ANALYTIC AND GEOMETRIC PROPERTIES OF CERTAIN UNIVALENT FUNCTIONS

Ph.D. Thesis

By

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DISCIPLINE OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

March 2020

ANALYTIC AND GEOMETRIC PROPERTIES OF CERTAIN UNIVALENT FUNCTIONS

A THESIS

Submitted in partial fulfillment of the requirements for the award of the degree

of

DOCTOR OF PHILOSOPHY

by

Vibhuti Arora

DISCIPLINE OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

March 2020

INDIAN INSTITUTE OF TECHNOLOGY INDORE
CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **ANALYTIC AND GEOMETRIC PROPERTIES OF CERTAIN UNIVALENT FUNCTIONS** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DISCIPLINE OF MATHEMATICS, Indian Institute of Technology Indore,** is an authentic record of my own work carried out during the time period from July 2015 to March 2020 under the supervision of Dr. Swadesh Kumar Sahoo, Associate Professor, Indian Institute of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Vibhuti
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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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ACKNOWLEDGEMENTS

I would like to acknowledge all those people who have contributed in some way to complete this dissertation. No book is one person's effort alone. For this one, too, I have many to thank: First of all, I am grateful to God for the blessing and for giving me the strength and wisdom to achieve this dream.

It is my great pleasure to express my sincere respect and most heartfelt gratitude to my supervisor Dr. Swadesh Kumar Sahoo (Associate Professor, Discipline of Mathematics, IIT Indore) for his constant guidance, timely suggestions, and encouragement throughout my work. I hope for his continue guidance. It is my pleasure to work under the supervision of such a nice person. I cannot imagine having a better mentor for my Ph.D. He has not only shaped me into a researcher but also helped me to become a better person. My deeply thanks to his family members for providing me homely comfort and care during my Ph.D.

I am highly grateful to Prof. Saminathan Ponnusamy for the valuable suggestions and the discussions during my visit to IIT Madras that have helped me in better understanding and improving the research work.

My warm and sincere thanks to Prof. Pradeep Mathur, Director IIT Indore for being so considerate and for the excellent facilities provided to accomplish the research work at the Institute.

I am grateful to my PSPC members Dr. Ashisha Kumar and Dr. Ankhi Roy for their valuable suggestions and remarks during the evaluation of my research progress every year. I wish to thank all the faculty members of the Discipline of Mathematics at IIT Indore for their inspirational words.

A special thanks to my family for their whole-hearted sacrifices and patience. Without their support none of this would have been possible and I deeply express my love and gratitude to my father Mr. Ganesh Chandra Arora, mother Mrs. Manju Lata Arora, brother Mr. Vikalp Arora due to him I got interest in Mathematics, and bhabhi Mrs. Vinita Arora. Lots of love for my sweet little nieces Vedika and Vamika.

I am very much indebted to my fiancé, Shankey Garg, who encouraged me in every possible way. His infallible love and support has always been my strength.

I am also thankful to the IIT Indore for providing me research fellowship under TA category throughout the tenure of my research work which helped me to perform my work comfortably. My sincere thanks to all my M.Sc. faculty members especially Dr. Nidhi Khandelwal, University of Rajasthan.

I express my earnest thanks to the journal referees and thesis examiners for careful reading and for their valuable inputs to improve the quality of the thesis.

I would like to express my thanks to my group mates Arstu and Shankey Garg for their generous help and co-operation. With such good friends, the journey of my Ph.D. became quite smooth. I am also thankful to my seniors of the research group Dr. Manas Ranjan Mohapatra (his fruitful guidance and suggestions as a senior are unforgettable), Dr. Sarita Agrawal, Dr. Navneet Lal Sharma, Dr. Nijjwal Karak, and Dr. Sudhananda Maharana for their trustworthy help, company, and moral support during the research work. I sincerely thank to my seniors of mathematics department Dr. Istkhar Ali, Dr. Rupsha Roy, Nitin Bisht, Dr. Linia Anie Sunny, Dr. Harish, Dr. Anupam Gumber, Prince Kanhya, Gyan Swarup Nag with whom I started my work and all my colleagues in the Discipline of Mathematics, IIT Indore for their friendship. Many thanks to Jitendra bhaiya for his constant help whenever required. A special thanks to my friends Aarti, Anita, Kriti, Jamuna, Mayuri, Latha, and Amith.

Last but not least, there are many people who have directly or indirectly contributed in making this journey successful. I wish I could thank them all, but time compel me to stop here.

Dedicated

to My

Family whose Prayers, Efforts

and

Wishes are an inspiration.

LIST OF PUBLICATIONS

1. V. Arora, S. K. Sahoo, *Meromorphic functions with small Schwarzian derivative*, Stud. Univ. Babeş-Bolyai Math., **63** (2018), no. 3, 355–370.
2. S. Agrawal, V. Arora, M. R. Mohapatra, and S. K. Sahoo, *Area problem for univalent functions in the unit disk with quasiconformal extension to the plane*, Bull. Iranian Math. Soc., **45** (2019), no. 4, 1061-1069.
3. V. Arora, S. Ponnusamy, and S. K. Sahoo, *Successive coefficients for spirallike and related functions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat., **113**(4) (2019), 2969–2979.
4. V. Arora and S. K. Sahoo, *Area estimates of images of disks under analytic functions,* Submitted.
5. V. Arora, S. K. Sahoo, and S. Singh, *Approximation of certain non-vanishing analytic functions in a parabolic region.* Submitted.

ABSTRACT

KEYWORDS: Analytic functions, Approximations of analytic functions, Area problem, Close-to-convex functions, Convex functions, Dirichlet finite, Karush-Kuhn-Tucker conditions, Meromorphic functions, Meromorphically convex functions, Quadratic programming, Quasiconformal extension, Quasiconformal mappings, Schwarzian derivative, Spirallike functions, Starlike functions, Successive coefficients, Univalent functions.

Most of the work in the present thesis is concerned with the class of functions analytic and univalent in the unit disk with the standard normalization. The theory of univalent functions builds a relation between analytic structure and geometric behaviour of complex function theory.

In **Chapter 1** we give a short literature survey of univalent function theory and state some main results of this thesis. This chapter also provides basic definitions, properties and some results which are useful in later chapters.

Chapter 2 deals with the area problem which extends the Yamashita's extremal problem for the class of normalized analytic univalent functions defined in the unit disk. We determine the area of the image of the subdisk of radius r, $0 < r \leq 1$, under a function z/f when f varies over the class of normalized analytic univalent functions in the unit disk with quasiconformal extension to the entire complex plane. Further, we construct a new function which is an extremal function for the above area problem and also an extension of the Koebe function $z/(1-z)^2$.

In addition to the above, the area problems are also studied in **Chapter 3**. In this chapter, we estimate areas of images of the subdisks of radius r, $0 < r \leq 1$, under non-vanishing analytic functions of the form $(z/f)^{\mu}$, $\mu > 0$, in principal powers, when f ranges over certain classes of analytic and univalent functions in the unit disk. We found that most of the estimations are sharp in nature by constructing some extremal functions.

Chapter 4 focuses on coefficient problems for univalent functions. We consider the family of all analytic and univalent functions in the unit disk of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. We determine the difference of the moduli of successive coefficients, that is $\big||a_{n+1}| - |a_n|\big|$, for f belonging to the family of γ-spirallike functions of order α. Our particular results include the case of starlike and convex functions of order α and other related class of functions.

Chapter 5 is devoted to the family of all meromorphic functions g having a simple pole at the origin and locally univalent in the puncture disk $\mathbb{D}_0 := \{z \in \mathbb{C} : 0 < |z| < 1\}$. We obtain a sufficient condition for g to be meromorphically convex of order α, $0 \leq \alpha < 1$, in terms of the fact that the absolute value of the well-known Schwarzian derivative of g is bounded above by a smallest positive root of a non-linear equation. We also consider a family of functions f of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ analytic and locally univalent in the unit disk, and show that f is belonging to a family of functions convex in one direction if Schwarzian derivative of f is bounded above by a small positive constant depending on the second coefficient a_2. In particular, we show that such functions f are also contained in the starlike and close-to-convex family.

Finally, in **Chapter 6** we consider a family of analytic functions f defined on the unit disk so that the values of zf'/f lie on a parabolic region in the right-half plane. A subfamily of this family is constructed by considering a sufficient condition for functions to be in the original family in terms of the Taylor coefficients of z/f. The main objective of this chapter is to find a best approximation of non-vanishing analytic functions of the form z/f by functions z/g with members g from the above said subfamily. A technique for solving a semi-infinite quadratic programming problem has been used to calculate the best approximation.

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NOTATION

$\mathcal{A}$	class of normalized analytic functions in $\mathbb{D}$
$A \subset B$	A is a subset of B
$A \subsetneq B$	A is a proper subset of B
$(a)_n$	the Pochhammer symbol or shifted factorial
$\mathbb{C}$	complex plane
$\mathcal{C}$	class of convex functions
$\mathcal{C}(\alpha)$	class of convex functions of order $\alpha,\ 0 \leq \alpha \leq 1$
$\mathbb{D}$	unit disk
$\mathbb{D}_0$	puncture disk $\{z \in \mathbb{C} : 0 < \lvert z\rvert < 1\}$
$\mathbb{D}_r$	disk of radius $r\,(z : \lvert z\rvert < r,\ 0 < r \leq 1)$
${}_2F_1$	Guassian Hypergeometric Function
$f \prec g$	f is subordinate to g
∂G	boundary of G
$\overline{G}$	closure of G
G^c	complement of G
$\mathcal{H}$	class of analytic functions in $\mathbb{D}$
$\mathcal{H}_a$	class of analytic functions in $\mathbb{D}$ which take origin into $a \in \mathbb{C}$
$\mathcal{K}$	class of close-to-convex functions
$\mathrm{Log}(z)$	the principal value of the logarithmic function $\log z$ for $z \neq 0$
$\mathcal{S}$	class of univalent functions
$\mathcal{S}^*$	class of starlike functions
$\mathcal{S}^*(\alpha)$	class of starlike functions of order $\alpha,\ 0 \leq \alpha \leq 1$
$\mathcal{S}_\gamma(\alpha)$	class of γ-spirallike functions of order $\alpha,\ 0 \leq \alpha \leq 1$
S_f	Schwarzian derivative of f
$\mathrm{Re}\, z$	real part of z
$\mathrm{Im}\, z$	imaginary part of z

Greek Symbols

Ω	$\{z : \lvert z\rvert > 1\}$
$\Delta(r, f)$	area of the image of $\mathbb{D}_r$ under analytic function f
Γ	gamma function

CHAPTER 1

INTRODUCTION

Geometric Function Theory is a classical area of complex analysis to study geometric properties of analytic functions. In some sense, it is about study of relationships between the geometries of various domains in the complex plane. Theory of univalent functions is one of the most interesting topics of Geometric Function Theory which was originated by Koebe [39] in 1907. From the introduction of the Bieberbach conjecture in 1916, until its proof given by de Branges in 1985, a lot of methods and concepts have been developed in the field of univalent function theory.

One of the classical problems in univalent function theory is to consider the class of functions f for which the area of the image of the disks of radius r centered at origin under f is bounded. We consider such type of area problems for certain analytic functions f, for which the problem was not studied before.

The univalency of an analytic function is an important problem in geometric function theory, and there are many necessary and sufficient conditions for univalency in the literature. Bieberbach's conjecture is one of the most popular necessary conditions for the class of univalent functions. On the other hand, the problem of estimating bounds for successive coefficients is also another interesting necessary condition which was studied with an idea to solve the Bieberbach conjecture. We investigate similar problem for certain subclasses of univalent functions, as the problem is still open for the whole class of univalent analytic functions. Such investigations sometimes may lead to a new technique to deal the main open problem or related problems. The study of necessary and sufficient conditions for functions to be univalent in terms of Schwarzian derivative are also attracted by a number of mathematicians. Similarly, we also find some sufficient conditions for f when Schwarzian derivative is bounded by a small constant.

If a function is not univalent, then, in practical problems, it is of interest to find a best approximation of it by a univalent function. We intend to compute best approximations of non-vanishing analytic functions of the form z/f in a parabolic region.

In this chapter, we discuss preliminary results and definitions along with a brief description of the work explained in the later chapters. These include meromorphic functions, univalent functions, basic properties of several subclasses of univalent functions, several well known interesting problems on univalent functions, and some main results of the thesis. This chapter also defines the objective of the thesis.

The following section deals with basic literature on analytic univalent functions in the unit disk and importance of considering subclasses of univalent functions.

1.1. Analytic univalent functions

A *domain* is an open connected set in the complex plan $\mathbb{C}$. An analytic function F is said to be *univalent* (or one-one) in a domain $D \subset \mathbb{C}$ if it never takes the same value twice: $F(z_1) \neq F(z_2)$ for all $z_1 \neq z_2$ in D. An analytic function F is said to be *locally univalent* at a point z_0 if it is univalent in some neighborhood of z_0. As an application of Rouche's Theorem (see [96, p. 198]), it is well-known that if F is analytic on D, then $F'(z_0) \neq 0$ if and only if F is locally univalent at z_0. An analytic univalent function F defined in the unit disk $\mathbb{D} := \{z : |z| < 1\}$ has the Taylor series expansion of the form

$$F(z) = A_0 + A_1 z + A_2 z^2 + \cdots . \tag{1.1}$$

Since univalent functions don't possess zero derivatives, $F'(0) = A_1 \neq 0$. So we can divide by A_1 and rewrite (1.1) as

$$f(z) := \frac{F(z) - A_0}{A_1} = z + a_2 z^2 + a_3 z^3 + \cdots , \tag{1.2}$$

where $a_n = A_n / A_1$. We observe that if F is univalent then so is f and vice versa. Thus studying functions of the form (1.2) is sufficient to study general functions of the form (1.1). To normalize an analytic function we use the most usual set of conditions $f(0) = 0$, $f'(0) = 1$, however, other normalizations may also be possible.

The well known Riemann Mapping Theorem was formulated by Riemann in his Ph.D. thesis in 1851 with an incomplete proof. The first complete proof was given by Carathéodory in 1912. The Riemann Mapping Theorem states that for every simply connected domain (i.e. a domain whose complement is connected in the extended complex plane) $D \subsetneq \mathbb{C}$, there is an analytic univalent function $f : \mathbb{D} \to D$ such that f is onto.

Therefore, statements about univalent functions in arbitrary simply connected domains D can be translated into statements about univalent functions in $\mathbb{D}$. For this reason, mathematicians working in this field prefer to study univalent functions in detail in the unit disk. Hence, the unit disk is usually considered as a standard domain for the theory of univalent functions.

Let us denote the family of all analytic functions f defined in $\mathbb{D}$ of the form (1.2) by $\mathcal{A}$. The family of all univalent functions $f \in \mathcal{A}$ is denoted by $\mathcal{S}$. That is,

$$\mathcal{S} = \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ is analytic univalent, } f(0) = 0, \text{ and } f'(0) = 1\}.$$

The *Koebe function* is an important example for the class $\mathcal{S}$ and is defined for $z \in \mathbb{D}$ by

$$k(z) := \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n.$$

It maps $\mathbb{D}$ onto the entire plane excluding the part of the negative real axis $(-\infty, -1/4]$. The Koebe function and its rotations are solutions to many extremal problems for the class $\mathcal{S}$.

The study of the family $\mathcal{S}$ became popular when the Bieberbach conjecture [15] was first posed in 1916 which states that the Taylor coefficients a_n of functions $f \in \mathcal{S}$ which are of the form (1.2) satisfy the inequality $|a_n| \leq n$ and furthermore, equality could only occur if f is some rotation of the Koebe function, i.e. if $f(z) = k_\theta(z) = e^{-i\theta}k(e^{i\theta}z)$. For $n = 2$, the proof of $|a_2| \leq 2$ was given in 1916 by Biebarbach himself. In 1923 Löwner [48] proved $|a_3| \leq 3$ using parametric representation of slit mappings, and in the intervening years it was also proved for $n = 4, 5,$ and 6. In 1925, Littlewood [46] proved that $|a_n| < e \cdot n$ for $n \geq 2$, and this result was refined by Bazilevic [12] in 1951 to be $|a_n| < (e \cdot n)/2 + 1.51,\ n \geq 2$. Biebarbach's conjecture has been attracted by many mathematicians and has inspired to develop important new methods in geometric function theory. One way to encounter this conjecture is to analyze it for some special univalent functions which generate certain subclasses of $\mathcal{S}$. The Bieberbach conjecture was remained as a challenge to all mathematicians until it was solved by de Branges [16] in 1985. Since then, the conjecture is known as the *de Branges Theorem.* In the sequel, there are several intresting results proved in the literature to attempt the Bieberbach conjecture. We refer to the standard books [21, 24, 26, 73] for more details about this.

Closely related to the class $\mathcal{S}$ is the class Σ. By Σ, we denote the class of functions of the form

$$g(z) = z + b_0 + \frac{b_1}{z} + \cdots = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n} \tag{1.3}$$

that are analytic and univalent in the domain $\Omega := \{z : |z| > 1\}$, except for simple pole at infinity with residue 1. The class Σ' denotes the collection of functions g in Σ such that $g(z) \neq 0$ in Ω. It is easy to verify that each $f \in \mathcal{S}$ is associated with a function $g \in \Sigma'$ through the relation $g(z) = \{f(1/z)\}^{-1}$, which gives

$$g(z) = z - a_2 + (a_2^2 - a_3)\frac{1}{z} + \cdots \quad \text{for } z \in \Omega\ .$$

So, there exists a one-to-one correspondence between $\mathcal{S}$ and Σ' (see [21, p 28]). Using a simple geometric argument, Gronwall [28] in 1914 proved the classical area theorem which says that the coefficients of $g \in \Sigma$ satisfy the sharp inequality

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1. \tag{1.4}$$

The first three coefficients of $g \in \Sigma$ satisfy the inequalities $|b_1| \leq 1$, $|b_2| \leq 2/3$ and $|b_3| \leq 1/2 + e^{-6}$. For the class Σ, the problem of finding the bounds of b_n for $n \geq 4$ are still open.

In the following section, we recall certain well-known classes of functions that will help in relating the classes of functions on which our problems are studied.

1.2. Some special subclasses of univalent functions

In this section, we consider some special subclasses of univalent functions defined by simple geometric properties. These classes can more often be characterized by simple mathematical inequalities. A domain $D \subset \mathbb{C}$ is said to be *starlike* with respect to a point $z_0 \in D$ if the line segment joining z_0 to every other point $z \in D$ lies entirely in D. A function $f \in \mathcal{A}$ is called *starlike* if $f(\mathbb{D})$ is a starlike domain with respect to origin. The class of univalent starlike functions is denoted by $\mathcal{S}^*$. This class obeys a very nice analytic characterization that $f \in \mathcal{S}^*$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$

In 1920, Nevanlinna [58] first proved the Bieberbach conjecture for starlike functions.

A domain $D \subset \mathbb{C}$ is said to be *convex* if the line segment joining any two arbitrary points of D lies entirely in D; that is, if it is starlike with respect to each points of D. A function $f \in \mathcal{A}$ is said to be *convex* in $\mathbb{D}$ if $f(\mathbb{D})$ is a convex domain. The class of all univalent convex functions is denoted by $\mathcal{C}$. Analytically, a function $f \in \mathcal{C}$ if and only if

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$

Every convex function is evidently starlike. Thus $\mathcal{C} \subsetneq \mathcal{S}^* \subsetneq \mathcal{S}$. From the above strict inclusion relations, it is evident that there are functions in $\mathcal{S}$ which neither belong to $\mathcal{S}^*$ nor belong to $\mathcal{C}$. However, there is another close analytic connection between convex and starlike functions. Alexander [6] in 1915 first observed that $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$. This is named as the Alexander Theorem. Note that if $l(z) = z/(1-z)$ then $zl'(z) = k(z)$ and $l \in \mathcal{C}$ whereas $k \in \mathcal{S}^* \setminus \mathcal{C}$. The function $l(z)$ maps $\mathbb{D}$ onto the half-plane $\text{Re}\{w\} > -1/2$. The function l plays the role of extremal function for many problems in the class $\mathcal{C}$. For $f \in \mathcal{C}$ of the form (1.2) we have the sharp inequality $|a_n| \leq 1$ for all $n \in \mathbb{N}$ which was proved by Löwner [47] in 1917.

Natural generalizations of $\mathcal{S}^*$ and $\mathcal{C}$ are respectively the so-called the class of starlike functions of order α and convex functions of order α. These classes were generated by Robertson [84] in 1936. A function $f \in \mathcal{A}$ is said to be *starlike of order* α, denoted by $\mathcal{S}^*(\alpha)$ for $0 \leq \alpha < 1$, if

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad \text{for } z \in \mathbb{D}.$$

In particular, we have $\mathcal{S}^*(0) = \mathcal{S}^*$. The class $\mathcal{S}^*(\alpha)$ is meaningful even if $\alpha < 0$, although the univalency may be destroyed in this situation.

A function $f \in \mathcal{A}$ is called *convex of order* α, denoted by $\mathcal{C}(\alpha)$, if, for some $0 \leq \alpha < 1$, $zf'(z)$ belongs to $\mathcal{S}^*(\alpha)$; i.e.

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad \text{for } z \in \mathbb{D}. \tag{1.5}$$

Clearly, if $\alpha = 0$, $\mathcal{C}(0) = \mathcal{C}$. Recall that the class $\mathcal{S}^*(1/2)$ contains the class $\mathcal{C}$ given by Marx [51] and Strhohäcker [95] (see also [54, p. 57]).

There is a beautiful and simple sufficient condition for univalency due to Nashiro [59] (1934 − 35) and Warschawski [98] (1935), and then onward the result is known as the Nashiro-Warschawski Theorem. This says, if a function h is analytic in a convex domain

D and $\mathrm{Re}(e^{i\theta}h'(z)) > 0$, then h is univalent in D, see also [21]. Let f be a function in $\mathcal{A}$. We say that f is *close-to-convex* on $\mathbb{D}$ if there exists a real number $\theta \in (-\pi/2, \pi/2)$ and a convex function g on $\mathbb{D}$ such that

$$\mathrm{Re}\left(e^{i\theta}\frac{f'(z)}{g'(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}. \tag{1.6}$$

In fact it is an equivalent statement of Nashiro-Warschawski Theorem. Note that the condition $\mathrm{Re}(f'/g') > 0$ is equivalent to $\mathrm{Re}(h') > 0$, if we take $h(w) = f(g^{-1}(w))$ where w lies in a convex domain D. Using Alexander's Theorem, we can replace (1.6) by the requirement that

$$\mathrm{Re}\left(e^{i\theta}\frac{zf'(z)}{p(z)}\right) > 0 \quad \text{for } z \in \mathbb{D},$$

here p is a starlike function in the unit disk. We denote the class of close-to-convex functions by $\mathcal{K}$. Obviously $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$. Let $f \in \mathcal{A}$ be locally univalent. Then, according to Kaplan's Theorem [36], it follows that f is close-to-convex if and only if for each r $(0 < r < 1)$ and for each pair of real numbers θ_1 and θ_2 with $\theta_1 < \theta_2$,

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) d\theta > -\pi, \quad z = re^{i\theta}.$$

The notion of starlike domains and starlike functions can be extended by using logarithmic spirals instead of line segments. A *logarithmic γ-spiral* (or *γ-spiral*), $\gamma \in (-\pi/2, \pi/2)$, is a curve in the complex plane given by

$$w(t) = w_0 e^{-te^{i\gamma}} \quad \text{for } t \in \mathbb{R},$$

where $w_0 \in \mathbb{C} \setminus \{0\}$. A domain D containing the origin is said to be *γ-spirallike* with $|\gamma| < \pi/2$ if for all point $w_0 \neq 0$ in D, the arc of the γ-spiral joining w_0 to the origin lies entirely in D. Such a domain is simply connected. A function $f \in \mathcal{S}$ is said to be *γ-spirallike* if $f(\mathbb{D})$ is a γ-spirallike domain. We use $\mathcal{S}_\gamma$ to denote the subclass of $\mathcal{S}$ consisting of γ-spirallike functions and they do not necessarily belong to the starlike family $\mathcal{S}^*$. Obviously $\mathcal{S}_0 = \mathcal{S}^*$. Analytically, a γ-spirallike function f is characterized by the relation

$$\mathrm{Re}\left(e^{-i\gamma}\frac{zf'(z)}{f(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$

The class $\mathcal{S}_\gamma$ was first introduced by Špaček [94] (see also [21]). It is easy to see that the function $k(z) = z(1-z)^{-2e^{i\gamma}\cos\gamma}$ belongs to the class $\mathcal{S}_\gamma$.

There is one natural generalization of γ-spirallike functions which leads to a useful criterion for univalency. The family $\mathcal{S}_\gamma(\alpha)$ of *γ-spirallike functions of order α* is defined by

$$\mathcal{S}_\gamma(\alpha) := \left\{ f \in \mathcal{A} : \mathrm{Re}\left(e^{-i\gamma} \frac{zf'(z)}{f(z)} \right) > \alpha \cos\gamma,\ z \in \mathbb{D} \right\}, \tag{1.7}$$

where $0 \leq \alpha < 1$ and $\gamma \in (-\pi/2, \pi/2)$. Each function in $\mathcal{S}_\gamma(\alpha)$ is univalent in $\mathbb{D}$ (see [45]). Clearly, $\mathcal{S}_\gamma(\alpha) \subset \mathcal{S}_\gamma(0) \subset \mathcal{S}$ whenever $0 \leq \alpha < 1$. Moreover, $\mathcal{S}_0(\alpha) =: \mathcal{S}^*(\alpha)$. More literature on spirallike functions can be found in [4, 45].

For two analytic functions f and g in $\mathbb{D}$, we say that f is *subordinate* to g if $f(z) = g(w(z))$, $|z| < 1$, for some analytic function w in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$. We express this symbolically by $f \prec g$. Note that if g is univalent then the condition $f \prec g$ is equivalent to the conditions $f(0) = g(0)$ and $\{f(z) : |z| < r < 1\} \subset \{g(z) : |z| < r < 1\}$.

We consider the following generalization of γ-spirallike functions. For normalized analytic functions f in $\mathbb{D}$, we consider the class

$$\mathcal{S}^*(A,B) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz},\ z \in \mathbb{D} \right\}$$

for $-1 \leq B \leq 0$, $A \in \mathbb{C}$ and $A \neq B$. Geometrically, when $f \in \mathcal{S}^*(A,B)$, we mean that the values of zf'/f lie on the disk of radius $(|B-A|r)/(1-B^2r^2)$ with center $(1-\overline{A}Br^2)/(1-B^2r^2)$ for $|z| = r < 1$. The class $\mathcal{S}^*(A,B)$ was initially considered by Janowski [33] for the restriction $-1 \leq B < A \leq 1$ and further extensively studied in the literature (see for instance [78, 82]). Note that for $0 \leq \alpha < 1$ we have

$$\mathcal{S}^*(1-2\alpha, -1) = \mathcal{S}^*(\alpha) \quad \text{and} \quad \mathcal{S}^*((1-\alpha)e^{2i\gamma} - \alpha, -1) = \mathcal{S}_\gamma(\alpha),$$

where $\gamma \in (-\pi/2, \pi/2)$.

