EQUIVALENT CRITERIA FOR THE RIEMANN HYPOTHESIS AND RADEMACHER-TYPE EXACT FORMULA

Ph.D. Thesis

By

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DEPARTMENT OF MATHEMATICS

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MEGHALI



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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **EQUIVALENT CRITERIA FOR THE RIEMANN HYPOTHESIS AND RADEMACHER-TYPE EXACT FORMULA** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DEPARTMENT OF MATHEMATICS**, **Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from December 2020 to June 2024 under the supervision of **Dr. Bibekananda Maji**, Associate Professor, Department of Mathematics, IIT Indore.

The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

Meghali 9/11/2026

Signature of the student with date

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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MEGHALI has successfully given her Ph.D. Oral Examination held on 9/11/2026.

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Signature of Thesis Supervisor with date

(**DR. BIBEKANANDA MAJI**)

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ABSTRACT

This thesis is devoted to two principal themes in analytic number theory: Equivalent criteria for the Riemann hypothesis (RH) for general class of L-functions, and the derivation of Rademacher-type exact formulas together with higher-order Turán inequalities for some restricted partition functions. Both parts are unified by analytic techniques that lie at the intersection of complex analysis, modular forms, and number theory, continuing the tradition initiated by Riemann, Hardy, and Ramanujan.

The first part of the thesis concerns Hardy-Littlewood-Riesz-type equivalent criteria for the Riemann hypothesis. In 1916, Riesz established that RH is equivalent to the bound

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \exp\left(-\frac{x}{n^2}\right) = O_\epsilon\left(x^{-\frac{3}{4}+\epsilon}\right), \quad \text{as } x \to \infty,$$

and around the same time, Hardy and Littlewood derived another equivalent criterion while correcting an identity of Ramanujan. In this thesis, we establish a one-variable generalization of an identity of Hardy and Littlewood and as an application, we provide Riesz-type criteria for the Riemann hypothesis. We further establish a character analogue of our result and prove that the generalized Riemann hypothesis (GRH) for Dirichlet L-functions $L(s,\chi)$ is equivalent to the following asymptotic estimate. Let $k \geq 1$ and ℓ be positive real numbers. For any $\epsilon > 0$, we have

$$\sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^k} \exp\left(-\frac{x}{n^\ell}\right) = O_{\epsilon,k,\ell}\left(x^{-\frac{k}{\ell}+\frac{1}{2\ell}+\epsilon}\right), \quad \text{as } x \to \infty,$$

where χ is a primitive Dirichlet character modulo q. This bound generalizes the previous bounds given by Riesz, and Hardy-Littlewood. Analogous criteria are also established for L-functions associated with cusp forms, along with a generalization of the identity of Ramanujan, Hardy, and Littlewood for a general class of L-functions. These results provide a unified analytic framework for Riesz-type equivalences in a broader class of L-functions.

The second part of the thesis is devoted to the circle method and its applications in the theory of partitions. Building upon the Hardy–Ramanujan–Rademacher approach, we derive an exact, convergent series representation for the cubic over-

partition function and obtain an explicit error term that implies its log-concavity. Utilizing a result of Griffin, Ono, Rolen, and Zagier, we further establish higher-order Turán inequalities for cubic overpartition function. In addition, we obtain log-subadditivity and generalized log-concavity properties for cubic overpartitions inspired by the work of Bessenrodt-Ono and DeSalvo-Pak on the ordinary partition function. Subsequently, we obtain a Rademacher-type exact formula for the r-colored ℓ-regular partition function for any integers $r \geq 1$ and $\ell \geq 2$, thereby extending the earlier works of Hua and Hagis. As an application, we derive higher order Turán inequalities for the r-colored ℓ-regular partition function using a result of Griffin, Ono, Rolen, and Zagier. Furthermore, as additional consequences, we establish Rademacher-type exact formulas and higher order Turán inequalities for the r-colored distinct part partition function and for the sum of minimal excludants over ordinary partitions and overpartitions.

Keywords: Riemann zeta function, Dirichlet L-function, L-functions, Non-trivial zeros, Möbious function, Riemann hypothesis, Generalized Riemann hypothesis, Equivalent criteria, Circle method, Cubic overpartitions, Rademacher-type exact formula, Log-concavity, Higher-order Turán inequality, r-colored partitions, ℓ-regular partitions.

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Chapter 1

INTRODUCTION

1.1 A journey through the mystique of the Riemann zeta function

In the vast realm of mathematics, the Riemann zeta function, denoted as $\zeta(s)$, stands as one of the most profound objects in the modern number theory. It has captivated mathematicians, not only for its elegance but also for the fundamental insights it seems to hold about the fabric of numbers. At its most basic, the Riemann zeta function is defined as the infinite series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

converges absolutely and uniformly in any compact subset of $\mathrm{Re}(s) > 1$ and hence defines an analytic function in this domain. The study of Riemann zeta function has spanned over a century, evolving from an initial curiosity about the sums of reciprocals of powers of integers to a central figure in the understanding of prime number distribution. Long before Bernhard Riemann explored its depths in the 19th century, pieces of this puzzle had already emerged. In the 14th century, scholars understood that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Then, in the 18th century, Leonhard Euler shocked the mathematical world by discovering that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and went on to find beautiful expressions for even positive integer values of the zeta function using Bernoulli numbers. Mainly, he showed that

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}, \quad n \in \mathbb{N}.$$

But it was Riemann who, in 1859, transformed this curiosity into something profound. In a brief yet groundbreaking paper [103], he extended the zeta function into the complex domain and proposed what would become one of the most famous and

challenging problems in mathematics: the Riemann hypothesis (RH). This conjecture posits that all the non-trivial zeros of the Riemann zeta function lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. To contrast, the trivial zeros occur at the negative even integers and are well-understood. But the non-trivial ones-those lying somewhere in the critical strip $0 < \mathrm{Re}(s) < 1$, were still shrouded in mystery.

1.1.1 Modular relations and Riesz-type criteria

In the study of Riemann hypothesis and its generalizations to a broader class of L-functions, the presence of deep symmetries in their functional equations is a theme that occurs frequently. Among all these symmetries, the modular transformations play a significant role. These transformations connect two variables α and β via a function f such that $f(\alpha) = f(\beta)$ with $\alpha\beta =$ constant. Then by converting expressions in terms of α into the equivalent ones in terms of β, modular transformations enables us to do the unified analysis of the dual asymptotic behaviours.

This structural duality has profound consequences. In many cases, the modular relation satisfied by the coefficients of the Dirichlet series mirrors the functional equation of the corresponding L-function. Chandrasekharan and Narasimhan, in their fundamental work [28], established that such formulations are not just formal analogies, but are often equivalent formulations of the functional equation itself. In the context of this thesis, which seeks equivalent criteria for the Riemann hypothesis for general class of L- functions, understanding and utilizing these modular symmetries becomes essential. They not only guide the analytic continuation and symmetric properties of L-functions, but also shape the equivalent conditions under which the non-trivial zeros lie on the critical line.

Over the years, numerous equivalent formulations for the Riemann hypothesis have been proposed, each offering a distinct analytic or arithmetic prospective on this central problem in Number Theory. Among these, Littlewood established the following bound equivalent to the Riemann hypothesis.

$$\sum_{1\le n\le x} \mu(n) = O(x^{1/2+\epsilon}), \quad \text{for any } \epsilon > 0.$$

Another noteworthy criterion was established by Riesz [104] in 1916, which links the Riemann hypothesis to the decay rate of the series involving the Möbious function $\mu(n)$. Riesz showed that the Riemann hypothesis is equivalent to the bound

$$P_2(x) := \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \exp\left(-\frac{x}{n^2}\right) = O_\epsilon\left(x^{-\frac{3}{4}+\epsilon}\right), \quad \text{as } x \to \infty, \tag{1.1.1}$$

for any positive ϵ. Motivated by this result, Hardy and Littlewood [69, p. 161] established another equivalent criterion for the Riemann hypothesis while addressing a

discrepancy in an identity of Ramanujan. Mainly, they showed that

$$P_1(x) := \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \exp\left(-\frac{x}{n^2}\right) = O_\epsilon\left(x^{-\frac{1}{4}+\epsilon}\right), \quad \text{as } x \to \infty. \tag{1.1.2}$$

They were inspired from the following identity found in Ramanujan's notebooks [98, p. 312], [15, Equation (37.3), p. 470], where he mentioned that for any $x > 0$,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \exp\left(-\frac{x}{n^2}\right) = \sqrt{\frac{\pi}{x}} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \exp\left(-\frac{\pi^2}{n^2 x}\right). \tag{1.1.3}$$

Ramanujan, during his stay at Trinity College, Cambridge, showed the above identity to Hardy and Littlewood from his Second Notebook [98, p. 312]. Unfortunately, all the identities on page 312 are incorrect. However, Berndt later [15, p. 468-469] presented a compelling numerical explanation that offers a detailed explanation of the inaccuracy in the identity (1.1.3). Hardy and Littlewood [69, p. 156, Section 2.5] established a corrected version of (1.1.3). They proved that, for $x > 0$,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \exp\left(-\frac{x}{n^2}\right) = \sqrt{\frac{\pi}{x}} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \exp\left(-\frac{\pi^2}{n^2 x}\right) - \frac{1}{2\sqrt{\pi}} \sum_{\rho} \left(\frac{\pi}{\sqrt{x}}\right)^{\rho} \frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)}, \tag{1.1.4}$$

where the right-side sum over ρ runs through the non-trivial zeros of the Riemann zeta function $\zeta(s)$, assuming all such zeros are simple. Simplicity hypothesis asserts that all the non-trivial zeros of $\zeta(s)$ are simple. This identity reflects a deeper modular symmetry. It connects asymptotic expansions - a hallmark of modular type transformations. A more symmetric form of the above identity is the following. For two positive real numbers α, β with $\alpha\beta = \pi$, we have

$$\sqrt{\alpha} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \exp\left(-\left(\frac{\alpha}{n}\right)^2\right) - \sqrt{\beta} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \exp\left(-\left(\frac{\beta}{n}\right)^2\right) = -\frac{1}{2\sqrt{\beta}} \sum_{\rho} \frac{\Gamma\left(\frac{1-\rho}{2}\right) \beta^{\rho}}{\zeta'(\rho)}. \tag{1.1.5}$$

The convergence of the sum over ρ on the right-hand side of (1.1.4) and (1.1.5) is not immediately evident. While numerical evidence suggests rapid convergence, regorous justification is known only under additional assumptions. In particular, convergence can be shown by grouping the zeros that lie sufficiently close in the critical strip. That is, if ρ_1 and ρ_2 are two non-trivial zeros of $\zeta(s)$, they are said to lie in the same bracket if they satisfy

$$|\operatorname{Im}(\rho_1) - \operatorname{Im}(\rho_2)| < \exp\left(-\frac{A_0 \operatorname{Im}(\rho_1)}{\log(\operatorname{Im}(\rho_1))}\right) + \exp\left(-\frac{A_0 \operatorname{Im}(\rho_2)}{\log(\operatorname{Im}(\rho_2))}\right), \tag{1.1.6}$$

for some positive constant A_0. A more detailed discussion of this bracketing method and related convergence issues involving $\zeta'(\rho)$ in the denominator has been provided in [79, p. 783].

The identity (1.1.5) has attracted the attention of many mathematicians over

the years. Readers are encouraged to see Berndt [15, p. 470], Paris and Kaminski [93, p. 143], and Titchmarch [110, p. 219] for more information related to the identity (1.1.5). Interestingly, Bhaskaran [17] found a connection between Fourier reciprocity and Wiener's Tauberian theory. Ramanujan himself indicated a nice generalization of (1.1.5) associated to a pair of reciprocal functions, and this was worked out by Hardy and Littlewood [69, p. 160, Section 2.5].

An elegant character analogue of (1.1.5) was established by Dixit [45] in 2012. Later, Dixit, Roy and Zaharescu [48] studied a one-variable generalization of (1.1.5) and derived equivalent criteria for the Riemann hypothesis, and also for the generalized Riemann hypothesis. They [47] also found an identity analogous to (1.1.5) for L-functions associated to Hecke eigenforms. Recently, Dixit, Gupta and Vatwani [46] obtained a generalization for the Dedekind zeta function and also established an equivalent criterion for the extended Riemann hypothesis. In 2023, Banerjee and Kumar [11] derived an analogous identity corresponding to L-functions associated to the primitive cusp forms over $\Gamma_0(N)$ and found Riesz-Hardy-Littlewood type equivalent criteria for the corresponding L-function. Very recently, Gupta and Vatwani [60] further studied this identity for L-functions in Selberg class and obtained Riesz-type equivalent criteria. Interested readers can also see the work of Báez-Duarte [10], where a sequential Riesz-like criteria for the Riemann hypothesis was presented. This phenomenon was further studied by Cislo and Wolf [34]. In a very recent work, Bansal and Maji [12] showed that the identity (1.1.4) is equivalent to the functional equation of $\zeta(s)$.

Now we mention a character analogue of (1.1.5) that was obtained by Dixit [45, Theorem 1.9]. Mainly, he proved the following identity.

Theorem 1.1.1. *Let α and β be two positive real numbers with $\alpha\beta = 1$. Let χ be an odd primitive Dirichlet character modulo q. We have*

$$\begin{aligned}&\alpha\sqrt{\alpha}\sqrt{G(\chi)}\bigg(\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^2}\exp\left(-\frac{\pi\alpha^2}{qn^2}\right)-\frac{q}{4\pi\alpha^2}\sum_{\rho}\frac{\Gamma(\frac{2-\rho}{2})}{L'(\rho,\chi)}\left(\frac{\pi}{q}\right)^{\frac{\rho}{2}}\alpha^{\rho}\bigg)\\ &=\beta\sqrt{\beta}\sqrt{G(\chi)}\bigg(\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^2}\exp\left(-\frac{\pi\beta^2}{qn^2}\right)-\frac{q}{4\pi\beta^2}\sum_{\rho}\frac{\Gamma(\frac{2-\rho}{2})}{L'(\rho,\overline{\chi})}\left(\frac{\pi}{q}\right)^{\frac{\rho}{2}}\beta^{\rho}\bigg),\end{aligned}\tag{1.1.7}$$

where ρ runs over the non-trivial zeros of $L(s,\chi)$ and $L(s,\overline{\chi})$ are assumed to be simple.

Here we remark that the convergence of the above two infinite series over all the non-trivial zeros of $L(s,\chi)$ and $L(s,\overline{\chi})$ can be shown under the similar bracketing condition (1.1.6). Corresponding to an even character χ, an identity similar to (1.1.7) was also obtained by Dixit [45, Theorem 1.10].

In Chapter 3 of this thesis, we investigate a one variable generalization of Ramanujan–Hardy–Littlewood identity (1.1.4) and use it to derive a family of equivalent criteria for the Riemann hypothesis. In Chapter 4, we further establish a character analogue and derive a one-variable generalization of the above identity (1.1.7) of Dixit. Furthermore, the generalized character analogue identity motivates us to derive equivalent criteria for the generalized Riemann hypothesis (GRH) for the Dirichlet L-function. Our identity not only recovers the known results of Hardy and Littlewood as special cases but also leads to analogous identities for Dirichlet L-functions establishing equivalent formulations for Generalized Riemann hypothesis (GRH). Moreover, in Chapter 5, we further extend our analysis to a more broader framework of a general class of L-functions, which encompasses a wide variety of Dirichlet series satisfying functional equation of a specific type.

1.2 Introduction to circle method

In 1918, without access to modern computers or even a calculator, Hardy and Ramanujan derived an astonishingly accurate formula to estimate the number of integer partitions of a number like 100 – a number with 9 digits of possible partitions, using little more than contour integrals, modular transformations and deep intuitions. What is perhaps more remarkable is that their result predicted the exponential growth of this combinatorial function with uncanny precision. This was not a lucky guess or a numerical trick, but the beginning of what would become one of the most powerful analytic tools in number theory – The Circle Method.

This breakthrough emerged from one of the most fascinating collaborations in the history of Mathematics of deeply intuitive Indian mathematician Srinivasa Ramanujan and the rigorous Cambridge analyst G.H. Hardy. Ramanujan had come to England just a few years prior, after sending Hardy a letter containing numerous enigmatic yet strikingly elegant formulas. Hardy instantly recognized the raw genius in Ramanujan's untrained but deeply insightful work. Their partnership, though tragically short due to Ramanujan's early death in 1920, transformed various areas of mathematics. Among their many discoveries, the asymptotic formula for the partition function [71] for both its elegance and novel analytic approach that inspired it.

The core idea of the circle method arises naturally from the use of generating functions and complex analytic techniques. For the partition function $p(n)$, the generating function was first given by Euler [52] way back in 1748,

$$f(q) := \sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

Here and throughout this thesis, q is a complex number with $|q| < 1$ and $(a;q)_\infty$ denotes the q-Pochhammer symbol defined as

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

The coefficient of $p(n)$ can be extracted using Cauchy's integral formula:

$$p(n) = \frac{1}{2\pi i}\int_C \frac{f(q)}{q^{n+1}}\mathrm{d}q,$$

where C denotes a positively oriented contour around the origin, typically taken as the unit circle $C : |q| = r < 1$. The analytic challenge is to evaluate or approximate this integral for large n.

Hardy and Ramanujan observed that $f(q)$ exhibits modular-type transformations when $q = e^{2\pi i\tau}$ with $\mathrm{Im}(\tau) > 0$ due to the transformation properties of the Dedekind eta-function. This observation allowed them to approximate the generating function near the points of the form $q = e^{2\pi i\tau\frac{h}{k}}$, where h and k are coprime integers. They divided the unit circle into small arcs centered at these points-called major arcs and the remaining portions-called minor arcs, where the function behaves more irregularly but can be bounded by analytic estimates. The integral over the major arcs contributes the dominant asymptotic term, while that over the minor arcs gives a small error term. This decomposition and the subsequent asymptotic evaluation produced the celebrated Hardy-Ramanujan formula [71],

$$p(n) \sim \frac{1}{4n\sqrt{3}}\exp\left(\pi\sqrt{\frac{2n}{3}}\right), \quad \text{as} \quad n\to\infty. \tag{1.2.1}$$

What made their technique revolutionary was the way they approximated the behaviour of the integrand near the point $q = e^{2\pi i\tau\frac{h}{k}}$ corresponding to the rational number $\frac{h}{k}$. Though their arguments were not completely rigorous as per today's standards, but they introduced the essential blueprint for the circle method. Later in 1937, Rademacher built upon their foundation, replacing asymptotics with exact formulas for the partition function. Using the modular properties of the generating function and refining the analytic machinery introduced by Hardy and Ramanujan, Rademacher [99, 100] obtained an absolutely convergent series for $p(n)$,

$$p(n) = 2\pi\left(\frac{1}{6\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}}\right)^{\frac{3}{2}}\sum_{k=1}^{\infty}\frac{A_k(n)}{k}I_{\frac{3}{2}}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right), \tag{1.2.2}$$

where I_ν denotes the modified Bessel function of the first kind and $A_k(n)$ is a Kloosterman-type sum involving exponential terms, defined as

$$A_k(n) = \sum_{\substack{h \bmod k\\(h,k)=1}} e^{i\pi s(h,k)-2\pi i n\frac{h}{k}},$$

with $s(h,k)$ being the Dedekind sum given by

$$s(h,k) := \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right). \tag{1.2.3}$$

It is worth noting that Rademacher [99, Equation (1.8)] first used the path considered by Hardy and Ramanujan to derive the above exact formula (1.2.2) for $p(n)$. Later, he [100] modified the path using Ford circles based on Farey fractions in the unit interval, which we will briefly discuss in Section 2.4. The first term of the series (1.2.2) aligns with the asymptotic formula (1.2.1) established by Hardy and Ramanujan. Their proof fundamentally relied on the fact that the generating function for the partition function has a connection with the Dedekind eta function, which is a half integral weight modular form and its transformation formula played a crucial role. Along with Zuckerman [102], Rademacher extended his method for the Fourier coefficients of certain modular forms of positive weight. Building upon this approach, Zuckerman [114] derived exact convergent series for the Fourier coefficients, at any cusp, of all weakly holomorphic modular forms of negative weight associated with any congruence subgroup of $SL_2(\mathbb{Z})$. In 2012, Bringmann and Ono [22] further extended the work of Rademacher and Zuckerman for the coefficients of harmonic Maass forms of weight less than or equal to half.

Meanwhile, Hardy and Littlewood were working on applying this technique to a wider class of problems in additive number theory. Their chief target was Waring's problem, an ancient question which asks whether every natural number n can be expressed as the sum of fixed number of k-th powers of natural numbers. For example, can every natural number be expressed as sum of four squares? or nine cubes? While there are partial answers to these questions since the time of Lagrange, Hardy and Littlewood aimed at quantifying how many such representations existed for large n and under what conditions. Their adaptation of the circle method to this setting was immensely fruitful. They developed what is now known as the Hardy-Littlewood circle method [70], which provided not only the asymptotic formulas but also profound insight into the structure of additive number theory. Their work laid the foundation for major advances, including progress on the Goldbach problem and related additive questions.

This generalization required dealing with exponential sums and intricate estimates on minor arc contributions, but the core idea remained, transform the problem into an integral over the unit circle, isolate the dominant contributions, and bound the rest. Their work not only resolved many long lasting problems but also inspired the host of related developments in additive number theory leading to the solution of ternary Goldbach problem and beyond.

Over time, the circle method evolved, enriched by contributions from mathematicians like Davenport, Vaughan and more recently, Bringmann and Ono. Its reach extended to mock theta functions, and automorphic representations. The method has been adapted to study ranks and cranks of partitions, generating functions associated with L-functions and many instances of modular type objects that encode deep arithmetic information. The circle method continues to serve as a unifying analytic framework for understanding deep arithmetic and combinatorial phenomena.

What makes the circle method enduring is not merely its power to produce results, but its flexibility and universality. It provides a bridge between discrete arithmetic structures and analytic tools, revealing hidden regularities in seemingly chaotic number–theoretic data. In this thesis, we apply the circle method to obtain Rademacher-type exact formulas for cubic overpartitions and r-colored ℓ-regular partitions. These results illustrate how the foundational ideas of Hardy, Ramanujan, and Rademacher continue to inspire and shape contemporary research in analytic number theory.

1.2.1 Rademacher-type exact formula

Hardy–Ramanujan-type asymptotic formula (1.2.1) and Rademacher-type exact formula (1.2.2) for various partition functions have been investigated by several mathematicians including Andrews [6], Bringmann-Ono [21], Bringmann-Mahlburg [19], Grosswald [58, 59], Hagis [61]-[68], Hua [73], Iseki [75], Niven [88], and Sills [107]. For the modern framework of Rademacher-type expansions for the coefficients of harmonic Maass forms, we refer the reader to [20].

Over time, numerous analogues and generalizations of the partition function have been introduced. One such example is the *cubic partition function* introduced by Chan [25] in relation to Ramanujan's cubic continued fraction. He further studied Ramanujan-type congruences [26, 27] modulo prime powers. This function is denoted by $a(n)$, which counts the number of ways n can be written as sum of natural numbers where even numbers can appear in two different colors. For example, the number 3 has 4 such partitions, namely, $3,\ 2_R+1,\ 2_B+1$ and $1+1+1$, where the subscripts R and B denote the colors Red and Blue, respectively. The generating function for $a(n)$ is given by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}.$$

The arithmetic properties of the cubic partition function, particularly Ramanujan-style congruences for various primes have been investigated extensively; for further details, we refer the reader to [5, 32, 72, 82, 85, 106] and the references therein. In

2024, Mauth [84, Theorem 3.1] applied Zuckerman's technique to derive an exact formula for $a(n)$.

In recent years, the study of overpartitions, first introduced by Corteel and Lovejoy [36], has become an active area of research in partition theory. Overpartitions are partitions in which the first occurrence (or equivalently last occurrence) of each distinct part may be overlined. This concept has yielded numerous combinatorial and analytic applications, especially in the theory of mock modular forms, ranks, and cranks. A Rademacher-type exact formula for overpartitions had already been established by Zuckerman [114, p. 321, Equation (8.53)] way back in 1939, which was later explicitly given by Sills [107],

$$\overline{p}(n) = \frac{\pi}{4\sqrt{2}n^{\frac{3}{4}}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{B_k(n)}{k} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{n}}{k}\right),$$

where

$$B_k(n) = \sum_{\substack{h \bmod k \\ (h,k)=1}} e^{\pi i(2s(h,k)-s(2h,k))-2\pi i n\frac{h}{k}},$$

and $s(h,k)$ is the Dedekind sum defined in (1.2.3).

An overpartition analogue of the cubic partition function was introduced by Kim [80], where the first occurrence of a part in a cubic partition may be overlined. If parts are repeated, only one of them is allowed to be over lined. For example, there are 12 cubic overpartitions of 3, namely, $3,\ \overline{3},\ 2_R+1,\ \overline{2}_R+1,\ 2_R+\overline{1},\ \overline{2}_R+\overline{1},\ 2_B+1,\ \overline{2}_B+1,\ 2_B+\overline{1},\ \overline{2}_B+\overline{1},\ 1+1+1$, and $\overline{1}+1+1$. The function counting such partitions is denoted by $\overline{a}(n)$ and its generating function is given by

$$\sum_{n=0}^{\infty} \overline{a}(n)q^n = \frac{(-q;q)_\infty(-q^2;q^2)_\infty}{(q;q)_\infty(q^2;q^2)_\infty}. \tag{1.2.4}$$

Despite the significant progress in exact formulas for various partition functions, a Rademacher-type exact formula for the cubic overpartition function $\overline{a}(n)$ has remained elusive. In Chapter 6 of the thesis, we establish an exact formula for $\overline{a}(n)$ expressed as a sum of two convergent series of Bessel functions weighted by Kloosterman-type sums. This result not only extends Rademacher's classical work but also illustrates the modular richness inherent in cubic overpartitions.

A natural generalization of $p(n)$ is the r-colored ordinary partition function $p^{(r)}(n)$ for any $r \in \mathbb{N}$. Numerous mathematicians have studied Ramanujan-type congruence properties of $p^{(r)}(n)$, see [14, 42]. Recently, for $1 \leq r \leq 24$, Pribitkin and Williams [96] obtained an exact formula for r-colored partitions using the duality between modular forms of weight $\frac{-r}{2}$ and $2+\frac{r}{2}$. Motivated by their work, we aim to establish a Rademacher-type exact formula for the r-colored partition function for

any $r \in \mathbb{N}$. Mainly, we prove the following result. For $r \in \mathbb{N}$ and $n > \frac{r}{24}$, we have

$$p^{(r)}(n) = 2\pi \sum_{k=1}^{\infty} \frac{1}{k} \sum_{m=0}^{\lfloor \frac{r}{24} \rfloor} p^{(r)}(m) A(n) I_{1+\frac{r}{2}} \left(\frac{4\pi}{k} \sqrt{\left(\frac{r}{24} - m\right)\left(n - \frac{r}{24}\right)} \right) \left(\frac{\frac{r}{24} - m}{n - \frac{r}{24}} \right)^{\frac{1}{2}+\frac{r}{4}},$$

where

$$A(n) := A_{k,r,m}(n) := \sum_{\substack{h \bmod k \\ \gcd(h,k)=1}} e^{ir\pi s(h,k) + \frac{2\pi i}{k}(mh' - nh)},$$

with $hh' \equiv -1 (\mathrm{mod}\ k)$ and $s(h,k)$ is defined in (1.2.3). However, later we came to know that the above formula has already been obtained by Iskander, Jain and Talvola [76, Theorem 1.1] for *any real* $r > 0$.

In 1942, Hua [73] obtained a Rademacher-type exact formula for partitions into distinct parts. Let $p_d(n)$ be the distinct part partition function. Hua showed that

$$p_d(n) = \frac{1}{\sqrt{2}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} E_k(n) \frac{\mathrm{d}}{\mathrm{d}n} J_0 \left(\frac{i\pi}{k\sqrt{3}} \sqrt{n + \frac{1}{24}} \right), \tag{1.2.5}$$

where

$$E_k(n) = \sum_{h \bmod k} e^{\pi i (s(h,k) - s(2h,k)) - 2\pi i n \frac{h}{k}},$$

and J_0 is the Bessel function of the first kind.

In Chapter 7 of this thesis, we extend Hua's result by establishing a Rademacher-type exact formula for r-colored distinct part partition function, which we denote by $p_d^{(r)}(n)$.

Moreover, for any positive integer $\ell \geq 2$, we denote ℓ-regular partitions of n as $b_\ell(n)$ that counts the number of partitions of n where parts are not divisible by ℓ. Many mathematicians studied Ramanujan-type congruence properties of $b_\ell(n)$ for different values of ℓ. Curious readers can see [13, 23, 39, 40, 94] and references therein. A Rademacher-type exact formula for ℓ-regular partitions was established by Hagis [68] in 1971. One of the main aims of this thesis is to generalize the result of Hagis by deriving an exact formula for r-colored ℓ-regular partition function $b_\ell^{(r)}(n)$, whose generating function is given by

$$\sum_{n=0}^{\infty} b_\ell^{(r)}(n) q^n = \frac{(q^\ell; q^\ell)_\infty^r}{(q;q)_\infty^r} = \prod_{n=1}^{\infty} \left(\frac{1 - q^{\ell n}}{1 - q^n} \right)^r, \tag{1.2.6}$$

where $(A;B)_\infty := \prod_{n=0}^{\infty}(1 - AB^n)$ for $|B| < 1$ and $A \in \mathbb{C}$.

1.2.2 Higher order Turán inequalities

In addition to exact formulas and asymptotics, another important direction in the study of partition functions is the investigation of Turán inequalities. A sequence

$\{t(n)\}$ of real numbers is said to be log-concave if it satisfies the Turán inequality:

$$t(n)^2 \geq t(n-1)t(n+1), \quad \text{for all} \quad n \geq 1.$$

The study of log-concavity and Turán inequalities play a pivotal role in combinatorics, number theory, and are deeply connected to the theory of real entire functions in the Laguerre–Pólya class, as well as to aspects of the Riemann hypothesis as discussed in [38, 43, 57, 109]. Beyond log-concavity, the study of higher-order Turán inequalities finds a natural framework within the theory of Jensen polynomials.

For a real sequence $\{t(n)\}$, the Jensen polynomial of degree d and shift n is defined as

$$J_t^{d,n}(X) := \sum_{i=0}^{d} \binom{d}{i} t(n+i)\, X^i. \tag{1.2.7}$$

We say that $\{t(n)\}$ satisfies the degree d Turán inequality at n if the Jensen polynomial $J_t^{d,n-1}(X)$ is hyperbolic i.e., has only real roots.

Mathematicians have been interested in studying Turán inequalities for various partition functions due to their deep connections with log-concavity, real-rootedness of polynomials, and asymptotic behaviour of combinatorial sequences. Log-concavity is a common property observed in many sequences arising in combinatorics. Prominent examples include the binomial coefficients, the Stirling numbers, and the Bessel numbers, see [108]. For the ordinary partition function $p(n)$, the following inequality:

$$p(n)^2 > p(n-1)p(n+1)$$

was first established by Nicolas [87] for all $n > 25$. This result was later reproved by DeSalvo and Pak [41] using Lehmer's estimate on the error term of the Hardy–Ramanujan–Rademacher formula for $p(n)$.

Recently, Chen, Jia and Wang [30] established the hyperbolicity of $J_p^{3,n-1}(X)$ for all $n \geq 94$. Further, in the same paper, they conjectured that for any integer $d \geq 1$ there exists an integer $N_p(d)$ such that $J_p^{d,n-1}(X)$ is hyperbolic for $n \geq N_p(d)$.

This conjecture was settled by Griffin, Ono, Rolen and Zagier [57, Theorem 5]. Moreover, they proved the hyperbolicity of Jensen polynomials for a wider class of sequences, namely for the Fourier coefficients of weakly holomorphic modular forms over the full modular group $SL_2(\mathbb{Z})$. They further connected their result via a more general phenomenon, in which Jensen polynomials for a class of sequences can be modelled by the Hermite polynomials $H_d(X)$, which is defined as

$$\sum_{d=0}^{\infty} H_d(X)\frac{t^d}{d!} = e^{-t^2+Xt} = 1 + Xt + \frac{X^2-2}{2!}t^2 + \frac{X^3-6X}{3!}t^3 + \cdots.$$

They proved the following beautiful result.

Theorem 1.2.1. *[57, Theorem 3 and 8] Let $\{\alpha(n)\}, \{A(n)\}, \{\delta(n)\}$ be sequences of*

positive real numbers, with $\delta(n)$ tending to 0. For integers $j \geq 0$, $d \geq 3$, suppose that there are real numbers $g_3(n), g_4(n), \ldots, g_d(n)$, for which

$$\log\left(\frac{\alpha(n+j)}{\alpha(n)}\right) = A(n)j - \delta(n)^2 j^2 + \sum_{i=3}^{d} g_i(n) j^i + o(\delta(n)^d), \quad \text{as } n \to \infty,$$

with $g_i(n) = o(\delta(n)^i)$ for each $3 \leq i \leq d$. Then we have

$$\lim_{n\to\infty}\left(\frac{\delta(n)^{-d}}{\alpha(n)} J_{\alpha}^{d,n}\left(\frac{\delta(n)X - 1}{\exp(A(n))}\right)\right) = H_d(X).$$

It is well-known that the Hermite polynomials have distinct real roots and this property of a polynomial with real coefficients is invariant under any linear transformation, which implies that Jensen polynomials are hyperbolic for large values of n.

Log-concavity for overpartitions was established by Engel [50] in 2017. Subsequently, Liu and Zhang [83] studied higher-order Turán inequalities for overpartitions. More recently, log-concavity, third-order Turán inequalities, and strict log-subadditivity for cubic partitions were examined by Li, Peng, and Zhang [33]. In 2024, Dong and Ji [49] established log-concavity and the third-order Turán inequality for the distinct partition function. Higher-order Turán inequalities for MacMahon's plane partitions were investigated by Ono, Pujahari, and Rolen [90].

The Hardy-Ramanujan-type asymptotic formula does not directly yield the log-concavity property for the corresponding partition function. In contrast, the Rademacher-type exact formula provides refined error terms that can be employed to establish log-concavity. Motivated by this, in Chapter 6 we first establish a Rademacher-type exact formula and, as an application, prove the second-order Turán inequality for cubic overpartitions. Furthermore, by applying the above result of Griffin, Ono, Rolen, and Zagier, we establish higher-order Turán inequalities for the cubic overpartition function $\overline{a}(n)$.

In 2021, Craig and Pun [37] examined the higher order Turán inequalities associated with the ℓ-regular partition function. In 2024, Dong and Ji [49] also studied the same for the partition function into distinct parts. In Chapter 7, we utilize the result of Griffin, Ono, Rolen and Zagier to obtain higher order Turán inequalities for r-colored ℓ-regular partitions.

1.3 Organization of the thesis

This thesis is organized into several chapters, each devoted to a specific aspect of the problems under investigation.

In Chapter 1, we provide a detailed overview of the two central themes of the thesis. We begin with a historical and conceptual introduction to the Riemann zeta function and the Riemann hypothesis, emphasizing modular relations and Riesz–Hardy–Littlewood type equivalent criteria. We then introduce the circle method, outlining its development from the work of Hardy and Ramanujan to Rademacher's exact formulas, and discuss its significance in partition theory. This chapter sets the stage for the analytic techniques and problems addressed throughout the thesis.

In Chapter 2, we collect the necessary background material and standard results that will be used repeatedly in later chapters. This includes properties of special functions, modular forms, Dirichlet characters, Bessel functions, and derivation of transformation formulas for cubic overpartitions and r-colored ℓ-regular partition function.

Chapter 3 is devoted to a one-variable generalization of the Ramanujan–Hardy–Littlewood identity. Using this generalization, we derive a family of Riesz-type equivalent criteria for the Riemann hypothesis. These results extend classical formulations and highlight the role of modular-type transformations in encoding information about the zeros of the Riemann zeta function.

In Chapter 4, we establish character analogues of the generalized identities developed earlier. We derive corresponding one-variable identities for Dirichlet L-functions and use them to obtain equivalent criteria for the GRH. These results recover known criteria as special cases.

Chapter 5 extends the analysis to a broader setting of L-functions. We show that the methods developed earlier continue to apply in this general framework and lead to equivalent criteria for the Riemann hypothesis for a wide class of Dirichlet series satisfying suitable functional equations.

In Chapter 6, we apply the circle method to establish a Rademacher-type exact formula for the cubic overpartition function. As an application of this exact formula, we prove the second-order Turán inequality and, using the framework of Jensen polynomials, derive higher-order Turán inequalities for cubic overpartitions.

Finally, Chapter 7 is devoted to r-colored ℓ-regular partitions. We first obtain a Rademacher-type exact formula for this partition functions, generalizing earlier results of Hua and Hagis. We then employ the theory of Jensen polynomials and the results of Griffin, Ono, Rolen, and Zagier to establish higher-order Turán inequalities for these functions.

The thesis concludes with a brief summary of the main results and a discussion of possible directions for future research.

Chapter 2

PRELIMINARIES

2.1 The Riemann zeta function

The Riemann zeta function, developed from Euler's evaluations and Riemann's analytic continuation, occupies a central role in number theory, especially due to the Riemann hypothesis. A fundamental property of the Riemann zeta function is its Euler product representation, valid for $\mathrm{Re}(s) > 1$:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

This formula expresses $\zeta(s)$ as product over all primes p, highlighting the deep connection between the zeta function and primes. It directly encodes the unique factorization of the prime numbers and serves as a bridge between multiplicative number theory and complex analysis.

Differentiating logarithmically, we obtain for $\mathrm{Re}(s) > 1$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

where $\Lambda(n)$ is the Van Mangoldt function defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, \quad p \text{ prime}, k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.1}$$

This representation is the starting point of deep analytic techniques relating zeros of $\zeta(s)$ to prime number theorems. The prime number theorem has a central position in analytic number theory and describes the asymptotic distribution of prime numbers. Early numerical investigations by Gauss in 1792 and later by Legendre in 1798 led them independently to conjecture a precise law governing the density of primes. After nearly a century of effort, this conjecture was finally proved in 1896 through the independent works of Hadamard and de la Vallée Poussin, marking a major milestone

in the development of the subject. If $\pi(x)$ denotes the number of primes not exceeding a real number $x > 0$, then the prime number theorem asserts that

$$\pi(x) \sim \frac{x}{\log(x)}, \quad \text{as} \quad x \to \infty.$$

For $x > 0$ and $n \in \mathbb{N}$, Chebyshev's $\psi(x)$-function is defined as

$$\psi(x) = \sum_{n \leq x} \Lambda(n),$$

where $\Lambda(n)$ is defined in (2.1.1). Then as $x \to \infty$, the prime number theorem is equivalent to the asymptotic relation

$$\psi(x) \sim x.$$

Riemann [103] showed that the distribution of primes can be well understood from the location of the non-trivial zeros of the Riemann zeta function. Although initially defined only for $\mathrm{Re}(s) > 1$, the Riemann zeta function admits a meromorphic continuation to the entire complex plane except for a simple pole at $s = 1$ with residue 1. A key structural result is the functional equation of $\zeta(s)$ relating values at s with values at $1 - s$, given as

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s),$$

where $\Gamma(s)$ denotes the gamma function, an extension of factorial function to complex numbers. Defining the completed zeta function

$$\xi(s) := \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

one obtains the symmetric relation

$$\xi(s) = \xi(1-s).$$

This symmetry of the functional equation is critical in understanding the behaviour of the zeros of $\zeta(s)$. The zeros of $\zeta(s)$ fall into two categories: trivial zeros, located at negative even integers $-2, -4, -6, \cdots$, arising from the poles of $\Gamma(s/2)$ in the functional equation and non-trivial zeros, which lie in the critical strip $0 < \mathrm{Re}(s) < 1$. The Riemann hypothesis states that all the non-trivial zeros of $\zeta(s)$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. Despite substantial numerical evidence and partial results, the hypothesis remains unproven and is one of the most important open problems in Mathematics. The distribution of these zeros has profound implications on the error term in the prime number theorem. In 1895, Van Mangoldt established

$$\psi(x) - x = -\sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log\left(1 - x^{-2}\right),$$

where ρ runs over the non-trivial zeros of $\zeta(s)$. This provides a more explicit connection between the prime number theorem and the zeros of $\zeta(s)$.

2.2 Dirichlet L-functions

In analytic number theory, the study of arithmetic functions twisted by multiplicative characters leads to a generalization of Riemann zeta function, namely the Dirichlet L-functions. These functions play a fundamental role in understanding the distribution of prime numbers in arithmetic progressions and form a cornerstone in understanding the exploration of the generalized Riemann hypothesis (GRH).

A Dirichlet character modulo q is a completely multiplicative arithmetic function $\chi : \mathbb{Z} \to \mathbb{C}$ with several important properties. Firstly, it is periodic with period q, that is, $\chi(n+q) = \chi(n)$ for all $n \in \mathbb{Z}$. Also, $\chi(n) = 0$ if $(n, q) \neq 1$ and it is non-zero otherwise. As already mentioned, it is a completely multiplicative function, which means $\chi(mn) = \chi(m)\chi(n)$ for all integers m, n. Among these characters, the principal character χ_0 modulo q is defined as $\chi_0(n) = 1$ whenever $(n, q) = 1$ and it takes value zero elsewhere. All other characters are called non-principal characters.

Dirichlet characters can be viewed as group homomorphisms from the multiplicative group of units $(\mathbb{Z}/q\mathbb{Z})^\times$ to the complex unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. The set of all Dirichlet characters modulo q forms a finite abelian group under pointwise multiplication with principal character serving as the identity element. The order of this group is given by the Euler's totient function $\phi(q)$.

For prime modulus p, Dirichlet character corresponds to the characters of the cyclic group $(\mathbb{Z}/p\mathbb{Z})^\times$. In case of composite modulus, characters can be constructed using Chinese remainder theorem from characters modulo prime powers. One of the important examples is the non-principal characters modulo 4, denoted by $\chi_{4,2}$ and defined as

$$\chi_{4,2}(n) = \begin{cases} 0, & \text{if } 2 \mid n, \\ 1, & \text{if } n \equiv 1 \pmod 4, \\ -1, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

This character plays a key role in expressing certain Epstein zeta functions which we will see in the next subsection.

Given a Dirichlet character χ modulo q, the Dirichlet L-function is defined as

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \text{for } \operatorname{Re}(s) > 1.$$

This function generalizes the Riemann zeta function, as for the principal character χ_0 modulo 1, it reduces to $\zeta(s)$. Analogous to Riemann zeta function, the Dirichlet

L-function also admits an Euler product representation over primes as:

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \quad \text{for } \operatorname{Re}(s) > 1.$$

This expression for $L(s,\chi)$ reflects the multiplicative nature of the character χ and connects the series to the prime numbers filtered by χ.

The Dirichlet L-function can be meromorphically continued to the entire complex plane except for a simple pole at $s = 1$, when χ is a principal character. In case of non-principal characters, Dirichlet L-function extends as an entire function. Similar to $\zeta(s)$, associated to a primitive character χ modulo $q > 1$, $L(s,\chi)$ obeys the following symmetrical functional equation:

$$\left(\frac{q}{\pi}\right)^{\frac{a+s}{2}} \Gamma\left(\frac{a+s}{2}\right) L(s,\chi) = \frac{G(\chi)}{i^a\sqrt{q}} \left(\frac{q}{\pi}\right)^{\frac{1-s+a}{2}} \Gamma\left(\frac{1-s+a}{2}\right) L(1-s,\overline{\chi}), \tag{2.2.1}$$

where $G(\chi)$ denotes the Gauss sum and the constant a is defined as

$$a := \begin{cases} 0, & \text{if } \chi \text{ is even,} \\ 1, & \text{if } \chi \text{ is odd.} \end{cases} \tag{2.2.2}$$

Here, $\bar{\chi}$ denotes the complex conjugate of character χ. Regarding the zeros of these functions, the generalized Riemann hypothesis (GRH) extends the classical Riemann hypothesis by stating that for every non-principal Dirichlet character χ, all the non-trivial zeros of $L(s,\chi)$ lie on the critical line $\operatorname{Re}(s) = \frac{1}{2}$.

This conjecture has profound implications in number theory, including the distribution of primes in arithmetic progressions and improved error estimated for prime number theorems for such progressions. Dirichlet L-functions were instrumental in proving Dirichlet's theorem on arithmetic progressions, which states that any arithmetic progression of the form $a, a+q, a+2q, \cdots$ with $(a,q) = 1$ contains infinitely many primes. The study of the zeros of the Dirichlet L-functions improves the understanding of the error terms in counting primes in arithmetic progressions.

Moreover, Dirichlet L-function serves as prototypes for more general L-functions appearing in the Langlands program and have deep connections with automorphic forms, representation theory and algebraic geometry, underscoring their central role in modern number theory. In the next section, we will see a more general class of L-functions.

2.3 A general class of L-functions

Definition 1. *Let $a(n)$ and $b(n)$ be two arithmetical functions, which are not identically zero and $a(1)b(1) \neq 0$. Associated with these functions, we define the two*

Dirichlet series as

$$\phi(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad \textit{for} \quad \mathrm{Re}(s) = \sigma > \sigma_a,$$

$$\psi(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}, \quad \textit{for} \quad \mathrm{Re}(s) = \sigma > \sigma_b.$$

Here, σ_a and σ_b denote the finite abscissae of absolute convergence for $\phi(s)$ and $\psi(s)$, respectively. It is assumed that both $\phi(s)$ and $\psi(s)$ can be analytically continued to the entire $\mathbb{C}$ except for a finite number of poles. Now, for a specified $\delta > 0$, we assume that $\phi(s)$ and $\psi(s)$ satisfy the following functional equation:

$$c^s \Gamma(As + B)\phi(s) = \nu c^{\delta - s} \Gamma(A(\delta - s) + B)\psi(\delta - s), \tag{2.3.1}$$

where $|\nu| = 1$, $A > 0$, $B \geq 0$ and $c \in \mathbb{R}^+$.

Examples from the above class of L-functions

The Riemann zeta function and the Dirichlet L-function trivially fall under the category of the above class of L-functions. A few more examples that belong to this class of L-functions are given below.

Example 1. *Let $r \in \mathbb{Z}$ and $\sigma_r(n)$ denote the sum of r^{th} powers of positive divisors of n. Then the L-function associated to $\sigma_r(n)$ is*

$$\sum_{n=1}^{\infty} \frac{\sigma_r(n)}{n^s} = \zeta(s)\zeta(s - r), \quad \text{for} \quad \mathrm{Re}(s) = \sigma > \max\{1, r + 1\},$$

and if r is an odd positive integer, this Dirichlet series satisfies the functional equation:

$$(2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s - r) = (-1)^{\frac{r+1}{2}}(2\pi)^{-(r+1-s)}\Gamma(r + 1 - s)\zeta(r + 1 - s)\zeta(1 - s), \tag{2.3.2}$$

which fits into the framework of (2.3.1) for $a(n) = b(n) = \sigma_r(n), c = \frac{1}{2\pi}, A = 1, B = 0, \delta = r + 1$ and $\nu = (-1)^{\frac{r+1}{2}}$. One can refer [28, Example 2] for further insight into the functional equation (2.3.2).

