2021.
- [16] B. D. Johnson and K. A. Okoudjou. Frame potential and finite abelian groups. In *Radon Transforms, Geometry, and Wavelets*, volume 464 of

- Contemporary Mathematics*, pages 137–148. American Mathematical Society, Providence, RI, 2008.
- [17] Sahil kalra and Niraj K.Shukla. Ramanujan sums in signal recovery and uncertainty principle. *Applied and Computational Harmonic Analysis*, 82, 2026.
 - [18] Sahil Kalra and Niraj K. Shukla. Uncertainty principle for convolutional tight frames and its applications in signal recovery. *Sampling Theory, Signal Processing, and Data Analysis*, 23(2):Paper No. 21, 2025.
 - [19] Pranav Kulkarni and Palghat P. Vaidyanathan. Periodicity-aware signal denoising using capon-optimized Ramanujan filter banks and pruned ramanujan dictionaries. *IEEE Transactions on Signal Processing*, 71:494–511, 2023.
 - [20] J.-C. Lin. Least-squares channel estimation for mobile OFDM communication on time-varying frequency-selective fading channels. *IEEE Transactions on Vehicular Technology*, 57(6):3538–3550, 2008.
 - [21] S. F. Mason, C. R. Berger, S. Zhou, and P. Willett. Detection, synchronization, and doppler scale estimation with multicarrier waveforms in underwater acoustic communication. *IEEE Journal on Selected Areas in Communications*, 26(9):1638–1649, 2008.
 - [22] Hongxia Miao, Feng Zhang, Ran Tao, and Mugen Peng. When ramanujan sums meet affine fourier transform. *Signal Processing*, 206:108907, 2023.
 - [23] P.P.Vaidyanathan and Srikanth Tenneti. Properties of ramanujan filter banks. 2015.
 - [24] P.P.Vaidyanathan and Srikanth Tenneti. Srinivasa ramanujan and signal problems. *Philos. Trans R. Soc.A*, 378, 2020.

- [25] L. Shen, M. Papadakis, I. A. Kakadiaris, I. Konstantinidis, D. Kouri, and D. Hoffman. Image denoising using a tight frame. *IEEE Transactions on Image Processing*, 15(5):1254–1263, 2006.
- [26] J.-W. Tao, C.-Z. Yang, and C.-W. Xu. Estimation of PRI stagger in case of missing observations. *IEEE Transactions on Geoscience and Remote Sensing*, 58(11):7982–8001, 2020.
- [27] Palghat P. Vaidyanathan. Ramanujan sums in the context of signal processing—Part I: Fundamentals. *IEEE Transactions on Signal Processing*, 62:4145–4157, 2014.
- [28] Palghat P. Vaidyanathan. Ramanujan sums in the context of signal processing—Part II: FIR representations and applications. *IEEE Transactions on Signal Processing*, 62:4158–4172, 2014.
- [29] Palghat P. Vaidyanathan and Srikanth V. Tenneti. Srinivasa ramanujan and signal-processing problems. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 378(2163):20180446, 2020.