We also consider the family $\mathcal{S}_p(\alpha)$, $-1 \leq \alpha \leq 1$, studied in [88], associated with parabolic regions:

$$\mathcal{S}_p(\alpha) = \left\{ f \in \mathcal{S} : \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \mathrm{Re}\frac{zf'(z)}{f(z)} - \alpha,\ z \in \mathbb{D} \right\}. \tag{1.8}$$

In the notation $\mathcal{S}_p$, the set $\mathcal{S}$ comes from *schlicht* and the symbol p, not a parameter, comes from *parabolic*. Geometrically, $f \in \mathcal{S}_p(\alpha)$ if and only if the function $zf'(z)/f(z)$, $z \in \mathbb{D}$, satisfy the parabolic inequality

$$(\mathrm{Im}(zf'(z)/f(z)))^2 \leq (1-\alpha)[2\mathrm{Re}(zf'(z)/f(z)) - (1+\alpha)].$$

Clearly, if $-1 \leq \alpha \leq 1$ then $\mathcal{S}_p(\alpha) \subset \mathcal{S}^*$. This is the reason, we can call a function $f \in \mathcal{S}_p(\alpha)$ as a *parabolic starlike function of order* α. Note that if $\alpha < -1$, then the family $\mathcal{S}_p(\alpha)$ must contain non-univalent functions, see [88]. Setting $\mathcal{S}_p := \mathcal{S}_p(0)$, the family of *parabolic starlike functions*. It is appropriate to state that the family $\mathcal{S}_p$ is connected to another family of functions, namely, the family of *uniformly convex functions. Indeed, due to [49, Theorem 2] and [89, Theorem 1], the family $\mathcal{S}_p$ consists of functions $f = zF'$, where F is uniformly convex, i.e. for every circular arc $\gamma \in \mathbb{D}$ centered at ζ the image arc $F(\gamma)$ is convex. One can refer to [25, Theorem 1] for an analytic characterization of uniformly convex functions in $\mathbb{D}$ and more properties on this and its related families can be found from the survey [8]. The class $\mathcal{S}_p$ has also been studied in [62, 81].*

Conformal mappings play extremely important role in complex analysis, as well as in many areas of physics and engineering. The class of conformal mappings turned out to be too restrictive for some problems. For instance, Liouvilles theorem says that the only conformal mappings in $\mathbb{R}^n$, $n \geq 3$, are the Mobius transformations. Hence, the theory of conformal mappings in plane does not directly generalize to the higher dimensions. Thus, a natural generalization of conformal mapping is introduced, namely, quasiconformal mapping. We provide the definition of quasiconformal mappings and some classes of functions associated with this in the next section.

1.3. Quasiconformal mapping

In response to the classical Grötzsch problem raised in 1928, Ahlfors introduced the notion so-called "quasiconformal mappings" in 1935. Quasiconformal mappings are nothing but natural generalizations of conformal mappings. There are several equivalent definitions of quasiconformal mappings in the literature; see for instance [3, 41]. We adopt the following definition of Ahlfors. Let $K \geq 1$. A C^1 homeomorphism f from one region to another is called K-quasiconformal if $D_f \leq K$ where

$$D_f = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \text{ and } \frac{K-1}{K+1} = k < 1\,, \tag{1.9}$$

$f_{\overline{z}} = \partial f/\partial \overline{z}$ and $f_z = \partial f/\partial z$. D_f is called the *dilatation* of f at the point z. Note that f is conformal if and only if $D_f = 1$. Therefore, 1-quasiconformal mappings are nothing but conformal mappings. For basic properties of quasiconformal mappings, we refer to [41].

Let k be defined as in (1.9). We denote $\Sigma(k)$ by the class of all functions $g \in \Sigma$ that admit K-quasiconformal extension to the unit disk $\mathbb{D}$, and $\Sigma_0(k)$ is obtained from $\Sigma(k)$ by assuming $g(0) = 0$. Similarly, let us denote $\mathcal{S}(k)$ by the class of all functions $f \in \mathcal{S}$ that admit K-quasiconformal extension to the plane. Clearly, $f \in \mathcal{S}(k)$ if and only if $1/f(1/\zeta) \in \Sigma_0(k)$.

Every function defined on $\mathbb{D}$ may not be analytic. It may have singularities inside $\mathbb{D}$. The functions for which poles are the only singularities are of independent interest. Such functions are important for several reasons. We present such functions briefly in the next section.

1.4. Meromorphic functions in $\mathbb{D}$ with a simple pole

Recall that a function h which is analytic in a region, except possibly at poles, is said to be *meromorphic* in that region. Hence, analytic functions are by default meromorphic without poles. In this thesis, we consider the family of all meromorphic functions h of the form

$$h(z) = \frac{1}{z} + b_0 + b_1 z + b_2 z^2 + \cdots$$

defined in $\mathbb{D}$. Clearly, h has a simple pole at the origin, and hence it is analytic in the punctured disk $\mathbb{D}_0 := \mathbb{D} \setminus \{0\}$. Let us denote this family of meromorphic functions by $\mathcal{B}$. The set of all univalent functions in $\mathcal{B}$ is usually denoted by Σ_0.

Let us now recall the definition of the *Schwarzian derivative*. Let h be a *meromorphic function* and $h'(z) \neq 0$ in $\mathbb{D}$ (in other words, we say, h is locally univalent in $\mathbb{D}$), then the Schwarzian derivative of h at z is defined as

$$S_h(z) = \left(\frac{h''}{h'}\right)' - \frac{1}{2}\left(\frac{h''}{h'}\right)^2.$$

It is appropriate here to recall from texts that $S_h = 0$ if and only if h is a Möbius transformation (see for instance, [41, p 51]). A quick observation which can easily be verified that

$$h \in \mathcal{B} \iff f = 1/h \in \mathcal{A}. \tag{1.10}$$

A simple computation through (1.10) yields the useful relation

$$S_h(z) = S_f(z)$$

for all locally univalent meromorphic functions $h \in \mathcal{B}$ and $f = 1/h \in \mathcal{A}$. Note that if $f \in \mathcal{A}$ is univalent then (1.10) leads to the useful coefficient relation $|a_2^2 - a_3| = |S_h(0)|/6$; see [21, p. 263].

The remaining section concerns about the definition of a subclasses of the class $\mathcal{B}$, namely, the meromorphically starlike and convex functions of order α having simple pole at $z = 0$. If $h \in \mathcal{B}$ satisfies $h(z) \neq 0$ in $\mathbb{D}_0$ and

$$-\mathrm{Re}\left(\frac{zh'(z)}{h(z)}\right) > \alpha \quad \text{for } z \in \mathbb{D},\ 0 \leq \alpha < 1,$$

then h is said to be *meromorphically starlike of order* α. A function $h \in \mathcal{B}$ is said to be *meromorphically starlike (of order* 0*)* if and only if complement of $h(\mathbb{D}_0)$ is starlike with respect to the origin (see [24, p. 265, Vol. 2]). Note that meromorphically starlike functions are univalent and hence they lie in the class Σ_0. Similarly, if $h \in \mathcal{B}$ satisfies $h(z) \neq 0$ in $\mathbb{D}_0$ and

$$-\mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > \alpha \quad \text{for } z \in \mathbb{D},\ 0 \leq \alpha < 1, \tag{1.11}$$

then h is said to be *meromorphically convex of order* α. If $\alpha = 0$, the inequality (1.11) is equivalent to the definition of meromorphically convex functions. That is, h maps $\mathbb{D}$ onto the complement of a convex region [22, 60]. In this case, we say h is *meromorphically convex*. Note that meromorphically convex functions are also univalent and hence they lie in the class Σ_0. For more geometric properties of these classes, we refer to the standard books [24, 54].

1.5. Outline of the thesis

This thesis consists of seven chapters and each of the remaining chapters presents solution to a number of problems. In the thesis we consider the following problems:

- Area problem

- Successive coefficient problem
- Sufficient conditions invloving Schwarzian derivative
- Approximation problem

1.5.1. Area problem

For an analytic function f in $\mathbb{D}$, we denote by $\Delta(r, f)$, the area of the image of $\mathbb{D}_r$ under f counting multiplicities. If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1},$$

then as an application of the classical Parseval-Gutzmer formula, the Dirichlet integral of f has the area formula [24, Vol 1, pp. 25-26]

$$\Delta(r, f) = \int\int_{\mathbb{D}_r} |f'(z)|^2 dxdy = \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}. \tag{1.12}$$

Estimating the area $\Delta(r, f)$ is called the *area problem for functions of type f*. We call f the *Dirichlet-finite* if $\Delta(1, f) < \infty$. In such situation, we call the quantity $\Delta(1, f)$ as *Dirichlet-finite area of* $f(\mathbb{D})$. The area $\Delta(r, f)$ may not be bounded for all $f \in \mathcal{S}$ as can be seen from the fact that

$$\Delta(r, k) = \pi r^2 (r^4 + 4r^2 + 1)(1 - r^2)^{-4} \to \infty$$

as $r \to 1$, where $k(z)$ is the classical Koebe function. However, surprisingly, as an application of the classical Area Theorem and Bieberbach's Theorem, Yamashita [99] proved in 1990 that $\Delta(r, z/f)$ is bounded for all $f \in \mathcal{S}$. Indeed, he proved that

Theorem A. *[99, Theorem 1] We have*

$$\max_{f\in\mathcal{S}} \Delta\left(r, \frac{z}{f(z)}\right) = 2\pi r^2 (r^2 + 2),$$

for $0 < r \leq 1$. *The maximum is attained only for a suitable rotation of the Koebe function.*

We also call the problem of type Theorem 1 as Yamashita's extremal problem for the class $\mathcal{S}$ or area problem for functions of type z/f when $f \in \mathcal{S}$. Further, in the same paper, he stated a conjecture that $\Delta(r, z/f) \leq \pi r^2$ for all functions $f \in \mathcal{C}$. In 2013, this conjecture was settled in [65]. Indeed, [65] solves the area problem for a wider class, namely, the class $\mathcal{S}^*(\alpha)$ of starlike functions f of order α, $0 \leq \alpha < 1$. Subsequently, in 2014, Ponnusamy and Wirths [82] solved Yamashita's extremal problem for the class $\mathcal{S}_\gamma(\alpha)$ of γ-spirallike functions of order α. Further, in [78], Yamashita's extremal problem for the classes $\mathcal{S}^*(A,0)$ and $\mathcal{S}^*(A,B)$ are proved in the following forms:

Theorem B. *Let $f \in \mathcal{S}^*(A,0)$, $0<|A|\le 1$. Then we have*

$$\Delta\left(r,\frac{z}{f(z)}\right) \le \pi|A|^2r^2{}_0F_1(2;|A|^2r^2).$$

The inequality becomes equality only for the rotations of $k_{A,0}(z)=ze^{Az}$.

Theorem C. *Let $f \in \mathcal{S}^*(A,B)$ for $-1 \le B < 0$ and $A \ne B$. Then we have*

$$\Delta\left(r,\frac{z}{f(z)}\right) \le \pi|\overline{A}-B|^2r^2{}_2F_1\left(\frac{A}{B},\frac{\overline{A}}{B};2;B^2r^2\right),$$

where the equality holds only for the rotations of $k_{A,B}(z)=z(1+Bz)^{A/B-1}$, $B\ne 0$.

Related work in this direction can also be found in [66, 78, 90]. Our objective is to extend the extremal problem of Yamashita from the class $\mathcal{S}$ to itself with quasiconformal extension to the whole complex plane and to extend Theorem B and Theorem C for analytic functions of type $(z/f)^\mu$. Being motivated by the above discussion, we establish Chapter 2 and 3.

Chapter 2 deals with the area problem for functions of type z/f for f in the class $\mathcal{S}$ with quasiconformal extension to the whole complex plane and the motivation to study such problems comes from a conjecture of Yamashita [99] which is settled in [65]. We are interested to discuss the following extremal problem of determining the upper bound of $\Delta(r,z/f)$ where $f\in\mathcal{S}(k)$.

Theorem 1.1. *For $0<r\le 1$, we have*

$$\max_{f\in\mathcal{S}(k)}\Delta\left(r,\frac{z}{f(z)}\right) = 2\pi r^2k^2(2+r^2).$$

The maximum is attained only for a suitable rotation of the function

$$
(1.13) \qquad f(z) = \begin{cases} \dfrac{z}{1-2kz+kz^2}, & \text{for } |z|<1, \\[2ex] \dfrac{z\overline{z}}{\overline{z}-2kz\overline{z}+kz}, & \text{for } |z|\geq 1. \end{cases}
$$

Remark 1.2. *Observe that Theorem 1.1 is a natural extension of Theorem A. In fact, for $k=1$, Theorem 1.1 is equivalent to Theorem A.*

Chapter 3 is again about the area problem. We estimate the areas of images of $\mathbb{D}_r$ under non-vanishing analytic functions of the form $(z/f)^\mu$, $\mu>0$, in principal powers, when f ranges over certain classes of analytic and univalent functions in $\mathbb{D}$. One of our results is for the class $\mathcal{S}^*(A,B)$ and is of the following form:

Theorem 1.3. *Let $f\in\mathcal{S}^*(A,B)$ for $-1\leq B<0$ and $A\neq B$. Then we have*

$$
\Delta\left(r, \left(\frac{z}{f(z)}\right)^\mu\right) \leq \pi|A-B|^2\mu^2r^2 {}_2F_1\left(\left(\frac{A}{B}-1\right)\mu+1, \left(\frac{\overline{A}}{B}-1\right)\mu+1; 2; B^2r^2\right).
$$

The inequality becomes equality only for the rotation of $k_{A,B}(z)=z(1+Bz)^{A/B-1}$, $B\neq 0$.

1.5.2. Successive Coefficient Problem

In general, the coefficient problem is to determine the size of a_n in the complex plane. Recall that one of the popular necessary conditions for a functions f of the form (1.2) to be in $\mathcal{S}$ is the sharp inequality $|a_n|\leq n$ for $n\geq 2$, which was first conjectured by Bieberbach in 1916 and proved by de Branges in 1985. On the other hand, the problem of estimating sharp bound for successive coefficients, namely, $\big||a_{n+1}|-|a_n|\big|$, is also an interesting necessary condition for a function to be in $\mathcal{S}$. This problem was first studied by Goluzin [23] with an idea to solve the Bieberbach conjecture. Several results are known in this direction. For example, Hamilton [29] proved that $\overline{\lim}_{n\to\infty}\big||a_{n+1}|-|a_n|\big|\leq 1$. Prior to this paper, Hayman [30] proved in 1963 that

$$
(1.14) \qquad \big||a_{n+1}|-|a_n|\big| \leq A, \quad n=1,2,3,\ldots,
$$

where $A\geq 1$ is an absolute constant, for functions f in $\mathcal{S}$ of the form (1.2). Milin [52,53] found a simpler approach, which led to the bound $A\leq 9$ and Ilina [32] improved this to $A\leq 4.26$. It is still an open problem to find the minimal value of A which works for all

$f \in \mathcal{S}$, however, the best known bound as of now is 3.61 which is due to Grinspan [27] (see also [53]). The fact that A in (1.14) cannot be replaced by 1 may be seen from the work of [91]. On the other hand, sharp bound is known only for $n = 2$ (see [21, Theorem 3.11]), namely

$$-1 \leq |a_3| - |a_2| \leq 1.029\ldots.$$

Since Schaeffer and Spencer [91] showed that for each $n \geq 2$ there corresponds an odd function $h(z) = z + a_3 z^3 + \cdots$ in $\mathcal{S}$ with all of its coefficients real such that $|a_{2n+1}(h)| > 1$, it is also clear that the constant A in (1.14) must be greater than 1 for odd functions in the class $\mathcal{S}$. Note that for the Koebe function $k(z) = z/(1-z)^2$ and its rotation $e^{-i\theta}k(e^{i\theta}z)$, we have $\big||a_{n+1}| - |a_n|\big| = 1$ for $n \geq 1$.

Concerning the class $\mathcal{S}^*$, Leung [42] (see also [44]) in 1978 has proved that $A = 1$ for starlike functions that was first conjectured by Pommerenke in [73]. More precisely, we have

Theorem D. [42] *For every $f \in \mathcal{S}^*$ given by* (1.2)*, we have*

$$\big||a_{n+1}| - |a_n|\big| \leq 1, \quad n = 1, 2, 3, \ldots.$$

Equality occurs for fixed n only for the function

$$\frac{z}{(1-\gamma z)(1-\zeta z)}$$

for some γ and ζ with $|\gamma| = |\zeta| = 1$.

We remark that, as an application of the triangular inequality, Theorem D leads to $|a_n| \leq n$ for $n \geq 2$ which is the well known coefficient inequality for starlike functions. This is one of reasons for studying the successive coefficients problem in the univalent function theory. From the above discussion, we understand the importance of finding the minimal value of A for functions to be in $\mathcal{S}$. Later, the problem of finding the minimal value of A was considered for certain other subfamilies of univalent functions such as convex, close-to-convex, and spirallike functions. Among other things, Hamilton in [29] has shown some bound for successive coefficients for spirallike functions and for the class of starlike functions of non-positive order. For convex functions, recently Li and Sugawa [44] obtained the sharp upper bound which is $|a_{n+1}| - |a_n| \leq 1/(n+1)$ for $n \geq 2$, and for $n = 2, 3$ sharp lower bounds are $1/2$ and $1/3$, respectively. For $n \geq 4$, it is still an open problem to find the best lower bound for convex functions. These information clearly shows the level of difficulty in determining the bound on the successive coefficients problem and encourage us to study such problem on other subclasses of $\mathcal{S}$. This leads to the results presented in Chapter 4.

In **Chapter 4** we find successive coefficient bounds for functions $f \in \mathcal{S}_\gamma(\alpha)$ which shows that Theorem D continues to hold for γ-spirallike functions. More generally, as a generalization and the extension of Leung's result (Theorem D), we prove the following result whose proof will be presented in Section 3.3.

Theorem 1.4. *For every $f \in \mathcal{S}_\gamma(\alpha)$ of the form* (1.2)*,*

$$\big||a_{n+1}| - |a_n|\big| \leq \exp(-M\alpha\cos\gamma)$$

for some constant $M > 0$ depending on f and n, and for $n \geq 2$.

Note that for $\alpha = 0$, the above theorem extends the result of Leung [42] from starlike to γ-spirallike functions and hence Theorem 1.4 contains the result of Hamilton [29]. For a ready reference, we recall it here. However we get his result as a consequence of a general result with an alternate proof.

Corollary 1.5. *Let $f \in \mathcal{S}_\gamma(0)$ for some $|\gamma| < \pi/2$, and be of the form* (1.2)*. Then*

$$\big||a_{n+1}| - |a_n|\big| \leq 1 \quad \text{for } n \geq 2.$$

1.5.3. Sufficient condition involving Schwarzian derivative

Recall that the de Branges theorem gives a necessary condition for a function f to be in $\mathcal{S}$ in terms of its Taylor's coefficient. On the other hand, several important sufficient conditions for functions to be in $\mathcal{S}$ were also introduced by several researchers to generate its subclasses having interesting geometric properties. In fact, various new families have been introduced, for example, the family of convex functions, starlike functions, close-to-convex functions, etc. Later, counterpart of this development for the family Σ_0 of meromorphic univalent functions were also studied extensively. We refer to the standard books by Duren [21], Goodman [24], Lehto [41], and Pommerenke [74] for the literature on the topic. Therefore, the study of sufficient conditions for functions to be in $\mathcal{S}$, in particular, in its subfamilies are important in this context. We mainly deal with such properties in terms of the well-known Schwarzian derivative of locally univalent functions.

The study of necessary and sufficient conditions for functions to be univalent, in particular to be starlike, convex, close-to-convex, in terms of Schwarzian derivatives is attracted to number of mathematicians. A surprising fact is that most of such necessary conditions are proved using standard theorems in complex variables, whereas sufficient conditions are proved through initial value problems of differential equations; see for instance [21, 41]. The conditions of the form

$$|S_h(z)| \leq \frac{C_0}{(1-|z|^2)^2}, \tag{1.15}$$

for a positive constant C_0, have been most popular to many mathematicians. For instance, Nehari in 1949 first proved that if h is an analytic and locally univalent function in $\mathbb{D}$ satisfying (1.15) with $C_0 = 2$ then h is univalent in $\mathbb{D}$. This condition becomes necessary when the constant $C_0 = 6$; see [55]. Hille [31] showed that the constant 2 in the sufficient condition of Nehari is the best possible constant. Related problems are also investigated in [56, 57, 72]. Thus, applications of the Schwarzian derivative can be seen in second order linear differential equations, univalent functions, and also in Teichmüller spaces [21, 74].

Another form of sufficient condition for univalency in terms of Schwarzian derivative, attracted by many researchers in this field, is

$$|S_h(z)| \leq 2C_1, \tag{1.16}$$

for some positive constant C_1. Note that if $S_h(z)$ is uniformly bounded in $\mathbb{C}$, then the Schwarzian derivative is still well defined. Hence the assumption that h is locally univalent at a point z (or $h'(z) \neq 0$), in (1.16) is not chosen; see also Tichmarsh [96, p. 198]. If $h \in \mathcal{A}$ satisfies (1.16) with $C_1 = \pi^2/4$, then it is proved by Nehari [55] that h is univalent in $\mathbb{D}$. Gabriel [22] studied a sufficient condition for a function $h \in \mathcal{A}$ to be starlike in the form (1.16) for some optimal constant C_1. Sufficient condition in the form (1.16) for convexity of order α is investigated by Chiang in [17]. However, the best possible constant is not yet known in this case. Kim and Sugawa in [38] obtained a sufficient condition in the form (1.16) for starlikeness of order α by fixing the second coefficient of the function.

Gabriel modified Nehari's technique to show univalency and convexity property of functions $h \in \mathcal{B}$ and proved the following:

Theorem E. *[22, Theorem 1] If $h \in \mathcal{B}$ satisfies*

$$|S_h(z)| \le 2c_0 \quad \text{for } |z| < 1, \tag{1.17}$$

where c_0 is the smallest positive root of the equation

$$2\sqrt{x} - \tan\sqrt{x} = 0,$$

then h is univalent in the punctured disk and maps the interior of each circle $|z| = r < 1$ onto the exterior of a convex region. The constant c_0 is the largest possible constant satisfying (1.17).

An analog to this result for meromorphically convex functions of order α is one of our main results which is stated in Theorem 1.6.

Chapter 5 deals with functions whose Schwarzian derivatives are bounded above by some constant, that is, functions satisfying (1.16). We obtain some sufficient condition for starlike, close-to-convex and meromorphically convex functions and one of the main results is stated in the following form:

Theorem 1.6. *Let $0 \le \alpha < 1$. If $h \in \mathcal{B}$ satisfies*

$$|S_h(z)| \le 2c_\alpha \quad \text{for } |z| < 1, \tag{1.18}$$

where c_α is the smallest positive root of the equation

$$2\sqrt{x} - (1+\alpha)\tan\sqrt{x} = 0 \tag{1.19}$$

depending on α, then

a. *h is meromorphically convex of order α; and*
b. *the quantity c_α is the largest possible constant satisfying* (1.18).

In particular, if $\alpha = 0$, Theorem 1.6 reduces to Theorem E.

1.5.4. Approximation Problem

Intuitively, in one hand, researchers started finding the largest disk $\mathbb{D}_r \subset \mathbb{D}$ (or the largest $r < 1$) for which a function $f \in \mathcal{S}$ also belongs to $\mathcal{S}^*$. Such a number r is known as the *radius of starlikeness* in $\mathcal{S}$. Similarly, the radius of convexity was also studied in the literature (see [21] for the best radii of starlikeness and convexity). In the later periods, radii problems for several other families of analytic univalent functions were studied by several authors (see for instance, [20,62,76,79,85]). On the other hand, if a function $f \in \mathcal{S}$ does not belong to $\mathcal{S}^*$ then it is of interest to find a *best approximation* of f by a function $g \in \mathcal{S}^*$. This problem was first considered in 2012 by Pascu and Pascu [69] connecting through a measure of the non-univalency of an analytic function. In the same paper, they derived a method for constructing the best starlike univalent approximations of analytic functions, suitable for both practical problems and numerical implementation. However, the problem of constructing the best starlike univalent approximations of analytic functions was handled by considering the sufficient condition $\sum_{n=2}^{\infty} n|a_n| \leq 1$ for a function $f \in \mathcal{A}$ to be in $\mathcal{S}^*$. In general, sufficient conditions for functions to be in a particular family are helpful to generate functions in that family. In a similar way, in [37, 70], Pascu and his co-authors have respectively studied locally univalent approximations and convex approximations of analytic functions.

To generate functions in a particular family, the best way is to look for suitable sufficient conditions for functions to be in that family. In this context, we recall the following sufficient conditions for functions in the family $\mathcal{S}_p(\alpha)$. For the sake of simplification we use the notation

$$A(n,\alpha) = \frac{2n+1-\alpha}{1-\alpha}, \quad n \geq 1. \tag{1.20}$$

Lemma 1.7. [76] *Let z/f be a non-vanishing analytic function in $\mathbb{D}$ of the form*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad b_n \in \mathbb{C}. \tag{1.21}$$

Then the condition

$$\sum_{n=1}^{\infty} A(n,\alpha)|b_n| \leq 1$$

is sufficient for f to be in the family $\mathcal{S}_p(\alpha)$, where the quantity $A(n,\alpha)$ is defined by (1.20).

Using Lemma 1.7, we now define the following subfamily of $\mathcal{S}_p(\alpha)$:

$$\mathcal{F}_\alpha = \left\{ f \in \mathcal{S} : \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} A(n,\alpha)|b_n| \leq 1,\ -1 \leq \alpha \leq 1 \right\}. \tag{1.22}$$

In order to find the best approximation of an analytic function of type (1.21) by a function z/g, $g \in \mathcal{F}_\alpha$, here we consider a distance between two functions $f, g \in \mathcal{S}$ using the idea of the L^2-norm as follows:

$$d(f,g) := \left(\int_{\mathbb{D}} \left| \frac{z}{f(z)} - \frac{z}{g(z)} \right|^2 dx\,dy \right)^{1/2}, \quad z = x+iy.$$

Note that since z/f and z/g are non-vanishing analytic functions in $\mathbb{D}$, the integral is well-defined and hence the space $(\mathcal{S}, d)$ becomes a metric space. Further, if $f \in \mathcal{S}$ then we define a distance from f to $\mathcal{F}_\alpha$ in the following form:

$$d_\alpha(f, \mathcal{F}_\alpha) := \inf_{g \in \mathcal{F}_\alpha} d(f,g). \tag{1.23}$$

This measures how far is the function f from being in the family $\mathcal{F}_\alpha$ (see Theorem 6.3 for the details). Note that, if $f \in \mathcal{F}_\alpha$ then $d_\alpha(f, \mathcal{F}_\alpha) = 0$.

Chapter 6 focuses on determining the best approximation of an analytic function in the family $\mathcal{F}_\alpha$. For this, we introduce and solve a semi-infinite quadratic programming (Theorem 6.2). With the help of Theorem 6.2, we investigate the following approximation problem.

Theorem 1.8. *Let $f \in \mathcal{S}$ be a function of the form* (1.21) *and assume that*

$$\lim_{n\to\infty} \frac{b_n}{n^2} = 0.$$

The following two properties hold:

- **(i)** *If $\sum_{n=1}^\infty A(n,\alpha)|b_n| \le 1$ then $d_\alpha(f, \mathcal{F}_\alpha) = 0$ and the minimum for the quantity $d_\alpha(f, \mathcal{F}_\alpha)$ is attained by the function $g = f \in \mathcal{F}_\alpha$.*
- **(ii)** *If $\sum_{n=1}^\infty A(n,\alpha)|b_n| > 1$ then*

$$d_\alpha(f, \mathcal{F}_\alpha) = \left(\pi \sum_{n \in \mathcal{I}^c} \frac{|b_n|^2}{n+1} + \pi \frac{\left(\sum_{n\in\mathcal{I}} A(n,\alpha)|b_n| - 1 \right)^2}{\sum_{n\in\mathcal{I}} A^2(n,\alpha)(n+1)} \right)^{1/2},$$

where $\mathcal{I} = \{i_1, i_2, \ldots, i_N\}$ and $(i_n)_{n=1,2,\ldots,|\mathcal{P}|}$ is a permutation of the indices in $\mathcal{P} = \{n \ge 1 : b_n > 0\}$ such that

$$\beta_{i_n} := \frac{2|b_{i_n}|(1-\alpha)}{(i_n+1)(2i_n+1-\alpha)}, \quad n = 1, 2, \ldots, |\mathcal{P}|$$

is a non-increasing sequence. The minimum for the quantity $d_\alpha(f, \mathcal{F}_\alpha)$ *is attained for the function* $g \in \mathcal{F}_\alpha$*, where* $z/g(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ *with*

$$c_n = \begin{cases} \left(|b_n| - \dfrac{A(n,\alpha)(n+1)\Big(\sum_{m\in\mathcal{I}} A(m,\alpha)|b_m| - 1\Big)}{\sum_{m\in\mathcal{I}} A^2(m,\alpha)(m+1)}\right)e^{i\arg b_n}, & n \in \mathcal{I}; \\ 0, & n \in \mathcal{I}^c. \end{cases}$$

Finally, **Chapter 7** deals with concluding remarks and provides some direction for future study.