An intriguing observation about Example 1 is that, due to the Riemann hypothesis of $\zeta(s)$, the function $\zeta(s)\zeta(s - r)$ expects to have its non-trivial zeros lying on two distinct vertical lines $\mathrm{Re}(s) = \frac{1}{2}$ and $\mathrm{Re}(s) = r + \frac{1}{2}$. However, it offers surprising insights into the Riemann hypothesis for $\zeta(s)$ by presenting an entirely novel form of equivalent criteria, as demonstrated in Theorem 5.2.13. The next example corresponds to the cusp form of weight 12.

Example 2. *Let $\tau(n)$ denote the Ramanujan tau function, defined via the Fourier*

expansion of the discriminant modular form

$$\Delta(q) = \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}, \quad |q| < 1.$$

Here, $\Delta(q)$ is a cusp form of level 1 and weight 12. The associated L-function is formed using normalized coefficients $\tau_0(n) = \tau(n) n^{-\frac{11}{2}}$ and is defined as

$$L(\Delta, s) = \sum_{n=1}^{\infty} \frac{\tau_0(n)}{n^s} \quad \text{for} \quad \mathrm{Re}(s) = \sigma > 1.$$

This function admits analytic continuation to the whole complex plane as an entire function. Moreover, $L(\Delta, s)$ satisfies the following nice-looking functional equation:

$$(2\pi)^{-s} \Gamma\left(s + \frac{11}{2}\right) L(\Delta, s) = (2\pi)^{-(1-s)} \Gamma\left(\frac{13}{2} - s\right) L(\Delta, 1-s), \tag{2.3.3}$$

which aligns with the general form (2.3.1), with the parameters $a(n) = b(n) = \tau_0(n), c = \frac{1}{2\pi}, A = 1, B = \frac{11}{2}$, $\delta = 1$ and $\nu = 1$. It is conjectured that all the non-trivial zeros of $L(\Delta, s)$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$, aligning with the general form of Riemann hypothesis for L-functions associated with modular forms. This conjecture remains one of the key open problems in the theory of automorphic L-functions.

Bochner [18] showed that the functional equation (2.3.3) is equivalent to the modular relation

$$\sum_{n=1}^{\infty} \tau(n) e^{-nx} = \left(\frac{2\pi}{x}\right)^{12} \sum_{n=1}^{\infty} \tau(n) e^{-\frac{4\pi^2 n}{x}}.$$

More general setting for L-functions associated to cusp forms are stated below.

Example 3. *Let $f(z)$ be a holomorphic Hecke eigenform of weight ω for the full modular group $SL(2, \mathbb{Z})$. The Fourier expansion of $f(z)$ at the cusp $i\infty$ is given by*

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{\omega-1}{2}} e^{2\pi i n z},$$

and $\lambda_f(n)$ is the normalized nth Fourier coefficient, known to be multiplicative and satisfy the Deligne bound $|\lambda_f(p)| \leq 2$ for all primes p, as a consequence of the proof of the Weil conjectures. Then the L-function associated to $f(z)$ is defined by

$$L(f, s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_{p:\ \text{prime}} \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1} \quad \text{for} \quad \mathrm{Re}(s) > 1. \tag{2.3.4}$$

This Dirichlet series extends to an entire function and satisfies a symmetric functional equation, discovered by Hecke, of the form

$$\Lambda(f, s) = \Lambda(f, 1-s), \tag{2.3.5}$$

where $\Lambda(f, s) := (2\pi)^{-s} \Gamma\big(s + \frac{\omega-1}{2}\big) L(f, s)$. The functional equation (2.3.5) clearly synchronizes in the general form framework (2.3.1) with the parameters $a(n) = b(n) = \lambda_f(n), c = \frac{1}{2\pi}, A = 1, B = \frac{\omega-1}{2}$, $\delta = 1$ and $\nu = 1$. The Riemann hypothesis for $L(f, s)$

posits that all the non-trivial zeros of $L(f,s)$ are situated on the critical line $\mathrm{Re}(s)=\frac{1}{2}$. *This is a central aspect of the grand Riemann hypothesis and remains an area of deep investigation, connecting the analytic properties of $L(f,s)$ to the arithmetic of modular forms.*

Example 4. *Let $r_j(n)$ denotes the number of representations of a positive integer n as sum of j squares, that is*

$$r_j(n) := \#\left\{(x_1,\ldots,x_j)\in\mathbb{Z}^j : x_1^2+\cdots+x_j^2=n\right\}.$$

The corresponding Dirichlet series, often referred to as the Epstein zeta function for the identity matrix of size j is given by

$$\zeta_{id}(j,s) := \sum_{n=1}^{\infty}\frac{r_j(n)}{n^s}, \quad \text{for} \quad \mathrm{Re}(s)>\frac{j}{2}.$$

This function satisfies the following functional equation, originally due to Epstein [51]

$$\pi^{-s}\Gamma(s)\zeta_{id}(j,s)=\pi^{s-\frac{j}{2}}\Gamma\left(\frac{j}{2}-s\right)\zeta_{id}\left(j,\frac{j}{2}-s\right),$$

and it admits meromorphic continuation to the entire complex plane, with a simple pole at $s=\frac{j}{2}$. This is a special case of general functional equation (2.3.1) *with $a(n)=b(n)=r_j(n), c=\frac{1}{\pi}, A=1, B=0, \delta=\frac{j}{2}$ and $\nu=1$. When $j=2$, it has been conjectured that all the non-trivial zeros of $\zeta_{id}(2,s)$ lies on the critical line* $\mathrm{Re}(s)=\frac{1}{2}$.

Exploring whether a given subset $\mathcal{A}$ of natural numbers $\mathbb{N}$ can be used to represent every natural number n as a sum of elements from $\mathcal{A}$, and if so, determining the number of such representations, constitutes a fundamental and enduring topic in number theory. In the context of Example 4, the subset $\mathcal{A}$ is the collection of all perfect squares. Typically, the Epstein zeta functions are studied in the broader setting of positive definite quadratic forms. For a positive definite symmetric matrix $M\in\mathbb{R}^{j\times j}$, the Epstein zeta function is defined as

$$\zeta_M(j,s)=\sum_{\kappa\in\mathbb{Z}^j/\{0\}}(\kappa^T M\kappa)^{-s} \quad \text{for} \quad \mathrm{Re}(s)>\frac{j}{2}.$$

Example 4 corresponds to the special case, with M being identity matrix. In particular, for $j=2,4,6$ and 8, the Epstein zeta function can be expressed in terms of Riemann zeta function and certain Dirichlet L-functions. An elegant formula for $\zeta_{id}(2,s)$ due to Zucker [113, (1.1)] is

$$\zeta_{id}(2,s)=4\zeta(s)\beta(s), \tag{2.3.6}$$

where $\beta(s)$ denotes the Dirichlet beta function, defined as

$$\beta(s)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^s}=L(s,\chi_{4,2}),$$

where $\chi_{4,2}$ is the non-principal character modulo 4. The generalized Riemann hypoth-

esis (GRH) for Dirichlet L- functions implies that all the non-trivial zeros of $L(s, \chi_{4,2})$ lie on the critical line $\mathrm{Re}(s) = 1/2$. Consequently, under the assumption of GRH for $\beta(s)$, the Epstein Riemann hypothesis for $\zeta_{id}(2, s)$ will also hold true. However, for higher dimensions $j \geq 3$, it has been shown that the corresponding Epstein zeta functions $\zeta_{id}(j, s)$, fail to satisfy the Riemann hypothesis. In particular, a detailed analytical account for $j = 4$ is provided by Travenec and Samaj [111, (1.4)-(1.6)], with accompanying numerical evidences presented in [111, Section 5] supporting the conclusion that the non-trivial zeros of $\zeta_{id}(j, s)$ do not, in general lie solely on the critical line for $j \geq 4$.

We now turn our attention to another important family of L-functions arising from algebraic number theory, namely the Dedekind zeta functions. These are natural generalization of the Riemann zeta function to the number fields.

Example 5. *Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $D > 0$. The Dedekind zeta function associated to K is defined as*

$$\zeta_K(s) := \sum_{n=1}^{\infty} \frac{\mathsf{a}_n}{n^s} \quad \text{for} \quad \mathrm{Re}(s) > 1, \tag{2.3.7}$$

where a_n denotes the number of integral ideals in the ring of integers O_K of K with norm equal to n. The series (2.3.7) *admits meromorphic continuation to the whole complex plane with a simple pole at $s = 1$ and satisfies the following functional equation:*

$$\left(\frac{2\pi}{\sqrt{D}}\right)^{-s} \Gamma(s)\zeta_K(s) = \left(\frac{2\pi}{\sqrt{D}}\right)^{-(1-s)} \Gamma(1-s)\zeta_K(1-s), \tag{2.3.8}$$

due to [91, p. 266]. This fits in (2.3.1) *for $c = \frac{\sqrt{D}}{2\pi}, A = 1, B = 0, \delta = 1$ and $\nu = 1$. The extended Riemann hypothesis for the Dedekind zeta function posits that all the non-trivial zeros of $\zeta_K(s)$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. This hypothesis generalizes the classical Riemann hypothesis for $\zeta(s)$, and its validity has deep implications for the distribution of prime ideals over number fields.*

2.4 Farey Sequences and Ford Circles

In order to apply the circle method to derive an exact formula for the partition functions, it is essential to carefully choose an appropriate path of integration in the complex plane. This path is typically constructed using Farey fractions and Ford circles. In this section, we briefly recall these classical concepts from the work of Hardy-Ramanujan [71] and Rademacher [101].

A Farey sequence of order n, denoted as F_n, is an ascending sequence of reduced

fractions in $[0,1]$, whose denominator do not exceed n. Formally, one can write

$$F_n = \left\{ \frac{h}{k} \in \mathbb{Q} \cap [0,1] \;\middle|\; 0 \leq h \leq k \leq n,\ \gcd(h,k) = 1 \right\}.$$

The first few sequences are

$$F_1 = \left\{ \frac{0}{1}, \frac{1}{1} \right\}$$
$$F_2 = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\}$$
$$F_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\}$$
$$F_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}$$
$$F_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\}.$$

The following lemma establishes a fundamental property of Farey fractions.

Lemma 2.4.1. *If $\frac{h_1}{k_1}$ and $\frac{h_2}{k_2}$ are two adjacent Farey fractions, then*

$$\frac{h_2}{k_2} - \frac{h_1}{k_1} = \frac{1}{k_1 k_2}.$$

This is a well-known result, first noted by Farey [53] and later given its first proof by Cauchy [24]. A remarkably simple proof of this is due to Hurwitz [74].

We can associate a geometrical object to each Farey fraction $\frac{h}{k}$, known as the Ford circle, denoted by $C(h,k)$, in the complex plane with centre at $\left(\frac{h}{k}, \frac{1}{2k^2}\right)$ and radius $\frac{1}{2k^2}$. That is,

$$C(h,k) := \left\{ z \in \mathbb{C} \;\middle|\; \left| z - \left(\frac{h}{k} + \frac{i}{2k^2} \right) \right| = \frac{1}{2k^2} \right\}. \tag{2.4.1}$$

Several properties of Ford circles are noteworthy to mention. The Ford circle $C(h,k)$ lies entirely in the upper half plane and the real axis acts as a tangent at $(\frac{h}{k}, 0)$ as shown in Figure 2.1.

No two Ford circles intersect. Consider two Ford circles with centres $\left(\frac{h}{k}, \frac{1}{2k^2}\right)$ and $\left(\frac{h'}{k'}, \frac{1}{2k'^2}\right)$. Let d be the distance between their centres. Then

$$d^2 = \left(\frac{h'}{k'} - \frac{h}{k} \right)^2 + \left(\frac{1}{2k'^2} - \frac{1}{2k^2} \right)^2.$$

Depending on whether $d^2 - \left(\frac{1}{2k'^2} + \frac{1}{2k^2}\right)^2$ is positive, negative or zero, we will have our circle as non-intersecting, intersecting or touching at exactly one point respectively. In this case,

$$d^2 - \left(\frac{1}{2k'^2} + \frac{1}{2k^2} \right)^2 = \left(\frac{h'}{k'} - \frac{h}{k} \right)^2 + \left(\frac{1}{2k'^2} - \frac{1}{2k^2} \right)^2 - \left(\frac{1}{2k'^2} + \frac{1}{2k^2} \right)^2$$

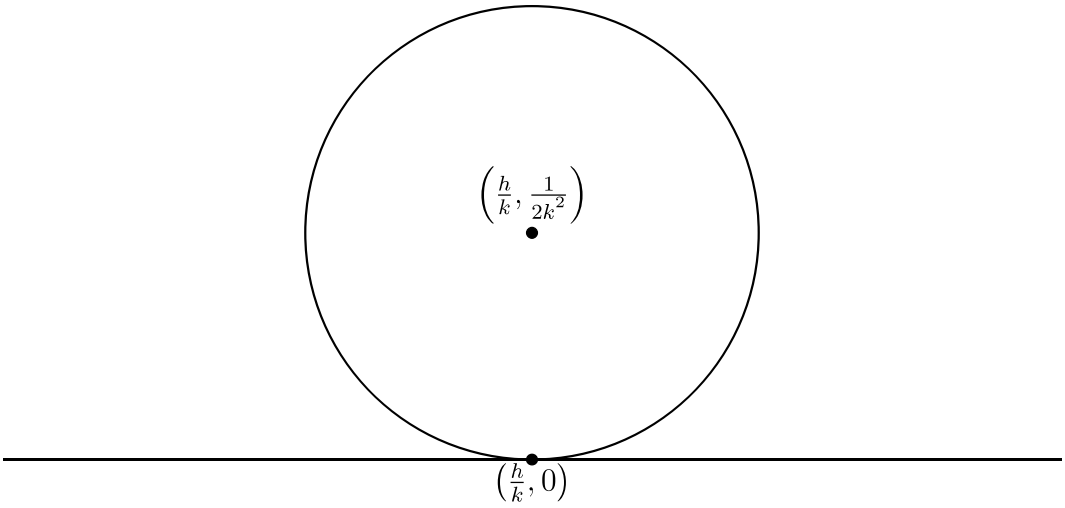


Figure 2.1

$$= \frac{(h'k - k'h)^2 - 1}{k'^2 k^2}.$$

In general, $h'k - k'h \geq 1$, which means circles are either non-intersecting or are tangent to each other. Thus, if $\frac{h}{k}$ and $\frac{h'}{k'}$ are two adjacent Farey fractions, then utilizing Lemma 2.4.1, it follows that the two Ford circles are tangent to each other. Hence, one can conclude that if $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ are three adjacent fractions in a Farey sequence, then $C(h, k)$ will be tangent to $C(h_1, k_1)$ and $C(h_2, k_2)$. Let α_1 and α_2 be the points of tangency. Readers can see the Figure 2.2 for clarity. One can easily

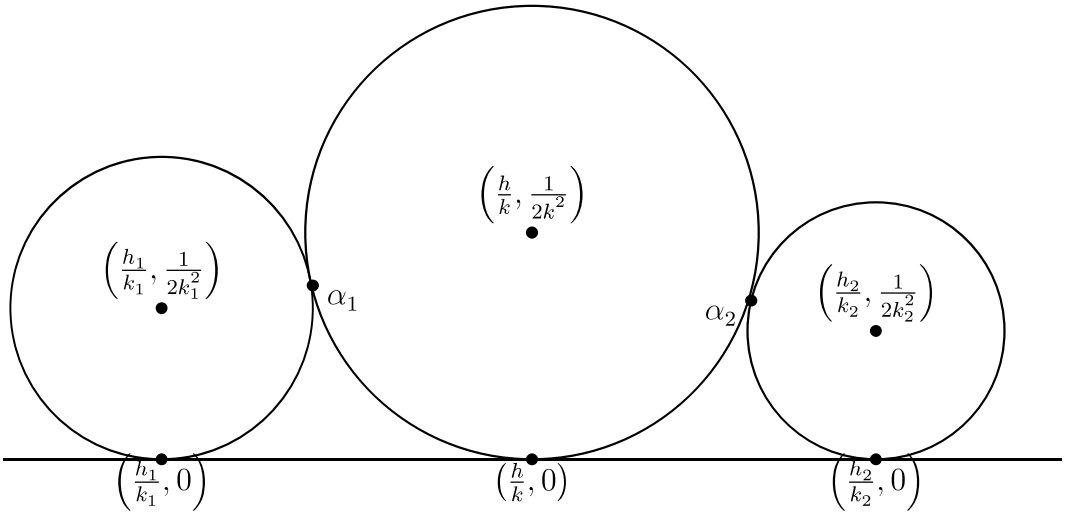


Figure 2.2

show that

$$\alpha_1 = \frac{h}{k} - \frac{k_1}{k(k^2+k_1^2)} + i\frac{1}{k^2+k_1^2}, \quad \alpha_2 = \frac{h}{k} + \frac{k_2}{k(k^2+k_2^2)} + i\frac{1}{k^2+k_2^2}.$$

The Farey sequence provides a systematic way to dissect the unit interval into subintervals corresponding to rational cusps, while Ford circles are non-overlapping regions associated to each cusp. While applying the Hardy–Ramanujan–Rademacher circle method for the generating function of cubic overpartitions and r-colored ℓ-regular partition function, we choose Rademacher's path of contour integration in which the path is carefully chosen to follow arcs of Ford circles up to a height determined by the Farey sequence of order N. This decomposition of the path of integration allows to find precise approximations of the generating function near each cusp, which is the cornerstone of deriving a Rademacher-type exact formula.

2.5 Transformation Formulas

We first state an important result about Dedekind sums, which will be a key ingredient in our derivation of the transformation formula.

Lemma 2.5.1. *If h_1 is an integer such that $hh_1 \equiv 1 \pmod{k}$, then $s(h,k) = s(h_1,k)$.*

Proof. See [78, Equation (68.5)] for a proof of this result. □

2.5.1 Transformation formula for the generating function of r-colored ℓ-regular partitions

Hardy and Ramanujan [71, Lemma 4.31] established a transformation formula for the generating function of $p(n)$, which was crucial in obtaining Rademacher's exact formula for $p(n)$. The same transformation formula will also be useful to obtain an exact formula for r-colored ℓ-regular partition function. Thus, for clarity of the reader, here we state the transformation formula for $f(q) = \frac{1}{(q)_\infty}$. For $\frac{h}{k} \in \mathbb{Q}$, $\mathrm{Re}(z) > 0$, they proved that

$$f\left(e^{2\pi i\left(\frac{h}{k}+\frac{iz}{k^2}\right)}\right) = \sqrt{\frac{z}{k}} e^{i\pi s(h,k)} e^{\frac{\pi}{12k}\left(\frac{k}{z}-\frac{z}{k}\right)} f\left(e^{2\pi i\left(\frac{h'}{k}+\frac{i}{z}\right)}\right), \tag{2.5.1}$$

with $hh' \equiv -1 \pmod{k}$. We now derive an analogous transformation formula for $R\left(e^{2\pi i\left(\frac{h}{k}+\frac{iz}{k^2}\right)}\right)$, where $R(q)$ is the generating function for r-colored ℓ-regular partition function $b_\ell^{(r)}(n)$, given by (1.2.6). Note that

$$R(q) = (L(q))^r, \tag{2.5.2}$$

where

$$L(q) = \frac{f(q)}{f(q^\ell)} = \frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty},$$

which is the generating function for ℓ-regular partition function $b_\ell(n)$. Before deriving the transformation formula for $R(q)$, we first establish the same for $L(q)$. To this end, we require a transformation formula for $f(q^\ell)$. Quite interestingly, we will see that the transformation formula for $f(q^\ell)$ is a piecewise function consisting of $d(\ell)$ cases, where $d(\ell)$ denotes the number of positive divisors of ℓ.

It is well known that the Dedekind eta function $\eta(\tau)$ has a close connection with the partition generating function. Mainly, for $\tau \in \mathbb{H}$, it is defined as

$$\eta(\tau) = e^{\frac{i\pi\tau}{12}} \prod_{m=1}^{\infty}(1-e^{2\pi im\tau}) = \frac{e^{\frac{i\pi\tau}{12}}}{f(e^{2\pi i\tau})}.$$

For a divisor Q of ℓ, we replace τ by $\frac{\ell}{Q}\left(\frac{h}{k}+\frac{iz}{k}\right)$ to see that

$$f\left(e^{2\pi i\frac{\ell}{Q}\left(\frac{h}{k}+\frac{iz}{k}\right)}\right) = e^{\frac{i\pi\ell}{12Q}\left(\frac{h}{k}+\frac{iz}{k}\right)}\eta^{-1}\left(\frac{\ell}{Q}\left(\frac{h}{k}+\frac{iz}{k}\right)\right). \tag{2.5.3}$$

We now recall a transformation formula for $\eta(\tau)$. For $\tau' \in \mathbb{H}$,

$$\eta\left(\frac{a\tau'+b}{c\tau'+d}\right) = \epsilon(a,b,c,d)\sqrt{\frac{c\tau'+d}{i}}\eta(\tau'), \quad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}), \tag{2.5.4}$$

where $\epsilon(a,b,c,d)$ is defined as

$$\epsilon(a,b,c,d) = \exp\left(\frac{i\pi}{12}\Phi\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right),$$

with

$$\Phi\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{cases} b+3, & \text{for } \ c=0, d=1, \\ -b-3, & \text{for } \ c=0, d=-1, \\ \frac{a+d}{c} - 12 \ \ \text{sign}(c)s(d,|c|), & \text{for } \ c\neq 0, \end{cases}$$

and $s(h,k)$ is the Dedekind sum defined in (1.2.3). To utilize the above transformation formula for $\eta(\tau)$, we require

$$\frac{a\tau'+b}{c\tau'+d} = \frac{\ell}{Q}\left(\frac{h}{k}+\frac{iz}{k}\right),$$

which suggest that, one should consider $\tau' = \frac{h_Q}{k} + \frac{i}{\ell kz}$ for some $h_Q \in \mathbb{Z}$ and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{\ell}{Q}h & -\frac{\frac{\ell}{Q}hh_Q+1}{k} \\ k & -h_Q \end{bmatrix}. \tag{2.5.5}$$

In order for $\frac{\frac{\ell}{Q}hh_Q+1}{k}$ to be an integer, we impose the condition that $\frac{\ell}{Q}hh_Q \equiv -1 (\text{mod } k)$.

Note that $c = k \neq 0$ implies that $\text{sign}(k) = 1$. Therefore, we have

$$\epsilon(a,b,c,d) = \exp\left(\frac{i\pi}{12}\left(\frac{\frac{\ell}{Q}h - h_Q}{k} - 12s(-h_Q,k)\right)\right). \tag{2.5.6}$$

Now we make use of Lemma 2.5.1 to see that $s(-h_Q,k) = s\left(\frac{\ell h}{Q},k\right)$. Hence, utilizing (2.5.4), (2.5.5) and (2.5.6) in (2.5.3), we get

$$f\left(e^{2\pi i\frac{\ell}{Q}\left(\frac{h}{k}+\frac{iz}{k}\right)}\right) = \sqrt{\frac{\ell z}{Q}}e^{i\pi s\left(\frac{\ell h}{Q},k\right)+\frac{\pi}{12k}\left(\frac{Q}{\ell z}-\frac{\ell z}{Q}\right)}f\left(e^{2\pi i\left(\frac{h_Q}{k}+\frac{iQ}{\ell kz}\right)}\right).$$

Now, replace k by $\frac{k}{Q}$ and then z by $\frac{z}{k}$ to have

$$f\left(e^{2\pi i\ell\left(\frac{h}{k}+\frac{iz}{k^2}\right)}\right) = \sqrt{\frac{\ell z}{kQ}}e^{i\pi s\left(\frac{\ell h}{Q},\frac{k}{Q}\right)+\frac{\pi}{12k}\left(\frac{Q^2k}{\ell z}-\frac{\ell z}{k}\right)}f\left(e^{2\pi iQ\left(\frac{h_Q}{k}+\frac{iQ}{\ell z}\right)}\right), \tag{2.5.7}$$

where $\frac{\ell}{Q}hh_Q \equiv -1 (\text{mod } \frac{k}{Q})$. Here, we must pay attention to the fact that for each divisor Q of ℓ with $\gcd(k,\ell) = Q$, we will get a different transformation formula. At this juncture, dividing (2.5.1) by (2.5.7), we obtain

$$L\left(e^{2\pi i\left(\frac{h}{k}+\frac{iz}{k^2}\right)}\right) = \sqrt{\frac{Q}{\ell}}e^{i\pi\left(s(h,k)-s\left(\frac{\ell h}{Q},\frac{k}{Q}\right)\right)+\frac{\pi}{12z}\left(1-\frac{Q^2}{\ell}\right)+\frac{\pi z}{12k^2}(\ell-1)}\frac{f\left(e^{2\pi i\left(\frac{h'}{k}+\frac{i}{z}\right)}\right)}{f\left(e^{2\pi iQ\left(\frac{h_Q}{k}+\frac{iQ}{\ell z}\right)}\right)}.$$

Finally, raising r-th power on both sides, we derive the following transformation formula for $R(q)$.

Lemma 2.5.2. *Let r,k and $\ell \geq 2$ be positive integers and $\gcd(\ell,k) = Q$. Let h_Q be an integer such that $\frac{\ell}{Q}hh_Q \equiv -1 (\text{mod } \frac{k}{Q})$, then we have the following transformation formula for $R(q)$:*

$$R\left(e^{2\pi i\left(\frac{h}{k}+\frac{iz}{k^2}\right)}\right) = \left(\frac{Q}{\ell}\right)^{\frac{r}{2}}e^{i\pi r\left(s(h,k)-s\left(\frac{\ell h}{Q},\frac{k}{Q}\right)\right)+\frac{\pi r}{12z}\left(1-\frac{Q^2}{\ell}\right)+\frac{\pi rz}{12k^2}(\ell-1)}\frac{F\left(e^{2\pi i\left(\frac{h'}{k}+\frac{i}{z}\right)}\right)}{F\left(e^{2\pi iQ\left(\frac{h_Q}{k}+\frac{iQ}{\ell z}\right)}\right)},$$

where $hh' \equiv -1 (\text{mod } k)$ and $F(q) = f(q)^r = (q)_\infty^{-r}$.

2.5.2 Transformation formula for generating function of cubic overpartition function

Here, we establish transformation formula for $f(q^2)$ and $f(q^4)$, with $q = e^{2\pi i\left(\frac{h}{k}+\frac{iz}{k^2}\right)}$, $\frac{h}{k} \in \mathbb{Q}, \text{Re}(z) > 0$, which will play crucial role in obtaining the exact formula for cubic overpartitions. By substituting $\ell = 2$ in (2.5.7), the transformation formula for $f(q^2)$

becomes

$$f\left(e^{4\pi i\left(\frac{h}{k}+\frac{iz}{k^2}\right)}\right)=\begin{cases}\sqrt{\frac{2z}{k}}e^{i\pi s(2h,k)}e^{\frac{\pi}{12k}\left(\frac{k}{2z}-\frac{2z}{k}\right)}f\left(e^{2\pi i\left(\frac{h_2}{k}+\frac{i}{2z}\right)}\right), & \text{if } k \text{ odd},\\ \sqrt{\frac{z}{k}}e^{i\pi s(h,\frac{k}{2})}e^{\frac{\pi}{6k}\left(\frac{k}{z}-\frac{z}{k}\right)}f\left(e^{4\pi i\left(\frac{h_3}{k}+\frac{i}{z}\right)}\right), & \text{if } k \text{ even},\end{cases} \tag{2.5.8}$$

where $2hh_2 \equiv -1 \pmod{k}$ and $hh_3 \equiv -1 \pmod{\frac{k}{2}}$. In a similar way we find the transformation formula for $f(q^4)$ by substituting $\ell = 4$ in (2.5.7),

$$f\left(e^{8\pi i\left(\frac{h}{k}+\frac{iz}{k^2}\right)}\right)=\begin{cases}\sqrt{\frac{4z}{k}}e^{i\pi s(4h,k)}e^{\frac{\pi}{12k}\left(\frac{k}{4z}-\frac{4z}{k}\right)}f\left(e^{2\pi i\left(\frac{h_4}{k}+\frac{i}{4z}\right)}\right), & \text{if } k \text{ odd},\\ \sqrt{\frac{2z}{k}}e^{i\pi s\left(2h,\frac{k}{2}\right)}e^{\frac{\pi}{12k}\left(\frac{k}{z}-\frac{4z}{k}\right)}f\left(e^{4\pi i\left(\frac{h_5}{k}+\frac{i}{2z}\right)}\right), & \text{if } k\equiv 2(\text{mod } 4),\\ \sqrt{\frac{z}{k}}e^{i\pi s\left(h,\frac{k}{4}\right)}e^{\frac{\pi}{3k}\left(\frac{k}{z}-\frac{z}{k}\right)}f\left(e^{8\pi i\left(\frac{h_6}{k}+\frac{i}{z}\right)}\right), & \text{if } k\equiv 0(\text{mod } 4),\end{cases} \tag{2.5.9}$$

where $4hh_4 \equiv -1 \pmod{k}$, $2hh_5 \equiv -1 \pmod{\frac{k}{2}}$, and $hh_6 \equiv -1 \pmod{\frac{k}{4}}$.

2.6 Some well-known results

In this section, we state a few well-known results, which will play a vital role in proving our main identities. We begin with an important summation formula, namely, Euler's summation formula.

Lemma 2.6.1. *Let $\{a_n\}$ be a sequence of complex number and $f(t)$ be a continuously differentiable function on $[1, x]$. Consider $A(x) := \sum_{1\le n\le x} a_n$. Then we have*

$$\sum_{1\le n\le x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)\mathrm{d}t. \tag{2.6.1}$$

Proof. Proof of this result can be found in [86, p. 17]. □

The next result gives important information about the asymptotic expansion of the gamma function.

Lemma 2.6.2. *In a vertical strip $\alpha \le \sigma \le \beta$, we have*

$$|\Gamma(\sigma+iT)| = \sqrt{2\pi}|T|^{\sigma-1/2}e^{-\frac{1}{2}\pi|T|}\left(1+O\left(\frac{1}{|T|}\right)\right) \quad \text{as} \quad |T|\to\infty. \tag{2.6.2}$$

This is popularly known as Stirling's formula. We now turn our attention to Meijer G-function [89, p. 415, Definition 16.17], a highly versatile special function that generalizes many classical special functions, including exponential, logarithmic, Bessel and hypergeometric functions. This function was introduced by Cornelis Simon

Meijer in 1930s and has since become an important tool in the theory of integral transforms and differential equations due to its unifying nature.

To formally define the Meijer G-function, let m, n, p, q be non-negative integers satisfying the inequalities $0 \leq m \leq q$, $0 \leq n \leq p$. Let $a_1, \cdots, a_p$ and $b_1, \cdots, b_q$ be complex parameters with the restriction $a_i - b_j \notin \mathbb{N}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. This condition ensures that the integrand's poles do not overlap in a way that would make the integrand ill-defined. The Meijer G-function is defined by the following line integral:

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix} \middle| z\right) := \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s) z^s}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} \mathrm{d}s, \quad (2.6.3)$$

where the line of integration L is taken to be a suitable path in complex plane running from $-i\infty$ to $+i\infty$, which separates the poles of the factors $\Gamma(1 - a_j + s)$ from those of the factors $\Gamma(b_j - s)$. The choice of the contour is essential for the convergence and correct analytic properties of the function. This integral converges under the conditions $p + q < 2(m + n)$ and $|\arg(z)| < \left(m + n - \frac{p+q}{2}\right)\pi$.

A fundamental result associated with the Meijer G-function is the Slater's theorem [89, p. 415, Equation 16.17.2] which expresses it as a finite linear combination of generalized hypergeometric functions. Specifically, if $p \leq q$ and $b_j - b_k \notin \mathbb{Z}$ for $j \neq k$, $1 \leq j, k \leq m$, then the Meijer G-function admits the following hypergeometric representation:

$$\begin{aligned} &G_{p,q}^{m,n}\left(\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix} \middle| z\right) \quad (2.6.4) \\ &= \sum_{k=1}^{m} A_{p,q,k}^{m,n}(z) {}_pF_{q-1}\left(\begin{matrix} 1 + b_k - a_1, \cdots, 1 + b_k - a_p \\ 1 + b_k - b_1, \cdots, *, \cdots, 1 + b_k - b_q \end{matrix} \middle| (-1)^{p-m-n} z\right), \end{aligned}$$

where ${}_pF_{q-1}$ denotes the generalized hypergeometric function and $*$ indicates the omission of the term $1 + b_k - b_k$ in the denominator and the coefficient $A_{p,q,k}^{m,n}(z)$ is given by:

$$A_{p,q,k}^{m,n}(z) := \frac{z^{b_k} \prod_{j=1, j\neq k}^{m} \Gamma(b_j - b_k) \prod_{j=1}^{n} \Gamma(1 + b_k - a_j)}{\prod_{j=m+1}^{q} \Gamma(1 + b_k - b_j) \prod_{j=n+1}^{p} \Gamma(a_j - b_k)}.$$

Now we state a result, from the theory of Bessel functions, which will be useful to prove our main result.

Lemma 2.6.3. *[112, p. 181, Equation (1)] For $c > 0$ and $\mathrm{Re}(\nu) > 0$, an integral representation of the modified Bessel function of the first kind is given by:*

$$I_\nu(z) = \frac{(z/2)^\nu}{2\pi i} \int_{c-\infty}^{c+i\infty} t^{-\nu-1} e^{t + \frac{z^2}{4t}} \mathrm{d}t. \quad (2.6.5)$$

Chapter 3

RIESZ-TYPE CRITERIA FOR THE RIEMANN HYPOTHESIS

All the results presented in this chapter are taken from a joint work with Agarwal and Maji [1]. In this chapter, we establish a one-variable generalization of the Ramanujan–Hardy–Littlewood identity (1.1.4), and as an application of our result, we obtain Riesz-type equivalent criteria for the Riemann hypothesis. In particular, we are able to recover the equivalent criterion (1.1.1) given by Riesz, as well as the bound (1.1.2) given by Hardy and Littlewood.

3.1 Preliminaries

In this section, we first state and prove a few results, which will play a vital role in proving our main identity. For $k \geq 1$, we first define a function:

$$P_k(x) := \sum_{n=1}^{\infty} \frac{\mu(n)}{n^k} \exp\left(-\frac{x}{n^2}\right). \tag{3.1.1}$$

To obtain the equivalent criteria (1.1.2) for the Riemann hypothesis, Hardy and Littlewood used the following crucial identity due to Riesz, namely,

$$\int_0^{\infty} x^{-s-1} P_1(x) \mathrm{d}x = \frac{\Gamma(-s)}{\zeta(2s+1)}, \tag{3.1.2}$$

valid in the region $0 < \mathrm{Re}(s) < 1$. The below result is a one variable generalization of this identity and one of the crucial ingredients in providing family of equivalent criteria for the Riemann hypothesis.

Lemma 3.1.1. *Let $k > 1$. In the region $\frac{1-k}{2} < \mathrm{Re}(s) < 0$, we have*[1]

$$\int_0^\infty x^{-s-1} P_k(x) \mathrm{d}x = \frac{\Gamma(-s)}{\zeta(2s+k)}.$$

Proof. Using the definition of $P_k(x)$, one can show that

$$P_k(x) := \sum_{n=1}^{\infty} \frac{\mu(n)}{n^k} \exp\left(-\frac{x}{n^2}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^k} \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{m! n^{2m}} = \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{m! \zeta(k+2m)}. \tag{3.1.3}$$

Here interchange of summations is possible since one can see that, for $k > 1$,

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left| \frac{\mu(n)}{n^{k+2m}} \frac{(-1)^m x^m}{m!} \right| \leq \zeta(k) e^x < \infty.$$

Also $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$ for $\mathrm{Re}(s) > 1$, [8, p. 228, example 1]. For $\frac{1-k}{2} < \mathrm{Re}(s) < 0$, use the series expansion of $\zeta(2s+k)$ to write

$$\begin{aligned} \zeta(2s+k) \int_0^\infty x^{-s-1} P_k(x) \mathrm{d}x &= \sum_{n=1}^{\infty} \frac{1}{n^k} \int_0^\infty \frac{x^{-s-1}}{n^{2s}} P_k(x) \mathrm{d}x \\ &= \sum_{n=1}^{\infty} \frac{1}{n^k} \int_0^\infty x^{-s-1} P_k\left(\frac{x}{n^2}\right) \mathrm{d}x \\ &= \int_0^\infty x^{-s-1} \sum_{n=1}^{\infty} \frac{1}{n^k} P_k\left(\frac{x}{n^2}\right) \mathrm{d}x. \end{aligned} \tag{3.1.4}$$

Here in the second last step we replaced x by x/n^2 and in the final step, we interchanged the order of summation and integration by an appeal to the Weierstrass M-test, since $\left|\frac{1}{n^k} P_k\left(\frac{x}{n^2}\right)\right| \leq \frac{\zeta(k)}{n^k}$ for $k > 1$. Now using (3.1.3), one can find that

$$\sum_{n=1}^{\infty} \frac{1}{n^k} P_k\left(\frac{x}{n^2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^k} \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{m! n^{2m} \zeta(k+2m)} = \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{m!} = e^{-x}. \tag{3.1.5}$$

On substituting (3.1.5) in (3.1.4), we get

$$\int_0^\infty x^{-s-1} P_k(x) \mathrm{d}x = \int_0^\infty x^{-s-1} e^{-x} \mathrm{d}x = \frac{\Gamma(-s)}{\zeta(2s+k)}, \tag{3.1.6}$$

for $\mathrm{Re}(s) < 0$. This completes the proof of Lemma 3.1.1. □

Now we state an important result, which provides us a bound for $1/\zeta(s)$.

Lemma 3.1.2. *Suppose, for every non-trivial zero ρ of $\zeta(s)$, there exist a sequence of arbitrary larger positive numbers T with $|T - \mathrm{Im}(\rho)| > \exp\left(-\frac{C_0 \mathrm{Im}(\rho)}{\log(\mathrm{Im}(\rho))}\right)$, where A is some positive constant. Then,*

$$\frac{1}{|\zeta(\sigma + iT)|} < e^{C_1 T},$$

[1] Lemma 2.4 in [1] has been corrected in this thesis, namely, we have proved that it is valid in the region $\frac{1-k}{2} < \mathrm{Re}(s) < 0$.

for some constant $0 < C_1 < \pi/4$.

Proof. One can find a proof of this result in [110, p. 219]. An important point to note that the assumption of the bracketing condition (1.1.6) guarantees the existence of such a sequence. □

We are now ready to state the main results of this chapter.

3.2 Main Results

Theorem 3.2.1. *Let* $k \geq 1$ *be a real number. Assume that all non-trivial zeros of* $\zeta(s)$ *are simple. Then for any positive* x*, we have*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^k} \exp\left(-\frac{x}{n^2}\right) = \frac{\Gamma(\frac{k}{2})}{x^{\frac{k}{2}}} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} {}_1F_1\left(\frac{k}{2}; \frac{1}{2}; -\frac{\pi^2}{n^2 x}\right) + \frac{1}{2} \sum_{\rho} \frac{\Gamma(\frac{k-\rho}{2})}{\zeta'(\rho)} x^{-\frac{(k-\rho)}{2}}, \tag{3.2.1}$$

where the sum over ρ *runs through all non-trivial zeros of* $\zeta(s)$ *and satisfies the condition* (1.1.6)*.*

Remark 1. *Substituting* $k = 1$ *in* (3.2.1)*, and replacing* x *by* π^2/x*, one can immediately obtain* (1.1.4)*.*

Letting $x \to \infty$ in (3.2.1) and assuming the absolute convergence of the right side series over the non-trivial zeros of $\zeta(s)$, one can immediately see that

$$P_k(x) := \sum_{n=1}^{\infty} \frac{\mu(n)}{n^k} \exp\left(-\frac{x}{n^2}\right) = O\left(x^{-\frac{k}{2}+\frac{1}{4}}\right),$$

under the assumption of the Riemann hypothesis. This observation motivated us to obtain the following equivalent criteria for the Riemann hypothesis.