CHAPTER 2

AREA PROBLEM FOR UNIVALENT FUNCTIONS WITH QUASICONFORMAL EXTENSION

In this chapter[1], we extend the problem of Yamashita to the functions belonging to the family $\mathcal{S}$ having quasiconformal extension to the entire complex plane. We observe that the modified Koebe function studied in [40] does not play an extremal role in our investigation. However, we construct a new function which also extends the Koebe function $z/(1-z)^2$ to the K-quasiconformal setting and show that it plays the extremal role in our problem. Section 2.2 is devoted to the comparison of the areas obtained in Section 2.1 for our extremal function with the modified Koebe function.

2.1. Preliminaries and proof of the main result

We remark that if $f \in \mathcal{S}$ then z/f is non-vanishing and hence, $f \in \mathcal{S}$ may be expressed as

$$f(z) = \frac{z}{F_f(z)}, \quad \text{where } F_f(z) = 1 + \textstyle\sum_{n=1}^{\infty} c_n z^n, \ z \in \mathbb{D}.$$

Yamashita in [99] considered the area problem for functions of type F_f for $f \in \mathcal{S}$, and proved that the area of $F_f(\mathbb{D}_r)$ is bounded (see Theorem A).

Area theorem is so important in the theory of univalent functions which says that the function $g \in \Sigma$ satisfies the sharp inequality (1.4). Lehto [40] generalized the area theorem by assuming the additional hypothesis that g admits a quasiconformal extension to the closed unit disk, where the resultant inequality is sharp. For updated research work related to the area theorem, readers can refer to [14,18]. To consider the Yamashita problem for functions in $\mathcal{S}$ having quasiconformal extension to the entire complex plane, the following theorem of Lehto [40] is useful.

[1]This chapter forms by the paper S. Agrawal, V. Arora, M. R. Mohapatra, and S. K. Sahoo in Bull. Iranian Math. Soc., 45 (2019), no. 4, 1061-1069.

Theorem F. *Let $g \in \Sigma(k)$ be of the form* (1.3)*. Then*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq k^2. \tag{2.1}$$

The equality holds for the function

$$g(z) = \frac{1}{z} + a_0 + a_1 z, \quad z \in \mathbb{D},$$

with $|a_1| = k$. Moreover, its k-quasiconformal extension is given by setting

$$g(z) = \frac{1}{z} + a_0 + \frac{a_1}{\overline{z}}, \quad z \in \overline{\Omega}.$$

We also need an immediate consequence of Theorem F, proved by Lehto in the same paper, which gives the sharp bound for second coefficient of functions in $\mathcal{S}$ having quasiconformal extension to the plane. The consequence is stated as follows, which provides the sharp bound of the second coefficient of $f \in \mathcal{S}(k)$. Note that the definition of $\mathcal{S}(k)$ is provided in Section 1.3 of Chapter 1.

Theorem G. *[40, Corollary 3] For a function $f \in \mathcal{S}(k)$ of the form* (1.2) *with $f(\infty) = \infty$, we have $|a_2| \leq 2k$.*

Using Theorem F and Theorem G, we now prove our main result.

2.1.1. Proof of Theorem 1.1

Let $f \in \mathcal{S}(k)$ be of the form (1.2). Then

$$\frac{1}{f(\frac{1}{z})} = z - a_2 + (a_2^2 - a_3)\frac{1}{z} + \ldots = z + b_1 + \frac{b_2}{z} + \ldots \text{ (say)}.$$

Substituting $1/z$ by z and multiplying z, we obtain

$$F_f(z) = \frac{z}{f(z)} = 1 - a_2 z + (a_2^2 - a_3)z^2 + \ldots = 1 + b_1 z + b_2 z^2 + \ldots$$

It is clear that $b_1 = -a_2$. Now, we compute

$$\begin{aligned}\frac{1}{\pi}\Delta(r, F_f) &= \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} \\ &= |b_1|^2 r^2 + \sum_{n=2}^{\infty} n|b_n|^2 r^{2n} \\ &= |-a_2|^2 r^2 + 2r^4 \sum_{n=1}^{\infty} \frac{n+1}{2} |b_{n+1}|^2 r^{2n-2}.\end{aligned}$$

Using the estimate for a_2 from Theorem G, we obtain

$$\frac{1}{\pi}\Delta(r, F_f) \leq 4r^2k^2 + 2r^4 \sum_{n=1}^{\infty} n|b_{n+1}|^2.$$

Then by Theorem F, we have

$$\frac{1}{\pi}\Delta(r, F_f) \leq 4r^2k^2 + 2r^4k^2 = 2r^2k^2(r^2+2).$$

Now, it remains to consider the sharpness part. For $|z| < 1$, consider the function $f(z) = z/(1 - 2kz + kz^2)$. So, $f_{\overline{z}} = 0$. That is, f is conformal in $\mathbb{D}$. Since $F_f(z) = 1 - 2kz + kz^2$, by (1.12) we obtain

$$\frac{1}{\pi}\Delta(r, F_f) = \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} = 4r^2k^2 + 2r^4k^2 = 2r^2k^2(r^2+2).$$

For $|z| \geq 1$, let

$$f(z) = \frac{z\overline{z}}{\overline{z} - 2kz\overline{z} + kz}.$$

An easy calculation shows that

$$f_{\overline{z}} = \frac{z(\overline{z} - 2kz\overline{z} + kz) - z\overline{z}(1 - 2kz)}{(\overline{z} - 2kz\overline{z} + kz)^2} = \frac{kz^2}{(\overline{z} - 2kz\overline{z} + kz)^2}$$

and

$$f_z = \frac{\overline{z}(\overline{z} - 2kz\overline{z} + kz) - z\overline{z}(-2k\overline{z} + k)}{(\overline{z} - 2kz\overline{z} + kz)^2} = \frac{\overline{z}^2}{(\overline{z} - 2kz\overline{z} + kz)^2}.$$

Thus, $|f_{\overline{z}}/f_z| = k$.

Both the functions defined in (1.13) agree on the boundary $\partial\mathbb{D}$ of $\mathbb{D}$. The proof is complete. □

Remark 2.1. *It is easy to check that for $f \in \mathcal{S}(k)$, $\Delta(1, F_f) \leq 6\pi k^2$ and hence F_f is Dirichlet finite.*

2.2. Comparison of areas

Recall the modified Koebe function from [40] which is defined by

$$g(z) = \begin{cases} \dfrac{z}{(1+ke^{i\phi}z)^2}, & \text{for } |z|<1, \\[2ex] \dfrac{z\overline{z}}{(\sqrt{\overline{z}}+ke^{i\phi}\sqrt{z})^2}, & \text{for } |z|\geq 1. \end{cases} \tag{2.2}$$

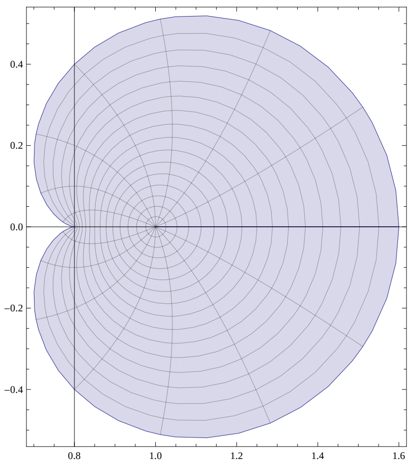
A simple computation yields

$$\Delta(r, F_g) = 2r^2k^2(k^2r^2+2)\pi,$$

which geometrically describes the area of $F_g(\mathbb{D})$. Note that

$$2r^2k^2(k^2r^2+2)\pi = \Delta(r, F_g) < \Delta(r, F_f) = 2r^2k^2(r^2+2).$$

To see the graphical and numerical comparisons of the Dirichlet finites $\Delta(1, F_g)$ and $\Delta(1, F_f)$, we end this section with the following observations (Table 2.1; Figs. 2.1, 2.2, 2.3, 2.4). First we show the graphs of F_f and F_g, where f and g are defined by (1.13) and (2.2) respectively, for different values of k. Observe that as $k \to 1$ the graphs of F_g are approaching to those of F_f.



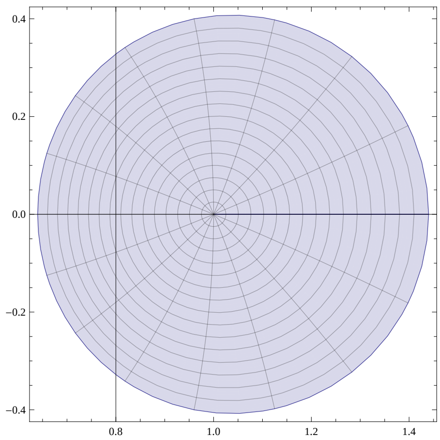


FIGURE 2.1. Graphs of F_f and F_g for $k = 0.2$

Second, for these choices of k, Table 2.1 compares the area $\Delta(1, F_g)$, of the image of $\mathbb{D}$ under F_g, and the area $\Delta(1, F_f)$, of the image of $\mathbb{D}$ under F_f.

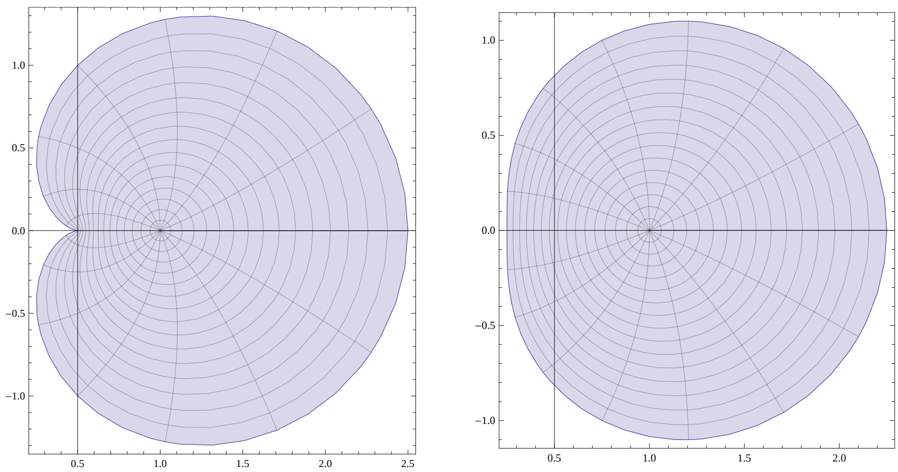


FIGURE 2.2. Graphs of F_f and F_g for $k = 0.5$

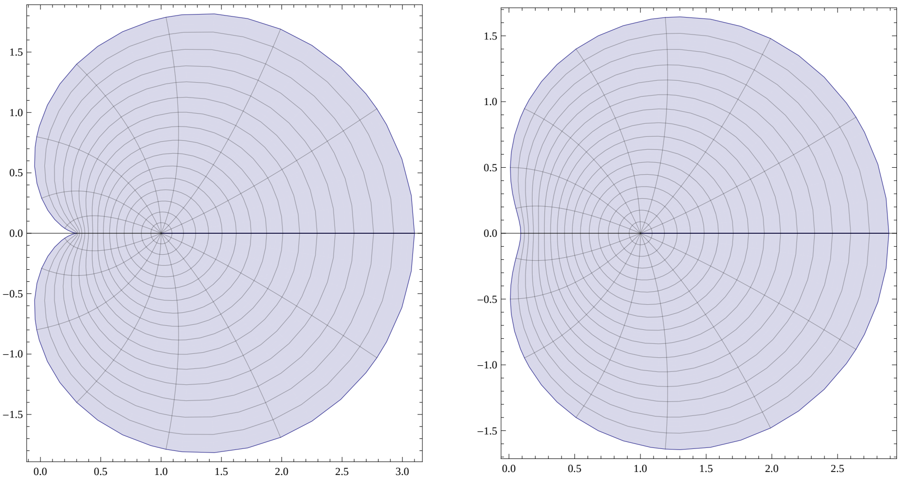


FIGURE 2.3. Graphs of F_f and F_g for $k = 0.7$

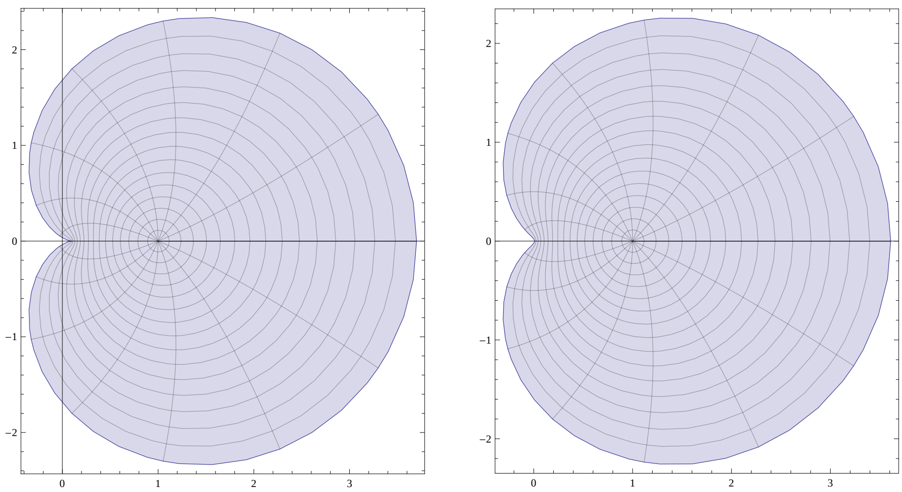


FIGURE 2.4. Graphs of F_f and F_g for $k = 0.9$

k	$\Delta(1, F_g)$	$\Delta(1, F_f)$
0.2	0.1632π	0.24π
0.5	1.125π	1.5π
0.7	2.4402π	2.94π
0.9	4.5522π	4.86π
1	6π	6π

TABLE 2.1. Comparison of areas of $F_f(\mathbb{D})$ and $F_g(\mathbb{D})$.

CHAPTER 3

AREA ESTIMATES OF IMAGES OF DISKS

As we discussed in Chapter 1, Yamashita's extremal problem (or equivalently area problem) for functions of type z/f has been studied for several well-known subclasses of the class $\mathcal{S}$. However, area problems for functions of type $(z/f)^\mu$, $\mu > 0$, have not been considered before although such type of functions are studied in different contexts in the literature; see for instance [76] and references therein. Considering area problems for functions of type $(z/f)^\mu$, where f is in some subclasses of analytic univalent functions, is our main objective of this Chapter[1].

If $f \in \mathcal{S}$ then z/f is a non-vanishing analytic function in $\mathbb{D}$. Let $\mu > 0$. Now, we can write the function $(z/f)^\mu$ of the form

$$\left(\frac{z}{f(z)}\right)^\mu = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}, \tag{3.1}$$

where $(z/f)^\mu$ represents principal powers. Non-vanishing analytic functions of type (3.1) were first considered by Goodman in [24, p. 193, Vol. 2] and later by others; see for instance [76].

3.1. Preliminaries and main results

To state our main results, we need some preparation. Let ${}_2F_1(a, b; c; z)$ denote the Gaussian hypergeometric function defined by

$${}_2F_1(a, b; c; z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad z \in \mathbb{D},$$

where $(a)_n$ denotes the Pochhammer symbol $(a)_n := a(a+1)\cdots(a+n-1)$ for $n \in \mathbb{N}$ and a, b, c are complex numbers such that $c \neq 0, -1, -2, \ldots$. According to the well-known

[1]The results of this chapter will appear in: V. Arora and S.K. Sahoo, Area estimates of images of disks under analytic functions, submitted

Gauss formula, if $\mathrm{Re}\,(c-a-b)>0$ then

$$(3.2)\qquad {}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}<\infty.$$

Similarly, the function ${}_0F_1(c;z)$ is defined as

$${}_0F_1(c;z)=\sum_{n=0}^{\infty}\frac{1}{(c)_n}\frac{z^n}{n!},\quad z\in\mathbb{D},$$

where c is a complex number other than $0,-1,-2,\ldots$.

For a better clarity in our presentation, we divide this section into several subsections consisting of different families of functions from $\mathcal{A}$ and state main results associated with those classes of functions.

3.1.1. The class $\mathcal{S}$

Let us start discussing certain basic observations. Considering first the Koebe function k, we write the series expansion of $(z/k)^{\mu}$ to obtain

$$\left(\frac{z}{k(z)}\right)^{\mu}=(1-z)^{2\mu}=\sum_{n=0}^{\infty}\frac{(-2\mu)_n}{n!}z^n.$$

In this situation, the area formula (1.12) for the function of type $(z/k)^{\mu}$ simplifies to

$$\begin{aligned}\Delta\left(r,\left(\frac{z}{k(z)}\right)^{\mu}\right)&=\pi\sum_{n=1}^{\infty}n\left(\frac{(-2\mu)_n}{n!}\right)^2r^{2n}=\pi r^2\sum_{n=1}^{\infty}n\left(\frac{(-2\mu)_n}{n!}\right)^2r^{2n-2}\\&=\pi r^2\sum_{n=1}^{\infty}\frac{((-2\mu)_n)^2}{n!(n-1)!}r^{2n-2}=\pi r^2\sum_{n=0}^{\infty}\frac{((-2\mu)_{n+1})^2}{n!(n+1)!}r^{2n}\\&=\pi r^2(-2\mu)^2\sum_{n=0}^{\infty}\frac{((-2\mu+1)_n)^2}{(1)_n(2)_n}r^{2n}\\&=\pi r^2(-2\mu)^2{}_2F_1(-2\mu+1,-2\mu+1;2;r^2).\end{aligned}$$

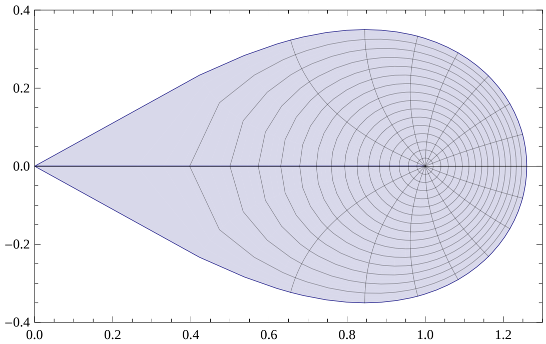
Clearly, the case $\mu=1$ leads to the equality part of Theorem A. This observation motivates us to investigate the following problem:

Problem 3.1. *If $f\in\mathcal{S}$ with $\mu>0$, then*

$$\Delta\left(r,\left(\frac{z}{f(z)}\right)^{\mu}\right)\le 4\pi r^2\mu^2{}_2F_1(-2\mu+1,-2\mu+1;2;r^2).$$

Equality holds only for the rotations of the Koebe function $k(z)=z/(1-z)^2$.

Figure 3.1 and Table 3.1 respectively describe the image domains $(z/k)^{\mu}(\mathbb{D})$ for a couple of particular choices of μ and their respective areas computed numerically.



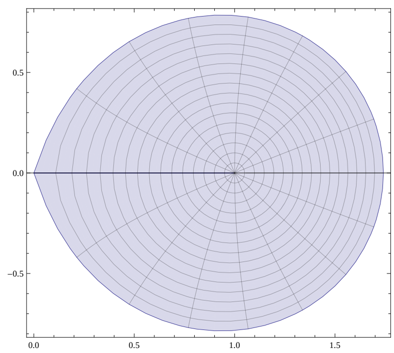


FIGURE 3.1. Images of the unit disk under $(z/k)^{1/6}$ and $(z/k)^{2/5}$

Area of $(z/k)^{1/6}(\mathbb{D})$	$\left(\dfrac{\pi}{9}\right) {}_2F_1(2/3, 2/3; 2; 1) \approx 0.593$
Area of $(z/k)^{2/5}(\mathbb{D})$	$\left(\dfrac{16\pi}{25}\right) {}_2F_1(1/5, 1/5; 2; 1) \approx 2.071$

TABLE 3.1. Dirichlet-finite areas of $(z/k)^{1/6}(\mathbb{D})$ and $(z/k)^{2/5}(\mathbb{D})$

In this chapter, we deal Problem 3.1 only for the case $\mu = 1/2$ (see Theorem 3.2 stated below whose proof is given in Section 3.3). For the remaining positive values of μ (i.e. for $0 < \mu \neq 1/2$), at this moment, we do not have any solutions. Investigation for a complete solution to this problem may lead to new techniques in this development.

Theorem 3.2. *If $f \in \mathcal{S}$ has the form*

$$\sqrt{\frac{z}{f(z)}} = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D},$$

then we have

$$\Delta\left(r, \sqrt{\frac{z}{f(z)}}\right) \leq \pi r^2.$$

The equality holds only for the rotations of the Koebe function.

However, in the following sections, we intend to provide solutions to Problem 3.1 for certain subclasses of analytic functions other than the class $\mathcal{S}$ and see how the areas are estimated in those classes of functions. We start with the following class.

3.1.2. The class $\mathcal{S}^*(A,B)$

Recall the definition of $\mathcal{S}^*(A,B)$ as

$$\mathcal{S}^*(A,B)=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\frac{1+Az}{1+Bz},\ z\in\mathbb{D}\right\},$$

where $-1\leq B\leq 0$, $A\in\mathbb{C}$ and $A\neq B$. An extremal function belonging to the class $\mathcal{S}^*(A,B)$ is obtained in the following way: let $f\in\mathcal{S}^*(A,B)$ and we set $g(z)=(z/f(z))^\mu$. Then g is analytic in $\mathbb{D}$, $g(0)=1$, $g(z)\neq 0$ in $\mathbb{D}$ and a simplification of the logarithm derivative obtains

$$\frac{zf'(z)}{f(z)}=1-\frac{zg'(z)}{\mu g(z)}.$$

Since $f\in\mathcal{S}^*(A,B)$, by the definition we write

$$1-\frac{zg'(z)}{\mu g(z)}\prec\frac{1+Az}{1+Bz},\quad z\in\mathbb{D},$$

or, equivalently,

$$\frac{zg'(z)}{g(z)}\prec\frac{\mu(B-A)z}{1+Bz}=:p(z)\,,\quad z\in\mathbb{D}.$$

Note that $p(0)=0$ and $p(z)$ is clearly convex in $\mathbb{D}$ being $p(\mathbb{D})$ a half-plane. Then

$$\left(\frac{z}{f(z)}\right)^\mu=g(z)\prec\exp\left(\int_0^z\frac{p(t)}{t}\,dt\right)=\left(\frac{z}{k_{A,B}(z)}\right)^\mu,$$

which follows from [54, Corollary 3.1d.1, p. 76]. It is a simple exercise to compute that

$$k_{A,B}(z)=\begin{cases} z(1+Bz)^{(A/B-1)}, & \text{for } B\neq 0,\\ ze^{Az}, & \text{for } B=0,\end{cases}$$

and see that $k_{A,B}(z)\in\mathcal{S}^*(A,B)$. Note that, in most of the situations, the function $k_{A,B}$ acts as a role of an extremal function for the class $\mathcal{S}^*(A,B)$. This can also be seen from several well-known results on this class of functions available in the literature (see for instance, [33, 77]) and our main results stated below.

Our main objective is to extend Theorem B and Theorem C for analytic functions of type $(z/f)^\mu$ defined by (3.1). We also present their consequences resulting to area problems for functions of type $(z/f)^\mu$ when f ranges over the classical classes of analytic functions other than the class $\mathcal{S}$. In this setting, first we state an extension of Theorem B.

Theorem 3.3. *Let $f \in \mathcal{S}^*(A,0)$, $0 < \mu|A| \le 1$. Then we have*

$$\Delta\left(r, \left(\frac{z}{f(z)}\right)^{\mu}\right) \le \pi|A|^2\mu^2r^2{}_0F_1(2;|A|^2\mu^2r^2).$$

The inequality becomes equality only for the rotation of $k_{A,0}(z) = ze^{Az}$.

Indeed, we see that $\mu = 1$ brings Theorem 3.3 back to Theorem B. The following Figure 3.2 and Table 3.2 respectively describe areas of image domains $(z/k_{A,0})^{\mu}(\mathbb{D})$ for some A, μ, and their areas computed numerically with the help of Theorem 3.3.

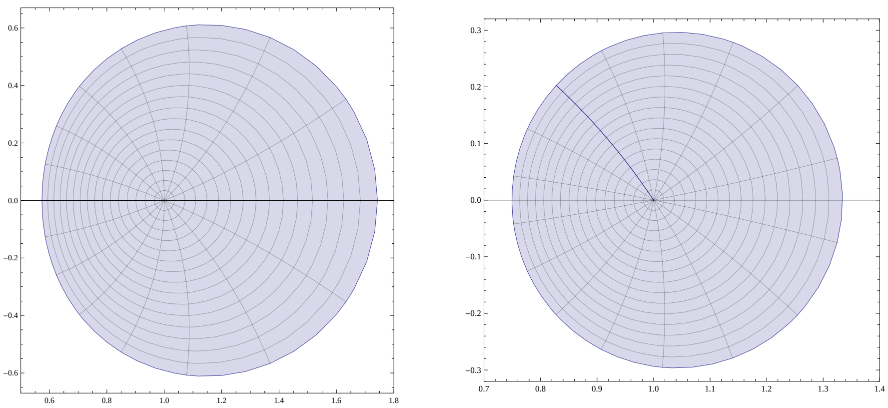


FIGURE 3.2. Images of the unit disk under $(z/k_{5/6,0})^{2/3}$ and $(z/k_{(2-3i)/5,0})^{2/5}$

Area of $(z/k_{5/6,0})^{2/3}(\mathbb{D})$	$\left(\frac{25\pi}{81}\right){}_0F_1(2;25/81) \approx 1.127$
Area of $(z/k_{(2-3i)/5,0})^{2/5}(\mathbb{D})$	$\left(\frac{52\pi}{625}\right){}_2F_1(2;52/625) \approx 0.272$

TABLE 3.2. Dirichlet-finite areas of $(z/k_{5/6,0})^{2/3}(\mathbb{D})$ and $(z/k_{(2-3i)/5,0})^{2/5}(\mathbb{D})$

Similarly, the generalization of Theorem C for $\mathcal{S}^*(A,B)$ presented in Chapter 1 by Theorem 1.3. In Figure 3.3 given below, we describe the image domains $(z/k_{A,B})^{\mu}(\mathbb{D})$ for some choices of A, B and μ, whereas, their areas numerically are given in Table 3.3.