Theorem 3.2.2. *For* $k > 1$*, the Riemann hypothesis is equivalent to the bound*

$$P_k(x) = O_\epsilon\left(x^{-\frac{k}{2}+\frac{1}{4}+\epsilon}\right), \quad \text{as } x \to \infty, \tag{3.2.2}$$

for any positive ϵ*.*

Remark 2. *The above result is also true for* $k = 1$ *due to Hardy and Littlewood [69, p. 161, Equation (2.544)].*

Remark 3. *Letting* $k = 2$ *gives the bound* (1.1.1) *of Riesz.*

3.3 Proof of Main results

Proof of Theorem 3.2.1. It is well-known that the gamma function $\Gamma(s)$ is the Mellin transform of e^{-x}. Therefore, e^{-x} is the inverse Mellin transform of $\Gamma(s)$. That is,

$$e^{-x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)x^{-s}\mathrm{d}s,$$

valid for any $c > 0$. If we shift the line of integration to the line $\mathrm{Re}(s) = c$ with $c \in (-1, 0)$, then we have

$$e^{-x} - 1 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)x^{-s}\mathrm{d}s, \tag{3.3.1}$$

as we will encounter a pole at $s = 0$. First, we recall that the prime number theorem is equivalent to the fact that $\sum_{n=1}^{\infty}\frac{\mu(n)}{n} = 0$. One can easily show that, for any $k \in \mathbb{N}$, the series $\sum_{n=1}^{\infty}\frac{\mu(n)}{n^k}\exp\left(-\frac{x}{n^2}\right)$ is convergent. In fact, for $k \geq 2$, it is absolutely convergent, whereas for $k = 1$, it is conditionally convergent. However, our proof is valid for any $k \geq 1$. With the help of (3.3.1), we can write

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^k}\exp\left(-\frac{x}{n^2}\right) &= \sum_{n=1}^{\infty}\frac{\mu(n)}{n^k} - \sum_{n=1}^{\infty}\frac{\mu(n)}{n^k}\left(1-\exp\left(-\frac{x}{n^2}\right)\right)\\ &= \sum_{n=1}^{\infty}\frac{\mu(n)}{n^k} + \sum_{n=1}^{\infty}\frac{\mu(n)}{n^k}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\left(\frac{x}{n^2}\right)^{-s}\mathrm{d}s\\ &= \sum_{n=1}^{\infty}\frac{\mu(n)}{n^k} + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)}{\zeta(k-2s)}x^{-s}\mathrm{d}s, \end{aligned}\tag{3.3.2}$$

where in the ultimate step we interchanged the summation and integration since the series $\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{k-2s}}$ is absolutely and uniformly convergent in any compact subset of the domain $\mathrm{Re}(k-2s) > 1$. Now we shall try to simplify the line integral present in (3.3.2). Let us denote this line integral as

$$I_k(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)}{\zeta(k-2s)}x^{-s}\mathrm{d}s. \tag{3.3.3}$$

First, we observe that the integrand has simple poles at non-positive integers due to the simple poles of $\Gamma(s)$. Again, for any $m \in \mathbb{N}$, $\frac{k}{2} + m$ is also a simple pole of the integrand since $\zeta(k-2s)$ has trivial zeros at $-2m$. Another important observation is that $\zeta(k-2s)$ has infinitely many non-trivial zeros in the critical strip $\frac{k}{2}-1 < \mathrm{Re}(s) < \frac{k}{2}$. If one assumes the Riemann hypothesis, then we can say all the non-trivial zeros of $\zeta(k-2s)$ will lie on the line $\mathrm{Re}(s) = \frac{k}{2} - \frac{1}{4}$. However, to obtain our main identity, we do not require the assumption of the Riemann hypothesis. Nevertheless, as we are interested in collecting the contributions of the residual terms corresponding to the non-trivial zeros of $\zeta(k-2s)$, we have to shift the line of integration to the line

$\mathrm{Re}(s) = \lambda$, where $\lambda \in \left(\frac{k}{2}, \frac{k}{2}+1\right)$. Thus, we consider the contour $\mathcal{C}$ determined by the line segments $[\lambda - iT, \lambda + iT], [\lambda + iT, c + iT], [c + iT, c - iT]$, and $[c - iT, \lambda - iT]$, where $-1 < c < 0$ and $\frac{k}{2} < \lambda < \frac{k}{2} + 1$, and T is some large positive number. Now appealing to Cauchy's residue theorem, we have

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\Gamma(s)}{\zeta(k-2s)} x^{-s} \mathrm{d}s = R_0 + \mathcal{R}_T(x), \tag{3.3.4}$$

where R_0 denotes the residual term due to the simple pole of $\Gamma(s)$ at $s = 0$, and $\mathcal{R}_T(x)$ denotes the sum of the residual terms corresponding to the non-trivial zeroes ρ of $\zeta(k-2s)$ with $|\,\mathrm{Im}(\rho)| < T$. Next, we shall show that the horizontal integrals

$$H_1(x;T) := \frac{1}{2\pi i}\int_{\lambda+iT}^{c+iT} \frac{\Gamma(s)}{\zeta(k-2s)} x^{-s}\mathrm{d}s, \quad \text{and} \quad H_2(x;T) := \frac{1}{2\pi i}\int_{c-iT}^{\lambda-iT} \frac{\Gamma(s)}{\zeta(k-2s)} x^{-s}\mathrm{d}s$$

vanish as T approaches to infinity. Here, we employ Stirling's formula for $\Gamma(s)$ i.e., Lemma 2.6.2, and along with Lemma 3.1.2, we find that

$$|H_1(x;T)| \ll \int_{\lambda}^{c} |T|^{C} \exp\left(2C_1 T - \frac{\pi}{2}|T|\right) \mathrm{d}T,$$

where C_1 is some constant, and $0 < C_1 < \pi/4$. This immediately implies that $H_1(x;T)$ vanishes as $T \to \infty$. In a similar way, one can show that $H_2(x;T)$ goes to zero as $T \to \infty$. Now letting $T \to \infty$ in (3.3.4), we arrive at

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\Gamma(s)}{\zeta(k-2s)} x^{-s}\mathrm{d}s = \frac{1}{2\pi i}\int_{\lambda-iT}^{\lambda+iT} \frac{\Gamma(s)}{\zeta(k-2s)} x^{-s}\mathrm{d}s - R_0 - \mathcal{R}(x), \tag{3.3.5}$$

where $\mathcal{R}(x)$ is the sum of the residual terms corresponding to the non-trivial zeros of $\zeta(k-2s)$. One can easily calculate the residue at $s = 0$,

$$R_0 = \lim_{s\to 0} \frac{s\Gamma(s)}{\zeta(k-2s)} x^{-s} = \frac{1}{\zeta(k)}. \tag{3.3.6}$$

If we assume that all the non-trivial zeros of $\zeta(s)$ are simple, then we get

$$\mathcal{R}(x) = \sum_{\rho} \lim_{s\to\frac{k-\rho}{2}} \frac{\left(s - \frac{k-\rho}{2}\right)\Gamma(s)}{\zeta(k-2s)} x^{-s} = -\frac{1}{2}\sum_{\rho} \frac{\Gamma(\frac{k-\rho}{2})}{\zeta'(\rho)} x^{-\frac{k-\rho}{2}}, \tag{3.3.7}$$

where the sum runs through all the non-trivial zeros $\zeta(s)$. More generally, if we do not assume the simplicity hypothesis, that is, if n_ρ is the multiplicity of ρ, then

$$\mathcal{R}(x) = \sum_{\rho} \frac{1}{(n_\rho - 1)!} \lim_{s\to\frac{k-\rho}{2}} \frac{\mathrm{d}^{n_\rho - 1}}{\mathrm{d}s^{n_\rho-1}} \left\{ \left(s - \frac{k-\rho}{2}\right)^{n_\rho} \frac{\Gamma(s)}{\zeta(k-2s)x^s} \right\}.$$

Substituting (3.3.6) and (3.3.7) in (3.3.5) and combining with (3.3.2) and (3.3.3), we obtain

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^k} \exp\left(-\frac{x}{n^2}\right) = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} \frac{\Gamma(s)}{\zeta(k-2s)} x^{-s}\mathrm{d}s + \frac{1}{2}\sum_{\rho} \frac{\Gamma(\frac{k-\rho}{2})}{\zeta'(\rho)} x^{-\frac{k-\rho}{2}}. \tag{3.3.8}$$

At this juncture, we shall try to simplify the right vertical integral

$$V(x,k) := \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} \frac{\Gamma(s)}{\zeta(k-2s)} x^{-s}\mathrm{d}s, \tag{3.3.9}$$

where $\frac{k}{2} < \lambda < \frac{k}{2} + 1$. Here we make use of the symmetric form of the functional equation of $\zeta(s)$, that is,

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Replace s by $k-2s$ and simplify to see

$$\frac{1}{\zeta(k-2s)} = \frac{\pi^{\frac{1}{2}-k+2s}\Gamma\left(\frac{k-2s}{2}\right)}{\Gamma\left(\frac{1-k+2s}{2}\right)\zeta(1-k+2s)}. \tag{3.3.10}$$

Plugging (3.3.10) in (3.3.9), one can see that

$$V(x,k) := \frac{\pi^{\frac{1}{2}-k}}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\Gamma(s)\Gamma\left(\frac{k}{2}-s\right)}{\Gamma\left(\frac{1-k}{2}+s\right)\zeta(1-k+2s)}\left(\frac{x}{\pi^2}\right)^{-s}\mathrm{d}s. \tag{3.3.11}$$

Here we can check that $1 < \mathrm{Re}(1-k+2s) < 3$ since $\frac{k}{2} < \mathrm{Re}(s) = \lambda < \frac{k}{2}+1$. Thus, we can write

$$\frac{1}{\zeta(1-k+2s)} = \sum_{n=1}^{\infty}\frac{\mu(n)n^{k-1}}{n^{2s}}.$$

Making use of the above series expansion in (3.3.11) and then interchanging the order of summation and integration, (3.3.11) reduces to

$$V(x,k) := \pi^{\frac{1}{2}-k}\sum_{n=1}^{\infty}\mu(n)n^{k-1}I(X_n,k), \tag{3.3.12}$$

where

$$I(X_n,k) := \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\Gamma(s)\Gamma\left(\frac{k}{2}-s\right)}{\Gamma\left(\frac{1-k}{2}+s\right)}X_n^{-s}\mathrm{d}s, \tag{3.3.13}$$

and $X_n = \frac{n^2x}{\pi^2}$. Now our main goal is to simplify the integral $I(X_n,k)$. We shall express this integral in terms of the Meijer G-function. We know that $\Gamma(s)$ has poles at $s = 0,-1,-2,\cdots$, and the poles of $\Gamma\left(\frac{k}{2}-s\right)$ are at $\frac{k}{2}, \frac{k}{2}-1, \frac{k}{2}-2, \cdots$. Thus, we can easily observe that the line of integration $\mathrm{Re}(s) = \lambda$ does not separate the poles of $\Gamma(s)$ with the poles of $\Gamma\left(\frac{k}{2}-s\right)$ since $\frac{k}{2} < \lambda < \frac{k}{2}+1$. Therefore, we shift the line of integration $\mathrm{Re}(s) = \lambda$ to a new line $\mathrm{Re}(s) = \lambda_1$, where $0 < \lambda_1 < \frac{k}{2}$, so that this new line of integration does separate the poles $\Gamma(s)$ from the poles of $\Gamma\left(\frac{k}{2}-s\right)$. With the help of this new line of integration, we construct a contour $\mathcal{C}_1$ determined by line segments $[\lambda-iT, \lambda+iT]$, $[\lambda+iT, \lambda_1+iT]$, $[\lambda_1+iT, \lambda_1-iT]$, and $[\lambda_1-iT, \lambda-iT]$. Again, utilizing the residue theorem, we have

$$\frac{1}{2\pi i}\int_{\lambda-iT}^{\lambda+iT} + \int_{\lambda+iT}^{\lambda_1+iT} + \int_{\lambda_1+iT}^{\lambda_1-iT} + \int_{\lambda_1-iT}^{\lambda-iT}\frac{\Gamma(s)\Gamma\left(\frac{k}{2}-s\right)}{\Gamma\left(\frac{1-k}{2}+s\right)}X_n^{-s}\mathrm{d}s = \operatorname*{Res}_{s=\frac{k}{2}}\frac{\Gamma(s)\Gamma\left(\frac{k}{2}-s\right)}{\Gamma\left(\frac{1-k}{2}+s\right)}X_n^{-s}. \tag{3.3.14}$$

Letting $T \to \infty$ and using Stirling's formula (2.6.2) for $\Gamma(s)$, one can show that both of integrals with horizontal paths will vanish. Therefore, (3.3.14) becomes

$$\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\Gamma(s)\Gamma\left(\frac{k}{2}-s\right)}{\Gamma\left(\frac{1-k}{2}+s\right)}X_n^{-s}\mathrm{d}s = \frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1-i\infty}\frac{\Gamma(s)\Gamma\left(\frac{k}{2}-s\right)}{\Gamma\left(\frac{1-k}{2}+s\right)}X_n^{-s}\mathrm{d}s - \frac{\Gamma(\frac{k}{2})X_n^{-\frac{k}{2}}}{\sqrt{\pi}}. \tag{3.3.15}$$

We have already seen that the line of integration $\mathrm{Re}(s) = \lambda_1$ does separate the poles of $\Gamma(s)$ from the poles of $\Gamma\left(\frac{k}{2}-s\right)$. Thus, making use of the definition (2.6.3) of the Meijer G-function, with $m = n = p = 1, q = 2$, and $a_1 = 1, b_1 = \frac{k}{2}, b_2 = \frac{1+k}{2}$, we can write

$$\frac{1}{2\pi i}\int_{\lambda_1-i\infty}^{\lambda_1-i\infty}\frac{\Gamma(s)\Gamma\left(\frac{k}{2}-s\right)}{\Gamma\left(\frac{1-k}{2}+s\right)}X_n^{-s}\mathrm{d}s = G_{1,2}^{1,1}\left(\begin{matrix}1\\ \frac{k}{2},\frac{k+1}{2}\end{matrix}\Big|\frac{1}{X_n}\right). \tag{3.3.16}$$

Here one can easily check that $p+q < 2(m+n)$ and $|\arg(1/X_n)| < \left(m+n-\frac{p+q}{2}\right)\pi$. Thus all the necessary conditions for the convergence of the above Meijer G-function are satisfied. Now invoking Slater's theorem (2.6.4), one can show that

$$G_{1,2}^{1,1}\left(\begin{matrix}1\\ \frac{k}{2},\frac{k+1}{2}\end{matrix}\Big|\frac{1}{X_n}\right) = \frac{\Gamma\left(\frac{k}{2}\right)}{X_n^{\frac{k}{2}}\sqrt{\pi}}{}_1F_1\left(\frac{k}{2};\frac{1}{2};-\frac{1}{X_n}\right). \tag{3.3.17}$$

Substituting (3.3.17) in (3.3.16) and together with (3.3.15) and (3.3.13), we see that the right vertical integral $V(x,k)$ (3.3.12) takes the final shape as

$$V(x,k) = \frac{\Gamma\left(\frac{k}{2}\right)}{\pi^k}\sum_{n=1}^{\infty}\frac{\mu(n)n^{k-1}}{X_n^{\frac{k}{2}}}\left\{{}_1F_1\left(\frac{k}{2};\frac{1}{2};-\frac{1}{X_n}\right)-1\right\}.$$

Now putting $X_n = \frac{n^2x}{\pi^2}$, the above expression reduces to

$$\begin{aligned} V(x,k) &= \frac{\Gamma\left(\frac{k}{2}\right)}{x^{\frac{k}{2}}}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left\{{}_1F_1\left(\frac{k}{2};\frac{1}{2};-\frac{\pi^2}{n^2x}\right)-1\right\} \\ &= \frac{\Gamma\left(\frac{k}{2}\right)}{x^{\frac{k}{2}}}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left\{{}_1F_1\left(\frac{k}{2};\frac{1}{2};-\frac{\pi^2}{n^2x}\right)\right\}. \end{aligned} \tag{3.3.18}$$

In the last step, we utilized the fact that $\sum_{n=1}^{\infty}\frac{\mu(n)}{n} = 0$, which is equivalent to the prime number theorem. Finally, in view of (3.3.18) and (3.3.9), and together with (3.3.8), we complete the proof of Theorem 3.2.1. □

Proof of Theorem 3.2.2. First, let us assume that the Riemann hypothesis is true. Under the assumption of the Riemann hypothesis, Littlewood showed that, for any $\epsilon > 0$,

$$A(x) := \sum_{1\le n\le x}\mu(n) = O_\epsilon(x^{\frac{1}{2}+\epsilon}). \tag{3.3.19}$$

We invoke Euler's partial summation formula (2.6.1), with $a_n = \mu(n)$ and $f(t) = t^{-k}$,

to see that

$$M(\ell;n) := \sum_{m=\ell}^{n} \frac{\mu(m)}{m^k} = A(n)f(n) - A(\ell-1)f(\ell-1) - \int_{\ell-1}^{n} A(t)f'(t)\mathrm{d}t. \quad (3.3.20)$$

Making use of (3.3.19) in (3.3.20) and simplifying, one can easily show that

$$M(\ell;n) = O_\epsilon(\ell^{\frac{1}{2}-k+\epsilon}), \quad (3.3.21)$$

uniformly in n. Our main aim is to find an estimate for the following sum

$$P_k(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^k} \exp\left(-\frac{x}{n^2}\right).$$

We separate the sum into two parts, that is,

$$\begin{aligned} P_k(x^2) &= \sum_{n=1}^{\ell-1} \frac{\mu(n)}{n^k} \exp\left(-\frac{x^2}{n^2}\right) + \sum_{n=\ell}^{\infty} \frac{\mu(n)}{n^k} \exp\left(-\frac{x^2}{n^2}\right) \\ &:= S_1(x^2) + S_2(x^2), \end{aligned} \quad (3.3.22)$$

where $\ell = [x^{1-\epsilon}] + 1$. First, we shall try to find an estimate for $S_2(x^2)$. Using the representation (3.3.20) of $M(l;n)$, for each integer $N > \ell$, one can check that

$$\begin{aligned} \sum_{n=\ell}^{N} \frac{\mu(n)}{n^k} \exp\left(-\frac{x^2}{n^2}\right) &= \frac{\mu(\ell)}{\ell^k} \exp\left(-\frac{x^2}{\ell^2}\right) + \sum_{\ell<n\le N} (M(l;n) - M(l;n-1)) \exp\left(-\frac{x^2}{n^2}\right) \\ &= \sum_{n=\ell}^{N-1} M(l;n) \left(e^{-\frac{x^2}{n^2}} - e^{-\frac{x^2}{(n+1)^2}}\right) + M(l;N)e^{-\frac{x^2}{N^2}}. \end{aligned} \quad (3.3.23)$$

Letting $N \to \infty$ in (3.3.23) and utilizing (3.3.21), we arrive at

$$S_2(x^2) = S_3(x^2) + O_\epsilon\left(\ell^{\frac{1}{2}-k+\epsilon}\right), \quad (3.3.24)$$

where

$$S_3(x^2) := \sum_{n=\ell}^{\infty} M(l;n) \left(e^{-\frac{x^2}{n^2}} - e^{-\frac{x^2}{(n+1)^2}}\right).$$

To estimate $S_3(x^2)$, we consider a function $f(y) = e^{-\frac{x^2}{y^2}}$. Using the mean value theorem, one can find $y_n \in (n, n+1)$ such that $f(n)-f(n+1) = -f'(y_n) = -\frac{2x^2}{y_n^3} e^{-\frac{x^2}{y_n^2}}$. Utilizing this fact and together with (3.3.21), we get

$$|S_3(x^2)| \ll_\epsilon \ell^{\frac{1}{2}-k+\epsilon} \sum_{n=\ell}^{\infty} \frac{2x^2}{y_n^3} e^{-\frac{x^2}{y_n^2}}.$$

Further, we divide the sum into two parts in the following way,

$$|S_3(x^2)| \ll_\epsilon \ell^{\frac{1}{2}-k+\epsilon} \left(\sum_{n=\ell}^{[x]} + \sum_{n=[x]+1}^{\infty}\right) \frac{x^2}{y_n^3} e^{-\frac{x^2}{y_n^2}}. \quad (3.3.25)$$

The finite sum can be trivially bounded by

$$\sum_{n=\ell}^{[x]} \frac{x^2}{y_n^3} e^{-\frac{x^2}{y_n^2}} \ll \sum_{n=\ell}^{\infty} \frac{x^2}{n^3} \ll \frac{x^2}{l^2} \ll x^{2\epsilon}, \quad (3.3.26)$$

as $\ell = [x^{1-\epsilon}] + 1$. Now the tail of the series in (3.3.25) can be estimated by

$$\sum_{n=[x]+1}^{\infty} \frac{x^2}{y_n^3} e^{-\frac{x^2}{y_n}} \ll \sum_{n>x} \frac{x^2}{n^3} \ll O(1). \tag{3.3.27}$$

Here and in the previous inequality, we have used the fact that $\sum_{n>x} n^{-s} = O(x^{1-s})$ for $\mathrm{Re}(s) > 1$. Note that all the big-O estimates are for $x \to \infty$. Substituting (3.3.26) and (3.3.27) in (3.3.25), we obtain

$$|S_3(x^2)| \ll_\epsilon x^{\frac{1}{2}-k+\epsilon}.$$

Finally, substitute this bound of $S_3(x^2)$ in (3.3.24) to see

$$|S_2(x^2)| = O_\epsilon\left(x^{\frac{1}{2}-k+\epsilon}\right). \tag{3.3.28}$$

Now we shall concentrate on $S_1(x^2)$. We write

$$|S_1(x^2)| = \left|\sum_{n=1}^{\ell-1} \frac{\mu(n)}{n^k} \exp\left(-\frac{x^2}{n^2}\right)\right| \leq \sum_{n=1}^{\ell-1} \exp\left(-\frac{x^2}{l^2}\right).$$

This implies that

$$S_1(x^2) = O\left(x^{1-\epsilon} \exp(-x^{2\epsilon})\right), \tag{3.3.29}$$

as $\ell = [x^{1-\epsilon}] + 1$. In view of (3.3.29) and (3.3.28), one can observe that due to exponential decay, the bound for $S_1(x^2)$ goes to zero much faster than the bound for $S_2(x^2)$ as $x \to \infty$. Therefore, (3.3.22) yields that

$$P_k(x^2) = O_\epsilon\left(x^{\frac{1}{2}-k+\epsilon}\right).$$

Finally, replacing x by $\sqrt{x}$, one can deduce (3.2.2).

Now we prove the converse part. The case $k = 1$ is proved by Hardy and Littlewood [69, p. 161, Equation (2.544)]. Thus we prove for $k > 1$. Assuming the bound (3.2.2) for $P_k(x)$, we shall show that the Riemann hypothesis is true. From Lemma 3.1.1, we know that

$$\zeta(2s+k) \int_0^\infty x^{-s-1} P_k(x) \mathrm{d}x = \Gamma(-s) \tag{3.3.30}$$

valid in the region $\frac{1-k}{2} < \mathrm{Re}(s) < 0$. Next, we try to extend the region of validity of this lemma on the left half plane. For a large positive integer N, we separate the above integral in the following way:

$$\zeta(2s+k) \left(\int_0^N + \int_N^\infty\right) x^{-s-1} P_k(x) \mathrm{d}x = \Gamma(-s).$$

Employing the bound (3.2.2) for $P_k(x)$, one can easily show that $\int_N^\infty x^{-s-1} P_k(x) \mathrm{d}x$ is analytic if $\mathrm{Re}(s) > \frac{1}{4} - \frac{k}{2}$. The integral $\int_0^N x^{-s-1} P_k(x) \mathrm{d}x$ is analytic for $\mathrm{Re}(s) < 0$. As $\zeta(2s+k)$ has a pole only at $s = \frac{1-k}{2}$, thus left hand side of (3.3.30) is analytic in the region $\frac{1}{4} - \frac{k}{2} < \mathrm{Re}(s) \leq \frac{1-k}{2}$ except at $s = \frac{1-k}{2}$. Also, $\Gamma(-s)$ has no pole in this region which allows us to extend the analyticity of the identity (3.3.30) in the region $\frac{1}{4} - \frac{k}{2} < \mathrm{Re}(s) \leq \frac{1-k}{2}$ except at $s = \frac{1-k}{2}$. Now we observe that the right side of

(3.3.30) is never zero in the region $\frac{1}{4} - \frac{k}{2} < \mathrm{Re}(s) \leq \frac{1-k}{2}$, which implies that $\zeta(2s+k)$ has no zero in the same region. Equivalently, we can say that $\zeta(s)$ has no zero in the region $\frac{1}{2} < \mathrm{Re}(s) \leq 1$ and thus the functional equation of $\zeta(s)$ implies that it has no zero for $0 \leq \mathrm{Re}(s) < \frac{1}{2}$. This proves that all the non-trivial zeros of $\zeta(s)$ lie on $\mathrm{Re}(s) = \frac{1}{2}$. This completes the proof of Theorem 3.2.2. $\square$

Chapter 4

HARDY-LITTLEWOOD-RIESZ-TYPE EQUIVALENT CRITERIA FOR THE GENERALIZED RIEMANN HYPOTHESIS

This chapter is based on the paper [54]. The main goal of the current chapter is to establish a character analogue of the identity (3.2.1) and derive a one-variable generalization of the identity (1.1.7) of Dixit. Furthermore, the generalized character analogue identity motivates us to derive equivalent criteria for the generalized Riemann hypothesis for Dirichlet L-functions.

4.1 Preliminaries

In this section, we collect all the necessary results which are essential to prove our main results.

Lemma 4.1.1. *Let χ be any primitive character of conductor q and $m \geq 2$ be an integer such that $\chi(-1) = (-1)^m$, then*

$$\frac{(m-2)!\, q^{m-2} G(\chi)}{2^{m-1}\pi^{m-2} i^{m-2}} L(m-1, \bar{\chi}) = L'(2-m, \chi).$$

Proof. One can find the proof of this lemma in [4, Lemma 3.1]. □

Lemma 4.1.2. *Suppose there exists a sequence of arbitrarily large positive numbers T with $|T - \mathrm{Im}(\rho)| > \exp\left(-\frac{A_0 \mathrm{Im}(\rho)}{\log(\mathrm{Im}(\rho))}\right)$, for every non-trivial zero of $L(s, \chi)$. Then,*

$$\frac{1}{|L(\sigma + iT)|} < e^{A_1 T},$$

for some constant $0 < A_1 < \pi/4$.

Proof. The existence of such sequence of real numbers is guaranteed from the bracketing condition (1.1.6) on the non-trivial zeros of $L(s,\chi)$. We can obtain a proof of this result along the lines given in [110, p. 219]. □

The next lemma gives us a two-variable generalization of identity (3.1.2). This lemma plays a crucial role in establishing equivalent criteria for the GRH. For $k \geq 1$ and $\ell > 0$, we define a function

$$P_{k,\ell,\chi}(x) := \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^k} \exp\left(-\frac{x}{n^\ell}\right).$$

Lemma 4.1.3. *Let* $q \in \mathbb{N}$ *and* χ *be a principal Dirichlet character modulo* q. *Let* $k = 1$ *and* $\ell > 0$ *be positive real numbers. In the region* $0 < \mathrm{Re}(s) < 1$, *we have*[1]

$$\int_0^\infty x^{-s-1} P_{1,\ell,\chi}(x)\mathrm{d}x = \frac{\Gamma(-s)}{L(\ell s+1,\chi)}.$$

Proof. Using the definition of $P_{1,\ell,\chi}(x)$, one can see that

$$\begin{aligned} P_{1,\ell,\chi}(x) := \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n} \exp\left(-\frac{x}{n^\ell}\right) &= \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n} \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{m! n^{\ell m}} \\ &= \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n} \left(1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^m}{m! n^{\ell m}}\right) \\ &= \begin{cases} \frac{1}{L(1,\chi)} + \sum_{m=1}^{\infty} \frac{(-1)^m x^m}{m! L(1+\ell m,\chi)}, & \text{if } \chi \text{ non-principal,} \\ \sum_{m=1}^{\infty} \frac{(-1)^m x^m}{m! L(1+\ell m,\chi)}, & \text{if } \chi \text{ principal.} \end{cases} \end{aligned} \tag{4.1.1}$$

Note that when χ is principal character, we know $\frac{1}{L(1,\chi)} = 0$. Also we utilize $\sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^s} = \frac{1}{L(s,\chi)}$ for $\mathrm{Re}(s) > 1$, [8, p. 229, example 3]. For $0 < \mathrm{Re}(s) < 1$, we utilize the series representation of $L(\ell s+1,\chi)$, to see that

$$\begin{aligned} L(\ell s+1,\chi) \int_0^\infty x^{-s-1} P_{1,\ell,\chi}(x)\mathrm{d}x &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \int_0^\infty \frac{x^{-s-1}}{n^{\ell s}} P_{1,\ell,\chi}(x)\mathrm{d}x \\ &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \int_0^\infty x^{-s-1} P_{1,\ell,\chi}\left(\frac{x}{n^\ell}\right)\mathrm{d}x \\ &= \int_0^\infty x^{-s-1} \sum_{n=1}^{\infty} \frac{\chi(n)}{n} P_{1,\ell,\chi}\left(\frac{x}{n^\ell}\right)\mathrm{d}x. \end{aligned} \tag{4.1.2}$$

[1] Lemma 3.3 in [54] contains an error which has been corrected in this thesis. Mainly, we divided the lemma into two cases, namely, Lemma 4.1.3 and Lemma 4.1.4 corresponding to $k = 1$ and $k > 1$, respectively. Moreover, Lemma 4.1.3 holds for $0 < \mathrm{Re}(s) < 1$ whereas Lemma 4.1.4 holds for $\frac{1-k}{\ell} < \mathrm{Re}(s) < 0$.

In the penultimate step, we replaced x by x/n^{ℓ}. Moreover, the interchange of summation and integration was possible due to the dominated convergence theorem as we can see from the equation (4.1.3) that the infinite sum can be bounded by exponential function, which is an integrable function in $(0,\infty)$. Employ (4.1.1) to see that

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n}P_{1,\ell,\chi}\left(\frac{x}{n^{\ell}}\right)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n}\left(\frac{1}{L(1,\chi)}+\sum_{m=1}^{\infty}\frac{(-1)^m x^m}{m!n^{\ell m}L(1+\ell m,\chi)}\right)$$
$$=\begin{cases}e^{-x}, & \text{if } \chi \text{ non-principal,}\\ e^{-x}-1, & \text{if } \chi \text{ principal.}\end{cases} \tag{4.1.3}$$

At this situation, combining (4.1.3) for χ principal and (4.1.2), we arrive at

$$L(\ell s+1,\chi)\int_0^{\infty}x^{-s-1}P_{1,\ell,\chi}(x)\mathrm{d}x=\int_0^{\infty}x^{-s-1}(e^{-x}-1)\mathrm{d}x=\frac{-1}{s}\int_0^{\infty}x^{(1-s)-1}e^{-x}\mathrm{d}x$$
$$=-\frac{\Gamma(1-s)}{s},$$

which is valid in $0<\mathrm{Re}(s)<1$. Here in the above calculation, we have used integration by parts and the fact that $\lim_{x\to 0}\frac{e^{-x}-1}{x^s}=0$ as $0<\mathrm{Re}(s)<1$. Using functional equation of $\Gamma(s)$, we can clearly see that $-\frac{\Gamma(1-s)}{s}=\Gamma(-s)$. This completes the proof of Lemma 4.1.3. □

Remark 4. *Note that if χ is non-principal, then combining* (4.1.3) *with* (4.1.2)*, we will have*

$$L(\ell s+1,\chi)\int_0^{\infty}x^{-s-1}P_{1,\ell,\chi}(x)\mathrm{d}x=\int_0^{\infty}x^{-s-1}e^{-x}\mathrm{d}x=\Gamma(-s),$$

which is valid for $\mathrm{Re}(s)<0$*. However, at the beginning of* (4.1.2)*, we started with* $\mathrm{Re}(s)>0$*. This justifies why χ needs to be principal character in Lemma 4.1.3.*

Lemma 4.1.4. *Let χ be a Dirichlet character. Let $k>1$ and $\ell>0$ be positive real numbers. In the region $\frac{1-k}{\ell}<\mathrm{Re}(s)<0$, we have*

$$\int_0^{\infty}x^{-s-1}P_{k,\ell,\chi}(x)\mathrm{d}x=\frac{\Gamma(-s)}{L(\ell s+k,\chi)}.$$

Proof. Making use of the definition of $P_{k,\ell,\chi}(x)$, we see that

$$P_{k,\ell,\chi}(x):=\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^k}\exp\left(-\frac{x}{n^{\ell}}\right)=\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^k}\sum_{m=0}^{\infty}\frac{(-1)^m x^m}{m!n^{\ell m}}$$
$$=\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^k}\left(1+\sum_{m=1}^{\infty}\frac{(-1)^m x^m}{m!n^{\ell m}}\right)$$
$$=\frac{1}{L(k,\chi)}+\sum_{m=1}^{\infty}\frac{(-1)^m x^m}{m!L(k+\ell m,\chi)}. \tag{4.1.4}$$

For $\frac{1-k}{\ell} < \mathrm{Re}(s) < 0$, utilizing the series representation of $L(\ell s+k,\chi)$, one can write

$$\begin{aligned} L(\ell s+k,\chi)\int_0^\infty x^{-s-1}P_{k,\ell,\chi}(x)\mathrm{d}x &= \sum_{n=1}^\infty \frac{\chi(n)}{n^k}\int_0^\infty \frac{x^{-s-1}}{n^{\ell s}}P_{k,\ell,\chi}(x)\mathrm{d}x \\ &= \sum_{n=1}^\infty \frac{\chi(n)}{n^k}\int_0^\infty x^{-s-1}P_{k,\ell,\chi}\left(\frac{x}{n^\ell}\right)\mathrm{d}x \\ &= \int_0^\infty x^{-s-1}\sum_{n=1}^\infty \frac{\chi(n)}{n^k}P_{k,\ell,\chi}\left(\frac{x}{n^\ell}\right)\mathrm{d}x. \end{aligned} \tag{4.1.5}$$

Here again, we replaced x by x/n^ℓ and the interchange of summation and integration was possible due to the dominated convergence theorem. Employ (4.1.4) to see that

$$\begin{aligned} \sum_{n=1}^\infty \frac{\chi(n)}{n^k}P_{k,\ell,\chi}\left(\frac{x}{n^\ell}\right) &= \sum_{n=1}^\infty \frac{\chi(n)}{n^k}\sum_{m=0}^\infty \frac{(-1)^m x^m}{m!n^{\ell m}L(k+\ell m,\chi)} \\ &= \sum_{n=1}^\infty \frac{\chi(n)}{n^k}\left(\frac{1}{L(k,\chi)}+\sum_{m=1}^\infty \frac{(-1)^m x^m}{m!n^{\ell m}L(k+\ell m,\chi)}\right) = e^{-x}. \end{aligned} \tag{4.1.6}$$

At this juncture, combining (4.1.6) and (4.1.5), we see that

$$L(\ell s+k,\chi)\int_0^\infty x^{-s-1}P_{k,\ell,\chi}(x)\mathrm{d}x = \int_0^\infty x^{-s-1}e^{-x}\mathrm{d}x = \Gamma(-s),$$

which is valid in the region $\frac{1-k}{\ell} < \mathrm{Re}(s) < 0$. This completes the proof of the identity. □

4.2 Main results

Theorem 4.2.1. *Let χ be a primitive character modulo q and $k \geq 1$ be any real number. Assume that all the non-trivial zeroes of $L(s,\chi)$ are simple. Then, for any positive real number x, we have*

$$\begin{aligned} \sum_{n=1}^\infty \frac{\chi(n)\mu(n)}{n^k}\exp\left(-\frac{\pi x^2}{qn^2}\right) &= \frac{i^a\sqrt{q}}{G(\chi)}\left(\frac{q}{\pi x^2}\right)^{\frac{k+a}{2}}\left(\frac{\pi}{q}\right)^{a+\frac{1}{2}}\frac{\Gamma(\frac{k+a}{2})}{\Gamma(a+\frac{1}{2})} \\ &\quad\times\sum_{n=1}^\infty \frac{\overline{\chi(n)}\mu(n)}{n^{1+a}}{}_1F_1\left(\frac{k+a}{2};a+\frac{1}{2};-\frac{\pi}{qn^2x^2}\right) \\ &\quad+\frac{1}{2}\sum_\rho \frac{\Gamma(\frac{k-\rho}{2})}{L'(\rho,\chi)}\left(\frac{q}{\pi x^2}\right)^{\frac{k-\rho}{2}}, \end{aligned} \tag{4.2.1}$$

where ρ runs over the non-trivial zeroes of $L(s,\chi)$, which satisfy the bracketing conditions (1.1.6).

Remark 5. *The above theorem can be thought of as a character analogue of* (3.2.1). *Substituting $q=1$ i.e., considering χ to be the trivial character and replacing x by*

$x/\sqrt{\pi}$ *in* (4.2.1)*, one can immediately obtain* (3.2.1)*.*

Remark 6. *Theorem 4.2.1 is a one-variable generalization of Theorem 1.1.1. At a glance, they may look totally different from each other, but substituting* $k = 2$ *and considering* χ *be an odd character in* (4.2.1) *and doing a little more simplification, we can derive Theorem 1.1.1.*

The bound (3.2.2), which gives equivalent criteria for the Riemann hypothesis, motivates us to expect that, for any $\epsilon > 0$,

$$P_{k,\chi}(x) := \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^k} \exp\left(-\frac{x}{n^2}\right) = O_{\epsilon,k}\left(x^{-\frac{k}{2}+\frac{1}{4}+\epsilon}\right), \quad \text{as } x \to \infty, \tag{4.2.2}$$

must be the equivalent criteria for the generalized Riemann hypothesis. More generally, we obtain the following equivalent criteria.

Theorem 4.2.2. *Let* ℓ *be a positive real number. For* $k = 1$*,* χ *principal Dirichlet character and for* $k > 1$*,* χ *being any Dirichlet character, the generalized Riemann hypothesis is equivalent to the bound*

$$P_{k,\ell,\chi}(x) := \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^k} \exp\left(-\frac{x}{n^\ell}\right) = O_{\epsilon,k,\ell}\left(x^{-\frac{k}{\ell}+\frac{1}{2\ell}+\epsilon}\right), \quad \text{as } x \to \infty. \tag{4.2.3}$$

Letting $\ell = 2$ in (4.2.3), one can immediately see the bound (4.2.2).

4.3 Proof of Main results

Proof of Theorem 4.2.1. Recall that the inverse Mellin transform of $\Gamma(s)$ suggests that, for $-1 < c < 0$, one can obtain

$$e^{-x} - 1 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) x^{-s} \mathrm{d}s. \tag{3.3.1}$$

Let $L(s,\chi)$ be the Dirichlet L-function associated to the primitive Dirichlet character χ modulo q. Given that x is a positive real number and letting $k \geq 1$ be any real number, we can write by making use of (3.3.1),

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^k} \exp\left(-\frac{\pi x^2}{qn^2}\right) &= \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^k} + \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^k} \left(\exp\left(-\frac{\pi x^2}{qn^2}\right) - 1\right) \\
&= \frac{1}{L(k,\chi)} + \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^k} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \left(\frac{\pi x^2}{qn^2}\right)^{-s} \mathrm{d}s \\
&= \frac{1}{L(k,\chi)} + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)}{L(k-2s,\chi)} \left(\frac{\pi x^2}{q}\right)^{-s} \mathrm{d}s.
\end{aligned} \tag{4.3.1}$$

Here, change in the order of summation and integration in the final step, is valid due to absolute convergence of the infinite series, since $\mathrm{Re}(k-2s) > 1$. We must try to analyze the following vertical line integral:

$$V_{k,\chi}^{(1)}(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)}{L(k-2s,\chi)}\left(\frac{\pi x^2}{q}\right)^{-s} \mathrm{d}s. \tag{4.3.2}$$

At first, we shall find the poles of the integrand. We know that $\Gamma(s)$ has simple poles at non-positive integers. Again, trivial zeros of $L(k-2s,\chi)$ will also contribute to the simple poles of the integrand. Employing the functional equation of $L(s,\chi)$, one can detect that the trivial zeros of $L(k-2s,\chi)$ are at $\frac{k+a+2n}{2}$ for any $n \in \mathbb{N}\cup\{0\}$, where the constant a is defined as in (2.2.2). Another crucial observation is that there will be infinitely many poles of the integrand due to the non-trivial zeros of $L(k-2s,\chi)$ in the critical strip $\frac{k-1}{2} < \mathrm{Re}(s) < \frac{k}{2}$. As we are interested to evaluate the contributions of the poles corresponding to these non-trivial zeros of $L(k-2s,\chi)$, we must shift the line of integration $\mathrm{Re}(s) = c$ to $\mathrm{Re}(s) = d$ with $\frac{k+1}{2} < d < \frac{k}{2}+1$. Mainly, we construct a rectangular contour $\mathfrak{C}$ with corner points $c \pm iT, d \pm iT$, where T is some large positive real number. Now, making use of Cauchy's residue theorem, we see that

$$\frac{1}{2\pi i}\int_{\mathfrak{C}} \frac{\Gamma(s)}{L(k-2s,\chi)}\left(\frac{\pi x^2}{q}\right)^{-s} \mathrm{d}s = R_0 + R_{\frac{k+a}{2}} + \mathcal{R}_{T,\chi}(x), \tag{4.3.3}$$

where R_0 and $R_{\frac{k+a}{2}}$ denote the residual terms corresponding to $s=0$ and $s=\frac{k+a}{2}$ respectively, and the residual term $\mathcal{R}_{T,\chi}(x)$ corresponds to the non-trivial zeros ρ of $L(k-2s,\chi)$ with $|\,\mathrm{Im}(\rho)| < T$. Without much effort, we can evaluate residual terms R_0 and $R_{\frac{k+a}{2}}$. One can find out that

$$R_0 = \frac{1}{L(k,\chi)}, \quad R_{\frac{k+a}{2}} = -\frac{\Gamma\left(\frac{k+a}{2}\right)\left(\frac{\pi x^2}{q}\right)^{-\frac{k+a}{2}}}{2L'(-a,\chi)}. \tag{4.3.4}$$

Again, if we assume that all the non-trivial zeros of $L(s,\chi)$ are simple, then we have

$$\mathcal{R}_{T,\chi}(x) = \sum_{|\,\mathrm{Im}(\rho)|<T} \lim_{s\to\frac{k-\rho}{2}} \frac{\left(s-\frac{k-\rho}{2}\right)\Gamma(s)}{L(k-2s,\chi)}\left(\frac{\pi x^2}{q}\right)^{-s} = -\frac{1}{2}\sum_{|\,\mathrm{Im}(\rho)|<T} \frac{\Gamma(\frac{k-\rho}{2})}{L'(\rho,\chi)}\left(\frac{\pi x^2}{q}\right)^{-\frac{k-\rho}{2}},$$

where the sum runs through all the non-trivial zeros of $L(s,\chi)$. Now one of our main goals is to show that the following horizontal integrals

$$H_{T,\chi}^{(1)}(x) := \frac{1}{2\pi i}\int_{d+iT}^{c+iT} \frac{\Gamma(s)}{L(k-2s,\chi)}\left(\frac{\pi x^2}{q}\right)^{-s} \mathrm{d}s,$$

$$H_{T,\chi}^{(2)}(x) := \frac{1}{2\pi i}\int_{c-iT}^{d-iT} \frac{\Gamma(s)}{L(k-2s,\chi)}\left(\frac{\pi x^2}{q}\right)^{-s} \mathrm{d}s.$$

will vanish as $T \to \infty$. Now, we shall make use of Stirling's formula for $\Gamma(s)$, that is, for $c \leq \sigma \leq d$,

$$|\Gamma(\sigma + iT) \ll |T|^{\sigma - \frac{1}{2}} e^{-\frac{\pi}{2}|T|}, \quad \text{as } |T| \to \infty,$$

and together with the bound for $1/L(s,\chi)$, i.e., using Lemma 4.1.2, one can see that

$$|H^{(1)}_{T,\chi}(x)| \ll |T|^{\sigma - \frac{1}{2}} e^{2A_1 T - \frac{\pi}{2}|T|},$$

where $0 < A_1 < \pi/4$. This immediately implies that the horizontal integral vanishes as $T \to \infty$. Similarly, one can show that the same holds for the other horizontal integral $H^{(2)}_{T,\chi}(x)$. Thus letting $T \to \infty$ in (4.3.3) and considering the fact that the horizontal integrals tend to zero and in view of (4.3.1) and (4.3.2), we reach

$$\sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^k} \exp\left(-\frac{\pi x^2}{qn^2}\right) = V^{(2)}_{k,\chi}(x) - R_{\frac{k+a}{2}} - \mathcal{R}_\chi(x), \tag{4.3.5}$$

where the residual term $\mathcal{R}_\chi(x)$ contains the contribution of all the non-trivial zeros of $L(s,\chi)$, that is,

$$\mathcal{R}_\chi(x) = -\frac{1}{2} \sum_{\rho} \frac{\Gamma(\frac{k-\rho}{2})}{L'(\rho,\chi)} \left(\frac{\pi x^2}{q}\right)^{-\frac{k-\rho}{2}}, \tag{4.3.6}$$

and the right vertical integral is denoted by

$$V^{(2)}_{k,\chi}(x) := \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma(s)}{L(k-2s,\chi)} \left(\frac{\pi x^2}{q}\right)^{-s} \mathrm{d}s, \tag{4.3.7}$$

where $\frac{k+1}{2} < d < \frac{k}{2} + 1$. At this juncture, we shall try to simplify the above integral. Here we make use of the functional equation (2.2.1) of $L(s,\chi)$. Replace s by $k - 2s$ in (2.2.1) and simplify to see that

$$\frac{1}{L(k-2s,\chi)} = \frac{i^a}{G(\chi)} \frac{\left(\frac{q}{\pi}\right)^{k-2s} \Gamma\left(\frac{k+a}{2} - s\right)}{\Gamma\left(\frac{1+a-k}{2} + s\right) L(1-k+2s,\overline{\chi})}. \tag{4.3.8}$$

Putting (4.3.8) in (4.3.7), we obtain

$$V^{(2)}_{k,\chi}(x) = \frac{1}{\epsilon(\chi)} \left(\frac{q}{\pi}\right)^{k-\frac{1}{2}} \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma(s)\Gamma(\frac{k+a}{2} - s)}{\Gamma(\frac{1+a-k}{2} + s)} \frac{\left(\frac{qx^2}{\pi}\right)^{-s}}{L(1-k+2s,\overline{\chi})} \mathrm{d}s, \tag{4.3.9}$$

where $\epsilon(\chi) = \frac{G(\chi)}{i^a\sqrt{q}}$. At this point, one can verify that the argument of $L(1-k+2s,\bar{\chi})$ lies in the interval $2 < \mathrm{Re}(1-k+2s) < 3$ since $\frac{k+1}{2} < \mathrm{Re}(s) = d < \frac{k}{2} + 1$. Thus, we can write

$$\frac{1}{L(1-k+2s,\overline{\chi})} = \sum_{n=1}^{\infty} \frac{\mu(n)\overline{\chi(n)}}{n^{1-k+2s}}. \tag{4.3.10}$$

Substituting the expression (4.3.10) in (4.3.9) and then interchanging the integration and summation, equation (4.3.9) takes the shape

$$V^{(2)}_{k,\chi}(x) = \frac{1}{\epsilon(\chi)} \left(\frac{q}{\pi}\right)^{k-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\mu(n)\overline{\chi(n)}}{n^{1-k}} J(X_{n,q}, k, a), \tag{4.3.11}$$

where

$$J(X_{n,q},k,a):=\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\frac{\Gamma(s)\Gamma\left(\frac{k+a}{2}-s\right)}{\Gamma\left(\frac{1+a-k}{2}+s\right)}X_{n,q}^{-s}\mathrm{d}s, \tag{4.3.12}$$

and $X_{n,q}:=\frac{q\,(nx)^2}{\pi}$. At this moment, our final attempt is to simplify the above integral and write it in terms of well known functions. Here we shall use the definition (2.6.3) of the Meijer G-function. Evaluating the poles of $\Gamma(s)$ and $\Gamma\left(\frac{k+a}{2}-s\right)$, one can verify that the line of integration $\mathrm{Re}(s)=d$ does not separate the poles of $\Gamma\left(\frac{k+a}{2}-s\right)$ from the poles of $\Gamma(s)$ since $\frac{k+1}{2}<d<\frac{k}{2}+1$. So, we draw a new line of integration $\mathrm{Re}(s)=d_1$ with $0<d_1<\frac{k}{2}$. Now we can easily check that this new line $\mathrm{Re}(s)=d_1$ does separate the poles of $\Gamma\left(\frac{k+a}{2}-s\right)$ from the poles of $\Gamma(s)$. Once again, considering a rectangular contour with corners $d\pm iT, d_1\pm iT$, we apply Cauchy's residue theorem. Letting $T\to\infty$ and simplifying, we show that

$$\begin{aligned}J(X_{n,q},k,a)&=\frac{1}{2\pi i}\int_{d_1-i\infty}^{d_1+i\infty}\frac{\Gamma(s)\Gamma\left(\frac{k+a}{2}-s\right)}{\Gamma\left(\frac{1+a-k}{2}+s\right)}X_{n,q}^{-s}\mathrm{d}s+\mathrm{Res}_{s=\frac{k+a}{2}}\frac{\Gamma(s)\Gamma\left(\frac{k+a}{2}-s\right)}{\Gamma\left(\frac{1+a-k}{2}+s\right)}X_{n,q}^{-s}\\&=\frac{1}{2\pi i}\int_{d_1-i\infty}^{d_1+i\infty}\frac{\Gamma(s)\Gamma\left(\frac{k+a}{2}-s\right)}{\Gamma\left(\frac{1+a-k}{2}+s\right)}X_{n,q}^{-s}\mathrm{d}s-\frac{\Gamma\left(\frac{k+a}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right)}X_{n,q}^{-\frac{k+a}{2}}.\end{aligned} \tag{4.3.13}$$

As the line of integration $\mathrm{Re}(s)=d_1$ separates the poles of $\Gamma(s)$ from the poles of $\Gamma\left(\frac{k+a}{2}-s\right)$, we use the definition (2.6.3) of the Meijer G-function, with $m=n=p=1, q=2$, and $a_1=1, b_1=\frac{k+a}{2}, b_2=\frac{1-a+k}{2}$. Thus, we can write

$$\frac{1}{2\pi i}\int_{d_1-i\infty}^{d_1+i\infty}\frac{\Gamma(s)\Gamma\left(\frac{k+a}{2}-s\right)}{\Gamma\left(\frac{1+a-k}{2}+s\right)}X_{n,q}^{-s}\mathrm{d}s=G_{1,2}^{1,1}\left(\begin{matrix}1\\ \frac{k+a}{2},\frac{1-a+k}{2}\end{matrix}\Big|\frac{1}{X_{n,q}}\right). \tag{4.3.14}$$

Note that $b_1-b_2=a-\frac{1}{2}\notin\mathbb{Z}$, so we can employ Slater's theorem (2.6.4). Therefore, using Slater's theorem (2.6.4), we get

$$G_{1,2}^{1,1}\left(\begin{matrix}1\\ \frac{k+a}{2},\frac{1-a+k}{2}\end{matrix}\Big|\frac{1}{X_{n,q}}\right)=\frac{\Gamma\left(\frac{k+a}{2}\right)}{X_{n,q}^{\frac{k+a}{2}}\Gamma(a+\frac{1}{2})}{}_1F_1\left(\frac{k+a}{2};a+\frac{1}{2};-\frac{1}{X_{n,q}}\right). \tag{4.3.15}$$

Substituting (4.3.15) in (4.3.14) and in view of (4.3.13), (4.3.12) and (4.3.11), one can see that the final expression of the right vertical integral becomes

$$\begin{aligned}V_{k,\chi}^{(2)}(x)&=\frac{1}{\epsilon(\chi)}\left(\frac{q}{\pi}\right)^{k-\frac{1}{2}}\frac{\Gamma\left(\frac{k+a}{2}\right)}{\Gamma(a+\frac{1}{2})}\sum_{n=1}^{\infty}\frac{\mu(n)\overline{\chi(n)}}{n^{1-k}X_{n,q}^{\frac{k+a}{2}}}\left({}_1F_1\left(\frac{k+a}{2};a+\frac{1}{2};-\frac{1}{X_{n,q}}\right)-1\right)\\&=\frac{i^a\sqrt{q}}{G(\chi)x^{k+a}}\left(\frac{q}{\pi}\right)^{\frac{k-1-a}{2}}\frac{\Gamma\left(\frac{k+a}{2}\right)}{\Gamma(a+\frac{1}{2})}\sum_{n=1}^{\infty}\frac{\mu(n)\overline{\chi(n)}}{n^{1+a}}\left({}_1F_1\left(\frac{k+a}{2};a+\frac{1}{2};-\frac{\pi}{q(nx)^2}\right)-1\right),\end{aligned}$$

where in the last step we have substituted $\epsilon(\chi)=\frac{G(\chi)}{i^a\sqrt{q}}$ and $X_{n,q}=\frac{q(nx)^2}{\pi}$. Now we employ Lemma 4.1.1 to simplify further. Replace m by $a+2$ in Lemma 4.1.1 to get

$$\frac{(a)!q^aG(\chi)}{2^{a+1}\pi^a i^a}L(1+a,\bar{\chi})=L'(-a,\chi).$$

Substituting this expression and simplifying, we reach

$$V_{k,\chi}^{(2)}(x) = \frac{i^a\sqrt{q}}{G(\chi)}\left(\frac{q}{\pi x^2}\right)^{\frac{k+a}{2}}\left(\frac{\pi}{q}\right)^{a+\frac{1}{2}}\frac{\Gamma(\frac{k+a}{2})}{\Gamma(a+\frac{1}{2})}\sum_{n=1}^{\infty}\frac{\overline{\chi(n)}\mu(n)}{n^{1+a}}{}_1F_1\left(\frac{k+a}{2};a+\frac{1}{2};-\frac{\pi}{qn^2x^2}\right)$$
$$-\frac{\Gamma\left(\frac{k+a}{2}\right)\left(\frac{\pi x^2}{q}\right)^{-\frac{k+a}{2}}}{2L'(-a,\chi)}. \quad (4.3.16)$$

The last term in the above expression will be cancelled with the residual term $R_{\frac{k+a}{2}}$ present in (4.3.4). Thus, finally substituting (4.3.16) in (4.3.5) and together with residual terms (4.3.4) and (4.3.6), we complete the proof of Theorem 4.2.1.

□

Remark 7. *Here we would like to point out that the convergence of the series present in* (4.2.1) *is not direct and straightforward. This series is a Dirichlet character analogue of the series present in the identity of Hardy and Littlewood* (1.1.4).

The bracketing condition (1.1.6) *tells that, if ρ_1 and ρ_2 are two non-trivial zeros of $L(s,\chi)$ such that for some positive constant C,*

$$|\operatorname{Im}(\rho_1)-\operatorname{Im}(\rho_2)| < \exp\left(-\frac{C\operatorname{Im}(\rho_1)}{\log(\operatorname{Im}(\rho_1))}\right)+\exp\left(-\frac{C\operatorname{Im}(\rho_2)}{\log(\operatorname{Im}(\rho_2))}\right),$$

then the terms corresponding to these zeros will be considered in the same bracket. Note that the expression $\exp\left(-\frac{C\operatorname{Im}(\rho_1)}{\log(\operatorname{Im}(\rho_1))}\right)$ tends to zero as $\operatorname{Im}(\rho_1)$ tends to infinity. It is highly probable that the non-trivial zeros are far apart than the above condition.

Thus, due to this assumption of the bracketing conditions of the ordinates of the non-trivial zeros of the $L(s,\chi)$, we will be able to find a sequence of real numbers T, tending to infinity, such that $|T-\operatorname{Im}(\rho)| > \exp\left(-\frac{A_0\operatorname{Im}(\rho)}{\log(\operatorname{Im}(\rho))}\right)$, for every non-trivial zero of $L(s,\chi)$. The existence of such sequence is exactly same as the assumption of Lemma 4.1.2 and it was one of the crucial lemmas to prove Theorem 4.2.1. Moreover in Theorem 4.2.1, we have assumed the simplicity hypothesis, which implies $|L'(\rho,\chi)| > 0$, but till today we do not know much about the lower bound for $|L'(\rho,\chi)|$. Due to this reason, it is difficult to prove the absolute convergence of the series present in (4.2.1). *To know more about this delicacy of convergence of such kind of series, readers are encouraged to see [110, p. 219], [79, p. 783, Remark 4].*

Proof of Theorem 4.2.2. Firstly, assuming the bound (4.2.3) for $P_{k,\ell,\chi}(x)$, we shall show that all the non-trivial zeros of $L(s,\chi)$ lie on $\operatorname{Re}(s)=1/2$. We prove this by considering two different cases, namely for $k=1$ and $k>1$. We first prove the result for $k>1$. Invoking Lemma 4.1.4, one has

$$L(\ell s+k,\chi)\int_0^{\infty} x^{-s-1}P_{k,\ell,\chi}(x)\mathrm{d}x = \Gamma(-s), \quad (4.3.17)$$

valid in the region $\frac{1-k}{\ell} < \mathrm{Re}(s) < 0$. Now we shall attempt to extend the region of validity of the above identity (4.3.17) on the left half plane, mainly, in the region $\frac{1}{2\ell} - \frac{k}{\ell} < \mathrm{Re}(s) < 0$ except for a possible pole at $s = \frac{1-k}{\ell}$. We choose a large positive real number R, and then write

$$L(\ell s + k, \chi) \left(\int_0^R + \int_R^\infty \right) x^{-s-1} P_{k,\ell,\chi}(x) \mathrm{d}x = \Gamma(-s).$$

Utilizing the bound (4.2.3) for $P_{k,\ell,\chi}(x)$, we can clearly see that the unbounded part is analytic in the domain $\mathrm{Re}(s) > \frac{1}{2\ell} - \frac{k}{\ell}$, whereas using (4.1.4) we have trivial bound for $P_{k,\ell,\chi}(x) = O(e^x)$, then one can show that the finite part is analytic for $\mathrm{Re}(s) < 0$. Therefore, in particular, the identity (4.3.17) is analytic in the strip $\frac{1}{2\ell} - \frac{k}{\ell} < \mathrm{Re}(s) \leq \frac{1-k}{\ell}$ except for a possible pole at $s = \frac{1-k}{\ell}$. Note that $L(\ell s + k, \chi)$ has a simple pole at $s = \frac{1-k}{\ell}$ when χ is a principal character, otherwise it is an entire function. We know that $\Gamma(s)$ never vanishes, which indicates that $L(\ell s + k, \chi)$ has no zero in the strip $\frac{1}{2\ell} - \frac{k}{\ell} < \mathrm{Re}(s) \leq \frac{1-k}{\ell}$. This is equivalent to saying that $L(s, \chi)$ does not vanish in the strip $\frac{1}{2} < \mathrm{Re}(s) \leq 1$ and thus the functional equation of $L(s, \chi)$ suggests that it has no zero in the strip $0 \leq \mathrm{Re}(s) < \frac{1}{2}$ except for a possible trivial zero at $s = 0$. This proves that all the non-trivial zeros of $L(s, \chi)$ will lie on the critical line $\mathrm{Re}(s) = 1/2$.

For the case $k = 1$ and χ principal, the similar arguments works except the fact that we use Lemma 4.1.3 which is valid in the strip $0 < \mathrm{Re}(s) < 1$ and we extend the region of analyticity of

$$L(\ell s + 1, \chi) \left(\int_0^R + \int_R^\infty \right) x^{-s-1} P_{1,\ell,\chi}(x) \mathrm{d}x = \Gamma(-s). \tag{4.3.18}$$

to $-\frac{1}{2\ell} < \mathrm{Re}(s) \leq 0$ except for a possible pole at $s = 0$. This can be done by utilizing the bound (4.2.3) for $P_{k,\ell,\chi}(x)$ for $k = 1$, we can clearly see that the unbounded part is analytic in the domain $\mathrm{Re}(s) > \frac{-1}{2\ell}$, whereas using (4.1.1) we have trivial bound for $P_{1,\ell,\chi}(x) = O(e^x)$, then one can show that the finite part is analytic for $\mathrm{Re}(s) \leq 0$. Since $\Gamma(s)$ is never vanishing implying that $L(\ell s + 1, \chi)$ is non-zero in this region which is same as saying that $L(s, \chi)$ is non-vanishing in $\frac{1}{2} < \mathrm{Re}(s) \leq 1$ and by the symmetry of the functional equation, $L(s, \chi)$ no zero in the region $0 \leq \mathrm{Re}(s) < \frac{1}{2}$ except for a possible trivial zero at $s = 0$. This proves that all the non-trivial zeros of $L(s, \chi)$ lie on the critical line $\mathrm{Re}(s) = 1/2$.

Now we shall try to show the converse part, that is, we assume that the generalized Riemann hypothesis for $L(s, \chi)$ is true. In [78, Proposition 5.14], we find that the generalized Riemann hypothesis is equivalent to the following bound, that is, for any $\epsilon > 0$,

$$S(x) := \sum_{1 \leq n \leq x} \chi(n)\mu(n) = O_\epsilon \left(x^{\frac{1}{2}+\epsilon} \right). \tag{4.3.19}$$

This can be proved along the lines given in Titchmarch [110, 370]. Now we employ Euler's partial summation formula, i.e., Lemma 2.6.1 with $a_n = \chi(n)\mu(n)$ and $f(t) = t^{-k}$, to see that

$$T(m;n) := \sum_{j=m}^{n} \frac{\chi(j)\mu(j)}{j^k} = S(n)f(n) - S(m-1)f(m-1) - \int_{m-1}^{n} S(t)f'(t)\mathrm{d}t. \tag{4.3.20}$$

Substituting (4.3.19) in (4.3.20), we can readily see that

$$T(m;n) = O_{\epsilon,k}\left(m^{\frac{1}{2}-k+\epsilon}\right), \tag{4.3.21}$$

uniformly in n. We are interested to find the bound for the following infinite series

$$P_{k,\ell,\chi}(x) = \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^k} \exp\left(-\frac{x}{n^\ell}\right).$$

Now for the sake of simplicity, we replace x by x^ℓ and divide the sum into two parts, that is,

$$P_{k,\ell,\chi}(x^\ell) := Y_1(x^\ell) + Y_2(x^\ell), \tag{4.3.22}$$

where

$$Y_1(x^\ell) := \sum_{n=1}^{m-1} \frac{\chi(n)\mu(n)}{n^k} \exp\left(-\frac{x^\ell}{n^\ell}\right), \quad Y_2(x^\ell) = \sum_{n=m}^{\infty} \frac{\chi(n)\mu(n)}{n^k} \exp\left(-\frac{x^\ell}{n^\ell}\right),$$

and $m = [x^{1-\epsilon}] + 1$. First, we trivially bound $Y_1(x^\ell)$. By using $|\chi(n)\mu(n)| \leq 1$, one has

$$|Y_1(x^\ell)| \leq \sum_{n=1}^{m-1} \exp\left(-\frac{x^\ell}{m^\ell}\right).$$

As $m = [x^{1-\epsilon}] + 1$, it follows that

$$Y_1(x^\ell) = O\left(x^{1-\epsilon}\exp(-x^{\ell\epsilon})\right). \tag{4.3.23}$$

Now we shall concentrate on the evaluation of the bound for $Y_2(x^\ell)$. Utilizing the definition (4.3.20) of $T(m;n)$, we derive that, for any integer $N > m$,

$$\begin{aligned}\sum_{n=m}^{N} \frac{\chi(n)\mu(n)}{n^k} \exp\left(-\frac{x^\ell}{n^\ell}\right) &= \sum_{n=m}^{N-1} T(m;n)\left[\exp\left(-\frac{x^\ell}{n^\ell}\right) - \exp\left(-\frac{x^\ell}{(n+1)^\ell}\right)\right] \\ &\quad + T(m;N)\exp\left(-\frac{x^\ell}{N^\ell}\right).\end{aligned}$$

Allowing $N \to \infty$ in the above equation and invoking the bound (4.3.21) for $T(m,n)$, one can deduce that

$$Y_2(x^\ell) = Z(x^\ell) + O_{\epsilon,k}\left(m^{\frac{1}{2}-k+\epsilon}\right), \tag{4.3.24}$$

where

$$Z(x^\ell) := \sum_{n=m}^{\infty} T(m;n)\left[\exp\left(-\frac{x^\ell}{n^\ell}\right) - \exp\left(-\frac{x^\ell}{(n+1)^\ell}\right)\right]$$

$$= \sum_{n=m}^{\infty} T(m;n)\left[F(n) - F(n+1)\right],$$

with $F(y) := \exp\left(-\frac{x^\ell}{y^\ell}\right)$. To simplify further, we make use of mean value theorem. We write $F(n+1) - F(n) = F'(c_n) = \frac{\ell x^\ell}{c_n^{\ell+1}} \exp\left(-\frac{x^\ell}{c_n^\ell}\right)$ for some $n < c_n < n+1$. Substituting this expression and in view of the bound (4.3.20), we can show that

$$\begin{aligned}
\left|Z(x^\ell)\right| &\ll_{\epsilon,\ell} m^{\frac{1}{2}-k+\epsilon} \sum_{n=m}^{\infty} \frac{x^\ell}{c_n^{\ell+1}} \exp\left(-\frac{x^\ell}{c_n^\ell}\right) \\
&\ll_{\epsilon,\ell} m^{\frac{1}{2}-k+\epsilon} \sum_{n=m}^{\infty} \frac{x^\ell}{n^{\ell+1}} \\
&\ll_{\epsilon,\ell} m^{\frac{1}{2}-k+\epsilon} \frac{x^\ell}{m^\ell} \ll_{\epsilon,\ell} x^{\frac{1}{2}-k+\epsilon'}, \qquad (4.3.25)
\end{aligned}$$

as $m \sim x^{1-\epsilon}$. At this moment, plugging (4.3.25) in (4.3.24), we see that

$$|Y_2(x^\ell)| = O_\epsilon\left(x^{\frac{1}{2}-k+\epsilon}\right). \qquad (4.3.26)$$

Finally, substituting the bounds (4.3.23) and (4.3.26) for $Y_1(x^\ell)$ and $Y_2(x^\ell)$ in (4.3.22) and comparing, we can conclude that

$$P_{k,\ell,\chi}(x^\ell) = O_{\epsilon,k,\ell}\left(x^{\frac{1}{2}-k+\epsilon}\right).$$

At the end, replace x by $x^{\frac{1}{\ell}}$ to obtain (4.2.2). This completes the proof of Theorem 4.2.2. □

This chapter is motivated from the identity (1.1.4) of Hardy and Littlewood. In 2012, Dixit [45] obtained the identity (1.1.7), which is a character analogue of the identity (1.1.4). In the present chapter, we have established a one-variable generalization of Dixit's identity, namely, Theorem 4.2.1. This generalization also provides a character analogue of the identity (3.2.1). Motivated from the work of Riesz, Hardy and Littlewood, we have established the following equivalent criteria for the generalized Riemann hypothesis for $L(s,\chi)$: For $k \geq 1, \ell > 0$, we have

$$\sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^k} \exp\left(-\frac{x}{n^\ell}\right) = O_\epsilon\left(x^{-\frac{k}{\ell}+\frac{1}{2\ell}+\epsilon}\right), \quad \text{as } x \to \infty.$$

This bound generalizes the bounds given by Riesz, and Hardy and Littlewood as well as our previous bound (3.2.2). The above bound motivates us to predict a more general observation. Let us consider

$$L(f,s) = \sum_{n=1}^{\infty} \frac{A_f(n)}{n^s}, \text{ for } \mathrm{Re}(s) > 1, \qquad (4.3.27)$$

be a "nice" L-function. Mainly, we are assuming that $L(f,s)$ obeys the grand Riemann hypothesis [78, p. 113–114], which says that all the non-trivial zeros of $L(f,s)$ in the critical strip $0 < \mathrm{Re}(s) < 1$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. Let us suppose

that [78, p. 114],

$$1/L(f,s) = \sum_{n=1}^{\infty} \frac{\mu_f(n)}{n^s}, \text{ for } \mathrm{Re}(s) > 1. \tag{4.3.28}$$

Moreover, we assume that $L(f,s)$ has a pole at $s=1$ which indicates that $\sum_{n=1}^{\infty} \frac{\mu_f(n)}{n}$ converges to zero. Littlewood's prediction says that the grand Riemann hypothesis [78, Proposition 5.14] is equivalent to the following bound:

$$\sum_{n \le x} \mu_f(n) \ll_{\epsilon,f} x^{\frac{1}{2}+\epsilon}, \tag{4.3.29}$$

for any $\epsilon > 0$. The above bound played a significant role to prove (8.1). This motivates us to give the following conjecture.

Conjecture 4.3.1. *Let $k \geq 1$ and $\ell > 0$ be two real numbers. Let $L(f,s)$ be a "nice" L-function. The grand Riemann hypothesis for $L(f,s)$ is equivalent to the bound*

$$\sum_{n=1}^{\infty} \frac{\mu_f(n)}{n^k} \exp\left(-\frac{x}{n^\ell}\right) = O_{\epsilon,f,k,\ell}\left(x^{-\frac{k}{\ell}+\frac{1}{2\ell}+\epsilon}\right), \quad \text{as } x \to \infty. \tag{4.3.30}$$

In the next chapter, we give a proof of the above Conjecture.

Chapter 5

EQUIVALENT CRITERIA FOR THE RIEMANN HYPOTHESIS FOR A GENERAL CLASS OF L-FUNCTIONS

This chapter is based on our paper [55]. In this chapter, one of our main goals is to prove Conjecture 4.3.1. Moreover, we also establish a generalization of the Ramanujan–Hardy–Littlewood identity (1.1.4) for a more general class of L-functions, defined in the Section 2.3. Furthermore, we obtain an entirely novel form of equivalent criteria for the Riemann hypothesis of $\zeta(s)$, see Theorem 5.2.13.

5.1 Preliminaries

In this section, we state a few results that will be instrumental in proving our main findings. We begin with a result for the growth of the summatory function of an arithmetical function.

Lemma 5.1.1. *Let $a(n)$ be an arithmetical function such that $a(n) = O(\Phi(n))$, where $\Phi(n)$ is an increasing function for $x \geq x_0$. Suppose that the Dirichlet series $A(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ is absolutely convergent for $\mathrm{Re}(s) > c_0$ and*

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{\sigma}} \leq \frac{1}{(\sigma - c_0)^{\gamma}} \quad \text{as} \quad \sigma \to \mathrm{c}_0^{+},$$

for some $\gamma > 0$. Then for any $c > c_0$, one has

$$\sum_{n \leq x} a(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} A(w) x^{w} \frac{\mathrm{d}w}{w} + O\left(\frac{x^{c}}{T(c - c_0)^{\gamma}}\right)$$

$$+ O\left(\frac{\Phi(2x)x\log(2x)}{T}\right) + O\left(\Phi(2x)\right),$$

where T is some large positive real number.

Proof. This result can be found in [77, p. 486, Equation (A.10)]. □

Using the above identity, under the assumption of the Riemann hypothesis for $\zeta(s)$, we obtain the below result which is an analogue of Littlewood's bound (4.3.29).

Lemma 5.1.2. *Let r be a non-negative integer. Under the assumption of the Riemann hypothesis for $\zeta(s)$, we have*

$$\sum_{n\leq x}\sigma_r^{-1}(n) = O\left(x^{\frac{1}{2}+r+\epsilon}\right).$$

Proof. First, assuming RH for the Riemann zeta function, we will prove this bound. In Lemma 5.1.1, we consider $a(n) = \sigma_r^{-1}(n) = \mu N^r * \mu$. Then

$$|a(n)| = \left|\sum_{d|n}\mu(d)d^r\mu\left(\frac{n}{d}\right)\right| \leq \sum_{d|n} d^r \leq \sum_{d|n} n^r = n^r d(n) = O(n^{r+\epsilon}),$$

for any $\epsilon > 0$. This suggests us to take $\Phi(n) = n^{r+\epsilon}$. Thus, with $A(s) = \frac{1}{\zeta(s)\zeta(s-r)}$ and $c = 1 + r + \epsilon$, we see that

$$\sum_{n\leq x}\sigma_r^{-1}(n) = \frac{1}{2\pi i}\int_{1+r+\epsilon-iT}^{1+r+\epsilon+iT}\frac{1}{\zeta(w)\zeta(w-r)}\frac{x^w}{w}\mathrm{d}w + O\left(\frac{x^{1+r+\epsilon}}{T}\right)$$
$$+ O\left(\frac{x^{r+\epsilon}x\log(2x)}{T}\right) + O\big(x^{r+\epsilon}\big).$$

Now we change the line of integration from $\mathrm{Re}(s) = 1 + r + \epsilon$ to $\mathrm{Re}(s) = \frac{1}{2} + r + \epsilon$. Then we have

$$\sum_{n\leq x}\sigma_r^{-1}(n) = \frac{1}{2\pi i}\left[\int_{\frac{1}{2}+r+\epsilon+iT}^{1+r+\epsilon+iT} + \int_{\frac{1}{2}+r+\epsilon-iT}^{\frac{1}{2}+r+\epsilon+iT} + \int_{1+r+\epsilon-iT}^{\frac{1}{2}+r+\epsilon-iT}\right]\frac{1}{\zeta(w)\zeta(w-r)}\frac{x^w}{w}\mathrm{d}w$$
$$+ O\left(\frac{x^{1+r+\epsilon}}{T}\right) + O(x^{r+\epsilon}). \tag{5.1.1}$$

Under the assumption of RH, we know $\frac{1}{\zeta(\sigma+it)} = O(|t|^\epsilon)$ for $\sigma > 1/2$. Using this bound and under simplification, one can check that

$$\left|\int_{\frac{1}{2}+r+\epsilon+iT}^{1+r+\epsilon+iT}\frac{1}{\zeta(w)\zeta(w-r)}\frac{x^w}{w}\mathrm{d}w\right| \ll \int_{\frac{1}{2}+r+\epsilon}^{1+r+\epsilon} x^\sigma T^{2\epsilon-1}d\sigma \ll T^{2\epsilon-1}x^{2+r+\epsilon},$$
$$\left|\int_{\frac{1}{2}+r+\epsilon-iT}^{\frac{1}{2}+r+\epsilon+iT}\frac{1}{\zeta(w)\zeta(w-r)}\frac{x^w}{w}\mathrm{d}w\right| \ll \int_{-T}^{T}|t|^{2\epsilon}x^{\frac{1}{2}+r+\epsilon}\frac{dt}{|t|} \ll T^{2\epsilon}x^{\frac{1}{2}+r+\epsilon},$$
$$\left|\int_{1+r+\epsilon-iT}^{\frac{1}{2}+r+\epsilon-iT}\frac{1}{\zeta(w)\zeta(w-r)}\frac{x^w}{w}\mathrm{d}w\right| \ll \int_{1+r+\epsilon}^{\frac{1}{2}+r+\epsilon} x^\sigma T^{2\epsilon-1}d\sigma \ll T^{2\epsilon-1}x^{2+r+\epsilon}.$$

Utilizing these bounds in (5.1.1), we arrive

$$\sum_{n\leq x}\sigma_r^{-1}(n) = O\left(T^{2\epsilon-1}x^{2+r+\epsilon}\right) + O\left(T^{2\epsilon}x^{\frac{1}{2}+r+\epsilon}\right) + O\left(\frac{x^{1+r+\epsilon}}{T}\right) + O\big(x^{r+\epsilon}\big)$$
$$= O\left(T^{2\epsilon-1}x^{2+r+\epsilon}\right) + O\left(T^{2\epsilon}x^{\frac{1}{2}+r+\epsilon}\right).$$

Taking $T = x^{\frac{3}{2}}$, the result follows. □

The forthcoming result will serve as one of the vital components in establishing equivalent criteria for the grand Riemann hypothesis. For $k \geq 1$, $\ell > 0$, we define the function

$$P_{k,f,\ell}(x) := \sum_{n=1}^{\infty}\frac{\mu_f(n)}{n^k}\exp\left(-\frac{x}{n^\ell}\right).$$

Lemma 5.1.3. *Let $L(f,s)$ be a nice L-function and its inverse defined in* (4.3.27) *and* (4.3.28)*, respectively. For $k > 1$ and $\ell > 0$, we have*[1]

$$\int_0^\infty x^{-s-1}P_{k,f,\ell}(x)\mathrm{d}x = \frac{\Gamma(-s)}{L(f,\ell s+k)},$$

valid in the region $\frac{1-k}{\ell} < \mathrm{Re}(s) < 0$.

Proof. The proof follows the same line of argument as Lemma 4.1.4. Thus, we omit the details for the reader. □

Lemma 5.1.4. *Let $L(f,s)$ be a nice L-function with pole at $s = 1$. For $k = 1$ and ℓ be any positive real number, we have*

$$\int_0^\infty x^{-s-1}P_{1,f,\ell}(x)\mathrm{d}x = \frac{\Gamma(-s)}{L(f,\ell s+1)},$$

valid in the region $0 < \mathrm{Re}(s) < 1$.

The next two results are analogues of Lemma 5.1.3 and Lemma 5.1.4 for $\zeta(s)\zeta(s-r)$. In this context, we first define the following function.
Let r be a non-negative integer and $k \geq r+1$ be any positive real number. We define

$$P_{k,\sigma,\ell}(x) := \sum_{n=1}^{\infty}\frac{\sigma_r^{-1}(n)}{n^k}\exp\left(-\frac{x}{n^\ell}\right).$$

Lemma 5.1.5. *Let $k > r+1$ and $\ell > 0$. Then for $\frac{1-k+r}{\ell} < \mathrm{Re}(s) < 0$, we have*

$$\int_0^\infty x^{-s-1}P_{k,\sigma,\ell}(x)\mathrm{d}x = \frac{\Gamma(-s)}{\zeta(k+\ell s)\zeta(k+\ell s-r)}.$$

[1]Lemma 4.4 in [55] contains an error which has been corrected in this thesis. Mainly, we divided the lemma into two cases, namely, Lemma 5.1.3 and Lemma 5.1.4 corresponding to $k > 1$ and $k = 1$, respectively. Moreover, Lemma 5.1.3 holds for $\frac{1-k}{\ell} < \mathrm{Re}(s) < 0$, whereas Lemma 5.1.4 holds for $0 < \mathrm{Re}(s) < 1$.

Proof. The proof follows along the same line as of Lemma 4.1.4. Thus, we skip the details here. □

Lemma 5.1.6. *Let $k = r + 1$ and $\ell > 0$. Then for $0 < Re(s) < 1$, we have*

$$\int_0^\infty x^{-s-1} P_{r+1,\sigma,\ell}(x)\mathrm{d}x = \frac{\Gamma(-s)}{\zeta(r+1+\ell s)\zeta(1+\ell s)}.$$

Proof. The proof goes along the same path as of Lemma 5.1.4. Thus, we omit the details here. □

The next result gives an information about the behaviour of general L-functions in a vertical strip.

Lemma 5.1.7. *Any L-function is polynomially bounded in vertical strips $\sigma_1 < \mathrm{Re}(s) < \sigma_2$, and $\mathrm{Im}(s) \geq 1$.*

Proof. We refer [78, Lemma 5.2] for the proof. □

Before stating our main theorems, we need to define the Dirichlet inverse of an arithmetical function $a(n)$. Let $a(n)$ be an arithmetical function with $a(1) \neq 0$, then there exists a unique arithmetical function $a^{-1}(n)$, known as Dirichlet inverse of $a(n)$, such that

$$a * a^{-1} = a^{-1} * a = I.$$

Furthermore, $a^{-1}(n)$ is recursively defined as

$$a^{-1}(1) = \frac{1}{a(1)}, \quad a^{-1}(n) = \frac{-1}{a(1)} \sum_{\substack{d|n \\ d<n}} a\left(\frac{n}{d}\right) a^{-1}(d) \quad \text{for} \quad n > 1.$$

A proof of this standard result can be found in [8, p. 30].

5.2 Main Results

In this section, we state the main results of this chapter.

Theorem 5.2.1. *Let $a(n)$ and $b(n)$ be two arithmetical functions whose corresponding L-functions are $\phi(s)$ and $\psi(s)$ and satisfy the functional equation (2.3.1). Assume that all the non-trivial zeros of $\phi(s)$ are simple. Then for any $k \geq \delta + \frac{B}{A}$ and $x > 0$, we have*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{a^{-1}(n)}{n^k} \exp\left(-\frac{x}{n^{1/A}}\right) \\
&= \frac{c^{-\frac{A\delta+2B}{A}}}{\nu x^{Ak+B}} \frac{\Gamma(Ak+B)}{\Gamma(A\delta+2B)} \sum_{n=1}^{\infty} \frac{b^{-1}(n)}{n^{\frac{A\delta+B}{A}}} \left[{}_1F_1\left(\begin{matrix} Ak+B \\ A\delta+2B \end{matrix} \Big| -\frac{1}{(c^2 n)^{1/A} x} \right) - 1 \right]
\end{aligned}$$

$$- R_t + A \sum_{\rho} \frac{\Gamma(A(k-\rho))}{\phi'(\rho)} x^{-A(k-\rho)}, \tag{5.2.1}$$

where ρ runs through the non-trivial zeros of $\phi(s)$ and the convergence of the infinite series over ρ adhere to the bracketing condition (1.1.6)*, and R_t represents the residual term corresponding to the trivial zeros of $\phi(k-\frac{s}{A})$ with $0 \leq \mathrm{Re}\left(k-\frac{s}{A}\right) \leq Ak+B$.*

Remark 8. *Setting $a(n) = b(n) = 1$ and thus $a^{-1}(n) = b^{-1}(n) = \mu(n)$, $\phi(s) = \psi(s) = \zeta(s)$ in Theorem 5.2.1 with $A = \frac{1}{2}, B = 0, c = \frac{1}{\sqrt{\pi}}, \delta = 1$ and $\nu = 1$, one can easily obtain the identity* (3.2.1)*.*

Remark 9. *Let χ be a primitive Dirichlet character modulo q and $L(s,\chi)$ be the associated Dirichlet series. Letting $a(n) = \chi(n), b(n) = \overline{\chi(n)}$ and thus $a^{-1}(n) = \chi(n)\mu(n), b^{-1}(n) = \overline{\chi(n)}\mu(n)$, $\phi(s) = L(s,\chi), \psi(s) = L(s,\overline{\chi})$ in Theorem 5.2.1 with $A = \frac{1}{2}, B = \frac{a}{2}, c = \sqrt{\frac{q}{\pi}}, \delta = 1$ and $\nu = \frac{G(\chi)}{i^a\sqrt{q}}$, and upon simplification, we can derive* (4.2.1)*.*

Now we shall state a few more interesting results coming from Theorem 5.2.1. First, we state an ingenious modular relation, which is a perfect generalization of Ramanujan–Hardy–Littlewood identity (1.1.5).

Corollary 5.2.2. *Assuming all the hypothesis of the Theorem 5.2.1 to be true. Let α, β be positive real numbers such that $\alpha\beta = \frac{1}{c^{2/A}}$. Then we have*

$$\begin{aligned}
&\sqrt{\nu}\alpha^{\frac{A\delta+2B}{2}} \sum_{n=1}^{\infty} \frac{b^{-1}(n)}{n^{\delta+B/A}} \exp\left(-\frac{\alpha}{n^{1/A}}\right) - \sqrt{\nu}\beta^{\frac{A\delta+2B}{2}} \sum_{n=1}^{\infty} \frac{a^{-1}(n)}{n^{\delta+B/A}} \exp\left(-\frac{\beta}{n^{1/A}}\right) \\
&= \frac{\alpha^{\frac{A\delta+2B}{2}}}{\sqrt{\nu}} \sum_{n=1}^{\infty} \frac{b^{-1}(n)}{n^{\delta+B/A}} + \sqrt{\nu}\beta^{\frac{A\delta+2B}{2}} R_t - A\sqrt{\nu} \sum_{\rho} \frac{\Gamma(A(\delta-\rho)+B)}{\phi'(\rho)} \beta^{\frac{A\delta-2B}{2}}.
\end{aligned}$$

The next result is corresponding to the Epstein zeta function, defined in Example 4.

Corollary 5.2.3. *Let $k \geq 1$ and x be any positive real number. Let $r_2(n)$ be the number of ways n can be written as sum of two squares and the associated L-function is defined in Example 4 for $j = 2$. Then, we have*

$$\sum_{n=1}^{\infty} \frac{r_2^{-1}(n)}{n^k} \exp\left(-\frac{x}{n}\right) = \frac{\pi\Gamma(k)}{x^k} \sum_{n=1}^{\infty} \frac{r_2^{-1}(n)}{n} {}_1F_1\left(\begin{matrix}k\\1\end{matrix}\middle| -\frac{\pi^2}{nx}\right) + \sum_{\rho} \frac{\Gamma(k-\rho)}{\zeta_{id}'(2,\rho)} x^{-(k-\rho)}. \tag{5.2.2}$$

Here, ρ runs over all the non-trivial zeros of $\zeta_{id}(2,s)$ and are assumed to be simple. The convergence of the series over ρ follows under the assumption of the bracketing condition (1.1.6)*.*

A few numerical evidences for the above result is given in Table 5.1.

Remark 10. *In particular, letting $k = 1$ and $x = \beta > 0$ with $\alpha\beta = \pi^2$ in Corollary 5.2.3, we obtain the following beautiful identity,*

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\frac{r_2^{-1}(n)}{n}\exp\left(-\frac{\alpha}{n}\right) - \sqrt{\beta}\sum_{n=1}^{\infty}\frac{r_2^{-1}(n)}{n}\exp\left(-\frac{\beta}{n}\right) = -\frac{1}{\sqrt{\beta}}\sum_{\rho}\frac{\Gamma(1-\rho)}{\zeta_{id}^{'}(2,\rho)}\beta^{\rho}.$$

The next identity is associated to the Dedekind zeta function over an imaginary quadratic field.

Corollary 5.2.4. *Let $k \geq 1$ be any real number. If a_n counts the number of integral ideals of norm n of an imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$ and the associated L-function is defined as in* (2.3.7) *and satisfies the functional equation* (2.3.8)*, then for any $x > 0$, we have*

$$\sum_{n=1}^{\infty}\frac{\mathsf{a}_n^{-1}}{n^k}\exp\left(-\frac{x}{n}\right) = \frac{2\pi\Gamma(k)}{\sqrt{D}x^k}\sum_{n=1}^{\infty}\frac{\mathsf{a}_n^{-1}}{n}{}_1F_1\left(\begin{matrix}k\\1\end{matrix}\Big|-\frac{4\pi^2}{Dnx}\right) + \sum_{\rho}\frac{\Gamma(k-\rho)}{\zeta_K^{'}(\rho)}x^{-(k-\rho)},$$

where ρ runs over the non-trivial zeros of $\zeta_K(s)$ and convergence of the infinite series over ρ follows from the bracketing condition (1.1.6)*.*

Remark 11. *In particular, substituting $k = 1$ and $x = \beta > 0$ and $\alpha\beta = \frac{4\pi^2}{D}$ in Corollary 5.2.4 gives the following identity[2] of Dixit, Gupta and Vatwani [46, Cor. 3.4],*

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\frac{\mathsf{a}_n^{-1}}{n}\exp\left(-\frac{\alpha}{n}\right) - \sqrt{\beta}\sum_{n=1}^{\infty}\frac{\mathsf{a}_n^{-1}}{n}\exp\left(-\frac{\beta}{n}\right) = -\frac{1}{\sqrt{\beta}}\sum_{\rho}\frac{\Gamma(1-\rho)}{\zeta_K^{'}(\rho)}\beta^{\rho}.$$

The next result corresponds to $\phi(s) = \zeta(s)\zeta(s-r)$, defined in Example 1, an important example of a Dirichlet series lying in the general class of L-functions, which lacks the Riemann hypothesis.

Theorem 5.2.5. *Let $\sigma_r(n)$ denote the sum of r^{th} powers of positive divisors of n. Then the corresponding L-function as defined in Example 1 satisfies the functional equation* (2.3.2)*. Let $k \geq r+1$, where r is an odd positive integer. Then for any positive x, we have*

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\sigma_r^{-1}(n)}{n^k}\exp\left(-\frac{x}{n}\right) &= \frac{(-1)^{\frac{r+1}{2}}(2\pi)^{r+1}}{x^k}\frac{\Gamma(k)}{\Gamma(r+1)}\sum_{n=1}^{\infty}\frac{\sigma_r^{-1}(n)}{n^{r+1}}{}_1F_1\left(\begin{matrix}k\\r+1\end{matrix}\Big|-\frac{4\pi^2}{nx}\right)\\ &+\sum_{m=1}^{\left[\frac{r}{2}\right]}\frac{\Gamma(k-r+2m)}{\zeta(r-2m)\zeta(2m+1)}\frac{(-1)^m 2(2\pi)^{2m}}{(2m)!}x^{r-k-2m}\\ &+\sum_{\rho}\frac{x^{\rho-k}}{\zeta^{'}(\rho)}\left[\frac{\Gamma(k-\rho)}{\zeta(\rho-r)}+\frac{x^r\Gamma(k-r-\rho)}{\zeta(r+\rho)}\right],\end{aligned}$$

[2] In [46, Cor. 3.4], there is an extra 1/2 on the right side of the identity.

where ρ runs over non-trivial zeros of $\zeta(s)$ and are assumed to be simple. The convergence of the associated series follows under the assumption of bracketing condition (1.1.6).

A numerical verification of the identity is given in Table 5.2.

In particular, letting $k = r + 1$ and $\alpha\beta = 4\pi^2$ in Theorem 5.2.5, we get the following identity.

Corollary 5.2.6. *Let r be an odd positive integer and α, β be positive real numbers with $\alpha\beta = 4\pi^2$. Then we have*

$$\begin{aligned}\alpha^{\frac{r+1}{2}} \sum_{n=1}^{\infty} \frac{\sigma_r^{-1}(n)}{n^{r+1}} \exp\left(-\frac{\alpha}{n}\right) - (-\beta)^{\frac{r+1}{2}} \sum_{n=1}^{\infty} \frac{\sigma_r^{-1}(n)}{n^{r+1}} \exp\left(-\frac{\beta}{n}\right) \\ = 2\alpha^{\frac{r-1}{2}} \sum_{m=1}^{[\frac{r}{2}]} \frac{(-1)^m}{\zeta(r-2m)\zeta(2m+1)} \left(\frac{\beta}{2\pi}\right)^{2m} \\ + \alpha^{\frac{r+1}{2}-k} \sum_{\rho} \frac{\alpha^{\rho}}{\zeta^{'}(\rho)} \left[\frac{\Gamma(r+1-\rho)}{\zeta(\rho-r)} + \frac{\alpha^r \Gamma(1-\rho)}{\zeta(r+\rho)}\right].\end{aligned}$$

The next result, for L-functions associated to cusp forms, we state as a separate theorem as it might have a significant importance in the literature.

Theorem 5.2.7. *Let $L(f, s)$ be the L-function associated to a cusp form f defined in* (2.3.4). *If we assume all the non-trivial zeros of $L(f, s)$ to be simple, then for $k \geq \frac{w+1}{2}$, $x > 0$, we have*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\lambda_f^{-1}(n)}{n^k} \exp\left(-\frac{x}{n}\right) = \frac{(2\pi)^{\omega}}{x^{k+\frac{\omega-1}{2}}} \frac{\Gamma(k+\frac{\omega-1}{2})}{\Gamma(\omega)} \sum_{n=1}^{\infty} \frac{\lambda_f^{-1}(n)}{n^{\frac{\omega+1}{2}}} {}_1F_1\left(\begin{matrix} k+\frac{\omega-1}{2} \\ \omega \end{matrix}\Big| -\frac{4\pi^2}{nx}\right) \\ + \sum_{\rho} \frac{\Gamma(k-\rho)x^{-(k-\rho)}}{L^{'}(f,\rho)}, \qquad (5.2.3)\end{aligned}$$

where ρ runs through non-trivial zeros of $L(f, s)$ and satisfy the bracketing condition (1.1.6).

By substituting $k = \frac{\omega+1}{2}$ in Theorem 5.2.7, we obtain an elegant modular relation depicted as follows.

Corollary 5.2.8. *Assuming all the hypothesis in Theorem 5.2.7, for $\alpha, \beta > 0$ with $\alpha\beta = 4\pi^2$, we have*

$$\alpha^{\frac{\omega}{2}} \sum_{n=1}^{\infty} \frac{\lambda_f^{-1}(n)}{n^{\frac{\omega+1}{2}}} \exp\left(-\frac{\alpha}{n}\right) - \beta^{\frac{\omega}{2}} \sum_{n=1}^{\infty} \frac{\lambda_f^{-1}(n)}{n^{\frac{\omega+1}{2}}} \exp\left(-\frac{\beta}{n}\right) = -\frac{1}{\sqrt{\beta}} \sum_{\rho} \frac{\Gamma(\frac{\omega+1}{2}-\rho)\beta^{\rho}}{L^{'}(f,\rho)}.$$

The next corollary gives a modular relation involving Ramanujan tau function.

Corollary 5.2.9. *Let α and β be two positive real numbers with $\alpha\beta = 4\pi^2$. Then we have*

$$\alpha^6 \sum_{n=1}^{\infty} \frac{\tau_0^{-1}(n)}{n^{\frac{13}{2}}} \exp\left(-\frac{\alpha}{n}\right) - \beta^6 \sum_{n=1}^{\infty} \frac{\tau_0^{-1}(n)}{n^{\frac{13}{2}}} \exp\left(-\frac{\beta}{n}\right) = -\frac{1}{\sqrt{\beta}} \sum_{\rho} \frac{\Gamma(\frac{13}{2} - \rho)\beta^{\rho}}{L^{'}(\Delta, \rho)}, \tag{5.2.4}$$

where ρ runs through non-trivial zeros of $L(\Delta, s)$.

It is interesting to note that the above identity has already been obtained by Dixit, Roy and Zaharescu [47, Corollary 1.2]. One can also obtain Corollary 5.2.8 by letting $z \to 0$ in Theorem 1.1 of [47]. However, our Theorem 5.2.7 is different from Theorem 1.1 of Dixit et. al [47].

5.2.1 Equivalent criteria for the grand Riemann hypothesis

Motivated from the equivalent criteria for the Riemann hypothesis of $\zeta(s)$ due to Hardy-Littlewood and Riesz, in this subsection, we state equivalent criteria for the Riemann hypothesis for a general class of L-functions, defined in (4.3.27) and satisfying Littlewood's bound (4.3.29). In particular, we give equivalent criteria for the Riemann hypothesis for L-functions lying in the general class of L-functions that may obey Riemann hypothesis.

Theorem 5.2.10. *Conjecture 4.3.1 holds true.*

As we know that the Epstein zeta function $\zeta_{id}(2, s)$ and L-functions $L(f, s)$ associated to Hecke eigenforms are particular examples of nice L-functions that are lying in the general class of L- functions. Thus, as an immediate implication, we have the following results.

Corollary 5.2.11. *Let $\zeta_{id}(2, s)$ be the Epstein zeta function defined in Example 4. For $k \geq 1, \ell > 0$, the Riemann hypothesis for $\zeta_{id}(2, s)$ is equivalent to*

$$\sum_{n=1}^{\infty} \frac{r_2^{-1}(n)}{n^k} \exp\left(-\frac{x}{n^{\ell}}\right) = O_{\epsilon,k,\ell}\left(x^{-\frac{k}{\ell}+\frac{1}{2\ell}+\epsilon}\right),$$

for any $\epsilon > 0$, as $x \to \infty$.

Corollary 5.2.12. *Let f be a holomorphic Hecke eigenform of weight ω over the full modular group $SL_2(\mathbb{Z})$ and $L(f, s)$ be the L-function defined in (2.3.4). For $k \geq \frac{\omega+1}{2}, \ell > 0$, the Riemann hypothesis for $L(f, s)$ is equivalent to the following bound:*

$$\sum_{n=1}^{\infty} \frac{\lambda_f^{-1}(n)}{n^k} \exp\left(-\frac{x}{n^{\ell}}\right) = O_{\epsilon,k,f,\ell}\left(x^{-\frac{k}{\ell}+\frac{1}{2\ell}+\epsilon}\right),$$

for any $\epsilon > 0$ as $x \to \infty$.

In the next result, we state a new equivalent criteria for the Riemann hypothesis of $\zeta(s)$. This criteria significantly differ from the previous known equivalent criteria for RH due to Hardy-Littlewood and Riesz. The heuristic of this bound comes from the identity mentioned in Theorem 5.2.5.

Theorem 5.2.13. *Let r be a non-negative integer and $k \geq r+1, \ell > 0$. The Riemann hypothesis for $\zeta(s)$ is equivalent to the bound*

$$P_{k,\sigma,\ell}(x) := \sum_{n=1}^{\infty} \frac{\sigma_r^{-1}(n)}{n^k} \exp\left(-\frac{x}{n^\ell}\right) = O_{\epsilon,k,r,\ell}\left(x^{-\frac{k}{\ell}+\frac{1+2r}{2\ell}+\epsilon}\right), \tag{5.2.5}$$

for any $\epsilon > 0$, as $x \to \infty$.

5.3 Proof of Main Result

Proof of Theorem 5.2.1. Shifting the line of integration from $\mathrm{Re}(s) > 0$ to $\mathrm{Re}(s) = d \in (-1, 0)$ in the inverse mellin transform of $\Gamma(s)$, we have

$$e^{-x} - 1 = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Gamma(s) x^{-s} \mathrm{d}s. \tag{5.3.1}$$

We choose k_0 such that the following series $\sum_{n=1}^{\infty} \frac{a^{-1}(n)}{n^k}$, $\sum_{n=1}^{\infty} \frac{b^{-1}(n)}{n^k}$ are convergent for $k = k_0$ and absolutely convergent for all $k > k_0$. Then for $k \geq k_0$, $x > 0$, we use (5.3.1) to write

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{a^{-1}(n)}{n^k} \exp\left(-\frac{x}{n^{1/A}}\right) &= \sum_{n=1}^{\infty} \frac{a^{-1}(n)}{n^k} + \sum_{n=1}^{\infty} \frac{a^{-1}(n)}{n^k} \left(\exp\left(-\frac{x}{n^{1/A}}\right) - 1\right) \\
&= \frac{1}{\phi(k)} + \sum_{n=1}^{\infty} \frac{a^{-1}(n)}{n^k} \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Gamma(s) \left(\frac{x}{n^{1/A}}\right)^{-s} \mathrm{d}s \\
&= \frac{1}{\phi(k)} + \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma(s)}{\phi(k - s/A)} x^{-s} \mathrm{d}s.
\end{aligned} \tag{5.3.2}$$

Since $\mathrm{Re}(k - s/A) > k_0$, hence $1/\phi(k - s/A) = \sum_{n=1}^{\infty} \frac{a^{-1}(n)}{n^{k-s/A}}$ is absolutely and uniformly convergent in any compact subset of the domain. Thus, interchanging the order of summation and integration is justified. We now intend to solve the following vertical integral,

$$J_{k,A}(x) := \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma(s)}{\phi(k - s/A)} x^{-s} \mathrm{d}s.$$

We shall investigate the poles of the integrand function in the first place. One can observe that $\Gamma(s)$ has simple poles at non-positive integers. Let S be the set of all the trivial zeros of $\phi(s)$ and to collect the contribution of the residual term due to non-trivial zeros of $\phi(k - s/A)$, we change the line of integration from $\mathrm{Re}(s) = d \in (-1, 0)$ to $\mathrm{Re}(s) = \lambda \in (Ak + B, Ak + B + \epsilon)$ with $0 < \epsilon < 1$. For that, we

consider a rectangular contour $\mathcal{C}$ as in Figure 5.1 determined by the line segments $[\lambda - iT, \lambda + iT], [\lambda + iT, d + iT], [d + iT, d - iT]$, and $[d - iT, \lambda - iT]$, where T is some large positive number.