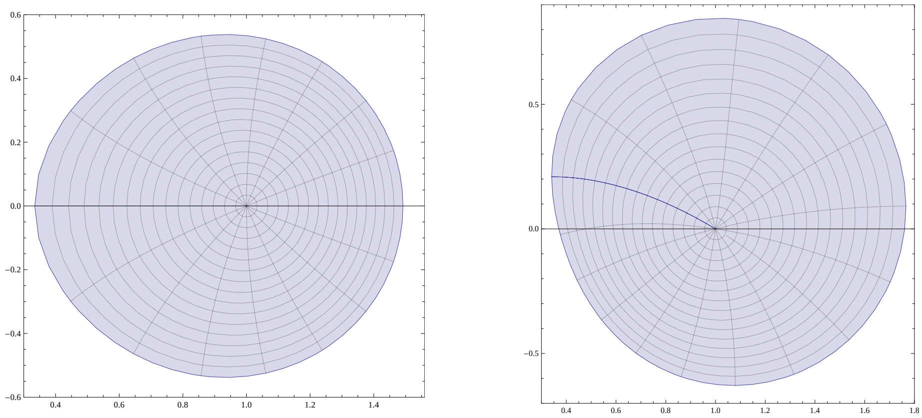


FIGURE 3.3. Images of the unit disk under $(z/k_{5/6,-4/5})^{1/3}$ and $(z/k_{(2-3i)/5,-3/5})^{3/5}$

Area of $(z/k_{5/6,-4/5})^{1/3}(\mathbb{D})$	$\left(\frac{2401\pi}{8100}\right) {}_2F_1(23/72, 23/72; 2; 16/25) \approx 0.970$
Area of $(z/k_{(2-3i)/5,-3/5})^{3/5}(\mathbb{D})$	$\left(\frac{306\pi}{625}\right) {}_2F_1(3i/5, -3i/5; 2; 9/25) \approx 1.647$

TABLE 3.3. Dirichlet-finite areas of $(z/k_{5/6,-4/5})^{1/3}(\mathbb{D})$ and $(z/k_{(2-3i)/5,-3/5})^{3/5}(\mathbb{D})$

3.1.3. The class $\mathcal{S}^*(\alpha)$

As noted in Chapter 1, for a given $0 \leq \alpha < 1$, a function $f \in \mathcal{A}$ is called starlike of order α, denoted by $f \in \mathcal{S}^*(\alpha)$, if $\mathrm{Re}(zf'(z)/f(z)) > \alpha$ for $z \in \mathbb{D}$. Since $\mathcal{S}^*(\alpha) = \mathcal{S}^*(1-2\alpha, -1)$, the substitutions $A = 1 - 2\alpha$ and $B = -1$ bring Theorem 1.3 into the form

Corollary 3.4. *Let $f \in \mathcal{S}^*(\alpha)$, $0 \leq \alpha < 1$. Then we have*

$$\Delta\left(r, \left(\frac{z}{f(z)}\right)^{\mu}\right) \leq 4\pi r^2 \mu^2 (1-\alpha)^2 {}_2F_1(2\mu(\alpha-1)+1, 2\mu(\alpha-1)+1; 2; r^2).$$

The inequality becomes equality only for the rotation of $k_\alpha(z) = z/(1-z)^{2(1-\alpha)}$.

It is now appropriate to remark that when $\mu = 1$, Corollary 3.4 coincides with [65, Theorem 3]. For some specific choices of α and μ, we here present a figure and a table respectively describing the image domains $(z/k_\alpha)^\mu(\mathbb{D})$ and their areas.

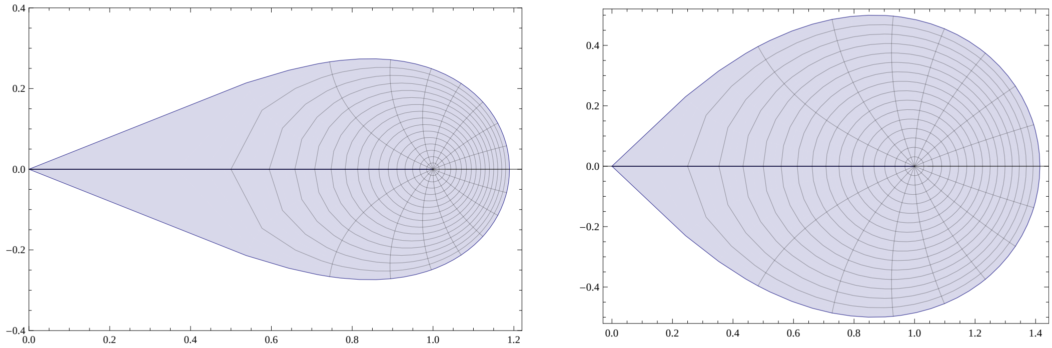


FIGURE 3.4. Images of the unit disk under $(z/k_{1/2})^{1/4}$ and $(z/k_{1/4})^{1/3}$

Area of $(z/k_{1/2})^{1/4}(\mathbb{D})$	$(\pi/16)_2F_1(3/4,3/4;2;1) \approx 0.424$
Area of $(z/k_{1/4})^{1/3}(\mathbb{D})$	$(\pi/4)_2F_1(1/2,1/2;2;1) = 1$

TABLE 3.4. Dirichlet-finite areas of $(z/k_{1/2})^{1/4}(\mathbb{D})$ and $(z/k_{1/4})^{1/3}(\mathbb{D})$

3.1.4. The class $\mathcal{C}$

As stated in Chapter 1, $f \in \mathcal{C}$ are characterized by the condition $\mathrm{Re}\,(1 + zf''(z)/f'(z)) > 0$ for $z \in \mathbb{D}$. Recall from the literature that the class $\mathcal{S}^*(1/2)$ contains the class $\mathcal{C}$ of normalized convex univalent functions (see for instance [54, p. 57]). Thus, if we choose $\alpha = 1/2$ in Corollary 3.4, as a consequence of it one obtains

Corollary 3.5. *Let $f \in \mathcal{C}$ be of the form* (3.1)*. Then we have*

$$\Delta\left(r, \left(\frac{z}{f(z)}\right)^{\mu}\right) \leq \pi r^2 \mu^2 {}_2F_1(1-\mu, 1-\mu; 2; r^2).$$

The equality holds only for the rotations of the function $l(z) = z/(1-z)$.

Corollary 3.5, for $\mu = 1$, was initially a conjecture raised by Yamashita in 1990 (see [99, p. 439]) and it was settled after twenty three years by Obradović et al. (see [65, Theorem 2]). However, none of the techniques so far developed are applicable to solve the area problem for functions of type z/f when f ranges over the class $\mathcal{C}(\alpha)$ of convex functions of order α, $0 \leq \alpha < 1$, that is, f satisfies $\mathrm{Re}\,(1 + zf''(z)/f'(z)) > \alpha$, $z \in \mathbb{D}$. Figure 3.5 and Table 3.5 describe maximal areas of the domains $(z/f)^2\mathbb{D}_r$, $r = 1, 0.5$, when f ranges over the class $\mathcal{C}$.

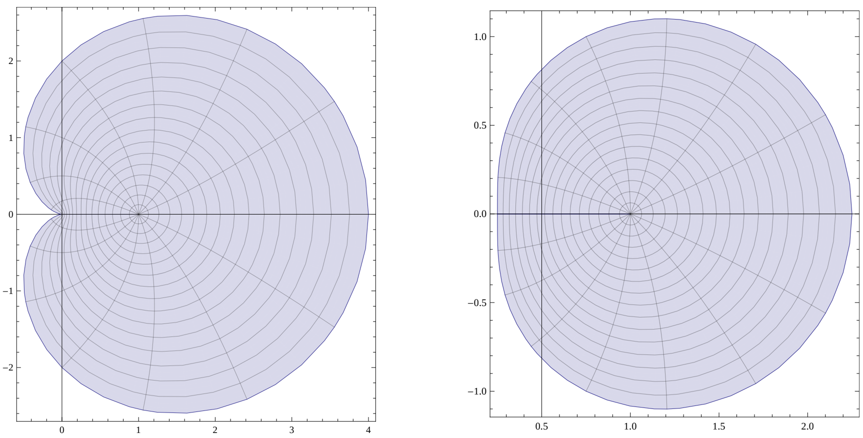


FIGURE 3.5. Images of $\mathbb{D}$ and $\mathbb{D}_{0.5}$ under $(z/l)^2$

Area of $(z/l)^2(\mathbb{D})$	$4\pi {}_2F_1(-1,-1;2;1) \approx 18.850$
Area of $(z/l)^2(\mathbb{D}_{0.5})$	$\pi {}_2F_1(-1,-1;2;0.25) \approx 3.534$

TABLE 3.5. Areas of $(z/l(z))^2(\mathbb{D}_r)$, $r = 1, 0.5$

3.1.5. The class $\mathcal{S}_\gamma(\alpha)$

Recall from Chapter 1 that for $\alpha \in [0,1)$ and $\gamma \in (-\pi/2, \pi/2)$, the class $\mathcal{S}_\gamma(\alpha)$ of γ-spirallike functions of order α is defined by (1.7). Also as noted in Section 1.5.1, Yamashita's area problem for functions of type z/f when f ranges over the class $\mathcal{S}_\gamma(\alpha)$ was proved in [82]. Indeed, by choosing $A = (1-\alpha)e^{2i\gamma} - \alpha$ and $B = -1$ in Theorem 1.3, we have a more general result as follows.

Corollary 3.6. *Let $f \in \mathcal{S}_\gamma(\alpha)$ for $0 \le \alpha < 1$ and $\gamma \in (-\pi/2, \pi/2)$. Then we have*

$$\Delta\left(r, \left(\frac{z}{f(z)}\right)^\mu\right) \le \pi|\delta|^2 r^2 {}_2F_1(1-\delta, 1-\overline{\delta}; 2; r^2),$$

with $\delta = \mu(1-\alpha)(1+e^{2i\gamma}) = 2\mu(1-\alpha)e^{i\gamma}\cos\gamma$. The inequality becomes equality only for the rotations of $k_\gamma(\alpha)(z) = z/(1-z)^{2(1-\alpha)e^{i\gamma}\cos\gamma}$.

If one chooses $\mu = 1$, then Corollary 3.6 reduces to [82, Theorem 3].

3.1.6. Dirichlet Finite

We end this section by verifying the cases for which the function $(z/f)^\mu$ is Dirichlet finite when $f \in \mathcal{S}^*(A,B)$. First, recall from Theorem 1.3 that if $f \in \mathcal{S}^*(A,B)$ for

$-1 \leq B < 0$ and $A \neq B$, then we have

$$\Delta\left(r, \left(\frac{z}{f(z)}\right)^{\mu}\right) \leq E_{\mu,A,B}(r),$$

where $E_{\mu,A,B}(r)$ can be rewritten as

$$E_{\mu,A,B}(r) = \pi|A-B|^2\mu^2r^2\left(1+\sum_{n=1}^{\infty}\frac{|((A/B-1)\mu+1)_n|^2}{(1)_n(2)_n}B^{2n}r^{2n}\right).$$

We notice that the coefficients of the series are all non-negative with some are in positive powers. Thus, the function $E_{\mu,A,B}(r)$ is an increasing function of real variable r, $0 < r \leq 1$. This observation shows that

$$E_{\mu,A,B}(r) \leq E_{\mu,A,B}(1) = \pi|A-B|^2\mu^2{}_2F_1((A/B-1)\mu+1, (\overline{A}/B-1)\mu+1; 2; B^2).$$

Now, for $B = -1$, with the help of (3.2), the last expression becomes

$$E_{\mu,A,-1}(r) \leq E_{\mu,A,-1}(1) = \pi|A+1|^2\mu^2\frac{\Gamma(2\mu(1+\operatorname{Re} A))}{\Gamma(1+(A+1)\mu)\Gamma(1+(\overline{A}+1)\mu)} < \infty,$$

if $2\mu(1+\operatorname{Re} A) > 0$ i.e., if $\operatorname{Re} A > -1$. This implies that if $f \in \mathcal{S}^*(A,-1)$ then the function $(z/f)^{\mu}$ is Dirichlet finite for $\operatorname{Re} A > -1$. In particular, $(z/f)^{\mu}$ is Dirichlet finite for $f \in \mathcal{S}^*(\alpha)$, $0 \leq \alpha < 1$, as well as for $f \in \mathcal{S}_{\gamma}(\alpha)$, $0 \leq \alpha < 1$ and $-\pi/2 < \gamma < \pi/2$. Similarly, from Theorem 3.3 we have the estimate

$$E_{\mu,A,0}(r) \leq E_{\mu,A,0}(1) = \pi|A|^2\mu^2\sum_{n=0}^{\infty}\frac{1}{(1)_n(2)_n}(|A|\mu)^{2n} = \pi|A|^2\mu^2{}_0F_1(2; |A|^2\mu^2).$$

3.2. Preliminary Results

Recall the following result (see [24, Theorem 11, p. 193, Vol-2] and also [76]) which is required for proving Theorem 3.2.

Lemma 3.7. *Let $\mu > 0$ and $f \in \mathcal{S}$ be in the form* (3.1)*. Then we have*

$$\sum_{n=1}^{\infty}(n-\mu)|b_n|^2 \leq \mu.$$

A refinement of [78, Lemma 3.1] for functions $f \in \mathcal{S}^*(A,B)$ of the form (3.1) is the following lemma. This is used to prove Theorem 3.3 and Theorem 1.3.

Lemma 3.8. *Let $\mu > 0$ and $f \in \mathcal{S}^*(A,B)$, $-1 \le B \le 0$ and $A \neq B$, be of the form* (3.1). *Then we have*

$$\sum_{n=1}^{\infty}(n^2 - |(B-A)\mu - Bn|^2)|b_n|^2 \le |B-A|^2\mu^2.$$

Proof. Let $f \in \mathcal{S}^*(A,B)$ and setting

$$g(z) = \left(\frac{z}{f(z)}\right)^{\mu} = 1 + \sum_{n=1}^{\infty} b_n z^n.$$

The logarithmic derivative of $g(z)$ leads to

$$\frac{zg'(z)}{g(z)} = \mu\left(1 - \frac{zf'(z)}{f(z)}\right).$$

Rewrite the above equation and use the definition of $\mathcal{S}^*(A,B)$, we obtain

$$1 - \frac{1}{\mu}\frac{zg'(z)}{g(z)} = \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}.$$

Then by the definition of subordination, there exist a function $w : \mathbb{D} \to \mathbb{D}$ analytic in the unit disk such that

$$\frac{1}{\mu}\frac{zg'(z)}{g(z)} = \frac{(B-A)zw(z)}{1+Bzw(z)},$$

and so

$$g'(z) = [(B-A)\mu g(z) - Bzg'(z)]w(z). \tag{3.3}$$

Writing this in series form, we obtain

$$\sum_{k=1}^{\infty} kb_k z^{k-1} = \left((B-A)\mu\sum_{k=0}^{\infty} b_k z^k - Bz\sum_{k=1}^{\infty} kb_k z^{k-1}\right)w(z),$$

or equivalently,

$$\sum_{k=1}^{n} kb_k z^{k-1} + \sum_{k=n+1}^{\infty} kb_k z^{k-1} = \left(\sum_{k=0}^{\infty}\Big((B-A)\mu b_k - Bkb_k\Big)z^k\right)w(z).$$

By Clunie's method [19] (see also [65]), for any $n \in \mathbb{N}$, we derive the inequality

$$\sum_{k=1}^{n} k^2|b_k|^2 r^{2k-2} \le \sum_{k=0}^{n-1} |(B-A)\mu - Bk|^2|b_k|^2 r^{2k},$$

for $0 < r \le 1$. A simplification leads to

$$\sum_{k=1}^{n-1} |b_k|^2(k^2 - |(B-A)\mu - Bk|^2 r^2)r^{2k-2} + n^2|b_n|^2 r^{2n-2} \le |B-A|^2\mu^2.$$

Multiplying by r^2, we obtain

$$\sum_{k=1}^{n-1} |b_k|^2 (k^2 - |(B-A)\mu - Bk|^2 r^2) r^{2k} + n^2 |b_n|^2 r^{2n} \le |B-A|^2 \mu^2 r^2. \tag{3.4}$$

Allowing $r = 1$ and $n \to \infty$ in (3.4), we obtain our desired inequality. □

If we choose $A = 1 - 2\alpha$ and $B = -1$, as an immediate consequence of Lemma 3.8, we have a generalized version of [65, Lemma 1] for the functions $f \in \mathcal{S}^*(\alpha)$ but of the form (3.1).

Corollary 3.9. *If a function f of the form (3.1) with $\mu > 0$ is in $\mathcal{S}^*(\alpha)$, $0 \le \alpha < 1$, we then have the following inequality*

$$\sum_{n=1}^{\infty} (n - (1-\alpha)\mu) |b_n|^2 \le (1-\alpha)\mu.$$

The following lemma plays a crucial role in the proof of Theorem 3.3.

Lemma 3.10. *Let $f \in \mathcal{S}^*(A, 0)$, with $0 < \mu|A| \le 1$. If f is of the form (3.1) and*

$$h(z) = e^{-A\mu z} = 1 + \sum_{n=1}^{\infty} (-1)^n c_n z^n,$$

then for any $n_0 \in \mathbb{N}$ we have the valid inequality

$$\sum_{k=1}^{n_0} k |b_k|^2 r^{2k} \le \sum_{k=1}^{n_0} k |c_k|^2 r^{2k}, \tag{3.5}$$

for $0 < r \le 1$.

Proof. First we check that the function $h(z) = e^{-A\mu z}$ satisfies the differential equation $h'(z) = -A\mu h(z)$. By comparing with (3.3), we conclude that the equality in (3.4) holds for $b_k = (-1)^k c_k$ when $n \to \infty$. Rewrite the equation (3.4) in the form

$$\sum_{k=1}^{n-1} |b_k|^2 (k^2 - |A_0|^2 r^2) r^{2k} + n^2 |b_n|^2 r^{2n} \le |A_0|^2 r^2, \tag{3.6}$$

where $A_0 = A\mu$. Following the ideas from [78, Lemma 3.2], we devided the remaining proof into two steps.

Step 1: Cramer's Rule. We consider the inequalities corresponding to (3.6) for $n = 1, \dots, n_0$ and multiply the n-th inequality by a factor λ_{n,n_0}. Here we are choosing λ_{n,n_0} in such a way that the addition of the left sides of the modified inequalities results the left side of (3.5). To find the factors λ_{n,n_0}, we obtain the following system of linear equations

$$k = k^2\lambda_{k,n_0} + \sum_{n=k+1}^{n_0} \lambda_{n,n_0}(k^2 - {A_0}^2 r^2), \quad k = 1, \dots, n_0. \tag{3.7}$$

The solution of this system is uniquely determined since the matrix of this system is upper triangular matrix with positive integers as diagonal elements. We can get the solution of the system (3.7) in the form

$$\lambda_{n,n_0} = \frac{((n-1)!)^2}{(n_0!)^2} Det\, A_{n,n_0}$$

using Cramer's rule, where A_{n,n_0} is the $(n_0 - n + 1) \times (n_0 - n + 1)$ matrix constructed as follows:

$$A_{n,n_0} = \begin{bmatrix} n & n^2 - {A_0}^2 r^2 & \cdots & n^2 - {A_0}^2 r^2 \\ n+1 & (n+1)^2 & \cdots & (n+1)^2 - {A_0}^2 r^2 \\ \vdots & \vdots & \vdots & \vdots \\ n_0 & 0 & \cdots & n_0^2 \end{bmatrix}.$$

Determinants of these matrices can be obtained by expanding according to Laplace's rule with respect to the last row, wherein the first coefficient is n_0 and the last one is n_0^2. The rest entries are zeros. This expansion and a little mathematical induction results in the following formula. If $k \le n_0 - 1$, then

$$\lambda_{k,n_0} = \lambda_{k,n_0-1} - \frac{1}{n_0}\left(1 - \frac{{A_0}^2 r^2}{k^2}\right) \prod_{m=k+1}^{n_0-1} \left(\frac{{A_0}^2 r^2}{m^2}\right)$$

For fixed $k \in \mathbb{N}$ and $n_0 \ge k$, we see that the sequence $\{\lambda_{k,n_0}\}$ is strictly decreasing, i.e. $\lambda_{k,n_0} - \lambda_{k,n_0-1} < 0$ with

$$\lambda_k = \lim_{n_0 \to \infty} \lambda_{k,n_0} = \frac{1}{k} - \left(1 - \frac{{A_0}^2 r^2}{k^2}\right) \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{m=k+1}^{n-1} \left(\frac{{A_0}^2 r^2}{m^2}\right).$$

To prove that $\lambda_{k,n_0} > 0$ for all $n_0 \in \mathbb{N}$, $1 \le k \le n_0$, it is adequate to show that $\lambda_k \ge 0$ for $k \in \mathbb{N}$. This will be done in step 2. But before that we want to remarks that the proof of the said inequality is sufficient for the proof of the theorem, since, as we remarked, for (3.6) equality holds for $b_k = (-1)^k c_k$.

Step 2: Positivity of Multipliers. In this step we need to show $\lambda_k \geq 0$ for $k \in \mathbb{N}$, or equivalently

$$\sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{m=k+1}^{n-1} \left(\frac{{A_0}^2 r^2}{m^2}\right) \leq \frac{1}{k\left(1 - \dfrac{{A_0}^2 r^2}{k^2}\right)} = \frac{1}{k} \sum_{n=0}^{\infty} \left(\frac{{A_0}^2 r^2}{k^2}\right)^n$$

which is indeed easy to prove. This completes the proof of Lemma 3.10.

□

Lemma 3.11. *For $-1 \leq B < 0$ and $A \neq B$ let $f \in \mathcal{S}^*(A, B)$. If f is of the form* (3.1) *and*

$$h(z) = (1 + Bz)^{\mu(1-A/B)} = 1 + \sum_{n=1}^{\infty} (-1)^n c_n z^n,$$

then for any $n_0 \in \mathbb{N}$ we have the valid inequality

$$\sum_{k=1}^{n_0} k|b_k|^2 r^{2k} \leq \sum_{k=1}^{n_0} k|c_k|^2 r^{2k}, \tag{3.8}$$

for $0 < r \leq 1$.

Proof. Rewrite the equation (3.4) in the form

$$\sum_{k=1}^{n-1} (k^2 - |k - \phi|^2 B^2 r^2)|b_k|^2 r^{2k} + n^2 |b_n|^2 r^{2n} \leq B^2 |\phi|^2 r^2 \tag{3.9}$$

where $\phi = (B - A)\mu/B$. As the function $h(z) = (1 + Bz)^{\mu(1-A/B)}$ satisfies the differential equation

$$g'(z) = (B - A)\mu g(z) - Bzg'(z), \quad z \in \mathbb{D},$$

it is clear from a similar agument as in Lemma 3.10 that, in the inequality (3.9), equality is attained for $b_k = (-1)^k c_k$.

Rest of the proof is divided into two steps.

Step 1: Cramer's Rule. We consider the inequalities corresponding to (3.9) for $n = 1, \ldots, n_0$ and multiply the nth coefficient by a factor λ_{n,n_0}. These factors are chosen in such a way that the addition of the left sides of the modified inequalities results the left side of (3.8). To obtain the factors λ_{n,n_0} we get the following system of linear equations

$$k = k^2 \lambda_{k,n_0} + \sum_{n=k+1}^{n_0} \lambda_{n,n_0} (k^2 - |k - \phi|^2 B^2 r^2), \quad k = 1, \ldots, n_0. \tag{3.10}$$

Since the matrix of this system is an upper triangular matrix with positive integers as diagonal elements, the solution of this system is uniquely determined. Cramer's rule allows us to write the solution of the system is (3.10) in the form

$$\lambda_{n,n_0} = \frac{((n-1)!)^2}{(n_0!)^2} Det\, A_{n,n_0}$$

where A_{n,n_0} is the $(n_0 - n + 1) \times (n_0 - n + 1)$ matrix constructed as follows:

$$A_{n,n_0} = \begin{bmatrix} n & n^2 - |n-\phi|^2 B^2 r^2 & \cdots & n^2 - |n-\phi|^2 B^2 r^2 \\ n+1 & (n+1)^2 & \cdots & (n+1)^2 - |n+1-\phi|^2 B^2 r^2 \\ \vdots & \vdots & \vdots & \vdots \\ n_0 & 0 & \cdots & n_0^2 \end{bmatrix}.$$

The evauation of the determinants of these matrices can be done by expanding according to Laplace's rule with respect to the last row, wherein the first coefficient is n_0 and the last one is ${n_0}^2$. The rest of the entries are zeros. This expansion and a mathematical induction results in the following formula. If $k \le n_0 - 1$, then

$$\lambda_{k,n_0} = \lambda_{k,n_0-1} - \frac{1}{n_0}\left(1 - \left|1 - \frac{\phi}{k}\right|^2 B^2 r^2\right) \prod_{m=k+1}^{n_0-1} \left|1 - \frac{\phi}{m}\right|^2 B^2 r^2.$$

Let us use the abbreviation $V_k = V_k(A, B, \mu) = 1 - \left|1 - \phi/k\right|^2 B^2 r^2$, we get

$$\lambda_{k,n_0} = \lambda_{k,n_0-1} - \frac{1}{n_0} V_k \prod_{m=k+1}^{n_0-1} (1 - V_m). \tag{3.11}$$

Case (i): Suppose that V_k is negative. From the relation (3.11), we see that for fixed $k \in \mathbb{N}, k \le n_0 - 1$, the sequence $\{\lambda_{k,n_0}\}$ is strictly increasing, i.e.,

$$\lambda_{k,n_0} - \lambda_{k,n_0-1} > 0$$

so that

$$\lambda_{k,n_0} > \lambda_{k,n_0-1} > \cdots > \lambda_{k,k} = 1/k > 0,$$

and thus $\lambda_k \ge 0$ when $n_0 \to \infty$ as required.

Case (ii): Now, we consider that V_k is positive. For fixed $k \in \mathbb{N}$, $n_0 \geq k$, the sequence $\{\lambda_{k,n_0}\}$ is strictly decreasing, i.e. $\lambda_{k,n_0} - \lambda_{k,n_0-1} < 0$ with

$$\lambda_k = \lim_{n_0 \to \infty} \lambda_{k,n_0} = \frac{1}{k} - V_k \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{m=k+1}^{n-1} (1 - V_m). \tag{3.12}$$

For all $n_0 \in \mathbb{N}$, $1 \leq k \leq n_0$, to prove that $\lambda_{k,n_0} > 0$, it is sufficient to prove $\lambda_k \geq 0$ for $k \in \mathbb{N}$. This will be completed in Step 2. But before that we want to annotate that the proof of the said inequality is sufficient for the proof of the theorem, since, as we noted in the begining of the proof, equality is received for $b_k = (-1)^k c_k$.