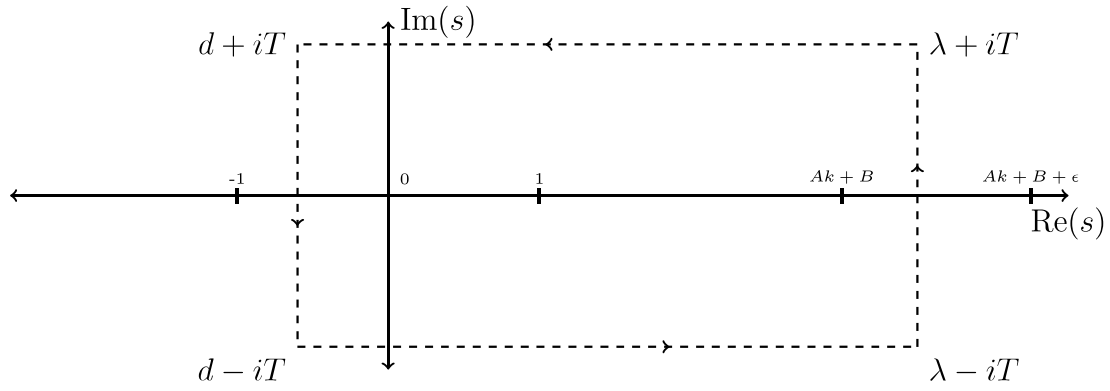


Figure 5.1

We now use Cauchy's residue theorem to get

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\Gamma(s)}{\phi(k-s/A)} x^{-s} \mathrm{d}s = R_0 + R_t + \sum_{|\operatorname{Im}(\rho)|<T} R_\rho(x), \tag{5.3.3}$$

where R_0 stands for the residual term corresponding to the simple pole of $\Gamma(s)$ at $s = 0$, R_t is the residual term due to trivial zeros of $\phi(k-s/A)$ with $0 \leq \operatorname{Re}\left(k - \frac{s}{A}\right) \leq Ak + B$ and $R_\rho(x)$ is the residual term corresponding to the non-trivial zero ρ of $\phi(k - s/A)$. Next goal is to show that the horizontal integrals

$$H_1(T, A) = \frac{1}{2\pi i}\int_{\lambda+iT}^{d+iT} \frac{\Gamma(s)}{\phi(k-s/A)} x^{-s} \mathrm{d}s,$$
$$H_2(T, A) = \frac{1}{2\pi i}\int_{d-iT}^{\lambda-iT} \frac{\Gamma(s)}{\phi(k-s/A)} x^{-s} \mathrm{d}s,$$

will vanish as $T \to \infty$. This follows if we make use of Stirling's formula for $\Gamma(s)$, that is, Lemma 2.6.2 along with Lemma 5.1.7. Thus, letting $T \to \infty$ in (5.3.3), we arrive at

$$\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} \frac{\Gamma(s)}{\phi(k-s/A)} x^{-s} \mathrm{d}s = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} \frac{\Gamma(s)}{\phi(k-s/A)} x^{-s} \mathrm{d}s - R_0 - R_t - \sum_{\rho} R_\rho(x), \tag{5.3.4}$$

where the sum over ρ runs through all the non-trivial zeros of $\phi(k-s/A)$. The residue at $s = 0$ can be evaluated as

$$R_0 = \lim_{s\to 0} \frac{s\Gamma(s)}{\phi(k-s/A)} x^{-s} = \frac{1}{\phi(k)}. \tag{5.3.5}$$

Considering the non-trivial zeros of $\phi(s)$ to be simple, we get

$$\sum_{\rho} R_{\rho}(x) = \sum_{\rho} \lim_{s \to A(k-\rho)} \frac{(s - A(k-\rho))\,\Gamma(s)}{\phi(k - s/A)} x^{-s} = -A \sum_{\rho} \frac{\Gamma(A(k-\rho))}{\phi'(\rho)} x^{-A(k-\rho)}. \tag{5.3.6}$$

Substituting (5.3.4), (5.3.5) and (5.3.6) in (5.3.2), we get

$$\sum_{n=1}^{\infty} \frac{a^{-1}(n)}{n^k} \exp\left(-\frac{x}{n^{1/A}}\right) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \frac{\Gamma(s)}{\phi(k - s/A)} x^{-s} \mathrm{d}s - R_t + A \sum_{\rho} \frac{\Gamma(A(k-\rho))}{\phi'(\rho)} x^{-A(k-\rho)}. \tag{5.3.7}$$

We shall now focus on the derivation of the line integral present above. Let us define

$$I_{k,A}(x) := \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \frac{\Gamma(s)}{\phi\left(k - \frac{s}{A}\right)} x^{-s} \mathrm{d}s. \tag{5.3.8}$$

Here, we bring into play the role of the functional equation for $\phi(s)$. Replacing s by $k - s/A$ in (2.3.1), it can be seen that

$$\frac{1}{\phi(k - \frac{s}{A})} = \frac{c^{2k - \delta - \frac{2s}{A}} \Gamma(Ak + B - s)}{\nu \Gamma(A\delta - Ak + B + s) \psi(\delta - k + \frac{s}{A})}. \tag{5.3.9}$$

Utilizing (5.3.9) in (5.3.8), we get

$$I_{k,A}(x) = \frac{1}{\nu} \frac{c^{2k-\delta}}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \frac{\Gamma(s)\Gamma(Ak + B - s)}{\Gamma(A(\delta - k) + B + s) \psi(\delta - k + \frac{s}{A})} (c^{2/A} x)^{-s} \mathrm{d}s. \tag{5.3.10}$$

Note that $\mathrm{Re}(\delta - k + \frac{s}{A}) > \delta + \frac{B}{A}$. Thus, to use the series expansion of $\frac{1}{\psi(\delta - k + \frac{s}{A})} = \sum_{n=1}^{\infty} \frac{b^{-1}(n)}{n^{s/A}} n^{k-\delta}$ in (5.3.10), constraints us to choose k_0 such that $k_0 = \delta + \frac{B}{A}$. Then interchanging the order of summation and integration leads us to obtain that,

$$I_{k,A}(x) = \frac{c^{2k-\delta}}{\nu} \sum_{n=1}^{\infty} \frac{b^{-1}(n)}{n^{\delta - k}} V_{k,A}(X_n), \tag{5.3.11}$$

where

$$V_{k,A}(X_n) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \frac{\Gamma(s)\Gamma(Ak + B - s)}{\Gamma(A(\delta - k) + B + s)} X_n^{-s} \mathrm{d}s, \tag{5.3.12}$$

and $X_n = (c^2 n)^{1/A} x$. We shall now focus on $V_{k,A}(X_n)$ and we aim at writing this vertical integral in terms of the Meijer G-function. However, it can be seen that the line of integration $Ak + B < \mathrm{Re}(s) = \lambda < Ak + B + \epsilon$ does not separate the poles of $\Gamma(s)$ from that of $\Gamma(Ak + B - s)$. Thus, to use the definition of the Meijer G-function (2.6.3), we relocate the line of integration from $\mathrm{Re}(s) = \lambda$ to $0 < \mathrm{Re}(s) = \beta < Ak + B$ to ensure that the line of integration separates the poles of $\Gamma(s)$ and $\Gamma(Ak + B - s)$. Finally, we consider a contour $\mathcal{D}_1$ as in Figure 5.2 determined by the line segments $[\lambda - iT, \lambda + iT]$, $[\lambda + iT, \beta + iT]$, $[\beta + iT, \beta - iT]$, and $[\beta - iT, \lambda - iT]$. Again, utilizing

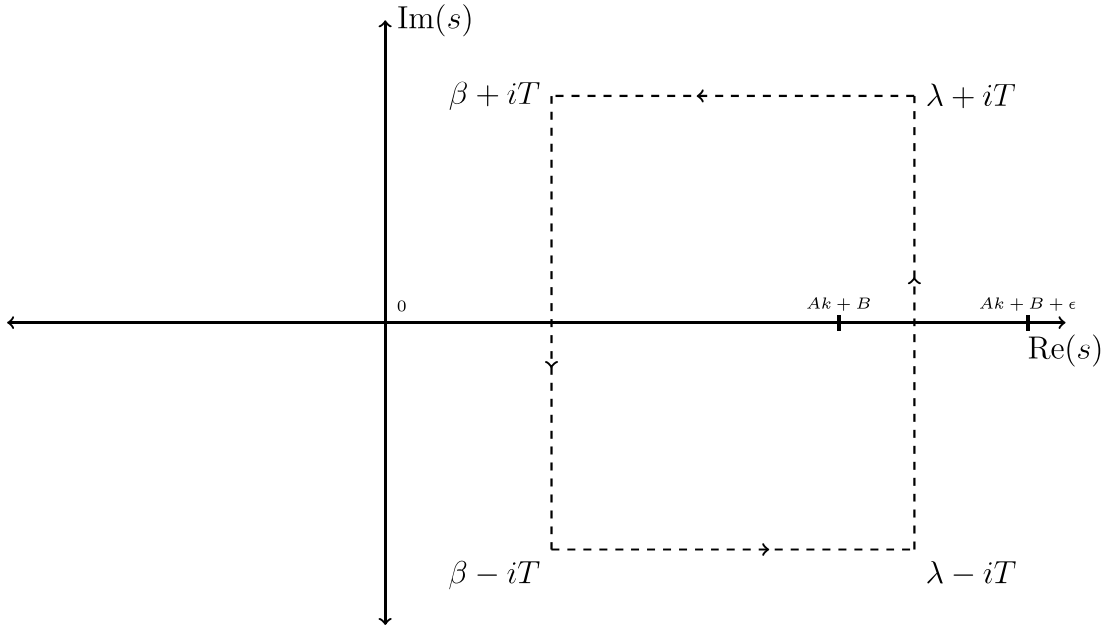


Figure 5.2

the Cauchy's residue theorem, we have

$$\frac{1}{2\pi i}\left(\int_{\lambda-iT}^{\lambda+iT}+\int_{\lambda+iT}^{\beta+iT}+\int_{\beta+iT}^{\beta-iT}+\int_{\beta-iT}^{\lambda-iT}\right)\frac{\Gamma(s)\Gamma(Ak+B-s)}{\Gamma(A(\delta-k)+B+s)}X_n^{-s}\mathrm{d}s=R_{Ak+B}, \tag{5.3.13}$$

where R_{Ak+B} denotes the residual term due to the simple pole of $\Gamma(Ak+B-s)$ at $s=Ak+B$. We can easily verify that

$$R_{Ak+B}=-\frac{\Gamma(Ak+B)}{\Gamma(A\delta+2B)}X_n^{-Ak-B}. \tag{5.3.14}$$

Letting $T\to\infty$, one can show that horizontal integrals vanish. Thus substituting (5.3.14) in (5.3.13) and using (5.3.12), it leads us to obtain that

$$V_{k,A}(X_n)=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{\Gamma(s)\Gamma(Ak+B-s)}{\Gamma(A(\delta-k)+B+s)}X_n^{-s}\mathrm{d}s-\frac{\Gamma(Ak+B)}{\Gamma(A\delta+2B)}X_n^{-Ak-B}. \tag{5.3.15}$$

As we know that the line of integration $\mathrm{Re}(s)=\beta$ separates the poles of $\Gamma(s)$ from the poles of $\Gamma(Ak+B-s)$, so we are all set to write the vertical integral, in (5.3.15), in terms of the Meijer G-function (2.6.3) with $n=p=m=1, q=2$ and $a_1=1, b_1=Ak+B$ and $b_2=1-A\delta+Ak-B$. Thus, we have

$$\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{\Gamma(s)\Gamma(Ak+B-s)}{\Gamma(A(\delta-k)+B+s)}X_n^{-s}\mathrm{d}s=G_{1,2}^{1,1}\left(\begin{matrix}1\\ Ak+B, 1-A\delta+Ak-B\end{matrix}\Big|\frac{1}{X_n}\right). \tag{5.3.16}$$

Here one can easily check that $p+q<2(m+n)$ and $|\arg(\frac{1}{X_n})|<\left(m+n-\frac{p+q}{2}\right)\pi$.

Thus all the necessary conditions for the convergence of the above Meijer G-function are satisfied. Now invoking Slater's theorem (2.6.4), one can see that

$$G_{1,2}^{1,1}\left(\begin{matrix}1\\ Ak+B, 1-A\delta+Ak-B\end{matrix}\Big|\frac{1}{X_n}\right)=\frac{1}{X_n^{Ak+B}}\frac{\Gamma(Ak+B)}{\Gamma(A\delta+2B)}{}_1F_1\left(\begin{matrix}Ak+B\\ A\delta+2B\end{matrix}\Big|-\frac{1}{X_n}\right). \tag{5.3.17}$$

Finally, substituting (5.3.17) in (5.3.16) and together with (5.3.15), (5.3.11), (5.3.8) and then combining all these terms in (5.3.7), we complete the proof of Theorem 5.2.1. □

Proof of Corollary 5.2.2. Letting $k=\delta+\frac{B}{A}$ and $x=\beta>0$ with $\alpha\beta=\frac{1}{c^{2/A}}$ in Theorem 5.2.1 and then multiplying by $\sqrt{\nu}\beta^{\frac{A\delta+2B}{B}}$ throughout and using the fact that $|\nu|=1$, one can complete the proof of this result. □

Proof of Corollary 5.2.3. The identity (5.2.2) represents a particular instance of the identity (5.2.1) with $a(n)=b(n)=r_2(n)$, where $r_2(n)$ is defined in Example 4 for $j=2$. Note that $\zeta_{id}(2,s)$ satisfies the following functional equation

$$\pi^{-s}\Gamma(s)\zeta_{id}(2,s)=\pi^{-(1-s)}\Gamma\left(1-s\right)\zeta_{id}\left(2,1-s\right).$$

This corresponds to (2.3.1) for $a(n)=b(n)=r_2(n), c=\frac{1}{\pi}, A=1, B=0, \delta=1$ and $\nu=1$. Now substituting these values in Theorem 5.2.1 and noting that the term R_t is zero as we do not encounter any trivial zeros of $\zeta_{id}(2,k-s)$ with $0\leq \mathrm{Re}(k-s)\leq k$, and also $\sum_{n=1}^{\infty}\frac{r_2^{-1}(n)}{n}=0$ since we know the relation (2.3.6), we complete the proof. □

Proof of Corollary 5.2.4. This scenario represents a special case of (5.2.1) for the Dedekind zeta function for the imaginary quadratic field $Q(\sqrt{-D})$, which is defined in Example 5. From the functional equation (2.3.8), it is clear that $c=\frac{\sqrt{D}}{2\pi}, A=1, B=0, \delta=1$ and $\nu=1$. Now putting these values in Theorem 5.2.1 and observing that $R_t=0$ since we do not encounter a trivial zero of $\zeta_{Q(\sqrt{-D})}(k-s)$ with $0\leq \mathrm{Re}(k-s)\leq k$, and $\sum_{n=1}^{\infty}\frac{\mathsf{a}_n^{-1}}{n}=0$, one can complete the proof. □

Proof of Theorem 5.2.5. This identity is a special case of the identity (5.2.1) with $a(n)=b(n)=\sigma_r(n), c=\frac{1}{2\pi}, A=1, B=0, \delta=r+1$ and $\nu=(-1)^{\frac{r+1}{2}}$ follows from Example 1. In this case $\phi(s)=\zeta(s)\zeta(s-r)$, where $r\geq 1$ odd integer. Thus, the trivial zeros of $\phi(s)$ are at $\{-2m, r-2m\}$, with $m\in\mathbb{N}$. Hence, the trivial zeros of $\phi(k-s)$ with $0\leq \mathrm{Re}(k-s)\leq k$ are at $k-r+2m$ such that $1\leq m\leq \lfloor\frac{r}{2}\rfloor$. Therefore, from (5.3.3), one can see that the residual term R_t becomes

$$R_t=\sum_{m=1}^{\lfloor\frac{r}{2}\rfloor}\lim_{s\to k-r+2m}\frac{(s-(k-r+2m))\Gamma(s)x^{-s}}{\zeta(k-s)\zeta(k-s-r)}$$

$$= -\sum_{m=1}^{\lfloor \frac{r}{2} \rfloor} \frac{\Gamma(k-r+2m)x^{r-k-2m}}{\zeta(r-2m)\zeta^{'}(-2m)}$$
$$= \sum_{m=1}^{\lceil \frac{r}{2} \rceil} \frac{\Gamma(k-r+2m)}{\zeta(r-2m)\zeta(2m+1)} \frac{(-1)^m 2(2\pi)^{2m}}{(2m)!} x^{r-k-2m}. \tag{5.3.18}$$

In the final step we utilized the identity

$$\zeta^{'}(-2m) = \frac{(-1)^m (2m)! \zeta(2m+1)}{2(2\pi)^{2m}}.$$

From (5.3.3), one can see that the integrand function $\frac{\Gamma(s)x^{-s}}{\phi(k-s/A)}$, and in this case $A = 1$ and $\phi(k-s) = \zeta(k-s)\zeta(k-s-r)$, so the residual term $R_\rho(x)$ will depend on the non-trivial zeros of $\zeta(k-s)$ as well as $\zeta(k-s-r)$. Thus, if we assume that the non-trivial zeros of $\zeta(s)$ are simple, then one can show that

$$\sum_{\rho} R_\rho(x) = -\sum_{\rho} \frac{x^{\rho-k}}{\zeta^{'}(\rho)} \left[\frac{\Gamma(k-\rho)}{\zeta(\rho-r)} + \frac{x^r \Gamma(k-r-\rho)}{\zeta(r+\rho)} \right], \tag{5.3.19}$$

where the sum over ρ runs through the non-trivial zeros of $\zeta(s)$. Again note that $\phi(s) = \zeta(s)\zeta(s-r)$ has a simple pole at $1+r$, thus we will have $\sum_{n=1}^{\infty} \frac{\sigma_r^{-1}(n)}{n^{r+1}} = 0$. Substituting the above values (5.3.18)-(5.3.19) of R_t and $R_\rho(x)$ in Theorem 5.2.1, we complete the proof of Theorem 5.2.5. □

Proof of Corollary 5.2.6. Substituting $k = r+1$, $x = \alpha$, $\alpha\beta = 4\pi^2$ and multiplying by $\alpha^{\frac{r+1}{2}}$ on both sides of Theorem 5.2.5, we obtain the result. □

Proof of Theorem 5.2.7. This identity represents a particular instance of (5.2.1), for the normalized Fourier coefficients of the holomorphic Hecke eigenform $f(z)$. Here, we take $a(n) = b(n) = \lambda_f(n)$ and the corresponding L-function $L(f,s)$ is defined in Example 3. In this case, we encounter a pole of the integrand $\frac{\Gamma(s)x^{-s}}{L(f,k-s)}$ due to the trivial zero of $L(f,k-s)$ at $s = k + \frac{\omega-1}{2}$ which satisfies the condition $0 \leq \mathrm{Re}(k-s) \leq k + \frac{w-1}{2}$. Thus, we get our residual term R_t to be

$$R_t = \lim_{s \to k+\frac{\omega-1}{2}} \frac{\left(s - \left(k + \frac{\omega-1}{2}\right)\right) \Gamma(s) x^{-s}}{L(f, k-s)}$$
$$= \lim_{s \to k+\frac{\omega-1}{2}} \frac{\left(s - \left(k + \frac{\omega-1}{2}\right)\right) \Gamma\left(k - s + \frac{\omega-1}{2}\right) \Gamma(s) (2\pi)^{1+2s-2k} x^{-s}}{\Gamma\left(1 - k + s + \frac{\omega-1}{2}\right) L(f, 1-k+s)}$$
$$= -\frac{(2\pi)^\omega}{x^{k+\frac{\omega-1}{2}}} \frac{\Gamma\left(k + \frac{\omega-1}{2}\right)}{\Gamma(\omega)} \frac{1}{L\left(f, \frac{\omega+1}{2}\right)}.$$

Here, in the penultimate step, we have used functional equation (2.3.5) for $L(f,s)$. Substituting this value of R_t along with $c = \frac{1}{2\pi}, A = 1, B = \frac{\omega-1}{2}$, $\delta = 1$ and $\nu = 1$ in Theorem 5.2.1 and using $\frac{1}{L\left(f, \frac{\omega+1}{2}\right)} = \sum_{n=1}^{\infty} \frac{\lambda_f^{-1}(n)}{n^{\frac{\omega+1}{2}}}$, we obtain (5.2.3). □

Proof of Corollary 5.2.8. Letting $k = \frac{\omega+1}{2}$, $x = \beta$, $\alpha\beta = 4\pi^2$ in Theorem 5.2.7 and multiplying throughout by $\beta^{\frac{\omega}{2}}$, we obtain the result. □

Proof of Corollary 5.2.9. This result is an immediate implication of Corollary 5.2.8 for the Ramanujan cusp form of weight 12. Thus, substituting $\lambda_f(n) = \tau_0(n)$ and $\omega = 12$, we get our modular relation (5.2.4). □

Proof of Theorem 5.2.10. Let $L(f, s)$ be a nice L-function defined as in (4.3.27). First, we begin by presupposing that the grand Riemann hypothesis (GRH) for $L(f, s)$ holds true. Then from (4.3.29), for any $\epsilon > 0$, we have

$$G(x) := \sum_{1\leq n\leq x} \mu_f(n) \ll_{\epsilon,f} x^{\frac{1}{2}+\epsilon}. \tag{5.3.20}$$

Now, we employ Euler's partial summation formula, i.e., Lemma 2.6.1 with $a(n) = \mu_f(n)$ and $f(t) = t^{-k}$ to see that

$$H(m, n) := \sum_{i=m}^{n} \frac{\mu_f(i)}{i^k} = G(n)f(n) - G(m-1)f(m-1) - \int_{m-1}^{n} G(t)f'(t)\mathrm{d}t. \tag{5.3.21}$$

Utilizing (5.3.20) in (5.3.21), one can obtain

$$H(m, n) = O_{\epsilon,k}\left(m^{\frac{1}{2}-k+\epsilon}\right). \tag{5.3.22}$$

The above bound is uniform in n. Our main objective is to determine the bound for the following infinite series, under the assumption of the GRH for $L(f, s)$,

$$P_{k,f,\ell}(x) := \sum_{n=1}^{\infty} \frac{\mu_f(n)}{n^k} \exp\left(-\frac{x}{n^\ell}\right).$$

For simplicity, we replace x by x^ℓ and separate the sum into finite and infinite part as

$$\begin{aligned} P_{k,f,\ell}(x^\ell) &= \sum_{n=1}^{m-1} \frac{\mu_f(n)}{n^k} \exp\left(-\frac{x^\ell}{n^\ell}\right) + \sum_{n=m}^{\infty} \frac{\mu_f(n)}{n^k} \exp\left(-\frac{x^\ell}{n^\ell}\right) \\ &:= Q_1(x^\ell) + Q_2(x^\ell). \end{aligned} \tag{5.3.23}$$

where $m = [x^{1-\epsilon}] + 1$. To get an estimate for $Q_1(x^\ell)$, we write

$$\begin{aligned} |Q_1(x^\ell)| &= \left|\sum_{n=1}^{m-1} \frac{\mu_f(n)}{n^k} \exp\left(-\frac{x^\ell}{n^\ell}\right)\right| \leq \sum_{n=1}^{m-1} |H(n, n)| \exp\left(-\frac{x^\ell}{m^\ell}\right) \\ &\ll \sum_{n=1}^{m-1} n^{\frac{1}{2}-k+\epsilon} \exp\left(-\frac{x^\ell}{m^\ell}\right) \ll_{k,\epsilon} m^{\frac{3}{2}-k+\epsilon} \exp\left(-\frac{x^\ell}{m^\ell}\right). \end{aligned}$$

This gives

$$Q_1(x^\ell) = O\left(\exp(-x^{\ell\epsilon}) x^{(1-\epsilon)(\frac{3}{2}-k+\epsilon)}\right), \tag{5.3.24}$$

as $m - 1 = [x^{1-\epsilon}]$. We shall now try to estimate $Q_2(x^\ell)$. For sufficiently large integer

$N > m$, we write

$$\sum_{n=m}^{N} \frac{\mu_f(n)}{n^k} \exp\left(-\frac{x^\ell}{n^\ell}\right) = \sum_{n=m}^{N-1} H(m,n)\left(\exp\left(-\frac{x^\ell}{n^\ell}\right) - \exp\left(-\frac{x^\ell}{(n+1)^\ell}\right)\right) + H(m,n)\exp\left(-\frac{x^\ell}{N^\ell}\right).$$

Allowing $N \to \infty$ and invoking the bound (5.3.22) for $H(m,n)$, it can be inferred that

$$Q_2(x^\ell) = Q_3(x^\ell) + O_{\epsilon,k}\left(m^{\frac{1}{2}-k+\epsilon}\right), \tag{5.3.25}$$

where

$$Q_3(x^\ell) = \sum_{n=m}^{N-1} H(m,n)\left(\exp\left(-\frac{x^\ell}{n^\ell}\right) - \exp\left(-\frac{x^\ell}{(n+1)^\ell}\right)\right).$$

Now to simplify further, we use Cauchy's mean value theorem with the function $T(z) = \exp\left(-\frac{x^\ell}{z^\ell}\right)$. One can find $z_n \in (n, n+1)$ such that

$$T(n) - T(n+1) = -T^{'}(z_n) = -\frac{\ell x^\ell}{z_n^{\ell+1}} \exp\left(-\frac{x^\ell}{z_n^\ell}\right).$$

Employing the above fact in conjunction with (5.3.22), we obtain

$$\begin{aligned} |Q_3(x^\ell)| &\ll_{\epsilon,\ell} m^{\frac{1}{2}-k+\epsilon} \sum_{n=m}^{\infty} \frac{x^\ell}{z_n^{\ell+1}} \exp\left(-\frac{x^\ell}{z_n^\ell}\right) \\ &\ll_{\epsilon,\ell} m^{\frac{1}{2}-k+\epsilon} \sum_{n=m}^{\infty} \frac{x^\ell}{n^{\ell+1}} \\ &\ll_{\epsilon,\ell} x^{\frac{1}{2}-k+\epsilon} \frac{x^\ell}{m^\ell} \ll_{\epsilon,\ell} x^{\frac{1}{2}-k+\epsilon'} \end{aligned} \tag{5.3.26}$$

as $m \sim x^{1-\epsilon}$. Now utilizing (5.3.26) in (5.3.25), we see that

$$|Q_2(x^\ell)| = O_{\epsilon,k,\ell}\left(m^{\frac{1}{2}-k+\epsilon}\right). \tag{5.3.27}$$

Finally, substituting the bounds (5.3.24) and (5.3.27) in (5.3.23), one can see that due to exponential decay, the bound for $Q_1(x^\ell)$ goes to zero much faster than the bound for $Q_2(x^\ell)$ as $x \to \infty$, and therefore

$$P_{k,f,\ell}(x^\ell) = O_{\epsilon,k,\ell}\left(x^{\frac{1}{2}-k+\epsilon}\right).$$

Replace x by $x^{1/\ell}$, we get the desired bound (4.3.30).

Now, turning to the converse aspect, that is, assuming the validity of the bound (4.3.30), we aim to show that the GRH for $L(f,s)$ follows. We split the proof in two cases, namely for $k > 1$ and $k = 1$. For $k > 1$, utilizing Lemma 5.1.3, we deduced that

$$L(f, \ell s + k) \int_0^\infty x^{-s-1} P_{k,f,\ell}(x) \mathrm{d}x = \Gamma(-s), \tag{5.3.28}$$

valid for $\frac{1-k}{\ell} < \mathrm{Re}(s) < 0$. We aim to extend the validity of (5.3.28) towards the left half plane. To accomplish this, we choose a sufficiently large positive real number R and rewrite the expression (5.3.28) as

$$L(f, \ell s + k) \left(\int_0^R + \int_R^\infty \right) x^{-s-1} P_{k,f,\ell}(x) \mathrm{d}x = \Gamma(-s).$$

Utilizing the bound (4.3.30) for $P_{k,f,\ell}(x)$, it is evident that unbounded part possesses analyticity for $\mathrm{Re}(s) > \frac{1}{2\ell} - \frac{k}{\ell}$ and the bounded integral is analytic for $\mathrm{Re}(s) < 0$. Consequently, identity (5.3.28) is analytic in the strip $\frac{1}{2\ell} - \frac{k}{\ell} < \mathrm{Re}(s) \leq \frac{1-k}{\ell}$ except for a possible pole at $s = \frac{1-k}{\ell}$. Considering the fact that $\Gamma(s)$ never attains zero, it implies that $L(f, \ell s + k)$ is non-vanishing in the strip $\frac{1}{2\ell} - \frac{k}{\ell} < \mathrm{Re}(s) \leq \frac{1-k}{\ell}$, which in turn implies that $L(f, s)$ does not possess any zero in the strip $\frac{1}{2} < \mathrm{Re}(s) \leq 1$. From the symmetry of the functional equation, one can conclude that $L(f, s)$ is non-vanishing in $0 \leq \mathrm{Re}(s) < \frac{1}{2}$ except for a possible trivial zero at $s = 0$. This proves that all the non-trivial zeros of $L(f, s)$ lies on the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

For $k = 1$, we utilize Lemma 5.1.4, which says that the identity

$$L(f, \ell s + 1) \int_0^\infty x^{-s-1} P_{1,f,\ell}(x) \mathrm{d}x = \Gamma(-s),$$

is valid for $0 < \mathrm{Re}(s) < 1$. Working along the same path as above, we extend its region of analyticity to $-\frac{1}{2\ell} < \mathrm{Re}(s) \leq 0$ expect for a possible pole at $s = 0$. Utilizing the fact that $\Gamma(s)$ is never vanishing, in a similar way we conclude that all the non-trivial zeroes of $L(f, s)$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. Thus, the GRH follows. This completes the proof of Conjecture 4.3.1. □

Proof of Theorem 5.2.13. For the direct part, we assume RH for $\zeta(s)$ to be true and we wish to obtain the bound (5.2.5) for $P_{k,\sigma,\ell}(x)$. The proof goes along the similar line of reasoning as in the proof of Theorem 5.2.10 except for that fact that we start with $\sum_{n \leq x} \sigma_r^{-1}(n) = O(x^{\frac{1}{2}+r+\epsilon})$, as established in Lemma 5.1.2 under the assumption of RH. Thus, we omit the proof here for the direct part.

For the converse part, assuming the validity of the bound (5.2.5) for $P_{k,\sigma,\ell}(x)$, that is,

$$P_{k,\sigma,\ell}(x) := \sum_{n=1}^\infty \frac{\sigma_r^{-1}(n)}{n^k} \exp\left(-\frac{x}{n^\ell}\right) = O_{\epsilon,k,r,\ell}\left(x^{-\frac{k}{\ell}+\frac{1+2r}{2\ell}+\epsilon}\right),$$

for any $\epsilon > 0$, we will try to show that all the non-trivial zeros of $\zeta(s)$ are located on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. We consider two cases, namely for $k > r+1$ and $k = r+1$. For $k > r + 1$, we utilize Lemma 5.1.5 to deduce that

$$\zeta(k + \ell s)\zeta(k + \ell s - r) \int_0^\infty x^{-s-1} P_{k,\sigma,\ell}(x) \mathrm{d}x = \Gamma(-s), \tag{5.3.29}$$

holds within the region $\frac{1-k+r}{\ell} < \mathrm{Re}(s) < 0$. Continuing along the path drawn in the proof of Theorem 5.2.10, one can extend the region of analyticity of (5.3.29) in $\frac{1}{2\ell} - \frac{k}{\ell} + \frac{r}{\ell} < \mathrm{Re}(s) \leq \frac{1-k+r}{\ell}$ except at $s = \frac{1-k+r}{\ell}$. Considering the fact that $\Gamma(s)$

never attains zero, it implies that $\zeta(k+\ell s)$ and $\zeta(k+\ell s-r)$ is non-vanishing in the strip $\frac{1}{2\ell}-\frac{k}{\ell}+\frac{r}{\ell}<\mathrm{Re}(s)\leq\frac{1-k+r}{\ell}$. Since ℓ is a positive integer, so $\mathrm{Re}(k+\ell s)>1$ and $\zeta(k+\ell s)$ is non-vanishing in the associated region and as a result, we are unable to derive any useful information. However, the factor $\zeta(k+\ell s-r)$ should also be non-vanishing in the region $\frac{1}{2}<\mathrm{Re}(k+\ell s-r)\leq 1$, which in turn implies that $\zeta(s)$ does not possess any zeros within the strip $\frac{1}{2}<\mathrm{Re}(s)\leq 1$, and thus the functional equation for $\zeta(s)$ implies that it is non-vanishing in the strip $0\leq\mathrm{Re}(s)<\frac{1}{2}$. This proves that all the non-trivial zeros of $\zeta(s)$ must lie on the line $\mathrm{Re}(s)=\frac{1}{2}$.

The proof for $k=1$ goes along the same line as above except that we will use Lemma 5.1.6 and we extend the region of analyticity of

$$\zeta(r+1+\ell s)\zeta(1+\ell s)\int_0^\infty x^{-s-1}P_{r+1,\sigma,\ell}(x)\mathrm{d}x=\Gamma(-s),$$

to $-\frac{1}{2\ell}<\mathrm{Re}(s)\leq 0$ except at $s=0$. □

5.4 Numerical evidences for Corollary 5.2.3 and Theorem 5.2.5

We used Mathematica software to make the following tables. To obtain this data, we considered the left-hand side sum over n with 200 terms and right-hand side sum over n with 200000 terms.

Table 5.1: Verification of Corollary 5.2.3

k	x	Left-hand side	Right-hand side
2	$e+1$	-0.0577422	-0.0577021
3	$\pi+1$	-0.0554663	-0.0554081
6	π^2	-0.0000785321	-0.0000787028
7	e^3	-8.02103×10^{-7}	-8.02821×10^{-7}

Table 5.2: Verification of Theorem 5.2.5

k	r	x	Left-hand side	Right-hand side
8	5	e	0.0147028	0.0147079
4	1	π	-0.0103086	-0.0103073
10	1	$\pi+1$	0.0155115	0.0155119
11	5	$e+1$	0.0213461	0.0213510

Chapter 6

RADEMACHER-TYPE EXACT FORMULA AND HIGHER ORDER TURÁN INEQUALITIES FOR CUBIC OVERPARTITIONS

All the results presented in this chapter are drawn from our joint work [2], carried out in collaboration with Agarwal and Maji. The main objective of this chapter is to establish a Rademacher-type exact formula for cubic overpartitions and, as an application, to derive an explicit error term that leads to their log-concavity. Furthermore, applying a result of Griffin, Ono, Rolen, and Zagier, we establish higher-order Turán inequalities for cubic overpartitions. In addition, we obtain log-subadditivity and generalized log-concavity properties for cubic overpartitions inspired by the work of Bessenrodt-Ono and DeSalvo-Pak on the ordinary partition function.

6.1 Preliminaries

We start this section by stating a result which gives an upper and lower bound for the modified Bessel function of first kind $I_2(s)$.

Lemma 6.1.1. *Define the expression $E_{I_2}(s)$ as follows:*

$$E_{I_2}(s) := 1 - \frac{15}{8s} + \frac{105}{128s^2} + \frac{315}{1024s^3} + \frac{10395}{32768s^4} + \frac{135135}{262144s^5}, \tag{6.1.1}$$

then for $s \geq 25$, the bounds for $I_2(s)$ are given by

$$\frac{e^s}{\sqrt{2\pi s}}\left(E_{I_2}(s) - \frac{31}{s^6}\right) \leq I_2(s) \leq \frac{e^s}{\sqrt{2\pi s}}\left(E_{I_2}(s) + \frac{31}{s^6}\right). \tag{6.1.2}$$

Proof. For a proof of this result we refer [33, Lemma 3.1]. □

We now present a key result that provides an upper bound for the tail of a series involving modified Bessel functions of the first kind. This bound plays a crucial role to obtain bounds for the error term of cubic overpartitions.

Lemma 6.1.2. *Let N be a positive integer. Then, we have*

$$\sum_{j=N+1}^{\infty} I_2\left(\frac{s}{j}\right) \leq \frac{2N^2}{s} I_1\left(\frac{s}{N}\right). \tag{6.1.3}$$

Proof. For details of the proof, one can see [33, Equation (2.2)]. □

The next result presents a lower bound for $I_2(s)$.

Lemma 6.1.3. *For all $s \geq 30$, we have*

$$I_2(s) \geq \frac{e^s}{\sqrt{2\pi s}} \left(1 + \frac{2}{s}\right)^{-1}. \tag{6.1.4}$$

Proof. We refer [33, Equation (3.14)] to get insights of the proof. □

6.2 Main results

We begin this section with an exact formula for the cubic overpartition function $\overline{a}(n)$, which mirrors the classical Rademacher series for the partition function.

Theorem 6.2.1. *Let $\overline{a}(n)$ be the number of cubic overpartitions of a non-negative integer n. Then, $\overline{a}(n)$ admits the following exact formula:*

$$\overline{a}(n) = \frac{3\pi}{16n\sqrt{2}} \sum_{\substack{k=1 \\ k \ odd}}^{\infty} \frac{A_k^{(1)}(n)}{k} I_2\left(\frac{\pi}{k}\sqrt{\frac{3n}{2}}\right) + \frac{\pi}{4n\sqrt{2}} \sum_{\substack{k=1 \\ k\equiv 2 \pmod 4}}^{\infty} \frac{A_k^{(2)}(n)}{k} I_2\left(\frac{\pi}{k}\sqrt{2n}\right), \tag{6.2.1}$$

where

$$A_k^{(1)}(n) = \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} e^{\pi i(2s(h,k)+s(2h,k)-s(4h,k))-2n\pi i\frac{h}{k}},$$

$$A_k^{(2)}(n) = \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} e^{\pi i\left(2s(h,k)+s\left(h,\frac{k}{2}\right)-s\left(2h,\frac{k}{2}\right)\right)-2n\pi i\frac{h}{k}},$$

and I_ν is the modified Bessel function of first kind and $s(h,k)$ is the Dedekind sum defined in (1.2.3).

Remark 12. *The formula* (6.2.1) *is structurally analogous to Rademacher's exact formula for the ordinary partition function $p(n)$, with the primary difference being the presence of two distinct series corresponding to k odd, and $k \equiv 2 \pmod 4$.*

As a direct consequence of Theorem 6.2.1, we deduce an asymptotic formula of $\overline{a}(n)$ for large n by analyzing the dominant contribution from the leading terms of the two series. This leads to the following asymptotic formula.

Corollary 6.2.2. *The cubic overpartition function $\overline{a}(n)$ satisfies the following estimate:*

$$\overline{a}(n) = \frac{3\pi}{16n\sqrt{2}} I_2\left(\pi\sqrt{\frac{3n}{2}}\right) + \mathcal{O}\left(\left(\frac{2\cdot 5^{\frac{5}{2}}}{\pi n^{\frac{7}{4}}} + \frac{1}{n^{\frac{5}{4}}}\right) e^{\pi\sqrt{\frac{n}{2}}}\right). \tag{6.2.2}$$

Further, we have

$$\overline{a}(n) \sim \frac{3^{\frac{3}{4}}}{2^{\frac{19}{4}} n^{\frac{5}{4}}} e^{\pi\sqrt{\frac{3n}{2}}}, \quad as \quad n \to \infty. \tag{6.2.3}$$

Before presenting the next result, we introduce a few variables which will be used frequently. We define

$$v := \pi\sqrt{\frac{3n}{2}}, \quad v^+ := \pi\sqrt{\frac{3(n+1)}{2}}, \quad v^- := \pi\sqrt{\frac{3(n-1)}{2}}. \tag{6.2.4}$$

In our pursuit of establishing Turán inequality for cubic overpartitions, the following result plays an important role.

Theorem 6.2.3. *For all $n \geq 2363$, we have*

$$\Upsilon_1(n) \leq \frac{\overline{a}(n+1)\overline{a}(n-1)}{(\overline{a}(n))^2} \leq \Upsilon_2(n),$$

where

$$\Upsilon_1(n) := 1 - \frac{9\pi^4}{16v^3} + \frac{45\pi^4}{16v^4} - \frac{309}{v^5} - \frac{535}{v^6} - \frac{405\pi^8}{2048v^6} - \frac{729\pi^{12}}{v^6},$$

$$\Upsilon_2(n) := 1 - \frac{9\pi^4}{16v^3} + \frac{45\pi^4}{16v^4} - \frac{308}{v^5} - \frac{286}{v^6} + \frac{81\pi^8}{256v^6} + \frac{729\pi^{12}}{16v^6}.$$

The next result gives the second order Turán inequality for $\overline{a}(n)$.

Theorem 6.2.4. *For $n \geq 10$, we have $\overline{a}^2(n) > \overline{a}(n+1)\overline{a}(n-1)$.*

More generally, we prove the following result, which in turn gives higher Turán inequalities for cubic overpartitions.

Theorem 6.2.5. *For any positive integer $d \geq 3$, $J_{\overline{a}}^{d,n}(X)$ is hyperbolic for all but finitely many values of n.*

Bessenrodt and Ono [16], in 2016, proved the following log-subadditivity result for the ordinary partition function i.e., for any integers $n, m > 1$ with $n + m > 8$, the inequality

$$p(n)p(m) \geq p(n+m),$$

is true and the equality holds only when $\{n, m\} = \{2, 7\}$. Inspired from this result, we state the following log-subadditivity result for the cubic overpartition function.

Theorem 6.2.6. *For any $n, m \geq 1$ and $\{n, m\} \neq \{1, 1\}, \{1, 3\}$, we have*

$$\overline{a}(n)\overline{a}(m) \geq \overline{a}(n+m).$$

Moreover, the equality holds only when $\{n, m\} = \{1, 2\}$.

In 2010, Chen [31] conjectured that the partition function $p(n)$ satisfies the generalized log-concavity property, namely, for integers $n > m > 1$,

$$p(n)^2 > p(n-m)p(n+m).$$

This conjecture was later proved by DeSalvo and Pak [41]. Motivated by this result, we prove an analogous result for the cubic overpartition function $\overline{a}(n)$ as follows:

Theorem 6.2.7. *For all $n > m > 1$, we have*

$$\overline{a}(n)^2 > \overline{a}(n-m)\overline{a}(n+m).$$

6.3 Proof of main results

6.3.1 Rademacher-type exact formula for cubic overpartitions

In this subsection, we present a detailed proof of Rademacher-type exact formula for cubic overpartitions by applying the circle method, leveraging the transformation formula derived earlier, and carefully analyzing contributions from different Farey arcs associated with Ford circles.