Step 2: Positivity of the multipliers. Set as an abbreviation

$$S_k = \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{m=k+1}^{n-1} (1 - V_k), \quad k \in \mathbb{N}.$$

We now prove that

$$S_k \leq \frac{1}{kV_k}$$

We see from (3.12) that

$$\lambda_k = \frac{1}{k} - S_k + (1 - V_k) S_k.$$

Again set for abbreviation

$$T_k = \frac{1}{k} + (1 - V_k) S_k.$$

It suffices to show that

$$T_k \leq \frac{1}{kV_k} \tag{3.13}$$

To prove (3.13), we use the following inequality and the identity

$$\frac{1}{nV_n} > \frac{1}{(n+1)V_{n+1}} \quad \text{and} \quad \frac{1}{nV_n} = \frac{1}{n} + \frac{1 - V_n}{nV_n} \tag{3.14}$$

which are valid for each $n \in \mathbb{N}$. Repeated application of (3.14) for $n = k, k+1, \ldots, P$ results the inequality

$$\frac{1}{kV_k} > \sum_{n=k}^{P} \frac{1}{n} \prod_{m=k}^{n-1} (1 - V_m) + \frac{\prod_{m=k}^{P} (1 - V_k)}{PV_P} = S_{k,P} + R_{k,P}, \quad \text{for } k \leq P.$$

Since $R_{k,P} > 0$, taking the limit as $P \to \infty$ we obtain

$$\frac{1}{kV_k} \geq \lim_{P\to\infty} S_{k,P} = \sum_{n=k}^{\infty} \frac{1}{n} \prod_{m=k}^{n-1} (1 - V_m),$$

and we complete the inequality (3.13). This completes the proof of Lemma 3.11. □

3.3. Proof of the main results

We begin with the proof of Theorem 3.2

3.3.1. Proof of Theorem 3.2

We first estimate

$$\begin{aligned}
\sum_{n=1}^{\infty} n|b_n|^2 r^{2n} &\leq r^2 \sum_{n=1}^{\infty} n|b_n|^2 \\
&= r^2 \left(\sum_{n=1}^{\infty} (2n-1)|b_n|^2 - \sum_{n=2}^{\infty} (n-1)|b_n|^2 \right) \\
&\leq r^2 \sum_{n=1}^{\infty} (2n-1)|b_n|^2 \\
&\leq r^2,
\end{aligned}$$

where the last inequality follows from Lemma 3.7 with $\mu = 1/2$ and hence

$$\Delta\left(r, \sqrt{\frac{z}{f(z)}}\right) = \pi \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} \leq \pi r^2 = \Delta\left(r, \sqrt{\frac{z}{k(z)}}\right),$$

completing the proof. □

3.3.2. Proof of Theorem 3.3

Consider the function $k_{A,0}(z) = ze^{Az}$. It is a simple exercise to compute that

$$\left(\frac{z}{k_{A,0}(z)}\right)^{\mu} = e^{-A\mu z} = \sum_{n=1}^{\infty} (-1)^n \frac{(A\mu z)^n}{n!}.$$

It directly follows from the area formula (1.12) that

$$\begin{aligned}\pi^{-1}\Delta\left(r,\left(\frac{z}{k_{A,0}(z)}\right)^{\mu}\right) &= \sum_{n=1}^{\infty} n\frac{|A\mu|^{2n}}{(n!)^2}r^{2n}\\ &= |A|^2\mu^2r^2\sum_{n=0}^{\infty}\frac{1}{(1)_n(2)_n}(|A|\mu r)^{2n}\\ &= |A|^2\mu^2r^2{}_0F_1(2;|A|^2\mu^2r^2),\end{aligned}$$

for $\mu|A|\leq 1$. Note that from the last identity, it is enough to prove the inequality $\Delta(r,(z/f)^{\mu})\leq\Delta(r,(z/k_{A,0})^{\mu})$ to conclude the proof. Therefore, we may deduce the inequality from Lemma 3.10 by letting $n_0\to\infty$

$$\Delta\left(r,\left(\frac{z}{f(z)}\right)^{\mu}\right)=\pi\sum_{k=1}^{\infty}k|b_k|^2r^{2k}\leq\pi\sum_{k=1}^{\infty}k|c_k|^2r^{2k}=\Delta\left(r,\left(\frac{z}{k_{A,0}(z)}\right)^{\mu}\right),$$

since $c_k=(A\mu)^k/k!$ in this case. This completes the proof of Theorem 3.3. □

3.3.3. Proof of Theorem 1.3

Consider the function $k_{A,B}(z)=z(1+Bz)^{A/B-1}$. An easy computation shows that

$$\left(\frac{z}{k_{A,B}(z)}\right)^{\mu}=(1+Bz)^{(1-A/B)\mu}=1+\sum_{n=1}^{\infty}(-1)^nc_nz^n=\sum_{n=1}^{\infty}(-1)^n\frac{(\gamma_0)_n}{n!}B^nz^n,$$

where $\gamma_0=(A/B-1)\mu$. Now applying the area formula (1.12) to obtain

$$\begin{aligned}\pi^{-1}\Delta\left(r,\left(\frac{z}{k_{A,B}(z)}\right)^{\mu}\right) &= \sum_{n=1}^{\infty}n\frac{|(\gamma_0)_n|^2}{(n!)^2}B^{2n}r^{2n}=\sum_{n=1}^{\infty}n\frac{(\gamma_0)_n(\overline{\gamma_0})_n}{(n!)^2}B^{2n}r^{2n}\\ &= |\gamma_0|^2B^2r^2\sum_{n=0}^{\infty}\frac{(\gamma_0+1)_n(\overline{\gamma_0}+1)_n}{(1)_n(2)_n}B^{2n}r^{2n}\\ &= |\gamma_0|^2B^2r^2{}_2F_1(\gamma_0+1,\overline{\gamma_0}+1;2;B^2r^2)\\ &= |A-B|^2\mu^2r^2{}_2F_1((A/B-1)\mu+1,(\overline{A}/B-1)\mu+1;2;B^2r^2).\end{aligned}$$

Note that it is enough to prove the inequality $\Delta(r,(z/f)^{\mu})\leq\Delta(r,(z/k_{A,B})^{\mu})$ to conclude the proof. Allowing $n_0\to\infty$ in Lemma 3.11, we get

$$\Delta\left(r,\left(\frac{z}{f(z)}\right)^{\mu}\right)=\pi\sum_{k=1}^{\infty}k|b_k|^2r^{2k}\leq\pi\sum_{k=1}^{\infty}k|c_k|^2r^{2k}=\Delta\left(r,\left(\frac{z}{k_{A,B}(z)}\right)^{\mu}\right)$$

and the proof of Theorem 1.3 is complete. □

Though Corollary 3.5 is a direct consequence of Corollary 3.4, we have a direct proof of this corollary for the case $\mu = 2$ only using the classical Bieberbach theorem for the second coefficient. For an independent interest, we present it here.

3.3.4. Proof of Corollary 3.5 for $\mu = 2$

It is well known that $\mathcal{C} \subsetneq \mathcal{S}^*(1/2)$. Thus, Corollary 3.9 is also true for $f \in \mathcal{C}$ with $\alpha = 1/2$ and $\mu = 2$. Hence, we obtain

$$\sum_{n=2}^{\infty}(n-1)|b_n|^2 \leq 1. \tag{3.15}$$

We already know that

$$\pi^{-1}\Delta\left(r, \left(\frac{z}{f(z)}\right)^2\right) = \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} = \sum_{n=2}^{\infty} 2(n-1)|b_n|^2 r^{2n} + \sum_{n=1}^{\infty}(2-n)|b_n|^2 r^{2n}.$$

Since $r \leq 1$, this simplifies to

$$\pi^{-1}\Delta\left(r, \left(\frac{z}{f(z)}\right)^2\right) \leq 2r^4 \sum_{n=1}^{\infty}(n-1)|b_n|^2 - \sum_{n=1}^{\infty}(n-2)|b_n|^2 r^{2n}.$$

By equation (3.15), we have

$$\pi^{-1}\Delta\left(r, \left(\frac{z}{f(z)}\right)^2\right) \leq 2r^4 + |b_1|^2 r^2 - \sum_{n=3}^{\infty}(n-2)|b_n|^2 r^{2n} \leq 2r^4 + |b_1|^2 r^2.$$

Note that for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}$ satisfying $(z/f(z))^2 = 1 + \sum_{n=1}^{\infty} b_n z^n$, we have $|b_1| = 2|a_2| \leq 2$ and this implies that

$$\Delta\left(r, \left(\frac{z}{f(z)}\right)^2\right) \leq \pi(2r^4 + 4r^2) = 4\pi r^2(1 + r^2/2) = 4\pi r^2 {}_2F_1(-1, -1; 2; r^2).$$

We now proceed to prove the equality part. It is well-known that $l \in \mathcal{C}$ and clearly

$$\left(\frac{z}{l(z)}\right)^2 = (1-z)^2 = 1 - 2z + z^2.$$

This, using the area formula (1.12), gives

$$\Delta\left(r, \left(\frac{z}{l(z)}\right)^2\right) = \pi(4r^2 + 2r^4),$$

concluding the proof. $\square$

3.4. Concluding Remarks

In the similar line of Lemma 3.7, Lemma 3.8 and Corollary 3.9, we have certain necessary conditions for functions to be in some other class of functions in the literature. Those are recalled in the following two subsections in which we intend to investigate the area problems for functions of type $(z/f)^\mu$, for $\mu > 0$. However, we find that the techniques used in the above section are not enough to find the sharp area estimates for these classes. Finding sharp estimates may lead to development of new techniques in function theory. Therefore, in the following subsections, we present some optimal area estimates using the available necessary conditions talked above. The sharpness parts remain open.

3.4.1. The class $\mathcal{S}_p(\alpha)$

As noted in Chapter 1, for $-1 \leq \alpha \leq 1$, the class $\mathcal{S}_p(\alpha)$, a subclass of the class of starlike functions of order α, is defined by (1.8). In general, it is not easy to find a function belonging to the class $\mathcal{S}_p(\alpha)$. However, to generate functions in a particular class, the best way is to collect sufficient conditions for functions to be in that class. For the class $\mathcal{S}_p(\alpha)$, we know that if

$$\sum_{n=1}^{\infty}[2n+\mu(1-\alpha)]|b_n| \leq \mu(1-\alpha) \tag{3.16}$$

holds for a non-vanishing analytic function f_p of the form (3.1) then $f_p \in \mathcal{S}_p(\alpha)$ (see [76, Theorem 2]). For constructing an example of a function in $\mathcal{S}_p(\alpha)$, for simplicity, we assume that $\mu = 1$ in (3.16). One can easily verify that the coefficients b_n, defined by

$$b_n = \begin{cases} \dfrac{1-\alpha}{3-\alpha}, & \text{for } n = 1, \\ 0, & \text{for } n \geq 2, \end{cases}$$

satisfy (3.16). This gives that

$$\frac{z}{f_p(z)} = 1 + \left(\frac{1-\alpha}{3-\alpha}\right) z \iff f_p(z) = \frac{(3-\alpha)z}{(3-\alpha)+(1-\alpha)z}$$

and $f_p \in \mathcal{S}_p(\alpha)$. Clearly, by (1.12), one computes that

$$\Delta\left(r, \frac{z}{f_p(z)}\right) = \pi r^2 \left(\frac{1-\alpha}{3-\alpha}\right)^2.$$

Now, for estimation of area problem, recall the following necessary condition proved in [76].

Lemma 3.12. *If a function $f \in \mathcal{S}_p(\alpha)$ is of the form* (3.1) *with $b_n \geq 0$, we then have*

$$\sum_{n=1}^{\infty}(2n - \mu(1-\alpha))b_n \leq \mu(1-\alpha).$$

Using this, we now prove the following area estimate for functions of type $(z/f)^\mu$, $\mu > 0$, when f ranges over the class $\mathcal{S}_p(\alpha)$.

Theorem 3.13. *Let $f \in \mathcal{S}_p(\alpha)$ and have the form* (3.1), *where $\mu(1-\alpha) \leq 1$, $b_n \geq 0$ for $n \geq 1$. Then*

$$\Delta\left(r, \left(\frac{z}{f(z)}\right)^{\mu}\right) \leq \pi r^2 \mu(1-\alpha).$$

Proof. Let $f \in \mathcal{S}_p(\alpha)$. Then we have

$$\pi^{-1}\Delta\left(r, \left(\frac{z}{f(z)}\right)^{\mu}\right) = \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} = r^2 \sum_{n=1}^{\infty} n b_n^2 r^{2n-2} \leq r^2 \sum_{n=1}^{\infty} n b_n^2. \tag{3.17}$$

Now by Lemma 3.12, $(2n - \mu(1-\alpha))b_n \leq \mu(1-\alpha)$ since $(2n - \mu(1-\alpha))b_n$ is positive which can be deduced from the hypothesis. Thus,

$$b_n \leq \frac{\mu(1-\alpha)}{2n - \mu(1-\alpha)}.$$

Again, since $n \geq 1$, using the hypothesis we obtain

$$\frac{\mu(1-\alpha)}{2n - \mu(1-\alpha)} \leq 1$$

and hence $0 \leq b_n \leq 1$ which gives $0 \leq nb_n \leq n$ and $n \leq 2n - \mu(1-\alpha)$. Therefore,

$$nb_n^2 \leq (2n - \mu(1-\alpha))b_n$$

thus

$$\sum_{n=1}^{\infty} nb_n^2 \leq \sum_{n=1}^{\infty}(2n - \mu(1-\alpha)b_n) \leq \mu(1-\alpha).$$

Plugging this into the equation (3.17), we obtain the desired inequality. $\square$

3.4.2. The class $\mathcal{N}$

This section deals with the class $\mathcal{N}$ of functions $f \in \mathcal{A}$ which satisfy the condition $|N_f(z)| \leq 1$ for $z \in \mathbb{D}$ where

$$N_f(z) = -z^3\left(\frac{z}{f(z)}\right)''' + f'(z)\left(\frac{z}{f(z)}\right)^2 - 1.$$

This class was introduced in [63]. It is interesting to observe that the Koebe function belongs to the class $\mathcal{N}$ although functions in $\mathcal{N}$ are not necessarily starlike but univalent in $\mathbb{D}$ (see [63]). Indeed, in [63], the following necessary condition was proved.

Lemma 3.14. *Let $f \in \mathcal{N}$ and have the form $z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$. Then we have*

$$\sum_{n=2}^{\infty}(n-1)^6|b_n|^2 \leq 1.$$

This gives us Yamashita's extremal problem for the class $\mathcal{N}$ which is stated below.

Theorem 3.15. *Let $f \in \mathcal{N}$ has the form $z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$. Then*

$$\Delta\left(r, \frac{z}{f(z)}\right) \leq \pi r^2(4 + 2r^2 + r^4).$$

Proof. Suppose $f \in \mathcal{N}$. Recall that, for $z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ we have

$$\begin{aligned}\pi^{-1}\Delta\left(r, \frac{z}{f(z)}\right) &= \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} \\ &= |b_1|^2 r^2 + 2|b_2|^2 r^4 + r^6 \sum_{n=3}^{\infty} n|b_n|^2 r^{2n-6}.\end{aligned}$$

Note that $f \in \mathcal{S}$ and hence $|b_1| = |a_2| \leq 2$. Also, by Lemma 3.14

$$|b_2|^2 \leq \sum_{n=2}^{\infty}(n-1)^6|b_n|^2 \leq 1$$

and

$$\sum_{n=3}^{\infty} n|b_n|^2 \leq \sum_{n=3}^{\infty}(n-1)^6|b_n|^2 \leq 1 - |b_2|^2 \leq 1.$$

It now follows that

$$\pi^{-1}\Delta\left(r, \frac{z}{f(z)}\right) \leq 4r^2 + 2r^4 + r^6,$$

concluding the proof. □

CHAPTER 4

ESTIMATES ON SUCCESSIVE COEFFICIENTS

In this chapter[1] our objective is to obtain results related to successive coefficients for starlike functions of order α, convex functions of order α, spirallike functions and functions in the close-to-convex family. The chapter is organized as follows. Section 4.1 deals with definition of some classes of functions, statement of some main results and some well known theorems. In Section 4.2, we state and prove a lemma which will be used in the proof of our main results in Section 4.3.

4.1. Definitions and results

We consider a family of functions that includes the class of convex functions as a proper subfamily. For $-\pi/2 < \gamma < \pi/2$, we say that $f \in \mathcal{C}_\gamma(\alpha)$ provided $f \in \mathcal{A}$ is locally univalent in $\mathbb{D}$ and $zf'(z)$ belongs to $\mathcal{S}_\gamma(\alpha)$, i.e.

$$\text{(4.1)} \qquad \mathrm{Re}\left\{ e^{-i\gamma} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \alpha \cos\gamma, \quad z \in \mathbb{D}.$$

We may set $\mathcal{C}_\gamma(0) =: \mathcal{C}_\gamma$ and observe that the class $\mathcal{C}_0(\alpha) =: \mathcal{C}(\alpha)$ consists of the normalized convex functions of order α. For general values of γ ($|\gamma| < \pi/2$), a function in $\mathcal{C}_\gamma(0)$ need not be univalent in $\mathbb{D}$. For example, the function $f(z) = i(1-z)^i - i$ is known to belong to $\mathcal{C}_{\pi/4} \backslash \mathcal{S}$. Robertson [87] showed that $f \in \mathcal{C}_\gamma$ is univalent if $0 < \cos\gamma \leq 0.2315\cdots$. Finally, Pfaltzgraff [71] has shown that $f \in \mathcal{C}_\gamma$ is univalent whenever $0 < \cos\gamma \leq 1/2$. This settles the improvement of range of γ for which $f \in \mathcal{C}_\gamma$ is univalent. On the other hand, in [93] it was also shown that functions in $\mathcal{C}_\gamma$ which satisfy $f''(0) = 0$ are univalent for all real values of γ with $|\gamma| < \pi/2$. For a general reference about these special classes we refer to [24].

[1]This chapter is based on the paper: V. Arora, S. Ponnusamy, and S. K. Sahoo in Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Mat., 113 (2019), no. 4, 2969-2979.

Theorem H. [44] *For every* $f \in \mathcal{C} := \mathcal{C}(0)$ *of the form* (1.2)*, the following inequality holds*

$$|a_{n+1}| - |a_n| \leq \frac{1}{n+1}$$

for $n \geq 2$*, and the extremal function is given by*

$$L_\phi(z) = \frac{1}{e^{i\phi} - e^{-i\phi}} \log\left(\frac{1 - e^{-i\phi}z}{1 - e^{i\phi}z}\right)$$

for $\phi = \pi/n$*, where a principal branch of logarithm is chosen.*

A straightforward application of Theorem 1.4 yields the following generalization of Theorem H for convex functions of order α and also for locally univalent functions that are not necessarily univalent in the unit disk $\mathbb{D}$.

Corollary 4.1. *Suppose that* $f \in \mathcal{C}_\gamma(\alpha)$ *for some* $\alpha \in [0, 1)$ *and* $-\pi/2 < \gamma < \pi/2$*. Then we have*

$$|a_{n+1}| - |a_n| \leq \frac{\exp(-M\alpha\cos\gamma)}{n+1}$$

for some constant $M > 0$ *depending on* f *and* n*. In particular, we have*

(1) *For* $f \in \mathcal{C}_\gamma(0)$*,*

$$|a_{n+1}| - |a_n| \leq \frac{1}{n+1}.$$

(2) *For* $f \in \mathcal{C}(\alpha)$ *we have*

$$|a_{n+1}| - |a_n| \leq \frac{\exp(-M\alpha)}{n+1}$$

for some constant $M > 0$ *depending on* f *and* n*.*

Proof. By the classical Alexander theorem, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to $\mathcal{C}_\gamma(\alpha)$ if and only if $zf'(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is $\mathcal{S}_\gamma(\alpha)$ and clearly, $b_n = na_n$. Thus, by Theorem 1.4, we have

$$(n+1)|a_{n+1}| - n|a_n| = |b_{n+1}| - |b_n| \leq \exp(-M\alpha\cos\gamma).$$

This gives,

$$|a_{n+1}| - |a_n| \leq |a_{n+1}| - \frac{n}{n+1}|a_n| \leq \frac{\exp(-M\alpha\cos\gamma)}{n+1}.$$

The proof of the corollary is complete. □

We would like to remark that Hamilton generalized Leung's result to the case of starlike functions of non-positive order and proved the following:

Theorem I. [29] *For a function $f \in \mathcal{S}^*(\alpha)$ for some $\alpha \leq 0$,*

$$\big||a_{n+1}| - |a_n|\big| \leq \frac{\Gamma(1-2\alpha+n)}{\Gamma(1-2\alpha)\Gamma(n+1)}.$$

Equality holds for the function $f(z) = z(1-z)^{2(\alpha-1)}$.

If a locally univalent analytic function f defined in $\mathbb{D}$ satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2} \ \text{ for } z \in \mathbb{D}$$

then by the Kaplan characterization it follows easily that f is close-to-convex in $\mathbb{D}$, and hence f is univalent in $\mathbb{D}$. This generates the following subclass of the class of close-to-convex (univalent) functions:

$$\mathcal{C}(-1/2) := \left\{f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2} \ \text{ for } z \in \mathbb{D}\right\}.$$

This class of functions is also studied recently by the authors in [9], and others in different contexts; for instance see [1, 43, 80] and references therein. Functions in $\mathcal{C}(-1/2)$ are not necessarily starlike but is convex in some direction as the function

$$f(z) = \frac{z - (z^2/2)}{(1-z)^2} \tag{4.2}$$

shows. Note that

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \mathrm{Re}\left(\frac{1+2z}{1-z}\right) > -\frac{1}{2} \ \text{ for } z \in \mathbb{D}$$

and thus $f \in \mathcal{C}(-1/2)$, but not starlike in $\mathbb{D}$.

Remark 4.2. *In Theorem 4.3, we see that Theorem D and Corollary 1.5 continue to hold for functions that are not necessarily starlike but is close-to-convex. At this place it is worth pointing out that there are functions that are γ-spirallike but not close-to-convex. It is also equally true that there exist close-to-convex functions but are not γ-spirallike. Theorem 4.3 is supplementary for this reasoning.*

Theorem 4.3. *Let $f \in \mathcal{C}(-1/2)$. Then*

$$|a_{n+1}| - |a_n| \leq 1.$$

The following result is an immediate consequence of Theorem 4.3 which solves the Robertson conjecture problem for the class $\mathcal{C}(-1/2)$. It is worth pointing out that in 1966 Robertson [86] conjectured that the Bieberbach Conjecture could be strengthened to

$$\big|n|a_n| - m|a_m|\big| \leq |n^2 - m^2| \quad \text{for all } m, n \geq 2,$$

however, two years latter Jenkins [34] showed that this inequality fails in the class $\mathcal{S}$.

Theorem 4.4. *Let $f \in \mathcal{C}(-1/2)$. Then for $n > m$ we have*

$$\big|n|a_n| - m|a_m|\big| \leq \frac{(n^2 - m^2) + (n - m)}{2} = \frac{(n - m)(n + m + 1)}{2}.$$

Equality holds for $f(z) = (z - (z^2/2))/(1 - z)^2$.

4.2. Preliminary result

The following lemma plays a crucial role in the proof of our main results.

Lemma 4.5. *Let $\varphi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$ such that $\mathrm{Re}\,\varphi(z) > \alpha$ in $\mathbb{D}$ for some $\alpha < 1$. Suppose that $\psi(z) = e^{i\gamma} \sum_{n=1}^{\infty} \lambda_n c_n z^n$ is analytic in $\mathbb{D}$, where $\lambda_n \geq 0$ and $\mathrm{Re}\,\psi(z) \leq M$ for some $M > 0$. Then we have the inequality*

$$\cos\gamma \sum_{n=1}^{\infty} \lambda_n |c_n|^2 \leq 2M(1 - \alpha).$$

Proof. Let us first prove the result for $\alpha = 0$. Consider the identity

$$4(\mathrm{Re}\,\varphi)(\mathrm{Re}\,\psi) = (\varphi + \overline{\varphi})(\psi + \overline{\psi}) = (\varphi\psi + \varphi\overline{\psi}) + \overline{(\varphi\psi + \varphi\overline{\psi})}$$

so that

$$4\int_{|z|=r} (\mathrm{Re}\,\varphi)(\mathrm{Re}\,\psi)\, d\theta = 2\mathrm{Re}\left(\int_{|z|=r} \varphi(z)\overline{\psi(z)}\, d\theta\right), \tag{4.3}$$

since (with $z = re^{i\theta}$)

$$\int_0^{2\pi} \varphi(z)\psi(z)\, d\theta = \int_{|z|=r} \varphi(z)\psi(z)\,\frac{dz}{iz} = 0, \tag{4.4}$$

by the Cauchy integral formula and the fact that $\psi(0) = 0$. Using the power series representation of $\varphi(z)$ and $\psi(z)$, it follows that (since $\overline{z} = r^2/z$ on $|z| = r$)

$$\begin{aligned}
\int_{|z|=r} \varphi(z)\overline{\psi(z)}\, d\theta &= e^{-i\gamma}\int_{|z|=r}\left[1+\sum_{n=1}^{\infty} c_n z^n\right]\left[\sum_{n=1}^{\infty}\overline{c_n}\lambda_n \frac{r^{2n}}{z^n}\right]\frac{dz}{iz} \\
&= 2\pi e^{-i\gamma}\sum_{n=1}^{\infty}\lambda_n |c_n|^2 r^{2n}. \qquad (4.5)
\end{aligned}$$

By (4.4), (4.5) and the assumption that $\mathrm{Re}\,\psi(z) \le M$ for some $M > 0$, the identity (4.3) reduces to

$$4\pi\cos\gamma\sum_{n=1}^{\infty}\lambda_n|c_n|^2 r^{2n} = 4\int_0^{2\pi}(\mathrm{Re}\,\varphi(z))(\mathrm{Re}\,\psi(z))\, d\theta \le 4M\int_0^{2\pi}\mathrm{Re}\,\varphi(z)\, d\theta = 8M\pi,$$

where we have used the fact that

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\,\varphi(z)\, d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\varphi(z)+\overline{\varphi(z)}}{2}\, d\theta \\
&= \frac{1}{4\pi}\left[\int_{|z|=r}\varphi(z)\,\frac{dz}{iz} + \overline{\int_{|z|=r}\varphi(z)\,\frac{dz}{iz}}\right] \\
&= \frac{1}{4}(2\pi+2\pi) = 1.
\end{aligned}$$

The desired result for the case $\alpha = 0$ follows by letting $r \to 1^-$ in the last inequality.

Finally, for the general case, we first observe that $\mathrm{Re}\,\Phi(z) > 0$, where

$$\Phi(z) = \frac{\varphi(z)-\alpha}{1-\alpha} = 1 + \sum_{n=1}^{\infty} d_n z^n, \quad d_n = \frac{c_n}{1-\alpha}.$$

Also, the given condition on ψ gives $\mathrm{Re}\,\Psi(z) \le \frac{M}{1-\alpha}$, where

$$\Psi(z) = e^{i\gamma}\sum_{n=1}^{\infty}\lambda_n d_n z^n = \frac{1}{1-\alpha}\left(e^{i\gamma}\sum_{n=1}^{\infty}\lambda_n c_n z^n\right) = \frac{1}{1-\alpha}\psi(z).$$

Applying the previous arguments for the pair $(\Phi(z), \Psi(z))$, one obtains that

$$\cos\gamma\sum_{n=1}^{\infty}\lambda_n|d_n|^2 = \frac{\cos\gamma}{(1-\alpha)^2}\sum_{n=1}^{\infty}\lambda_n|c_n|^2 \le \frac{2M}{1-\alpha}$$

so that $\cos\gamma\sum_{n=1}^{\infty}\lambda_n|c_n|^2 \le 2M(1-\alpha)$, as desired. □

Remark 4.6. *We remark that Lemma 4.5 for $\gamma = 0$ is obtained by MacGregor [50] (see also [42] and [21, p.178, Lemma]).*

4.3. Proof of the main results

We begin with the proof of Theorem 1.4

4.3.1. Proof of Theorem 1.4

Let $f \in \mathcal{S}_\gamma(\alpha)$. Then by the definition, we may consider φ by

$$\frac{1}{\cos\gamma}\left[e^{-i\gamma}\frac{zf'(z)}{f(z)} + i\sin\gamma\right] = \varphi(z)$$

so that

$$e^{-i\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right) = \cos\gamma\,(\varphi(z) - 1),$$

where $\mathrm{Re}\,\{\varphi(z)\} > \alpha$ and $\varphi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ is analytic in $\mathbb{D}$. We may rewrite the last equation as

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = e^{i\gamma}\cos\gamma\sum_{n=1}^{\infty} c_n z^{n-1} \tag{4.6}$$

which by simple integration gives

$$\log\left(\frac{f(z)}{z}\right) = e^{i\gamma}\cos\gamma\sum_{n=1}^{\infty}\frac{c_n z^n}{n}, \tag{4.7}$$

where we use the principal value of the logarithm such that log 1=0. By the Taylor series expansion of $\log(1 - \xi z)$ and (4.7), we get

$$\log(1 - \xi z)\frac{f(z)}{z} = \sum_{n=1}^{\infty}\frac{C_n - \xi^n}{n}z^n = \sum_{n=1}^{\infty}\alpha_n z^n, \tag{4.8}$$

where $C_n = e^{i\gamma}\cos\gamma\, c_n$ and

$$\alpha_n = \frac{C_n - \xi^n}{n} = \frac{e^{i\gamma}\cos\gamma\, c_n - \xi^n}{n}.$$

Also, for $|\xi| = 1$, we have

$$(1 - \xi z)\frac{f(z)}{z} = \sum_{n=0}^{\infty}\beta_n z^n, \quad \beta_n = a_{n+1} - \xi a_n. \tag{4.9}$$

From (4.8) and (4.9), it follows that

$$\exp\left(\sum_{n=1}^{\infty}\alpha_n z^n\right) = \sum_{n=0}^{\infty}\beta_n z^n,\ \beta_0 = 1.$$

Then, by the third Lebedev-Milin inequality (see [21, p. 143]), we have

$$|\beta_n|^2 \le \exp\left\{\sum_{k=1}^{n}\left(k|\alpha_k|^2 - \frac{1}{k}\right)\right\},$$

or equivalently

$$|a_{n+1} - \xi a_n|^2 \le \exp\left\{\sum_{k=1}^{n}\left(\frac{|C_k - \xi^k|^2}{k} - \frac{1}{k}\right)\right\}. \tag{4.10}$$

Now we consider

$$\psi(z) = e^{i\gamma}\sum_{k=1}^{n}\frac{c_k z^k}{k},$$

and let M be the maximum of $\mathrm{Re}\{\psi(z)\}$ on $|z| = 1$. Applying Lemma 4.5 with $\lambda_k = 1/k$ for $1 \le k \le n$ and $\lambda_k = 0$ for $k > n$, we obtain

$$\begin{aligned}\sum_{k=1}^{n}\left(\frac{|C_k - \xi^k|^2}{k} - \frac{1}{k}\right) &= \cos^2\gamma\sum_{k=1}^{n}\frac{|c_k|^2}{k} - 2\cos\gamma\sum_{k=1}^{n}\frac{\mathrm{Re}(e^{i\gamma}c_k\overline{\xi}^k)}{k}\\ &\le 2M(1-\alpha)\cos\gamma - 2\cos\gamma\,\mathrm{Re}\{\psi(\overline{\xi})\}.\end{aligned}$$

Choosing ξ (say ξ_0) so that $\mathrm{Re}\{\psi(\overline{\xi_0})\} = M$, we see that

$$\sum_{k=1}^{n}\left(\frac{|C_k - \xi_0^k|^2}{k} - \frac{1}{k}\right) \le 2M(1-\alpha)\cos\gamma - 2M\cos\gamma = -2M\alpha\cos\gamma.$$

Hence from (4.10), $|a_{n+1} - \xi_0 a_n| \le \exp(-M\alpha\cos\gamma)$ for some ξ_0 with $|\xi_0| = 1$. Since

$$\big||a_{n+1}| - |a_n|\big| \le |a_{n+1} - \xi_0 a_n| \le \exp(-M\alpha\cos\gamma),$$

the proof of our theorem is complete. □

Here we provide one example that associates to Theorem 1.4.

Example 4.7. *Consider the function $f(z) := f_{\gamma,\alpha}(z) = z/(1-z)^\beta$, where $\beta = 2(1-\alpha)\cos\gamma$. It is easy to check that $f \in \mathcal{S}_\gamma(\alpha)$,*

$$f(z) = z + \sum_{n=2}^{\infty}\frac{\Gamma(n+\beta)}{\Gamma(n+1)\Gamma(\beta)}z^n \quad \textit{and} \quad e^{-i\gamma}\frac{zf'(z)}{f(z)} = e^{-i\gamma} + 2(1-\alpha)\cos\gamma\frac{z}{1-z}.$$

Again consider the function

$$\varphi(z) = e^{-i\gamma}\frac{zf'(z)}{f(z)} = e^{-i\gamma} + 2(1-\alpha)\cos\gamma\sum_{n=1}^{\infty}z^n.$$

It is clear that $\operatorname{Re}(\varphi(z)) > \alpha\cos\gamma$. *Now, if we adopt the proof of Lemma 4.5 and Theorem 1.4 by assuming* $\psi(z) = 2(1-\alpha)\sum_{n=1}^{\infty} z^n$ *and* $\gamma = 0$, *then for* $f \in \mathcal{S}^*(\alpha)$ *we obtain*

$$\big||a_{n+1}| - |a_n|\big| \leq \exp(-\alpha M), \quad M = 2(1-\alpha)(\log n + 1).$$

4.3.2. Proof of Theorem 4.3

Let $f \in \mathcal{C}(-1/2)$. Then the function $g(z) = \sum_{n=1}^{\infty} b_n z^n = zf'(z)$, where $b_n = na_n$, belongs to $\mathcal{S}^*(-1/2)$. From Theorem I, we obtain that

$$\big||b_{n+1}| - |b_n|\big| = \big|(n+1)|a_{n+1}| - n|a_n|\big| = (n+1)\left||a_{n+1}| - \frac{n}{n+1}|a_n|\right| \leq n+1 \tag{4.11}$$

which implies that

$$|a_{n+1}| - |a_n| \leq \left||a_{n+1}| - \frac{n}{n+1}|a_n|\right| \leq 1,$$

and the proof is complete. □

Example 4.8. *Consider the function* f *defined by* (4.2), *namely,*

$$f(z) = \frac{z - z^2/2}{(1-z)^2} = \sum_{n=1}^{\infty} \frac{n+1}{2} z^n.$$

It is easy to check that f *satisfies the hypothesis of Theorem 4.3. For this function, we have*

$$|a_{n+1}| - |a_n| = \frac{n+2}{2} - \frac{n+1}{2} = \frac{1}{2} < 1.$$

Example 4.9. *Consider the function* f *defined by*

$$f(z) = \frac{z}{\sqrt{1-z^2}} = \sum_{n=1}^{\infty} \frac{\Gamma(n+1/2)}{\pi\Gamma(n+1)} z^{2n+1}.$$

A simple computation shows that $f \in \mathcal{C}(-1/2)$ *and for this function, we see that*

$$|a_{n+1}| - |a_n| = \frac{\Gamma(n+1/2)}{\pi\Gamma(n+1)} < 1,$$

so the result is compatible with Theorem 4.3.

4.3.3. Proof of Theorem 4.4

Let $f \in \mathcal{C}(-1/2)$. Then we have

$$\big|(k+1)|a_{k+1}| - k|a_k|\big| \leq k+1 \text{ for } k \geq 1,$$

by (4.11). Here $a_1 = 1$. Using the triangle inequality, we deduce that for $n \geq m$

$$\begin{aligned}\big||n|a_n| - m|a_m|\big| &= \left|\sum_{k=m}^{n-1}(k+1)|a_{k+1}| - k|a_k|\right| \\ &\leq \sum_{k=m}^{n-1}\big|(k+1)|a_{k+1}| - k|a_k|\big| \\ &= \sum_{k=m}^{n-1}(k+1) = \frac{(n^2-m^2)+(n-m)}{2}.\end{aligned}$$

Clearly the equality holds for $f \in \mathcal{C}(-1/2)$ defined by (4.2) in which the coefficient of z^n is $(n+1)/2$. $\square$

CHAPTER 5

MEROMORPHIC FUNCTIONS WITH SMALL SCHWARZIAN DERIVATIVE

The Purpose of this chapter[1] is to study the sufficient conditions of the form (1.16) involving Schwarzian derivative for meromorphically convex functions of order α and for functions in a family that are convex in one direction, in particular in the starlike and close-to-convex family.

5.1. Definitions and main results

For $\beta \geq 3/2$, we consider the class $\mathcal{C}_\beta$ introduced by Shah in [92] as follows:

$$\mathcal{C}_\beta = \left\{ f \in \mathcal{A} : \frac{-\beta}{2\beta - 3} < \mathrm{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) < \beta,\ z \in \mathbb{D} \right\}.$$

This originally follows from a sufficient condition for a function f to be convex in one direction studied by Umeraza in [97]. Note that the special cases $\mathcal{C}_{3/2}$ and $\mathcal{C}_\infty$ are contained in the family of starlike and close-to-convex functions respectively (see the detailed discussion below in this section). It is a natural question to ask for functions belonging to the family $\mathcal{C}_\beta$ for all $\beta \geq 3/2$. Such functions can be generated in view of [92, Theorem 12], which says that for all functions $f \in \mathcal{A}$ satisfying

$$\frac{\beta}{3 - 2\beta} < \mathrm{Re}\Big(\frac{zf'(z)}{f(z)}\Big) < \beta,$$

the Alexander transform of f belongs to the family $\mathcal{C}_\beta$, $\beta \geq 3/2$.

Chiang proved the following sufficient condition for convex functions of order α in terms of small Schwarzian derivative:

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[1]Results of this chapter are published in: V. Arora, S. K. Sahoo in Stud. Univ. Babeş-Bolyai Math., 63 (2018), no. 3, 355-370.

Theorem J. *[17, Theorem 2] Let $f \in \mathcal{A}$ and $|a_2| = \eta < 1/3$. Suppose that*

$$\sup_{z\in\mathbb{D}} |S_f(z)| = 2\delta,$$

where $\delta = \delta(\eta)$ satisfies the inequality

$$6\eta + 5\delta(1+\eta)e^{\delta/2} < 2.$$

Then f is convex of order

$$\frac{2 - 6\eta - 5(1+\eta)\delta e^{\delta/2}}{2 - 2\eta - (1+\eta)\delta e^{\delta/2}}.$$

Our aim in this section is to state results similar to Theorem J for certain functions convex in one direction, in particular, for functions in the family of starlike and close-to-convex functions.

We now state our second main result which provides a sufficient condition for functions to be in $\mathcal{C}_\beta$ with respect to its small Schwarzian derivative.

Theorem 5.1. *For $\beta \geq 3/2$, set*

$$\phi(\beta) = \min\left\{\frac{\beta-1}{\beta+1}, \frac{6(\beta-1)}{2(7\beta-9)}\right\} \quad \textit{and} \quad \psi(\beta) = \max\left\{\frac{\beta+3}{\beta+1}, \frac{11\beta-15}{7\beta-9}\right\}.$$

Let $f \in \mathcal{A}$ and $|a_2| = \eta < \phi(\beta)$. Suppose that

$$\sup_{z\in\mathbb{D}} |S_f(z)| = 2\delta,$$

where $\delta = \delta(\eta)$ satisfies the inequality

$$2\eta + \psi(\beta)\delta(1+\eta)e^{\delta/2} < 2\phi(\beta). \tag{5.1}$$

Then $f \in \mathcal{C}_\beta$. In particular, f is convex in one direction.

Recall the sufficient condition for starlike functions $f \in \mathcal{A}$ from [75, (16)] which tells us that

$$\mathrm{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) < \frac{3}{2} \implies \Big|\frac{zf'(z)}{f(z)} - \frac{2}{3}\Big| < \frac{2}{3}.$$

This generates the following subclass of the class of starlike functions:

$$\mathcal{C}_{3/2} := \left\{f \in \mathcal{A} :\, \mathrm{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) < \frac{3}{2}\right\}.$$

This particular class of functions is also studied in different contexts in [77].

The following corollary immediately follows from Theorem 5.1 for the class $\mathcal{C}_{3/2}$.

Corollary 5.2. *Let* $f \in \mathcal{A}$ *and* $|a_2| = \eta < 1/5$. *Suppose*

$$\sup_{z\in\mathbb{D}} |S_f(z)| = 2\delta$$

where $\delta = \delta(\eta)$ *satisfies the inequality*

$$10\eta + 9\delta(1+\eta)e^{\delta/2} < 2. \tag{5.2}$$

Then $f \in \mathcal{C}_{3/2}$. *In particular,* f *is starlike.*

We next recall what is close-to-convex function followed by a subclass of the class of close-to-convex functions and then state the corresponding result which is again an easy consequence of Theorem 5.1.

If a locally univalent analytic function f defined in $\mathbb{D}$ satisfies

$$\mathrm{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > -1/2,$$

then by the Kaplan characterization it follows easily that f is close-to-convex in $\mathbb{D}$ (here θ_1 and θ_2 are chosen as 0 and 2π respectively) and hence f is univalent in $\mathbb{D}$. This generates the following subclass of the class of close-to-convex (univalent) functions:

$$\mathcal{C}_\infty := \left\{ f \in \mathcal{A} :\ \mathrm{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > -\frac{1}{2} \right\}.$$

This class of functions is also studied recently by several authors in different contexts; for instance see [1, 13, 43, 80] and references therein.

Now we are ready to state our sufficient condition for functions f to be in $\mathcal{C}_\infty$ in terms of their Schwarzian derivatives bounded by small quantity.

Corollary 5.3. *Let* $f \in \mathcal{A}$ *and* $|a_2| = \eta < 3/7$. *Suppose that*

$$\sup_{z\in\mathbb{D}} |S_f(z)| = 2\delta$$

where $\delta = \delta(\eta)$ *satisfies the inequality*

$$14\eta + 11\delta(1+\eta)e^{\delta/2} < 6. \tag{5.3}$$

Then $f \in \mathcal{C}_\infty$ *and hence* f *is close-to-convex function.*

5.2. Preliminary results

Connection with a linear differential equation

In this section we study a relationship between Schwarzian derivative of a meromorphic function h and solution of a second order linear differential equation depending on h.

Recall the following lemma from Duren [21, p. 259].

Lemma 5.4. *For a given analytic function $p(z)$, a meromorphic function h has the Schwarzian derivative of the form $S_h(z) = 2p(z)$ if and only if $h(z) = w_1(z)/w_2(z)$ for any pair of linearly independent solutions $w_1(z)$ and $w_2(z)$ of the linear differential equation*

$$w'' + p(z)w = 0. \tag{5.4}$$

Note that an example satisfying Lemma 5.4 is described in the proof of Theorem 1.6(b). Assume now that $w_1(z)$ and $w_2(z)$ satisfy the following conditions:

$$w_1(0) = 1,\ w_2(0) = 0;$$
$$w_1'(0) = 0,\ w_2'(0) = 1.$$

Clearly $w_1(0)$ and $w_2(0)$ are linearly independent since the Wronskian $W(w_1(0), w_2(0))$ is non-vanishing. Recall that

$$h(z) = \frac{w_1(z)}{w_2(z)} = \frac{1}{z} + b_0 + b_1 z + \cdots . \tag{5.5}$$

Hence, a simple computation on logarithmic derivative of $h'(z)$ leads to

$$\frac{h''(z)}{h'(z)} = \frac{w_2(z)w_1''(z) - w_1(z)w_2''(z)}{w_2(z)w_1'(z) - w_1(z)w_2'(z)} - 2\frac{w_2'(z)}{w_2(z)}.$$

Since $w_1(z)$ and $w_2(z)$ satisfy (5.4), it follows that

$$\frac{h''(z)}{h'(z)} = -2\frac{w_2'(z)}{w_2(z)},$$

and hence we have the relation

$$\mathrm{Re}\Big(1 + \frac{zh''(z)}{h'(z)}\Big) = 1 - 2\mathrm{Re}\Big(\frac{zw_2'(z)}{w_2(z)}\Big). \tag{5.6}$$

The Function $2x - (1+\alpha)\tan x$.

For $0 \le \alpha < 1$, we set

$$r(x) := 2x - (1+\alpha)\tan x.$$

Derivative test for $r(x)$ tells us that $r(x)$ is decreasing in $(\arctan(\sqrt{(1-\alpha)/(1+\alpha)}), \pi/2)$.

Then the following lemma is useful.

Lemma 5.5. *Let $\beta < \pi/2$ be the smallest positive root of $r(x) = 2x - (1+\alpha)\tan x = 0$ for some $\alpha > 0$. Then*

$$\beta \ge \arctan\sqrt{(1-\alpha)/(1+\alpha)}$$

holds true.

Proof. Given that $r(\beta) = 0 = 2\beta - (1+\alpha)\tan\beta$. This gives

$$\alpha = \frac{2\beta}{\tan\beta} - 1. \tag{5.7}$$

On contrary, suppose that $0 < \beta < \arctan\sqrt{(1-\alpha)/(1+\alpha)} < \pi/2$. This implies that

$$\tan^2\beta < \frac{1-\alpha}{1+\alpha}.$$

Substituting the value of α in (5.7), we obtain

$$\tan^2\beta < \frac{\tan\beta}{\beta} - 1$$

equivalently,

$$\sec^2\beta < \frac{\tan\beta}{\beta} \iff 2\beta < \sin 2\beta,$$

which is a contradiction. Thus, the proof of our lemma is complete. □

Let c_α be the smallest positive root of the equation (1.19). Since $r(\sqrt{c_\alpha}) = 0$, it follows by Lemma 5.5 that

$$r(x) \begin{cases} \ge 0, & \text{for } 0 \le x \le \sqrt{c_\alpha}; \\ < 0, & \text{for } \sqrt{c_\alpha} < x < \pi/2. \end{cases} \tag{5.8}$$

If we replace x by $x\sqrt{c}$, $c > 0$, in (5.8), we obtain

$$r(x\sqrt{c}) = 2x\sqrt{c} - (1+\alpha)\tan(x\sqrt{c}) \ge 0 \text{ for } 0 \le x\sqrt{c} \le \sqrt{c_\alpha} \tag{5.9}$$

and

$$r(x\sqrt{c}) = 2x\sqrt{c} - (1+\alpha)\tan(x\sqrt{c}) < 0 \text{ for } \sqrt{c_\alpha} < x\sqrt{c} < \pi/2. \tag{5.10}$$

We may have the following two cases when $r(x\sqrt{c})$ is negative.

Case 1: If $c \le c_\alpha$, then (5.10) gives that $r(x\sqrt{c})$ is also negative in $[1, \pi/2\sqrt{c})$.

Case 2: If $c > c_\alpha$, then (5.10) gives that $r(x\sqrt{c})$ is also negative in $(\sqrt{c_\alpha/c}, 1)$.

In the sequel, we collect the following lemmas to be used in the proof of Theorem 1.6.

Lemma 5.6. *A function $h \in \mathcal{B}$ in the form* (5.5) *is meromorphically convex of order α if and only if $w_2(z)$ is starlike of order* $(\alpha+1)/2$.

Proof. Condition (5.6) is equivalent to

$$-\text{Re}\Big(1 + \frac{zh''(z)}{h'(z)}\Big) = -1 + 2\text{Re}\Big(\frac{zw_2'(z)}{w_2(z)}\Big),$$

which yields

$$-\text{Re}\Big(1 + \frac{zh''(z)}{h'(z)}\Big) > \alpha \iff \text{Re}\Big(\frac{zw_2'(z)}{w_2(z)}\Big) > \frac{\alpha+1}{2}\ .$$

Since $w_2(0) = 0$ and $w_2'(0) = 1$, $w_2(z)$ is starlike of order $(\alpha+1)/2$. Thus, completing the proof of our lemma. □

Remark 5.7. *A simple computation using the identity (5.6) yields*

$$\text{Re}\left(\frac{zw_1'(z)}{w_1(z)}\right) = \frac{1}{2} + \text{Re}\left(\frac{zh'(z)}{h(z)}\right) - \frac{1}{2}\text{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right).$$

Therefore, the function w_1 is not necessarily starlike when the function h is meromorphically convex.

Lemma 5.8. *For $0 \le \alpha < 1$, let c_α $(0 < c \le c_\alpha)$ be the root of the equation given by* (1.19). *Then we have*

$$\text{Re}(z\sqrt{c}\cot(z\sqrt{c})) > \frac{\alpha+1}{2}, \quad |z| < 1. \tag{5.11}$$

Proof. Substituting $z = x+iy$ in (5.11), we see that the desired inequality is equivalent to

$$2\text{Re}\Big(\sqrt{c}(x+iy)\frac{\cos(\sqrt{c}(x+iy))}{\sin(\sqrt{c}(x+iy))}\Big) > \alpha + 1.$$

This is, using the basic identities $2\mathrm{Re}\, w = w + \overline{w}$, $\cos(iy) = \cosh(y)$, and $\sin(iy) = i\sinh(y)$, we see that it is equivalent to proving

$$\begin{aligned} &2x\sqrt{c}\sin(\sqrt{c}x)\cos(\sqrt{c}x) + 2y\sqrt{c}\sinh(\sqrt{c}y)\cosh(\sqrt{c}y) \\ &\qquad\qquad > (1+\alpha)(\sin^2(\sqrt{c}x) + \sinh^2(\sqrt{c}y)). \end{aligned}$$

So, it suffices to prove the inequality

$$\begin{aligned} &\sin(\sqrt{c}x)\cos(\sqrt{c}x)[2\sqrt{c}x - (1+\alpha)\tan(\sqrt{c}x)] \\ &\qquad > \sinh(\sqrt{c}y)\cosh(\sqrt{c}y)[(1+\alpha)\tanh(\sqrt{c}y) - 2y\sqrt{c}y] \end{aligned} \tag{5.12}$$

for $0 < c \le c_\alpha$ and $x^2+y^2 < 1$. First consider the points $x,\ y$ in the first quadrant. Then we see that $\sin(\sqrt{c}x)$, $\cos(\sqrt{c}x)$, $\sinh(\sqrt{c}y)$ and $\cosh(\sqrt{c}y)$ are all positive since $c < c_\alpha < \pi^2/4$. Also $2x\sqrt{c} - (1+\alpha)\tan(\sqrt{c}x)$ is positive which follows from (5.9). On the other hand, $(1+\alpha)\tanh(\sqrt{c}y) - 2(\sqrt{c}y)$ is non-positive because $f(y) = (1+\alpha)\tanh(\sqrt{c}y) - 2(\sqrt{c}y)$ is decreasing, hence for $y \ge 0$ we obtain

$$f(y) = (1+\alpha)\tanh(\sqrt{c}y) - 2\sqrt{c}y \le 0.$$

Hence, the inequality (5.12) holds true in the first quadrant. Now if we replace x by $-x$ and y by $-y$ then the inequality (5.12) remains same in all the other quadrants of $\mathbb{D}$. The desired inequality thus follows. □

The following results of Gabriel are also useful.

Lemma 5.9. *[22, Lemma 4.1] If $w(z)$ satisfies (5.4) with $w(0) = 0$ and $w'(0) = 1$, then for $0 < \rho \le r < 1$ and for a fixed $\theta \in [0, 2\pi]$, we have*

$$|w(re^{i\theta})|^2 \mathrm{Re}\Big(\frac{re^{i\theta} w'(re^{i\theta})}{w(re^{i\theta})}\Big) = r\int_0^r |w'(\rho e^{\theta})|^2 d\rho - r\int_0^r \mathrm{Re}(\rho^2 e^{2i\theta} p(\rho e^{i\theta}))\frac{|w(\rho e^{i\theta})|^2}{\rho^2} d\rho. \tag{5.13}$$

Lemma 5.10. *[22, Lemma 4.2] Let $y(\rho)$ and $y'(\rho)$ be continuous real functions of ρ for $0 \le \rho < 1$. For small values of ρ let $y(\rho) = O(\rho)$. Then*

$$r\int_0^r [y'(\rho)]^2 d\rho - cr\int_0^r [y^2(\rho)] d\rho - r\sqrt{c}\cot(r\sqrt{c}) \cdot y^2(r) \ge 0 \tag{5.14}$$

for $0 < r < 1$ and $c > 0$. Equality holds for

$$y(\rho) = c^{-1/2}\sin(\rho\sqrt{c}), \quad c > 0.$$

5.3. Proof of the main results

5.3.1. Proof of Theorem 1.6

Given that $h \in \mathcal{B}$ satisfies (1.18) and c_α is the smallest positive root of the equation (1.19). A simple computation yields

$$\alpha = \frac{2\sqrt{c_\alpha} - \tan\sqrt{c_\alpha}}{\tan\sqrt{c_\alpha}}.$$

Differentiating α with respect to c_α, we obtain

$$\frac{d\alpha}{dc_\alpha} = \frac{\tan\sqrt{c_\alpha} - \sqrt{c_\alpha}\sec^2\sqrt{c_\alpha}}{\sqrt{c_\alpha}\tan^2\sqrt{c_\alpha}}.$$

Since $\tan x - x\sec^2 x \leq 0$ is equivalent to $\sin 2x \leq 2x$, which is always true for all $x \in \mathbb{R}$, it follows that c_α increases if and only if α decreases.

Now we proceed for completing the proof of (a) and (b).

a. In this part we prove that h is meromorphically convex of order α, $0 \leq \alpha < 1$, that is h satisfies (1.11).

Set $S_h(z) = 2p(z)$ for a given analytic function $p(z)$. Then by (1.18), it follows that $|p(z)| \leq c_\alpha$, and hence we have

$$\mathrm{Re}(z^2 p(z)) \leq c_\alpha |z|^2 \quad \text{for } |z| < 1.$$

By Lemma 5.4, the function has the form $h(z) = w_1(z)/w_2(z)$ for any pair of linearly independent solutions $w_1(z)$ and $w_2(z)$ of the linear differential equation (5.4). Clearly, the particular solution $w_2(z)$ satisfies the hypothesis of Lemma 5.9. Since $\mathrm{Re}(z^2 p(z)) \leq c_\alpha |z|^2$ holds, (5.13) implies

$$|w_2(re^{i\theta})|^2 \mathrm{Re}\Big(\frac{re^{i\theta} w_2'(re^{i\theta})}{w_2(re^{i\theta})}\Big) \geq r\int_0^r |w_2'(\rho e^{i\theta})|^2 d\rho - rc_\alpha \int_0^r |w_2(\rho e^{i\theta})|^2 d\rho, \tag{5.15}$$

for $0 < \rho \leq r < 1$ and for some fixed θ.