Proof of Theorem 6.2.1. Let $\bar{A}(q)$ denotes the generating function for $\overline{a}(n)$. Then from (1.2.4), we have

$$\overline{A}(q) := \sum_{n=0}^{\infty} \overline{a}(n)q^n = \frac{(-q;q)_\infty(-q^2;q^2)_\infty}{(q;q)_\infty(q^2;q^2)_\infty} = \frac{(q^4;q^4)_\infty}{(q;q)_\infty^2(q^2;q^2)_\infty} = \frac{(f(q))^2 f(q^2)}{f(q^4)},$$

where $f(q) = \frac{1}{(q;q)_\infty}$. Now applying Cauchy's integral formula, one has

$$\overline{a}(n) = \frac{1}{2\pi i}\int_C \frac{\overline{A}(q)}{q^{n+1}}\mathrm{d}q,$$

where $C : |q| = r < 1$. Substitute $q = e^{2\pi i\tau}$, with $\tau \in \mathbb{H}$, to get

$$\overline{a}(n) = \int_{\tau_0}^{\tau_0+1} e^{-2n\pi i\tau}\overline{A}(e^{2\pi i\tau})\mathrm{d}\tau, \tag{6.3.1}$$

where $\tau_0 \in \mathbb{H}$ and as the integrand is regular, so we can choose any path joining τ_0 to $\tau_0 + 1$. We consider our path based on Farey dissections with $\tau_0 = i$. Given any natural number N, we construct Ford circles belonging to Farey sequence of order N. Then for the Ford circle $C(h, k)$, defined in (2.4.1), we choose arc $\gamma_{h,k}$ in such a way

that it connects the tangency points α_1 and α_2, see Section 2.4. Here we denote α_1 and α_2 as $\frac{h}{k}+C'_{h,k}$ and $\frac{h}{k}+C''_{h,k}$, where

$$C'_{h,k} := -\frac{k_1}{k(k^2+k_1^2)} + i\frac{1}{k^2+k_1^2}, \quad \text{and} \quad C''_{h,k} := \frac{k_2}{k(k^2+k_2^2)} + i\frac{1}{k^2+k_2^2}.$$

Note that the arc $\gamma_{h,k}$ is chosen in a way so that it does not touches the real axis and the path of the integration is the collection of all such arcs. Thus, considering the above mentioned arcs $\gamma_{h,k}$ corresponding to the fractions $\frac{h}{k}$ in the Farey sequence of order N, the integral (6.3.1) takes the form

$$\overline{a}(n) = \sum_{\substack{0\le h<k\le N\\(h,k)=1}} \int_{\gamma_{h,k}} e^{-2n\pi i\tau}\ \overline{A}(e^{2\pi i\tau})\mathrm{d}\tau,$$

where $\gamma_{h,k}$ is an arc of the Ford circle $\left|\tau-\left(\frac{h}{k}+\frac{i}{2k^2}\right)\right| = \frac{1}{2k^2}$ not touching the x-axis. Substituting $\tau = \frac{h}{k}+\zeta$ in the above integral gives

$$\overline{a}(n) = \sum_{\substack{0\le h<k\le N\\(h,k)=1}} e^{-2n\pi i\frac{h}{k}} \int_{C'_{h,k}}^{C''_{h,k}} e^{-2n\pi i\zeta}\ \overline{A}\left(e^{2\pi i\left(\frac{h}{k}+\zeta\right)}\right)\mathrm{d}\zeta.$$

Replace ζ by $\frac{iz}{k^2}$ to arrive at

$$\overline{a}(n) = \sum_{\substack{0\le h<k\le N\\(h,k)=1}} \frac{i}{k^2} e^{-2n\pi i\frac{h}{k}} \int_{z'_{h,k}}^{z''_{h,k}} e^{\frac{2n\pi z}{k^2}}\ \overline{A}\left(e^{2\pi i\left(\frac{h}{k}+\frac{iz}{k^2}\right)}\right)\mathrm{d}z, \tag{6.3.2}$$

where

$$z'_{h,k} = \frac{k^2}{k^2+k_1^2} + \frac{ikk_1}{k^2+k_1^2}, \quad z''_{h,k} = \frac{k^2}{k^2+k_2^2} - \frac{ikk_2}{k^2+k_2^2},$$

lie on circle $|z-\frac{1}{2}| = \frac{1}{2}$. Now our goal is to write a transformation formula for the below function,

$$\overline{A}\left(e^{2\pi i\left(\frac{h}{k}+\frac{iz}{k^2}\right)}\right) = \frac{\left\{f\left(e^{2\pi i\left(\frac{h}{k}+\frac{iz}{k^2}\right)}\right)\right\}^2 f\left(e^{4\pi i\left(\frac{h}{k}+\frac{iz}{k^2}\right)}\right)}{f\left(e^{8\pi i\left(\frac{h}{k}+\frac{iz}{k^2}\right)}\right)}. \tag{6.3.3}$$

Hardy and Ramanujan [71, Lemma 4.31] obtained the following transformation formula for $f(q)$, with $hh_1 \equiv -1 \pmod{k}$,

$$f\left(e^{2\pi i\left(\frac{h}{k}+\frac{iz}{k^2}\right)}\right) = \sqrt{\frac{z}{k}} e^{i\pi s(h,k)} e^{\frac{\pi}{12k}\left(\frac{k}{z}-\frac{z}{k}\right)} f\left(e^{2\pi i\left(\frac{h_1}{k}+\frac{i}{z}\right)}\right). \tag{6.3.4}$$

Now employing (6.3.4), and the transformation formulas for $f(q^2)$ and $f(q^4)$ obtained in (2.5.8) and (2.5.9) respectively, in (6.3.3), we arrive at the following transformation

formula for $\overline{A}(q)$:

$$\overline{A}\left(e^{2\pi i\left(\frac{h}{k}+\frac{iz}{k^2}\right)}\right)=\begin{cases}\frac{z}{k\sqrt{2}}e^{i\pi\{2s(h,k)+s(2h,k)-s(4h,k)\}}e^{\frac{3\pi}{16z}}\dfrac{\left\{f\left(e^{2\pi i\left(\frac{h_1}{k}+\frac{i}{z}\right)}\right)\right\}^2 f\left(e^{2\pi i\left(\frac{h_2}{k}+\frac{i}{2z}\right)}\right)}{f\left(e^{2\pi i\left(\frac{h_4}{k}+\frac{i}{4z}\right)}\right)}, & \text{if } k \text{ odd},\\ \frac{z}{k\sqrt{2}}e^{i\pi\left\{2s(h,k)+s(h,\frac{k}{2})-s(2h,\frac{k}{2})\right\}}e^{\frac{\pi}{4z}}\dfrac{\left\{f\left(e^{2\pi i\left(\frac{h_1}{k}+\frac{i}{z}\right)}\right)\right\}^2 f\left(e^{4\pi i\left(\frac{h_3}{k}+\frac{i}{z}\right)}\right)}{f\left(e^{4\pi i\left(\frac{h_5}{k}+\frac{i}{2z}\right)}\right)}, & \text{if } k\equiv 2(\text{mod } 4),\\ \frac{z}{k}e^{i\pi\left\{2s(h,k)+s(h,\frac{k}{2})-s(h,\frac{k}{4})\right\}}\dfrac{\left\{f\left(e^{2\pi i\left(\frac{h_1}{k}+\frac{i}{z}\right)}\right)\right\}^2 f\left(e^{4\pi i\left(\frac{h_3}{k}+\frac{i}{z}\right)}\right)}{f\left(e^{8\pi i\left(\frac{h_6}{k}+\frac{i}{z}\right)}\right)}, & \text{if } k\equiv 0(\text{mod } 4).\end{cases} \tag{6.3.5}$$

We now apply the transformation formula (6.3.5) in (6.3.2). The resulting sum is then partitioned into three components written as follows:

$$\overline{a}(n)=S_0+S_1+S_2. \tag{6.3.6}$$

In this decomposition, S_0 comprises the contribution coming from $k\equiv 0(\text{mod } 4)$, and S_1 accounts for the terms with k odd, S_2 includes those with $k\equiv 2(\text{mod } 4)$. We start by analyzing S_1. Implementing the transformation formula for k odd in (6.3.2), we get

$$S_1=\frac{i}{\sqrt{2}}\sum_{\substack{k=1\\ k \text{ odd}}}^{N}\frac{A_k^{(1)}(n)}{k^3}\int_{z'_{h,k}}^{z''_{h,k}} ze^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}}\frac{\left\{f\left(e^{2\pi i\left(\frac{h_1}{k}+\frac{i}{z}\right)}\right)\right\}^2 f\left(e^{2\pi i\left(\frac{h_2}{k}+\frac{i}{2z}\right)}\right)}{f\left(e^{2\pi i\left(\frac{h_4}{k}+\frac{i}{4z}\right)}\right)}\mathrm{d}z, \tag{6.3.7}$$

where

$$A_k^{(1)}(n)=\sum_{\substack{h=0\\(h,k)=1}}^{k-1} e^{\pi i(2s(h,k)+s(2h,k)-s(4h,k))-2n\pi i\frac{h}{k}}.$$

Our goal now is to simplify the integral present in (6.3.7) and denote it as J_1. For this purpose, we first estimate the bound of few terms present in the integral. For $\mathrm{Re}(z)\to 0^+$, $\mathrm{Re}(1/z)$ goes to infinity. Thus, one can check that

$$\begin{aligned}\left\{f\left(e^{2\pi i\left(\frac{h_1}{k}+\frac{i}{z}\right)}\right)\right\}^2&=1+\mathcal{O}\left(e^{-2\pi\mathrm{Re}\left(\frac{1}{z}\right)}\right),\\ f\left(e^{2\pi i\left(\frac{h_2}{k}+\frac{i}{2z}\right)}\right)&=1+\mathcal{O}\left(e^{-\pi\mathrm{Re}\left(\frac{1}{z}\right)}\right),\\ f\left(e^{2\pi i\left(\frac{h_4}{k}+\frac{i}{4z}\right)}\right)&=1+\mathcal{O}\left(e^{-\frac{\pi}{2}\mathrm{Re}\left(\frac{1}{z}\right)}\right).\end{aligned}$$

Combine these bounds to see that

$$\frac{\left\{f\left(e^{2\pi i\left(\frac{h_1}{k}+\frac{i}{z}\right)}\right)\right\}^2 f\left(e^{2\pi i\left(\frac{h_2}{k}+\frac{i}{2z}\right)}\right)}{f\left(e^{2\pi i\left(\frac{h_4}{k}+\frac{i}{4z}\right)}\right)} = 1+\mathcal{O}\left(e^{-\frac{\pi}{2}\mathrm{Re}\left(\frac{1}{z}\right)}\right).$$

Utilizing the above bound, J_1 becomes

$$\begin{aligned} J_1 &= \int_{z'_{h,k}}^{z''_{h,k}} z e^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}}\left(1+\mathcal{O}\left(e^{-\frac{\pi}{2}\mathrm{Re}\left(\frac{1}{z}\right)}\right)\right)\mathrm{d}z \\ &= \int_{z'_{h,k}}^{z''_{h,k}} z e^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}}\mathrm{d}z+\mathcal{O}\left(\int_{z'_{h,k}}^{z''_{h,k}}|z|e^{\frac{2n\pi}{k^2}\mathrm{Re}(z)}e^{-\frac{5\pi}{16}\mathrm{Re}\left(\frac{1}{z}\right)}|\mathrm{d}z|\right). \end{aligned}$$

In the first integral above, we add and subtract the integral over the remaining portion of the circle $|z-\frac{1}{2}|=\frac{1}{2}$, namely, the arc from 0 to $z'_{h,k}$ and $z''_{h,k}$ to 0. Since the integrand of J_1 is regular, so to bound the second integral, we choose the straight line path connecting $z'_{h,k}$ and $z''_{h,k}$. Along this straight line, we have $|z|\leq\frac{\sqrt{2}k}{N}$ and $0<\mathrm{Re}(z)\leq\frac{2k^2}{N^2}$. Furthermore, the length of the straight line path is bounded above by $|z'_{h,k}|+|z''_{h,k}|\leq\frac{2\sqrt{2}k}{N}$. This bound combined with the earlier estimates on the integrand allows us to effectively control the contribution of the integral J_1 in the subsequent analysis. Thus, we have

$$\begin{aligned} J_1 &= \int_{K^-} z e^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}}\mathrm{d}z-\int_0^{z'_{h,k}} z e^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}}\mathrm{d}z-\int_{z''_{h,k}}^0 z e^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}}\mathrm{d}z+\mathcal{O}\left(\frac{k}{N}e^{\frac{4n\pi}{N^2}}\int_{z'_{h,k}}^{z''_{h,k}}|\mathrm{d}z|\right) \\ &= \int_{K^-} z e^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}}\mathrm{d}z-\int_0^{z'_{h,k}} z e^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}}\mathrm{d}z-\int_{z''_{h,k}}^0 z e^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}}\mathrm{d}z+\mathcal{O}\left(\frac{k^2}{N^2}e^{\frac{4n\pi}{N^2}}\right), \end{aligned} \tag{6.3.8}$$

where K^- denote the circle $|z-\frac{1}{2}|=\frac{1}{2}$ oriented in the clockwise direction. On the arc from 0 to $z'_{h,k}$, we have $|z|\leq\frac{\sqrt{2}k}{N}$, $0<\mathrm{Re}(z)\leq\frac{2k^2}{N^2}$, $\mathrm{Re}\left(\frac{1}{z}\right)=1$ and the arc length is bounded above by $\frac{\pi}{2}\frac{\sqrt{2}k}{N}$. Consequently, this gives

$$\int_0^{z'_{h,k}} z e^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}}\mathrm{d}z=\mathcal{O}\left(\frac{k^2}{N^2}e^{\frac{4n\pi}{N^2}}\right). \tag{6.3.9}$$

Analogously, the integral along the segment from $z''_{h,k}$ to 0 can be estimated by

$$\int_{z''_{h,k}}^0 z e^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}}\mathrm{d}z=\mathcal{O}\left(\frac{k^2}{N^2}e^{\frac{4n\pi}{N^2}}\right). \tag{6.3.10}$$

Finally utilizing the bounds (6.3.9) and (6.3.10) in (6.3.8), we arrive at

$$J_1=\int_{K^-} z e^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}}\mathrm{d}z+\mathcal{O}\left(\frac{k^2}{N^2}e^{\frac{4n\pi}{N^2}}\right). \tag{6.3.11}$$

Incorporating the expression for J_1 obtained in (6.3.11) into (6.3.7), we obtain

$$S_1 = \frac{i}{\sqrt{2}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{N} \frac{A_k^{(1)}(n)}{k^3} \int_{K^-} z e^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}} \mathrm{d}z + \mathcal{O}\left(\frac{e^{\frac{4n\pi}{N^2}}}{N^2} \sum_{\substack{k=1 \\ k \text{ odd}}}^{N} \frac{1}{k} \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} 1 \right)$$

$$= \frac{i}{\sqrt{2}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{N} \frac{A_k^{(1)}(n)}{k^3} \int_{K^-} z e^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}} \mathrm{d}z + \mathcal{O}\left(\frac{e^{\frac{4n\pi}{N^2}}}{N} \right). \tag{6.3.12}$$

In the above estimate, we have employed the trivial bound $|A_k^{(1)}(n)| \leq k$. Proceeding in a similar way, we can obtain analogous estimate for S_2 as

$$S_2 = \frac{i}{\sqrt{2}} \sum_{\substack{k=1 \\ k \equiv 2 \bmod 4}}^{N} \frac{A_k^{(2)}(n)}{k^3} \int_{K^-} z e^{\frac{2n\pi z}{k^2}+\frac{\pi}{4z}} \mathrm{d}z + \mathcal{O}\left(\frac{e^{\frac{4n\pi}{N^2}}}{N} \right), \tag{6.3.13}$$

where

$$A_k^{(2)}(n) = \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} e^{\pi i \left(2s(h,k)+s\left(h,\frac{k}{2}\right)-s\left(2h,\frac{k}{2}\right)\right)-2n\pi i\frac{h}{k}}.$$

However, to calculate S_0, we need to use transformation formula (6.3.5) corresponding to $k \equiv 0 (\mathrm{mod}\ 4)$. Note that there is no term involving $e^{\mathrm{Re}(\frac{1}{z})}$ in the transformation formula and hence the corresponding integral has no dominant contribution. Therefore, the evaluation of S_0 goes in the error term. Mainly, one can show that

$$S_0 = \mathcal{O}\left(\frac{e^{\frac{4n\pi}{N^2}}}{N} \right). \tag{6.3.14}$$

Now, combining (6.3.12), (6.3.13) and (6.3.14) in (6.3.6), we obtain

$$\overline{a}(n) = \frac{i}{\sqrt{2}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{N} \frac{A_k^{(1)}(n)}{k^3} \int_{K^-} z e^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}} \mathrm{d}z + \frac{i}{\sqrt{2}} \sum_{\substack{k=1 \\ k \equiv 2 \bmod 4}}^{N} \frac{A_k^{(2)}(n)}{k^3} \int_{K^-} z e^{\frac{2n\pi z}{k^2}+\frac{\pi}{4z}} \mathrm{d}z$$

$$+ \mathcal{O}\left(\frac{e^{\frac{4n\pi}{N^2}}}{N} \right).$$

Since the left-hand side is independent of N, taking the limit as $N \to \infty$ forces the error term on the right-hand side to vanish. Consequently, we have

$$\overline{a}(n) = \frac{i}{\sqrt{2}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{A_k^{(1)}(n)}{k^3} \int_{K^-} z e^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}} \mathrm{d}z + \frac{i}{\sqrt{2}} \sum_{\substack{k=1 \\ k \equiv 2 \bmod 4}}^{\infty} \frac{A_k^{(2)}(n)}{k^3} \int_{K^-} z e^{\frac{2n\pi z}{k^2}+\frac{\pi}{4z}} \mathrm{d}z. \tag{6.3.15}$$

This provides an exact formula for $\overline{a}(n)$, with the sole remaining task being to verify the convergence of the infinite series. To this end, note that along the circle K^-, we have $|z| \leq 1$, $\mathrm{Re}(z) \leq 1$ and $\mathrm{Re}\left(\frac{1}{z}\right) = 1$, which leads to

$$\left| \int_{K^-} z e^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}} \mathrm{d}z \right| \leq \pi e^{2n\pi+\frac{3\pi}{16}}.$$

Thus, we have

$$\left|\frac{i}{\sqrt{2}}\sum_{\substack{k=1\\ k \text{ odd}}}^{\infty}\frac{A_k^{(1)}(n)}{k^3}\int_{K^-} ze^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}}\mathrm{d}z\right| \leq \frac{\pi}{\sqrt{2}}e^{2n\pi+\frac{3\pi}{16}}\sum_{\substack{k=1\\ k \text{ odd}}}^{\infty}\frac{1}{k^2}.$$

Here again, we have utilized the bound $|A_k^{(1)}(n)| \leq k$. Since the series on the right hand side above is convergent, it follows that the first sum in (6.3.15) is absolutely convergent. An identical argument works for the convergence of second sum as well. Our next objective is to express both integrals in (6.3.15) in terms of classical special functions. We will illustrate this procedure for the integral in the first sum; the same method can be applied verbatim to the integral in the second sum. To achieve this, we perform a change of variable $z = \frac{1}{\omega}$, which maps the circular contour K^- onto the vertical line $\mathrm{Re}(\omega) = 1$, running from $1 - i\infty$ to $1 + i\infty$. Under this substitution, the integral reduces to

$$\int_{K^-} ze^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}}\mathrm{d}z = -\int_{1-i\infty}^{1+i\infty}\omega^{-3}e^{\left(\frac{2n\pi}{k^2\omega}+\frac{3\pi\omega}{16}\right)}\mathrm{d}\omega.$$

Next, substitute $\frac{3\pi\omega}{16} = t$ in the integral on the right-hand side. Under this change of variable, the real part of the line of integration becomes $\mathrm{Re}(t) = \dfrac{3\pi}{16}$. Hence, we get

$$\int_{K^-} ze^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}}\mathrm{d}z = -\left(\frac{3\pi}{16}\right)^2\int_{\frac{3\pi}{16}-i\infty}^{\frac{3\pi}{16}+i\infty} t^{-3}e^{\left(t+\frac{3n\pi^2}{8k^2t}\right)}\mathrm{d}t. \tag{6.3.16}$$

One can observe that the integral derived above closely resembles the classical integral representation of the modified Bessel function of the first kind, namely, for $c > 0$, $\mathrm{Re}(\nu) > 0$,

$$I_\nu(z) = \frac{(z/2)^\nu}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-\nu-1}e^{\left(t+\frac{z^2}{4t}\right)}\mathrm{d}t. \tag{6.3.17}$$

Now invoking (6.3.17), one can simplify the integral in (6.3.16) in terms of $I_\nu(z)$ as

$$\int_{K^-} ze^{\frac{2n\pi z}{k^2}+\frac{3\pi}{16z}}\mathrm{d}z = -\frac{3\pi ik^2}{16n}I_2\left(\frac{\pi}{k}\sqrt{\frac{3n}{2}}\right). \tag{6.3.18}$$

Similarly, the integral in the second sum of (6.3.15) can be simplified in terms of the modified Bessel function $I_\nu(z)$, namely,

$$\int_{K^-} ze^{\frac{2n\pi z}{k^2}+\frac{\pi}{4z}}\mathrm{d}z = -\frac{\pi ik^2}{4n}I_2\left(\frac{\pi}{k}\sqrt{2n}\right). \tag{6.3.19}$$

Substituting (6.3.18) and (6.3.19) into (6.3.15) establishes the desired Rademacher-type exact formula for cubic overpartition function $\overline{a}(n)$. This completes the proof of Theorem 6.2.1. □

Proof of Corollary 6.2.2. We start by separating the term corresponding to $k = 1$

from the exact formula (6.2.1) for the cubic overpartition, which gives

$$\overline{a}(n) = M_{\overline{a}}(n) + E_{\overline{a}}(n), \tag{6.3.20}$$

where

$$M_{\overline{a}}(n) = \frac{3\pi}{16n\sqrt{2}} I_2\left(\pi\sqrt{\frac{3n}{2}}\right), \tag{6.3.21}$$

and

$$E_{\overline{a}}(n) = \frac{3\pi}{16n\sqrt{2}} \sum_{\substack{k=3\\ k \text{ odd}}}^{\infty} \frac{A_k^{(1)}(n)}{k} I_2\left(\frac{\pi}{k}\sqrt{\frac{3n}{2}}\right) + \frac{\pi}{4n\sqrt{2}} \sum_{\substack{k=1\\ k\equiv 2 \pmod 4}}^{\infty} \frac{A_k^{(2)}(n)}{k} I_2\left(\frac{\pi}{k}\sqrt{2n}\right).$$

Our first goal is to obtain an upper bound for $|E_{\overline{a}}(n)|$. As we can bound $A_k^{(1)}(n)$ and $A_k^{(2)}(n)$ trivially by k for any $k \geq 1$ and $n \geq 0$, so we have

$$\begin{aligned}
|E_{\overline{a}}(n)| &\leq \frac{3\pi}{16n\sqrt{2}} \sum_{\substack{k=3\\ k \text{ odd}}}^{\infty} I_2\left(\frac{\pi}{k}\sqrt{\frac{3n}{2}}\right) + \frac{\pi}{4n\sqrt{2}} \sum_{\substack{k=1\\ k\equiv 2 \pmod 4}}^{\infty} I_2\left(\frac{\pi}{k}\sqrt{2n}\right) \\
&\leq \frac{3\pi}{16n\sqrt{2}} \sum_{k=3}^{\infty} I_2\left(\frac{\pi}{k}\sqrt{\frac{3n}{2}}\right) + \frac{\pi}{4n\sqrt{2}} \sum_{k=6}^{\infty} I_2\left(\frac{\pi}{k}\sqrt{2n}\right) + \frac{\pi}{4n\sqrt{2}} I_2\left(\pi\sqrt{\frac{n}{2}}\right).
\end{aligned} \tag{6.3.22}$$

Here, in the last step, we have separated the term corresponding to $k = 2$. Next we utilize (6.1.3) to bound the first two series as

$$\sum_{k=3}^{\infty} I_2\left(\frac{\pi}{k}\sqrt{\frac{3n}{2}}\right) \leq \frac{8}{\pi}\sqrt{\frac{2}{3n}} I_1\left(\frac{\pi}{2}\sqrt{\frac{3n}{2}}\right), \tag{6.3.23}$$

and

$$\sum_{k=6}^{\infty} I_2\left(\frac{\pi}{k}\sqrt{2n}\right) \leq \frac{50}{\pi\sqrt{2n}} I_1\left(\frac{\pi}{5}\sqrt{2n}\right). \tag{6.3.24}$$

Thus, substituting (6.3.23) and (6.3.24) in (6.3.22) and further utilizing the exponential upper bound for $I_\nu(s) \leq \sqrt{\frac{2}{\pi s}} e^s$, for any $\nu, s > 0$, we reach at

$$|E_{\overline{a}}(n)| \leq \frac{6^{\frac{1}{4}}}{\pi n^{\frac{7}{4}}} e^{\frac{\pi}{2}\sqrt{\frac{3n}{2}}} + \frac{5^{\frac{5}{2}}}{\pi(2n)^{\frac{7}{4}}} e^{\frac{\pi}{5}\sqrt{2n}} + \frac{1}{2^{\frac{7}{4}} n^{\frac{5}{4}}} e^{\pi\sqrt{\frac{n}{2}}}.$$

As $\frac{1}{\sqrt{2}} > \frac{\sqrt{3}}{2\sqrt{2}} > \frac{\sqrt{2}}{5}$ and exponential is an increasing function, hence we have

$$|E_{\overline{a}}(n)| \leq \frac{1}{2^{\frac{7}{4}}} \left(\frac{2 \cdot 5^{\frac{5}{2}}}{\pi n^{\frac{7}{4}}} + \frac{1}{n^{\frac{5}{4}}}\right) e^{\pi\sqrt{\frac{n}{2}}}, \tag{6.3.25}$$

while combining the terms, we also utilized the fact that $6^{\frac{1}{4}} < \frac{5^{\frac{5}{2}}}{2^{\frac{7}{4}}}$. This completes the proof of (6.2.2). Finally, employing the below classical asymptotic expansion for the modified Bessel function of the first kind, that is,

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad \text{as} \quad x \to \infty,$$

in (6.3.21) and simplifying yields the asymptotic relation (6.2.3).

□

6.3.2 Turán inequalities for cubic overpartitions

Here, we prove various results which eventually lead us to obtain the second order Turán inequality i.e., log-concavity for the cubic overpartition function. At the end, we also present higher Turán inequalities for the cubic overpartition function.

The first result in this direction provides explicit upper and lower bounds for $\overline{a}(n)$.

Lemma 6.3.1. *For all $n \geq 393$, we have*

$$M_{\overline{a}}(n)\left(1-\frac{1}{n^6}\right) \leq \overline{a}(n) \leq M_{\overline{a}}(n)\left(1+\frac{1}{n^6}\right),$$

where

$$M_{\overline{a}}(n) = \frac{3\pi}{16n\sqrt{2}} I_2\left(\pi\sqrt{\frac{3n}{2}}\right). \tag{6.3.26}$$

Proof. First, let us define

$$G_{\overline{a}}(n) := \frac{1}{M_{\overline{a}}(n)}\left(\frac{1}{2^{\frac{7}{4}}}\left(\frac{2\cdot 5^{\frac{5}{2}}}{\pi n^{\frac{7}{4}}}+\frac{1}{n^{\frac{5}{4}}}\right)e^{\pi\sqrt{\frac{n}{2}}}\right). \tag{6.3.27}$$

Then, from (6.3.20) and (6.3.25), we see that

$$|\overline{a}(n) - M_{\overline{a}}(n)| = |E_{\overline{a}}(n)| \leq G_{\overline{a}}(n) M_{\overline{a}}(n).$$

Now, our main goal is to prove that, for $n \geq 393$,

$$G_{\overline{a}}(n) \leq \frac{1}{n^6}. \tag{6.3.28}$$

To get this upper bound for $G_{\overline{a}}(n)$, we first need a lower bound for $M_{\overline{a}}(n)$. This can be achieved by using a lower bound (6.1.4) for $I_2(s)$. Thus applying (6.1.4) in (6.3.27), for $n \geq 67$, one can see that

$$G_{\overline{a}}(n) \leq 0.027 e^{\pi\sqrt{n}\left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)}.$$

Hence proving (6.3.28) is equivalent to verifying the following inequality

$$0.027 e^{\pi\sqrt{n}\left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)} \leq \frac{1}{n^6}, \quad \text{for all } n \geq 393.$$

To do so, we define the function

$$H(x) := 0.027 e^{\pi\sqrt{x}\left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)} x^6.$$

One can check that, for all $x \geq 54.4516$, $H'(x) < 0$. This shows that $H(x)$ is decreasing on $(54.4516, \infty)$. As $H(393) < 1$, so $H(n) < H(393) < 1$ for all $n \geq 393$. This proves that

$$G_{\overline{a}}(n) \leq 0.027 e^{\pi\sqrt{n}\left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)} \leq \frac{1}{n^6},$$

for all $n \geq 393$, which confirms the validity of (6.3.28) and hence the proof of Lemma 6.3.1 is now complete. □

Remark 13. *We would also like to point out that the Lemma 6.3.1 can also be true for any $\alpha > 0$ such that*

$$M_{\overline{a}}(n)\left(1-\frac{1}{n^{\alpha}}\right) \leq \overline{a}(n) \leq M_{\overline{a}}(n)\left(1+\frac{1}{n^{\alpha}}\right), \quad \text{for } n \geq n_0(\alpha).$$

In the next result, we establish bounds for the product of modified Bessel functions of the first kind.

Lemma 6.3.2. *For $n \geq 2363$, the following bounds are true:*

$$\frac{v}{\sqrt{v^+v^-}}\Xi_{\overline{a}}(n) \leq \frac{I_2(v^-)I_2(v^+)}{\left(I_2(v)\right)^2} \leq \frac{v}{\sqrt{v^+v^-}}\Psi_{\overline{a}}(n),$$

where

$$\Xi_{\overline{a}}(n) := \left(1-\frac{9\pi^4}{16v^3}-\frac{27\pi^6}{8v^5}-\frac{405\pi^8}{2048v^7}\right)\left(1-\frac{309}{v^5}-\frac{535}{v^6}\right), \tag{6.3.29}$$

$$\Psi_{\overline{a}}(n) := \left(1-\frac{9\pi^4}{16v^3}+\frac{81\pi^8}{256v^6}\right)\left(1-\frac{308}{v^5}-\frac{286}{v^6}\right), \tag{6.3.30}$$

and v, v^+, v^- are defined as in (6.2.4).

Proof. First, we recall

$$v = \pi\sqrt{\frac{3n}{2}}, \quad v^+ = \pi\sqrt{\frac{3(n+1)}{2}}, \quad \text{and} \quad v^- = \pi\sqrt{\frac{3(n-1)}{2}}. \tag{6.3.31}$$

To attain the required bounds for $\frac{I_2(v^-)I_2(v^+)}{\left(I_2(v)\right)^2}$, we utilize an upper and lower bound given in (6.1.2) for $I_2(s)$. Mainly, for all $n \geq 44$, we get

$$\frac{v}{\sqrt{v^+v^-}}e^{v^++v^--2v}U_{\overline{a}}(n) \leq \frac{I_2(v^-)I_2(v^+)}{\left(I_2(v)\right)^2} \leq \frac{v}{\sqrt{v^+v^-}}e^{v^++v^--2v}V_{\overline{a}}(n),$$

where

$$U_{\overline{a}}(n) := \frac{\left(E_{I_2}(v^-)-\frac{31}{v^{-6}}\right)\left(E_{I_2}(v^+)-\frac{31}{v^{+6}}\right)}{\left(E_{I_2}(v)+\frac{31}{v^6}\right)^2}, \tag{6.3.32}$$

$$V_{\overline{a}}(n) := \frac{\left(E_{I_2}(v^-)+\frac{31}{v^{-6}}\right)\left(E_{I_2}(v^+)+\frac{31}{v^{+6}}\right)}{\left(E_{I_2}(v)-\frac{31}{v^6}\right)^2}, \tag{6.3.33}$$

and $E_{I_2}(s)$ is defined as in (6.1.1). We start by deriving an upper and lower bound for $e^{v^++v^--2v}$. In this context, we first derive bounds for v^+, v^- in terms of v. One can easily check that, for $n > 1$,

$$\sqrt{n}\left(1-\frac{1}{2n}-\frac{1}{8n^2}-\frac{1}{16n^3}-\frac{1}{n^3}\right) < \sqrt{n-1} < \sqrt{n}\left(1-\frac{1}{2n}-\frac{1}{8n^2}-\frac{1}{16n^3}\right),$$

and then multiply throughout by $\pi\sqrt{\frac{3}{2}}$ to see that

$$C_v \leq v^- \leq D_v, \tag{6.3.34}$$

where

$$C_v := v - \frac{3\pi^2}{4v} - \frac{9\pi^4}{32v^3} - \frac{27\pi^6}{128v^5} - \frac{27\pi^6}{8v^5},$$
$$D_v := v - \frac{3\pi^2}{4v} - \frac{9\pi^4}{32v^3} - \frac{27\pi^6}{128v^5}.$$

Similarly, one can verify that

$$A_v \leq v^+ \leq B_v, \tag{6.3.35}$$

where

$$A_v := v + \frac{3\pi^2}{4v} - \frac{9\pi^4}{32v^3} + \frac{27\pi^6}{128v^5} - \frac{405\pi^8}{2048v^7},$$
$$B_v := v + \frac{3\pi^2}{4v} - \frac{9\pi^4}{32v^3} + \frac{27\pi^6}{128v^5}.$$

Now combining (6.3.34) and (6.3.35), we obtain

$$-\frac{9\pi^4}{16v^3} - \frac{27\pi^6}{8v^5} - \frac{405\pi^8}{2048v^7} \leq v^+ + v^- - 2v \leq -\frac{9\pi^4}{16v^3}.$$

Next, we make use of the fact that for $s < 0$, $1 + s < e^s < 1 + s + s^2$, which gives

$$1 - \frac{9\pi^4}{16v^3} - \frac{27\pi^6}{8v^5} - \frac{405\pi^8}{2048v^7} \leq e^{v^+ + v^- - 2v} \leq 1 - \frac{9\pi^4}{16v^3} + \frac{81\pi^8}{256v^6}.$$

Our next aim is to look for a lower bound of $U_{\overline{a}}(n)$ and an upper bound for $V_{\overline{a}}(n)$. First, we write

$$U_{\overline{a}}(n) \geq \frac{\Theta}{\left(E_{I_2}(v) + \frac{31}{v^6}\right)^2}, \quad V_{\overline{a}}(n) \leq \frac{\Delta}{\left(E_{I_2}(v) - \frac{31}{v^6}\right)^2},$$

where Θ and Δ can be obtained by first expanding the numerators of $U_{\overline{a}}(n)$ and $V_{\overline{a}}(n)$ in (6.3.32), (6.3.33), and utilizing the bounds (6.3.34) and (6.3.35) for v^- and v^+, respectively. The exact expressions for Θ and Δ will be clear from the below calculations of the numerators of $U_{\overline{a}}(n)$ and $V_{\overline{a}}(n)$. Utilizing the definition (6.1.1) of $E_{I_2}(s)$, one can see that

$$\begin{aligned}
&\left(E_{I_2}(v^-) - \frac{31}{v^{-^6}}\right)\left(E_{I_2}(v^+) - \frac{31}{v^{+^6}}\right) \\
&= \frac{1}{v^{-^6}v^{+^6}}\left(v^{-^6} - \frac{15}{8}v^{-^5} + \frac{105}{128}v^{-^4} + \frac{315}{1024}v^{-^3} + \frac{10395}{32768}v^{-^2} + \frac{135135}{262144}v^- - 31\right) \\
&\quad \times \left(v^{+^6} - \frac{15}{8}v^{+^5} + \frac{105}{128}v^{+^4} + \frac{315}{1024}v^{+^3} + \frac{10395}{32768}v^{+^2} + \frac{135135}{262144}v^+ - 31\right) \\
&\geq \frac{1}{v^{-^6}v^{+^6}}\left(v^{-^6} - \frac{15}{8}v^{-^4}D_v + \frac{105}{128}v^{-^4} + \frac{315}{1024}v^{-^2}C_v + \frac{10395}{32768}v^{-^2} + \frac{135135}{262144}C_v - 31\right)
\end{aligned}$$

$$\times \left(v^{+^6} - \frac{15}{8}v^{+^4}B_v + \frac{105}{128}v^{+^4} + \frac{315}{1024}v^{+^2}A_v + \frac{10395}{32768}v^{+^2} + \frac{135135}{262144}A_v - 31\right) := \Theta,$$

$$\left(E_{I_2}(v^-) + \frac{31}{v^{-^6}}\right)\left(E_{I_2}(v^+) + \frac{31}{v^{+^6}}\right)$$
$$= \frac{1}{v^{-^6}v^{+^6}}\left(v^{-^6} - \frac{15}{8}v^{-^5} + \frac{105}{128}v^{-^4} + \frac{315}{1024}v^{-^3} + \frac{10395}{32768}v^{-^2} + \frac{135135}{262144}v^- + 31\right)$$
$$\times \left(v^{+^6} - \frac{15}{8}v^{+^5} + \frac{105}{128}v^{+^4} + \frac{315}{1024}v^{+^3} + \frac{10395}{32768}v^{+^2} + \frac{135135}{262144}v^+ + 31\right)$$
$$\leq \frac{1}{v^{-^6}v^{+^6}}\left(v^{-^6} - \frac{15}{8}v^{-^4}C_v + \frac{105}{128}v^{-^4} + \frac{315}{1024}v^{-^2}D_v + \frac{10395}{32768}v^{-^2} + \frac{135135}{262144}D_v + 31\right)$$
$$\times \left(v^{+^6} - \frac{15}{8}v^{+^4}A_v + \frac{105}{128}v^{+^4} + \frac{315}{1024}v^{+^2}B_v + \frac{10395}{32768}v^{+^2} + \frac{135135}{262144}B_v + 31\right) := \Delta.$$

Now we want to show that

$$\frac{\Theta}{\left(E_{I_2}(v) + \frac{31}{v^6}\right)^2} \geq 1 - \frac{309}{v^5} - \frac{535}{v^6}, \quad \frac{\Delta}{\left(E_{I_2}(v) - \frac{31}{v^6}\right)^2} \leq 1 - \frac{308}{v^5} - \frac{286}{v^6}, \quad \text{for all } n \geq 2363.$$

These are equivalent to verifying that

$$v^{12}\Theta - \left(1 - \frac{309}{v^5} - \frac{535}{v^6}\right)\left(v^6E_{I_2}(v) + 31\right)^2 \geq 0, \quad \text{for all } n \geq 2363, \tag{6.3.36}$$

$$v^{12}\Delta - \left(1 - \frac{308}{v^5} - \frac{286}{v^6}\right)\left(v^6E_{I_2}(v) - 31\right)^2 \leq 0, \quad \text{for all } n \geq 2363. \tag{6.3.37}$$

Substituting expressions of A_v, B_v, C_v, D_v, v^+ and v^- in Θ and Δ and expanding, we write the left hand side of (6.3.36) and (6.3.37) as follows:

$$v^{12}\Theta - \left(1 - \frac{309}{v^5} - \frac{535}{v^6}\right)\left(v^6E_{I_2}(v) + 31\right)^2 = \frac{\sum_{i=0}^{25}\alpha_i v^i}{v^6\left(v^4 - \frac{9\pi^4}{4}\right)^3}, \tag{6.3.38}$$

$$v^{12}\Delta - \left(1 - \frac{308}{v^5} - \frac{286}{v^6}\right)\left(v^6E_{I_2}(v) - 31\right)^2 = \frac{\sum_{j=0}^{25}\beta_j v^j}{v^6\left(v^4 - \frac{9\pi^4}{4}\right)^3}, \tag{6.3.39}$$

where $\alpha_i's$ and $\beta_j's$ are some real numbers. One can easily verify that, for all $n \geq 2$, $0 \leq i \leq 20$ and $0 \leq j \leq 21$,

$$-|\alpha_i|v^i \geq -|\alpha_{21}|v^{21}, \quad |\beta_j|v^j \leq |\beta_{22}|v^{22}.$$

Thus, we have

$$\begin{aligned}
\sum_{i=0}^{25}\alpha_i v^i &\geq -\sum_{i=0}^{21}|\alpha_i|v^i + \alpha_{22}v^{22} + \alpha_{23}v^{23} + \alpha_{24}v^{24} + \alpha_{25}v^{25} \\
&\geq -22|\alpha_{21}|v^{21} + \alpha_{22}v^{22} + \alpha_{23}v^{23} + \alpha_{24}v^{24} + \alpha_{25}v^{25} := g_1(v), \\
\sum_{j=0}^{25}\beta_j v^j &\leq \sum_{j=0}^{22}|\beta_j|v^j + \beta_{23}v^{23} + \beta_{24}v^{24} + \beta_{25}v^{25} \\
&\leq 23|\beta_{22}|v^{22} + \beta_{23}v^{23} + \beta_{24}v^{24} + \beta_{25}v^{25} := g_2(v),
\end{aligned}$$

where

$$\alpha_{21} = -\frac{2677185}{2048} - \frac{545363523\pi^4}{262144} - \frac{8505\pi^6}{8192} + \frac{15795\pi^8}{1024},$$
$$\alpha_{22} = \frac{242745}{128} + \frac{31185\pi^4}{16384},$$
$$\alpha_{23} = -\frac{5775}{32} - \frac{14175\pi^4}{4096},$$
$$\alpha_{24} = -\frac{2991}{4} + \frac{3915\pi^4}{512},$$
$$\alpha_{25} = 309 - \frac{405\pi^4}{128},$$
$$\beta_{22} = \frac{13095}{16} + \frac{31185\pi^4}{16384} - \frac{6075\pi^6}{512},$$
$$\beta_{23} = \frac{2265}{8} - \frac{14175\pi^4}{4096} + \frac{405\pi^6}{64},$$
$$\beta_{24} = -745 + \frac{3915\pi^4}{512},$$
$$\beta_{25} = 308 - \frac{405\pi^4}{128}.$$

As the largest real root of $g_1'(v)$ is ≈ 39.421, it follows that $g_1'(v) > 0$ for all $v \geq 40$. Consequently, $g_1(v)$ is strictly increasing for $v \geq 40$. Moreover, as $g_1(41.23) > 0$, we deduce that $g_1(v) > g_1(41.23) > 0$ for all $v \geq 41.23$. This, in turn, establishes that $g_1(v) > 0$ for all $n \geq 115$. In a similar manner, the largest real root of $g_2'(v)$ occurs at ≈ 179.206. Since the leading coefficient of $g_2(v)$ is negative, we conclude that $g_2'(v) < 0$ whenever $v \geq 180$, and thus $g_2(v)$ is strictly decreasing. Moreover, as $g_2(187) < 0$, it follows that $g_2(v)$ remains negative for all $v \geq 187$. Consequently, we obtain $g_2(v) < 0$ for all $n \geq 2363$. Utilizing these facts in (6.3.38) and (6.3.39) along with the observation $v^4 - \frac{9\pi^4}{4} > 0$ for all $n > 1$, we finish the proof of (6.3.36) and (6.3.37). This completes the proof of Lemma 6.3.2. □

Proof of Theorem 6.2.3. Using Lemma 6.3.1, we obtain the inequality

$$\Omega(n)\frac{\left(1-\frac{1}{(n+1)^6}\right)\left(1-\frac{1}{(n-1)^6}\right)}{\left(1+\frac{1}{n^6}\right)^2} \leq \frac{\overline{a}(n+1)\overline{a}(n-1)}{(\overline{a}(n))^2} \leq \Omega(n)\frac{\left(1+\frac{1}{(n+1)^6}\right)\left(1+\frac{1}{(n-1)^6}\right)}{\left(1-\frac{1}{n^6}\right)^2}, \tag{6.3.40}$$

where

$$\Omega(n) := \frac{M_{\overline{a}}(n+1)M_{\overline{a}}(n-1)}{(M_{\overline{a}}(n))^2}.$$

Next, by using (6.3.46) for $M_{\overline{a}}(n)$, we see that

$$\Omega(n) = \frac{n^2}{(n+1)(n-1)} \frac{I_2\left(\pi\sqrt{\frac{3(n+1)}{2}}\right) I_2\left(\pi\sqrt{\frac{3(n-1)}{2}}\right)}{I_2\left(\pi\sqrt{\frac{3n}{2}}\right)^2}.$$

Rewriting the above expression in terms of v, v^+, v^- (6.3.31), we obtain

$$\Omega(n) = \frac{v^4}{{v^+}^2 {v^-}^2} \frac{I_2(v^+)I_2(v^-)}{(I_2(v))^2}.$$

Applying the bounds from Lemma 6.3.2, we deduce that, for all $n \geq 2363$,

$$\frac{v^5}{(v^+v^-)^{\frac{5}{2}}} \Xi_{\overline{a}(n)} \leq \Omega(n) \leq \frac{v^5}{(v^+v^-)^{\frac{5}{2}}} \Psi_{\overline{a}(n)},$$

where $\Xi_{\overline{a}(n)}$ and $\Psi_{\overline{a}(n)}$ are defined as in (6.3.29) and (6.3.30), respectively. Moreover, for all $n \geq 7$, the factor $\frac{v^5}{(v^+v^-)^{\frac{5}{2}}}$ can be bounded as

$$1 + \frac{45\pi^4}{16v^4} + \frac{3645\pi^8}{512v^8} \leq \frac{v^5}{(v^+v^-)^{\frac{5}{2}}} \leq 1 + \frac{45\pi^4}{16v^4} + \frac{8\pi^8}{v^8}.$$

Consequently, for all $n \geq 2363$, we have

$$\left(1 + \frac{45\pi^4}{16v^4} + \frac{3645\pi^8}{512v^8}\right) \Xi_{\overline{a}(n)} \leq \Omega(n) \leq \Psi_{\overline{a}(n)} \left(1 + \frac{45\pi^4}{16v^4} + \frac{8\pi^8}{v^8}\right).$$

Now employing (6.3.29), (6.3.30) and after further simplification, it yields

$$1 - \frac{9\pi^4}{16v^3} + \frac{45\pi^4}{16v^4} - \frac{309}{v^5} - \frac{27\pi^6}{8v^5} - \frac{535}{v^6} - \frac{\pi^8}{v^6} \leq \Omega(n) \leq 1 - \frac{9\pi^4}{16v^3} + \frac{45\pi^4}{16v^4} - \frac{308}{v^5} - \frac{286}{v^6} + \frac{81\pi^8}{256v^6}. \tag{6.3.41}$$

To complete the proof, it remains to estimate the multiplicative corrections arising from (6.3.40), namely,

$$\frac{\left(1 - \frac{1}{(n+1)^6}\right)\left(1 - \frac{1}{(n-1)^6}\right)}{\left(1 + \frac{1}{n^6}\right)^2}, \quad \text{and} \quad \frac{\left(1 + \frac{1}{(n+1)^6}\right)\left(1 + \frac{1}{(n-1)^6}\right)}{\left(1 - \frac{1}{n^6}\right)^2}.$$

Expressing these in terms of v, v^+, v^- and we require a lower bound for

$$\frac{\left(1 - \frac{729\pi^{12}}{64{v^+}^{12}}\right)\left(1 - \frac{729\pi^{12}}{64{v^-}^{12}}\right)}{\left(1 + \frac{729\pi^{12}}{64v^{12}}\right)^2},$$

and an upper bound for

$$\frac{\left(1 + \frac{729\pi^{12}}{64{v^+}^{12}}\right)\left(1 + \frac{729\pi^{12}}{64{v^-}^{12}}\right)}{\left(1 - \frac{729\pi^{12}}{64v^{12}}\right)^2}.$$

Using series expansions for the denominators of the above two expressions and the bounds (6.3.34) and (6.3.35) for v^-, v^+, one obtains the following estimates valid for

all $n \geq 1$,

$$\frac{\left(1-\frac{729\pi^{12}}{64v^{+12}}\right)\left(1-\frac{729\pi^{12}}{64v^{-12}}\right)}{\left(1+\frac{729\pi^{12}}{64v^{12}}\right)^2} \geq 1-\frac{729\pi^{12}}{v^6}, \tag{6.3.42}$$

$$\frac{\left(1+\frac{729\pi^{12}}{64v^{+12}}\right)\left(1+\frac{729\pi^{12}}{64v^{-12}}\right)}{\left(1-\frac{729\pi^{12}}{64v^{12}}\right)^2} \leq 1+\frac{729\pi^{12}}{v^6}. \tag{6.3.43}$$

Finally, combining (6.3.41), (6.3.42) and (6.3.43) with (6.3.40), we arrive at the desired bounds, thereby completing the proof of Theorem 6.2.3. □

Proof of Theorem 6.2.4. Utilizing Theorem 6.2.3, one can observe that

$$1-\frac{\overline{a}(n+1)\overline{a}(n-1)}{(\overline{a}(n))^2} \geq \frac{9\pi^4}{16v^3}-\frac{45\pi^4}{16v^4}+\frac{308}{v^5}+\frac{286}{v^6}-\frac{81\pi^8}{256v^6}-\frac{729\pi^{12}}{16v^6}, \tag{6.3.44}$$

for $n \geq 2363$. Now we consider the polynomial

$$g_3(v)=\frac{9\pi^4}{16}v^3-\frac{45\pi^4}{16}v^2+308v+286-\frac{81\pi^8}{256}-\frac{729\pi^{12}}{16}.$$

A direct computation shows that the real roots of $g_3'(v)$ are ≈ 0.715853 and ≈ 2.61748. Hence, $g_3'(v)>0$ for all $v \geq 3$, which implies that $g_3(v)$ is increasing for $v \geq 3$. Since $g_3(93.28)>0$, it follows that $g_3(v)>0$ when $v \geq 93.28$. This concludes that $g_3(v)>0$ whenever $n \geq 588$. Consequently, we obtain

$$\frac{g_3(v)}{v^6}=\frac{9\pi^4}{16v^3}-\frac{45\pi^4}{16v^4}+\frac{308}{v^5}+\frac{286}{v^6}-\frac{81\pi^8}{256v^6}-\frac{729\pi^{12}}{16v^6}>0 \tag{6.3.45}$$

for all $n \geq 588$. Finally, using (6.3.45) in (6.3.44), we prove the log concavity for $\overline{a}(n)$ when $n \geq 2363$. Using Mathematica, one can check that $\overline{a}(n)$ also satisfies log-concavity for $10 \leq n \leq 2363$ and hence $\overline{a}(n)$ attains log concavity for all $n \geq 10$. This finishes the proof of Theorem 6.2.4. □

6.3.3 Higher order Turán inequalities for cubic overpartitions

Proof of Theorem 6.2.5. For any $j \in \mathbb{N} \cup \{0\}$, utilize (6.2.3) to see

$$\overline{a}(n+j) \sim \frac{3^{\frac{3}{4}}}{2^{\frac{19}{4}}(n+j)^{\frac{5}{4}}}e^{\pi\sqrt{\frac{3(n+j)}{2}}}, \quad \text{as} \quad n \to \infty.$$

This gives

$$\frac{\overline{a}(n+j)}{\overline{a}(n)} \sim \frac{n^{\frac{5}{4}}}{(n+j)^{\frac{5}{4}}}e^{\pi\sqrt{\frac{3(n+j)}{2}}-\pi\sqrt{\frac{3n}{2}}}.$$

Thus, taking log on both sides, we have

$$\log\left(\frac{\overline{a}(n+j)}{\overline{a}(n)}\right) \sim \pi\sqrt{\frac{3}{2}}\left(\sqrt{n+j}-\sqrt{n}\right) - \frac{5}{4}\log\left(\frac{n+j}{n}\right).$$

Further, one can simplify the above right hand side expression as follows:

$$\begin{aligned}\pi\sqrt{\frac{3}{2}}\left(\sqrt{n+j}-\sqrt{n}\right) - \frac{5}{4}\log\left(\frac{n+j}{n}\right) &= \pi\sqrt{\frac{3}{2}}\sum_{i=1}^{\infty}\binom{1/2}{i}\frac{j^i}{n^{i-1/2}} + \frac{5}{4}\sum_{i=1}^{\infty}\frac{(-1)^i j^i}{in^i}\\ &= \left(\frac{\pi}{2}\sqrt{\frac{3}{2n}} - \frac{5}{4n}\right)j - \left(\frac{\pi}{8}\sqrt{\frac{3}{2n^3}} - \frac{5}{8n^2}\right)j^2\\ &\quad + \sum_{i=3}^{\infty}\left(\frac{\pi}{n^{i-1/2}}\sqrt{\frac{3}{2}}\binom{1/2}{i} + \frac{5(-1)^i}{4in^i}\right)j^i.\end{aligned}$$

Now we consider $A(n) = \frac{\pi}{2}\sqrt{\frac{3}{2n}} - \frac{5}{4n}$, $\delta(n) = \sqrt{\frac{\pi}{8}\sqrt{\frac{3}{2n^3}} - \frac{5}{8n^2}}$, and for $i \geq 3$,

$$g_i(n) = \frac{\pi}{n^{i-1/2}}\sqrt{\frac{3}{2}}\binom{1/2}{i} + \frac{5(-1)^i}{4in^i}.$$

Note that for $3 \leq i \leq d$, one can easily show that

$$\lim_{n\to\infty}\frac{g_i(n)}{(\delta(n))^i} = \lim_{n\to\infty}\frac{\frac{\pi}{n^{i-1/2}}\sqrt{\frac{3}{2}}\binom{1/2}{i} + \frac{5(-1)^i}{4in^i}}{\left(\frac{\pi}{8}\sqrt{\frac{3}{2n^3}} - \frac{5}{8n^2}\right)^{i/2}} = 0.$$

Moreover, we can check that $\lim_{n\to\infty}\frac{g_i(n)}{(\delta(n))^d} = 0$, for all $i \geq d+1$. Thus, the sequences $\{\overline{a}(n)\}, \{A(n)\}, \{\delta(n)\}, \{g_i(n)\}$ satisfy hypotheses of Theorem 1.2.1. Hence, it follows that the Jenson polynomials associated with cubic overpartition function can be expressed in terms of Hermite polynomials which are hyperbolic for sufficiently large n. This completes the proof of Theorem 6.2.5.