Putting $w_2(\rho e^{i\theta}) = u_2(\rho, \theta) + iv_2(\rho, \theta)$. For a constant ray θ, w_2 will become a function of ρ only. Note that $u_2(\rho)$ and $v_2(\rho)$ satisfies the hypothesis of Lemma 5.10. We obtain the following two inequalities after substituting $u_2(\rho)$ and $v_2(\rho)$ in (5.14) and replacing c by c_α

$$r\int_0^r [u_2'(\rho)]^2 d\rho - c_\alpha r\int_0^r [u_2^2(\rho)] d\rho - \sqrt{c_\alpha} r\cot(\sqrt{c_\alpha} r)\cdot u_2^2(r) \geq 0, \tag{5.16}$$

and

$$r\int_0^r [v_2'(\rho)]^2 d\rho - c_\alpha r\int_0^r [v_2^2(\rho)]d\rho - \sqrt{c_\alpha} r\cot(\sqrt{c_\alpha} r)\cdot v_2^2(r) \geq 0. \tag{5.17}$$

Since $w_2(\rho e^{i\theta}) = u_2(\rho,\theta) + iv_2(\rho,\theta)$, addition of (5.16) and (5.17) leads to

$$r\int_0^r |w_2'(\rho e^{i\theta})|^2 d\rho - rc_\alpha \int_0^r |w_2(\rho e^{i\theta})|^2 d\rho \geq \sqrt{c_\alpha} r\cot(\sqrt{c_\alpha} r)|w_2|^2. \tag{5.18}$$

Comparing (5.15) with (5.18), we obtain

$$|w_2(re^{i\theta})|^2 \mathrm{Re}\Big(\frac{zw_2'(re^{i\theta})}{w_2(re^{i\theta})}\Big) \geq \sqrt{c_\alpha} r\cot(\sqrt{c_\alpha} r)|w_2(re^{i\theta})|^2,$$

that is,

$$\mathrm{Re}\Big(\frac{zw_2'(z)}{w_2(z)}\Big) \geq \sqrt{c_\alpha} r\cot(\sqrt{c_\alpha} r) \quad \text{for } |z| = r < 1. \tag{5.19}$$

It follows from Lemma 5.8 that

$$\sqrt{c_\alpha} r\cot(\sqrt{c_\alpha} r) = \mathrm{Re}(\sqrt{c_\alpha} r\cot(\sqrt{c_\alpha} r)) > \frac{\alpha+1}{2}. \tag{5.20}$$

Comparison of (5.19) with (5.20) yields

$$\mathrm{Re}\Big(\frac{zw_2'(z)}{w_2(z)}\Big) > \frac{\alpha+1}{2},$$

and hence it follows from Lemma 5.6 that h is meromorphically convex of order α.

b. We prove that the quantity c_α is the largest possible constant satisfying (1.18), i.e. we can not replace c_α by a larger quantity. We prove this by contradiction. If we replace c_α by a larger number $c = c_\alpha + \epsilon$ for some $\epsilon > 0$, then we observe that there exists a function $h \in \mathcal{B}$ satisfying

$$|S_h(z)| \leq 2(c_\alpha + \epsilon), \quad |z| < 1, \tag{5.21}$$

but h is not meromorphically convex of order α. For this, we consider the function

$$h(z) = \frac{w_1(z)}{w_2(z)}, \quad |z| < 1,$$

with the two linearly independent solutions

$$w_1(z) = \cos(\sqrt{c}z) \ \text{ and } \ w_2(z) = \frac{\sin(\sqrt{c}z)}{\sqrt{c}}$$

of the differential equation $w'' + cw = 0$. Clearly, by a simple computation, the function $h(z) = \sqrt{c}\cot(\sqrt{c}z)$ satisfies $S_h(z) = 2c$. It remains to show that this function h is not meromorphically convex of order α, equivalently, by definition, we prove that

$$-\mathrm{Re}\Big(1 + \frac{z_0 h''(z_0)}{h'(z_0)}\Big) \leq \alpha$$

for some $z_0 \in \mathbb{D}$. By Lemma 5.6, it is equivalently to proving

$$\mathrm{Re}\Big(\frac{z_0 w_2'(z_0)}{w_2(z_0)}\Big) = \mathrm{Re}\Big(\frac{\sqrt{c}z_0\cos(\sqrt{c}z_0)}{\sin(\sqrt{c}z_0)}\Big) \leq \frac{\alpha+1}{2}. \tag{5.22}$$

for some non-zero $z_0 \in \mathbb{D}$, since for $z_0 = 0$ the relation (5.22) contradicts to the assumption $\alpha < 1$. Substituting $0 \neq z_0 = x_0 + iy_0 \in \mathbb{D}$ in (5.22) and simplifying, we obtain

$$\begin{aligned} 2x_0\sqrt{c}\sin(\sqrt{c}x_0)\cos(\sqrt{c}x_0) + 2y_0\sqrt{c}\sinh(\sqrt{c}y_0)\cosh(\sqrt{c}y_0) \\ \leq (1+\alpha)(\sin^2(\sqrt{c}x_0) + \sinh^2(\sqrt{c}y_0)), \end{aligned}$$

or

$$\begin{aligned} \sin(\sqrt{c}x_0)\cos(\sqrt{c}x_0)[2x_0\sqrt{c} - (1+\alpha)\tan(\sqrt{c}x_0)] \\ \leq \sinh(\sqrt{c}y_0)\cosh(\sqrt{c}y_0)[(1+\alpha)\tanh(\sqrt{c}y_0) - 2(\sqrt{c}y_0)], \end{aligned}$$

for $0 < c = c_\alpha + \epsilon$ and $x_0^2 + y_0^2 < 1$. Choose $y_0 = 0$. Then to obtain our desired inequality, we have to find $x_0 \in (-1, 1)$, $x_0 \neq 0$, such that

$$\sin(\sqrt{c}x_0)\cos(\sqrt{c}x_0)[2x_0\sqrt{c} - (1+\alpha)\tan(\sqrt{c}x_0)] \leq 0 \tag{5.23}$$

holds. Now, we see that $\sin(\sqrt{c}x_0)$ and $\cos(\sqrt{c}x_0)$ are positive in $(0, \pi/2\sqrt{c})$, and $2x_0\sqrt{c} - (1+\alpha)\tan(\sqrt{c}x_0)$ is negative in $(\sqrt{c_\alpha}/\sqrt{c}, \pi/2\sqrt{c})$, where the latter part follows by (5.10). Therefore, (5.23) holds true for some x_0 in the intersection

$$(0, \pi/2\sqrt{c}) \cap (\sqrt{c_\alpha}/\sqrt{c}, 1) \subset (0, 1),$$

since $c_\alpha < c$. This completes the proof of our first main theorem. □

In the following example, we construct a function meromorphically convex of order α satisfies the hypothesis of Theorem 1.6.

Example 5.11. *For a constant $c > 0$, consider the function h defined by*

$$h(z) = \frac{w_1(z)}{w_2(z)} = \sqrt{c}\cot(\sqrt{c}z),$$

where $w_1(z) = \cos(\sqrt{c}z)$ and $w_2(z) = (1/\sqrt{c})\sin(\sqrt{c}z)$ that satisfy the differential equation

$$w'' + 2cw = 0.$$

By Lemma 5.4, it follows that $S_h(z) = 2c$. Now, for any such constant $c \leq c_\alpha$, where c_α is the smallest positive root of the equation (1.19), one clearly sees that

$$|S_h(z)| \leq 2c_\alpha.$$

Next, by comparing with Lemma 5.8, we see that

$$\mathrm{Re}\Big(\frac{zw_2'(z)}{w_2(z)}\Big) = \mathrm{Re}(z\sqrt{c}\cot(\sqrt{c}z)) > \frac{1+\alpha}{2}.$$

This is equivalent to saying that h is meromorphically convex of order α, by Lemma 5.6. Thus, Theorem 1.6 is satisfied by the function $h(z) = \sqrt{c}\cot(\sqrt{c}z)$.

5.3.2. Proof of Theorem 5.1

We adopt the idea from the proof of [17, Theorem 2]. Suppose that $u(z)$ and $v(z)$ are two linearly independent solutions of the differential equation (5.4) with $S_f(z) = 2p(z)$, where $u(0) = v'(0) = 0$ and $u'(0) = v(0) = 1$. Then by a similar analysis as in the proof of [17, Theorem 2], we obtain

$$f(z) = \frac{u(z)}{cu(z) + v(z)},$$

where $c = -a_2$. An easy computation yields

$$1 + \frac{zf''(z)}{f'(z)} = 1 - 2z\frac{cu'(z) + v'(z)}{cu(z) + v(z)}. \tag{5.24}$$

Now, by the hypothesis, it is easy to see that

$$\phi(\beta) = \min\left\{\frac{\beta - 1}{\beta + 1}, \frac{6(\beta - 1)}{2(7\beta - 9)}\right\} < 1 \text{ and } \psi(\beta) = \max\left\{\frac{\beta + 3}{\beta + 1}, \frac{11\beta - 15}{7\beta - 9}\right\} > 1.$$

Also, we note that

$$2\eta + (1+\eta)\delta e^{\delta/2} < 2\eta + \psi(\beta)\delta(1+\eta)e^{\delta/2} < 2\phi(\beta) < 2$$

follows from the assumption (5.1). Hence $\eta + (1+\eta)\delta e^{\delta/2}/2 < 1$. Now [17, (13)] also satisfied by our hypothesis. Thus, it follows from the similar argument as in the proof of [17, Theorem 2] that

$$\left|\frac{cu'(z)+v'(z)}{cu(z)+v(z)}\right| \leq \frac{2(\eta+(1+\eta)\delta e^{\delta/2})}{2-2\eta-(1+\eta)\delta e^{\delta/2}},$$

which yields

$$\operatorname{Re}\left(\frac{z(cu'(z)+v'(z))}{cu(z)+v(z)}\right) > -\left|\frac{z(cu'+v')}{cu+v}\right| > -\frac{2(\eta+(1+\eta)\delta e^{\delta/2})}{2-2\eta-(1+\eta)\delta e^{\delta/2}}, \tag{5.25}$$

and

$$\operatorname{Re}\left(\frac{z(cu'(z)+v'(z))}{cu(z)+v(z)}\right) \leq \left|\frac{z(cu'+v')}{cu+v}\right| < \frac{2(\eta+(1+\eta)\delta e^{\delta/2})}{2-2\eta-(1+\eta)\delta e^{\delta/2}}. \tag{5.26}$$

The relations (5.24), (5.25) and (5.26) together lead to

$$\frac{2-6\eta-5(1+\eta)\delta e^{\delta}/2}{2-2\eta-(1+\eta)\delta e^{\delta}/2} < \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < \frac{2+2\eta+3(1+\eta)\delta e^{\delta}/2}{2-2\eta-(1+\eta)\delta e^{\delta}/2}.$$

The hypothesis (5.1) thus obtains

$$\frac{2+2\eta+3(1+\eta)\delta e^{\delta}/2}{2-2\eta-(1+\eta)\delta e^{\delta}/2} < \beta$$

and

$$\frac{2-6\eta-5(1+\eta)\delta e^{\delta}/2}{2-2\eta-(1+\eta)\delta e^{\delta}/2} > \frac{-\beta}{2\beta-3},$$

completing the proof. □

Remark 5.12. *The constant $\phi(\beta)$ in the statement of Theorem 5.1 is not sharp. For instance, the function $f(z) = \dfrac{2z-z^2}{2(1-z)^2} \in \mathcal{C}_\infty$ for which $|a_2| = 3/2 > 1$.*

In the following example we construct a function that agree with Theorem 5.1 for some $\beta \geq 3/2$.

Example 5.13. *For any constant c with $|c| < 3/7$, consider the function f defined by*

$$f(z) = \frac{z}{1-cz}, \quad |z| < 1.$$

We show that $f \in \mathcal{C}_{5/2}$ and it satisfies the hypothesis of Theorem 5.1.

First, we note that f *is a Möbius transformation and hence* $S_f = 0$*. Therefore, it trivially satisfies the hypothesis of Theorem 5.1.*

Secondly, an easy computation yields

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1+cz}{1-cz}.$$

From this, we have

$$\mathrm{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) = \frac{1-|c|^2|z|^2}{|1-cz|^2}.$$

By the usual triangle inequalities, it follows that

$$\frac{1-|c||z|}{1+|c||z|} \leq \frac{1-|c|^2|z|^2}{|1-cz|^2} \leq \frac{1+|c||z|}{1-|c||z|}.$$

Since $|c| < 3/7$*, for* $|z| < 1$*, it is easy to verify that*

$$\frac{1+|c||z|}{1-|c||z|} < \frac{5}{2} \quad \textit{and} \quad -\frac{5}{4} < \frac{1-|c||z|}{1+|c||z|}$$

hold true. Thus, $f \in \mathcal{C}_{5/2}$*.*

CHAPTER 6

APPROXIMATION OF ANALYTIC FUNCTIONS

As we discussed about approximation problem in Chapter 1, Section 1.5.4, the purpose of this chapter[1] is to consider non-vanishing analytic functions of the form z/f, $f \in \mathcal{S}$, and study their best approximations by functions z/g when g belongs to the family $\mathcal{F}_\alpha$ defined by (1.22).

Clearly, z/f with $f \in \mathcal{S}$ has a Taylor series expansion of the form (1.21). Such forms of non-vanishing analytic functions have been widely studied by many authors for different purposes. For instance, in 1990, Yamashita [99] proved that the maximal area of the images of $\mathbb{D}$ under non-vanishing analytic functions of the form z/f, $f \in \mathcal{S}$, is bounded. For more research works on non-vanishing analytic functions of the form z/f, one can refer to [7, 62, 64, 65, 76, 78].

6.1. Preliminaries

We recall the following result from [69] which describes the L^2-norm of an analytic function $f : \mathbb{D} \to \mathbb{C}$ in terms of the coefficients of the Taylor series of f. This plays an important role in the proof of our main results.

Lemma 6.1. *If $f : \mathbb{D} \to \mathbb{C}$ is analytic and has series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{D},$$

then

$$\int_{\mathbb{D}} |f(x+iy)|^2 dx\, dy = \pi \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}.$$

In the spirit of [37, 69, 70] we determine the best approximation of an analytic function of the form z/f, for $f \in \mathcal{S}$, by functions belonging to the family $\mathcal{F}_\alpha$. The method is based

[1]Results of this chapter are based on paper: V. Arora, S. K. Sahoo, and S. Singh, Approximation of certain non-vanishing analytic functions in a parabolic region, submitted.

on solving a particular quadratic problem with an infinite number of variables using the classical Karush-Kuhn-Tucker (KKT) conditions.

6.2. Approximation by functions in the family $\mathcal{F}_\alpha$

In this section, we intend to find the best approximation of an analytic function of the form z/f by a function z/g, where $g \in \mathcal{F}_\alpha$. In view of (1.23) and Lemma 6.1, it is enough to consider the quadratic problem of finding

$$\inf \sum_{n=1}^{\infty} \frac{(c_n - b_n)^2}{n+1}, \tag{6.1}$$

where (b_n) is a given sequence of non-negative real numbers and the infimum is taken over all non-negative sequence (c_n) of real numbers satisfying

$$\sum_{n=1}^{\infty} A(n,\alpha) c_n \leq 1, \tag{6.2}$$

with $A(n,\alpha)$ as given in (1.20). The following result is our first main result which gives a solution of the quadratic problem of finding the quantity given by (6.1) satisfying the condition (6.2):

Theorem 6.2. *Let $A(n,\alpha)$ be given by (1.20). If (b_n) is a sequence of non-negative real numbers with*

$$\sum_{n=1}^{\infty} A(n,\alpha) b_n > 1$$

and

$$\lim_{n\to\infty} \frac{b_n}{n^2} = 0,$$

then there exists an integer $N \geq 1$ such that the minimum of the quadratic problem (6.1) and (6.2) is

$$\sum_{n\in\mathcal{I}^c} \frac{b_n^2}{n+1} + \frac{\left(\sum_{n\in\mathcal{I}} A(n,\alpha) b_n - 1\right)^2}{\sum_{n\in\mathcal{I}} A^2(n,\alpha)(n+1)},$$

which is attained for the sequence (c_n) given by

$$c_n = \begin{cases} b_n - \mu_N \dfrac{A(n,\alpha)(n+1)}{2}, & n \in \mathcal{I}; \\ 0, & n \in \mathcal{I}^c, \end{cases} \tag{6.3}$$

where

$$\mu_N = 2\frac{\sum_{n\in\mathcal{I}} A(n,\alpha)b_n - 1}{\sum_{n\in\mathcal{I}} A^2(n,\alpha)(n+1)},$$

$\mathcal{I} = \{i_1, i_2, \ldots, i_N\}$ *and* $(i_n)_{n=1,2,\ldots,|\mathcal{P}|}$ *is a permutation of the indices in* $\mathcal{P} = \{n \geq 1 : b_n > 0\}$ *such that*

$$\beta_{i_n} := \frac{2b_{i_n}}{(i_n+1)A(i_n,\alpha)}, \quad n = 1, 2, \ldots, |\mathcal{P}|$$

is a non-increasing sequence. Moreover,

$$N = \min\{n \geq 1 : \beta_{i_{n+1}} \leq \mu_{i_n} \leq \beta_{i_n}\},$$

where

$$\mu_{i_n} = 2\frac{\sum_{m=1}^{n} A(i_m,\alpha)b_{i_m} - 1}{\sum_{m=1}^{n} A^2(i_m,\alpha)(i_m+1)}.$$

Proof. Consider the quadratic problem of finding

$$\inf \sum_{n=1}^{\infty} \frac{(b_n - c_n)^2}{n+1},$$

where $(b_n)_{n\geq 1}$ is a given sequence of non-negative real numbers, and the infimum is taken over all non-negative sequence $(c_n)_{n\geq 1}$ of real numbers satisfying $\sum_{n=1}^{\infty} A(n,\alpha)c_n \leq 1$.

Note that for $\mathcal{P} = \{n \geq 1 : b_n > 0\}$, the above problem becomes

$$\inf \sum_{n=1}^{\infty} \frac{(b_n - c_n)^2}{n+1} = \inf \sum_{n\in\mathcal{P}} \frac{(b_n - c_n)^2}{n+1},$$

where the infimum is taken over all non-negative sequence $(c_n)_{n\geq 1}$ of real numbers satisfying $\sum_{n=1}^{\infty} A(n,\alpha)c_n \leq 1$. The following two possibilities arise.

Case 1: If $\sum_{n=1}^{\infty} A(n,\alpha)b_n \leq 1$. Then the infimum in (1.23) becomes 0 and it attains for $c_n = b_n$, $n \geq 1$.

Case 2: If $\sum_{n=1}^{\infty} A(n,\alpha)b_n > 1$. Then the Lagrangian becomes

$$L = \sum_{n=1}^{\infty} \frac{(b_n - c_n)^2}{n+1} + \mu\left(\sum_{n=1}^{\infty} A(n,\alpha)\, c_n - 1\right)$$

with the Karush-Kuhn-Tucker conditions (see [35, 69])

$$\frac{\partial L}{\partial c_n} = \frac{2(c_n - b_n)}{n+1} + \mu A(n,\alpha) \geq 0, \quad n \geq 1; \tag{6.4}$$

$$\frac{\partial L}{\partial \mu} = \sum_{n=1}^{\infty} A(n,\alpha)\, c_n - 1 \leq 0; \tag{6.5}$$

$$c_n \frac{\partial L}{\partial c_n} = c_n \left(\frac{2(c_n - b_n)}{n+1} + \mu A(n,\alpha) \right) = 0, \quad n \geq 1; \tag{6.6}$$

$$\mu(Ax - b) = \mu \left(\sum_{n=1}^{\infty} A(n,\alpha)\, c_n - 1 \right) = 0; \tag{6.7}$$

$$c_n \geq 0, \quad n \geq 1; \tag{6.8}$$

$$\mu \geq 0. \tag{6.9}$$

From the equation (6.7), we see that either $\mu = 0$ or $\sum_{n=1}^{\infty} A(n,\alpha)c_n = 1$ holds. If $\mu = 0$, then from the equation (6.6) we obtain $c_n = 0$ or $c_n = b_n$ and the equation (6.4) shows that $c_n \geq b_n$. If $c_n = 0$, then $b_n \leq 0$. But, by the hypothesis $b_n \geq 0$. Thus, we conclude that $c_n = b_n$ for all $n \geq 1$. However, this contradicts (6.5) which can be seen from the assumption of the case. Thus, in this case we must have $\mu > 0$ and $\sum_{n=1}^{\infty} A(n,\alpha)c_n = 1$.

Now rewrite the above conditions (6.4)-(6.9) as follows:

$$\frac{2(c_n - b_n)}{n+1} + \mu A(n,\alpha) \geq 0, \quad n \geq 1; \tag{6.10}$$

$$c_n \left(\frac{2(c_n - b_n)}{n+1} + \mu A(n,\alpha) \right) = 0, \quad n \geq 1; \tag{6.11}$$

$$\sum_{n=1}^{\infty} A(n,\alpha) c_n = 1; \tag{6.12}$$

$$c_n \geq 0, \quad n \geq 1; \tag{6.13}$$

$$\mu > 0. \tag{6.14}$$

The equation (6.11) shows that either

$$c_n = 0 \text{ or } c_n = b_n - \frac{\mu A(n,\alpha)(n+1)}{2}.$$

We denote by $\mathcal{I}^c$ the set of indices $n \geq 1$ for which $c_n = 0$ and

$$\mathcal{I} = \{1, 2, 3, \dots\} - \mathcal{I}^c = \left\{ n \geq 1 : c_n = b_n - \frac{\mu A(n, \alpha)(n+1)}{2} \right\}.$$

Hence by (6.12), we now obtain

$$\begin{aligned} 1 &= \sum_{n=1}^{\infty} A(n, \alpha)\, c_n \\ &= \sum_{n \in \mathcal{I}} A(n, \alpha) \left(b_n - \frac{\mu A(n, \alpha)(n+1)}{2} \right) \\ &= \sum_{n \in \mathcal{I}} A(n, \alpha)\, b_n - \frac{\mu}{2} \sum_{n \in \mathcal{I}} A^2(n, \alpha)(n+1). \end{aligned}$$

This simplifies to

$$\mu = 2 \frac{\sum_{n \in \mathcal{I}} A(n, \alpha) b_n - 1}{\sum_{n \in \mathcal{I}} A^2(n, \alpha)(n+1)} > 0. \tag{6.15}$$

Also from the equations (6.10) and (6.13), we have

$$\mu \leq \beta_n \text{ for } n \in \mathcal{I}; \text{ and } \mu \geq \beta_n \text{ for } n \in \mathcal{I}^c, \tag{6.16}$$

where

$$\beta_n := \frac{2 b_n}{(n+1) A(n, \alpha)}.$$

By the hypothesis, we observe that (β_n) is a sequence of positive numbers such that $\lim_{n \to \infty} \beta_n = 0$. Therefore, we can find a permutation of (i_n) of the indices in $\mathcal{P}$ such that (β_{i_n}) is a non-increasing sequence with $\lim_{n \to \infty} \beta_{i_n} = 0$. The set of indices $\mathcal{I}$ must be finite, otherwise if we pass to the limit $n \to \infty$ in the first inequality of (6.16) along a sequence of indices in $\mathcal{I}$, we obtain that $\mu \leq 0$, contradicting (6.14). Therefore, we can write $\mathcal{I} = \{i_1, i_2, i_3, \dots, i_N\}$ due to finiteness of $\mathcal{I}$.

Claim: $N = \min\{n \geq 1 : \beta_{i_{n+1}} \leq \mu_n \leq \beta_{i_n}\}$ where

$$\mu_n = 2 \frac{\sum_{m=1}^{n} A(i_m, \alpha) b_{i_m} - 1}{\sum_{m=1}^{n} A^2(i_m, \alpha)(i_m + 1)}.$$

In view of the hypothesis

$$\sum_{n=1}^{\infty} A(n, \alpha) b_n = \sum_{n=1}^{\infty} A(i_n, \alpha) b_{i_n} > 1.$$

Hence, there exists a smallest integer $n_0 \geq 1$ such that

$$\sum_{n=1}^{n_0} A(i_n, \alpha) b_{i_n} > 1 \quad \text{and} \quad \sum_{n=1}^{k} A(i_n, \alpha) b_{i_n} \leq 1$$

for all $k \leq n_0 - 1$. Note that for $n_0 = 1$, we have

$$\mu_1 = 2\frac{A(i_1, \alpha) b_{i_1} - 1}{A^2(i_1, \alpha)(i_1 + 1)} \leq \frac{2b_{i_1}}{(i_1 + 1) A(i_1, \alpha)} = \beta_{i_1}.$$

If $n_0 > 1$, then by the choice of n_0 we obtain

$$\mu_{n_0 - 1} = 2\frac{\sum_{m=1}^{n_0-1} A(i_m, \alpha) b_{i_m} - 1}{\sum_{m=1}^{n_0-1} A^2(i_m, \alpha)(i_m + 1)} \leq 0 \leq \frac{2b_{i_{n_0}}}{(i_{n_0} + 1) A(i_{n_0}, \alpha)} = \beta_{i_{n_0}}.$$

Using the mediant inequality

$$\frac{a}{b} \leq \frac{c}{d} \implies \frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}, \tag{6.17}$$

where b, $d > 0$, it follows that

$$\begin{aligned}
\mu_{n_0-1} &\leq \mu_{n_0} \\
&= 2\frac{\sum_{m=1}^{n_0-1} A(i_m, \alpha) b_{i_m} - 1 + A(i_{n_0}, \alpha) b_{i_{n_0}}}{\sum_{m=1}^{n_0-1} A^2(i_m, \alpha)(i_m + 1) + A(i_{n_0}, \alpha)^2 (i_{n_0} + 1)} \\
&\leq \frac{2b_{i_{n_0}}}{(i_{n_0} + 1) A(i_{n_0}, \alpha)} = \beta_{i_{n_0}}.
\end{aligned}$$

Thus, we proved that $\mu_{n_0} \leq \beta_{i_{n_0}}$ for $n_0 \geq 1$.

Next we consider the following two cases in order to complete the proof.

Case (i): $\beta_{i_{n_0+1}} \leq \mu_{n_0}$.

Since (β_{i_n}) is a non-increasing sequence and $\mu_{n_0} \leq \beta_{i_{n_0}}$, it follows that

$$\beta_{i_{n_0+1}} \leq \mu_{n_0} \leq \beta_{i_{n_0}} \leq \beta_{i_n}, \quad n \in \{1, 2, \dots, n_0\},$$

and

$$\beta_{i_n} \leq \beta_{i_{n_0+1}} \leq \mu_{n_0} \leq \beta_{i_{n_0}}, \quad n \in \{n_0 + 1, n_0 + 2, \dots\}.$$

In this case, we can then choose $N = n_0$ and $\mathcal{I} = \{i_1, i_2, \dots, i_{n_0}\}$.

Case(ii): $\beta_{i_{n_0+1}} > \mu_{n_0}$.

Using the observation (6.17), we again have

$$\mu_{n_0} \leq \mu_{n_0+1} \leq \beta_{i_{n_0+1}}.$$

Then either $\beta_{i_{n_0+2}} \leq \mu_{n_0+1}$ or $\beta_{i_{n_0+2}} > \mu_{n_0+1}$. If $\beta_{i_{n_0+2}} \leq \mu_{n_0+1}$, proceeding as in Case (i) above, we can choose $N = n_0 + 1$ and $\mathcal{I} = \{i_1, i_2, \ldots, i_{n_0+1}\}$. If $\beta_{i_{n_0+2}} > \mu_{n_0+1}$, then from the observation (6.17) we obtain

$$\mu_{n_0} \leq \mu_{n_0+1} \leq \mu_{n_0+2} \leq \beta_{i_{n_0+2}}.$$

Continuing as above, either at some point we can find an integer $k \geq 1$ such that

$$\beta_{i_{n_0+k+1}} \leq \mu_{n_0+k} \leq \beta_{i_{n_0+k}} \tag{6.18}$$

and then $N = n_0 + k$, or else we have

$$0 < \mu_{n_0} \leq \mu_{n_0+1} \leq \mu_{n_0+2} \cdots \mu_{n_0+k} \leq \beta_{i_{n_0+k}}. \tag{6.19}$$

However, since (β_{i_n}) is a non-increasing sequence of positive real numbers with $\lim_{n\to\infty} \beta_{i_n} = 0$, the inequalities in (6.19) cannot hold for every $k \geq 0$. It follows that we can always find an integer k for which (6.18) holds.

Now we justify the applicability of the Karush-Kuhn-Tucker conditions for an infinite (instead of a finite) number of variables c_n to find the minimum in the quadratic problem (6.1) and (6.2). Note that for any integer $m \geq 1$, we have

$$\inf \sum_{n=1}^{\infty} \frac{(b_n - c_n)^2}{n+1} \geq \inf \sum_{n=1}^{m} \frac{(b_n - c_n)^2}{n+1}, \tag{6.20}$$

where both the minimum quantities are taken over all non-negative sequence (c_n) of real numbers satisfying $\sum_{n=1}^{\infty} A(n,\alpha) c_n \leq 1$. Solving the Karush-Kuhn-Tucker conditions as above for finite dimensional problem of computing the second infimum in (6.20) with $\sum_{n=1}^{m} A(n,\alpha) c_n \leq 1$, it follows that for $m \geq \max\{i_1, i_2, \ldots, i_N\}$, the second infimum in (6.20) is attained for the sequence (c_n) given by

$$c_n = \begin{cases} b_n - \mu_N \dfrac{A(n,\alpha)(n+1)}{2}, & n \in \mathcal{I}; \\ 0, & n \in \mathcal{I}^c \cap \{1, 2, \ldots, m\}. \end{cases}$$

This argument can be applied for any arbitrary $m \geq \max\{i_1, i_2, \ldots, i_N\}$. From (6.20), we obtain

$$\inf \sum_{n=1}^{\infty} \frac{(b_n - c_n)^2}{n+1} \geq \sum_{n \in \mathcal{I}^c \cap \{1,\ldots,m\}} \frac{b_n^2}{n+1} + \frac{\left(\sum_{n\in\mathcal{I}} A(n,\alpha) b_n - 1\right)^2}{\sum_{n\in\mathcal{I}} A^2(n,\alpha)(n+1)}.$$

Since the above inequality is true for $m > \max\{i_1, i_2, \dots, i_N\}$, letting $m \to \infty$, we obtain that

$$\inf \sum_{n=1}^{\infty} \frac{(b_n - c_n)^2}{n+1} \geq \lim_{m\to\infty} \sum_{n\in\mathcal{I}^c\cap\{1,\dots,m\}} \frac{b_n^2}{n+1} + \frac{\left(\sum_{n\in\mathcal{I}} A(n,\alpha)b_n - 1\right)^2}{\sum_{n\in\mathcal{I}} A^2(n,\alpha)(n+1)}$$

$$= \sum_{n\in\mathcal{I}^c} \frac{b_n^2}{n+1} + \frac{\left(\sum_{n\in\mathcal{I}} A(n,\alpha)b_n - 1\right)^2}{\sum_{n\in\mathcal{I}} A^2(n,\alpha)(n+1)}.$$

From the above inequality, it follows that the infimum of the quadratic problem (6.1) and (6.2) is attained for the sequence (c_n) defined by (6.3) and this justifies the use of the Karush-Kuhn-Tucker conditions in infinite dimensional setting. This completes the proof of our theorem. □

An application of Theorem 6.2 is to determine the best approximation of an analytic function in the family $\mathcal{F}_\alpha$ stated in Chapter 1 (Theorem 1.8).

The family $\mathcal{S}_p(\alpha)$ can be characterized in terms of $d_\alpha(\cdot, \mathcal{S}_p(\alpha))$ as follows.

Theorem 6.3. *For $f \in \mathcal{S}$, $d_\alpha(f, \mathcal{S}_p(\alpha)) = 0$ iff $f \in \mathcal{S}_p(\alpha)$.*

Proof. If $f \in \mathcal{S}_p(\alpha)$, then we clearly get $d_\alpha(f, \mathcal{S}_p(\alpha)) = 0$. On the other hand, if $d_\alpha(f, \mathcal{S}_p(\alpha)) = 0$, then we can find a sequence of functions $(f_n)_{n\geq 1}$ in $\mathcal{S}_p(\alpha)$ such that

$$\int_{\mathbb{D}} \left| \frac{z}{f(z)} - \frac{z}{f_n(z)} \right|^2 dx\, dy < \frac{\pi}{n}, \quad n \geq 1. \tag{6.21}$$

Consider the power series expansion of z/f and z/f_n of the form

$$\frac{z}{f(z)} = 1 + \sum_{m=1}^{\infty} b_m z^m \text{ and } \frac{z}{f_n(z)} = 1 + \sum_{m=1}^{\infty} b_{n,m} z^m.$$

It follows from Lemma 6.1 and the inequality (6.21) that

$$\sum_{m=1}^{\infty} \frac{|b_m - b_{n,m}|^2}{m+1} < \frac{1}{n}. \tag{6.22}$$

We now proceed to prove that the sequence z/f_n converges to z/f uniformly on every compact subset of $\mathbb{D}$. For any arbitrarily $\xi \in \mathbb{D}_r := \{z : |z| < r\}$, $0 < r < 1$, an easy computation and analysis show that

$$\left| \frac{\xi}{f(\xi)} - \frac{\xi}{f_n(\xi)} \right| = \left| \sum_{m=1}^{\infty} (b_m - b_{n,m})\xi^m \right| \leq \sum_{m=1}^{\infty} |b_m - b_{n,m}| r^m.$$

Now using the classical Cauchy-Schwarz inequality and (6.22), we find that

$$\left|\frac{\xi}{f(\xi)}-\frac{\xi}{f_n(\xi)}\right| \leq \left(\sum_{m=1}^{\infty}\frac{|b_m-b_{n,m}|^2}{m+1}\right)^{1/2}\left(\sum_{m=1}^{\infty}(m+1)r^{2m}\right)^{1/2} \leq \frac{1}{\sqrt{n}}\left(\sum_{m=1}^{\infty}(m+1)r^{2m}\right)^{1/2}.$$

However, the last expression in the above inequality approaches to 0 as $n\to\infty$. This shows that the sequence z/f_n converges to z/f uniformly on $\overline{\mathbb{D}_r}$ for every $0<r<1$ and hence the sequence z/f_n converges to z/f uniformly on every compact subset of $\mathbb{D}$. Thus, the sequence f_n converges to f uniformly on every compact subset of $\mathbb{D}$. As we know that $f_n\in\mathcal{S}_p(\alpha)$, it satisfies

$$\left|\frac{zf_n'(z)}{f_n(z)}-1\right| \leq \mathrm{Re}\frac{zf_n'(z)}{f_n(z)}-\alpha,\ z\in\mathbb{D}.$$

Taking the limit $n\to\infty$, we conclude that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \mathrm{Re}\frac{zf'(z)}{f(z)}-\alpha,\ z\in\mathbb{D}.$$

Thus $f\in\mathcal{S}_p(\alpha)$, completing the proof. □

6.2.1. Proof of Theorem 1.8

It is easy to check that, if $\sum_{n=1}^{\infty}A(n,\alpha)|b_n|\leq 1$, then the infimum in (1.23) becomes 0 and it attains for $c_n=b_n$, $n\geq 1$.

Assume that $\sum_{n=1}^{\infty}A(n,\alpha)|b_n|>1$. Lemma 6.1 together with the triangle inequality lead to

$$d_\alpha(f,\mathcal{F}_\alpha)=\inf_{g\in\mathcal{F}_\alpha}\left(\int_{\mathbb{D}}\left|\frac{z}{f(z)}-\frac{z}{g(z)}\right|^2 dx\,dy\right)^{1/2} = \left(\pi\inf\sum_{n=0}^{\infty}\frac{|b_n-c_n|^2}{n+1}\right)^{1/2} \tag{6.23}$$

$$\geq \left(\pi\inf\sum_{n=0}^{\infty}\frac{(|b_n|-|c_n|)^2}{n+1}\right)^{1/2},$$

where the last two infimum quantities are taken over all sequences $(c_n)_{n\geq 2}$ of complex numbers satisfying $\sum_{n=1}^{\infty}A(n,\alpha)|c_n|\leq 1$.

We now apply Theorem 6.2, by replacing b_n with $|b_n|$ and by replacing c_n with $|c_n|$, to obtain the last infimum which is attained for the sequence (c_n) given by

$$|c_n|=\begin{cases}\left(|b_n|-A(n,\alpha)(n+1)\dfrac{\left(\sum_{m\in\mathcal{I}}A(m,\alpha)|b_m|-1\right)}{\sum_{m\in\mathcal{I}}A^2(m,\alpha)(m+1)}\right), & n\in\mathcal{I};\\[2ex] 0, & n\in\mathcal{I}^c.\end{cases}$$

The equality holds in (6.23) if $\arg b_n = \arg c_n$, i.e. the infimum is attained for the sequence of complex number $c_n = |c_n| e^{i \arg b_n}$ on $\mathcal{I}$ and $c_n = 0$ on $\mathcal{I}^c$.

It remains to prove that $g \in \mathcal{F}_\alpha$. For this, we consider

$$\sum_{n=1}^{\infty} A(n,\alpha)\,|c_n| = \sum_{n\in\mathcal{I}} \left[A(n,\alpha)\left(|b_n| - A(n,\alpha)(n+1)\frac{\left(\sum_{m\in\mathcal{I}} A(m,\alpha)|b_m| - 1\right)}{\sum_{m\in\mathcal{I}} A^2(m,\alpha)(m+1)} \right) \right]$$

$$= \sum_{n\in\mathcal{I}} A(n,\alpha)|b_n| - \left(\sum_{m\in\mathcal{I}} A(m,\alpha)|b_m| - 1 \right) = 1.$$

Hence $g \in \mathcal{F}_\alpha$ with $z/g(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Further, we have

$$d_\alpha(f, \mathcal{F}_\alpha) = \left(\pi \inf \sum_{n=0}^{\infty} \frac{(|b_n| - |c_n|)^2}{n+1} \right)^{1/2}$$

$$= \left(\pi \sum_{n\in\mathcal{I}^c} \frac{|b_n|^2}{n+1} + \pi \frac{\left(\sum_{n\in\mathcal{I}} A(n,\alpha)|b_n| - 1\right)^2}{\sum_{n\in\mathcal{I}} A^2(n,\alpha)(n+1)} \right)^{1/2},$$

and the proof is complete.

Here we provide an example that associates with Theorems 6.2, 6.3, and 1.8.

Example 6.4. *Consider the univalent function $f_a : \mathbb{D} \to \mathbb{C}$ defined by*

$$f_a(z) = \frac{z}{1 + z + az^2},$$

where $a \in \mathbb{C}$ is a non-zero constant such that $|a| < 1$. Then $z/f_a = 1 + z + az^2 = 1 + \sum_{n=1}^{\infty} b_n z^n$ and $b_1 = 1$, $b_2 = a$, $b_n = 0$ for $n \geq 3$. Applying Theorems 6.2 and 1.8 for an arbitrarily fixed α, $-1 \leq \alpha \leq 1$, we conclude what follows next.

By the choice of a and α, we have $(5-\alpha)|a| \geq 0 > -2$. Then Theorem 6.2 gives that $\mathcal{P} = \{1, 2\}$ and $i_n = n$ for $n \geq 1$. It arises with the following cases:

If $|a| \leq 3(5-\alpha)/(3-\alpha)^2$, then we have $N = 1$, $\mathcal{I} = \{i_1\} = \{1\}$.

If $|a| > 3(5-\alpha)/(3-\alpha)^2$, then $N = 2$, $\mathcal{I} = \{i_1, i_2\} = \{1, 2\}$. A simple computation gives

$$d_\alpha(f_a, \mathcal{F}_\alpha) = \inf_{g\in\mathcal{F}_\alpha} d(f_a, g) = \begin{cases} \left(\dfrac{\pi|a|^2}{3} + \dfrac{2\pi}{(3-\alpha)^2} \right)^{1/2} & \text{if } |a| \leq \dfrac{3(5-\alpha)}{(3-\alpha)^2}; \\ \left(\dfrac{\pi(2 + (5-\alpha)|a|)^2}{2(3-\alpha)^2 + 3(5-\alpha)^2} \right)^{1/2} & \text{if } |a| > \dfrac{3(5-\alpha)}{(3-\alpha)^2}, \end{cases}$$

where the infimum is attained for the function $g = g_a \in \mathcal{F}_\alpha$ defined by

$$\frac{z}{g_a(z)} = \begin{cases} 1 + \dfrac{1-\alpha}{3-\alpha} z & \text{if } |a| \le \dfrac{3(5-\alpha)}{(3-\alpha)^2}; \\ 1 + \left(1 - \dfrac{2(3-\alpha)(2+(5-\alpha)|a|)}{2(3-\alpha)^2 + 3(5-\alpha)^2}\right) z & \\ \quad + \left(|a| - \dfrac{3(5-\alpha)(2+(5-\alpha)|a|)}{2(3-\alpha)^2+3(5-\alpha)^2}\right) e^{i\arg(a)} z^2 & \text{if } |a| > \dfrac{3(5-\alpha)}{(3-\alpha)^2}. \end{cases}$$

Fig. 6.1 and Fig. 6.2 describe the images of $\mathbb{D}$ under f_a and g_a for some particular values of a and α where f_a is the given function and g_a is the corresponding function for which the minimum distance is attained.

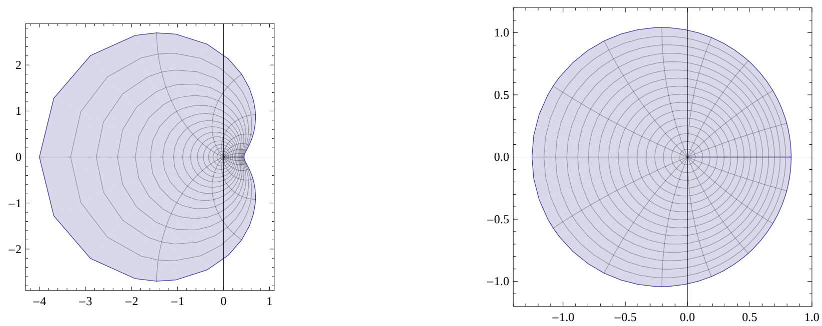

FIGURE 6.1. Images of $\mathbb{D}$ under f_a and g_a for $a = 1/4$, $\alpha = 1/2$.

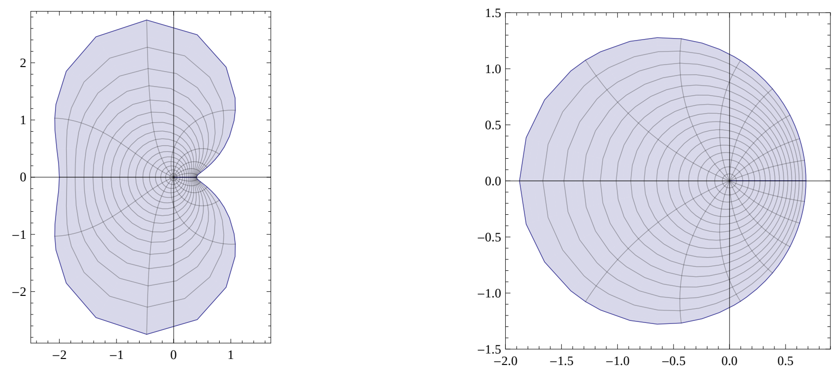

FIGURE 6.2. Images of $\mathbb{D}$ under f_a and g_a for $a = 1/2$, $\alpha = -3/4$.

6.3. Approximation by functions in the class $\mathcal{U}_\alpha$ and $\mathcal{P}_{2\alpha}$

We have seen that a sufficient condition for functions to be in that family played an important role. In this section, we recall some classes of functions and corresponding sufficient conditions of the form (1.21). Also, with the help of these sufficient conditions, here we generate some subclasses for which the Quadratic problem and approximation problem can be done. Although we are not much concerned about the proof as it follows similarly using the idea described in Theorem 6.2 and Theorem 1.8.

A function $f \in \mathcal{A}$ is said to be in $\mathcal{U}(\alpha)$ if

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^2 - 1 \right| \leq \alpha, \quad z \in \mathbb{D}$$

for some $\alpha \geq 0$. Aksentév [5] proved the inclusion $\mathcal{U}(\alpha) \subset \mathcal{S}$ for $0 \leq \alpha \leq 1$. Many properties of $\mathcal{U}(\alpha)$ and its various generalizations have been investigated in the literature, we refer for example [7, 67, 83] and the references therein. The following lemma gives the coefficient condition for functions in $\mathcal{U}(\alpha)$.

Lemma 6.5. [76] *Let $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be a non-vanishing analytic function in $\mathbb{D}$ satisfying the coefficient condition*

$$\sum_{n=1}^{\infty} (n-1)|b_n| \leq \alpha.$$

Then the function f defined by the equation $z/f = \phi$ is in $\mathcal{U}(\alpha)$.

The above lemma generate the following subclass of the class $\mathcal{U}(\alpha)$:

$$\mathcal{U}_\alpha = \left\{ f \in \mathcal{S} : \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} (n-1)|b_n| \leq \alpha,\ 0 \leq \alpha \leq 1 \right\} \subset \mathcal{U}(\alpha).$$

Consider the quadratic problem of finding

$$\inf \sum_{n=1}^{\infty} \frac{(c_n - b_n)^2}{n+1}, \tag{6.24}$$

where (b_n) is a given sequence of non-negative real numbers and the infimum is taken over all non-negative sequence (c_n) of real numbers satisfying

$$\sum_{n=1}^{\infty} \frac{(n-1)c_n}{\alpha} \leq 1. \tag{6.25}$$

A technique which is adopted in Theorem 6.2 and Theorem 1.8 further leads to the solution of Quadratic problem and approximation problem for the class $\mathcal{U}_\alpha$ presented below in Theorem 6.6 and Theorem 6.7.

Theorem 6.6. *If* (b_n) *is a sequence of non-negative real numbers with*

$$\sum_{n=1}^{\infty} \frac{n-1}{\alpha} b_n > 1$$

and

$$\lim_{n\to\infty} \frac{b_n}{n^2} = 0,$$

then there exists an integer $N \geq 1$ *such that the minimum of the quadratic problem* (6.24) *and* (6.25) *is*

$$\sum_{n\in\mathcal{I}^c} \frac{b_n^2}{n+1} + \frac{\left(\sum_{n\in\mathcal{I}}((n-1)b_n/\alpha) - 1\right)^2}{\sum_{n\in\mathcal{I}}((n-1)/\alpha)^2(n+1)}$$

which is attained for the sequence (c_n) *given by*

$$c_n = \begin{cases} b_n - \dfrac{\mu_N(n-1)(n+1)}{2\alpha}, & n \in \mathcal{I}; \\ 0, & n \in \mathcal{I}^c, \end{cases}$$

where

$$\mu_N = 2\frac{\sum_{n\in\mathcal{I}}((n-1)/\alpha)b_n - 1}{\sum_{n\in\mathcal{I}}((n-1)/\alpha)^2(n+1)},$$

$\mathcal{I} = \{i_1, i_2, \ldots, i_N\}$ *and* $(i_n)_{n=1,2,\ldots,|\mathcal{P}|}$ *is a permutation of the indices in* $\mathcal{P} = \{n \geq 1 : b_n > 0\}$ *such that*

$$\beta_{i_n} := \frac{2\alpha b_{i_n}}{(i_n+1)(i_n-1)}, \quad n = 1, 2, \ldots, |\mathcal{P}|$$

is a non-increasing sequence. Moreover,

$$N = \min\{n \geq 1 : \beta_{i_{n+1}} \leq \mu_{i_n} \leq \beta_{i_n}\},$$

where

$$\mu_{i_n} = 2\frac{\sum_{m=1}^{n}((i_m-1)/\alpha)b_{i_m} - 1}{\sum_{m=1}^{n}((i_m-1)/\alpha)^2(i_m+1)}.$$

Next, we present an application of Theorem 6.6. Here, we determine the best approximation of analytic function in the subclass $\mathcal{U}_\alpha$.

Theorem 6.7. *Let* $f \in \mathcal{S}$ *be a function of the form*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D},$$

and assume that

$$\lim_{n\to\infty} \frac{b_n}{n^2} = 0.$$

If $\sum_{n=1}^{\infty}(n-1)|b_n|/\alpha \leq 1$ *then* $dist(z/f,\mathcal{U}_\alpha) = 0$ *and the minimum is attained for the function* $g = f \in \mathcal{U}_\alpha$.

If $\sum_{n=1}^{\infty}(n-1)|b_n|/\alpha > 1$*then*

$$d_\alpha(f,\mathcal{U}_\alpha) = \left(\pi \sum_{n\in\mathcal{I}^c} \frac{|b_n|^2}{n+1} + \pi \frac{\left(\sum_{n\in\mathcal{I}}((n-1)|b_n|/\alpha) - 1\right)^2}{\sum_{n\in\mathcal{I}}((n-1)/\alpha)^2(n+1)}\right)^{1/2},$$

where $\mathcal{I}$ *is as given in Theorem 6.6 with* $|b_n|$ *instead of* b_n. *The minimum value of* $dist(z/f,\mathcal{U}_\alpha)$ *is attained for the function* $z/g(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, $g \in \mathcal{U}_\alpha$, *where*

$$c_n = \begin{cases} \left(|b_n| - \dfrac{(n-1)(n+1)}{\alpha} \dfrac{\sum_{m\in\mathcal{I}}((m-1)/(\alpha))|b_m| - 1}{\sum_{m\in\mathcal{I}}((m-1)/\alpha)^2(m+1)}\right) e^{i\arg b_n}, & n \in \mathcal{I}; \\ 0, & n \in \mathcal{I}^c. \end{cases}$$

The next gives the characterization of $\mathcal{U}_\alpha$ and derived by using the same techniques of Theorem 6.3.

Theorem 6.8. *For* $f \in \mathcal{S}$, $dist(z/f,\mathcal{U}_\alpha) = 0$ *iff* $f \in \mathcal{U}_\alpha$.

In [61], the authors studied the subclass $\mathcal{P}(2\alpha)$ of $\mathcal{U}(\alpha)$, consisting of functions f for which

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq 2\alpha, \quad z \in \mathbb{D}.$$

Close connections between the classes $\mathcal{P}(2\alpha)$ and $\mathcal{U}(\alpha)$ is given by $\mathcal{P}(2\alpha) \subset \mathcal{U}(\alpha)$, see [61]. Now, recall the following coefficient conditions for function in $\mathcal{P}(2\alpha)$.

Lemma 6.9. [76] *Let* $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ *be a non-vanishing analytic function in* $\mathbb{D}$ *and* $f = z/\phi$. *Then if*

$$\sum_{n=2}^{\infty} n(n-1)|b_n| \leq 2\alpha$$

then we have $f \in \mathcal{P}(2\alpha)$.

We consider the subclass $\mathcal{P}_{2\alpha}$ of $\mathcal{P}(2\alpha)$ generated via sufficient conditions proved in Lemma 6.9:

$$\mathcal{P}_{2\alpha} = \left\{ f \in \mathcal{S} : \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} n(n-1)|b_n| \leq 2\alpha,\ 0 \leq \alpha \leq 1 \right\} \subset \mathcal{P}(2\alpha).$$

Consider the problem of finding

$$\inf \sum_{n=1}^{\infty} \frac{(c_n - b_n)^2}{n+1} \tag{6.26}$$

where (b_n) is a given sequence of non-negative real numbers and the infimum is taken over all non-negative sequence (c_n) of real numbers satisfying

$$\sum_{n=1}^{\infty} \frac{n(n-1)c_n}{2\alpha} \leq 1. \tag{6.27}$$

Theorem 6.10. *If (b_n) is a sequence of non-negative real numbers with*

$$\sum_{n=1}^{\infty} \frac{n(n-1)}{2\alpha} b_n > 1$$

and

$$\lim_{n\to\infty} \frac{b_n}{n^3} = 0,$$

there exists an integer $N \geq 1$ such that the minimum of the quadratic problem (6.26) *and* (6.27) *is*

$$\sum_{n \in \mathcal{I}^c} \frac{b_n^2}{n+1} + \frac{\left(\sum_{n\in\mathcal{I}}(n(n-1)b_n/2\alpha) - 1\right)^2}{\sum_{n\in\mathcal{I}}(n(n-1)/2\alpha)^2(n+1)},$$

which is attained for the sequence (c_n) given by

$$c_n = \begin{cases} b_n - \dfrac{\mu_N n(n-1)(n+1)}{4\alpha}, & n \in \mathcal{I}; \\ 0, & n \in \mathcal{I}^c, \end{cases}$$

where

$$\mu_N = 2\frac{\sum_{n\in\mathcal{I}}(n(n-1)/2\alpha)b_n - 1}{\sum_{n\in\mathcal{I}}(n(n-1)/2\alpha)^2(n+1)},$$

$\mathcal{I} = \{i_1, i_2, \ldots, i_N\}$ *and* $(i_n)_{n=1,2,\ldots,|\mathcal{P}|}$ *is a permutation of the indices in* $\mathcal{P} = \{n \geq 1 : b_n > 0\}$ *such that*

$$\beta_{i_n} := \frac{4\alpha b_{i_n}}{i_n(i_n+1)(i_n-1)} \quad n = 1, 2, \ldots, |\mathcal{P}|$$

is a non-increasing sequence. Moreover,

$$N = \min\{n \geq 1 : \beta_{i_{n+1}} \leq \mu_{i_n} \leq \beta_{i_n}\},$$

where

$$\mu_{i_n} = 2\frac{\sum_{m=1}^{n}(i_m(i_m-1)/2\alpha)b_{i_m} - 1}{\sum_{m=1}^{n}(i_m(i_m-1)/2\alpha)^2(i_m+1)}.$$

Theorem 6.11. *Let $f \in \mathcal{S}$ be a function of the form*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D},$$

and assume that

$$\lim_{n\to\infty} \frac{b_n}{n^3} = 0.$$

If $\sum_{n=1}^{\infty} n(n-1)|b_n|/2\alpha \le 1$ then $dist(z/f, \mathcal{P}_{2\alpha}) = 0$ and the minimum is attained for the function $g = f \in \mathcal{P}_{2\alpha}$.
If $\sum_{n=1}^{\infty} n(n-1)|b_n|/2\alpha > 1$ then

$$d_\alpha(f, \mathcal{P}_{2\alpha}) = \left(\pi \sum_{n\in\mathcal{I}^c} \frac{|b_n|^2}{n+1} + \pi \frac{\left(\sum_{n\in\mathcal{I}}(n(n-1)|b_n|/2\alpha) - 1\right)^2}{\sum_{n\in\mathcal{I}}(n(n-1)/2\alpha)^2(n+1)}\right)^{1/2}$$

where $\mathcal{I}$ is as given in Theorem 6.10. The minimum value of $dist(z/f, \mathcal{P}_{2\alpha})$ is attained for the function $z/g(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, $g \in \mathcal{P}_{2\alpha}$, where

$$c_n = \begin{cases} \left(|b_n| - \dfrac{n(n-1)(n+1)}{2\alpha} \dfrac{\sum_{m\in\mathcal{I}}(m(m-1)/(2\alpha))|b_m| - 1}{\sum_{m\in\mathcal{I}}(m(m-1)/2\alpha)^2(m+1)}\right)e^{i\arg b_n}, & n \in \mathcal{I}; \\ 0, & n \in \mathcal{I}^c. \end{cases}$$

Theorem 6.12. *For $f \in \mathcal{S}$, $dist(z/f, \mathcal{P}_{2\alpha}) = 0$ iff $f \in \mathcal{P}_{2\alpha}$.*

The proof being similar to the proof of the characterization of the class $\mathcal{F}_\alpha$ (Theorem 6.3) and so we omit the proofs.

CHAPTER 7

CONCLUSION AND FUTURE DIRECTIONS

Chapter 1 gives a brief introduction to the basic definitions and some concepts of the theory of analytic-univalent functions. In Chapter 2 and Chapter 3, we are particularly interested to solve area problems. In particular, we study in Chapter 2 about the Yamashita extremal problem for $f \in \mathcal{S}$ with quasiconformal extension to the whole complex plane. Area problems for functions of type $(z/f)^{\mu}$, $\mu > 0$ for f belonging to some subclasses of $\mathcal{S}$ (eg. $\mathcal{S}^*$, $\mathcal{S}^*(\alpha)$, $\mathcal{C}$, $\mathcal{S}^*(A, B)$) are considered in Chapter 3. For $f \in \mathcal{S}$, the area problem of type $(z/f)^{\mu}$ is solved in this thesis only for $\mu = 1/2$ and for the remaining positive values of μ (i.e. for $0 < \mu \neq 1/2$), this problem is still open. Also, one can think to estimate areas of images of $\mathbb{D}_r$ under non-vanishing analytic functions of the form $(z/f)^{\mu}$ when f is in the class $\mathcal{S}$ with quasiconformal extension to the whole complex plane.

The problem of finding bounds for successive coefficients i.e. $\big||a_{n+1}| - |a_n|\big|$, proposed by Goluzin [23] with an idea to solve the Bieberbach conjecture, is still open for the class $\mathcal{S}$. We discussed this problem in Chapter 4 for some subclasses of $\mathcal{S}$, in particular for the class of γ-spirallike functions of order α, the classes of starlike and convex functions of order α, and other related classes of functions, but sharpness of these results remain open. It would be interesting to see improved version of our results in which the upper bounds are depending upon an absolute constant M.

The sufficient conditions for functions to be univalent, starlike, close-to-convex and meromorphically convex functions, in terms of Schwarzian derivatives of the form $|S_h(z)| \leq 2C_1$ are considered in Chapter 5. One of our main results on this deals with the sharpness of a sufficient condition for meromorphic convex functions of order α. Next, the sufficient conditions for functions in a family that are convex in one direction, in particular the starlike and close-to-convex family, are not sharp. So, further work in this direction is possible to prove the sharpness. Also, one can find the necessary conditions for these subclasses when the absolute value of Schwarzian derivative is bounded by some constant.

Main purpose of Chapter 6 is to find a best approximation of non-vanishing analytic functions of the form z/f in a parabolic region using the technique of semi-infinite quadratic programming problems. We have seen that a sufficient condition for functions to be in that family played an important role. It would be quite interesting to make investigations on approximation problems for subfamilies which could be generated out of sufficient conditions of other type (not necessarily in terms of Taylor's coefficients) for several classical families of analytic univalent functions. Such investigations may lead to introduction of new techniques in the literature.

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