□

6.3.4 Log-subadditivity and general log-concavity for cubic overpartitions

In this subsection, we establish the log-subadditivity and a general log-concavity for the cubic overpartition function $\overline{a}(n)$. Before proving Theorem 6.2.6, we state a lemma that will be useful in deriving the log-subadditivity for $\overline{a}(n)$. In Remark 13, we noted that in Lemma 6.3.1, the term $\frac{1}{n^6}$ can be replaced by $\frac{1}{n^\alpha}$ for any $\alpha > 0$. Here, we present a result corresponding to $\alpha = \frac{1}{2}$. For the sake of clarity, we also provide a proof, since it requires a sharper bound for $I_2(s)$.

Lemma 6.3.3. *For all $n \geq 11$, we have*

$$M_{\overline{a}}(n)\left(1 - \frac{1}{\sqrt{n}}\right) \leq \overline{a}(n) \leq M_{\overline{a}}(n)\left(1 + \frac{1}{\sqrt{n}}\right),$$

where

$$M_{\overline{a}}(n) = \frac{3\pi}{16n\sqrt{2}} I_2\left(\pi\sqrt{\frac{3n}{2}}\right). \tag{6.3.46}$$

Proof. Following the line of argument used in Lemma 6.3.1, our main goal here is to prove that, for $n \geq 11$,

$$G_{\overline{a}}(n) \leq \frac{1}{\sqrt{n}},$$

where $G_{\overline{a}}(n)$ is defined in (6.3.27). Now we employ more effective bounds [33, Equations (4.24)-(4.25)] for $I_2(s)$, which asserts that, for all $s \geq 10$,

$$I_2(s) > \frac{e^s}{\sqrt{2\pi s}}\left(1 - \frac{2}{s}\right), \tag{6.3.47}$$

$$I_2(s) < \frac{e^s}{\sqrt{2\pi s}}\left(1 - \frac{15}{8s} + \frac{2}{s^2}\right). \tag{6.3.48}$$

We also make use of an elementary inequality, valid for all $s \geq 10$,

$$\left(1 - \frac{2}{s}\right)\left(1 + \frac{3}{s}\right) = 1 + \frac{s-6}{s^2} > 1. \tag{6.3.49}$$

Utilizing (6.3.47) and (6.3.49) in (6.3.27), we obtain, for all $n \geq 7$,

$$G_{\overline{a}}(n) \leq 66e^{\pi\sqrt{n}\left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)}.$$

Finally, applying the same argument as in Lemma 6.3.1, one can deduce that

$$66e^{\pi\sqrt{n}\left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)} \leq \frac{1}{\sqrt{n}},$$

for all $n \geq 11$. This completes the proof of Lemma 6.3.3. □

Proof of Theorem 6.2.6. We begin by obtaining an explicit upper and lower bound for $\overline{a}(n)$. From Lemma 6.3.3, we have

$$\overline{a}(n) \geq \frac{3\pi}{16n\sqrt{2}} I_2\left(\pi\sqrt{\frac{3n}{2}}\right)\left(1 - \frac{1}{\sqrt{n}}\right).$$

Substituting the bound (6.3.47) of $I_2(s)$ into the above expression, we obtain

$$\overline{a}(n) \geq \frac{3^{\frac{3}{4}}}{2^{\frac{19}{4}} n^{\frac{5}{4}}} e^{\pi\sqrt{\frac{3n}{2}}}\left(1 - \frac{2}{\pi}\sqrt{\frac{2}{3n}}\right)\left(1 - \frac{1}{\sqrt{n}}\right) \geq \frac{3^{\frac{3}{4}}}{2^{\frac{19}{4}} n^{\frac{5}{4}}} e^{\pi\sqrt{\frac{3n}{2}}}\left(1 - \frac{8}{5\sqrt{n}}\right). \tag{6.3.50}$$

Here we have used the fact that $1 + \frac{2}{\pi}\sqrt{\frac{2}{3}} < \frac{8}{5}$. Similarly, applying the upper bound (6.3.48) for $I_2(s)$ yields,

$$\overline{a}(n) \leq \frac{3^{\frac{3}{4}}}{2^{\frac{19}{4}} n^{\frac{5}{4}}} e^{\pi\sqrt{\frac{3n}{2}}}\left(1 + \frac{1}{\sqrt{n}}\right). \tag{6.3.51}$$

The inequalities (6.3.50)-(6.3.51) are true for $n \geq 11$ and we have additionally verified by direct computation in Mathematica that they also hold for $1 \leq n \leq 10$. Let us suppose that $1 \leq m \leq n$ and $n = Bm$ with $B \geq 1$. Note that $B \in \mathbb{Q}$, whereas

$n, m \in \mathbb{N}$. Utilizing (6.3.50), we obtain

$$\overline{a}(m)\overline{a}(Bm) \geq \frac{3^{\frac{3}{2}}}{2^{\frac{19}{2}} B^{\frac{5}{4}} m^{\frac{5}{2}}} e^{\pi\sqrt{\frac{3}{2}}\left(\sqrt{m}+\sqrt{Bm}\right)} \left(1 - \frac{8}{5\sqrt{m}}\right)\left(1 - \frac{8}{5\sqrt{Bm}}\right),$$

while employing (6.3.51), we deduce

$$\overline{a}(m + Bm) \leq \frac{3^{\frac{3}{4}}}{2^{\frac{19}{4}} (m + Bm)^{\frac{5}{4}}} e^{\pi\sqrt{\frac{3(m+Bm)}{2}}} \left(1 + \frac{1}{\sqrt{m + Bm}}\right).$$

Hence, apart from finitely many exceptional cases, it suffices to determine conditions on m and B such that

$$e^{\pi\sqrt{\frac{3}{2}}\left(\sqrt{m}+\sqrt{Bm}-\sqrt{m+Bm}\right)} \geq \frac{m^{\frac{5}{4}} 2^{\frac{19}{4}}}{3^{\frac{3}{4}}} \left(\frac{B}{B+1}\right)^{\frac{5}{4}} \frac{\left(1 + \frac{1}{\sqrt{m+Bm}}\right)}{\left(1 - \frac{8}{5\sqrt{m}}\right)\left(1 - \frac{8}{5\sqrt{Bm}}\right)}. \tag{6.3.52}$$

For notational convenience, we define

$$L_{\overline{a}}(B) := \pi\sqrt{\frac{3}{2}}\left(\sqrt{m} + \sqrt{Bm} - \sqrt{m + Bm}\right),$$

$$H_{\overline{a}}(B) := \left(\frac{B}{B+1}\right)^{\frac{5}{4}} \frac{\left(1 + \frac{1}{\sqrt{m+Bm}}\right)}{\left(1 - \frac{8}{5\sqrt{m}}\right)\left(1 - \frac{8}{5\sqrt{Bm}}\right)}.$$

Taking logarithm on both sides of (6.3.52) yields an equivalent condition, namely,

$$L_{\overline{a}}(B) \geq \log\left(\frac{m^{\frac{5}{4}} 2^{\frac{19}{4}}}{3^{\frac{3}{4}}}\right) + \log\left(H_{\overline{a}}(B)\right). \tag{6.3.53}$$

As a function of B, one can check that $L_{\overline{a}}(B)$ is increasing for any $m \geq 1$, whereas $H_{\overline{a}}(B)$ is decreasing for $m \geq 3$. Consequently, we have

$$L_{\overline{a}}(B) \geq L_{\overline{a}}(1), \quad \text{and} \quad \log\left(H_{\overline{a}}(1)\right) \geq \log\left(H_{\overline{a}}(B)\right).$$

Thus, for all $m \geq 3$, it suffices to verify

$$L_{\overline{a}}(1) \geq \log\left(\frac{m^{\frac{5}{4}} 2^{\frac{19}{4}}}{3^{\frac{3}{4}}}\right) + \log\left(H_{\overline{a}}(1)\right). \tag{6.3.54}$$

By direct evaluation of $L_{\overline{a}}(1)$ and $H_{\overline{a}}(1)$, we confirm that (6.3.54) holds for all $m \geq 8$. Therefore, to complete the argument, it remains to check (6.3.53) for finitely many values $3 \leq m \leq 7$ and for these integers, let B_m denote the real number satisfying

$$L_{\overline{a}}(B_m) = \log\left(\frac{m^{\frac{5}{4}} 2^{\frac{19}{4}}}{3^{\frac{3}{4}}}\right) + \log\left(H_{\overline{a}}(B_m)\right).$$

The corresponding values of B_m for $3 \leq m \leq 7$ are displayed in the below Table 6.1.

Table 6.1: Values of B_m

m	3	4	5	6	7
B_m	369.385...	6.011...	2.548...	1.558...	1.105...

By the preceding discussion, if $n = Bm \geq m$ is an integer with $B > B_m$, then (6.3.53) is satisfied, thereby establishing the theorem in these cases. Thus, the only remaining possibilities are finitely many exceptional pairs of integers with $3 \leq m \leq 7$ and $1 \leq \frac{n}{m} = B \leq B_m$. These cases can be directly verified using Mathematica. This completes the proof of Theorem 6.2.6 for $n, m \geq 3$. Finally, corresponding to $m = 1, 2$, it remain to show that,

$$2\overline{a}(n) > \overline{a}(n+1), \forall\ n \neq 1, 3, \tag{6.3.55}$$

$$4\overline{a}(n) > \overline{a}(n+2), \forall\ n \neq 1. \tag{6.3.56}$$

To prove (6.3.55), we utilize the bounds (6.3.50) and (6.3.51) for $\overline{a}(n)$, which is equivalent to show that the following function

$$\log(2) + \pi\sqrt{\frac{3}{2}}\left(\sqrt{n} - \sqrt{n+1}\right) - \frac{5}{4}\log\left(\frac{n}{n+1}\right) - \log\left(1 + \frac{1}{\sqrt{n+1}}\right) + \log\left(1 - \frac{8}{5\sqrt{n}}\right)$$

is positive. One can check that, the above function is positive for all $n \geq 42$, which proves (6.3.55) in this range. Moreover, for $1 \leq n \leq 41$, except for $n = 1, 3$, we find that (6.3.55) holds via numerical computation on Mathematica. In a similar manner, one can prove (6.3.56). This finishes the proof of Theorem 6.2.6. □

Proof of Theorem 6.2.7. It is known that log-concavity is equivalent to strong log-concavity, readers can see [105]. Moreover, if $\{b(k)\}$ satisfies $b(k)^2 > b(k-1)b(k+1)$ for $k \geq k_0 > 0$, then one can show that

$$b(\ell - i)b(k + i) > b(k)b(\ell)$$

holds for all $k_0 \leq k < \ell$ and $0 < i < \ell - k$. Furthermore, Theorem 6.2.4 establishes that the cubic overpartition function $\overline{a}(n)$ satisfies strict log-concave for $n \geq 10$. Thus, by substituting $k = n - m,\ \ell = n + m$ and $i = m$, we obtain

$$\overline{a}(n)^2 > \overline{a}(n - m)\overline{a}(n + m) \tag{6.3.57}$$

for all $n > m > 1$ and $n - m \geq 10$. Accordingly, it suffices to establish (6.3.57) in the remaining cases, i.e., $1 \leq n - m \leq 9$. To resolve this, we aim to show that

$$\overline{a}(n)^2 \geq \overline{a}(m+1)^2 > \overline{a}(9)\overline{a}(9 + 2m) \geq \overline{a}(n - m)\overline{a}(n + m), \tag{6.3.58}$$

holds for all $1 \leq n - m \leq 9$ and $m \geq 37$. The remaining finitely many cases, namely those with $1 \leq n - m \leq 9$ and $m < 37$ can be checked numerically. Since $n \geq m + 1$, it follows immediately that

$$\overline{a}(n)^2 \geq \overline{a}(m+1)^2.$$

Moreover, as $n - m \leq 9$ and $n + m \leq 9 + 2m$, so we have

$$\overline{a}(9)\overline{a}(9 + 2m) \geq \overline{a}(n - m)\overline{a}(n + m).$$

Thus, both the first and third inequalities in (6.3.58) hold trivially. It remains to

establish that

$$\overline{a}(m+1)^2 > \overline{a}(9)\overline{a}(9+2m), \tag{6.3.59}$$

for all $m \geq 37$. Taking logarithm on both sides of (6.3.59), we see that it is equivalent to showing

$$2\log(\overline{a}(m+1)) - \log(\overline{a}(9)) - \log(\overline{a}(9+2m)) > 0, \tag{6.3.60}$$

for all $m \geq 37$. To address this, we employ a lower and upper bound for $\overline{a}(n)$ derived in (6.3.50) and (6.3.51), respectively. Utilizing these estimates together with the fact that $\overline{a}(9) = 470$, we obtain that the left-hand side of (6.3.60) is bounded below by

$$\begin{aligned} &2\log\left(\frac{3^{\frac{3}{4}}}{(m+1)^{\frac{5}{4}}2^{\frac{19}{4}}}\right) + 2\pi\sqrt{\frac{3(m+1)}{2}} + 2\log\left(1 - \frac{8}{5\sqrt{m+1}}\right) - \log(470) \\ &\quad - \log\left(\frac{3^{\frac{3}{4}}}{(9+2m)^{\frac{5}{4}}2^{\frac{19}{4}}}\right) - \pi\sqrt{\frac{3(9+2m)}{2}} - \log\left(1 + \frac{1}{\sqrt{9+2m}}\right), \end{aligned}$$

for all $m \geq 10$. A straightforward calculation shows that this expression is positive for all $m \geq 40$, thereby proving (6.3.60) in this range. For the remaining values of m, namely, for $m = 37, 38, 39$, the inequality (6.3.59) can be verified numerically using Mathematica. Thus, (6.3.59) holds for all $m \geq 37$, which establishes (6.3.58). This completes the proof of Theorem 6.2.7. □

6.4 Numerical verification of Theorem 6.2.1

To illustrate the accuracy of our exact formula for $\overline{a}(n)$, we provide a numerical verification by computing explicit values of $\overline{a}(n)$ using the first five terms of both series in the formula (6.2.1) and comparing them with the actual cubic overpartition counts.

Table 6.2: Verification of Theorem 6.2.1

n	Exact value	Value from our result
22	110012	110011.99958
12	2020	2020.0026
18	24962	24961.9983
87	1166034258272	1166034258271.996

Further, we compute $\overline{a}(100)$ numerically by considering contributions of the first five terms of both series present in (6.2.1):

13080870093246.877

$$
\begin{aligned}
&+957724.348\\
&-49.363\\
&-0.170\\
&+0.203\\
&+0.005\\
&+0.040\\
&+0.001\\
&+0.001\\
&+0.001\\
\hline
&= 13080871050921.943
\end{aligned}
$$

The exact value $\overline{a}(100) = 13080871050922$.

Chapter 7

RADEMACHER-TYPE EXACT FORMULA AND HIGHER ORDER TURÁN INEQUALITIES FOR r-COLORED ℓ-REGULAR PARTITIONS

All the results presented in this chapter are taken from our work [3]. In this chapter, we employ the circle method to obtain a Rademacher-type exact formula for r-colored ℓ-regular partitions for any $r \in \mathbb{N}$ and $\ell \geq 2$. As an application, we derive higher order Turán inequalities for the r-colored ℓ-regular partition function using a result of Griffin, Ono, Rolen, and Zagier. Furthermore, as additional consequences, we establish Rademacher-type exact formulas and higher order Turán inequalities for the r-colored distinct part partition function and for the sum of minimal excludants over ordinary partitions and overpartitions.

7.1 Preliminaries

The first result in this section gives Weil's bound for Kloosterman's sum. This bound plays a crucial role in proving the convergence of an infinite series.

Lemma 7.1.1. *Let a, b, c be integers with $c > 0$. Then we have*

$$\sideset{}{'}\sum_{x \bmod c} e^{2\pi i \frac{ax+b\bar{x}}{c}} = \mathcal{O}(c^{\frac{1}{2}} d(c) \gcd(a, b, c)^{\frac{1}{2}}),$$

where $\bar{x}$ denotes the multiplicative inverse of x modulo c, i.e. $x\bar{x} \equiv 1 \pmod{c}$ and

$d(c)$ denotes the number of divisors of c. Here, the symbol $\sum'$ indicates that the summation is restricted to a reduced residue system modulo c.

Proof. We refer [78, Corollary 11.12] for the proof of this result. □

Prior to stating the next result, we introduce several notations that will be used in the subsequent discussion. First, we define

$$\omega(h,k) = e^{\pi i s(h,k)}.$$

We then set

$$\omega(h,k,\ell,r) = \left[\frac{\omega(h,k)}{\omega\left(\frac{\ell h}{Q}, \frac{k}{Q}\right)}\right]^r, \tag{7.1.1}$$

where $Q = \gcd(k,\ell)$. Now one can write $k = QK$ and $\ell = QT$ with $\gcd(K,T) = 1$. Then the above expression becomes

$$\omega(h,k,\ell,r) = \left[\frac{\omega(h,k)}{\omega\left(Th,K\right)}\right]^r. \tag{7.1.2}$$

Our objective is to estimate a sum involving $\omega(h,k,\ell,r)$. To this end, we begin with the following result associated to $\omega(h,k)$, the proof of which can be found in [67].

Proposition 7.1.2. *If k is odd, then*

$$\omega(h,k) = \left(\frac{h}{k}\right) i^{(k-1)/2} e^{\frac{2\pi i p(h-h')}{gk}}. \tag{7.1.3}$$

If k is even, then

$$\omega(h,k) = \left(\frac{k}{h}\right) i^{b(k+1)/2} e^{\frac{2\pi i p(h-h')}{gk}}. \tag{7.1.4}$$

Here

$$g = \begin{cases} \gcd(3,k) & \text{if } k \text{ odd}, \\ 8\gcd(3,k) & \text{if } k \text{ even}, \end{cases}$$

and h' satisfies $hh' \equiv -1 \pmod{gk}$, and p satisfies the relation $fp \equiv 1 \pmod{gk}$ where $f = \frac{24}{g}$. In (7.1.4), $b \equiv h' \pmod 8$. The symbol $\left(\frac{h}{k}\right)$ denotes the Jacobi symbol.

Our goal is now to derive an analogue of Proposition 7.1.2 for $\omega(h,k,\ell,r)$. To accomplish this, we utilize basic properties of the Jacobi symbol and observe that the Proposition 7.1.2 remains valid when parameters h, h', k, g, f, p, b are replaced by Th, h^*, K, G, F, P, B, respectively. The proof is divided into three cases.

Case 1: Suppose k is odd, then K is necessarily odd as well. From (7.1.2) and (7.1.3), we obtain

$$\omega(h,k,\ell,r) = \left[\frac{\left(\frac{h}{k}\right) i^{(k-1)/2} e^{\frac{2\pi i p(h-h')}{gk}}}{\left(\frac{Th}{K}\right) i^{(K-1)/2} e^{\frac{2\pi i P(Th-h^*)}{GK}}}\right]^r. \tag{7.1.5}$$

Here

$$g=\gcd(3,k),\quad hh'\equiv -1 \pmod{gk},\quad fp\equiv 1 \pmod{gk},\quad f=24/g,$$
$$G=\gcd(3,K),\quad Thh^*\equiv -1 \pmod{GK},\quad FP\equiv 1 \pmod{GK},\quad F=24/G.$$

Choosing T' such that $TT'\equiv 1 \pmod{GK}$, we deduce that $h^*\equiv T'h' \pmod{GK}$. Note that

$$\begin{aligned}
&Thh^*\equiv -1 \pmod{GK}\\
&\Rightarrow T'Thh^*\equiv -T' \pmod{GK}\\
&\Rightarrow hh^*\equiv -T' \pmod{GK} \qquad (\text{since } TT'\equiv 1 \pmod{GK})\\
&\Rightarrow h'hh^*\equiv -T'h' \pmod{GK}\\
&\Rightarrow h^*\equiv T'h' \pmod{GK} \qquad (\text{since } hh'\equiv -1 \pmod{gk} \text{ and } GK\mid gk).
\end{aligned}$$

If $g=JG$ with $J=1$ or 3, then $F=\frac{24}{G}=\frac{24J}{g}=Jf$. One can see that $P\equiv Ap \pmod{GK}$, where $JA\equiv 1 \pmod{GK}$. Moreover, using properties of Jacobi symbol, we have

$$\frac{\left(\frac{h}{k}\right)}{\left(\frac{Th}{K}\right)}=\frac{\left(\frac{h}{QK}\right)}{\left(\frac{Th}{K}\right)}=\frac{\left(\frac{h}{Q}\right)\left(\frac{h}{K}\right)}{\left(\frac{T}{K}\right)\left(\frac{h}{K}\right)}=\frac{\left(\frac{h}{Q}\right)}{\left(\frac{T}{K}\right)}.$$

Hence, substituting the above expression in (7.1.5) and upon simplification, we obtain that

$$\omega(h,k,\ell,r)=\left[\frac{\left(\frac{h}{Q}\right)}{\left(\frac{T}{K}\right)}\right]^r i^{\frac{r(k-K)}{2}}e^{\frac{2\pi ipr(Uh+Vh')}{gk}},$$

where

$$U=1-JA\ell,\quad V=JAT'Q-1. \tag{7.1.6}$$

The term $\left[\left(\frac{h}{Q}\right)/\left(\frac{T}{K}\right)\right]^r$ has absolute value one and depends only on k and ℓ when $h\equiv a \pmod{Q}$ with $\gcd(a,Q)=1$.

Case 2: Suppose that both k and K are even. Then, from (7.1.2) and (7.1.4), we have

$$\omega(h,k,\ell,r)=\left[\frac{\left(\frac{k}{h}\right)i^{b(k+1)/2}e^{\frac{2\pi ip(h-h')}{gk}}}{\left(\frac{K}{Th}\right)i^{B(K+1)/2}e^{\frac{2\pi iP(Th-h^*)}{GK}}}\right]^r.$$

The parameters satisfy

$$g=8\gcd(3,k),\quad hh'\equiv -1(\text{mod } gk),\quad fp\equiv 1(\text{mod } gk),\quad f=\frac{24}{g},\quad b\equiv h'(\text{mod } 8),$$
$$G=8\gcd(3,K),\quad Thh^*\equiv -1(\text{mod } GK),\quad FP\equiv 1(\text{mod } GK),\quad F=\frac{24}{G},\quad B\equiv h^*(\text{mod } 8).$$

Let us consider T' such that $TT'\equiv 1 \pmod{GK}$, then one can check that $h^*\equiv T'h' \pmod{GK}$ and hence $B\equiv h^*\equiv T'b \pmod 8$ since $8\mid G$. If $g=JG$ ($J=1$ or 3) then $F=Jf$ and $P\equiv Ap \pmod{GK}$ where A is defined as in Case 1. Writing $Q=2^\alpha Q^*$

with $\alpha \geq 0$ and Q^* odd, we observe that

$$\frac{\left(\frac{k}{h}\right)}{\left(\frac{K}{Th}\right)} = \frac{\left(\frac{QK}{h}\right)}{\left(\frac{K}{Th}\right)} = \frac{\left(\frac{Q}{h}\right)\left(\frac{K}{h}\right)}{\left(\frac{K}{T}\right)\left(\frac{K}{h}\right)} = \frac{\left(\frac{Q}{h}\right)}{\left(\frac{K}{T}\right)} = \frac{\left(\frac{2^\alpha}{h}\right)\left(\frac{Q^*}{h}\right)}{\left(\frac{K}{T}\right)} = \frac{\left(\frac{2^\alpha}{h}\right)\left(\frac{h}{Q^*}\right)}{\left(\frac{K}{T}\right)}(-1)^{\frac{(h-1)(Q^*-1)}{4}},$$

where

$$\left(\frac{2^\alpha}{h}\right) = \begin{cases} 1, & \text{if } \alpha \text{ is even,} \\ (-1)^{(h^2-1)/8}, & \text{if } \alpha \text{ is odd.} \end{cases}$$

Note that h is odd since $hh' \equiv -1 (\text{mod } 8)$. Therefore, we finally have

$$\omega(h,k,\ell,r) = \left[\frac{\left(\frac{2^\alpha}{h}\right)\left(\frac{h}{Q^*}\right)}{\left(\frac{K}{T}\right)}\right]^r (-1)^{\frac{r(h-1)(Q^*-1)}{4}} i^{\frac{rb}{2}\{(k+1)-T'(K+1)\}} e^{\frac{2\pi i p r(Uh+Vh')}{gk}},$$

with U and V are defined as in (7.1.6). Note that the quotients of Jacobi symbols in the above expression has absolute value one and depends only on k and ℓ under the constraints $h \equiv a \pmod{Q}$ where $\gcd(a,Q) = 1$, and $h \equiv d \pmod 8$ with d odd.

Case 3: The final case remains when k is even and K is odd. Proceeding along the same line as we did in **Case 1** and **Case 2**, one can show that

$$\omega(h,k,\ell,r) = \left[\frac{\left(\frac{k}{h}\right) i^{b(k+1)/2} e^{\frac{2\pi i p(h-h')}{gk}}}{\left(\frac{Th}{K}\right) i^{(K-1)/2} e^{\frac{2\pi i P(Th-h^*)}{GK}}}\right]^r,$$

where

$$g = 8\gcd(3,k), \quad 7hh' \equiv -1 (\text{mod } gk), \qquad fp \equiv 1 (\text{mod } gk), \qquad f = 24/g, b \equiv h' (\text{mod } 8)$$
$$G = \gcd(3,K), \quad Thh^* \equiv -1 (\text{mod } GK), \quad FP \equiv 1 (\text{mod} GK), \quad F = 24/G.$$

Letting T' such that $TT' \equiv 1 \pmod{GK}$, we can show that $h^* \equiv T'h' \pmod{GK}$. Write $g = JG$ ($J = 8$ or 24), then $F = Jf$ and $P \equiv Ap \pmod{GK}$ where $JA \equiv 1 \pmod{GK}$. Writing $Q = 2^\alpha Q^*$, with Q^* odd, we see that

$$\frac{\left(\frac{k}{h}\right)}{\left(\frac{Th}{K}\right)} = \frac{\left(\frac{QK}{h}\right)}{\left(\frac{Th}{K}\right)} = \frac{\left(\frac{Q}{h}\right)\left(\frac{K}{h}\right)}{\left(\frac{T}{K}\right)\left(\frac{h}{K}\right)} = \frac{\left(\frac{Q}{h}\right)}{\left(\frac{T}{K}\right)}(-1)^{(h-1)(K-1)/4} = \frac{\left(\frac{2^\alpha}{h}\right)\left(\frac{Q^*}{h}\right)}{\left(\frac{T}{K}\right)}(-1)^{(h-1)(K-1)/4}$$
$$= \frac{\left(\frac{2^\alpha}{h}\right)\left(\frac{h}{Q^*}\right)}{\left(\frac{T}{K}\right)}(-1)^{\frac{(h-1)(K-Q^*)}{4}}.$$

Using the above expression, we conclude that

$$\omega(h,k,\ell,r) = \left[\frac{\left(\frac{2^\alpha}{h}\right)\left(\frac{h}{Q^*}\right)}{\left(\frac{T}{K}\right)}\right]^r (-1)^{\frac{r(h-1)(K-Q^*)}{4}} i^{br(k+1)/2 - r(K-1)/2} e^{\frac{2\pi i p r(Uh+Vh')}{gk}},$$

where U and V are defined as in (7.1.6). The expression $\left[\frac{\left(\frac{2^\alpha}{h}\right)\left(\frac{h}{Q^*}\right)}{\left(\frac{T}{K}\right)}\right]^r$ has absolute value one and depends only on k, ℓ and if $h \equiv a \pmod{Q}$ where $\gcd(a,Q) = 1$, and $h \equiv d \pmod 8$ where d is odd.

We now summarize the above three cases in the following proposition.

Proposition 7.1.3. *Let $\omega(h,k,\ell,r)$ be the function defined in* (7.1.1). *Considering the variables defined above, we have*

$$\omega(h,k,\ell,r) = A(h,k,\ell,r)e^{2\pi iqr\left(\frac{Uh+Vh'}{gk}\right)},$$

where $A(h,k,\ell,r)$ is some complex number with $|A(h,k,\ell,r)| = 1$ and it depends only on k and ℓ if $h \equiv a \pmod{Q}$ where $\gcd(a,Q) = 1$ and also, if k is even, $h \equiv d \pmod 8$ where d is odd. If $g = JG$ ($J = 1,3,8,24$) and $JA \equiv 1 \pmod{GK}$, then $U = 1 - JA\ell$ and $V = JAT'Q - 1$.

We are now prepared to state a bound for a Kloosterman-type sum that will be employed in the subsequent analysis.

Theorem 7.1.4. *Let $\gcd(k,\ell) = Q$, and suppose $h \equiv a \pmod{Q}$, $\gcd(a,Q) = 1$. Assume further that $hh' \equiv -1 \pmod k$ and that $s_1 \le h' \pmod k < s_2$ for integers satisfying $0 \le s_1 < s_2 \le k$. Let $t \le n$; and M is a fixed integer, then the exponential sum*

$$X := W_{\ell,r,M}(n,k) = \sideset{}{'}\sum_{h \bmod k} \omega(h,k,\ell,r)e^{-\frac{2\pi i(hn-h'M)}{k}}$$

admits the estimate

$$X := W_{\ell,r,M}(n,k) = \mathcal{O}\left(n^{1/2}k^{1/2+\epsilon}\right),$$

where the implied constant in the $\mathcal{O}$-term depends only on ℓ. Here, the symbol $\sum'$ indicates that the summation is restricted to a reduced residue system modulo k, possibly subject to additional stated conditions.

Proof. The proof of this result follows the same line of reasoning as that given by Hagis [68, p. 378]. However, for the sake of completeness and clarity, we provide a brief outline below. Using a property of the Dedekind sum, one can see that the function $\omega(h,k,\ell,r)$ has period k when viewed as a function of h. If we change the sum over mod k in X to mod gk and select h' so that $hh' \equiv -1 (\bmod\ gk)$, then employing Proposition 7.1.3, we obtain

$$X = g^{-1}\sideset{}{'}\sum_{h \bmod gk} A(h,k,\ell,r)e^{\frac{2\pi if(h)}{gk}},$$

where $f(h) = (qrU - gn)h + (qrV + gM)h'$. If k is even, we split X into four parts, X_1, X_3, X_5, X_7, so that in X_d we have $h \equiv d \pmod 8$ as well as $h \equiv a \pmod Q$. Therefore, $X = X_1 + X_3 + X_5 + X_7$, where

$$X_d = A_d\sideset{}{'}\sum_{h \bmod gk} e^{\frac{2\pi if(h)}{gk}} \tag{7.1.7}$$

with $|A_d| = g^{-1} \leq 1$. If k is odd, then (7.1.7) holds if we identify X with X_d and ignore the restriction $h \equiv d \pmod 8$. For two fixed integers s_1 and s_2, let us define a function $\beta(s)$ as follows:

$$\beta(s) = \begin{cases} 1, & \text{if } s_1 \leq s \pmod k < s_2, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\beta(s)$ is a periodic function with period k. Thus, $\beta(s)$ can be written as finite Fourier series,

$$\beta(s) = \sum_{j=0}^{k-1} \beta_j e^{\frac{2\pi isj}{k}}, \quad \text{where} \quad \beta_j = \frac{1}{k} \sum_{s=0}^{k-1} \beta(s) e^{\frac{-2\pi isj}{k}}.$$

One can prove that $\sum_{j=0}^{k-1} \beta_j = \mathcal{O}(k^{\varepsilon})$ for any $\varepsilon > 0$. We can now drop the restriction $s_1 \leq h' \pmod k < s_2$ and write

$$\begin{aligned} X_d &= A_d \sum_{h \bmod gk}{}' \beta(h') e^{\frac{2\pi i f(h)}{gk}} \\ &= A_d \sum_{j=0}^{k-1} \beta_j \sum_{h \bmod gk}{}' e^{\frac{2\pi i \left((qrU - gn)h + (qrV + gM + gj)h' \right)}{gk}}. \end{aligned}$$

If k is odd, then $\sum'$ represents a Kloosterman sum. When k is even, we write $Q = 2^{\alpha} Q^*$, where $\alpha \geq 0$ and Q^* is odd. It is straightforward to verify that if $\alpha \leq 1$, the two congruence conditions

$$\text{(I) } h \equiv a \pmod Q \quad \text{and} \quad \text{(II) } h \equiv d \pmod 8$$

are jointly equivalent to a single condition of the form

$$h \equiv a^* \pmod{8Q^*}.$$

If $\alpha = 2$ and $a \not\equiv d \pmod 4$, then the sum $\sum'$ is empty. However, if $a \equiv d \pmod 4$, conditions (I) and (II) again combine into a single congruence

$$h \equiv a^* \pmod{8Q^*}.$$

For $\alpha \geq 3$, if $a \not\equiv d \pmod 8$, the sum $\sum'$ is empty; whereas if $a \equiv d \pmod 8$, conditions (I) and (II) reduce to

$$h \equiv a \pmod Q.$$

Hence, in all cases, either $\sum'$ is empty or it is a Kloosterman sum. Finally, applying Weil's bound for Kloosterman sums i.e., Lemma 7.1.1, we obtain

$$X = \mathcal{O}(n^{1/2} k^{1/2+\epsilon}).$$

□

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7.2 Main Results

We divide this section in two folds. First, we present a Rademacher-type exact formula for the r-colored ℓ-regular partition function along with its applications. At the end, we discuss higher order Turán inequalities for r-colored ℓ-regular partition function and their corresponding applications.

The first result gives a Rademacher-type exact formula for r-colored ℓ-regular partitions.

Theorem 7.2.1. *Let $r \geq 1, \ell \geq 2$ be two positive integers and $b_\ell^{(r)}(n)$ be the number of r-colored ℓ-regular partitions of n and $p^{(r)}(n)$ be the number r-colored ordinary partitions of n. Let $a^{(r)}(n)$ be the coefficient of q^n in the power series expansion of $(q)_\infty^r$. An exact formula for $b_\ell^{(r)}(n)$ is given by*

$$b_\ell^{(r)}(n) = \sum_{\substack{Q|\ell \\ Q^2<\ell}} \left(\frac{Q}{\ell}\right)^{\frac{r}{2}} \sum_{\substack{k=1 \\ (k,\ell)=Q}}^{\infty} \sum_{m,s=0}^{\left\lfloor \frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor}{}' p^{(r)}(m)a^{(r)}(s)C(n)$$
$$\times \frac{\mathrm{d}}{\mathrm{d}n} J_0\left(\frac{4i\pi}{k}\sqrt{\left(\frac{r}{24}\left(1-\frac{Q^2}{\ell}\right)-\left(m+\frac{Q^2 s}{\ell}\right)\right)\left(n+\frac{r}{24}(\ell-1)\right)}\right), \quad (7.2.1)$$

where

$$C(n) := C_{k,r,\ell,Q,m,s}(n) = \sum_{\substack{h \bmod k \\ \gcd(h,k)=1}} e^{i\pi r(s(h,k)-s(\frac{\ell h}{Q},\frac{k}{Q}))+\frac{2\pi i}{k}(mh'+sQh_Q-nh)},$$

with $hh' \equiv -1 (\mathrm{mod}\ k)$ and $\frac{\ell}{Q}hh_Q \equiv -1 (\mathrm{mod}\ \frac{k}{Q})$. Here, the $\sum'$ means the sum is running over all those m and s, which satisfy $m + \frac{Q^2 s}{\ell} < \frac{r}{24}\left(1-\frac{Q^2}{\ell}\right)$.

As an immediate implication of the above result, we have the following asymptotic result for $b_\ell^{(r)}(n)$.

Corollary 7.2.2. *As $n \to \infty$, we have*

$$b_\ell^{(r)}(n) \sim \frac{1}{\sqrt{2}}\left(\frac{1}{\ell}\right)^{\frac{r}{2}}\left(\frac{r}{24}\left(1-\frac{1}{\ell}\right)\right)^{\frac{1}{4}}\left(\frac{1}{n}\right)^{\frac{3}{4}} e^{\left(4\pi\sqrt{\frac{nr}{24}\left(1-\frac{1}{\ell}\right)}\right)}. \quad (7.2.2)$$

Substituting $r = 1$ in Theorem 7.2.1, we recover a Rademacher-type exact formula for ℓ-regular partitions obtained by Hagis [68, Theorem 6].

Corollary 7.2.3. *Let $b_\ell(n)$ be the number of ℓ-regular partitions of n. An exact*

formula for $b_\ell(n)$ is given by

$$b_\ell(n) = \sum_{\substack{Q|\ell \\ Q^2<\ell}} \left(\frac{Q}{\ell}\right)^{\frac{1}{2}} \sum_{\substack{k=1 \\ (k,\ell)=Q}}^{\infty} {\sum_{m,s=0}^{\left\lfloor \frac{1}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor}}' p(m)a^{(1)}(s)D(n)$$
$$\times \frac{\mathrm{d}}{\mathrm{d}n} J_0\left(\frac{4i\pi}{k}\sqrt{\left(\frac{1}{24}\left(1-\frac{Q^2}{\ell}\right)-\left(m+\frac{Q^2 s}{\ell}\right)\right)\left(n+\frac{1}{24}(\ell-1)\right)}\right),$$

where

$$D(n) := C_{k,1,\ell,Q,m,s}(n) = \sum_{\substack{h \bmod k \\ \gcd(h,k)=1}} e^{i\pi\left(s(h,k)-s\left(\frac{\ell h}{Q},\frac{k}{Q}\right)\right)+\frac{2\pi i}{k}(mh'+sQh_Q-nh)},$$

with $hh' \equiv -1 (\mathrm{mod}\ k)$ and $\frac{\ell}{Q}hh_Q \equiv -1 (\mathrm{mod}\ \frac{k}{Q})$. Here, the $\sum'$ means the sum is running over all those m, s which satisfy $m + \frac{Q^2 s}{\ell} < \frac{1}{24}\left(1-\frac{Q^2}{\ell}\right)$.

Further, letting $\ell = 2$ in Theorem 7.2.1, we obtain the following Rademacher-type exact formula for r-colored distinct part partition function.

Corollary 7.2.4. *Let $p_d^{(r)}(n)$ be the number of r-colored distinct part partitions of n, where parts are distinct but allowed to appear in r different colors. A Rademacher-type exact formula for $p_d^{(r)}(n)$ is given by*

$$p_d^{(r)}(n) = \frac{1}{2^{\frac{r}{2}}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \sum_{\substack{m,s=0 \\ 2m+s<\frac{r}{24}}}^{\lfloor \frac{r}{24} \rfloor} p^{(r)}(m)a^{(r)}(s)E(n)\frac{\mathrm{d}}{\mathrm{d}n} J_0\left(\frac{4\pi i}{k}\sqrt{\left(\frac{r}{48}-\frac{s}{2}-m\right)\left(\frac{r}{24}+n\right)}\right),$$

where

$$E(n) := C_{k,r,2,1,m,s}(n) = \sum_{\substack{h \bmod k \\ \gcd(h,k)=1}} e^{i\pi r(s(h,k)-s(2h,k))+\frac{2\pi i}{k}(sh_1+mh'-nh)},$$

with $hh' \equiv -1 (\mathrm{mod}\ k)$, $2hh_1 \equiv -1 (\mathrm{mod}\ k)$.

Remark 14. *Substituting $r = 1$ in Corollary 7.2.4, we can easily recover the exact formula (1.2.5) for the distinct part partition function $p_d(n)$.*

Remark 15. *Letting $r = 2$ in Corollary 7.2.4, we get back a result of Grabner and Knopfmacher [56, Theorem 3], namely, a Rademacher-type exact formula for a restricted partition function which is the sum of the smallest gaps in all unrestricted partitions. This aforementioned function was later studied by Andrews and Newmann [7] and named as $\sigma mex(n)$, sum of the minimal excludants over all the partitions of n. Mainly, Grabner and Knopfmacher showed that*

$$\sigma mex(n) = \frac{\pi}{2\sqrt{6}\sqrt{n+\frac{1}{12}}} \sum_{k=1}^{\infty} \frac{F_{2k-1}(n)}{2k-1} I_1\left(\frac{\pi}{2k-1}\sqrt{\frac{2}{3}\left(n+\frac{1}{12}\right)}\right),$$

where I_1 denotes the modified Bessel function of the first kind and

$$F_k(n) := C_{k,2,2,1,0,0}(n) = \sum_{\substack{h \bmod k \\ \gcd(h,k)=1}} e^{2\pi i(s(h,k)-s(2h,k))-2n\pi i\frac{h}{k}}.$$

Letting $r = 3$ in Corollary 7.2.4, we obtain an exact formula for $\sigma\overline{mex}(n)$, which is the sum of minimal excludants over all the overpartitions of n, studied by Aricheta and Donato [9, Definition 1].

Corollary 7.2.5. *A Rademacher-type exact formula for $\sigma\overline{mex}(n)$ is*

$$\sigma\overline{mex}(n) = \frac{\pi}{4\sqrt{2}\sqrt{n+\frac{1}{8}}} \sum_{k=1}^{\infty} \frac{G_{2k-1}(n)}{2k-1} I_1\left(\frac{\pi}{2k-1}\sqrt{n+\frac{1}{8}}\right), \tag{7.2.3}$$

where

$$G_k(n) := C_{k,3,2,1,0,0}(n) = \sum_{\substack{h \bmod k \\ \gcd(h,k)=1}} e^{3\pi i(s(h,k)-s(2h,k))-2n\pi i\frac{h}{k}}.$$

The final result gives higher order Turán inequalities for the r-colored ℓ-regular partition function $b_\ell^{(r)}(n)$.

Theorem 7.2.6. *For any positive integer $d \geq 1$, $J^{d,n}_{b_\ell^{(r)}}(X)$ is hyperbolic for all but finitely many values of n.*

7.3 Proof of main results

Proof of Theorem 7.2.1. Let $b_\ell^{(r)}(n)$ be the number of r-colored ℓ-regular partitions of n. From the generating function (2.5.2) for $b_\ell^{(r)}(n)$ and use of Cauchy's integral formula gives

$$b_\ell^{(r)}(n) = \frac{1}{2\pi i}\int_C \frac{R(q)}{q^{n+1}}\mathrm{d}q,$$

where $R(q) = \frac{F(q)}{F(q^\ell)}$ and $F(q) = f(q)^r$ and $C : |q| = r' < 1$. Substitute $q = e^{2\pi i\tau}$ with $\mathrm{Im}(\tau) > 0$. Then, we have

$$b_\ell^{(r)}(n) = \int_{\tau_0}^{\tau_0+1} R(e^{2\pi i\tau})e^{-2\pi in\tau}\mathrm{d}\tau, \tag{7.3.1}$$

where $\tau_0 \in \mathbb{H}$ and any path from τ_0 to $\tau_0 + 1$ is allowed. The path we choose is same as Rademacher's path based on Farey dissection, interested reader can see [100] for more details.

We construct Ford circles belonging to Farey sequence of order N. For the Ford circle $C(h,k)$, we will choose arc $\gamma_{h,k}$ in such a way that it connects the tangency points α_1 and α_2 and it does not touches the real axis. Here we denote them as

$\frac{h}{k} + C'_{h,k}$ and $\frac{h}{k} + C''_{h,k}$, where

$$C'_{h,k} := -\frac{k_1}{k(k^2+k_1^2)} + i\frac{1}{k^2+k_1^2}, \quad \text{and} \quad C''_{h,k} := \frac{k_2}{k(k^2+k_2^2)} + i\frac{1}{k^2+k_2^2}.$$

Thus, considering the path of integration as the union of arcs $\gamma_{h,k}$, the integral (7.3.1) becomes

$$b_\ell^{(r)}(n) = \sum_{\substack{0\le h<k\le N\\ \gcd(h,k)=1}} \int_{\gamma_{h,k}} R(e^{2\pi i\tau}) e^{-2\pi i n\tau} \mathrm{d}\tau,$$

where $\frac{h}{k}$ is a fraction from Farey sequence of order N. Now, we substitute $\tau = \frac{h}{k} + \zeta$ to see that

$$b_\ell^{(r)}(n) = \sum_{\substack{0\le h<k\le N\\ \gcd(h,k)=1}} \int_{\zeta'_{h,k}}^{\zeta''_{h,k}} R\left(e^{2\pi i\left(\frac{h}{k}+\zeta\right)}\right) e^{-2\pi i n\left(\frac{h}{k}+\zeta\right)} \mathrm{d}\zeta.$$

We further substitute $\zeta = \frac{iz}{k^2}$. Under these substitutions, the circle $C(h,k) : \left|\tau - \left(\frac{h}{k} + \frac{1}{2k^2}\right)\right| = \frac{1}{2k^2}$ becomes $\left|z - \frac{1}{2}\right| = \frac{1}{2}$. Since $z = \frac{k^2}{i}\zeta$ and z varies from $z'_{h,k}$ to $z''_{h,k}$ with

$$z'_{h,k} = \frac{k^2}{k^2+k_1^2} + \frac{ikk_1}{k^2+k_1^2},$$

and

$$z''_{h,k} = \frac{k^2}{k^2+k_2^2} - \frac{ikk_2}{k^2+k_2^2}.$$

Thus, the integral becomes

$$b_\ell^{(r)}(n) = \sum_{\substack{0\le h<k\le N\\ \gcd(h,k)=1}} \frac{i}{k^2} e^{-2\pi i n\frac{h}{k}} \int_{z'_{h,k}}^{z''_{h,k}} R\left(e^{2\pi i\left(\frac{h}{k}+\frac{iz}{k^2}\right)}\right) e^{2\pi n\frac{z}{k^2}} \mathrm{d}z.$$

Now, we employ the transformation formula for $R\left(e^{2\pi i\left(\frac{h}{k}+\frac{iz}{k^2}\right)}\right)$ i.e. Lemma 2.5.2 in the above integral to see that

$$b_\ell^{(r)}(n) = \sum_{Q|\ell} \left(\frac{Q}{\ell}\right)^{\frac{r}{2}} \sum_{\substack{0\le h<k\le N\\ \gcd(h,k)=1\\ \gcd(k,\ell)=Q}} \frac{i}{k^2} e^{i\pi r\left(s(h,k)-s\left(\frac{\ell h}{Q},\frac{k}{Q}\right)\right)-2n\pi i\frac{h}{k}}$$

$$\times \int_{z'_{h,k}}^{z''_{h,k}} e^{\left(\frac{\pi r}{12z}\left(1-\frac{Q^2}{\ell}\right)+\frac{\pi r z}{12k^2}(\ell-1)+2\pi n\frac{z}{k^2}\right)} \frac{F\left(e^{2\pi i\left(\frac{h'}{k}+\frac{i}{z}\right)}\right)}{F\left(e^{2\pi i Q\left(\frac{h_Q}{k}+\frac{iQ}{\ell z}\right)}\right)} \mathrm{d}z.$$

Next, we make $\mathrm{Re}(z)$ sufficiently small so that $\mathrm{Re}\left(\frac{1}{z}\right)$ becomes large. This necessitates splitting the series into two parts depending whether $\left|e^{\frac{\pi r}{12z}\left(1-\frac{Q^2}{\ell}\right)}\right| \to 0$ or not. This

behaviour is determined by whether $1-\frac{Q^2}{\ell}$ is negative or not. Thus, we write

$$b_\ell^{(r)}(n) = T_1 + T_2, \tag{7.3.2}$$

where T_1 is the sum with $Q^2 < \ell$ and T_2 is with $Q^2 \geq \ell$. First, we shall evaluate T_2. We know $F(q) = (q)_\infty^{-r}$. Hence, for $\mathrm{Re}(z) \to 0$, one can see that

$$F\left(e^{2\pi i\left(\frac{h'}{k}+\frac{i}{z}\right)}\right) = \sum_{m=0}^{\infty} p^{(r)}(m) e^{2m\pi i\left(\frac{h'}{k}+\frac{i}{z}\right)} = 1 + \mathcal{O}\left(e^{-2\pi\,\mathrm{Re}(1/z)}\right) = \mathcal{O}(1).$$

Similarly, one can check that

$$\frac{1}{F\left(e^{2\pi iQ\left(\frac{h_Q}{k}+\frac{iQ}{\ell z}\right)}\right)} = \mathcal{O}(1).$$

From Theorem 7.1.4, we have

$$\sum_{\substack{0\leq h<k\leq N\\ \gcd(h.k)=1\\ \gcd(k,\ell)=Q}} e^{i\pi r\left(s(h,k)-s\left(\frac{\ell h}{Q},\frac{k}{Q}\right)\right)-2n\pi i\frac{h}{k}} = \mathcal{O}\left(n^{\frac{1}{2}}k^{\frac{1}{2}+\epsilon}\right).$$

Utilizing the above bounds, we get

$$\begin{aligned} T_2 &= \sum_{\substack{Q|\ell\\ Q^2\geq\ell}} \left(\frac{Q}{\ell}\right)^{\frac{r}{2}} \sum_{\substack{0\leq h<k\leq N\\ \gcd(h,k)=1\\ \gcd(k,\ell)=Q}} \frac{i}{k^2} e^{i\pi r\left(s(h,k)-s\left(\frac{\ell h}{Q},\frac{k}{Q}\right)\right)} e^{-2n\pi i\frac{h}{k}} \\ &\quad \times \int_{z'_{h,k}}^{z''_{h,k}} e^{\frac{\pi r}{12z}\left(1-\frac{Q^2}{\ell}\right)+\frac{\pi r z}{12k^2}(\ell-1)+2\pi n\frac{z}{k^2}} \frac{F\left(e^{2\pi i\left(\frac{h'}{k}+\frac{i}{z}\right)}\right)}{F\left(e^{2\pi iQ\left(\frac{h_Q}{k}+\frac{iQ}{\ell z}\right)}\right)} \mathrm{d}z. \\ &= \mathcal{O}\left(\sum_{k=1}^{N} \frac{\left(n^{\frac{1}{2}}k^{\frac{1}{2}+\epsilon}\right)}{k^2} \int_{z'_{h,k}}^{z''_{h,k}} e^{\frac{\pi r}{12k^2}(\ell-1)\mathrm{Re}(z)+\frac{2n\pi}{k^2}\mathrm{Re}(z)} \mathrm{d}z\right), \end{aligned}$$

since $\left|e^{-\frac{\pi r}{12}\left(\frac{Q^2}{\ell}-1\right)\mathrm{Re}\left(\frac{1}{z}\right)}\right| \leq 1$. The above integrand is regular, so we can choose any path joining $z'_{h,k}$ and $z''_{h,k}$. We consider the path as the chord from $z'_{h,k}$ to $z''_{h,k}$ and on this chord, we have $0 < \mathrm{Re}(z) \leq \frac{2k^2}{N^2}$, and length of the path of integration is less than $\frac{2\sqrt{2}k}{N}$. Thus, using these two bounds, we get

$$\begin{aligned} T_2 &= \mathcal{O}\left(\sum_{k=1}^{N} \frac{1}{k^{\frac{3}{2}-\epsilon}} e^{\frac{4\pi}{N^2}\left(n+\frac{r}{24}(\ell-1)\right)} \frac{2\sqrt{2}k}{N}\right) \\ &= \mathcal{O}\left(\frac{1}{N} e^{\frac{4\pi}{N^2}\left(n+\frac{r}{24}(\ell-1)\right)} \sum_{k=1}^{N} \frac{1}{k^{\frac{1}{2}-\epsilon}}\right). \end{aligned} \tag{7.3.3}$$

Now, our next aim is to evaluate T_1, the series corresponding to $Q^2 < \ell$. Note that

T_1 will give the main term of our result. We write

$$F\left(e^{2\pi i\left(\frac{h'}{k}+\frac{i}{z}\right)}\right) = \sum_{m=0}^{\left\lfloor \frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor} p^{(r)}(m)e^{2\pi im\left(\frac{h'}{k}+\frac{i}{z}\right)} + \sum_{m=\left\lfloor \frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor+1}^{\infty} p^{(r)}(m)e^{2\pi im\left(\frac{h'}{k}+\frac{i}{z}\right)}$$

$$= \sum_{m=0}^{\left\lfloor \frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor} p^{(r)}(m)e^{2\pi im\left(\frac{h'}{k}+\frac{i}{z}\right)} + \mathcal{O}\left(e^{-2\pi\left(\left\lfloor \frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor+1\right)\mathrm{Re}\left(\frac{1}{z}\right)}\right).$$

Further, we can write

$$\frac{1}{F\left(e^{2\pi iQ\left(\frac{h_Q}{k}+\frac{iQ}{\ell z}\right)}\right)} = \sum_{s=0}^{\infty} a_r(s)e^{2\pi isQ\left(\frac{h_Q}{k}+\frac{iQ}{\ell z}\right)}$$

$$= \sum_{s=0}^{\left\lfloor \frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor} a_r(s)e^{2\pi isQ\left(\frac{h_Q}{k}+\frac{iQ}{\ell z}\right)} + \sum_{s=\left\lfloor \frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor+1}^{\infty} a_r(s)e^{2\pi isQ\left(\frac{h_Q}{k}+\frac{iQ}{\ell z}\right)}$$

$$= \sum_{s=0}^{\left\lfloor \frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor} a_r(s)e^{2\pi isQ\left(\frac{h_Q}{k}+\frac{iQ}{\ell z}\right)} + \mathcal{O}\left(e^{\frac{-2\pi Q^2}{\ell}\left(\left\lfloor \frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor+1\right)\mathrm{Re}\left(\frac{1}{z}\right)}\right).$$

Thus, combining above two bounds, we get

$$\frac{F\left(e^{2\pi i\left(\frac{h'}{k}+\frac{i}{z}\right)}\right)}{F\left(e^{2\pi iQ\left(\frac{h_Q}{k}+\frac{iQ}{\ell z}\right)}\right)} = \sum_{m,s=0}^{\left\lfloor \frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor} p^{(r)}(m)a_r(s)e^{\frac{2\pi i}{k}(mh'+sQh_Q)}e^{\frac{-2\pi}{z}\left(m+\frac{Q^2}{\ell}s\right)}$$

$$+ \mathcal{O}\left(e^{\frac{-2\pi Q^2}{\ell}\left(\left\lfloor \frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor+1\right)\mathrm{Re}\left(\frac{1}{z}\right)}\right). \tag{7.3.4}$$

Note that

$$T_1 = \sum_{\substack{Q|\ell \\ Q^2<\ell}} \left(\frac{Q}{\ell}\right)^{\frac{r}{2}} \sum_{\substack{0\le h<k\le N \\ \gcd(h,k)=1 \\ \gcd(k,\ell)=Q}} \frac{i}{k^2} e^{i\pi r\left(s(h,k)-s\left(\frac{\ell h}{Q},\frac{k}{Q}\right)\right)}e^{-2n\pi i\frac{h}{k}}$$

$$\times \int_{z'_{h,k}}^{z''_{h,k}} e^{\frac{\pi r}{12z}\left(1-\frac{Q^2}{\ell}\right)+\frac{\pi rz}{12k^2}(\ell-1)+2\pi n\frac{z}{k^2}} \frac{F\left(e^{2\pi i\left(\frac{h'}{k}+\frac{i}{z}\right)}\right)}{F\left(e^{2\pi iQ\left(\frac{h_Q}{k}+\frac{iQ}{\ell z}\right)}\right)}\mathrm{d}z.$$

Substituting (7.3.4) in T_1 yields

$$T_1 = \mathcal{K}_1 + \mathcal{K}_2, \tag{7.3.5}$$

where

$$\mathcal{K}_1 := \sum_{\substack{Q|\ell \\ Q^2<\ell}} \left(\frac{Q}{\ell}\right)^{\frac{r}{2}} \sum_{\substack{0\le h<k\le N \\ \gcd(h,k)=1 \\ \gcd(k,\ell)=Q}} \frac{i}{k^2} e^{i\pi r\left(s(h,k)-s\left(\frac{\ell h}{Q},\frac{k}{Q}\right)\right)-2\pi i n\frac{h}{k}} \int_{z'_{h,k}}^{z''_{h,k}} e^{\left(\frac{\pi r}{12z}\left(1-\frac{Q^2}{\ell}\right)+\frac{\pi r z}{12k^2}(\ell-1)+2\pi n\frac{z}{k^2}\right)}$$

$$\times \sum_{m,s=0}^{\left\lfloor \frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor} p^{(r)}(m)a_r(s)e^{\frac{2\pi i}{k}(mh'+sQh_Q)} e^{\frac{-2\pi}{z}\left(m+\frac{Q^2}{\ell}s\right)}\mathrm{d}z, \qquad (7.3.6)$$

and

$$\mathcal{K}_2 := \sum_{\substack{Q|\ell \\ Q^2<\ell}} \left(\frac{Q}{\ell}\right)^{\frac{r}{2}} \sum_{\substack{0\le h<k\le N \\ \gcd(h,k)=1 \\ \gcd(k,\ell)=Q}} \frac{i}{k^2} e^{i\pi r\left(s(h,k)-s\left(\frac{\ell h}{Q},\frac{k}{Q}\right)\right)-2\pi i n\frac{h}{k}}$$

$$\times \int_{z'_{h,k}}^{z''_{h,k}} e^{\left(\frac{\pi r}{12z}\left(1-\frac{Q^2}{\ell}\right)+\frac{\pi r z}{12k^2}(\ell-1)+2\pi n\frac{z}{k^2}\right)} \mathcal{O}\left(e^{\frac{-2\pi Q^2}{\ell}\left(\left\lfloor \frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor+1\right)\mathrm{Re}\left(\frac{1}{z}\right)}\right)\mathrm{d}z,$$

which eventually simplifies as an error term upon employing Theorem 7.1.4. As before, we consider the path of integration to be the chord from $z'_{h,k}$ to $z''_{h,k}$ and on this chord, we have $0 \le \mathrm{Re}(z) \le \frac{2k^2}{N^2}$, length of the path of integration is less than $\frac{2\sqrt{2}k}{N}$ and $\mathrm{Re}(1/z) \ge 1$. Using these bounds, the error term becomes

$$\mathcal{K}_2 = \mathcal{O}\left(\frac{1}{N} e^{\frac{4\pi}{N^2}\left(n+\frac{r}{24}(\ell-1)\right)} \sum_{k=1}^{N} \frac{1}{k^{\frac{1}{2}-\epsilon}}\right). \qquad (7.3.7)$$

Now, we are left to evaluate the term $\mathcal{K}_1$ (7.3.6), which can be rewritten as

$$\mathcal{K}_1 = \sum_{\substack{Q|\ell \\ Q^2<\ell}} \left(\frac{Q}{\ell}\right)^{\frac{r}{2}} \sum_{\substack{k=1 \\ \gcd(k,\ell)=Q}}^{N} \frac{i}{k^2} \sum_{m,s=0}^{\left\lfloor \frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor} p^{(r)}(m)a_r(s)C(n)$$

$$\times \int_{z'_{h,k}}^{z''_{h,k}} e^{\frac{2\pi}{z}\left(\frac{r}{24}\left(1-\frac{Q^2}{\ell}\right)-\left(m+\frac{Q^2 s}{\ell}\right)\right)+\frac{2\pi z}{k^2}\left(n+\frac{r}{24}(\ell-1)\right)}\mathrm{d}z, \qquad (7.3.8)$$

where

$$C(n) := C_{k,r,\ell,Q,m,s}(n) = \sum_{\substack{h \bmod k \\ \gcd(h,k)=1}} e^{i\pi r\left(s(h,k)-s\left(\frac{\ell h}{Q},\frac{k}{Q}\right)\right)} e^{\frac{2\pi i}{k}(mh'+sQh_Q-nh)}.$$

As we make $\mathrm{Re}(z)$ small, so $\mathrm{Re}\left(\frac{1}{z}\right)$ will become large. Thus, for $\frac{r}{24}\left(1-\frac{Q^2}{\ell}\right) \le \left(m+\frac{Q^2 s}{\ell}\right)$, we get an error term whose evaluation is same as we did for $\mathcal{K}_2$. Thus, we leave the details for the reader. Therefore, we are left with the sum runs over all those m, s for which $\frac{r}{24}\left(1-\frac{Q^2}{\ell}\right) > \left(m+\frac{Q^2 s}{\ell}\right)$. Hence, the sum $\mathcal{K}_1$ becomes

$$\mathcal{K}_1 = \sum_{\substack{Q|\ell \\ Q^2<\ell}} \left(\frac{Q}{\ell}\right)^{\frac{r}{2}} \sum_{\substack{k=1 \\ \gcd(k,\ell)=Q}}^{N} \frac{i}{k^2} \sum_{m,s=0}^{\left\lfloor \frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor}{}' p^{(r)}(m)a_r(s)C(n)J^*_{h,k}$$

$$+ \mathcal{O}\left(\frac{1}{N} e^{\frac{4\pi}{N^2}\left(n+\frac{r}{24}(\ell-1)\right)} \sum_{k=1}^{N} \frac{1}{k^{\frac{1}{2}-\epsilon}}\right).$$

where

$$J^*_{h,k} = \int_{z'_{h,k}}^{z''_{h,k}} e^{\frac{2\pi}{z}\left(\frac{r}{24}\left(1-\frac{Q^2}{\ell}\right)-\left(m+\frac{Q^2 s}{\ell}\right)\right)+\frac{2\pi z}{k^2}\left(n+\frac{r}{24}(\ell-1)\right)} \mathrm{d}z,$$

and $\sum'$ means the sum is running over all those m, s which satisfy $m + \frac{Q^2 s}{\ell} < \frac{r}{24}\left(1 - \frac{Q^2}{\ell}\right)$. Now, our aim is to evaluate the integral $J^*_{h,k}$ and to do that, we slightly change the our path of integration. We begin by completing the circle $\left|z - \frac{1}{2}\right| = \frac{1}{2}$ and then we subtract the integral over additional arcs, which are the arcs from 0 to $z'_{h,k}$ and $z''_{h,k}$ to 0. Thus, we have

$$J^*_{h,k} = \left(\int_{K^{(-)}} - \int_0^{z'_{h,k}} - \int_{z''_{h,k}}^{0}\right) e^{\frac{2\pi}{z}\left(\frac{r}{24}\left(1-\frac{Q^2}{\ell}\right)-\left(m+\frac{Q^2 s}{\ell}\right)\right)+\frac{2\pi z}{k^2}\left(n+\frac{r}{24}(\ell-1)\right)} \mathrm{d}z,$$

where $K^{(-)}$ is the circle $\left|z - \frac{1}{2}\right| = \frac{1}{2}$ in the negative direction. First, we focus on the arc from 0 to $z'_{h,k}$, where $\mathrm{Re}\left(\frac{1}{z}\right) = 1$, $\mathrm{Re}(z) \leq \frac{2k^2}{N^2}$, $|z| < |z'_{h,k}| < \frac{k\sqrt{2}}{N}$ and length of this arc is less than $\frac{k\pi}{\sqrt{2}N}$. This gives

$$\left|\int_0^{z'_{h,k}} e^{\frac{2\pi}{z}\left(\frac{r}{24}\left(1-\frac{Q^2}{\ell}\right)-\left(m+\frac{Q^2 s}{\ell}\right)\right)+\frac{2\pi z}{k^2}\left(n+\frac{r}{24}(\ell-1)\right)} \mathrm{d}z\right|$$

$$\leq e^{2\pi\left(\frac{r}{24}\left(1-\frac{Q^2}{\ell}\right)-\left(m+\frac{Q^2 s}{\ell}\right)\right)+\frac{4\pi}{N^2}\left(n+\frac{r}{24}(\ell-1)\right)} \frac{k\pi}{\sqrt{2}N}.$$

Thus, we have

$$\int_0^{z'_{h,k}} e^{\frac{2\pi}{z}\left(\frac{r}{24}\left(1-\frac{Q^2}{\ell}\right)-\left(m+\frac{Q^2 s}{\ell}\right)\right)+\frac{2\pi z}{k^2}\left(n+\frac{r}{24}(\ell-1)\right)} \mathrm{d}z = \mathcal{O}\left(e^{\frac{4\pi}{N^2}\left(n+\frac{r}{24}(\ell-1)\right)} \frac{k}{N}\right).$$

The similar argument gives the same bound for integral over $z''_{h,k}$ to 0. Utilizing these bounds in (7.3.8) and together with Theorem 7.1.4, we arrive at

$$\mathcal{K}_1 = \sum_{\substack{Q|\ell \\ Q^2<\ell}} \left(\frac{Q}{\ell}\right)^{\frac{r}{2}} \sum_{\substack{k=1 \\ \gcd(k,\ell)=Q}}^{N} \frac{i}{k^2} \sum_{m,s=0}^{\left\lfloor \frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor}{}' p^{(r)}(m)a_r(s)C(n) \tag{7.3.9}$$

$$\times \int_{K^{(-)}} e^{\frac{2\pi}{z}\left(\frac{r}{24}\left(1-\frac{Q^2}{\ell}\right)-\left(m+\frac{Q^2 s}{\ell}\right)\right)+\frac{2\pi z}{k^2}\left(n+\frac{r}{24}(\ell-1)\right)} \mathrm{d}z + \mathcal{O}\left(\frac{1}{N} e^{\frac{4\pi}{N^2}\left(n+\frac{r}{24}(\ell-1)\right)} \sum_{k=1}^{N} \frac{1}{k^{\frac{1}{2}-\epsilon}}\right).$$

Finally, substituting (7.3.7) and (7.3.9) in (7.3.5) and combining with (7.3.3) in

(7.3.2), we get

$$b_\ell^{(r)}(n)=\sum_{\substack{Q|\ell\\ Q^2<\ell}}\left(\frac{Q}{\ell}\right)^{\frac{r}{2}}\sum_{\substack{k=1\\ \gcd(k,\ell)=Q}}^{N}\frac{i}{k^2}\sideset{}{'}\sum_{m,s=0}^{\left\lfloor\frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor}p^{(r)}(m)a_r(s)C(n) \tag{7.3.10}$$

$$\times\int_{K^{(-)}}e^{\frac{2\pi}{z}\left(\frac{r}{24}\left(1-\frac{Q^2}{\ell}\right)-\left(m+\frac{Q^2s}{\ell}\right)\right)+\frac{2\pi z}{k^2}\left(n+\frac{r}{24}(\ell-1)\right)}\mathrm{d}z+\mathcal{O}\left(\frac{1}{N}e^{\frac{4\pi}{N^2}\left(n+\frac{r}{24}(\ell-1)\right)}\sum_{k=1}^{N}\frac{1}{k^{\frac{1}{2}-\epsilon}}\right).$$

Observe that the left hand side is independent of N and the error term on the right hand side goes to zero as $N\to\infty$, $\sum_{k=1}^{N}\frac{1}{k^s}\sim N^{1-s}$, for $s>0, s\neq 1$. Thus, letting $N\to\infty$ on both sides of (7.3.10), we have

$$b_\ell^{(r)}(n)=\sum_{\substack{Q|\ell\\ Q^2<\ell}}\left(\frac{Q}{\ell}\right)^{\frac{r}{2}}\sum_{\substack{k=1\\ \gcd(k,\ell)=Q}}^{\infty}\frac{i}{k^2}\sideset{}{'}\sum_{m,s=0}^{\left\lfloor\frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor}p^{(r)}(m)a_r(s)C(n)$$

$$\times\int_{K^{(-)}}e^{\frac{2\pi}{z}\left(\frac{r}{24}\left(1-\frac{Q^2}{\ell}\right)-\left(m+\frac{Q^2s}{\ell}\right)\right)+\frac{2\pi z}{k^2}\left(n+\frac{r}{24}(\ell-1)\right)}\mathrm{d}z. \tag{7.3.11}$$

This essentially gives the exact formula for $b_\ell^{(r)}(n)$. However, we need to check the convergence of the above infinite series. One can see that

$$\left|\int_{K^{(-)}}e^{\frac{2\pi}{z}\left(\frac{r}{24}\left(1-\frac{Q^2}{\ell}\right)-\left(m+\frac{Q^2s}{\ell}\right)\right)+\frac{2\pi z}{k^2}\left(n+\frac{r}{24}(\ell-1)\right)}\mathrm{d}z\right|\leq\pi e^{\frac{2\pi r}{24}\left(1-\frac{Q^2}{\ell}\right)+\frac{2\pi}{k^2}\left(n+\frac{r}{24}(\ell-1)\right)}$$

as $\mathrm{Re}\left(\frac{1}{z}\right)=1$ and $\mathrm{Re}(z)\leq 1$ on the circle $K^{(-)}$. Using Theorem 7.1.4 and upon simplification, one can show that

$$b_\ell^{(r)}(n)=\mathcal{O}\left(\sum_{k=1}^{\infty}\frac{1}{k^{\frac{3}{2}-\epsilon}}\right),$$

which is absolutely convergent for any small $0<\epsilon<1/2$. The only task left is to write the integral present in (7.3.11) in a simplified form. In (7.3.11), we substitute $z=\frac{1}{\omega}$. The image of the circle $\left|z-\frac{1}{2}\right|=\frac{1}{2}$ under this transformation is $\mathrm{Re}(\omega)=1$. Thus, we have

$$b_\ell^{(r)}(n)=\sum_{\substack{Q|\ell\\ Q^2<\ell}}\left(\frac{Q}{\ell}\right)^{\frac{r}{2}}\sum_{\substack{k=1\\ \gcd(k,\ell)=Q}}^{\infty}\frac{i}{k^2}\sideset{}{'}\sum_{m,s=0}^{\left\lfloor\frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor}p^{(r)}(m)a_r(s)\mathcal{C}(n)$$

$$\times\int_{1-i\infty}^{1+i\infty}\frac{-1}{\omega^2}e^{2\pi\omega\left(\frac{r}{24}\left(1-\frac{Q^2}{\ell}\right)-\left(m+\frac{Q^2s}{\ell}\right)\right)+\frac{2\pi}{\omega k^2}\left(n+\frac{r}{24}(\ell-1)\right)}\mathrm{d}\omega,$$

Substituting $\left(\frac{\pi r}{12}\left(1-\frac{Q^2}{\ell}\right)-2\pi\left(m+\frac{Q^2s}{\ell}\right)\right)\omega=t$ and comparing it with (2.6.5), we

get

$$b_\ell^{(r)}(n)=\sum_{\substack{Q|\ell\\ Q^2<\ell}}\left(\frac{Q}{\ell}\right)^{\frac{r}{2}}\sum_{\substack{k=1\\ \gcd(k,\ell)=Q}}^{\infty}\frac{2\pi}{k^2}\sum_{m,s=0}^{\left\lfloor\frac{r}{24}\left(\frac{\ell}{Q^2}-1\right)\right\rfloor}{}'\,p^{(r)}(m)a_r(s)C(n)\sqrt{\frac{\frac{\pi r}{12}\left(1-\frac{Q^2}{\ell}\right)-2\pi\left(m+\frac{Q^2s}{\ell}\right)}{\frac{\pi r}{12k^2}(\ell-1)+\frac{2n\pi}{k^2}}}$$

$$\times I_1\left(\frac{4\pi}{k}\sqrt{\left(\frac{r}{24}\left(1-\frac{Q^2}{\ell}\right)-\left(m+\frac{Q^2s}{\ell}\right)\right)\left(n+\frac{r}{24}(\ell-1)\right)}\right). \tag{7.3.12}$$

Further, utilizing

$$I_\nu(z)=\frac{1}{i^\nu}J_\nu(iz),\quad J_1(z)=-\frac{\mathrm{d}}{\mathrm{d}z}J_0(z),$$

we get the desired result (7.2.1). □

Proof of Corollary 7.2.2. We first separate the main term $M(n)$ corresponding to $Q=1,k=1,m=s=0$ in (7.3.12) and denote the contribution of the remaining terms by $E(n)$. This yields

$$b_\ell^{(r)}(n)=M(n)+E(n),$$

where

$$M(n)=\left(\frac{1}{\ell}\right)^{\frac{r}{2}}2\pi\sqrt{\frac{\frac{r}{24}\left(1-\frac{1}{\ell}\right)}{\frac{r}{24}(\ell-1)+n}}I_1\left(4\pi\sqrt{\frac{r}{24}\left(1-\frac{1}{\ell}\right)\left(n+\frac{r}{24}(\ell-1)\right)}\right). \tag{7.3.13}$$

Next, we employ the classical asymptotic expansion for the modified Bessel function of the first kind,

$$I_\nu(x)\sim\frac{e^x}{\sqrt{2\pi x}}\quad\text{as}\quad x\to\infty.$$

Making use of this asymptotic expansion into (7.3.13) gives

$$M(n)\sim\frac{1}{\sqrt{2}}\left(\frac{1}{\ell}\right)^{\frac{r}{2}}\left(\frac{r}{24}\left(1-\frac{1}{\ell}\right)\right)^{\frac{1}{4}}\left(\frac{1}{n+\frac{r}{24}(\ell-1)}\right)^{\frac{3}{4}}e^{4\pi\sqrt{\frac{r}{24}\left(1-\frac{1}{\ell}\right)\left(n+\frac{r}{24}(\ell-1)\right)}}$$

$$\sim\frac{1}{\sqrt{2}}\left(\frac{1}{\ell}\right)^{\frac{r}{2}}\left(\frac{r}{24}\left(1-\frac{1}{\ell}\right)\right)^{\frac{1}{4}}\left(\frac{1}{n}\right)^{\frac{3}{4}}e^{4\pi\sqrt{\frac{nr}{24}\left(1-\frac{1}{\ell}\right)}}$$

as $n\to\infty$. Using the same asymptotic expansion, one can see that, for any $k\geq 2$, the modified Bessel function present in $E(n)$ has the following asymptotic expansion:

$$I_1\left(\frac{4\pi}{k}\sqrt{\delta\left(n+\frac{r}{24}(\ell-1)\right)}\right)\sim\frac{e^{\frac{4\pi}{k}\sqrt{\delta\left(n+\frac{r}{24}(\ell-1)\right)}}}{\sqrt{\frac{8\pi^2}{k}\sqrt{\delta\left(n+\frac{r}{24}(\ell-1)\right)}}}\sim\frac{e^{\frac{4\pi}{k}\sqrt{\delta n}}}{\sqrt{\frac{8\pi^2}{k}\sqrt{\delta n}}},$$

where

$$\delta=\left(\frac{r}{24}\left(1-\frac{Q^2}{\ell}\right)-\left(m+\frac{Q^2s}{\ell}\right)\right).$$

Moreover, one can easily check that, for any $k \geq 2$,

$$\lim_{n\to\infty} \frac{e^{\frac{4\pi}{k}\sqrt{\delta n}}}{e^{4\pi\sqrt{\frac{nr}{24}\left(1-\frac{1}{\ell}\right)}}} = 0.$$

Here we have used the fact that $\sqrt{\delta} < \sqrt{\frac{r}{24}\left(1-\frac{1}{\ell}\right)}$ as $(m,s) \neq (0,0)$. This completes the proof of Corollary 7.2.2. □

Proof of Corollary 7.2.3. Substitute $r = 1$ in Theorem 7.2.1 to obtain the desired result. □

Proof of Corollary 7.2.4. By specializing Theorem 7.2.1 to the case $\ell = 2$, we arrive at the desired conclusion. □

Proof of Corollary 7.2.5. By substituting $r = 3$ in Corollary 7.2.4, we get an exact formula for $\sigma\overline{mex}(n)$. Further, we utilize

$$I_\nu(z) = \frac{1}{i^\nu} J_\nu(iz), \quad J_1(z) = -\frac{\mathrm{d}}{\mathrm{d}z} J_0(z),$$

to attain the desired form (7.2.3). □

Proof of Theorem 7.2.6. From (7.2.2), for any non-negative integer j, we have

$$\frac{b_\ell^{(r)}(n+j)}{b_\ell^{(r)}(n)} \sim \left(\frac{n}{n+j}\right)^{\frac{3}{4}} e^{4\pi\sqrt{\frac{r(n+j)}{24}\left(1-\frac{1}{\ell}\right)} - 4\pi\sqrt{\frac{rn}{24}\left(1-\frac{1}{\ell}\right)}}.$$

Taking natural logarithm on both sides, we have

$$\log\left(\frac{b_\ell^{(r)}(n+j)}{b_\ell^{(r)}(n)}\right) \sim 4\pi\sqrt{\frac{r}{24}\left(1-\frac{1}{\ell}\right)}\left(\sqrt{n+j}-\sqrt{n}\right) - \frac{3}{4}\log\left(\frac{n+j}{n}\right).$$

The above right side expression can be simplified as follows:

$$\begin{aligned}
&4\pi\sqrt{\frac{r}{24}\left(1-\frac{1}{\ell}\right)}\left(\sqrt{n+j}-\sqrt{n}\right) - \frac{3}{4}\log\left(\frac{n+j}{n}\right) \\
&= 4\pi\sqrt{\frac{r}{24}\left(1-\frac{1}{\ell}\right)}\sum_{i=1}^{\infty}\binom{1/2}{i}\frac{j^i}{n^{i-1/2}} + \frac{3}{4}\sum_{i=1}^{\infty}\frac{(-1)^i j^i}{in^i} \\
&= \left(2\pi\sqrt{\frac{r}{24n}\left(1-\frac{1}{\ell}\right)} - \frac{3}{4n}\right)j - \left(\frac{\pi}{2}\sqrt{\frac{r}{24n^3}\left(1-\frac{1}{\ell}\right)} - \frac{3}{8n^2}\right)j^2 \\
&+ \sum_{i=3}^{\infty}\left(4\pi\sqrt{\frac{r}{24}\left(1-\frac{1}{\ell}\right)}\binom{1/2}{i}\frac{1}{n^{i-1/2}} + \frac{3}{4}\frac{(-1)^i}{in^i}\right)j^i.
\end{aligned}$$

Now, let $A(n) = 2\pi\sqrt{\frac{r}{24n}\left(1-\frac{1}{\ell}\right)} - \frac{3}{4n}$, $\delta(n) = \sqrt{\frac{\pi}{2}\sqrt{\frac{r}{24n^3}\left(1-\frac{1}{\ell}\right)} - \frac{3}{8n^2}}$ and

$$g_i(n) = 4\pi\sqrt{\frac{r}{24}\left(1-\frac{1}{\ell}\right)}\binom{1/2}{i}\frac{1}{n^{i-1/2}} + \frac{3}{4}\frac{(-1)^i}{in^i},$$

for all $i \geq 3$. A simple calculation shows that, for $3 \leq i \leq d$, we have $\lim_{n\to\infty} \frac{g_i(n)}{(\delta(n))^i} = 0$. We can further check that $\lim_{n\to\infty} \frac{g_i(n)}{(\delta(n))^d} = 0$, for all $i \geq d+1$. Hence, it follows that the sequences $\{b_\ell^{(r)}(n)\}$, $\{A(n)\}$, $\{\delta(n)\}$ and $\{g_i(n)\}$ satisfy the assumptions of Theorem 1.2.1. Therefore, the Jensen polynomials corresponding to the r-colored ℓ-regular partition function $b_\ell^{(r)}(n)$ admit representations in terms of Hermite polynomials, which are hyperbolic for sufficiently large values of n. This concludes the proof of Theorem 7.2.6. □

7.4 Numerical Verification

To demonstrate the accuracy of our exact formula for $b_\ell^{(r)}(n)$, we perform a numerical verification by evaluating explicit values of $b_\ell^{(r)}(n)$ for fixed $\ell = 3$ and $\ell = 6$. These values are computed using the first five terms of the series given in Theorem 7.2.1 and are then compared with the corresponding exact values.

Table 7.1: Verification of Theorem 7.2.1 for $\ell = 3$

r	n	Exact value	Value from our result
50	7	420621700	420621699.99999
39	10	24030437457	24030437457
63	4	849807	849806.999999
76	5	30050020	30050019.999999

Table 7.2: Verification of Theorem 7.2.1 for $\ell = 6$

r	n	Exact value	Value from our result
75	5	28462590	28462589.9998
25	11	3755606050	3755606049.9999
12	22	299225122470	299225122470.0000
22	12	5175590618	5175590617.9999

We also present a verification table for the r-colored distinct partitions of n, denoted by $p_d^{(r)}(n)$, for various values of r and n, and compare the computed results with their exact values.

Table 7.3: Verification of Corollary 7.2.4

r	n	Exact value	Value from our result
20	5	46724	46723.99988778628
25	10	206841715	206841714.9999436
30	7	9603210	9603209.99997784
50	9	12141387350	12141387349.99999
73	6	261788950	261788949.9999971

Furthermore, we compute the first six terms of the 25-colored distinct partitions of 10 numerically by considering the contributions from the six-term series given in Corollary 7.2.4.

$$\begin{aligned} &+ 206841714.5985165 \\ &\quad + 0.412498 \\ &\quad - 0.0120247 \\ &\quad + 0.00107527 \\ &\quad - 0.000487608 \\ &\quad - 0.0000162379 \\ \hline &206841714.9995613 \end{aligned}$$

The exact value of $p_d^{(25)}(10)$ is 206841715. Thus, the error is ≈ 0.0004.

Chapter 8

CONCLUDING REMARKS

This thesis focuses on two themes in analytic number theory: Equivalent criteria for the Riemann hypothesis within a broader class of L-functions, and the derivation of Rademacher-type exact formulas, along with higher-order Turán inequalities, for cubic overpartitions and r-colored ℓ-regular partitions. These two themes were introduced in Chapter 1, with the required background developed in Chapter 2. In Chapter 3, we established a one-variable generalization of the Ramanujan–Hardy–Littlewood identity (1.1.4). Hardy and Littlewood [69, p. 161] pointed out that the absolute convergence of the series in (1.1.4), over the non-trivial zeros of $\zeta(s)$, is not so immediate even after the assumption of the Riemann hypothesis. We remark that the identity (1.1.4) and our identity (3.2.1) are both valid under the assumption of the simplicity hypothesis of the non-trivial zeros of $\zeta(s)$ and we do not require the assumption of the Riemann hypothesis. In [79, pp. 783], the authors explained in more details why the convergence of this kind of series with $\zeta'(\rho)$ in the denominator is more delicate.

The Ramanujan–Hardy–Littlewood identity (1.1.4) has been studied by many mathematicians in various directions. A Dirichlet character analogue of (1.1.4) was given by Dixit [45], and further a one-variable generalization involving ${}_1F_1$ hypergeometric function was obtained by Dixit, Roy and Zaharescu [48]. In the Chapter 4, we have established a one-variable generalization of Dixit's identity (1.1.7), namely, Theorem 4.2.1. This generalization also provides a character analogue of the identity (3.2.1). Motivated from the work of Riesz, Hardy and Littlewood, we have established the following equivalent criteria for the generalized Riemann hypothesis for $L(s,\chi)$: For $k \geq 1, \ell > 0$, we have

$$\sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^k} \exp\left(-\frac{x}{n^\ell}\right) = O_\epsilon\left(x^{-\frac{k}{\ell}+\frac{1}{2\ell}+\epsilon}\right), \quad \text{as } x \to \infty. \tag{8.1}$$

This bound generalizes the bounds given by Riesz, and Hardy and Littlewood as well

as our previous bound (3.2.2). The above bound motivates us to predict a more general observation. Let $L(f,s)$ be a "nice" L-function for which the grand Riemann hypothesis [78, p. 113] predicts that the non-trivial zeros will lie on the critical line $\mathrm{Re}(s) = 1/2$. Suppose we write $1/L(f,s) = \sum_{n=1}^{\infty} \frac{\mu_f(n)}{n^s}$, which is valid in some right half plane. In [78, Proposition 5.14], one can see that the grand Riemann hypothesis is equivalent to the following bound: For any $\epsilon > 0$,

$$\sum_{1 \leq n \leq x} \mu_f(x) \ll_{\epsilon,f} x^{1/2+\epsilon}.$$

Note that exactly the same bound in case of the generalized Riemann hypothesis was crucial to obtain the above bound (8.1). This motivates us to give the following conjecture.

Conjecture 8.1. *Let $k \geq 1$ and $\ell > 0$ be two real numbers. Let $L(f,s)$ be a "nice" L-function. The grand Riemann hypothesis for $L(f,s)$ is equivalent to the bound*

$$\sum_{n=1}^{\infty} \frac{\mu_f(n)}{n^k} \exp\left(-\frac{x}{n^\ell}\right) = O_{\epsilon,f,k,\ell}\left(x^{-\frac{k}{\ell}+\frac{1}{2\ell}+\epsilon}\right), \quad \text{as } x \to \infty.$$

In Chapter 5, we settled this conjecture, see Theorem 5.2.10. In particular, Theorem 5.2.12 gives equivalent criteria for the Riemann hypothesis for L-functions associated to cusp forms. We also obtain an entirely novel form of equivalent criteria for the Riemann hypothesis of $\zeta(s)$, see Theorem 5.2.13. Furthermore, we generalized the identity (1.1.4) and (1.1.5) of Ramanujan, Hardy and Littlewood for a general class of L-functions in Theorem 5.2.1 and Corollary 5.2.2 respectively. We further verified the results of Chapter 5 numerically using Mathematica software, see Table 5.1 and Table 5.2.

In next two chapters, we utilized the circle method to obtain Rademacher-type exact formulas for various restricted partition functions along with higher order Turán inequalities. The main objective of Chapter 6 is to apply the Hardy–Ramanujan–Rademacher circle method to derive a Rademacher-type infinite series representation for cubic overpartitions. Mainly, we showed that $\overline{a}(n)$ can be expressed as the sum of two absolutely convergent series involving Kloosterman-type sums and Bessel functions, see Theorem 6.2.1. Using bounds for Kloosterman-type sums and modified Bessel functions of the first kind, we obtained an effective estimate for the error term, which yields the log-concavity of the cubic overpartition function, see Theorem 6.2.4. Furthermore, by applying a general result of Griffin, Ono, Rolen, and Zagier, we established higher-order Turán inequalities for the cubic overpartition function, see Theorem 6.2.5. In addition, motivated from the work of Bressenrodt and Ono, we proved the log-subadditivity of the cubic overpartition function, see Theorem 6.2.6. We also deduced generalized log-concavity for cubic overpartitions, see Theorem 6.2.7,

inspired from the work of DeSalvo and Pak.

We would like to point out that the generating function (1.2.4) for the cubic overpartition function is a weakly holomorphic modular form of weight -1 over $\Gamma_0(4)$. Therefore, it would be interesting to derive our Theorem 6.2.1 using a result of Zuckerman [114] and Bringmann-Ono [22].

The investigation of the hyperbolicity of Jensen polynomials, particularly those associated with the partition function, has received significant attention in recent years. DeSalvo and Pak [41] established that $J_p^{2,n-1}$ is hyperbolic for $n \geq 25$. Subsequently, Chen, Jia and Wang [30] showed that $J_p^{3,n-1}$ is hyperbolic for $n \geq 94$ and conjectured that for each $d > 1$, there exists a minimal integer $N_p(d)$ such that $J_p^{d,n-1}$ is hyperbolic for all $n \geq N_p(d)$, which was confirmed by Griffin, Ono, Rolen and Zagier [57] by showing that $J_p^{d,n-1}$ is hyperbolic for every degree d for all sufficiently large n. Later, Larson and Wagner [81] established an explicit upper bound for $N_p(d)$. In particular, they proved that $N_p(d) \leq (3d)^{24d}(50d)^{3d^2}$ and further determined the exact values $N_p(4) = 206$ and $N_p(5) = 381$. More recently, Pandey [92] studied the Jensen polynomial associated to plane partition function and established an effective upper bound for $N_{\mathrm{PL}}(d)$.

Based on the numerical evidence from Mathematica, in case of cubic overpartitions function $\overline{a}(n)$, we conjecture the following.

Conjecture 8.2. *Let $N_{\overline{a}}(d)$ be the minimum integer such that $J_{\overline{a}}^{d,n-1}$ is hyperbolic for all $n \geq N_{\overline{a}}(d)$, then we have*

d	3	4	5	6	7
$N_{\overline{a}}(d)$	39	89	172	279	423

It would be also an exciting problem to find an effective upper bound for $N_{\overline{a}}(d)$.

Finally, we consider a natural generalization of the cubic overpartition function. Let $\overline{a}_{\ell,r}(n)$ denote the number of representations of n as a sum of natural numbers, where parts divisible by ℓ may appear in r different colors, and the first occurrence of each distinct part may be overlined. Its generating function is given by

$$\sum_{n=0}^{\infty} \overline{a}_{\ell,r}(n)q^n = \frac{(-q;q)_\infty(-q^\ell;q^\ell)_\infty^{r-1}}{(q;q)_\infty(q^\ell;q^\ell)_\infty^{r-1}}.$$

This framework recovers several known cases. For example, when $(r,\ell) = (2,2)$, we obtain the cubic overpartition function; for $r = 1$, it reduces to the classical overpartition function; and for $\ell = 2$ with arbitrary r, one obtains the generalized cubic overpartition function studied by Amdeberhan, Sellers, and Singh [5].

In a forthcoming work, we plan to investigate Rademacher-type exact formula for $\overline{a}_{\ell,r}(n)$ and explore log-concavity and higher order Turán inequalities for this broader class of partition function.

In Chapter 7, we employed the circle method to obtain a Rademacher-type exact formula for r-colored ℓ-regular partitions for any $r \in \mathbb{N}$ and $\ell \geq 2$. As an application, we derived higher order Turán inequalities for the r-colored ℓ-regular partition function using a result of Griffin, Ono, Rolen, and Zagier.

To emphasize the conceptual link between the two central themes of this thesis, we briefly recall a classical connection due to Pólya involving Jensen polynomials. This perspective illustrates how questions about the zero distribution of Riemann zeta function naturally lead to hyperbolicity of Jensen polynomials.

Building on unpublished notes of Jensen, Pólya [95] showed that the RH is equivalent to a hyperbolicity property of certain Jensen polynomials associated with the Riemann zeta function at its point of symmetry. More precisely, he proved that RH holds if and only if all Jensen polynomials corresponding to the sequence of Taylor coefficients $\{\hat{b}_n\}$ defined as

$$(-1+4z^2)\Lambda\left(\frac{1}{2}+z\right) = \sum_{n=0}^{\infty} \frac{\hat{b}_n(n)}{n!} z^{2n}. \tag{8.2}$$

are hyperbolic. Here, $\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \Lambda(1-s)$ and Jensen polynomials are defined in (1.2.7).

With this notation, the Riemann hypothesis is equivalent to the hyperbolicity of the polynomials $J_{\hat{b}}^{d,n}(X)$ for all non-negative integers d and n. Hyperbolicity for $J_{\hat{b}}^{d,n}(X)$ was known for $d \leq 3$ due to the work of Csordas et al. [38] and Dimitrov and Lucas [44]. Asymptotic results for $\{\hat{b}_n\}$ were obtained by Coffey [35] and Pustyl'nikov [97] which were further improved by Griffin et al. [57, Theorem 9]. They [57, Theorem 1] utilized this result to prove the hyperbolicity of $J_{\hat{b}}^{d,n}(X)$ for each degree d for sufficiently large n and more precisely, they showed the hyperbolicity for every non-negative n for $1 \leq d \leq 8$.

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